Towards a Hegelian philosophy of mathematics

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> There is at present no intelligible account of what the statements of pure mathematics are about. The philosophy of mathematics is in a mess!
>
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1 Introduction

The objective of this paper is to substantiate the claim that the well-known difficulties in modern mathematical logic require a Hegelian grounding for their resolution and that, in the terminology of Hegelian logic,[3] this grounding is *posited by the difficulties themselves.* To establish this, we need first to describe modern mathematical logic. A brief discussion of the historical background to this logic is given in the second section. The consummate concept of this logic is that of the *formal system*, in terms of which all modern foundational systems for mathematics receive their precise expression. Formal systems are discussed in the third section. The fourth section examines the remarkable positive achievements of the formal system approach to mathematical logic. This is followed by a section discussing the inadequacies of this approach in accounting satisfactorily for modern mathematics. It will then be argued in the final section that these inadequacies "sublate" themselves in a modern Hegelian logic in which the notions of classical Hegelian logic appear transformed in the present day context. A brief sketch of some of the ideas of this modern Hegelian logic is given in the final section.

2 Background on the foundations of mathematics

The problem of the nature of mathematics goes back to antiquity and became pressing at the various crisis periods in its history. The main such crises involved the discovery of irrational lengths in Greek geometry, the problem of infinitesimals in the Calculus and the set theoretic paradoxes at the end of the last century. In each such crisis, the normal intuitive approach of the working mathematician was overtaken by rigor, as it were the mathematical version of Puritanism. Such rigor in the case of Greek geometry led to the profoundly influential *Elements* of Euclid, in which mathematics is geometrically based at the expense of number.

The gulf between geometry and number was bridged by Fermat and Descartes in their development of analytic geometry in which for the first time, the power of what we now call *College algebra* was used to establish with relative ease the theorems of geometry. The notion of function arose out of analytic geometry and provided the setting for the *calculus*, which in its basic form, is about differentiating and integrating *functions*. This was the birth of modern *analysis*, and since we can write down formulas for functions whose graphs cannot be drawn (ones, for example, whose graphs have no tangents anywhere), geometrical intuition is no longer adequate as a basis for mathematics.

Since for a function f to be given, we need to think of the real numbers x which appear in the former's evaluations, $f(x)$, as ranging over the *set* of real numbers, this led to mathematics being based on *set theory*, the requisite theory being provided by Cantor's brilliant (and controversial) work on infinite sets. Thus for the foundations of mathematics, set theoretic intuition came to replace the previous geometric intuition, the geometrical notion of the real line[4] as a pencil-drawn line of finite length extended indefinitely in imagination being replaced by the notion of an infinite set of things (numbers). This intuition was then coupled with the logic needed for mathematical proof, and the program remained to ground all of mathematics on the logic of the statements of mathematics, these statements being about sets. At the end of the 19th century, this program seemed well on the way to fulfillment as Cantor's set theory was incorporated into the context of the logic of mathematical statements in the foundational work of Frege.

As is well-known, Frege's program ran into serious difficulties through

the occurence of the puzzling logical paradoxes of set theory - in this context, *paradox* is a euphemism for *contradiction*. One of these – the Russell-Zermelo paradox – destroyed the credibility of the Fregean system as a foundation for mathematics. It asks if the set of sets which are not members of themselves is itself a member of itself – either it is or it isn't, and in either case, a contradiction follows.

The foundational systems of this century arose as proposed solutions to the logical paradoxes. These systems are formulated in terms of *formal systems* in which formal logic achieves its most precise form, the earlier logical systems of Frege and of Russell and Whitehead's *Principia Mathematica* being "sloppy" by comparison. Since the Hegelian-inspired approach to logic advocated here will be motivated by the problems associated with formal systems, we now look more closely at the notion of such a system.

3 Mathematics in terms of formal systems

The exact definition[5] of a *formal system* is complex and we will describe only the absolute minimum needed for the purposes of this paper. The oversimplifications involved will not materially affect the validity of the discussion.

In a formal system F, there is (in Hegelian language) posited symbols x_i, standing for variables, where i ranges over the natural numbers regarded not as a "completed infinite". (So no controversial set theory is presupposed.) There are also posited symbols, indexed by natural numbers, standing for constants, functions and predicates and certain basic signs such as parentheses, and the ones commonly used for the logical connectives such as "not" ($\sim$), "or" ($\vee$) and "if...then" ($\supset$). The *terms* of F are built up by applying functions to previously defined terms, the basic terms being the variables and constants. The system F also allows for quantification, the two standard quantifiers being that of the *universal* ("for all") and the *existential* ("there exists").

The *well-formed formulae*, abbreviated to wff's, are built up recursively, mimicking the way that we form and combine sentences in normal language. Thus, for example, a subject-predicate sentence would be represented by $A(t)$ (t has property A) where A is a predicate symbol and t is a term.

There are then posited certain wff's called *axioms* together with *rules of inference* which "permit" us to write down new wff's given that previous

ones have been written down. The axiom set includes all tautologies. *Modus Ponens* is an example of a rule of inference : if A and $A \supset B$ have occurred earlier, then we can write down B. (That is, intuitively, given propositions A and B, if A is true and we know that if A is true then so is B, then (of course) we can also assert the truth of B.) A *formal deduction* is a finite sequence of wff's such that every element of the sequence is either an axiom or is a consequence of previous members of the sequence using a rule of inference. A *theorem* is the last wff in a formal deduction.

In the context of the formal system, we thus know exactly what is to count as a proof or not. Two important points need to be made here. Firstly, we have not formalized what the formal system *is about*. For example, in a set theoretic foundation (such as the Zermelo-Fraenkel formal system (abbreviated to **ZF**)) it doesn't say anything about the sets being studied. Indeed it can be studied without knowing that it is relevant to sets at all. This limitation of the formal approach is dealt with in the (very important) branch of mathematical logic called *Model Theory*. (We shall refer very briefly to models in §5 when we discuss the Löwenheim-Skolem theorem.)

Secondly, what one can say about the theorems of a formal system are, like informal mathematics, expressed in terms of the sentences of an ordinary language (such as English). Such a mathematical investigation is called *metamathematics*.[6] Of course, metamathematics, as a branch of mathematics, is itself formalizable.

Of particular importance in metamathematics is that of *consistency*. A formal system is called *consistent* if there does not exist a wff A such that both A and its negation $\sim A$ are theorems. An inconsistent system is useless for foundations since it is contradictory, i.e. every wff and its negation is a theorem! Frege's system, alluded to earlier, is inconsistent.

A number of competing foundational systems for mathematics have been developed in terms of formal systems. At its simplest level, we have the most basic formal system of the *statement calculus* reflecting the simpler logic of ordinary language. This simple system is imbedded in the *predicate calculi* in which quantification (corresponding to the use of *for all* and *there exists*) is allowed. In terms of basic mathematics, we have the system **S** which develops elementary arithmetic based on a version of the Peano axioms. The most popular formal system for set theory among mathematicians is the system **ZF** alluded to above. More on the lines of the type theory which

Russell and Whitehead developed in *Principia Mathematica* is that of **ST** (which stands for *simplified types*). Closest to the original Frege system is the system **NF** of Quine. A more recent approach is that of using category theory and topos theory as a foundation. In this approach, a function is regarded as an object rather than (as in the set theory approach) a set through its graph.

4 Positive aspects of the formal system approach to mathematics

A great strength of the formal systems approach to mathematics is that it *makes precise vague intuitions*. As W. S. Hatcher points out,[7] without this precision

> discussions about Foundations easily degenerate into quasiphilosophic and obscure debates in which no one really knows quite what the issue is.

Hatcher illustrates the merit of the formal approach by showing how it clarifies the philosophic debate between Intuitionistic arithmetic and ordinary arithmetic.

The Intuitionistic approach to mathematics was advocated by Brouwer, and gave rise to polemic between him and Hilbert. Brouwer's approach starts with the two forms of pure intuition (the outer spatial and the inner temporal) of Kant, and rejects the outer on the grounds that it leads to the a priori necessity of Euclidean Geometry which he considered as undermined by the existence of non-Euclidean geometries. The inner form of intuition was still, for him, valid a priori, and consonant with the Kantian emphasis on concepts as having content only when they operate in a manifold synthesis, the asserting of a mathematical truth is an abbreviation for : *I have completed a certain mathematical construction.*[8] This leads to a view of mathematics which rejects infinite sets as "completed infinities" (since we can't construct them) and also the law of the excluded middle as applying universally. Intuitionistic arithmetic is therefore arithmetic developed in a "constructivist" way. The obvious question then is : given that Intuitionistic arithmetic and ordinary arithmetic are philosophically and technically very different, are they *mathematically* different?

Here the precision of the formal system approach enters and settles the matter. A result of Gödel proves that Intuitionistic arithmetic and ordinary arithmetic *are* the same, have exactly the same theorems, so that at this level, the only difference between the two number theories *is* "philosophic". We could not have known this without the precision of the formal system.

Here lies the great strength of the formal approach. It is a strength which Hegel would have appreciated. In Hegelian terms, what is implicit, "in-itself", *must* become explictly determined (*bestimmt*) and we cannot rest content with imprecision.

This precision is also highlighted in the establishing in mathematical logic of independence theorems, such as the independence of the continuum hypothesis and of the axiom of choice from the rest of the axioms of **ZF**. This, of course, does not settle whether, for example, we should assume the continuum hypothesis or not, but either way, we are assured of not introducing a contradiction (relative to **ZF**).

In this paper, we will argue that the formal approach is ultimately inadequate to account for mathematics. But this in no way should be interpreted as a denigration of what formal mathematics has achieved. Rather, its remarkable honesty and precision leads to the showing up *by itself* of its own inadequacies to whose discussion we now turn. In this discussion, the limitations of this (justly admired) precision will become explicit: the precision is only precise given the *presupposed* adequacy of the formal approach in accounting for mathematics. Once this approach is seen to be inadequate, such precision distorts, rather than elucidates the fundamental philosophical issues involved, and what is needed in its place is precision in the *conceptual* sense to which the former precision is ancillary.

5 The inadequacies of the formal system approach to mathematics

There are five main points in this section: the circularity introduced by the necessity of metamathematics, the incongruity of formal systems with respect to normal mathematics, the different, partial philosophic insights underlying the various formal systems, the inadequacy of formal systems as exhibited in Gödel's incompleteness theorems and finally the absorption of

formal mathematics into normal mathematics.

Starting with the first of these points, suppose that we are studying an important branch of mathematics, let us say, set theory. We want to be sure that it does not contain a contradiction. To determine this, we first formalize it in formal system, for example, **ZF**, the Zermelo-Fraenkel system. To show consistency, we have to prove that in **ZF**, the negation of every wff that is a theorem is itself not a theorem (a metamathematical problem). *But this is itself a mathematical problem, not a problem of formal systems.* That is, to settle such a problem, we have to study the infinite *set* of theorems and show that this set does not contain the negation of one of its members. But to study an infinite set, we have to use the methods of infinite set theory. So we are presupposing the validity of infinite set theory (at least, the theory as it applies to denumerable sets) in order to decide if infinite set theory is consistent! This is circular. While it is true that, as was done in the Hilbert formalist program, we can study this metamathematical problem using the very weak methods of the basic, arithmetic, formal system **S**, which has minimal set theoretic commitment, this can only give consistency *relative to* **S** and the question of the consistency of **S** is itself meaningless in terms of formal systems: to investigate the consistency of elementary arithmetic, one has to use, and hence presuppose, the consistency of elementary arithmetic.[9]

Secondly, a foundational system, to be adequate, has to square with the mathematics whose structure it is supposed to be clarifying. It is here that we come across a troubling incongruity. The inadequacy of a formal system to capture what the mathematician "sees" is illustrated by the Löwenheim-Skolem theorem[10] on countable models. For our purposes, we can take a "model" of a formal system to be a realization of the system as the set logic of a certain collection of subsets of a given set D.

The above theorem shows, in particular, that the most popular foundation for mathematics **ZF** can be realised in the context of a denumerable set, i.e. a set of the "same size" as the set of natural numbers. However, by a theorem of Cantor, the set of real numbers is not denumerable. Rather, it is of a higher order of infinitude than that of the natural numbers. But the set of real numbers is constructed in **ZF** and hence in this model it will itself appear as denumerable, contradicting the Cantor theorem. So **ZF**, taken purely on its own merit, does not capture the mathematician's understanding of the real numbers.

Indeed, this "non-capturing" of what mathematical activity is *about* perhaps explains the revulsion of many mathematicians to the formal view of the subject, mathematicians such as Kronecker, Brouwer, Weil, Poincaré and (more recently) René Thom. Since the formal approach is not capturing what the mathematician "sees", it can miss important mathematical insights, and indeed it can be convincingly argued that the formalizing of the subject and overly concern about "rigor"[11] has actually impeded progress in mathematics.

A good example of this is provided by the most popular mathematics book ever written – the *Elements* of Euclid. This work can be regarded as the forerunner of formal mathematics with its emphasis on axioms and proofs. It is ironic that the brilliant achievement of Euclid effectively blocked the progress of mathematics in the subsequent centuries[12] as it hindered the introduction of algebra into geometry.

Thirdly, looked at more closely, the foundations of mathematics today presents us with a variety of formal systems, each embodying some *reasonable* intuition of what restriction has to be placed on the paradoxical notion of *set*. The type theories, for example, are based on the vicious circle principle, that anything involving all of a collection should not be one of the collection, and the wff's are carefully constructed so as not to violate the principle. On the other hand, **ZF** is based on the notion of smallness, that we are only allowed to use the sets and the set operations that mathematicians use in practice. (So, for example, the "set" used in the Russell-Zermelo paradox (§2) is "not admitted" in **ZF**.) The category approach questions the fundamental role played by *set* and embodies the intuition that a function is more than its graph, that it is an object in its own right.

In each of these systems, some partial philosophical insight into mathematics is achieving determinate expression. But these insights are different – how do we reconcile them? None of them is adequate by itself, and indeed that no formal system can in principle be adequate is the fourth point of this section.

As is well known, the Hilbert program (*supra*) foundered by reason of the theorems of Gödel.[13] It is clearly of the greatest importance that any formal system used in foundations be consistent. Unfortunately no formal system can in principle be proved to be consistent using its own resources (by Gödel's second incompleteness theorem). The objective of a "completely

rigorous" context for mathematics thus fails.

Another consequence of this result is the following. Since in mathematics, we can in principle ask meaningfully and investigate whether or not *any* given system is consistent, it follows that mathematics as an activity is *not* obtained by working within a fixed formal system (implicit or explicit) which lies behind the scenes (as it were) and explains what mathematicians are doing. (For if mathematical activity was controlled by a formal system, we could not meaningfully ask the *mathematical* question of the consistency of that formal system, since, to settle the question, the formal system would be investigating itself and the incompleteness theorem would apply.)

Accordingly, the philosophy of mathematics is, in the words of the introductory quote, *in a mess*. Mathematics continues with brilliant successes more than ever – for example, in its recent solutions of the four color conjecture, the Bieberbach conjecture of complex analysis, the astonishing classification of the finite simple groups and the most famous classical unsolved problem of mathematics, *Fermat's last theorem* in number theory – but despite these, we still don't know what the mathematician is talking about in his work or why the subject works the way that it does.

This sorry situation leads to only one conclusion – it is that the approach to mathematics based on "deducing" it within a formal system *alone* has failed, and does not give an adequate account of the activity of mathematics.

This does not mean, as we commented at the end of §4, that the results of the foundations of mathematics are discredited. On the contrary, theorems such as Gödel's incompleteness theorems are among the truly great results of twentieth century mathematics. And indeed, the fifth point of this section is that that is how the foundations of mathematics now is – it is a *branch* of mathematics and instead of gobbling up mathematics within it (as it were) and setting it out in its own categories, the roles have been reversed and mathematical logic is now just a part of the whole of mathematics.

As the mathematical logician E. Post, commenting on Gödel's work, claimed[14] in 1941 :

> It is to the writer's continuing amazement that ten years after Gödel's remarkable achievement, current views on the nature of mathematics are thereby affected only to the point of seeing the need of many formal systems, instead of a universal one. Rather has it seemed to us to be inevitable that these developments will

> result in a reversal of the entire axiomatic trend of the late nineteenth and early twentieth centuries, with a return to meaning and truth.

That the above "return to meaning and truth" effectively means a return to a modern Hegelian-style conceptual development is the main theme of this paper.

6 Towards a Hegelian philosophy of mathematics

We now describe how the above discussion of the problems with the formal approach to mathematics generates a Hegelian approach to mathematics, one in which the great results of mathematical logic are *aufgehoben*, sublated, their integrity preserved, while at the same time, the rigid separation of mechanical logic and mathematical intuition – that is, the obsessive abstractive operation of *Verstand*, the Understanding – is overcome. The transition will be the making explicit of the conceptual logic (in the Hegelian sense) which operates (usually in hidden form) in the thinking of the working mathematician, whose insights are not destroyed in rigorous proof but rather in that achieve their determination.

As was stressed in the Introduction, such a conceptual logic, to be genuine, has to arise out of the difficulties of mathematics and its logic and not be imposed arbitrarily "from the outside". Accordingly, this section will examine the five inadequacies of the formal approach, discussed in §5, and show how their problems are effectively asserting and providing the transition to such a logic.

Starting with the first inadequacy, we observed that in order to be able to reason precisely about the theorems in a formal system – in particularly, about the latter's consistency – we needed to presuppose the consistency of the basic formal system **S** of elementary arithmetic. This implied that consistency (in the Hilbert program) for a formal system is only *relative* to **S**. This is, of course, a very important observation since the formal systems needed for substantial parts of mathematics are a lot more complicated than **S**. But what does this say about **S**? – it says that *in order to investigate* **S**, *we have to presuppose* **S**.

This circularity is often regarded as "regrettable": but instead of ignoring

it and settling for consistency relative to **S**, we should accept what it asserts about mathematics. What it is effectively asserting is that arithmetic, and more generally, mathematics, belong to the world of the Hegelian Concept, the Concept "that comprehends itself"[15] and "that has itself... for its subject matter"[16] – elementary arithmetic investigates *itself* through *itself*. The key point here is that, more generally, mathematics is to be thought of as an immediacy developing itself. In its *activity*, it is not grounded on something else outside it (eg. a formal system). Instead, the great theorems of the subject are the revealing of what is covered up in that initial immediacy as mediated through proof. In this respect, it is to be compared with Hegel's logical procedure in which he starts from the most immediate, abstract category of Being, which then expresses what it truly is in its self-development, out of its initially hidden depths, through the stages of Essence, Concept and Idea, mediated in logical argument. The formal aspect of the subject is just mathematics itself becoming for-itself in this mediation.

The failure of the modern formal approach is due to the latter's rigid abstraction in terms of a hard division between a supposed world of mathematical "objects", intuitively conceived, from which the logician separates himself and of which he creates symbolic representations, like a painter giving a "literal" representation of a nature scene in his painting. Rather, the Being-in-itself and Being-for-itself of mathematics are indissolubly linked, and when they are rigidly divorced from one another (as in the formal system approach), the mathematical world becomes finitized (in the Hegelian sense), and this gives rise to the second inadequacy noted in §5.

The latter was that, effectively, formal systems *do not capture the full truth of mathematics*. An example of this was provided by the existence of countable models of **ZF**. The uncountable sets that the mathematician uses have not been captured in the formal system. This, then, is further confirmation that the putting to one side of mathematics which is then externally investigated in formal symbolism, is invalid, that rather it is mathematics itself that must investigate itself, that the examination of the nature of mathematics is itself part of the mathematical activity and presupposes its immediacy.

The integrity of mathematics then requires that there is no gulf fixed between "living", intuitive mathematics and its "external" mechanical realisation in a system in which one is confronted with a batch of symbol se-

quences (wff's) checkable by a machine, the meaning having been squeezed out like the juice out of an orange. Instead, the two sides are inseparable – without the intuitive, living side of mathematics, we don't know what we are supposed to be formalizing – and formalizing has (at least implicitly) *meaning* even though it involves (like arithmetic) mechanical checking. And equally, without some formal expression, the intuitive mathematics remains a vague dream. This inseparability must be made explicit, and when we do make it explicit, we are faced with the Hegelian Concept whose moments in their very positing collapse back into each other, point to each other.

The Hegelian Concept, however, does not have just two moments. It has three, those of Universality, Particularity and Singularity (*Allgemeinheit, Besonderheit, Einzelheit*). The unity of the intuitive and the formal, discussed above is initially just this moment of Universality as "indifferent" and abstract, and to become concrete, rational, it must specify itself in the moment of Particularity. Accordingly, we naturally look for signs of this specification in formal mathematical logic, a specification which in latter's "external" medium will have fallen apart into "bits."[17]

The third inadequacy of §5 exhibits this "fallen apart" specification in a strikingly explicit way. There, it was observed that the formal systems that have been developed to give a foundation for mathematics each embodies some *intuition* of what restriction has to be placed on the notion of *set*. The intuitions that inspire these formal systems are what are expressed appropriately in the specification of wff's and axioms for the systems. This variety of system intuitions *is* the particularization of the concept of mathematics as Universality, the former relating to the latter as species to their genus.

In Hegelian logic, the Concept is self-identical in its three moments: roughly speaking, the thinking through of any one of them brings to light the others. Since the intuitions are the concept of mathematics as Particularity, the Concept (in the Hegelian approach) will produce these "intuitions" underlying formal systems as its own and explicitly show how it is self-identical in them. In the process, the relations between these intuitions will be made explicit. In formal logic, by contrast, the intuitions just lie there unconnected.[18]

In this self-identity of the Concept in its particularization (and its other moments) we have the full "genuine infinite" (*das wahrhafte Unendliche*) of the Concept contrasting with the "bad infinite" (*die schlechte Unendlichkeit*),

the latter being the finitized form of the former[19] in which is endlessly posited the same repeating process in the impossible task of "attaining" the genuine infinite. The correct picture of the genuine infinite is, for Hegel, that of a circle that closes back on itself while that of the bad infinite is that of an infinite line which goes on "for ever", or of an infinite sequence (in Hegelian terms, an infinite progress (*Progreß*)) that never stops.

In the externalizing of formal systems, we are ultimately trapped in the bad infinite almost by definition – the genuine infinite cannot be expressed in terms of sequences of meaningless wff's linking onto one another, the allowable linkings being checked mechanically. The conceptual "in-itself" of the system, driving the system, which is a genuine infinite must eventually make explicit the bad infinite of the formal system, the latter's incongruity with its concept.

The explicit exhibiting of this bad infinite – coming out of the methods of mathematical logic itself – is to be found in the fourth inadequacy discussed in §5, that concerning Gödel's second incompleteness theorem. In the latter, a formal system tries to investigate itself in its own terms. This is nothing other than the reflexivity of the genuine infinite of Concept that we saw in the discussion above with the system **S**. The failure of the system to prove its own consistency is really saying that the bad infinite of the formal system is not true (= not genuine in the Hegelian sense) – it cannot reach back on itself to establish that it does not contain a contradiction. The bad infinite shows itself through the theorem inasmuch as we might try to deal with the problem by embedding the orginal formal system F within a larger one F_2 relative to which we could determine the consistency of F. (For example, a result of Gentzen shows that **S** is consistent relative to **ZF**.) But then, of course, this only pushes the consistency issue over to F_2 which in turn requires a larger formal system F_3 to establish its consistency, and this is a bad infinite in the sense of Hegel, the sequence of formal systems never stopping.

Given that the formal approach – with impressive honesty – has thus exposed inescapably its bad infinite and that mathematicians do not normally think in terms of Hegelian logic, the natural way out of this for them is that of a retreat to the genuine infinite as in-itself (implicit) in normal mathematical activity. Hence, as discussed in the fifth inadequacy of §5, *foundations becomes a branch of mathematics.* Indeed non-formal mathe-

matics, as it is practiced by mathematicians, lurked from the start behind the scene as a teleological directive in the formal enterprise, judging what is to be regarded as a successful theory. Thus we have the positing of the truth of the formal approach – it is incorporated back into mathematics as a part of the subject. The world of non-formal mathematics, then, never could be dissolved in principle to that of a formal system since mathematics itself is the judge of what is to be regarded as a successful system and cannot dissolve itself (in a non-Hegelian framework). Further, as in the earlier discussion of **S**, this reversal from the formal back to intuitive mathematics is the asserting of mathematics itself as infinite in the Hegelian sense, as concept in its self-developing.

We have indicated above how Hegelian logic provides an explanation for the five inadequacies of formal systems of §5, in which these inadequacies are not to be treated as difficulties to be lamented but rather understood *positively*, as asserting the reflexivity and infinitude of mathematics. As such, the formalism of mathematical logic is *restored*, not adopting a pose external to mathematics and putting it to rights, as it were, but as part of the activity of mathematics itself which it presupposes and from whose meaningfulness it is derived. It follows, therefore, that it is not possible to build up mathematics by fitting it together from simple parts like a child building a model with wooden bricks.

Instead true mathematical logic is the self-determining of the concept of mathematics, in which one starts at the simplest, most abstract stage of intuitive mathematics and teases out what it implicitly contains in a logical development (*Entwicklung*). In Hegelian logic, the starting point is, of course, abstract, indeterminate *Being*. A major difference from the formal approach to logic is that such a development is *conceptual* in character, the bringing out explicitly of the riches implicit in mathematical concepts and their interconnection. It is to be compared to the thinking expressed when one mathematician explains to another *why* his latest theorem is true, without any formal proof, and the other *sees* it. This seeing is *conceptual* in character, often inspired by a good, well-understood, example.

In this conceptual development, the implicit conceptual content of the formal approach will be made explicit. For example, it will explain *why* formal systems are set up the way that they are. Also, as described in the third inadequacy above, the development will make conceptual sense of the

insights which motivated the various foundational systems. Even at the very basic level, there is the well-known issue in mathematical logic of how the logical connectives are to be understood, that would be dealt with in such a development. The latter will not displace formal proof and calculation but rather make explicit, for example, the conceptual *finitude* of the formal system. This conceptual finitude, as determined in mathematical form, is a *mathematical* question whose resolution required the genius of Gödel and, of course, a great deal of formal proof and calculation.

Such a development requires much detailed argument, and the classical Hegelian logic on which it is based will need to be substantially adapted and supplemented. However, we will briefly indicate how Hegel's logic contains the key to three of the difficult problems of the philosophy of mathematics: these are how to deal with 1) the semantic paradoxes (such as the classical liar paradox), 2) the logical paradoxes (such as the Russell-Zermelo paradox) and 3) the applicability of mathematics. These all center round the infinite in one form or another, whether in the form 1) of self-reflexive statements,[20] or 2) of "large" infinite sets or 3) of the axiom of infinity (as in Russell's brand of logicism).

These have their resolution in the three major transitions of the infinite in Hegel's treatment of Being (*Sein*). The first is the infinite qualitative progress (*der qualitative unendliche Progreß*) in the Section on Quality, the second is the infinite quantitative progress (*der quantitative unendliche Progreß*) in the Section on Quantity and the third is the infinite of the specification of measure (*die Unendlichkeit der Spezifikation des Maßes*) in the Section on Measure.

In conclusion, this paper claims that the reflexivity, effectively asserted in the failure of the formal approach to mathematics, poses the need for a framework which only the logic of Hegel has the resources to cope with. Bertrand Russell's claim that all that Hegel says in the *Greater Logic*[21] about mathematics is "muddle-headed nonsense" is a testimony to how little he and his mentor McTaggart actually understood Hegel. History has passed its judgement on the logicism which Russell advocated with such zeal[22] and his obdurate adherence to the logicist cause despite its glaring problems contrasts strikingly with Hegel's much more modest assessment of his own work on logic.[23]

Instead, it is time to grapple with what Hegel was trying to do in his

treatment of mathematics. Of course a lot of the *details* of Hegel's view of mathematics belong to his time and sound *quaint(!)* (but no more so than Newton's discussions of fluxions), and Hegel did not have the benefit of coming after Weierstrass, Dedekind, Cantor and Frege. But Hegel's method of inquiry in terms of a conceptual development is the correct method for investigating the human activity that is called mathematics simply because it is intrinsic and lets the subject develop itself (as a genuine infinite, a circle closed on itself) rather than assuming an external "objective" pose in which it is put in front of the investigator and inspected "from the outside", only for it to creep in from the back in the very act of investigation.

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Notes

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[2]*Euclidean and non-Euclidean Geometries*, (San Fransisco: W. H. Freeman and Company, 1980), 245.

[3]Hegel's great work on logic is, of course, the *Wissenschaft der Logik*, G. Lasson, ed., (Leipiz: Verlag von Felix Meiner, Hamburg, 1967). The English translation of the *Wissenschaft* by A. V. Miller, *Science of Logic*, (Atlantic Highlands: Humanities Press International, 1989) will be abbreviated in these notes to *SL*.

[4]It is important to remember, when considering Hegel's treatment of mathematics that he, like the mathematicians of his period, thought of the real numbers geometrically as representing points on the real line.

[5]A good reference for the discussion of formal systems in this paper is the book by W. S. Hatcher, *Foundations of mathematics* (Philadelphia: W. B. Saunders Company, 1968).

[6]See the discussion of this in Stephen C. Kleene, *Introduction to metamathematics* (New York: van Nostrand, 1952), 62.

[7]*Ibid.*, 84.

[8]Arend Heyting, *Intuitionism*, (Amsterdam: North-Holland Publishing Company, 1971), 3.

[9]See the discussion of this by Hatcher, *Ibid.*, 209.

[10]This is a deep theorem, but can be made plausible by observing that the variables, constants, functions and predicates in a formal system are, as we saw in §2, indexed by natural numbers and so countability is built into the system while this restriction does not

apply in informal mathematics. In this respect, it is helpful to contrast the original Peano axiom for induction for the natural numbers with its counterpart in the formal system **S** that gives elementary arithmetic: in the former case, *any* of the uncountably many subsets of the natural numbers is allowed to be considered in the definition of an *inductive set*, whereas in the latter, only *countably* many such subsets are available (because there are only countably many wff's with one free variable to define them). See the discussion by Hatcher. *Ibid.*, 72. This is also ultimately the reason why the Gödel theorems hold.

[11]It is a well-known joke in applied circles that the rigor of the "pure" analyst with his $\epsilon - \delta$ arguments in limiting operations is actually a case of *rigor mortis*!

[12]Israel Kleiner comments that "It is paradoxical that the very perfection of classical Greek mathematics – the insistence on strict, logical deduction – likely contributed to its eventual downfall." This quote is taken from, "Rigor and proof in mathematics: a historical perspective," *Mathematics Magazine* 64 (1991), 293-294.

[13]A recent discussion of the Gödel theorems is given by Stuart Shanker (ed.), *Gödel's Theorem in focus* (New York: Croom Helm, 1988).

[14]John Stillwell, *Mathematics and its History* (New York: Springer-Verlag, 1989), 328.

[15]*SL*, 842.

[16]*SL*, 826.

[17]The Singularity moment is represented by the "objects" of mathematics and its logic, such as natural numbers, sets and functions.

[18]However, there are fascinating traces of connections between foundational systems, for example, in the imbedding of **ST** in **ZF**.

[19]This is discussed in detail by Hegel in the chapter on *Dasein* in *Wissenschaft der Logik*.

[20]The Hegelian genuine infinite is self-reflexive in the sense that it is "bent back into itself." *SL*, 149.

[21]ed. P. A. Schilpp, *The Philosophy of Bertrand Russell*, (Chicago: The library of living philosophers, Northwestern University, 1944), 11.

[22]H. Wang points out that the concern of philosophers with the reduction of mathematics to set theory "has come about partly as a result of the historical accident that Frege and Russell, rightly or wrongly, connected it with philosophy, and that at least one of them is such a good propagandist." *From mathematics to philosophy* (London: Routledge and Kegan Paul, 1974), 233-234.

[23]It is a tribute to Hegel's utter honesty that after writing his monumental work on logic, he had reservations about its adequacy. See the preface to the second edition of Hegel's *Wissenschaft der Logik*.