THE SUCCESSOR FUNCTION AND INDUCTION PRINCIPLE IN A HEGELIAN PHILOSOPHY OF MATHEMATICS

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But further, we must recognize the infinite force of the understanding in splitting the concrete into abstract determinatenesses and plumbing the depth of the difference, the force that at the same time is alone the power that effects their transition.

G. W. F. Hegel[2]

But Findlay is not correct about Hegel's reason for finding nothing valuable for the ``science of experience'' in formal logic. In *this* connection Hegel would ``welcome'' formal logic precisely because it recognizes its own vanishing character. That is not a ``more radical lack'' (as Flay says) but a considerable advance in self-conscious rationality.

H. S. Harris[3]

1. Introduction

The objective of this paper is to present the basis for a modern philosophy of number using Hegelian principles. Number in this paper will always mean *natural number*. The natural numbers are just the numbers used in everyday counting, namely 0,1,2,3,.....

At first sight, this might appear to be a pointless objective since, as is almost universally accepted today, the logic of mathematics is based on the approach of *formal systems*. In that approach, the classical approach to geometry of Euclid and the Aristotelian logic of the syllogism are refined and extended to give the formal basis for mathematical concepts such as set theory and elementary number theory. The philosophy of mathematics is then reduced to that of giving a philosophical basis for such formal systems. For example, a philosophicaljustification for the formal system **RT** for set theory is the ramified theory of types of Russell and Whitehead in *Principia Mathematica*. The Hegelian approach to mathematics can certainly not be regarded as expressable through a formal system, and what there is of Hegelianism in the 20th century philosophy of mathematics would equally certainly have to be regarded as implicit.

What we will argue in this paper is that the 20th century philosophy of mathematics gives rise inescapably to concepts that are *self-referential*. The common reaction to this is to accept the limitations of modern logic and to deal with a variety of formal systems in an ``open-minded'' way. This can only be called an evasion of the basic problem, viz. how to develop a logic of self-referential concepts within whose context the normal (non-self-referential) aspects of mathematics can be studied. The paper claims that the logic of Hegel is precisely the logic of such self-referential concepts and that what is required to complete the remarkable achievements of modern mathematical logic is a modern Hegelian logic incorporating, and extending as necessary, classical Hegelian logic.

The paper makes a contribution to this project by presenting a modern Hegelian logic of *number*. This is presented in the final section of the paper. This modern Hegelian logic of number takes as its starting point the classical Hegelian logic of *Sein*, *Dasein*, *Fürsichsein* and *Quantität*, which contains Hegel's philosophy of number, and is discussed in § 3 of the paper. However, Hegel's philosophy of number, viewed from a present day perspective, has two striking omissions. It contains no discussion of the successor function and the induction principle, both of fundamental importance for present day number theory. (Indeed, these only came into prominence after Hegel's death in the work of Dedekind and Peano.)

These will be discussed in more detail in § 2. But for the present, the successor function n →n' is the function that sends a natural number to its successor. For example, 2'=3 and 567'=568. Intuitively, it is just the ``next'' number. The induction principle (roughly) says that we can assert a fact about *all* natural numbers if we know that it is true for the first one (0) and that it's truth is preserved when we go from a number to its successor.

The present paper argues that a closer examination of the details of Hegel's argument shows that there is a logical gap (or more precisely, an unmediated immediacy) in Hegel's argument which, when examined and developed, give in a natural way, the successor function and induction principle missing in classical Hegelian logic. This development will be given in the paper which therefore presents a modern Hegelian logic of number in which these important concepts of modern number theory are included.

The argument of the paper in no way can be regarded as a complete modern Hegelian philosophy of number but does give a self-contained Hegelian account of a significant part of number theory. A more comprehensive account will be given elsewhere.

In the second section, we survey the modern approach to the concept of number expressed in the context of formal logic. In particular, this approach includes the successor function and induction principle of elementary number theory whose understanding in a Hegelian context will be the major objective of this paper.

In the metamathematical enterprize of modern logic, the concept of number is self-referential in character. The induction principle, as pointed out by H. Poincaré, shows this very clearly, in the sense that in order to investigate the consistency of the use of the principle, we have to use that principle informally. This gives an initial justification for developing a Hegelian approach to number since the the concept (*Begriff*) in Hegelian logic is self-referential in character. In fact, Hegelian logic can be regarded as the logic of self-referential (or *infinite*) concepts. In the second section, we discuss what the principle of induction in elementary mathematics is and Poincaré's criticism of its use in metamathematics.

The third section presents a short survey of Hegel's logic of number as it is presented in Hegel's *Wissenschaft der Logik*, I. This logic of number cannot be isolated from the earlier chapters of Hegel's logic on which it depends. These earlier chapters give the logic of *Sein*, *Dasein*, *Fürsichsein* and *Quantität*, and a discussion of these is included in the survey.

In the fourth section, we turn to the formal approach to number. We discuss

very briefly the modern formal system **S** for number in order to see what is required (in the formal sense) to meet the needs of intuitive mathematics, and to compare it with the Hegelian logic of number in § 3. A few comments are made about the Hegelian interpretation of the system. Comparison of the system with the Hegelian logic of § 3 shows that two fundamental principles of the system are lacking in Hegelian logic, those of the *successor function* and the *induction principle*.

Inspection of the argument of § 3 shows a gap in Hegel's argument which will give these two missing concepts. The gap occurs where quantity develops into quantum (finite number). At that stage, the earlier *dasein* concept (which includes the concept of the *finite*) enters in the relation of the two moments of quantity. It is at this stage that the logic of dasein needs to be developed, so that we don't just get the finite (in the form of finite numbers like 7 or 11) but we also get the infinite (*Unendlichkeit*) in the *quantitative* medium. The infinite in the logic of dasein is the *qualitative* infinite.

(As an aside, we note that Hegel discusses the infinite of quantum and the quantitative infinite later on in WL in connection with *ratios* of numbers. However, the infinite of quantum considered in this paper arises at an earlier stage in Hegel's development of number. We will not consider this later infinite of quantum in the present paper.)

Intuitively, one would expect the concept of the natural number to arise out of this quantitative infinite since (at the very simplest level) the set of natural numbers is *infinite*. One might also expect the successor function and induction principle to have their origins in this logic of the quantitative infinite, since the successor function is defined for all (infinitely many) natural numbers and the induction principle is used to make assertions about *all* natural numbers.

This is in fact the case, and the precise argument for it is given in the fifth and final section of the paper. Of central importance are the forms that the infinite progresses of dasein take in the quantitative context. These have a dual character owing to the moments of the one and the many (which are present in number from *fürsichsein* logic). Each of these gives number as infinitely determined in modern mathematical forms, the first giving number as *cardinal* and the second giving number as *ordinal*. This logic also gives the missing successor function and induction principle within the Hegelian context.

The presenting of the development of this last section is the main objective of the paper. It shows that the key ideas of the modern concept of number are not alien to Hegelianism, but rather that the latter, appropriately developed, encompasses these ideas within its wonderful conceptual framework. This incorporation of modern logic within the Hegelian compass thus gives this logic a conceptual justification rather than just requiring the axioms, rules of inference, rules for forming the wff's, etc of formal logic to be accepted uncritically, ``take it or leave it''. It also breaks the technical, narrow, scholastic specialization of modern logic and instead embeds it within the rich, wide-ranging range of concepts of Hegelian logic, in which, for example, the number concept relates to scientific concepts and applications.

II. Transition from the formal logic of number to the Hegelian logic of number

The mathematical philosophy of Hegel, with which a substantial part of the *Wissenschaft der Logik*[4] is concerned, has been almost entirely neglected in the present century. Betrand Russell, notable among the few mathematical philosphers of the century who read any Hegel, dismissed it scathingly[5].

It seems that Russell and the other mathematical philosophers who read any Hegel made no real effort to think through Hegel's work in any sympathetic way, no doubt because Hegelian philosophy is not reductionist in character but rather is a development, where what is later is not attached (as it were) by symbolic hooks to what went before but rather was there implicitly all the time.

In this section, we will consider briefly the modern formal approach to the logic of number and how it forces one into understanding number as a self-referential concept. As such, it cannot be understood through the axiomatic approach of modern logic in which concepts are built up axiomatically in terms of simpler components. What is self-referential refers to or presupposes its *whole* self and so is not given (as are the concepts of modern logic and jig-saw puzzles) through the external collocation of constituent parts. So a new approach is needed to the concept of number, and we argue that this is to be found in Hegelian logic.

Hegelian logic is the logic of the self-referential, found in its most complete form in the *Begriff*, the Concept. In that context, what corresponds to the constituent parts in a formal system are the *moments* of the Concept. But these are not fixed and static but are dynamic, conceptually developing within themselves so that each when developed gives the whole of the Concept.[6]

We begin by discussing modern logic.

In the heady days of mathematical philosophy following Frege, it seemed, in contrast to the Hegelian approach, as if mathematics and the logic of its arguments could be built up like a lego model, backed by a fragile philosophic underpinning. The latter disappeared (by Occam's razor)[7] in Hilbert's formalist program, in which the terms of the logical systems developed were no more to be thought of as attached to mathematical concepts (such as functions and sets) as to *tables, chairs and beer mugs*.[8]

This led to the notion of a sentence as a well-formed formula (abbreviated to *wff*), one of a family of strings of symbols constructed recursively, according to rules, and proofs were just finite sequences of such strings formed using certain preassigned wff's called *axioms* and certain preassigned rules of inference (permitting one to write down a certain wff in a proof given that others of the appropriate form preceded it in that proof) but with no reference to meaning. A theorem is just the last wff of a proof.

However, meaningful mathematics was still operative implicitly in this enterprise inasmuch as the former's concepts and modes of deduction were what dictated the set-up in the formal system, and determined in particular the contentless, ``mechanical'' checking from one step to the next through which a finite sequence of wff's is conferred the status of *proof*.[9]

Meaningful, intuitive mathematics returned explicitly in

Hilbert's[10] mathematical study of formal systems, called *metamathematics*. In this study, the symbols, wff's, theorems and proofs of theorems of a formal system are present to perceptual intuition and so,[11] we may say, can be studied as ``applied mathematics'' without any ontological assumptions about what they mean. The strings of symbols and finite sequences of such strings of the system do not mean anything in themselves, and surely no contradiction can arise in studying the combinatorial possibilities of such visible, strings of marks on a piece of paper (anymore than the study of moves in chess can be conceived of as giving a ``contradiction'').[12]

For Hilbert, the ``propositions'' of such systems studied in metamathematics are ``ideal'' and only become ``real'', contentual, when they are determined in a concrete mathematical context, in much the same way as the ``ideal'' proposition of algebra x+y=z becomes ``real'' in the arithmetical proposition: 2+3=5.[13] So if we can show that the formal system is *consistent*, i.e. does not allow a contradiction, i.e. that there is no theorem of the form pv(~p), then the contentual mathematics in which mathematicians are *really* interested, and is formalized in the system, is also consistent. No meaning is attached to the ideal propositions -- which even could in principle contain constructions which, when given the natural interpretation, are ``dubious'' in terms of mathematical philosophy -- and the ideal propositions relate to contentual mathematics rather as the wave function□□in quantum mechanics, which has no physical interpretation, is related to
important measurable physical quantities. We do not *have* to give interpretations of such constructions!

But there is a fundamental problem with this. Since the study of formal systems which is to show the consistency of contentual mathematics is *itself* a branch of contentual mathematics, such a study can only be regarded as valid if the mathematics used in it *is itself consistent*, and hence the formalist enterprise is in danger of being caught in a vicious circle. This distressing situation was somewhat alleviated by the insistence that one should use only very fundamental mathematics in metamathematics, viz. that of elementary number theory.[14]

The circularity in the formal approach referred to above took a particularly acute form over the problem of mathematical induction. Since the main purpose of the present paper is to present an account of induction (and the successor function which it involves) as an extension of Hegelian logic, we will present here a brief discussion of what mathematical induction is and the criticism of its ``circular'' use in metamathematics by Poincaré.[15]
The principle of induction provides an indispensable method for establishing the truth of a propositional function[16]
P(n) where n ranges over the natural numbers 0,1,2,...
For example, (following Hilbert[17])
one such propositional function P(n) that is useful in elementary mathematics is that which asserts that for all $n \geq 0$, the following equality holds:

$$0^2 + 1^2 + \ldots + (n-1)^2 + n^2 = [n(n+1)(2n+1)]/6. \quad (*)$$

Checking this for a particular n is a mechanical, arithmetic calculation. For example, if n=3, then (*) reduces to showing that

$$0^2 + 1^2 + 2^2 + 3^2 = [3 \cdot 4 \cdot (2 \cdot 3 + 1)]/6$$

and this is true since the left-hand side is 0+1+4+9=14, while the right-hand side is $[12\cdot 7]/6 = 14$.

Of course, when the truth of P(n) is asserted for all n, what is being asserted is the truth of the *infinitely many* propositions P(0),P(1),P(2),.... Since we cannot establish the truth of P(n) by checking the truth of *all* of the propositions P(0),P(1),P(2), ... (since it is not humanly possible to check infinitely many propositions!), some other approach has to be adopted.

This is that of *induction* which asserts the truth of P(n) for *all* n provided that P(0) is true and (the induction hypothesis) that for any natural number m, the truth of P(m) implies the truth of P(m+1).

The usual way of justifying this is to argue that for any n, we can go from 0 to n in finitely many steps by adding on 1 at each step. Since P(0) is true, then, by the induction hypothesis with m=0, so also will P(1) be true. But then the same hypothesis with m=1 gives that P(2) is true, and continuing the process, in a finite number of steps we will reach n and thus have established the truth of P(n). The process is often thought of in terms of a row of standing dominoes: the first is knocked over and sets off a chain reaction in which each domino knocks down the one behind it so that all are knocked over. (The important difference is, however, that the row of dominoes is not *infinite*.) In the case of above, the establishing of P(0) is a trivial arithmetic calculation, while to obtain the truth of (*) in the case n=m+1 from that of n=m, one adds $(m+1)2$ to both sides of (*) for the case n=m. The *left-hand side* of the resulting equation is then that of (*) for n=m+1, while the right-hand side is the sum of the *two* algebraic expressions $[m(m+1)(2m+1)]/6$ and $(m+1)^2$, and simple algebra shows that this sum is also *the right-hand side* of (*) for the case n=m+1.

One can say then that the principle of induction reduces the establishing of the truth of infinitely many propositions (*) to a single calculation in arithmetic and another single calculation in algebra. As we shall see in § 4, in the formal system **S** for elementary number theory, the principle assumes the form of the axiom that for any wff A(x) with exactly one free variable x, we have

$$(A(0)\&(x)(A(x)\supset A(x')))\supset(x)A(x).$$

Here x' is the *successor* to a natural number x and intuitively, the wff says that if A(0) is true and if for any natural number x, the truth of A(x) implies the truth of A(x'), then A(x) is true for *every* natural number x.

Now if intuitive elementary number theory is consistent, then its formal version S must be consistent as well, i.e. none of its theorems is of the form pv(~p), a contradiction. But how can this be shown? There are infinitely many theorems to check after all! In the very simple formal system of the *statement calculus* (sometimes called the *propositional calculus*) one can prove consistency by using the truth table method and one might think that for this system, there is no problem. But even here the induction principle is required to prove that there is no contradiction in the statement calculus. One shows first that every theorem is a tautology. Since a contradiction is not a tautology, we obtain the consistency of the statement calculus. But the proof that every theorem is a

tautology requires using the induction principle on proof length. (The *length* of a proof of a theorem is the number of wff's occurring in the proof.)[18]

There is no truth-table method for determining the theorems of more complex formal systems such as the system **S** for elementary number theory, the predicate calculus or Zermelo-Fraenkel set theory. In these contexts, we are immediately forced back to the simplest method available to establish information about infinitely many propositions, and that is *the principle of induction*. Indeed, when investigating **S**, latent number theory emerges[19] in the *length* of a wff and the *number* of steps in a proof. With what happens in the case of the statement calculus in mind, we might try to prove consistency by establishing it inductively for theorems which admit a proof of length n.[20] (Actually, Gödel's second incompleteness theorem shows that consistency for **S** cannot be proved in this way using only number-theoretic methods (such as the induction principle).)

The use of the induction principle is of special significance in the case of investigating **S** since the latter formal system formalizes elementary arithmetic and so *formalizes the induction principle itself*. Indeed, the induction principle is an axiom schema for **S**. So using that principle in the metamathematics of **S** to investigate the latter's consistency means that we are *using the principle of induction to investigate the consistency of the use of the principle of induction* and are caught in a vicious circle. This was Poincaré's objection to Hilbert's use of induction in metamathematics and seems inescapable.[21]

However, far from being a surprise and a disappointment, the circular, self-referential, nature of ``infinite'' reasoning so clearly demonstrated in the induction principle is exactly what would be expected from the point of view of Hegelian logic. Conceptual thinking is self-referential, the concept both in and for-itself. But such thinking is not trapped in a vicious circle nor is it some kind of ``mysticism'' (in the sense that this term is sometimes scornfully used, as indicating an abandonment of rational thinking). Rather, all is implicitly present right from the start and develops itself explicitly in our thinking.[22]

The same happens in non-formal mathematics where, for example, the theorems of Euclidean geometry are present implicitly for the geometer in his construction of lines, circles, etc. in the plane but which it is his task to bring out explicitly. In Hegelian logic, one is not dealing with abstract husks of meaning manipulated according to mechanically applied rules, but rather this logic moves from the most inchoate level of thinking -- that of abstract Being -- to the concrete, and its progress is to be compared to the gradual clearing of mist from a mountain, which was initially completely obscured in a blanket of fog, but as the wind gets up, we first discern a vague triangular shape which then becomes more and more distinct until finally we can see details such as a rock face or a ridge.

Hegelian logic moves from the immediate through its mediation and back to the original immediate but as structured in this mediation.[23]
In the technical language of Hegelian logic, the immediate has been *sublated* in the result. We will use the term *sublation* a number of times in the paper and for the convenience of the reader, include here a brief discussion of it.

Sublation (*Aufhebung*) is the ``immanent'' character of deduction in Hegelian logic in which a concept is immediately given as fixed, has this immediacy broken in the bringing out of what the concept as dynamic has implicitly within it, and is finally restored to immediacy again but in a developed form. Sublation is not subjective in the sense that the conceptual dynamics is only *for us* as *we* think about the concept, but rather is intrinsic to the concept itself.

Words, for T. S. Eliot,[24]

> Crack and sometimes break, under the burden,
> Under the tension, slip, slide, perish
> Decay with imprecision, will not stay in place,
> Will not stay still.

This also applies to concepts but with the qualification that in the Hegelian philosophy, the concepts are of such a kind that they are not destroyed but rather express their content in their development and so *preserve* themselves in sublation. The sublated form of a concept is determined by and contains implicitly the original immediate concept.

As J. Burbidge expresses it with particular reference to the logic of being, becoming and nothing[25] (that we will discuss in § 3),

> In that single thought, the various components of `becoming' -- being, nothing, and the way thought moves from one to another -- disappear into a unity. They are retained, yet cancelled; so the result is a new simple -- and unmediated -- concept. `Aufheben' names that moment of collapsing.

For Hegel, sublation is ``a fundamental determination which repeatedly occurs throughout the whole of philosophy'',[26] and every sublation is a ``negation of a negation'', the first negation being the dissolution of the original concept in the mediation, and the second that in which the latter is itself ``cancelled'' in the restoring of the (sublated) concept.

We want to stress here that sublation is an expression of the *self-referentiality* of the conceptual thinking that we mentioned above and that arose out of our discussion of formal logic. Indeed, through sublation, a concept refers *back* to itself. The resulting concept is the original one sublated, and is the original but as infused with the character of the mediation. That resulting concept becomes definite, stabilizing into immediacy, and the mediation becomes implicit.

From the perspective of sublation, the self-reference of mathematical concepts such as the induction principle in the metamathematics of number is not to be regarded as giving a ``vicious circle''. For example, in the case of the induction principle, the ``viciousness'' can only come about if what we assumed to start with was fixed and so came into collision with itself in its dynamical mediation through the formal system **S** which destroys the presupposed fixity. The induction principle as fixed is the same as its being abstract with indeterminate content, and the mediation determines this content. Hence the self-reference of the principle is just its determination, its ``truth'' in the Hegelian sense, what it is in-itself as well as for-itself.

The inability of the formalist enterprise, then, to sustain itself brings one back to mathematics as a meaningful activity, working within its own terms, a self-referential infinite (in Hegelian terms). But this does not mean that the mathematical logic developed since the late 19th century is nugatory. Mathematics cannot see itself as it did in the pre-Frege days. Rather, this logic is part of mathematics *own* development as for-itself, in which the illusory objectivity of a completely ``rigorous'' deduction of mathematics was itself mathematical in character, the mathematical springs of it being (as commented earlier) left implicit, behind the scenes. Mathematics itself is, then, self-reflexive and its formal logic is part of this self-reflexivity, in which mathematics is true to itself in its apparent, contentless, self-alienation and returns to itself as mediated.

The positive role of mathematical logic is of particular importance in the question of *where to start* in understanding the subject in Hegelian terms. For the pin-point precise, abstract forms, schemas and mechanical procedures of mathematical logic, as a reflection of living mathematics, contain within themselves the truth of the logic of this mathematics but in ``frozen'', conceptless form.[27]
The forms of mathematical logic are *die leblosen Knochen eines Skeletts* as Hegel says about the logic of his day.[28]
This criticism of formal logic is reflected in the argument of Hermann Weyl[29] who refers to Fichte and Husserl anyone who wishes to *gain insight* into logic rather than just formalize it. However, as we saw above in the case of the induction principle, the problem with the formalizing of mathematics is that of the self-referential, and Hegel as the philosopher of the self-referential is the source of insight into mathematics rather than Fichte and Husserl. So the positive aspect of modern mathematical logic in terms of developing a Hegelian philosophy of the subject is that of pointing to the truth of mathematics which lies deeply hidden, shut up behind the abstractions and operations of this logic but through which it *scheint*. Modern mathematical logic then forms the starting point for a more insightful approach to the subject based on Hegelian logical principles. This is why in our discussion of the principle of induction in the paper, we start in § 4 by looking at the induction principle *as it is in the formal system* ***S***.
An understanding of the principle in Hegelian terms is then presented *after* that in § 5.

The above discussion raises the wider issue of developing a modern Hegelian philosophy of mathematics in which the subject is infinite in the Hegelian sense, to be thought of as a circle bending back on itself.[30] We conclude this section by making a few comments about such a philosophy of mathematics and how it relates to modern mathematical logic and Hegel's own philosophy of mathematics.

Because of the positive role of formal mathematical logic as described above, such a modern Hegelian mathematical philosophy will not be vague and imprecise. Rather, the precise logic that is needed for such a philosophy will be even more precise than that of the formal logic, since the precision of the formal approach is not negated but sublated, and so all of its resources are present. The greater precision lies in the conceptual basis for intuitive mathematics, suppressed and left implicit in the formal approach, now coming out explicitly, allowed in its meaningfulness to penetrate the external formalism and understand itself in this. It is the making explicit of the

Begriff or concept of mathematics, the source of its value and significance, whose depth goes far beyond the cautious perspective of the normal working mathematician, and has its roots in a modern Hegelian philosophy, a determination of which it is. So the insights of formal logic, for example, in the making explicit of the roles of induction and inference in mathematics, is mediated through a Hegelian-based philosophy which answers *why* notions such as induction, or rules of inference such as *modus ponens*, are valid.

Before one can even contemplate the formidable task of the working out of such a mathematical philosophy, it is clearly important first to look at what Hegel himself had to say about mathematics and whether his views about it can form an adequate basis for this philosophy. For the present writer, there is much work to be done in developing and clarifying Hegel's views of mathematics before it can be adequate to the subject as it is today. It seems clear that Hegel's insights into mathematics were never fully integrated by him -- for example, in *WL*, discussions about mathematics are to be found in all three of its parts, Being, Essence and Concept, but he does not present a coherent account of the subject.

Perhaps this is an indication that Hegel realized that extremely difficult mathematical issues remained to be resolved before a more complete philosophy of mathematics could be contemplated.

Indeed, the enormous advances in mathematics and its logic since the time of Hegel's death make many of the mathematical views that Hegel discusses in *WL* seem at first sight to be irrelevant. However, among the philosophers of his age, Hegel was as well-informed about mathematics as anybody else, and it is certainly unfair to cricitize him for not being able to anticipate the future progress of the subject. What is of profound value in his view of mathematics is that it is integrated into a far-ranging logical conceptual scheme -- not just concepts that are of interest to a group of mathematical specialists -- and the emphasis in discussing Hegel's views of mathematics should be on trying to clarify what Hegel is trying to say about the subject, rather than on some of his formulations based (from a present day perspective) on outmoded mathematical views.

So, for example, Hegel's treatment of the problematic *infinitesimals* of the *calculus* cannot be accepted uncritically, since from a modern mathematical perspective, the problems which the logic of infinitesimals was designed to solve have obtained satisfactory *mathematical* resolution in the theory of limits, convergent series and integration theory. The same cannot be claimed, however, for the *philosophical basis* of these theories, and in this respect, what Hegel sees as operative in the notion of the infinitesimal -- that of the shift from quantity to quality and backwards -- is, the present author would claim, of great importance, and applies every bit as much to concepts in the modern mathematical treatment of *calculus* (such as the notion of a *limit* and the *completeness axiom* for the real number system).

So, to reiterate, the goal of a modern Hegelian philosophy of mathematics is to understand the remarkable creations of modern mathematics in terms of its *Begriff,* and for this, the logical framework of concepts that comprise Hegelian logic needs to be examined and developed further to be adequate for mathematics as it is today. In the next section, we will examine the Hegelian logic on which his philosophy of number is based. This account of number on its own is not adequate to serve as a basis for modern number theory, but it will be developed further in § 5 to give a more adequate philosophical basis for

the concept of number.

III. From dasein to quantum

In this section, we will present an account of the part of Hegel's argument which will be needed in § 5. This argument appears in the first two sections of *WL*, I, and deals with *Sein*, *Dasein*, *Fürsichsein* and *Quantität*. The account given here summarizes in a few pages over 200 pages of dense Hegelian argument and so can in no way be regarded as comprehensive. But it gives what is adequate for our purposes, and the author hopes that it will be helpful to the reader.

The reader will notice that the kind of arguments used by Hegel are very different from those of modern logic discussed earlier. The reason for this is that Hegelian logic is a logic of the self-referential, and as we saw earlier, the approach of modern logic is not adequate when confronted with the self-referential. (A good example below of Hegelian self-referential logic is to be found in the mutual presupposing of the moments of the one and the many.) A new kind of argument is therefore needed.

Related to this, modern logic is formal in character, the conceptual content having been expressed externally in symbolic strings with instructions for making new legitimate ones. In contrast, self-referential logic is conceptual in character -- one *thinks* through, analyzes, a concept only to find that the full concept is presupposed in the very enterprise of this thinking -- the full concept reappears (but as mediated). Any symbolization that is used in self-referential logic is only there as a means for elucidating conceptual thought.

It is to be stressed, however, that concepts are not disembodied entities floating about in an abstract conceptual world, and for this reason, Hegelian conceptual thinking presupposes the *concrete*. This concrete provides the *language* for such conceptual thinking but only in the sense of providing a ladder to a ``higher'' level of thought which brings out what is implicit in the concrete situation. In this connection, the reader will notice that the language that Hegel uses arises out of a number of relatively concrete contexts, such as issues in pre-Socratic philosophy, Kant's metaphysical construction of matter, classical physics and practical arithmetic. These contexts are to be seen as providing striking situations that illustrate the operation of the concepts under consideration. So, for example, the moments of attraction and repulsion in Hegel's discussion of the one and the many originally arise in classical physics, but the conceptual content of these moments in Hegelian logic is *not* restricted to that original context. That context instantiates a general logical structure which is then described in terms drawn from the context. Something similar to this happens in mathematics where a general concept is suggested by consideration of an interesting special case, and the terminology of the latter is taken over in the description of the concept. (An illustration of this is the notion of a *geodesic* in Riemannian geometry. This term is taken from the special context of great circles on a sphere, which arises in the consideration of *geodesy*, the branch of mathematics dealing with the shape and area of the earth.)

We now begin the account of the part of Hegel's logic relevant to the purposes of the paper.

In order to start logic with minimum presuppositions, Hegel begins with the

concept of *abstract being*. Indeed, the concept of abstract being is presupposed in *any* meaningful activity or thought, since these involve focussing on something that *is* in one form or another. (Without such focus, the activity or thought is incoherent.) The concept of abstract being is what is common to all such focussing, resulting when we abstract completely from the content.

As *abstract*, such being is *de facto nothing* for there is no content to it. It is the thought of emptiness. Each of (abstract) being and (abstract) nothing is not given as *fixed* but is the vanishing process in which each disappears into the other. *Becoming* is this vanishing process, and the whole process, taken as being, gives being that is no longer abstract but limited by the negation (coming from its intrinsic *nothing*) and containing within it the vanishing process. It is being that is now relatively determinate, relatively concrete (for it is *not* certain things).

The logic here can be compared with the notion of *friend* as expressed in the paradoxical saying that if you are friendly with everybody, then you are friendly with nobody. *Abstract* friendship is an empty concept, and only becomes determinate if there are people with whom you are *not* friendly.

This relatively determinate being is what Hegel calls

Dasein.

Dasein is thus not completely abstract but has determinate character or *quality*. It is *something* (*Etwas*). This determinacy is immediate in character. Indeed it is the immediate unity of being and nothing in the form of being.[31] As immediate, encapsulating within it negation, it is conceptually blurred at the edges -- in particular, it is not precisely specified by numbers. This negation is the source of its further determining through the categories of *dasein,* through concepts such as limit and limitation. Quality is immediately present in something, identical with the latter's own being, through which the something maintains itself relative to other somethings. The identity of a quality of something with the latter's being can be illustrated by the invitation to ``Feel the quality!'' when we are presented with a fine piece of cloth material. An example[32] that Hegel himself gives of the qualitative is that of a *herb*, where the healing quality of the herb is immediately identified with what the herb really is, its being. In the realm of essence, quality and something lose their immediacy and assume respectively the more developed forms of *property* and *thing*.[33]

For the present, we reiterate that quality as immediate, conceals suppressed within it nothing (or not-being) through which the something was determined, and the transitional becoming from being to nothing which gave rise to it through the sublation which gave *dasein.* Accordingly, the being side of the determinateness quality of *dasein* is fundamentally unstable and, in the dramatic language of Boehme, a quality ``is the mobility, boiling, springing and driving of a thing'', associated with (for example, in the case of heat) ``fierceness, wrestling and fighting''.[34]

This dynamic aspect of *dasein* is, as we shall see, reflected in Hegel's use of *dasein* in terms of a verb, the process of *daseining*. (This verbal aspect also applies to *fürsichsein*.)

This instability in *dasein* first appears in Hegel's discussion of *something* and *other*, where the transition is moved outside something as *dasein* into an opposing other, thus giving initially a superficial stability to something. However, the transition is *something's transition* and eventually, in Hegel's development, the identity of something and other becomes explicitly posited in *limit*, where the two of them ``meet'' at their boundary, *where each as well is as is not*.[35] *Dasein* with its limit explicitly posited or folded into itself, ``taking responsibility'' for its limit, is defined to be the *finite*. In *dasein*, the finite has its limit, and hence its non-being, explicitly *in itself*, no longer passed on to another, so that the finite is unstable. On the one hand, the finite is subject to the *ought*, the negative relation of *dasein* to its limit[36] so that the finite in itself transcends the bounding limit, is outside itself, is what it is not, its non-being. At the same time, in its limit, the finite determines itself as *limitation*, ``holding in'' the ought, giving it a limiting boundary against which the ought can work to transcend. Ought and limitation mutually presuppose each other since each only has meaning if the other is already present.

The qualitative opposition of these two moments gives rise to the *infinite* as follows. The explicit positing of the opposition of the ought and limitation destroys the finite which ceases to be. But this ceasing to be is the realization of the finite itself, the fulfilment and explicit positing of what the finite was implicitly in itself. Hence[37] the finite is realized and so has not ceased to be but has returned. (We have gone from *something* to its *other* something.) But the returned finite is itself subject to the ought-limitation opposition, and hence there is generated an infinite sequence (*Progreß)* of disappearing finites. The truth of this sequence is the *infinite*, which asserts the positive of the progress, i.e. that the ought-limitation conflict which destroys the finite is the affirmation of the finite, in which it is posited through its disappearance. This infinite affirming of the finite through its ceasing to be is a negation of a negation, and hence a sublation. It is called the *infinite,* or, more precisely, the *affirmative* infinite. (This infinite is to be distinguished from what Hegel calls the *genuine infinite* which is discussed below.)

As affirming the stability of the finite in its instability, the affirmative infinite is abstract, a simple identity shining in and constant throughout the flickering sequence of finites. In its immediacy, this identity ignores the vanishing finites. But on closer inspection, it is essentially dependent on the ceasing to be and reemerging finite that produced it and relative to which its constant affirmation stands as an *other*. The infinite is therefore *itself* a something opposed to *its* other (the finite) and therefore is *dasein*. By the above logic of *dasein*, this infinite is itself finite. This is turn gives the infinite progress of finites which as we saw results in the affirmative infinite. We are back where we started with the affirmative infinite, and the argument repeats *ad infinitum* to give a second infinite sequence in which finite and affirmative infinite alternate. The truth of this second sequence lies in what it is asserting. This is that the affirmative infinite has returned to itself through the finite, and conversely, the finite has returned to itself through the affirmative infinite. The positing of the resulting sublation gives the *genuine infinite* (*die wahrhafte Unendlichkeit*) which is the movement of the finite back to itself through the affirmative infinite and equally of the affirmative infinite back to itself through the finite.

The genuine infinite stabilizes in a new kind of being, that of *Fürsichsein* or, as it is usually translated into English, *being-for-self*.[38] In the genuine infinite, determinateness (in the form of the finite) is intrinsic to the process, and has its truth in the affirmative infinite which in turn is bent back into the finite. Accordingly the being *fürsichsein* of this genuine infinite ``continues itself right through the determinateness '', does not ``stop'' with either finite or affirmative infinite.[39] *Fürsichsein* sublates the movement of the genuine infinite within itself and is immediately related *to itself only*, is not given (as was *dasein*) as dependent on an other external to it. (The Leibnizian monad, which Hegel discusses in this context[40] is a good example of *fürsichsein*.) *Fürsichsein* is the *immediate one*. This immediate one, as asserting only itself, gives rise to the *abstract one* as follows.[41]

The otherness that was present in *dasein* in polarities such as something-other and ought-limitation is initially enveloped, sublated, in the immediate one, sunk within *fürsichsein* itself. As such, all that is left of the movement of the genuine infinite is the *pure* relation of *fürsichsein* to itself, *pure* because it has lost in sublation the specific content of the finite and affirmative infinite. It is ``the self-relation of the sublating''.[42] This pure self-relation is completely abstract since if we allowed the sublated otherness to become explicit, the otherness would fall away from *fürsichsein* and we would have reverted back to the *dasein* stage negating the latter's sublation in *fürsichsein*. *Fürsichsein* as this simple abstract self-relation is the *abstract one* -- complete, infinite simplicity.

To clarify the relation between *dasein* and this abstract one, we observe that there is no *self-relation* in *dasein* -- rather the being of *dasein* is a looking away from itself to *another*, being which has its footing only in its other. It is to be compared to a long time resident of a small, closely knit community, who affirms himself only through the otherness of what the rest of the community think and value, and who anxiously looks outwards to that for the acceptance and approval which grounds his stability, and is alarmed when he visits the big city and so is separated from his community, because his being founders with the consequent negation of this supporting otherness.

By contrast, the one is solitary, defiant, wrapped up in itself, nursing itself in itself[43] in its complete withdrawal from the other which for it is not. (Unlike the
dasein case, there is no other to fasten on to.) The one is Promethean and only *is* in its self-originating negating.

Hegel illustrates the one[44] by the concrete forms of abstract freedom, the ``pure'' ego and evil.
For example, abstract freedom is actually the negation of the self-relation which constitutes it since the finite through which it only is as *fürsichsein* is negated, the movement stilled in the negative attitude to the negation, a negative attitude[45] ``which in seeking to possess its own being destroys it'', and, instead of embracing the negativity of *fürsichsein* in the movement of the finite and infinite through each other, rather lets that go and therefore lets go itself. The one still has difference in it as it is the turning away from itself -- but it has no other to go to and so returns to its

abstract self. With the vanishing of the content of *fürsichsein*, there is nothing in the one: *Es ist nichts in ihm*.[46]

This nothing in the one is the *void* and is the ``relation of the negative to itself''. In the void, the difference of the one is posited since the one as void is distinct from the affirmative being of the one.[47]

But since the one is only self-related, the void falls outside it and the one and the void are related as something and other, i.e. have reverted to the stage of *dasein*. So these moments of *fürsichsein* -- the *one* as abstractly self-related and the *void* as the equally abstract emptiness of this one -- therefore now become *daseining* : *seine Momente daher nun daseiende werden*.[48]

Next, both of the moments of *fürsichsein* -- the abstract one and the abstract void -- become immediate ones again. An easy way to see this is that, because of the *daseining* discussed above, each is initially a *dasein* and *dasein* has been shown to develop into *fürsichsein* and hence into the immediate one. (So each of the moments is now an *immediate* one, not the same immediate one that we had earlier, but rather that as mediated through the abstract one and abstract void.) However, Hegel's argument for this proceeds by examining each of the one and the void more closely.[49]

The (abstract) void as the negative relation of the one to itself is a relation to an *affirmative* being, the one, and so is not *just* indeterminate negation but implicitly involves this affirmative being. When this affirmative aspect is explicitly brought out, the void is completed in an immediate one. Similarly, the (abstract) one as affirmative being whose negative self-relation is the void, gives the immediate one again when this self-relation is made explicit in the void. (This is to be compared with the earlier argument by which Hegel obtains the genuine infinite from the affirmative infinite -- the implicit dependence of the affirmative infinite on the finite, when made explicit, gives the genuine infinite.)

So the two moments of the one is each itself a one. Hence the one is through its moments intrinsically a *many*, a multiplicity, a potential infinite. More precisely, the ones embraced by the original one form an indefinite many since any one of the embraced ones is itself a double one through its moments. It is important to stress that the many is not *fixed* but dynamic, a fluid movement beyond any specified many. The many is not to be thought of in terms of an infinite set. Rather it is the qualitative sense of plurality, not numerical (since that level of Hegelian logic has not yet been reached). The meaning of the *many* in this context is close to that of ``many'' as used in the well-known counting system
1,2,3,... many of a ``primitive'' people, except that we are at an even more ``primitive'' stage since we don't yet have the numbers 1,2,3!

By the above argument, the many is the product of the negative relation of the one to itself. The one is the sublation of otherness[50]
and so can only relate to itself. But the otherness sublated is the many, presupposed by the one's self-relation but not posited. The one in its abstract, simple self-relation shuts out the plurality, but the one as many (not abstracted in self-relation) is a matrix of many ones. However, as far as

the ones of this many are concerned, they relate only externally to one another and are each intrinsically only self-related.

The negative relation of the one to itself, left behind in the one's affirmation,[51] is for Hegel *repulsion*,[52] the generation of the many ones, what was presupposed for the one's self-affirmation. Since the one is only the negative relation of itself to itself -- without that, there is no one and the one *is* only as this negating -- *the many are posited by the one*. So the many are[53] ``infinity posited in the immediacy of being''. (It could be argued that the infinite of finites whose infinite progress sublated to give the affirmative infinite earlier in the argument has now reemerged in immediate, potential form as the many.)

Next[54]
the many ones themselves are each every bit as much the one as the original and so each is also the many and repels the many from it. Repulsion is the mutual relation between the many ones and, as above, since it is the negative relation of the one to itself without which the one cannot be, the ones therefore *are* only in so far as they repel each other.[55]
And indeed, the repelling is really a *relating* of the ones by which each asserts its common, ideal identity as one, the *Mit-sich-zusammengehen* of the ones. This positing of the many as one is *attraction*. In attraction, the one is now the *real* one.[56]

We have seen that the relation of the one and the many has the *two moments* of repulsion and attraction. In fact[57] each of the moments presupposes itself and the other. Repulsion at the being level presupposes ones that repel, but the many ones presupposed in this are
the result of the repulsion of the one and hence repulsion presupposes itself. Also, attraction is the many treated as ideal so that they lose their diversity and coallesce in the same conceptual one, but these ones in themselves were ones to start with and so attraction had to be operative before they were ``there'' to attract one another. So attraction presupposes itself. Also repulsion presupposes ones, each of which only is (as mediated) in attraction, and conversely, the one of attraction only is through the repulsion of the ones through which their common ideal identity is established. So each moment presupposes the other as well.

Fürsichsein culminates in the one as *infinitely* self-related,[58] the mediation by which it repels itself from itself in the abstract otherness of the flux of the potentially infinite many, and which in its sublation as attraction, relates itself negatively to this otherness. It is infinite in the sense that it is a becoming which no longer has a beginning in
the two moments since, as shown above, they presuppose each other. As a simple immediacy, this infinite is the ``very process of becoming''[59] itself. It is at the same time the infinite mediation of the one as for-itself, the unstable, *daseining* moments of repulsion and attraction having collapsed in *Mit-sich-Zusammensinken*, their sinking together in the mediation. This infinite mediation as being is defined to be *quantity*.

Quantity is a sublation of *fürsichsein* in the sense that the former incorporates explicitly in its mediation the difference (one-many, repulsion-attraction) that was originally veiled within the immediacy of *fürsichsein*.[60]

From the above, quantity is *both* simple immediacy and infinite mediation.

In fact, it is an *equalizing* of mediacy and immediacy. Indeed, the above sublation that is quantity in its immediacy is precisely its mediation. Alternatively expressed, the repulsion that is the mediation presupposes the immediacy that is attraction and conversely. So quantity equalizes its determinations, its otherness (in repulsion) being equally itself (in attraction). Hence we have left the realm of *dasein* with its fundamental something-other polarity -- quantity is as much its other as itself. In particular, the *dasein* category of limit which separated something and other *has been transgressed*. (Indeed, such a ``moving beyond'' is intrinsic to quantity since, by definition, the latter is the becoming of the infinite mediation of the one.) Hence the limit of quantity is no limit,[61] and indeed quantity is, as Hegel expresses it, *indifferent to quality*. Quality has been neutralized in quantity.

In the sublation of quantity, the moments of attraction and repulsion of the one-many relation are further determined in quantity as the moments of *continuity* and *discreteness*. Continuity is the smooth continuation of the one in the transgressing of limit discussed above, not ``interrupted'' in the ``asunderness of plurality''.[62]

Continuity is the continuity of the many, their self-sameness, and the spreading (as it were) of the self-sameness through the many without let or hindrance (except that this ``spreading'' is conceptual, not temporal).

Since plurality is immediately involved in continuity, the latter presupposes the former as the expression of repulsion in the many. Repulsion as determined in quantity is the second moment of quantity, that of *discreteness*. Discreteness determines the ones of the many as taken all together, *the many taken as one*. Indeed,[63] quantity is in itself asunderness of identical ones. Continuity is this asunderness continuing itself through the ones, connecting them up. But this has no negation in it -- the asunderness of quantity is suppressed in continuity and the many are not explicitly distinguished within that moment. Continuity on its own results in the plurality as a *one in general*. The negation missing in continuity is captured in the discreteness moment in which quantity is expressed in its discontinuity, as *interrupting* the smooth all-embracing unification of continuity, the units standing out on their own.

Discreteness expresses quantity as a *many of a unity* while continuity expresses quantity as a *unity of a many*. As with the moments of the one and the many (attraction and repulsion), each of the two moments of quantity presupposes the other. In fact, each moment contains both moments, and when confronted with quantity in one form or the other, we don't notice this containment because the other moment is only implicit. For example, as Hegel points out, a continuous quantity such as space has the discrete moment implicit. This discrete moment becomes explicit when a point is posited in space (thus ``interrupting'' space). On the other hand, given a discrete pile of things (such as the items in a shop window), the continuity moment is present in that each item in the pile is treated the same *qua item* in the shop window. This is the source of the ``unity of a many'', the continuity moment.

The mutual involvement of the continuous and discrete is of great importance in mathematics. The science of mathematics can be roughly partitioned into the study of the *continuous*, in particular of the continuum or real number system -- as in analysis, differential equations and topology -- and the *discrete* as in the study of the natural numbers

(number theory) or combinatorics. However, the study of the continuous involves the discrete and conversely. So, for example, in quantum mechanics, the Schrödinger partial differential equation -- which belongs to the ``continuous'' in Hegel's sense -- determines *discrete* quantum effects. What comes out of Hegel's discussion of the continuous and discrete moments is an explanation of *why* this is the case -- it is that at the conceptual level, the two intrinsically refer to each other so that we cannot have one without the other.

The turbulent ``pulling in'' and ``forcing away'' of attraction and repulsion respectively have become quiescent in continuity and discreteness through the transgression of limit in quantity, each permeating the other conceptually. However, quantity in these moments is immediate in character, and so does not explicitly distinguish between them, and obviously, there *is* a difference between them. Immediately,[64]
quantity as a unity has to be thought of in terms of continuity, for that is what ties the plurality together and stops it from becoming diffuse and disconnected. When quantity is posited as this unity, in its moment of continuity, it stands over against itself as discrete, and we are in a something-other situation, i.e. *dasein has returned*. At the same time, an advance has been made since the two moments are now explicitly distinguished and quantity is no longer immediate.

Quantity now with its moments qualitatively in this something-other relation is determined, by the logic of *dasein,* as a *dasein with limit.*
As such
it is *a* quantity, and is called by Hegel *quantum.* In *dasein* logic, the something
with its posited limit[65] is the
finite. So quantum is the *finite* as determined in quantity. As we shall see, when the moments of quantum are made explicit (in unit and amount), quantum is, for Hegel, *number*, the (finite) number of elementary arithmetic. (In particular, the notion of quantum does not apply to the Cantor generalization of number as the cardinality of an *infinite* set.) The limit for quantum is, as described below, that of *dasein but as determined by quantity*. This quantitatively determined limit will also further determine, as we shall see, the quantity moments of continuity and discreteness to give the moments of *unit* and *amount* for number.

To this end, recall that the limit of quantity is no limit, is qualitatively indifferent. Since quantity has now been *daseined* into quanta, the limit of *quantum*, i.e. of quantity as *dasein*, is now a ``genuine'' limit, no longer a ``no limit''. Since limit as *dasein* sets off finite against finite, it follows that quantum limit sets off quantum against quantum. So the quanta are in a *dasein* relation to one another. (In particular, numbers stand opposed to and separate from other numbers.)

Next, quantum limit has two aspects, which relate to the moments of quantum. The first is inward directed, qualitatively determining the quantum in terms of its internal, quantitative structure. This establishes quantum against the other quanta, gives it substantiality. Since what holds quantum (as a quantity) together, gives it coherence, is its continuity moment, this aspect of limit determines that moment to give the quantum moment of *unit*. The second aspect is that of quantum limit as turned outwards, sublating the internal content of quantum in a positing which blocks off the other quanta as other to it. This is the positing of quantum in the quantitative moment of *discreteness*. This posits quantum as a determinate plurality which

excludes other determinate pluralities (quanta). This is the quantum moment of *amount*. We now examine in more detail these quantum moments.

Given quantum limit as described above, the moment which it immediately determines is the continuity moment. Indeed, we saw earlier that it is the continuity moment that transgresses the ``no limit'' limit of quantity. (The discrete moment is initially left implicit in the limit.) So quantum limit immediately determines quantum qua continuous moment as *continuous* quantum. As such,[66]
quantum is the positing of the many as a definite many of the *same* ones, their indistinguishable sameness being their ``continuity'' or identity. The restless going beyond, transgressing of limit in quantity has now in the finiteness of quantum been stilled, made complete in a *totality* of identical ones. We emphasize that the elements of the many are not allowed to be distinguished by features such as color, position or size. Their identity in the one limits them to being the same and interchangeable in every respect. The determination in this limit of the continuity moment of quantum is called *unit*.

Next the limit determines the mediatory discrete moment of quantum. Discreteness, as we saw earlier, was the many taken as one, the items standing out on their own, interrupting the smooth flow of the continuity moment, the many in their common apartness. Discreteness is further determined in the other moment of quantum, that of ``specific aggregate'',[67] the
separating character of discreteness through which the items ``stand out'' in quantity being limited through the finiteness of quantum. The limiting of the discrete moment is posited in quantum as ``manifold within itself'',[68]
reflecting the multiplicity of the ones in their discrete relation. This quantum moment is called by Hegel *amount*. Quantum as explicitly determined in these moments of unit and amount is defined to be *number*.

Quantum, concretely realized as a specific collection of things, is thus determined as number through these two moments. In particular, as a quantum determination, the *number* of the collection is obtained by making explicit these two moments implicit in the collection, and this is done through the process of *counting* or *tallying* the items in the quantum. (The finiteness of quantum ensures the completability in principle of the counting of a quantum.) Hegel does not seem to discuss the philosophical significance of counting. (A discussion of this significance will be considered in § 5.)

We noted earlier that the moments of continuity and discreteness presuppose each other in quantity. The same is also true of the unit-amount versions of these moments when quantity is determined as number. The unit is the continuity of the amount, that which determines each as eligible to be counted in the amount, and conversely, unit is determined by amount in which it is incorporated and which excludes other quanta and their items.[69]

Note that the limit of quantum does not act as a boundary between the units in the many of the quantum. In the continuity of unit, they are indifferent to one another, each indistinguishable from the other and (as Hegel puts it) ``affirmatively present'' in the amount. The determining of amount is realized through the breaking off of the counting process. While it is true that the last item counted limits the tally, any of the items could have been the last so that the others do not relate as other to this last, but limit ``pervades'' them all. As discussed earlier for quantum, number is only *daseined* against other numbers and there is no such *daseining* intrinsic to a

particular number within itself.

We noted above the *indifference* of the units to one another and the quiescent nature of the continuity/discreteness moments of number. This indifferent ``inert'' character of number renders it for Hegel *passive*: number ``must be actuated from without and so brought into a relation''.[70] The modes of this relation are the ``species of calculation'', e.g. addition, subtraction and multiplication.[71]

The ``inert'' character of number which underlies the mechanical manipulations of the ``species of calculation'' shows number to be abstract and ``external'' in terms of the concept. Indeed Hegel defines number to be the ``abstract thought of externality itself''[72]
and as such, the effort required in performing numerical calculation (by hand or mentally) consists ``in holding fast what is devoid of the concept and in combining it purely mechanically''.[73]

Despite this judgement, Hegel seems to be aware that there is more to number than just these ``mindless'' manipulations. With commendable open-mindedness, he allows that his exposition of the species of arithmetic may not ``exhibit their inner significance''.[74]

To clarify what this significance might be and what is missing in Hegel's account of number, another approach is needed, that of *formal arith*metic (developed after his death) wherein is precisely specified (albeit in ``external'' form) what is required for the study of number in mathematics. We therefore now turn to formal arithmetic.

III. The successor function and induction principle in formal arithmetic

In the second half of the 19th century, efforts were made by mathematicians to isolate those key properties of the set **N** of natural numbers (nonnegative integers) on which the operations of arithmetic are based. A list of postulates for **N** were given by R. Dedekind and later by G. Peano, and are normally referred to as the *Peano postulates*. These provide the bed-rock for arithmetic, and can be regarded as a formulation of what is essential for the elementary mathematics of number. These postulates suitably formulated give the fundamental formal system *S* of mathematical logic.[75]

We now very briefly summarize this system **S** in order to make explicit key ingredients of that system that are missing in the Hegelian view of number.[76]

In **S**, there is one binary predicate letter ``='', two binary function letters ``+'' and ``·'', one singular function letter `` ' '' and one constant letter ``0''. Write +(x,y), ·(x,y) and '(t) as x+y,xy and t' respectively. The only terms of **S** are individual variables, the constant letter 0 and expressions of the form x+y,xy and x' where x and y are terms. By a wff (well-formed formula) for **S** we mean an expression (concatenation of symbols) X such that X is either of the form (t=r) where t and r are terms *or* of the form (x)A *or* of the form (Ex)A where x is a variable and A is a wff *or* of the form (~A) or (AvB) where A,B are wff's.

Intuitively, +,. represent the addition and multiplication operations on **N** with 0 representing zero. For any nε**N**, the number n'ε**N** is the *successor* n+1 to n. So, for example, 0'=1, 2'=3 and 11'=12.

The rules of inference (e.g. *modus ponens*) and the *logical axioms*[77] are the same for any formal system. This leaves for consideration the *proper axioms*, i.e. the axioms that are not the logical ones, to which we now turn. There are twelve proper axioms.[78]

The first three of these axioms are *equality axioms*, e.g. $(x_1)(x_1=x_1)$ (i.e. every term equals itself) and $(x_1)(x_2)(x_3)((x_1=x_2 \vee x_2=x_3)\supset x_1=x_3)$ (i.e. if $x_1=x_2$ and $x_2=x_3$ then $x_1=x_3$). The fourth and seventh axioms assert that the successors of two terms are equal if and only if the terms are equal. Axioms 6 and 8 concern the special properties of the term 0, that 0 is not a successor and $x_1+0=x_1$. Axioms 9 and 11 express how the successor function relates to addition and multiplication, e.g. $(x_1)(x_2)((x_1+x_2)'=x_1+x_2')$ (i.e. for every pair x_1,x_2 of natural numbers, $(x_1+x_2)+1 = x_1 + (x_2+1)$). Axiom 10 asserts that a product of any x_1 by 0 is 0. Finally, the axiom schema Axiom 12 is the powerful principle of induction, discussed informally in § 2, that for every wff in **S** of the form A(x), the expression

$$(A(0)\&(x)(A(x)\supset A(x')))\supset(x)A(x)$$

is an axiom.[79]

In terms of Hegelian logic, the conceptual basis for **S** is much more advanced than the logic leading up to quantum that was discussed in § 3. For the latter is a logic of *Sein* (being), while a thinking approach to **S** shows the presence of categories of the later two parts *Wesen* (Essence) and *Begriff* (Concept) of *WL*. For example, interpreting **S** in terms of **N** as above, the formal system relies for a start on the basic logic of variables, quantifiers and propositional connectives which are appropriate to the logic of the *Concept* (where Hegel examines the judgment and syllogism). As far as the axioms of equality are concerned, they involve in Hegelian terms the *essence* category of *identity*. In fact, the first such axiom given above is just the identity A=A[80] of Fichtean metaphysics. Also the second equality axiom given above is, in the terminology used by Hegel, the *concept* category of the mathematical syllogism.[81]

In contrast, Hegel's discussion of arithmetic is solely in terms of *Sein* logic. In particular, addition and multiplication in elementary arithmetic are, as we saw in § 3, *being* categories for Hegel, used in Hegel's computational view of number. Yet, as noted above, the presentation of these functions in **S** involves variables and quantification, whereas in the Hegelian context for arithmetic, these are missing.

However, what is strikingly present in **S** but not in Hegelian logic are the notions of the successor function and the *induction principle*. These belong to the substance of number rather than general logic (such as the logic of quantification). Accordingly, by present day standards, the Hegelian ``quantum'' account of number has to be judged defective, and we are faced with the task of examining more carefully his account to see if these notions are implicitly present somewhere in it and their explicit development can be integrated into it. Alternatively expressed, we are looking for an immediacy in Hegel's argument which when mediated, results in the notions of

the successor function and the induction principle. A reason why one might expect that this is the case is the discussion in § 2 where the criticism of the principle by Poincaré shows that it has the *same* self-referential property as the being, für*sichsein*, of the genuine infinite, arising out of Hegel's treatment of *dasein* which we saw (in § 3) to be part of Hegel's argument leading up to number.

To specify this immediacy, we need to look more closely at Hegel's argument concerning the *origins* of quantum (number) and see if dasein occurs somewhere in it. If it does, then we can ask if Hegel develops the genuine infinite out of that dasein situation. If not, then developing the genuine infinite at that place in *WL* may give the notions of the successor function and induction principle missing in Hegelian logic.

Inspection of the argument in § 3 shows that indeed dasein concepts *do* occur at the start of Hegel's development of quantum. Indeed as we saw, quantum as *determinate* quantity arises through the *daseining* of the moments of quantity. Further, Hegel does not develop the infinite aspects of this, the quanta (numbers) that he considers all being finite.[82]

What we will show in the last section is that the genuine infinite is present at the level of the natural numbers. We will give a Hegelian-style development out of which arise the successor function and induction principle missing in classical Hegelian logic.

IV. A Hegelian approach to the successor function and induction principle

Let us start by recalling part of the argument in § 3. There, quantity was determined by the distinguishing of its moments, continuity and discreteness, which sets quantity against itself. The moments *dasein* quantity as finite, so that quantity becomes limited in quantum (number). In quantum, continuity is further determined in *unit* and discreteness in *amount*. While the different quanta are *daseined* against one another, each is internally not in opposition within itself, its units remaining indifferent to one another, yet the quantum limit is ``smeared'' over all of the units, each being equally the limit. (Each can be the last unit in the counting of the number, its determining as amount.)

As observed in § 3, the continuity moment is the immediate common determination of the many of quantum as *unit*. So each unit is not just limit, but also the continuity moment. And indeed the determination of unit as the continuity moment is the same as the limit determination, the continuity that holds the quantum together in the uniform determination of the many as units being what ennables the quantum to ``hold off'' the other quanta, *daseined* against it, through its limiting units.

The determination of continuity (and equivalently of limit) in the units is, as presented above, *itself* indeterminate. Indeed use of the adjective ``smeared'' indicates that continuity is divided in itself -- it has to apply equally to all of the units and to none in particular. As such, continuity is (in Hegelian parlance) *ideal* and in order for the continuity moment to be realized, made determinate, have its indefinite, ``vague'' universality specified, continuity must make good its claim that it is the unit by positing a unit *as its complete self*. This conferring of the whole continuity moment (or limit) of quantum on a unit breaks the indifference of the units among themselves by highlighting one. (It is also the initial determination of the *counting* process which, as observed earlier, Hegel leaves unanalysed

in his treatment of number.)

The unit so determined as embodying all of that moment is defined to be the *first*.

Since continuity is now concentrated on the first, it is withdrawn from the many units in the quantum, and since the units are indifferent to one another and the first has been *distinguished* from the rest by its positing as the continuity moment, the first falls outside the other units. Yet the first, which was initially only one of the many, has qualitatively in it all of the many since by definition, the first is the first of the quantum's totality, and so has intrinsic reference to that totality, so that the whole of quantum is implicitly present in immediate form in the first. So quantum limits itself in the first and its coherence (in continuity and limit) becomes concentrated in it, representative of it, the gathering of the many in the one.

The limiting of quantum in *first* can be illustrated in the choice of a person as a *representative* of a group, such as political body or company. The representative *stands for* all of the members of the group, is conferred ``group'' status. In his solitary individuality, he stands for them all.[83]

In the positing of the continuity moment of quantum as the first, the first has been withdrawn from the rest which therefore ceases to be a coherent quantum and is determined only according to its discrete moment, as a *many*. Accordingly, the moments of quantum are separated, the continuous in the *first* and the discrete in the *rest*, the first without discreteness and the rest without continuity, the first and the rest *daseined* against one another. In particular, as *dasein*, the rest is qualitatively determined as *finite*. The term ``rest'' used here is purely negative, what is *not* the first, and since it is not positive, it does not function as a unifying continuity substitute for the many left. The first loses its indifferent, quantitative, ``unit'' character, and is qualitatively determined in its new character of bearer of the continuity moment, while the rest, we may say, has ``fallen apart'' in indeterminate, negatively determined, vague plurality. The rest is incoherent.

However, on closer inspection, the first is posited as the continuity *of the rest* -- the rest are embraced implicitly in the one -- and this continuity posited in the first is, like the determining of the rest above, a *negative* determining, viz. the negative continuous determining of the many left over which are other to the first. But the first exhibits explicitly the continuity of the quantum and hence also of the rest included in the quantum. Hence the rest *does* have (in the first) a positive continuity moment, and is no longer just *negatively* determined only according to the discrete moment. When the rest has this continuity moment explicitly posited in it, taken back into it, the continuity that determined the first becomes only ideal and the first disappears since its defining character is no longer posited in it. Next, as we saw above, the rest is a finite, and now has *both* the discrete and continuous moments. Hence it is *quantum* again, quantity as finite, limited.

This is illustrated in the practical counting process, when we pick out one from a collection of items being counted, call it *first*, *discard the first* and then turn back to the rest *regarded as a new collection* looking for a second. The first is now *irrelevant* and can be disposed of

and what is left is a new quantum. As far as the counting of the original collection is concerned, the first no longer *is* and we can put it to one side or even lose it without affecting that counting. While the item still is physically ``there'', it no longer is determined by the quantum, and so for the latter is not. What *is* qua quantum is the rest.[84]

The above discussion of *first* and *rest* gives the conceptual grounds for the counting process. Next, we observe that in the positing of unit as first and amount as *rest*, each moment of quantum returns to itself in sublated, concrete form through the other.

To formulate this more precisely, recall that unit and amount are respectively the quantum determinations of the quantity moments of continuity and discreteness. In terms of continuity (unit) and discreteness (amount), the abstract *continuity* of the original quantum, posited in the first, becomes the concrete continuity of the rest through the process of making explicit the *discrete* moment in the rest. Conversely, the discrete moment of the initial quantum, which was incoherent in the rest through the separation of the continuity moment in the first, becomes concrete through the return of *continuity* to the rest. This is just an expression of Hegel's result, discussed in § 3, that *each of the moments, continuity and discreteness, presupposes the other and itself.*

The process described above of quantum determining itself in a first applies again to the rest as quantum, then to the rest left over after that and so on, giving rise to the bad infinite (*das Schlecht-Unendliche*).[85]

So, for example, the rest as quantum itself posits its continuity moment in a unit, the *second*, which separates off from the rest and realizes the indefinite continuity moment of the rest. The quantum thus self-divides conceptually in the separation of the continuity and discrete moments, and the second unit as the ideal continuity of the rest is then realized as the continuity of the new rest (i.e. the old rest without the second) so that the new rest becomes a quantum.

It is true that for a *specific* quantum, e.g. one with 10 units, the process will eventually end with *nothing* remaining. The negativity and finitude of quantum, whose inner dynamic is expressed in its self-division and the concentrating of its continuity, then appears very explicitly in the quantum *having all gone*. However, the determining of quantity as quantum is *itself* undetermined, unlimited, for in this determining, we do not restrict the size of quantum to (say) ≤1000 units. This undetermined determining is the *infinitude* of quantum, and this infinitude is present in concept in quantum just as the affirmative infinite is woven through the finite of *dasein* in the genuine infinite of dasein. So according to the *concept* of quantum, the self-dividing of quantum and its return to quantum is an infinite process: it is conceptually present in quantum since it is intrinsic to the concept of quantum *as such* rather than just applying to a specific quantum.

The inner negativity of quantum powering the bad infinite, whereby it destroys itself in its self-division, is the positing of the negativity of determined quantity (quantum) *in the discrete medium*. The positing of the negativity of quantity in the continuous medium of *space* is important in Hegel's Philosophy of Nature[86]
where it is nothing other than *time.* There, just as in the case of the units of quantum, the points as limit have ``the form of subsistence'', and

space is ``infected with negation'' in such a way that ``this negation falls apart into indifferent subsistence''. Nonetheless, the negation behind this indifference drives the spatial point out of itself through its temporal negativity in the line that its ``movement'' generates.

The above discussion of quantum is the discrete version of this, whereby the mutual indifference of its units to one another is their falling apart in the negation of quantum, and this negation becomes explicit and ineluctable in the driving of the bad infinite in the self-division of quantum and its perpetual return to itself as the quantized ``rest''. This ``bad infinite'' of quantum is just the repetition *quantum, quantum, quantum, ...* as described above, that is, the continuity moment of quantum is focussed on a unit, leaving the discrete moment concentrated on the rest which then becomes the *next* quantum with the unit discarded. Conceptually, quantum repeats itself just as the finite repeated itself in the first infinite progress of dasein.

In the *dasein* case discussed in § 3, the affirmative infinite was the positive affirming of the finite through its ceasing to be in the infinite progress, its intrinsic infinite refusal to stay down dead. Equally, the *affirmative infinite of quantum* affirms quantum through its conceptual, self-destructive ``chipping away'' at its units. This infinite is defined to be the *natural number* in immediate form. It is the positing of the counting process as for-itself, the abstract medium for counting as such.

The mediation of the abstractness of the affirmative infinite of *dasein* lay in bringing out the role of the finite. As we saw in § 3, the suppression into implicitness of the finite in the affirmative infinite -- a finite without whose ceasing to be, there *would* be no affirmative infinite -- is resolved in the *genuine infinite of dasein*. The latter is the return of the finite to itself through the affirmative infinite and equivalently of that infinite to itself through the finite, with the result that neither finite nor affirmative infinite can be conceived in separation from the other but each only *is* in the other.

The affirmative infinite of quantum -- the natural number in its immediacy -- is, like that of *dasein*, mediated, sublated in its *genuine infinite*. This mediation, which we will now discuss, provides, as we will see, the natural number with both its cardinal (counting) and ordinal (order) aspects. The genuine infinite of quantum, however, has a dual character lacking in the dasein genuine infinite. The reason for this is that in the case of quantum, the *two* moments (unit and amount) are involved -- such a duality is not present in the case of *dasein*.

The affirmative infinite of quantum identified the concept of quantum with itself through its division in *first* and *rest*. On closer inspection, this identification is at the level of *amount*, the discrete moment of number. Indeed, as commented above, the bad infinite of quantum which sublated in this infinite is in the *discrete* medium. This affirmative infinite is then the *affirmative infinite of amount*. In that infinite, the quantum concept is preserved through change of size, i.e. of ``amount'', with the conceptual process of identification, through the negativity of quantum in *first* and *rest*, being suppressed. It is true that the infinite character of the ``amount'' concept has been exhibited in the affirmative infinite of quantum, that the concept is identical in all of them, but this identity is abstract, with the unit moment in *its* infinite character not made explicit.

Indeed, the conceptual identification of quantum with itself through the *first* and *rest* is equally determined at the level of *unit*. This infinite determination at the level of unit is the *affirmative infinite of unit*, which we will now describe more precisely.

The positing of unit in *first* results in its discarding, as the rest is determined as quantum again. But unit reappears out of that new quantum in the positing of *second* and so on *ad infinitum*. This is a familiar infinite progress of unit -- that of counting 1,2,3,..... This infinite progress is sublated, as also happened in the cases of *dasein* and *amount* earlier, in an affirmative infinite, that of *unit*.

Indeed, recall that the choice of *first, second* etc. is not intrinsic to the units of the collection. We could have chosen, for example, the second unit as our first. When this is made explicit in the infinite progress of unit, the ordering on the units drops away, and instead this infinite progress asserts the *identifying of unit with itself through its dissolution*. This is the affirmative infinite of unit. From the unit perspective, the logic of *first* and *rest* above is, then, that of unit as *first*, which is discarded in *rest* as the new quantum and out of which comes unit as *second*. At the level of unit *qua* unit, i.e. independent of order of choice, this gives the conceptual infinite process intrinsic to unit in which unit identifies with itself through its dissolution.

We have, then, *two* affirmative infinites of quantum, that of amount and that of unit. But both are still abstract. In each, the other has fallen away. Yet each is the same in the sense that each is unthinkable without the other. The two infinites are conceptually equivalent. For unit identifies with unit only through quantum as amount identifying with quantum as amount and conversely. More precisely, when we identify ``first'' with ``second'' as unit, we are presupposing quantum as given initially and quantum *again* as arising out of what is left of quantum after the first has been taken away. (There is no ``second'' without the rest becoming quantum.) But conceptually, this is just quantum as identifying itself with itself (through unit loss) as discrete, in the amount moment. The converse is equally true in the continuous moment of unit, and the *genuine* infinite of quantum is the assertion of the truth of this equivalence.

This genuine infinite we will call the *genuine infinite of amount and unit*. It is *the two-fold identification of quantum as amount with itself through unit and of quantum as unit with itself through amount*. Each of amount and unit *is* only through the other and in the genuine infinite, each is conceptually infinite, self-referential. (In particular, like the point in Hegel's view of space, the unit in isolation is ultimately unthinkable.)[87] The immediacy of *natural number* as the affirmative infinite of quantum (as amount) is thus mediated in the *natural number as the genuine infinite of amount and unit.*

For the rest of this paper, the concept of natural number used will be this genuine infinite. In this genuine infinite, the concept **N** of natural number is thus doubly determined. This two-fold character of natural number, described above, determines the two fundamental aspects of natural number in mathematics, the natural number as *cardinal* and the natural number as *ordinal*. We will conclude the paper by describing these two aspects very briefly, and by showing how the second of these gives the conceptual framework for the

successor function and induction principle of elementary number theory. This will then provide a Hegelian basis for these fundamental determinations of elementary number theory.

Firstly, the genuinely infinite concept **N** of natural number is the identification of quantum with itself in the moment of amount through its unit loss. Now quantum is finite *quantity* and its sublation in **N** is a genuine infinite, and so it follows that the concept **N** is now *quantitatively determined* as ``infinite'' quantum, which is therefore no quantum but rather sublated quantum. Since quantum here is quantum as amount, infinite quantum is ``infinite amount''. In more familiar mathematical terms, the concept of infinite amount is just **N** as the *infinite set* familiar in everyday mathematics, a set that contains other infinite sets (such as that of prime numbers). It is the conceptual determination of **N** through which it asserts its self-identity in the augmentation or removal of a unit, infinite quantum which embraces within itself such increase or decrease. This self-identity can be clarified by a definition[88] of an *infinite set* in set theory: in set theory, a set X can be defined to be *infinite* if it is equivalent to itself with one element removed, i.e. if, given an element a of X, there exists a one-to-one map from the set X onto the set X with a removed. To illustrate this in the case where X is the infinite set **N** and a=0, the map $n \rightarrow (n+1)$ is one-to-one from **N** onto the set of *positive* integers (which is, of course, **N** with 0 removed). The connection of this set-theoretic notion of the infinite with that of the genuine infinite of quantum is that in the latter infinite, the *self-preserving* of quantum through the loss of a unit translates to saying that **N** is conceptually identified with itself, conceptually the same, in the loss of a unit.

In particular, the quest for a canonical ``realization'' of the natural numbers is illusory since any realization of **N** as a set gives rise to an equally valid, different, realization of **N** by removing an element from that set.[89] This quantitative determination of the natural number will be called the *cardinal* determination of **N**.

The cardinal determination of number is the concept of infinite quantum, quantum (as amount) identifying itself with quantum through unit. But secondly, **N** is also the concept of *infinite unit*, the identification of unit with itself through amount. More precisely, as described earlier, unit (as first) is initially the bearer of the continuity of quantum. But unit as such disappears with the emergence of continuity in the rest (incoherently determined only as amount), only to return in the focussing of the continuity of the rest on a unit (the second). So conceptually, unit identifies with itself through amount. This gives **N** as *ordinal,* the infinite concept of unit. This concept is the conceptual basis for the common practice in mathematics of writing **N** as the (ordered) sequence 0,1,2,3,.... Strictly, the latter practice is an expression of **N** as a bad infinite since it is incomplete in principle, proceding as if it were counting a quantum but then negating this in the *and so on* of the ..., which reflects the helplessness of the procedure in face of the infinitude of **N**. The genuine infinite of **N** is then left implicit in the mathematical awareness of having ``got the pattern'' of the sequence of natural numbers.[90]

By contrast, the genuine infinite of **N**, as described above, does not express itself in the impossible task of trying to write down an infinite sequence, but rather gives the conceptual foundation for the natural number. We will now illustrate this by addressing the central question of the paper : that

of providing a conceptual basis for the successor function and the induction principle in terms of Hegelian logic.

Firstly, number as ordinal is expressed immediately in the concept of *the successor function*. This is nothing other than number as infinite unit, determined through the movement of unit back to itself through quantum. It is the preservation of the infinite concept of number through this unit movement.

The successor function is sometimes expressed in mathematics in the form

$$n \rightarrow n'.$$

The conceptual interpretation of this form is as follows. Here, the n stands for infinite unit, the variable number, number in general, while the → indicates the *conceptual movement* of unit within itself through which it is number as ordinal. Since this movement is being expressed in external, symbolic terms, it has a temporal character to it, as when the → (arrow) sign is used in everyday circumstances to indicate (e.g.) the direction of traffic flow in a system of one-way streets. This temporal character is easily illustrated in terms of our earlier discussion of ``first'' and ``rest'': we pick a ``first'' from a collection of things, then go back to the collection, and then pick a ``second'': $1 \rightarrow 1'=2$ (unit-quantum-unit).

The concept of the successor function is thus present in a very obvious way in everyday life. It is the ``Next, please!'' as uttered by a teller looking up at a line of people waiting in a queue at a bank.

Of course, when we assert that the sucessor function is the return of unit into itself, this does not mean that it is the reaffirmation of *abstract* unit as such, blank identity: 2 does not equal 1. The concept of unit involved in the successor function is that of *infinite* unit. Thus unit is mediated *through quantum* so that it sublates with itself and, in essence terms, the sublation involves *difference*, unit having been reflected back into itself through quantum. This difference is indicated in the change from n (number in general) to n': for $n \neq n'$. This difference leaves the formulation $n \rightarrow n'$ of the successor function open to an infinite progress:

$$\ldots n \rightarrow n' \rightarrow n'' \rightarrow \ldots.$$

undoing the infinite concept of number. Accordingly, the successor function is by itself an inadequate form of the concept of number.

As we will see below, this inadequacy of the successor function is overcome in the *induction principle* which is the expression of the truth of the natural number according to its concept and in which the difference involved in the successor function is folded into the concept in the asserting of its truth. (In particular, the induction principle includes conceptually within it the successor function $n \rightarrow n'$ as well as **N** as an infinite set[91] considered earlier.)

It thus remains to deal with the induction principle. We recall (§ 2) that this asserts that if P(n) is a propositional function with n ranging over the natural numbers n, then P(n) is true for *all* such n if the following two facts hold:

(1) for any m, the truth of P(m) implies the truth of P(m');
(2) P(0) holds.

Now the principle has to be true for *all* such (meaningful) propositional

functions, and since these express the truth of the natural number concept, the principle has therefore to be a truth of this concept *as concept* (in the Hegelian sense), and the proposition P can be dispensed with. For the true propositional function for the natural number is one that preserves or follows the concept of natural number. And the content of that concept's truth is expressed only in such propositions.

The ordinal concept of number has two aspects to it, one of which is the truth of (1) and the other that of (2). We deal with (1) and (2) in turn.

Firstly, the movement n →n' is, as above, the successor function, and expresses the preservation of the infinite concept of number in the form of the movement of the ordinal unit back to itself through quantum. But as we observed above, on its own, this is an inadequate form of the truth of that concept since it involves the unsublated *difference* of number in the forms of n and n', and is open to an infinite progress. What is needed is a complete expression of the truth of the number concept, in which it identifies with itself in its infinitude and sublates this difference.

This is expressed in (1): *the truth of the natural number is the* self-preserving of its concept through the successor function, i.e. through the mediation of infinite unit. This return in the form of the self-identifying of a number as successor through this mediation gives the concept of number, and it is this concept that is implicitly being asserted when the *truth* of P(n) gives the *truth* of P(n'), for n,n' are identifying in their common truth as number expressed in P. *The proposition's truth is preserved through the concept of number* and the difference n≠n' is sublated in the common determination. But the concept of number has another aspect to it not present in (1) alone, and this will emerge in our discussion of (2).

Indeed, where does (2) come into all of this? To answer this, recall that the concept of the natural number developed in this paper is the genuine infinite of amount and unit. This is the mediation of the affirmative infinite of amount and the affirmative infinite of unit through each other. Each of these infinites arose as a sublation of an infinite progress, arising from ``first'' and ``rest''. At the level of unit, without the ``first'' (which implicitly contain ``rest''), there is no conceptual movement in unit and hence no genuine infinite of unit.

(2) above is precisely the positing of unit as ``first'' (as the number 0), the initial determining of quantum continuity in the unit. The truth of P(0) in (2) is therefore the asserting of the truth of the concept of number in its self-preserving as expressed in P through this *initial* positing, the breaking up of the immediacy of quantum through which the conceptual dyanmic of number comes to be.

The complete concept of number as ordinal is then *both* this initial positing *and* the conceptual identification of number with itself through the successor function n →n'. Without that initial positing, the n in *n→n'* is an abstraction since it has lost the infinite progress whose sublation is the concept of number expressed in n. On the other hand, without the conceptual identification, the intial positing on its own gives the incoherent, unsublated infinite progress of ``first'' and ``rest''. The initial positing *and* the successor function are each conceptually through the other, and together express the full truth of the concept of the natural

number.

This concludes our discussion of a Hegelian basis for the successor function and the principle of induction in elementary number theory.

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Notes

[1]Professor of Mathematics, University of Mississippi (E-mail: mmap@olemiss.edu). General motivation for a Hegelian philosophy of mathematics is given in the author's paper: *Towards a Hegelian Philosophy of Mathematics*, Idealistic Studies, 27(1997), 1-10. A Hegelian approach to the philosophy of number through the paradoxes of set theory is given in the author's paper: *Self-reference and the natural numbers as the logic of Dasein*, Hegel Studien, 32(1997), pp. 93-121.

[2] *Hegel's Science of Logic*, translated by A. V. Miller, (Humanities Press International, Inc., Atlantic Highlands, N.J., 1989), p. 610.

[3]H. S. Harris, *Hegel's Ladder I: The Pilgrimage of Reason*, (Hackett Publishing Company, Inc., Indianapolis/Cambridge, 1997), p. 607.

[4]This work will be abbreviated to *WL* in the sequel. The edition of *WL* used in the paper is that of G. Lasson: G. W. Hegel, *Wissenschaft der Logik*, I, II, (Vol. 56, 57, *Der Philosophischen Bibliothek Band*), Verlag von Felix Meiner, Hamburg, 1967. The English translation of *WL* by A. V. Miller (*Hegel's Science of Logic*, *ibid*) will be abbreviated to *SL*. In quotations from *SL*, I have followed the present day practice of rendering *Begriff* by *Concept* rather than by *Notion*. The English translation of Hegel's *Encyclopedia Logic*, trans. By T. F. Geraets, W. A. Suchting and H. S. Harris, Hackett Publishing Company, Inc., Indianapolis, 1991, will be abbreviated to *EL*.

[5]cf. p. 11 of *The Philosophy of Betrand Russell*, Ed. P. A. Schilpp, Northwestern University, 1944.

[6]``... because the positedness is immediately being-in-and-for-itself, the *different moments* of the Concept are themselves the *whole Concept* ...'', *SL*, p. 596.

[7]A good example of this is Gödel's replacing the individuals of Russell and Whitehead in *Principia Mathematica* by the natural numbers which Russell and Whitehead originally constructed using the individuals.

[8]This is a famous slogan of Hilbert's Göttingen school -- see for example the discussion in *Hilbert, Towards a Philosophy of Mathematics*, J. Fang, Paideia, N.Y., 1970, p. 84.

[9]The mechanical aspect of proof theory was interestingly foreseen by Hegel. In *SL*, p. 812 (*WL*, II, pp. 470-471), the *proof* of a theorem is defined as that which *contains the mediation of what the*

theorem enunciates as connected and Hegel comments (with reference to proof) that because ``the content determinations of the theorem are not at the same time posited as concept-determinations but as given *indifferent parts* standing in various external relationships to one another, it is only the *formal, external* concept in which the necessity manifests itself''.

[10]See, for example, his 1925 paper: *On the infinite*, trans. S. Bauer-Mengelberg, pp. 367-392, in: *From Frege to Gödel*, Ed. J. v. Heijenoort, Harvard University Press, Cambridge, Mass., 1967.

[11]``... contentual inference is replaced by manipulation of signs'' (Hilbert, *ibid*, p. 381).

[12]Note, though, that there are *infinitely many* terms in a formal system but there are not infinitely many single chess moves.

[13]The influence of the Kantian regulatory philosophy of the *ideal* on this view of Hilbert is apparent.

[14]Of course, in the incompleteness theorems of Gödel, even this can no longer be regarded as tenable, certain arithmetic statements which for the mathematician are either true or false either being no longer decidable in the formal system or the latter containing a contradiction.

[15]H. Poincaré, *Les mathématiques et la logique*, Revue de métaphysique et de morale, 13(1905), pp. 815-835; 14(1906), pp. 17-34. A helpful discussion of the issues involved in the induction controversy between Poincaré, Hilbert and Weyl is given in v. Heijenoort, *ibid*, pp. 480-2.

[16]In the formal context, this is called a wff with one free variable. However, in the non-formal context, we have preferred to use Russell's more intuitive terminology *propositional function*.

[17]D. Hilbert, *ibid*, pp. 372.

[18]It is significant that even the elementary logic of statements presupposes non-trivial number theory. Contrary to the logicist position, formal logic requires number and the latter is not reducible to the former.

[19]Of course, Gödel developed this theme in great depth in the proof of his incompleteness theorems.

[20]As another example, to prove the consistency of any first order predicate calculus, one uses the *associated propositional form* together with an induction argument on proof length. (See, for example, W. S. Hatcher, *Foundations of mathematics*, W. B. Saunders Company, Philadelphia, 1968, p. 37 and p. 208.)

[21]The induction principle as used in metamathematics is the informal induction principle of elementary number theory. The use of the induction principle *within* the formal system **S** is actually weaker than that in informal number theory since the principle within **S** only applies to a countably infinite family of propositional functions.

[22]cf. ``In contrast [to Being and Essence] the moment of the *Concept* is *development* through which only that is posited which is already implicitly present'', *EL*, p. 237.

[23]cf. the discussion by Hegel of the logical *method* that has resulted in the absolute idea as form, *SL*, pp. 825ff. (*WL*, II, pp. 484ff.).

[24]T. S. Eliot, *Four Quartets*, *Burnt Norton*.

[25]p. 177 in : J. Burbidge, *Real Process: How Logic and Chemistry combine in Hegel's Philosophy of Nature*, University of Toronto Press, Toronto, 1996.

[26]*SL*, p. 107.

[27]Formal logic can be described in terms that Hegel used to describe the Concept in its immediacy (*WL*, II, p. 236): *er nur Inneres ist, so ist er ebensosehr nur ein Äußeres*. This applies to formal logic in the sense that its source in living mathematical and logical argument, from which it has abstracted its forms and practices, is made ``inner'' by this logic, left hidden as if they had never been considered, while these practices are completely ``outer'', explicit and mechanical.

[28]*WL*, I, p. 9.

[29]H. Weyl, *The Continuum*, trans. S. Pollard and T. Bole, Thomas Jefferson University Press, Kirksville MO, 1987, p. 111.

[30]As Hegel says in *SL*, p. 777 (*WL*, II, p. 432) with reference to the Concept as self-consciousness, self-consciousness is ``the absolute relation-to-self that, as a separating judgement, makes itself into its own object and is solely this process whereby it makes itself into a circle.'' See also Hegel's views on the infinite as a circle in his discussion of *dasein*. (There are different views in the literature about how Hegel's term *Dasein* should be translated into English. In this paper, I have preferred to leave it untranslated as *dasein*, building up its meaning in the discussion of § 3. The Hegelian term *Fürsichsein* will also be left untranslated as *fürsichsein* for reasons discussed in Note 38.

[31]*SL*, p. 111; *WL*, I, p. 98.

[32]*SL*, p. 114; *WL*, I, p. 101.

[33]*SL*, p. 487; *WL*, II, p. 109.

[34]Jacob Boehme, *Aurora*, trans. John Sparrow, Kessinger Publishing Company, Kila, MT, 1992, pp. 40-41.

[35]*SL*, p. 127; *WL*, I, p. 114.

[36]*SL*, p. 132; *WL*, I, p. 120.

[37]*SL*, p. 136; *WL*, I, p. 124.

[38]As for *dasein*, we will leave *Fürsichsein* untranslated

as *fürsichsein*. Two reasons why rendering *Fürsichsein* as *being-for-self* is unsatisfactory are as follows. Firstly, the German term is coherent, unhyphenated, not like the composite expression *being-for-self*. Secondly, and related to the coherence of *Fürsichsein*, the latter has a verbal use, e.g. *fürsichseiende* (*WL*, I, p. 363), while the corresponding use of *being-for-self* in English is surely unacceptable.

[39]*SL*, p. 178; *WL*, I, p. 169.

[40]*SL*, p. 162; *WL*, I, p. 152.

[41]Hegel does not seem to distinguish an immediate one from its abstract version as we have done in this paper, but it seems to me to be required for his argument.

[42]*SL*, p. 163; *WL*, I, p. 154.

[43]cf. Burns's poem *Tam o' Shanter* where Tam's long-suffering wife Kate sits at home ``nursing her wrath to keep it warm'', waiting for her ``blethering, blustering, drunken blellum'' of a husband to return from his ``bousing at the nappy''.

[44]*SL*, p. 172; *WL*, I, p. 163.

[45]*SL*, p. 172; *WL*, I, p. 163.

[46]*WL*, I, p. 158.

[47]*SL*, p. 165; *WL*, I, p. 155.

[48]*WL*, I, p. 154.

[49]*SL*, p. 167; *WL*, I, p. 158.

[50]*SL*, p. 169; *WL*, I, p. 160.

[51]*SL*, p. 168; *WL*, I, p. 158.

[52]The *fürsichsein* categories, and in particular, repulsion, are realized in *Hegel's Philosophy of Nature*, trans. A. V. Miller, Oxford, Clarendon Press, 1970. In that context, repulsion can be regarded as Hegel's conceptual version of the ``big bang''. Indeed, the negative self-relation of the *fürsichsein* of the ``center'' is (as formal) a repulsion *into the many immobile centres (stars*) (pp. 60-61). So *matter, in filling space, erupts into an infinite plurality of masses* (p. 62).

[53]*SL*, p. 170; *WL*, I, p. 161.

[54]*SL*, p. 170; *WL*, I, p. 161.

[55]*SL*, p. 171; *WL*, I, p. 162.

[56]*das Setzen des Eins als solchen, des reellen Eins,* WL, I, p. 167.

[57]*SL*, p. 174; *WL*, I, p. 165.

[58]*SL*, p. 177; *WL*, I, p. 168.

[59]*SL*, p. 177; *WL*, I, p. 168.

[60]*SL*, p. 178; *WL*, I, p. 169.

[61]*SL*, p. 185; *WL*, I, p. 177.

[62]*SL*, p. 187; *WL*, I, p. 179.

[63]*SL*, p. 200.

[64]*SL*, p. 201; *WL*, I, p. 195.

[65]*SL*, p. 129; *WL*, I, p. 116.

[66]*SL*, pp. 202-203; *WL*, I, pp. 196-197.

[67]*SL*, p. 203; *WL*, I, p. 197.

[68]*SL*, p. 203; *WL*, I, p. 197.

[69]It follows from this that for Hegel, the unit of a quantum conceptually involves the whole quantum itself. So unit is never in isolation but always determined in the context of a quantum. This relates to the problem, considered by the early Academy and discussed by Aristotle (Metaphysics, Book XIII), concerning the ``comparability'' of units, in particular, whether the units in the ``ideal 2'' are non-comparable with the units in the ``ideal 3''. Presumably for Hegel, the two 1's in the number 2 are given an additional ``two-ish'' determination as comprising 2, which distinguishes them from the three 1's in the number 3.

[70]*SL*, p. 205; *WL*, I, p. 200.

[71]Another account of the species of calculation is given in *EL*, pp. 162-163.

[72]*SL*, p. 213; *WL*, I, p. 208.

[73]*SL*, p. 216; *WL*, I, p. 212.

[74]*SL*, p. 211; *WL*, I, p. 207.

[75]The fundamental, simple character of **S** made it the natural choice for that canonical system relative to which other formal systems might be shown to be consistent. Of course, such a hope was dashed by Gödel's incompleteness theorems.

[76]Good references for the system **S** are the books: Hatcher, *ibid*, pp. 69ff. and E. Mendelson, *Introduction to Mathematical Logic*, D. Van Nostrand Company, Inc., New York, 1964, Chapter 3.

[77]Hatcher, *ibid*, p. 31.

[78]If we wanted to economize, we could get dispense with the axioms 5 and 6 by using the others (Hatcher, *ibid*, pp. 73-74).

[79]More precisely (Hatcher, *ibid*, p. 71), A(0) and A(x') are obtained from A(x) by substituting 0,x' respectively for all of the *free* occurrences of x in A(x).

[80]*SL*, p. 409; *WL*, II, p. 24.

[81]*SL*, p. 679; *WL*, II, p. 326.

[82]The infinite in number does occur later in *WL* at a more advanced level than that of the natural number. This infinite in *WL* arises out of the consideration of fractions (rational numbers), in which a fractional number can be expressed in *infinitely* many ways as a fraction, e.g.

$$3/7=6/14=9/21=...$$

Here, Hegel anticipated the modern way of constructing the (non-negative) rational numbers from the natural integers in which the rational numbers are defined to be *equivalence classes* of pairs (m,n) of natural numbers with n>0, where (m,n) is defined to be equivalent to (p,q) if pn=mq.

[83]The position of the monarch in the state in Hegel's Philosophy of Right is on a different level from that of the *first* above. The monarch in the (rational) state is the moment of *individuality* (or *subjectivity*) of the latter's conceptual basis and so is at the level of the Concept, and cannot be separated off from the forms assumed in the state by the other two moments of the Concept (universality and particularity). (See *Hegel's Philosophy of Right*, trans. T. M. Knox, Oxford, 1967, pp. 179ff..) In contrast, the *first* above is at the level of being in which the ``cohesion'' or organic character of quantum is minimal, the units ``indifferent'' to one another. This disparity between this indifference and the continuity of quantum is what is resolved in the explicit positing of the latter in the *first*.

[84]Of course, when the counting is finished, we can reassemble the original quantum by piling the counted units together again, but we need to use some external feature (for example, such as color or unit placement on a table or even an attached tag (as with luggage in an airport)) to remind us that they belonged to the original quantum. The point is that the counted unit qua unit has escaped free from the control of quantum, has reverted to the stage of an independent ``one'' with no intrinsic reference to the quantum to which it once belonged. The ``indifference'' with which the units of quantum cohered in quantum has achieved its fulfillment in this quantum-independent one.

[85]*WL*, I, p.128.

[86]*ibid*, pp. 28-34.

[87]Note that while the genuine infinite of *dasein* is the mediation of the finite *and* affirmative infinite through each other, in contrast, the genuine infinite of quantum is the infinite mediation of the moments unit and quantum through each other. In the latter case, the affirmative infinite of quantum is thus *not* one of the alternating terms in a second infinite progress, but rather this progress is replaced by the *mediation* of that affirmative infinite. The affirmative infinite of quantum is the infinite of quantum as amount (and only implicitly of unit) in its immediacy, while in the genuine infinite, amount *and* unit *both* become genuinely infinite through each other.

[88]This a variant of a definition of *infinite set* due to Dedekind.

[89]This deals with a serious problem confronting Platonism in mathematics. If the natural numbers really are objective, then we should be able to exhibit them as canonical objects. But in mathematics, there are many different models of the natural numbers. Would the *real* natural numbers please stand up! A discussion of this with reference to the different set theoretic models of the natural numbers due to von Neumann and Zermelo is given in the delightful paper of Paul Benaceraff (*What numbers could not be*, repr. in: P. Benaceraff and H. Putnam (eds.), *Philosophy of Mathematics*, 2nd edition, Cambridge University Press, Cambridge, 1983). What the Hegelian based philosophy of number developed in the present paper shows is that number is a conceptual structure developed initially out of the concept of abstract being, and what particular ``beings'' or ``objects'' one hooks the numbers onto is irrelevant.

[90]Further, when this ``ordering'' of **N** is used in the counting of a quantum, it does not by itself give a particular empirical ordering on the units of the quantum since the concept of quantum as described above is independent of the particular units chosen -- we need to specify *which* ones to pick out for the first, second, etc.. The above ``bad infinite'' is not an ordering of *numbers* as if they were lying there before us waiting to be arranged in their proper order, but rather is the inadequate expression of the genuine infinite of **N** in the temporal medium of unit returning to unit recorded by number names, repeated until we get tired.

[91]The ordinal aspect of number, i.e. the identification of unit with itself through quantum, implicitly also asserts the cardinal determination of number discussed above. (This follows from the earlier discussion of the genuine infinite of quantum and unit.) This is reflected in the ``set theoretic'' version of the induction principle in which the propositional function is replaced (through its truth values) by an *inductive* subset of **N**.