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SOME BASIC FACTS ABOUT BICATEGORIES AND MODULES

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Abstract.

The notion of "collage of a category" is used to characterize bicategories of modules over a distributive bicategory. The characterization is obtained using the fact that the construction of modules is idempotent.

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Introduction

In the development of enriched category theory there has been a recent shift of emphasis from categories enriched over monoidal closed categories to categories enriched over bicategories.

This begun with a variety of examples, from automata theory ([4]), sheaf theory ([3], [14], [15]), geometry ([7]) and fibred categories ([2], [12]).

These examples do not fit in the framework of the classical enriched category theory.

They also show the relevance and motivate the study of $\mathcal{B}$-*mod*, the bicategory of $\mathcal{B}$-categories and modules.

A most important achievement in this direction is the theorem of Ross Street characterizing bicategories of the form $\mathcal{W}$-*mod* (for small $\mathcal{W}$) in terms of a correct notion of "cocomplete" bicategory (see [13]).

We believe that it is important to develop the tendency of considering relevant those properties which lift from base bicategories to bicategories of modules and that a conceptualization of this process is in order. Work has been done, for example, on the lifting of a tensor product on a bicategory ([6]).

Here we want to introduce another lifting which has antecedents both in Topos theory and in enriched category theory, namely the *change of base*. This notion encompasses geometric morphisms ([3], [7]) and the monoidal functors with a right adjoint as described in [8].

We define an analogue of the Kan-Yoneda formula (see [9]) expressing the contradiction between *small* and *cocomplete* bicategories.

The change of base is one of the two main themes which are developed in the present paper. The second theme, *idempotence*, is of a somewhat more conceptual nature.

It is well known in Topos theory, e.g. that the process of forming the sheaves on a site is idempotent. This process has been put in the framework of enriched category theory in [15], by considering the bicategory of relations on a site.

The further abstraction we are considering here is the idempotence of the construction of modules, i.e. technically that $(\mathcal{B}\text{-}mod)mod$ is biequivalent to $\mathcal{B}\text{-}mod$. Enriched category theory fits into general bicategory theory as follows: the characteristic property of the bicategories $\mathcal{B}$ of the form $\mathcal{W}\text{-}mod$ is that "categories can be made into objects" and this is our construction of the *collage of a category*.

Formally, it can be expressed by: $\mathcal{B}$ is biequivalent to $\mathcal{B}\text{-}mod$.

As often happens, a relevant concept gives rise to effective techniques, so that idempotence theorem leads us (in Sections 2 and 3) towards an alternative proof of Street's characterization theorem and related results, viewed in a somewhat different light. Idempotence itself has not been considered by Street, whose main interest was to produce a characterization of $\mathcal{W}\text{-}mod$ for *small* $\mathcal{W}$.

As it has been pointed out also by Lawvere, idempotence provides a very strong "abstract" motivation (as opposite to the concrete motivations of the examples listed above) for the study of categories enriched over a bicategory.

In fact, it is impossible even to state idempotence theorem in the framework of the "one object" case of monoidal categories (*V-mod* is already a bicategory).

The authors gratefully acknowledge discussions with Bill Lawvere (both in Sydney and in Milan) and with Ross Street and other members of the Sydney Category Seminar.

§1 Terminology and notations

In this section we recall briefly some standard notions of bicategory theory and establish our notation.

We assume the reader is acquainted with the notions of *bicategory*, *morphisms* and *homomorphisms* of bicategories, *(strong) transformations* and *optransformations* of morphisms, *modifications*, and *equivalent objects* in a bicategory, as defined e.g. in [1] and [11]. Following [11], we call $Hom(\mathcal{B},\mathcal{C})$ the bicategories of homomorphisms (strong transformations and modifications) between two bicategories $\mathcal{B}$ and $\mathcal{C}$ and $\mathcal{B}icat(\mathcal{B},\mathcal{C})$ the bicategory of morphisms (transformations and modifications) between them.

A *distributive bicategory* $\mathcal{B}$ is a locally cocomplete bicategory (i.e. $\mathcal{B}(u,v)$ is small cocomplete for each pair u,v of objects of $\mathcal{B}$) with colimits preserved by composition on both sides.

In the present paper bicategories are assumed to be distributive (unless otherwise stated). The "right" homomorphisms between two distributive bicategories are the *local colimit preserving* ones, which we will denote by $Hom_{\ell c}(\mathcal{B},\mathcal{C})$ when it is necessary to avoid ambiguity.

As for the notions of $\mathcal{B}$-*categories* (i.e. categories enriched over $\mathcal{B}$), $\mathcal{B}$-*functors* and $\mathcal{B}$-*modules* we refer to [15], [12] and [2], limiting ourselves here to recalling some terminology and indicating some notation.

The conventions adopted in the papers mentioned above do not always agree: we will follow here the conventions of [15]. If X is a $\mathcal{B}$-category and x is an object of X, the "underlying object" of x is denoted by ex and the hom between x and y (y in X) by $X(x,y): ex \to ey$.

A $\mathcal{B}$-module $\phi\colon X \nrightarrow Y$, evaluated at x in X and y in Y, is denoted $\phi(x,y)$ and our convention is that ϕ is contravariant in x and covariant in y.

The bicategory of $\mathcal{B}$-categories and modules is denoted by $\mathcal{B}$-*mod*.

A module with right adjoint is called a *map* (as is any arrow with right adjoint in a bicategory); the bicategory of $\mathcal{B}$-categories and *maps* is denoted by $\mathcal{B}$-*map*. Let us recall also the embedding $(\hat{}): \mathcal{B} \hookrightarrow \mathcal{B}$-*mod* which assigns to the object u the $\mathcal{B}$-category $\hat{u}$ with only one object (say u agiin) and with $\hat{u}(u,u) = 1_u$ and to each arrow $\alpha\colon u \to v$ the module $\hat{\alpha}(u,v) = \alpha$. Often we do not distinguish notationally between u and $\hat{u}$.

Finally, let us mention the fact that we will use, quite informally, the "bar" notation, usually used for denoting bijections, here to denote also equivalences (ee, e.g. the proof of Proposition 2.9) and notice a clash of conventions which is difficult to avoid: dealing with maps, f_* denotes the *left* adjoint to f^* (as established by Lawvere in his paper on Metric Spaces) whereas in Proposition 4.4, F^* has a right biadjoint F_*, as with geometric morphisms.

§2. Idempotence and the characterization theorem

Let $\mathcal{B}$ be a distributive bicategory. A *collage* of a $\mathcal{B}$-category X is an object v of $\mathcal{B}$ together with a module $\iota : X \nrightarrow \hat{v}$ which induces an equivalence of categories

$$\mathcal{B}(v,w) \simeq \mathcal{B}\text{-}mod(X,\hat{w})$$

for all objects w of $\mathcal{B}$.

Lemma 2.1. *An object v of a bicategory $\mathcal{B}$ is the collage of a $\mathcal{B}$-category X if and only if $\hat{v} \simeq X$ in $\mathcal{B}$-mod*

Proof. Suppose v is the collage of X. We want to construct a module $\rho : \hat{v} \nrightarrow X$ right adjoint to ι. By the equivalence above, we have a family of components corresponding to the representables

$$\frac{X(-,x) : X \nrightarrow \hat{ex}, \quad x \in X}{\rho(x) : v \to ex}$$

and there is an induced action of X on $\rho(x)$, $x \in X$, exhibiting ρ as a module with $\rho.\iota \cong 1_X$. This is the condition on the unit of the adjunction to be an equivalence. As for the counit, notice that $\iota\rho\iota \cong \iota 1_X \cong \iota$ and that this induces an isomorphism $\iota\rho \cong 1_{\hat{v}}$. The converse is obvious. □

Notice that we have shown that the module ι in the definition of collage is actually a *map*.

Proposition 2.2. *If a distributive bicategory $\mathcal{B}$ admits collages of small categories, then the embedding homomorphism $(\hat{}) : \mathcal{B} \hookrightarrow \mathcal{B}\text{-}mod$ is a biequivalence.*

Proof. The embedding homomorphism $(\hat{\ })$ is locally an equivalence of categories and for each X in $\mathcal{B}$-*mod*, Lemma 2.1 guarantees the existence of an equivalent object in the image of $(\hat{\ })$, namely the collage of X. □

The left biadjoint of $(\hat{\ })$ in the equivalence above will be denoted by $\mathrm{Coll}(\ ): \mathcal{B}\text{-}mod \to \mathcal{B}$.

Proposition 2.3. *The bicategory* $\mathcal{B}$-*mod admits collages of small categories.*

Proof. Given a $\mathcal{B}$-*mod* category X, we construct a $\mathcal{B}$-category $W = \mathrm{Coll}(X)$ as follows: the set of objects of W is the disjoint union $\coprod_{x \in X} ex$ of the $\mathcal{B}$-categories ex and, if $\alpha \in ex$, $\beta \in ey$, $W(\alpha,\beta) = X(x,y)(\alpha,\beta)$. The $\mathcal{B}$-*mod* module $\iota: X \nrightarrow \hat{W} = \mathrm{Coll}(X)$ has components $\iota_X: ex \to W$, all $x \in X$, which are $\mathcal{B}$-modules induced by the inclusion functors $ex \hookrightarrow W$. To obtain the desired equivalence, observe first that $\mathcal{B}\text{-}mod(W,V) \simeq (\mathcal{B}\text{-}mod)mod(\hat{W},\hat{V})$ and that the functor $(\ ).\iota: (\mathcal{B}\text{-}mod)mod(\hat{W},\hat{V}) \to (\mathcal{B}\text{-}mod)mod(X,\hat{V})$ is fully faithful. We have just to show that for each $\phi: X \nrightarrow \hat{V}$ there exists a $\hat{\psi}: \hat{W} \nrightarrow \hat{V}$ such that $\hat{\psi}.\iota \cong \phi$. Define ψ by $\psi(\alpha,\delta) = \phi_X(\alpha,\delta)$, where $\alpha \in ex$ and $\delta \in V$. Then

$$\begin{aligned}
\psi(\alpha,\delta).\iota_X &= \int^{\alpha' \in W} \psi(\alpha',\delta).\iota_X(\alpha,\alpha') \\
&= \int^{\alpha' \in W} \psi(\alpha',\delta).W(\alpha,\alpha') \\
\text{(Yoneda)} \quad &\cong \psi(\alpha,\delta) \\
&= \phi_X(\alpha,\delta) \quad . \qquad \square
\end{aligned}$$

Remark 2.4. A $\mathcal{B}$-category X can be thought of as a $\mathcal{B}$-*mod* category. The collage of X is then again X seen as an object of $\mathcal{B}$-*mod*.

We are now able to produce some of the most important results of bicategory theory as straightforward corollaries.

Corollary 2.5. (Idempotence theorem) *The bicategory* ($\mathcal{B}$-*mod*)*mod* *is biequivalent to* $\mathcal{B}$-*mod*. □

Corollary 2.6. (Characterization theorem) *A bicategory* $\mathcal{B}$ *is biequivalent to* $\mathcal{W}$-*mod, for some bicategory* $\mathcal{W}$, *iff it admits collages of small categories.* □

Another characterization theorem is given by the following further

Corollary 2.7. *A bicategory* $\mathcal{B}$ *is biequivalent to* $\mathcal{W}$-*mod, for some bicategory* $\mathcal{W}$, *iff the embedding homomorphism* $(\hat{\ }): \mathcal{B} \hookrightarrow \mathcal{B}$-*mod* *is a biequivalence.* □

Defintion 2.8. (Street [13]) Given bicategories $\mathcal{D}$ and $\mathcal{B}$ (not necessarily distributive) and a morphism $T: \mathcal{D} \to \mathcal{B}$, a *collage* for T is an object t of $\mathcal{B}$ together with an optransformation $\tau: T \to \Delta t$ (the constant morphism $\Delta: \mathcal{D} \to \mathcal{B}$ with $\Delta t(\alpha) = t$, for all objects d of $\mathcal{D}$) including an equivalence of categories

$$\mathcal{B}(t,w) \simeq \mathcal{B}icat(\mathcal{D}^{op},\mathcal{B}^{op})(T,\Delta w).$$

It is obvious that the *collage of a* $\mathcal{B}$-*category* X, as defined at the beginning of this section, is a particular case of *collage for a morphism*

of bicategories (take the suitable morphism from the chaotic category on the set X into $\mathcal{B}$). We want to show now that *the two notions are in fact equivalent.*

Proposition 2.9. *A distributive bicategory* $\mathcal{B}$ *admits collages of small categories iff it admits collages for morphisms of bicategories with small domain, (in the sense of the definition above).*

Proof. Given a morphism $T: \mathcal{D} \to \mathcal{B}$, we will construct a $\mathcal{B}$-category $\tilde{T}$ such that

$$\mathcal{B}\text{-}mod(\tilde{T},\hat{W}) \;=\; \mathcal{B}icat(\mathcal{D}^{op},\mathcal{B}^{op})(T,\,W).$$

The $\mathcal{B}$-category $\tilde{T}$ has the same objects as $\mathcal{D}$ and $ed = Td$ (all objects d in $\mathcal{D}$). Further, $\tilde{T}(d,d') = \mathrm{Colim}\, T_{dd'}: \mathcal{D}(d,d') \to \mathcal{B}(Td,Td')$. Notice first that the components $\phi_d: d \to w$ of a module $\phi: \tilde{T} \nrightarrow \hat{W}$ can be thought of as the components τ_d of an optransformation $\tau: T \to \Delta W$. As for the 2-cells involved, observe that those of the action of the module ϕ are in bijection with the 2-cells of the optransformation τ. We have

$$\frac{\phi_{d'}.\tilde{T}(d,d') \to \phi_d}{\phi_{d'}.\mathrm{Colim}\, T_{dd'} \to \phi_d} \quad \text{(by definition)}$$

$$\frac{\phi_{d'}.\mathrm{Colim}\, T_{dd'} \to \phi_d}{\mathrm{Colim}(\phi_{d'}.T_{dd'}) \to \phi_d} \quad \text{(colimits are preserved by composition)}$$

$$\frac{\mathrm{Colim}(\phi_{d'}.T_{dd'}) \to \phi_d}{\phi_{d'}.\alpha \to \phi_d,\ \ \alpha \in T_{dd'}} \quad \text{(cone)}$$

$$\frac{\phi_{d'}.\alpha \to \phi_d,\ \ \alpha \in T_{dd'}}{\tau_{d'}.\alpha \to \tau_d,\ \ \alpha \in T_{dd'}}$$

This calculation proves also that, conversely, any optransformation τ gives rise to a module. Moreover, since 2-cells between modules and

modifications between optransformations correspond bijectively (being both determined by the same family of 2-cells in $\mathcal{B}$ between the components), we obtain the desired equivalence of categories. So, if $\mathcal{B}$ has *collages of small categories*, it holds:

$$\mathcal{B}icat(\mathcal{D}^{op},\mathcal{B}^{op})(T,\Delta w) \simeq \mathcal{B}\text{-}mod(\tilde{T},\hat{w}) \simeq \mathcal{B}(Coll(\tilde{T}),w)$$

i.e. $\mathcal{B}$ has *collages for any morphism* $T\colon \mathcal{D} \to \mathcal{B}$. □

§3 Density

Let $\mathcal{B}$ be a distributive bicategory. A full subbicategory $\mathcal{W}$ of $\mathcal{B}$ is *dense* if every object of $\mathcal{B}$ is the collage of a $\mathcal{W}$-category.

Remark 3.1. Notice that from the definition and from Remark 2.4 it follows that $\mathcal{W}$ *is dense in* $\mathcal{W}$-*mod*.

Remark 3.2. A homomorphism $F: \mathcal{C} \to \mathcal{B}$ induces a homomorphism $F^*: \mathcal{C}\text{-}mod \to \mathcal{B}\text{-}mod$ defined as follows: if X is a $\mathcal{C}$-category, F^*X is the $\mathcal{B}$-category with the same objects as $\mathcal{C}$ and $F^*X(x,y) = F(X(x,y))$. Further, if $\phi: X \nrightarrow Y$ is a $\mathcal{C}$-module, the $\mathcal{B}$-module $F^*\phi: F^*X \nrightarrow F^*Y$ is given by $F^*\phi(x,z) = F(\phi(x,z))$ $(x \in X$ and $z \in Y)$.

Let us denote by $i: \mathcal{W} \hookrightarrow \mathcal{B}$ the inclusion of a full subbicateogry $\mathcal{W}$ in $\mathcal{B}$.

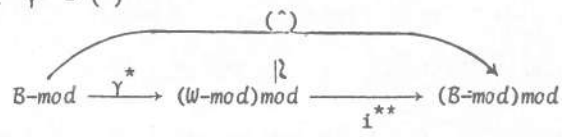
Proposition 3.3. *A full subbicategory* $\mathcal{W}$ *of* $\mathcal{B}$ *is dense iff the embedding* $\hat{(\ )}: \mathcal{B} \hookrightarrow \mathcal{B}\text{-}mod$ *factors through* $i^*: \mathcal{W}\text{-}mod \to \mathcal{B}\text{-}mod$ *(up to an equivalence).*

Proof. Let us construct a homomorphism $\gamma: \mathcal{B} \to \mathcal{W}\text{-}mod$ which assigns to each b in $\mathcal{B}$ a chosen $\mathcal{W}$-category X such that $Coll(X) = b$ and to each arrow $\alpha: b \to b'$ the module $\bar{\alpha}: X \nrightarrow Y$ (Y is chosen such that $Coll(Y) = b'$) obtained from $\hat{\alpha}: \hat{b} \nrightarrow \hat{b}'$ and the equivalences $\hat{b} \underset{\phi}{\overset{\psi}{\leftrightarrows}} X$, $\hat{b}' \underset{\phi'}{\overset{\psi'}{\leftrightarrows}} Y$ as $\bar{\alpha} = \phi'\hat{\alpha}\psi$. Then $i^*\gamma = \hat{(\ )}$is in $Hom(\mathcal{B}, \mathcal{B}\text{-}mod)$. The converse holds by Lemma 2.1. □

Proposition 3.4. *A full subbicategory $\mathcal{W}$ of $\mathcal{B}$ is dense iff the inclusion homomorphism $i^*: \mathcal{W}\text{-}mod \to \mathcal{B}\text{-}mod$ is a biequivalence of bicategories.*

Proof. Suppose $\mathcal{W}$ is dense in $\mathcal{B}$. The homomorphism i^* induces, for each pair of $\mathcal{W}$-categories X, X', an obvious isomorphism of categories $\mathcal{W}\text{-}mod(X,X') \simeq \mathcal{B}\text{-}mod(X,X')$. To show that every object of $\mathcal{B}\text{-}mod$ is equivalent to an object of $\mathcal{W}\text{-}mod$, notice first that an equivalence between two homomorphisms F and G in $\mathrm{Hom}(\mathcal{W},\mathcal{B})$ lifts to an equivalence between the induced homomorphisms F^*, G^* in $\mathrm{Hom}(\mathcal{W}\text{-}mod,\mathcal{B}\text{-}mod)$.

Hence the equivalence $i^*\gamma \simeq (\hat{\ })$ of Proposition 3.3 lifts to an equivalence $i^{**}\gamma^* \simeq (\hat{\ })$



Therefore, for any object X of $\mathcal{B}\text{-}mod$ we have $\hat{X} \simeq \gamma^*X$ and, since homomorphisms preserve equivalences, $X \simeq \mathrm{Coll}(\gamma^*X)$. But $\mathcal{W}\text{-}mod$ has collages, hence $\mathrm{Coll}(\gamma^*X)$ is an object of $\mathcal{W}\text{-}mod$.

Conversely, if i^* is a biequivalence, for every object of $\mathcal{B}\text{-}mod$ of the form $\hat{b}$ (where b is an object of $\mathcal{B}$) there exists a $\mathcal{W}$-category Y such that $\hat{b} \simeq Y$ in $\mathcal{B}\text{-}mod$. By Lemma 2.1 this is equivalent to say that b is the collage of Y. □

Notice, as a consequence, that $\mathcal{B}\text{-}map \simeq \mathcal{W}\text{-}map$.

We are now able to produce a "size sensitive" version of the characterization theorem (see [13]).

Theorem. (Street). *A bicategory $\mathcal{B}$ is biequivalent to $\mathcal{W}\text{-}mod$, for a* small *bicategory $\mathcal{W}$, iff $\mathcal{B}$ admits collages and there exists a* small *dense subbicategory of $\mathcal{B}$.*

Proof. Let $\mathcal{B}$ have collages and a dense subbicategory $\mathcal{W}$. Then we have by Proposition 3.4 and Proposition 2.2 that $\mathrm{Coll}(\ ).i^*: \mathcal{W}\text{-}mod \to \mathcal{B}$ is a biequivalence.

Conversely, if $\mathcal{B}$ is biequivalent to $\mathcal{W}\text{-}mod$ (for a small $\mathcal{W}$) just recall that $\mathcal{W}\text{-}mod$ has collages (Proposition 2.3) and that $\mathcal{W}$ is dense in $\mathcal{W}\text{-}mod$ (Remark 3.1). □

Another description of density may be given in terms of a suitable notion of *cover*.

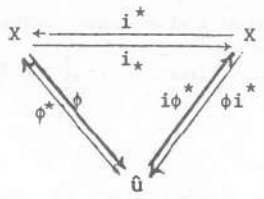
Let $\mathcal{B}$ be a bicategory, u an object of $\mathcal{B}$, $\mathcal{W}$ a full subbicategory of $\mathcal{B}$. Further, let X be a $\mathcal{W}$-category and $\phi: X \nrightarrow \hat{u}$ a map in $\mathcal{B}\text{-}mod$. We have the following:

Definition 3.5. *The pair* (X,ϕ) *is called a* $\mathcal{W}$*-cover of* u *if the counit* $\phi\phi^* \Rightarrow 1_{\hat{u}}$ *is invertible.*

Proposition 3.6. *A full subbicategory* $\mathcal{W}$ *of* $\mathcal{B}$ *is dense iff every object of* $\mathcal{B}$ *has a* $\mathcal{W}$*-cover.*

Proof. Suppose u is an object of $\mathcal{B}$ covered by the $\mathcal{W}$-category X. Let us construct a new $\mathcal{W}$-category X' with the same objects as X and with $X'(x,x') = \phi^*(x').\phi(x)$. We want to show that u is the collage of X', i.e. by Lemma 2.1, that $\hat{u} \simeq X'$.

There is a $\mathcal{B}$-functor $i: X \to X'$ which is the identity on the objects whose effect is provided by the unit $1_X \Rightarrow \phi^*\phi$. Hence we have a pair of adjoint bimodules $i_*(x,y) = X'(x,x)$ and $i^*(y,x) = X'(y,x)$:



We claim that the composite $i_*\phi^*$ (together with ϕi^*) is an equivalence in $\mathcal{B}$-*mod*. It is easily seen that $(i_*\phi^*)(\phi i^*) = 1_{X'}$, (i.e. that $i_*\phi^*(u).\phi i^*(x) \cong X'(x,y)$) just by checking that $\phi i^*(x) \cong \phi(x)$ and $i_*\phi^*(y) \cong \phi^*(y)$. To show that $\phi i^* i_*\phi^*(u) \cong 1_{\hat{u}}$ (where u denotes the one object of $\hat{u}$), observe first that

$$i^*i_*(x,y) \cong \int^{x'\in X'} i^*(x',y).i_*(x,x')$$

$$\cong \int^{x'\in X'} X'(x',y).i_*(x,x')$$

(Yoneda) $$\cong i_*(x,y) = X'(x,y) = \phi^*(y)\phi(x).$$

Hence

$$i^*i_* \cong \phi^*\phi$$

and

$$\phi i^* i_*\phi^*(u) = \phi\phi^*\phi\phi^*(u) = 1_{\hat{u}}.$$

The converse is obvious. □

§4 Change of base

We have already described in Remark 3.2 how a homomorphism of distributive bicategories $F: \mathcal{B} \to \mathcal{C}$ induces a homomorphism $F^*: \mathcal{B}\text{-}mod \to \mathcal{C}\text{-}mod$. (In fact, the construction works also for *morphisms*). Notice also that we have implicitly defined a restriction of F^*_1 as a homomorphism from $\mathcal{B}\text{-}map$ to $\mathcal{C}\text{-}map$.

Definition 4.1. *The homomorphism* F *is said to be* collage preserving *if whenever* $\iota: X \nrightarrow \hat{u}$ *exhibits* u *as a collage of* X, *then* $F^*\iota$ *exhibits* Fu *as a collage of* F^*X.

Proposition 4.2. *If the distributive bicategories* $\mathcal{B}$ *and* $\mathcal{C}$ *admit collages and the homomorphism* $F: \mathcal{B} \to \mathcal{C}$ *preserves local colimits, then* F *preserves collages.*

Proof. From the following commutative diagram

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{F} & \mathcal{C} \\ {\scriptstyle (\hat{})}\downarrow & & \downarrow{\scriptstyle (\hat{})} \\ \mathcal{B}\text{-}mod & \xrightarrow[F^*]{} & \mathcal{C}\text{-}mod \end{array}$$

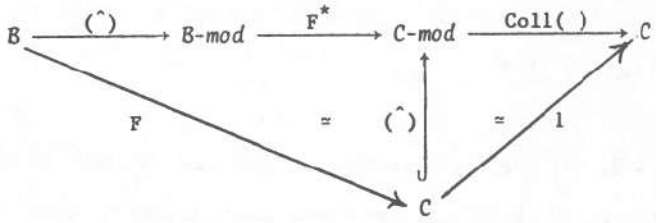
since the embeddings $(\hat{})$ are equivalences, we obtain the following equivalence of homomorphisms

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{F} & \mathcal{C} \\ {\scriptstyle \mathrm{Coll}(\ )}\uparrow & = & \uparrow{\scriptstyle \mathrm{Coll}(\ )} \\ \mathcal{B}\text{-}mod & \xrightarrow[F^*]{} & \mathcal{C}\text{-}mod \end{array}$$

which is just to say that F preserves collages. □

Proposition 4.3. *If the bicategory* C *admits collages, the bicategory* $Hom_{\ell c}(B,C)$ *is biequivalent to* $Hom_{\ell c}(B\text{-}mod,C)$.

Proof. The homomorphism $(\hat{\ })\cdot(-)\colon Hom_{\ell c}(B\text{-}mod,C) \to Hom_{\ell c}(B,C)$ is locally an equivalence of categories and for each homomorphism F there is an equivalent object in the image of $(\hat{\ })\cdot(-)$, as it is shown in the following diagram



where the equivalence in the left-hand triangle is that of Proposition 4.2. □

Theorem 4.4. *If* $F\colon B \to C$ *is a homomorphism of distributive bicategories and* B *is small, then* $F^*\colon B\text{-}map \to C\text{-}map$ *has a right biadjoint* F_* .

We do not give the proof of this theorem, which involves rather sophisticated technical details and will be the content of a subsequent paper.

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