Introduction to and Translation of Hegel's *Geometrical Studies*

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ABBREVIATIONS

The following abbreviations are used throughout.

Dok. *Dokumente zu Hegels Entwicklung,* herausgegeben von Johannes Hoffmeister, Frommann, Stuttgart*, 1936.*
GS *Geometrische Studien*, by G. W. F. Hegel, pp. 288-300 of *Dok..*
Hea. *The thirteen books of Euclid's Elements, Vol. 1,* translated by T. L. Heath, Dover Publications, New York, 1956.
JL *G. W. F Hegel The Jena System, 1804-5: Logic and Metaphysics,* translation edited by J. W. Burbidge and G. di Giovanni, Introduction and explanatory notes by H. S. Harris, McGill-Queen's University Press, Kingston and Montreal, 1986.
JS *Georg Wilhelm Friedrich Hegel Jenaer Systementwürfe II* , Herausgegeben Rolf-Peter Horstmann und Johann Heinrich Trede, Deutsche Forschungsgemeinschaft, Bonn-Bad Godesberg, 1971.
Lor. *Euklids Elemente, funfzehn Bücher,* translated into German by Johann Friedrich Lorenz, Halle, 1781.
NP *Enzyklopädie der philosophischen Wissenschaften* , *zweiter Teil, Naturphilosophie (1830)*, G. W. F. Hegel, herausgegeben von F. Nicolin und Otto Pöggeler, Verlag von Felix Meiner, Hamburg, 1959.
P Proclus: A Commentary on the first book of Euclid's *Elements*, translated by G. R. Morrow, Princeton University Press, Princeton, N. J., 1970.
PN *Hegel's Philosophy of Nature*, translation of Hegel's *Naturphilosophie (NP)* by A. V. Miller, Clarendon Press, Oxford, 1970.
SL *Hegel's Science of Logic*, translation of Hegel's *Wissenschaft der Logik* by A. V. Miller, Humanities Press International, Inc., Atlantic Highlands, NJ, 1990.
TS *Hegel's Development, Toward the Sunlight, 1770-1801* , by H. S. Harris, *Oxford, Clarendon Press, Oxford, 1972.*
WL *Georg Wilhelm Friedrich Hegel, Wissenschaft der Logik* , G. Lasson, ed., two volumes *WL I, WL II,* Verlag von Felix Meiner, Hamburg, 1967.

Euclid 1 stands for the first book of Euclid's *Elements.*

Introduction

The material of Hegel's *Geometrische Studien* (*GS*) poses serious difficulties for the interpreter. Much of the text is fragmentary, and its assertions cryptic. It is what has survived of a much longer

investigation by Hegel into geometry. The translators into English of the Jena Logic (*JS*), which Hegel wrote soon after completing *GS*, comment[1] that their ``manuscript is written in a language that is just as cumbersome and obscure as only Hegel (and especially the younger Hegel) would use''. This difficult situation is made even worse in *GS* where much of it is not even in the state of a fair copy (*reine Schrift*). In addition, to understand *GS*, one requires extensive background in classical Greek geometry and its philosophy, as well as in Kantian and post-Kantian philosophy. Yet even with that background, *GS* poses many problems for the reader. To help orient the reader in reading this English translation of *GS*, I have written a long introduction. This falls into two parts. The first part, **A**, presents a background to the philosophy of geometry presented in *GS*, and to the geometry out of which this philosophy arose. It's objective is to give an overview of *GS*. In this part, the great commentary on Book 1 of the Elements of Euclid by the Neo-Platonic philosopher Proclus is used as a framework for understanding the Hegelian work. The second part, **B**, gives a more detailed account of the main themes of *GS*. A number of the topics of **A** are, of course, revisited and discussed in more detail in **B**. While a certain amount of duplication of the material in **A** and **B** was inevitable, I felt that this approach in the Introduction was the best way to deal with the content of *GS*.

The philosophy of *GS* presents an astonishingly erudite account of Euclidean geometry, in which the great insights of the classical Greek philosophers and geometers are integrated into a modern, post-Kantian perspective. It's ideas are at first sight strange since most of the classical geometrical philosophy that inspired it has been forgotten over the last 100 years because of the influence of the dominant, set theoretically based, mathematical philosophy (in which Cantor and Frege played such formative roles). Present day mathematics has, however, in topology and differential geometry, developed its own geometrical intuition and insight, based on, but going far beyond, Euclidean geometry, while at the same time, incorporating the great insights of Cantor and Frege (when these insights are needed).

Given that almost all mathematicians are, and always have been, (mildly) Platonic in their view of mathematics – mathematics is about *something* – the need for a philosophy of mathematical objects is inescapable. Rather than just being content with producing axioms which will allow mathematicians to do what they would do anyway and hoping (*pace* Gödel) that some inspiration will ennable the ``correct'' axioms for set theory to ``force'' themselves on us, we need to confront the nature of these objects and elucidate the philosophical principles through which they are to be understood. Particularly pressing is the need to understand the most visually obvious mathematical objects, those of geometry, which express our intuition of space and which are fundamental for physical science – not just the objects of classical geometry: triangles, circles, planes, etc., but also of their profoundly developed, modern counterparts: Riemannian manifolds, geodesics, the curvature tensor, etc.. The author's hope is that the tantalizing fragments on the philosophy of geometry that are translated here, written by arguably the greatest of all philosophers, will prove stimulating in the search for a present day philosophy of mathematics.

A. General background to the philosophy of geometry in *GS*

It is generally agreed that the core and rationale for Hegel's philosophy is to be found in his Logic, a substantial part of which is concerned with mathematics and its nature. In the importance that Hegel gives to mathematics in his logical studies, he prefigured the dominant role that mathematics plays in present day logic, so that the latter is often called mathematical logic. While I think that it can be convincingly argued that modern logic takes this dominance too far, Hegel does not make this error, and his logic spans the full range of concepts required for thought in general: science, religion, political and ethical thought and of course mathematics itself. The need to consider the somewhat difficult issues of logic in general, and mathematics in particular, in Hegelian philosophy is inescapable. Through such consideration, the structure of thought in general becomes explicit, or equivalently, in Hegelian terms, the concept becomes for-itself. Without the logical perspective, Hegelian thought is in danger of becoming ``soft'' and imprecise, at the mercy of thought content in which what is essential and what is on the surface are not distinguished. To concentrate only on Hegel's Phenomenology or only on his political and religious thought surely has as a consequence the loss of many of Hegel's deepest insights.

Hegel's most serious, early, exposure to mathematics came through his reading of geometry in the *Elements* of Euclid. Hegel's study of and interest in geometry continued to the end of his life. He clearly thought very seriously about it, as is evidenced by his discussions of geometry in his mature logic *WL*. The importance of Hegel's geometrical thought for his logic of mathematics motivates its study in more detail, and the purpose of the present work is to discuss in detail, and give an English translation of, Hegel's most extensive and systematic discussion of geometry, what survives of his early manuscript *Geometrische Studien* (which we abbreviate to *GS*).

Hegel completed his work on *GS* in the Frankfurt period. Indeed, the part of *GS* dealing with the first twelve propositions of *Euclid 1* is dated 23 September, 1800 when Hegel was visiting the nearby city of Mainz. At that time, Hegel was based in Frankfurt, employed as a *Hofmeister* (private tutor). He left for Jena in the January of 1801 to join Schelling.

Some of the material of *GS* goes back much earlier than 1800.[2] It contains a philosophical and mathematical discussion of the first twelve propositions of *Euclid 1* (the first book of Euclid's *Elements*) and two other later propositions, an account of mathematical objects in general, an account of parallel lines and lastly some algebraic calculations with line lengths. The manuscript is at Harvard University. With minor changes, we follow the edition of *GS* given by Hoffmeister in *Dok*..[3] Helpful comments on *GS* are given in the latter's *Anmerkungen*.[4]

In his study of *Euclid 1*, Hegel used its German edition *Lor.* by Johann Friedrich Lorenz.[5] The Lorenz versions of the propositions of Euclid discussed by Hegel are given in the *Anmerkungen* of Hoffmeister. As

early as 1787, Hegel made at least one excerpt from Lorenz's edition of Euclid[6] and Rosenkranz records mathematical excerpts mainly from the work of Kästner, as well as from a school book for Geometry (presumably *Lor.*).[7] Hegel records in his *Tagebuch* (diary) that he studied excerpts on spherical trigonometry that he copied out of Lorenz's work. (This is a puzzling statement since there is no trigonometry in *Lor.*.)

According to the authoritative English translator of the *Elements*, T. L. Heath, Lorenz's work is ``the first attempt to reproduce Euclid in German word for word''[8] and Lorenz himself in the title page to his book says that his work is a translation out of the Greek. However, this is not quite accurate. The definitions, postulates and axioms of Lorenz given in his version of *Euclid 1* are not exactly the same as those of Euclid – for example, Lorenz has 12 axioms compared with Euclid's 5. The statements of the propositions given by Lorenz are translations of Euclid's statements. The proofs that Lorenz gives are (usually) much shorter and more readable versions of those of Euclid, and he sometimes adds additional proofs of his own: for example, he gives two proofs for Proposition 31 of Book 6. *Lor.* (in my opinion) is an excellent, lucid, though also uncompromising, treatment of the *Elements*, that manages to get the essential content of all of the 15 books of the *Elements* in a one volume work of 367 pages. Clearly Hegel liked the book, for he used the 1793 edition of Lorenz's book for his own school teaching of the ``Elements of Mathematics''.)

Diagrams used in some of the propositions discussed by Hegel from Lorenz's book are given by Hoffmeister in his *Anmerkungen*. In that connection, one of Lorenz's diagrams (that for Proposition 9) is misleading: the triangle ?ADE in that diagram appears as equilateral, whereas in fact in Euclid's original diagram, it is only isosceles. (As we will see, this seems to have caused Hegel a misunderstanding.) The other diagrams of Lorenz are the same as those of Euclid given in *Hea.*. Further, only the diagrams for Propositions 5,6,7,8,9,11 and 12 are given by Hoffmeister, and for consistency, it seemed better to reproduce the Euclidean diagrams for all of the propositions discussed by Hegel. For these reasons, the present translation reproduces the original diagrams of Euclid, as given in *Hea.*, for these propositions.

Hoffmeister says that the ``model'' (*die Vorlage*) on which Hegel develops *GS* was probably *Lor.*. I feel, however, that the influence of Lorenz on *GS* is quite minimal. Indeed, there is no philosophy in *Lor.,* but much in *GS*, and while, no doubt, Hegel first learned geometry from studying *Lor.,* he went far beyond it, in particular studying in depth for himself the works of the great Greek geometers and their geometrical philosophies. Hoffmeister[9] points out that the unsatisfactory nature of the Euclidean approach to geometry was recognized and worked on in the second half of the 18th century by mathematicians such as Karsten (1778), Hindenburg (1781), Schulz (1783) and Lambert (1786), but it is not clear to what extent their work influenced Hegel's treatment of geometry in *GS*.

We now begin our discussion of the philosophy of geometry developed by Hegel in *GS*. In my opinion, Hegel's geometrical philosophy is both original and relevant to present day interests in the philosophy of mathematics (as far as the nature of mathematical objects is concerned). However, like all philosophers, Hegel did not work in a vacuum, and relied heavily on the ideas of previous thinkers for his

own original insights. I believe that to understand what Hegel is trying to say in the often fragmentary assertions of *GS*, it is essential to view it against the background of classical Greek geometrical philosophy. (Indeed, more generally, the study of *GS* requires serious consideration of the pre-Fregean philosophy of mathematics, a philosophy which has been almost completely ignored in present-day philosophy of mathematics and well deserves reconsideration.)

In particular, to make sense of what Hegel is saying in *GS*, I believe that we need to set it against the background of the *NeoPlatonic* philosophy. This gives the most systematic presentation of classical Greek geometrical philosophy. In this context, we must not think of the NeoPlatonic philosophy of geometry in a narrow sense. Rather, it is to be regarded as the culmination of the complete Greek geometrical tradition, and embraces not only the thought of the great philosophers, such as Plato and Aristotle, but also of the great geometers, such as Pythagoras, Euclid, Archimedes and Apollonius.

Of course the Kantian and post-Kantian idealist philosophy, dominant in Germany in Hegel's early years, also is fundamental in understanding *GS* (especially with its emphasis on the concept). However, post-Kantian philosophy, as briefly discussed below, was itself influenced by the NeoPlatonic philosophy, and so the influences of these two philosophies on Hegel's geometrical thought are not incompatible.

This leads to what I believe is a central question for the student of *GS*: what are the main sources in classical Greek philososphy used by Hegel when writing *GS*? The problem is that nowhere in *GS* – or at least, what has survived of the original manuscript - is there any explicit reference to the NeoPlatonic tradition in mathematics. Even in his later logical works, *JS* and *WL*, there are very few explicit references to that tradition. Hegel's main source for Euclidean geometry, *Lor.*, does not help us with this question, since it does not discuss the philosophical issues involved with geometry.

In this introduction, I will argue that the main philosophical source for Hegel's philosophy in *GS* was probably the commentary on the first book of the *Elements* by the last great NeoPlatonic philosopher, Proclus. Further, Hegel's interest in NeoPlatonic mathematics was stimulated by the NeoPlatonic basis for much of the German idealist philosophy to which Hegel was exposed in his early years and no doubt also through his own intense studies of classical authors. We start by discussing very briefly the Proclus commentary on the *Elements.*

NeoPlatonism and Geometry

Proclus lived in the 5th century and in his later years, was head of the Academy in Athens. (As a result of the edict of Justinian, the Academy was closed down in 529 not long after the death of Proclus.) As Morrow comments in his translation *P* of the Proclus commentary[10], the latter contains ``priceless information ... regarding geometers and the history of geometry prior to Proclus' time''. T. S. Heath, whose (1908) three volume translation of, and commentary on, the *Elements of Euclid*, remains even today a major resource for Euclidean scholarship, goes on to say[11] that the commentary, together with the Collection of Pappus, is ``the more precious because the original works of the

forerunners of Euclid, Archimedes and Apollonius are lost''. In addition, and of particular importance for philosophy, Proclus gives a unique presentation of Greek geometry understood within the perspectives of the Platonic and Pythagorean philosophies.

The Proclian account of geometry (and more generally, of mathematics) is highly Pythagorean in character as well as Platonic. We can summarize it as follows in terms of the *Prologue* to *P*. The quest for the first principles for mathematics is (for Proclus) the search for ``the all-pervading principles that generate everything from themselves''. There is thus no clear distinction between pure and applied mathematics: the principles of mathematics go back to the principles of everything. There are two such principles: they are l*imit* (or end (p??as)) and the *unlimited* (or boundless, endless[12] (ape????)), that head the two columns of the Pythagorean categories. The *unlimited* is generative in character while *limit* acts as a bound for the unlimited and so gives rise to determinate things. According to Aristotle[13], under *limit*, the Pythagoreans listed categories such as *odd*, *one*, *male*, *straight*, *good* and *square*, while under the *unlimited*, they listed the corresponding categories such as *even*, *many*, *female*, *curved*, *evil*, and *oblong*. The pairings of corresponding geometrical categories in the two columns appear in the thought of Proclus, Euclid and Hegel. So for Proclus, the unlimited and limited for *line* are respectively the *straight line* and the *curved line* (especially the *circle*[14]). Also the reduction of the (unlimited) *oblong* (rectangle) to its limited counterpart, the *square*, at the end of the second book of Euclid's *Elements*, is seen, by Hegel[15], as ``an equation between the self-equal, or the square, and that which is in its own self unequal, the rectangle''. (The simplicity or *self-equality* of the Hegelian concept, exemplified in this reduction, will be discussed later.)

As in Plato's Philebus[16], everything (for Proclus) is a ``conjunction'' of the unlimited and limit, though some things lean more to one than the other. So for example, the point, as completely simple, fixed in its position, is all limit. But as we will see later, it does have an unlimited side. This side is implicit, ``secret'', but is made explicit in its expansion, the dialectic of which is expressed in the point-line-surface-solid development. On the other hand, the idea of ``body'' has more of the unlimited in it, and so is bounded from without (rather than intrinsically in itself) and is ``divisible to infinity in all directions''.[17] Above limit and the unlimited lies the *One*, the ineffable, incomprehensible, dialectical source of all that is, including the mathematical beings. Reality flows in its orders out (procession) from the One, and reverts back to it (recession).

Beneath these two principles, Proclus, following Plato's *divided line* imagery of the *Republic*, distinguishes three levels of reality: (1) the ``highest intelligence'' *noesis* (?ó?s??), (2) the level of discursive thought or understanding, *dianoia* (d?????a), and (3) the world of sense perception and its thought in the form of *opinion*, *doxa* (dó?a). The three levels decrease in order of clarity and comprehension, the top one (noesis) being ``immediate, total and sure comprehension'', to which, I suppose, the Kantian intuitive intellect, that was such strong motivation for Schelling and Hegel, corresponds. At the next level, that of the understanding (corresponding to the Kantian *Verstand*), is the world of *mathematics*, of geometry in particular.

These three levels are cognitive in character, and a natural, pressing philosophical problem for mathematics (even today) is: *where does the geometer gets his objects such as triangles, circles, squares etc., and how do these relate to their sensible counterparts?* So, for example, how does the mathematical triangle relate to one drawn with a pencil and ruler (or in the sand). The first triangle is exact and unchanging, having infinitely thin, perfectly straight sides, while the second is drawn through a temporal process, can be erased, is ``rough'', and the lines are not exactly straight and have (varying) positive breadths. The solution to this problem for Proclus is as follows. The objects of mathematics come[18] by projection of mind (soul) from the vast store of forms implicit within the noetic world and of which the world of sense perception only provides, in drawn diagrams, a reminder (a??µ??s??). (As Morrow comments in his introduction to *P*, this theory of projection anticipates the Kantian schematism of the categories.)

There are three levels at which the notion of *triangle* can be taken. There is first the *exact* mathematical triangle which is at the level of forms, completely simple in itself. Then there is the *mathematical* triangle which is a projection of the formal triangle in imagination (with some kind of extendedness and matter). Lastly, there is the *sensible* triangle which is a projection of the mathematical triangle with the rough and ready nature of the drawn figure. Through the medium of the sensible triangle, we can ``see'' its originals, the *genuine* mathematical triangle and the simple form behind that, and this explains why, in geometry, the proofs are accompanied by drawn figures. Against the background of this projection theory, Proclus works out in great detail in *P* his interpretation of *Euclid 1*.

NeoPlatonism in Hegel's Frankfurt period

Neoplatonism was, we may say, ``in the air'' at the end of the 18th and the beginning of the 19th centuries in German thought. In particular, in its aesthetic form, it features strongly in the work of Hölderlin (for example, in his *Hyperion* of 1797-1799) and Schelling (in the latter's *Bruno* of 1802). Hegel came to Frankfurt in 1797 as a private tutor (*Hofmeister*) at the invitation of his friend Hölderlin, who was also a *Hofmeister* there. Hölderlin greatly influenced Hegel[19] in their time together at Frankfurt, and so it is reasonable to attribute to Hölderlin a NeoPlatonic influence on *GS*.

In this connection, Klaus Düsing[20] discusses the influence of the aesthetic Platonism of Hölderlin on Hegel at that time. Aesthetic Platonism, in which the beautiful is given the highest ontological status, was developed in the Florentine academy by Marsilio Ficino and Leone Ebreo. Düsing points out that Hegel used an edition of Plato that contains Ficino's interpretation. Further, earlier in his life at the Tübingen *Stift*, Hegel, with Hölderlin and other friends, studied the writings of Plato (as well as those of Kant, Jacobi and Spinoza), and even made attempts to translate some of Plato's work.[21] So the sustained common interest of Hölderlin and Hegel in the work of Plato over a number of years is well documented.

However, the ecstatic, emotionally unstable, aesthetic idealism of Hölderlin's *Hyperion* seems almost completely absent from Hegel's work during the Frankfurt (or at any other) period.[22] In fact the bulk of his

writings in the Frankfurt period are on religious and political themes rather than on aesthetic ones. The striking exception is the *Ältesten Systemprogramm des deutschen Idealismus* (The earliest system-program of German idealism), with its enthusiastic endorsement of the aesthetic, in particular, of poetry (*Dichtung*). However, the authorship of the *Ältesten Systemprogramm* is controversial – Hegel, Hölderlin and Schelling have been suggested, with the consensus of scholars, including O. Pöggeler and H. S. Harris, settling for Hegel. [23]

Since also, as far as I can see, Hölderlin had little or no interest in Euclidean geometry, it seems to me that the NeoPlatonic background to *GS* has to come from elsewhere. In our discussion of this, we will first describe how the thought world to which Hegel belonged can be regarded as a transformed NeoPlatonism[24], and then secondly, will discuss the possible influence of Proclus on Hegelian geometry.

Post-Kantian philosophy as a transformed NeoPlatonism

The title of this section must not be taken to imply that post-Kantian philosophy in Germany is just a *repetition* of NeoPlatonic thought. The former is far more wide-ranging with many new ideas. What we are saying is that Neo-Platonic thought greatly influenced post-Kantian philosophy, and hence there is a basic common pattern of thought in these two philosophies. This explains to some extent the NeoPlatonic character of Hegelian geometry.

The role of the NeoPlatonic One is taken over in the Kantian philosophy by the transcendental unity of pure apperception, but with this crucial difference that the manifold content to which this unity applies is given in external intuition, and the unity is empty without that manifold content which it unifies in concepts, judgements and syllogisms. (There is nothing of this dependence on sensible intuition in classical NeoPlatonic thought - rather, as for Plato, sense perception is a transitory, derivative form of knowledge, whose primary function is to point back to the genuine world of pure forms on which any validity that sense perception has is ultimately based.)

The Kantian philosophy gave rise to antinomies through the role played in it by the *Ding-an-sich*, the reality, independent of the observer, that is the source *somehow* of the manifold content on which the unity of pure apperception goes to work and of the distinction between the *noumenal* and the *phenomenal*. Fichte's modification of the transcendental philosophy and the influence of Kant's third Kritik led, in post-Kantian thought, to the abolishment of the Ding-an-sich, and with the return of dialectical thought, a view of the world emerged that was closer to that of the NeoPlatonic. Concept and intuition were now no longer completely separated, and instead, in Schelling's form of the Fichtean system, cognition and being are integrated in the *absolute* which is ``*the point of indifference*'' (using the language of magnetism) or *absolute identity*.[25] Thought and being are then not to be conceived of as radically different but rather each is through the other and they are just different forms of the same ultimate reality or absolute. In philosophy, Schelling says, we meet with nothing ``but the Absolute – always just strictly the One and just this unique One in particular forms''.[26] Recalling that the source of the world in NeoPlatonic thought is also the *One*, we see that there is a very obvious NeoPlatonic basis for Schelling's identity philosophy.

And it is, I believe, this fundamental identity of basis for both the NeoPlatonic and the identity philosophies that ennables Hegelian geometry to use both. The Platonic and NeoPlatonic investigations into geometry, that depended on the method of division, the Pythagorean categories and classical dialectic, in Hegel's hands, became a post-Kantian philosophy of concepts functioning within an identity philosophy.

The relation between a NeoPlatonic philosophy of the One and the philosophy of the concept in the thinking of Hegel is made explicit in the *Jena Logic*, written soon after *GS*. In a difficult passage of *JL*[27], he distinguishes between the earlier concept that he has been discussing from the *absolute concept*. In the former, the reality or the ``becoming other'' of the concept is distinguished from its subjective taking back in *our* reflection. This taking back is an external *movement* imposed from without, a ``not-being'' that is regarded as ``the being of some other''. In the shift from the concept to the absolute concept, the reality of the concept as ``movement'' goes over into ``pure being or nothingness'' which vanishes ``in its realization''; being *per se* is only defined ``in the relation of thinking'' because it ``is equal to its contrary''. This absolute concept is, paradoxically, ``without concept''. In it, the nothing of the ``not-being'' is the equivalence of reality and thought. It is this absolute concept that plays the role of the NeoPlatonic One in Hegel's Jena logic.

Hegel develops his discussion of the absolute concept in his treatment of *cognition*. This includes a very condensed, early version of themes in *WL* - e.g. the Being-Nothing-Becoming logic with which it starts, the logic of the finite and the infinite, and also of the transition from Being into the reflection of Essence. Cognition, which I take to be the ``absolute concept'', is, Hegel says, the One in which *both* the unities of thought and being are posited. This is ``the positing of the numerical one as of something universal and divided, and of the taking back of this divided one into the negative unity''.

Clearly this is modelled on the procession and reversion of the One in NeoPlatonic philosophy. (It is also the One as the point of indifference of thought and being in the Identity philosophy.) But Hegel develops cognition far beyond classical NeoPlatonism. In particular, Hegel's philosophy of definition, construction and division, developed in the following pages of *JL* (and later, again, in *WL*), is derived from this unity of cognition. Through this, being is presented as remaining ``itself in multiplicity'', so that construction (as we have it in Euclidean geometry) is the concept dividing within itself but still remaining the same in the division (as, for example, in the production of the lines that criss-cross the original figure of a right-angled triangle and its three squares, in the Euclidean proof of the Pythagoras theorem).

While I do not have conclusive evidence that Hegel studied *P* in his thinking about geometry and while it may be the case that the influence of *P* on Hegel is only indirect (in particular through Kepler), I will argue that it does provide the correct setting for approaching *GS*. Without the background provided by *P*, it is difficult (at least for me) to make much sense of *GS*.

The possible influence of Proclus on Hegelian geometry

Hegel may have been first introduced to the Proclus commentary in his time (1788-1793) at the Tübingen Stift. There, Hegel was taught mathematics and physics by C. F. Pfleiderer who was apparently a good mathematician and according to Betzendörfer[28] an admirer of the Greek mathematicians. It seems certain that a serious scholar of Greek mathematics, such as Pfleiderer, would have studied the Proclus commentary and may well have brought it to the attention of his students, one of whom was Hegel.

Indeed, I think that it is almost certain that Hegel became aware of the commentary *P* for himself in the work of Kepler while he was at Tübingen. Kepler had been, like Hegel, a scholar at Tübingen, and both were Swabian and (somewhat unorthodox) Lutherans. Kepler was strongly influenced by *P* and makes frequent references to it. He even starts the 1st book of his famous work *Harmonices Mundi* with a quotation from *P*, and says in the introduction to that book that he would not have needed to write the work if Proclus had written a commentary on the 10 th book of Euclid.[29] Hegel would certainly have been aware of *P* since we know that he studied *Harmonices Mundi* in depth.[30] Hegel's admiration for and loyalty to Kepler is well-known, and it resulted in polemic against Newton (which appears even as early as 1801, in his dissertation on the planetary orbits).

My own tentative guess is that the influence of *P* on Hegel's geometrical thought in *GS* was probably not so much direct as through his reading of the works of Kepler. Indeed, I do not know of any quotation from *P* in the collected works of Hegel while on the other hand, Kepler's influence on Hegel is very explicit.

Moreover, there is no doubt that in later life at least, Hegel highly approved of the philosophy of Proclus. In his correspondence with the symbolist Creuzer[31], he praises Proclus, in particular, the latter's work the *Platonic Theology*, as ``the true turning point or transition from ancient to modern times, from ancient philosophy to Christianity''. In Hegel's *History of Philosophy* lectures[32], the work of Proclus is approvingly discussed, again with particular reference to the *Platonic Theology* as well as the commentary on the *Parmenides*. Hegel also refers in passing to the ``several mathematical works'' of Proclus. Hegel rated Proclus even higher than the great founder of NeoPlatonism, Plotinus.

It is reasonable then to suppose that directly or indirectly, Hegel came under the influence of the Proclus commentary on Euclid and developed what I think can be called the only NeoPlatonic philosophy of geometry of modern times, not as a reiteration of the geometry of Proclus but rather its use in interpreting and radically extending Kantian and post-Kantian geometry. (Of course, in his further investigations into mathematics (in *JS* and *WL*), Hegel went much further, developing a philosophy of mathematics that includes the more modern mathematics of (college) algebra, analytic geometry and the calculus.)

With the objective of understanding *GS* using a NeoPlatonic basis, we now briefly sketch some of the themes of the Proclus commentary and

describe the conceptual forms that these themes take in Hegel's own philosophy of geometry. (We note that there are also serious differences between the geometrical philosophies of Proclus and Hegel – these will be discussed later in this introduction.)

Proclus and Hegel: the philosophical basis for mathematics

Mathematics for Proclus proceeds on the basis of certain assumptions (e.g. axioms) which are unjustified within the subject. Rather, the justification for these lies in the ``unhypothetical science'' of Plato's *Republic*[33] which Proclus (following Aristotle) identifies with metaphysics (first philosophy). Hegel, in *NP*, also sees mathematics as based on hypotheses which are justified through philosophy:

> Man könnte noch weiter den Gedanken einer *philosophischen Mathematick* fassen, welche dasjenige aus Begriffen erkännte, was die gewöhnliche mathematische Wissenschaft aus vorausgesetzten Bestimmungen nach der Methode des Verstandes ableitet. [34]

While Hegel thinks that mathematics should not be mixed with the concept (*Begriff*) - mathematics is a science of the understanding (*Verstand*) - he also says that the concept can clarify the leading principles of Verstand itself, and of order and its necessity in arithmetic and in the theorems of geometry. Further, for Hegel, philosophy is definitely needed for a true account of mathematical determinations such as the infinite, infinitesimals and powers. In *GS*, conceptual (not mathematical) explanation is needed for the fundamental concepts of geometry, in particular (1) for the basic figures of geometry, such as circle, triangle and square, (2) the dialectical development of point, line, surface, solid, (3) the three dimensionality of space, and (4) the nature of ``angle''. As we will see, Hegel's explanation of these uses a post-Kantian version of the Pythagorean theory of *the unlimited and limit*, the latter theory being also used by Proclus in *P*.

For both Proclus and Hegel, philosophy (the unhypothetical science) also provides justification for the explicitly stated presuppositions on the basis of which geometry proceeds. These presuppositions, stated at the start of Euclidean geometry, are the *definitions*, *postulates* and *axioms* (common notions).[35]

But philosophy is not restricted only to the beginning of geometry, and in Hegel's geometrical philosophy, there are later mathematical claims of *Euclid 1* whose proofs need philosophical buttressing. Such proofs are based on the conceptual interplay of *limit/the unlimited*, the fundamental determinations of the geometry of Proclus. Examples of these arise (in *GS*) in *the theory of parallels* and the use of *the method of superposition*, and (not in *GS* but in *JL* and *WL*) in the *Pythagoras theorem*.

Hegel strictly limits such ``philosophical'' proofs. They are used only when the geometric proofs given by Euclid are unsatisfactory, and require that the underlying philosophical basis be made explicit. For the rest of the time, proof in Hegelian geometry is as for Euclid, and Hegel even comes up in *GS* with some geometrical proofs of his own.

A good example in *GS* of a geometrical situation where conceptual, philosophical explanation applies, and then leaves off so that proper mathematical proof can take over, appears in the treatment in *GS* of *parallelism*. There, given two parallel lines and two transversal lines between them of equal distance, we can say on conceptual grounds (using a theory of limits) that the corresponding *angles* which the transversals make with one of the parallel lines are the same. The reason for this, according to Hegel, is that ``the size of the line determines the angle''. In other words, a determination of equality between *lines* determines (on purely conceptual grounds) the same for *angles*. The equalities between other pairs of the angles made by the transversals and the parallel lines, however, is not justified philosophically: they are for geometry to prove in the Euclidean manner.

What determines philosophical significance in mathematics for Hegel (and this is true in his thinking from *GS* onwards) is *conceptual identity* in general, and its particular mathematical, quantitative form as *equality*. I believe that this is the same for Proclus (except that the *identity* and *equality* are derived through the forms from the *One* (rather than from the *concept*) and that he would also have understood this significance aesthetically). To illustrate this, Proclus, in *P*, using identity and equality (as well as, of course, the NeoPlatonic categories), gives what he thinks is the philosophical significance of the order[36] of the main figures of geometry as they appear in the first book of Euclid. (We will consider this in more detail below.) The order is derived from the ``perfection'' of the One expressed (we may say from a modern point of view) in the *symmetry* of geometrical figures. The One is self-equal (corresponding to the formal ``A=A'' of the Fichtean philosophy) and so the simplest, highest, geometrical forms of the One are also the most symmetrical, and also (from the identity of the Good, the One and the Beautiful in the Platonic philosophy) the most beautiful. For example, the paradigm triangle, for Proclus, is the equilateral triangle which is ``the most beautiful of triangles and most akin to the circle''[37], the circle being the primary closed curve. (Hegel's friend Hölderlin would also probably have understood it that way.) For Proclus, use of the *circle* in the construction of the *equilateral triangle* in Proposition 1 of *Euclid 1* ``seems to indicate in a likeness how the things that proceed from first principles receive perfection, identity, and equality from these principles''.[38]

The symmetry of a figure that determines its place in the order is expressed, for Proclus, in the identity and equality that the figure derives from its ``closeness'' to the One. So, we may say, the oneness of a figure is expressed through its symmetry by its *identifying* itself with itself through rotations, reflections, etc., and this entails *equalities* between the parts of the figure such as sides and angles. The more symmetry a figure has, the higher it ranks in the order. So, for example, the square is ``superior'' to the rectangle (i.e. precedes it in the order) since the former has more symmetry than the latter: the rotation of the square about its center through a right angle preserves the square, but this is not true for the rectangle in general. Yet every symmetry of the rectangle in general is one for the square since the square is a special kind of rectangle. This superiority is further seen in equality of the parts: the sides of a

square have more equality (all four equal) than do the sides of the rectangle (which are only equal in pairs).

As we will discuss in more detail below, there is, I believe, a similar order of figures in Hegelian geometry, but justified in terms of conceptual unity rather than in terms of order of closeness to an originating One. This order, as for Proclus, is also based on identity and equality. While the order is not given in detail in *GS*, an instructive example of it as it applies to triangles is given by Hegel (after his discussion of Proposition 5 of *Euclid 1*) in the following comment:

> the property of being isosceles is [at] a much more superficial level of unity for a triangle than is [the property of being] equilateral.

In explanation, we can say that the equilateral triangle is at a deeper (more conceptual) level of unity than is the isosceles triangle since the former has equality of all three sides while for the latter, only two sides are determined as equal, and the third side is arbitrary (unrelated to the other two). The equilateral triangle has more *symmetry* than the isosceles triangle, just as (in our discussion of the order of figures in *P*) the square has more symmetry than the rectangle. We could also say that the equilateral triangle possesses more *determination* in the relations of its limits (e.g. equality of its sides) than does the isosceles triangle.

So, consistent with the fundamental roles of the *unlimited/limit* in the thought of Pythagoras, Plato and Proclus, mathematical importance is, for Hegel a consequence of conceptual *determination* in terms of *equality*. We have seen this above, where the equilateral triangle is more significant than the isoceles triangle, and the square more significant than the rectangle. The more determined that a figure is, the more conceptual it is.

Determination of the circle is through the *equidistance* of its points from the center and for the straight line, through *constancy* of direction. For both Proclus and Hegel (and inspired by Plato [39]), the circle is primary among closed curves, while the constancy of direction of the straight line makes it primary among the non-closed curves. Both philosophers attach ``deeper'' (and related) significance to these two curves, beyond their foundational role in geometry. Proclus expatiates ecstatically on the simplicity and perfection of the circle and its character as *Nous*, while Hegel (in *WL*) sees in the circle the image (*Bild*) of the genuine infinite (*wahrhafte Unendlichkeit*).[40] (There is, however, nothing of this in *GS*.)

In mathematical conceptual determination lies, I think, Hegel's answer to the important question, largely ignored in present day mathematical philosophy: *what is the source of mathematical significance: why should one theorem be regarded as ``great'' while another is boringly ``routine''?* Significance in mathematics (for both Proclus and Hegel) comes from philosophy (in the form of the unhypothetical science).

We must emphasize again that the concept is not, for Hegel, a Proclian One (though it fulfills a similar unifying role). Nor, in terms of the identity philosophy, is it an ``absolute indifference'' point, sheer

blank identity, ``the night in which all cows are black'' [41], but the concept is as much *difference* as it is *unity*, which only is in its articulation – as (in *WL)*, e.g., in *whole and parts*, *force and expression*, *the syllogism and its terms*, *teleology and its means* – through which its self-identity is expressed and made actual. This identity in difference and difference in identity has its reality in what Hegel, in his mature logic, describes as the *Idea*, where the concept is completed in nature, in the apartness of space, which in turn leads back through nature's intrinsic logical development to the ``pure'' logical concept that it actualizes. (The process of the concept fulfilled in the Idea corresponds to the circular divine process for Proclus, the procession from the One through its forms into matter, and its reversion back.)

It has also to be stressed that the crux of NeoPlatonism, viz. how a perfectly simple, apart, *One* can give rise to *difference* (e.g. in the variety of forms) is not a problem for the Hegelian concept. For difference is intrinsic to that concept. The inadequate approach to difference in NeoPlatonic thought is paralleled, for Hegel, in Euclid's own imperfect view of difference. In Euclid's case, this philosophical inadequacy has unfortunate mathematical consequences with regard to his use of the *method of superposition* in geometry. (This is discussed in more detail below.) For our present purposes, two figures are proved congruent by *superposition* by imagining one being lifted up and laid on top of the other. The two figures should then exactly coincide in order for them to be congruent. The problem is*: how can you lift up ``pure'' triangles?* You can certainly lift up paper triangles, but they are inexact: they are not the triangles of pure geometry, which is an exact science.

As he makes clear in his discussion in *GS* of Proposition 5 of *Euclid 1*, the fundamental error with Euclid's use of the method of superposition is the latter's inadequate philosophical grasp of *difference*. Euclid, for Hegel, takes difference ``as if it were a thing'', in terms of the drawn figure. For Hegel, when difference is considered *conceptually*, then the problems with superposition do not arise. One rather has to consider what is required for a figure to be determinate. In this connection, a major problem with the geometer's approach – and one that Hegel criticizes Euclid for making – is in not distinguishing the concept, whose determination of a whole and its parts determines a geometrical figure, from the actual drawn figure with its parts spread out separately. The problems with the method of superposition, used by Euclid to prove congruence of triangles (and often literally impossible to achieve at the physical level anyway without breaking up the diagram in which the triangles are embedded), arise only because one is concentrating on the drawn figure – at the level of the *Idea* – rather than concentrating on the concept being imaged in the drawn figure. The necessity of this concept is the *real* reason for congruence.

Proclus and Hegel on point, line, surface and solid

To illustrate the role of philosophy (the unhypothetical science) in the foundations of geometry for Proclus and Hegel, we now discuss their views on the basic geometrical concepts, *point, line, surface and solid.* (Further discussion of Hegel's views on point, line, surface and solid will be given in **B** below.)

We first need to note that the German word *Fläche* that Hegel uses can mean either plane or, more generally, surface, and sometimes Hegel means one rather than the other. (Just as the straight line is the simplest kind of line, so the plane is the simplest kind of surface.) Below, the term *Fläche* will be translated as *plane* or *surface* depending on which Hegel seems mean. The plane is the surface that is flat. For Hegel, the plane is the simplest or paradigm surface, just as the straight line is the simplest or paradigm line. (Hegel's view of the paradigm character of the plane is confirmed in modern mathematics, where the surface is defined as a topological space that is locally just a deformation or bending of the plane.)

Appropriately, Euclid starts his *Elements* with the definition of a point. His definition is: *a point is that which has no part*. The explanation of this, for Proclus, is as follows. All levels of reality originate in the *One*. The One is completely simple, and reality is derived from its potentiality through the determinations of the categories of *limit* and *the unlimited* (as we saw earlier). So mathematical objects, intermediate between sensible and intellectual objects, announce ``their unitary and undivided and generative reality''[42], the One. ``Enmattered'' reality in the world is the *body* or *solid*, e.g. tables and trees, and to go back to the originating principle of the One in its geometric form, we have to go back through the sequence of *limitings* that gave rise to it.[43] The limit of the solid is the (curved) surface, as it is the *surface* that bounds the solid. Similarly, the bounding limit of the surface is the *line* (curved), while that of the line is the (end-)*point*, and there we come to a stop. The point is completely simple, purely determined by limit[44], *without parts*, and it is there that we have the One, or the simplicity of *Nous*, in geometric form.

For Proclus, the point is the ultimately simple element of geometry though not of mathematics as a whole. The number of arithmetic is ``purer'' and more fundamental than that of the point, since it has its existence in thought alone, and is ``free of any extraneous figure''. The point, on the other hand, is projected through imagination so that it has *position* in addition to its unitary character.

As completely determinate and simple, the point is (in terms of the basic Pythagorean categories) given as *limit*. As everything is, as we saw earlier, a combination of limit and the unlimited (for, to Proclus, all forms partake of both limit and the unlimited), the point does have an unlimited side. However this side is initially only *potential*, becoming explicit in the process through which point determines other figures. For Proclus[45], the point ``secretly possesses the nature of the unlimited and strives to be everywhere in the things that it bounds, it is present in them an infinite number of times''.[46]

So just as the One generates everything ``below it'', so, in its geometrical form as the point, it ``generates'' all of the (material) figures, their bounding being in terms of *limiting* through points, lines, and planes. Proclus describes this generation in Pythagorean[47] terms, as realized through movement, the ``flowing of a point'': so through its movement, the point generates the line, and in turn, the line through its movement generates the plane, and the plane through its movement generates the solid. We can think, for example, of a line rotating about a point on it and sweeping out a plane. This generation

reverses the direction of the *limiting* process above, where we went from solid, surface, line to point by taking boundaries. The generation process procedes through *unlimiting*, in which, for example, the point breaks out of its limitation and is no longer completely simple. (Proclus also discusses an alternative Pythagorean view of point-line-plane-solid in terms of the ``extendedness'' of the numbers 1,2,3,4.)

Hegel, in *GS* and indeed in his mature logic, argues for a dialectical version of this Pythagorean generation of point, line, plane and solid. He may well have come across this Pythagorean theory in his study at Frankfurt of the writings of Sextus Empiricus.[48] In *GS*, he develops the theory further within the conceptual framework of what he calls *the spatial relation*.

For the present, Hegel says in *GS*: ``The point is the simplest limitation of the infinite''. We can see this as a modern formulation of the view of Proclus that the point contains the potentiality of the unlimited. Hegel says of the *point* in the Jena Logic[49]:

> This bad infinity is posited immediately as absolute infinity, as one, as a point.

So in the Hegelian philosophy, the potential infinite of Proclus becomes the ``bad infinite'' in that it is the infinite as *unmediated*, and the ``striving'' through which, for Proclus, the point ``tries'' to be everywhere, takes the form in Hegelian logic of a dialectical development of point, line, plane and solid, in which the contradictory character of the concept of a point becomes fulfilled in the *solid*. So Hegel says in *WL*:

> So ist der Punkt diese Dialektik seiner selbst, zur Linie zu werden, die Linie die Dialektik, zur Fläche, die Fläche die, zum totalen Raume zu werden.[50]

In *GS*, the ``striving'' of the point is expressed by Hegel as the ``positing'' by the point of a line with its limits (end-points) as points.

Hegel gives us an explanation in *GS* of why we must stop at the solid in this development. The reason is that *the solid is not limit*, while point, line and surface are. It is limit that drives on the development, since what is limited refers to what is ``outside'' itself, has an implicit, unrealized infinite within it. It is this infinite which is expressed in the next stage of the development. Where there is no limit, there is nothing more to be expressed. For this reason, the limitless solid does not push forward to a fourth dimension, and space (for Hegel) is three-dimensional. (We will discuss this in more detail in **B**.)

One might wonder if the philosophy of Schelling influenced Hegel's view of the point in *GS*.[51] Indeed, as we saw earlier, in the identity philosophy of Schelling, ``point'' is used as a metaphor for the Absolute or the One, the point of *indifference* of cognition and being that is *as much subjective as objective*. (Of course the subjective is metaphorically present in the Proclus view of the point qua

``striving''.) But I have been unable to detect a clear influence of the identity philosophy in Hegel's view of the point in *GS*.

However, the *subjective* aspect of the point in Hegel's geometry becomes explicit in his geometrical thought a year after *GS*. In his 1801 dissertation, Hegel understands the *point* as *mind* (*Mens*)[52] and from this perspective, the point-line-plane-solid dialectic has its *conceptual* explanation as the development of mind in spatial terms, reality not just being objective but subjective (mental) as well. (This ``subjective'' aspect of geometry survives in Hegelian geometry in the *positing* of geometrical figures – see *infra.*)

Proclus and Hegel on geometrical figures

As we discussed above, for Proclus, the point as the One in geometry is primarily limit, but also has the unlimited potentially within it. It is ultimately the source of geometrical limiting and the infinite in the objects that it determines. These objects are the projections in imagination of geometrical forms and are the objects of study for geometry. In particular, in two dimensions, the objects are the *figures*, each of which is a specification of limit and the unlimited, the bounding lines determining the figure as *limiting*, but also determining the spatial region outside as the *unlimited*. The order of figures then follows the number series, based on the number *n* of bounding curves, and for each such *n*, the figures are listed in order according to the quality of their self-identity (symmetry) and equality as we discussed earlier. The first of these figures (for a given *n*) is the paradigm. The paradigm figures are those mathematicals which are ``closest'' to the form of the circle and hence also closest to the One – they are the most ``One-like'', the most symmetrical, determined through identity and equality. (If *n=1*, then the paradigm figure is the circle, if *n=2*, it is the semicircle, if *n=3*, then it is the equilateral triangle, and if *n=4*, then it is the square.)

Earlier, we discussed the striking correspondence between the geometrical philosophies of Proclus and Hegel. However, there are also serious differences between them as is particularly apparent in their views of geometricl figures. We will now discuss these differences.

The geometrical philosophy of Proclus is based on the originating One expressed at the highest level of *Nous* in the *forms*, and the mathematical objects (``mathematicals'') produced in turn from the forms by imaginative projection and studied through the understanding. The drawn geometrical figures are only reminders in sense perception of these objects and are indispensable helps for working with the figures. In contrast, Hegel's philosophy, as post-Kantian, is a philosophy of the *concept*. In Hegel's conceptual view of geometry[53] in *WL*, the concept of space is abstract, and conceptually posited in it are the ``shapes and configurations'' of geometry. Ontologically, these resemble the purposive *ends* of teleology and are to be distinguised from the *concrete* things of Nature and Spirit. Proclus, following Plato, interposes, as we have seen, between the form and the sensible figure the *mathematical* produced through imagination. From the above, Hegel's conceptual view of figures does not require mathematicals. Indeed, his sympathy with Aristotle and the latter's antipathy to the mathematicals of the early Academy suggest that the thought of a

separately existing mathematical with its strange kind of matter would have been unacceptable to Hegel.

For Proclus, the ``genuine'' triangle is the pure form. It is completely simple, and separated out into parts only at the level of the imaginatively projected mathematical triangle, and the latter is then represented as a drawn triangle of sense perception. But in the Hegelian position of *GS*, the triangle concept is from the start a whole of *interrelated* parts. For Hegel, the concept of a figure is the holding together of its parts (=limits), or the *necessity* of the figure, what gives it its coherence. Since whole and parts are together initially, Hegel's philosophy involves (as discussed above), *difference*, and he criticizes Euclid for the latter's imperfect understanding of difference. I think that Hegel would have criticized Proclus for the same reason. We now describe in more detail Hegel's view of figures in *GS*.

According to Hegel, ``the principle, the unity of the figure, is the relation of its limits''. This relation is the determining of the figure through its concept. The concept *makes determinate* the figure by the way that it *relates* the boundaries that it posits. We illustrate this by the cases of the circle, line and triangle. The principle of the *circle*, for Hegel, is its determining by *equidistance* from a fixed point (the center) (determining the actual circle as limit related to its center also as limit). The *straight line* is determined as limit by its principle, the *simplicity of direction* at each point (the geodesic property). The spatial relation that determines the *triangle* is the relating of three lines (the primary *limits* of the triangle), each pair determining subordinate limits in their intersection point and the angle at that point, and in totality, determining a bounded region of the plane. The spatial relation (a relation of limits) is thus determined according to the fundamental geometrical positing of point, line and plane (but not solid) together with the resulting limitation of the lines.

It is important not to see the determining of geometrical figures for Hegel (or for that matter, Proclus) in modern, Cantorian, set theoretic, terms. For example, for Hegel, the spatial relation for the triangle is given *conceptually*, not extensionally as a set of points[54]. Points on a line are, in general, only given *potentially*. The points that *are* determined are so, either through an initial positing (as the vertices of a triangle are given in a theorem proving something about triangles) or by explicit construction (as when a line constructed in a proof interects another line and so determines a point). This fits in well with the general structure of the concept in mature Hegelian logic in which, I think, the moments of universal, particular, singular, correspond respectively, for the concept of *triangle*, to the triangle itself, the edges and the vertices, the dialectic of these moments corresponding here to that of point, line and surface. The complete triangle is not, for Hegel, a *set* of points, but rather a conceptual relation of its limits that determines potentially its points and lines.

The Hegelian *spatial relation of limits*, translated into modern, set theoretic terms, gives each kind of figure as an equivalence *relation* on the plane. Let us illustrate this for the case of a circle. We say that two points in the plane are *equivalent* if they are equidistant

from the origin. This is an *equivalence relation* on the plane in the modern sense, and its equivalence classes (*sets*) are just the circles in the plane centered at the origin. The difference of this modern equivalence relation approach from that of the Hegelian spatial relation is obvious: in the modern approach, circles (and figures in general) are regarded as *sets* of points, rather than, as in the Hegelian position, given as conceptual wholes specified through a philosophy of limit and the unlimited, and implemented in sense perception through *construction*, e.g. by putting the point of a pair of compasses on a piece of paper, putting the pencil end at another point, and rotating. The resulting circle is given as *whole* rather than as a (highly infinite), ``fallen apart'', discrete set of points.

The order of the geometrical figures for Proclus and Hegel

(a) *Circle and straight line*

The One in geometry, for Proclus, is, as we have seen, the point. Like the One, the point, while primarily limit, is also potentially the unlimited. (Initially (in modern terms) it is ``zero dimensional''.) As we saw earlier, this potential is actualized in the ``flowing of the point'', through which it takes 1-dimensional form. This form assumes two primary determinations, according to the principles of limit and the unlimited. As *limit*, the point is the circle, and as *unlimited*, it is *the straight line*. These are the canonical figures with one boundary.

The circle is, I suppose, *limit* for Proclus in that it bounds two regions of space, the inside and the outside. More important for Proclus is the circle as reflecting the character of the One. For him, the circle ``returns to itself'' and ``of its own accord, wraps itself back into unity''.[55] One way to interpret these comments is as follows. As we go round the circle we come back to where we started: through its own nature it identifies itself with itself again, imitating the One whose nature it expresses. This is illustrated, for Proclus, by the ``circular revolution of the heavens''. However, at a deeper level, this ``return'' of the circle to itself follows from the dialectical character of the center in its linear expansion along the radii and its reflection back to itself. We discuss this in more detail below.

Unlike the circle, the straight line ``goes on for ever'' and it is never finished. In Euclidean geometry, we never deal with the complete line but just a line drawn ``long enough''. The line in Euclidean geometry is only potentially infinite, unlike the line whose form is completely given algebraically in modern mathematics by an equation: $ax+by+c=0$.

While initially, the circle is the point as limit, nevertheless, like all figures, it is, for Proclus, a ``mixture'' of limit and the unlimited. This mixed character follows from presence of the *point* in the definition of the circle. As a point, the geometrical ``One'', the center of the circle generates, by the dialectic of point-line-plane-solid, the *unlimited* as *straight lines*. These lines are just the *radii* of the circle, ``flowing'' from the center till they meet the boundary (=limit=the circle). As limit, the circle bounds this outwards flow, and reverses it backwards along the radii towards the center, the ``One''.[56] In this way the circle and its center realize in geometrical

form the One in its activities of progression (going out from the center along the radii) and reversion (returning back along the radii). (This philosophical interpretation of the circle is remarkably similar to the Fichtean image of the *Anstoss*.) Further, consistent with the fundamental role of equality and identity in the philosophical development of geometry in the thought of both Proclus and Hegel (discussed earlier), the circle is determined by *equality* since it has ``all its lines from the center equal''.[57]

However, as commented earlier, for Proclus, the form has to be distinguished from its mathematical projection. So, for example, the geometric circle, which is the *figure* of a circle, has to be distinguished from its noetic paradigm (its form). In the figure, the parts - the center, radii and circumference are all in different *places*. But they are all present at the *same* time. In the paradigm, however, not only are the parts present at the same time but also they are not distinguished spatially - ``up there, they [the parts] are all in one''.[58] (Spatial difference is not preserved in the ``genuine'' circle.)

There is no doubt that for Hegel, as for Proclus, the circle is of great philosophical (as well as geometrical) importance, and for similar reasons. It seems to me almost certain that Hegel discussed the circle in the original version of *GS*, but that discussion has not survived. However, in his discussion of the sphere in *GS*, Hegel refers to the role of the ``great circle'', which he says:

> has, in itself the returning line [a radius], [which has] in itself the returning plane [the plane of the circle;] as the circle has no [posited] point [on it], so it has no [straight] line [for such a line would have to have a bounding point]; within the circle lies its limit (the center, diameter[s]), external to it [the circle], which it [the circle] bounds.

We need to understand this cryptic passage against the background of the Proclus account of the circle. Hegel's view that the center and diameters together give the limit of the circle is initially puzzling since the center and the diameters are, as Hegel admits, ``external'' to the circle. One easily sees that the end-points of a line segment are the limits of the segment. But in the case of a circle, the center, one of Hegel's proposed limits, is at a radius's distance away from the points on the circle – so how can the center possibly be a limit of the circle? (This difficulty does not arise in the Proclus theory - I have not been able to find in *P* the Hegelian view that the center is a limit of the circle.)

However, help in understanding this is obtained from Proclus's view of the circle. Recall that for him, ``up there'', at the form level, the center, diameters and circumference are ``all in one''. Putting this in Hegel's conceptual (rather than Proclus's formal) framework, the center and the diameters *are in terms of philosophy* limits for the circle; they are the ``parts'' as determined by the concept of the circle just as much as the sides are the parts as determined by the triangle concept. The difference between the two cases lies in the fact that the limits (center and diameters) for the circle are not explicitly given for the circle of sense perception – there is just the circle on its own - whereas in the triangle case, the limits (the sides) are

explicitly presented with the triangle. (The limiting character of the center and diameter, not seen as limiting at the level of sense perception, belongs to Hegel's philosophy of ``negative being'' that we will discuss later.) But for Hegel they are constitutive for the concept of the circle. For him, the concept determines the genuine circle, just as for Proclus, the paradigmatic form gives the ``truly real circle''.

Next, in the philosophy of *GS*, a figure is, as we have seen, the relation of its limits. So how are we to understand the circle as the relation of its limits? We can say that it is through its limits that the symmetry, which is the self-equality of the circle, is realized. Recalling that the point is the limit of line and is potentially the line through its expansion, the center in its expansion is seen to be the *radius* or, if we think of the expansion in both directions, the *diameter*; the latter, in its turn, through rotation about the center sweeps out a circular region with its boundary[59] the circle (just as the rotation of the straight line sweeps out the plane). The circle is then the relation of its limits in the sense that by *relating* the center limit and the (generic) diameter limit through rotating the latter about the former, we get the circle. This would be my understanding of Hegel's claim (in the above quotation) that the center and diameters are the limits of the circle. Just as a point generates the line interval, so the center generates the diameter, and the two together generate the circle.[60]

The expansion process from the center along the radii, that gives the circle, is reversed through contraction of the circle back along the radii. This is, I think, how we should understand Hegel's comment on the ``returning line'' in the above quotation. The line through which the circle ``returns'' to the point is, of course, the radius. With this interpretation, Hegel's view of the circle is then seen to be a development of the Proclus view. But whereas, in the Proclus view, the circle is seen in terms of the procession from and reversion back to the One (as represented in the center), the Hegelian view is conceptual in character and relies on his theory of limit and negative being.

In Hegel's mature logic, the paradigmatic circle of Proclus assumes the (much more general) conceptual form of the *genuine infinite gone back into itself*. The circle, for Hegel, ``wird deren Bild der Kreis, die sich erreicht habende Linie, die geschlossen und ganz gegenwärtig ist, ohne Anfangspunkt und Ende''.[61] If we take *limit-the unlimited* in the thought of Proclus to correspond in Hegelian thought to the *Finite-Infinite*, the circle, as symbol of the One in its progression and reversion, in which the limited and unlimited are unified, corresponds to the genuine infinite (*die wahrhafte Unendlichkeit*) in which the finite is through the infinite and conversely. Further, in *WL*, Hegel goes beyond Proclus in identifying the conceptual basis for the circle with that of the greatest theorem of Euclidean geometry, the Pythagoras theorem. (This will be discussed in more detail below.)

In his development of the canonical geometrical figures with one boundary (the circle and the straight line) Proclus moves on to the *semicircle*, the canonical figure with two boundaries. The semicircle is bounded part by the circle and part by the straight line. As far as I know, the semicircle does not feature in Hegelian philosophy and so we move on to the figures with three boundaries.

(b) *The triangle*

The simplest form of a three boundary figure is the ***triangle***. Proclus, commenting on Definitions 19-22 of ***Euclid 1***, says[62] that the triangle is the first in the ``procession'' of rectilinear figures ``according to the endless number series from three to infinity''. So the rectilinear figures are determined by the ***number series*** rather than directly from the One. The sides of these figures are ***straight*** since at this stage, the simplest lines are the straight ones. The circle, I think, is not eligible as a side here since ***the unlimited*** dominates at the stage of three or more boundaries (the intermediate mixture of circle/straight line being at the second stage with the semicircle). The canonical figures with three or more sides have straight sides.

Starting with the triangle, for Hegel, as well as for Proclus, its determination as a triangle lies in its three sides (its ***limits***) and so identity for a triangle is expressed in terms of its sides. Proclus follows the classification of triangles given in Definition 20 of ***Euclid 1***, and sees the order of this classification determined by decreasing identity (=equality). This order is: (1) the paradigm, the ***equilateral triangle*** (all sides equal, the most equality), (2) two sides only equal (***isosceles***) and (3) none equal (***scalene***). Hegel in ***GS*** reaffirms with Proclus the priority of the equilateral triangle before the others.

The concept of ***angle*** for Hegel naturally comes into the discussion of triangles (since each triangle comes with three angles, one at each corner). This will be discussed in more detail later in this introduction. For the present, angle is to be understood in terms of ***limit*** and ***the unlimited***, the two concepts that, as we have seen, are foundational in Hegelian (as well as in Proclian) geometry. Angle is a limit in a mediate sense, being determined by the two lines enclosing the angle. For two reasons, I think, the ***first*** or ***primary*** limits of the triangle are its ***sides***. First, the sides obviously bound the triangle, determining it as limited against the backdrop of the unlimited plane, in a way that angles do not. Second, geometry starts with the point which then develops in the ***line***; the angle comes later and presupposes lines already ``there''. So what is linear (the sides) has logical priority over the angular.

As far as the kinds of angle are concerned, the right angle is of special importance for Euclid, being defined (Definition 10 of ***Euclid 1***) as the angle that a line makes with an other when it is equal to its adjacent angle. (Euclid's Postulate 4 asserts that all right angles are equal to one another so that there is only ***one*** right angle.) For Proclus (following the Pythagoreans[63]) the right angle is ``self-identity'', equal with itself through its complement. The right angle is also the ``measure of angles other than right'' in the sense that it determines the other two basic kinds of angle, ***acute*** (smaller than right) and ***obtuse*** (greater than right) – these other two angles are ``indefinite more-and-less''.

Hegel's view of the ***right angle*** is very close to that of Proclus and centers around the most important theorem in all of Euclidean geometry, the ***Pythagoras theorem*** (the last proposition of ***Euclid 1***). (Unfortunately, there is no surviving discussion of this theorem in

GS.) For Hegel, at least in his mature thought, the right angle is conceptually equivalent to ``equality with self''.[64] In *JS*, the right angle is the ``relation of being''[65] for the Pythagoras theorem, and in both *JS* and *WL*, the right angle's ``self-equality'' is associated with the square on the hypotenuse opposite it. This is one side of the Pythagorean equality, while the other side is the ``self-unequal, the remaining two sides''. For the latter, the two sides of the triangle bounding the right angle are ``conjoined as a sum'' through the addition of the areas of the squares on them.[66]

As we have seen, the natural, ``sensory'' definition of a triangle is that it is the region bounded by three straight lines. However, since for Hegel, the concept of a figure is to be distinguished from its sensory appearance – as we saw above in the case of circle, the limits determined by a mathematical concept may not appear as limits at the level of sense perception. One might, then, ask: *what is the non-sensory, ``real'' definition of a triangle*, its definition according to its concept? For Hegel, at least in his mature thought, the answer to this also lies in the Pythagoras theorem. Here again, a mathematical concept is quite different from what appears in sense perception: for the latter, a triangle is just a three sided figure, and Pythagoras's theorem is a remarkable theorem proved about a special kind of triangle.

This theorem is also, for Hegel, the ``real'' definition of the circle (the normal sensory definition being that of the locus of points at a given distance from a fixed point). In the case of the right angled triangle, the relation of the limits (sides) of the triangle is given by the Pythagoras theorem, and (presumably) for a general triangle, by that theorem's extension in trigonometry to the *cosine formula*. In the case of the circle its basis in the Pythagoras theorem becomes clear through its formula in analytic geometry: $x^2 + y^2 = r^2$.

(c) *The quadrilateral*

Turning to four-sided figures, Definition 22 of *Euclid 1* classifies the quadrilateral figures in accordance with decreasing order of identity. Proclus follows this classification though augmenting it. In his understanding of the Euclidean classification, recall that for Proclus (and Hegel), the right angle is a ``self-identity'', *equal* with itself through its complement, and so is also a source of identity. Each class of quadrilaterals is assumed to exclude the others. So, for example, the oblong excludes the square. The Euclidean classification is as follows: the *square* (equal sides, equal angles all right angles), *oblong* (right angle only, opposite sides equal), *rhombus* (equal sides, no right angles), parallelogram or *rhomboid* (opposite sides equal, opposite angles equal), *trapezoid* (the rest). Proclus accepts the Euclidean classification of the quadrilaterals, but refines it (under the influence of Posidonius) by interposing between the rhomboid and the trapezoid two kinds of trapezia.

Proclus says[67] that the square ``is the only area whose very nature exhibits an ideal with respect both to its sides and to its angles''. Equality is, of course, the key here: all of its sides are equal, all of its angles are equal and this angle is the right angle, which, as we have seen, is for him the ``self-identical''.

In his later logic, Hegel also interprets the priority or ``superiority'' of the square among quadrilaterals (in particular when compared with the rectangle=oblong) in terms of equality. For him, the square (like the right angle) is the *self-equal* (*dem sich selbst Gleichen*)[68], while the rectangle is the self-unequal, so that Euclid's reduction of a rectangle to a square, in the proof of Proposition 14 of Book 2 of the *Elements*, is yet another form of the conceptual identity present (for Hegel) in Pythagoras's theorem: namely *the equality of the self-equal to the self-unequal*. Hegel's understanding of triangles and circles in terms of the Pythagoras theorem was discussed above. Of course, Pythagoras's theorem involves squares as well, so that the conceptual identity expressed in this theorem, for Hegel, brings together triangles, circles, squares and rectangles - central figures in Euclidean geometry - under *one* determination. This, I take it, would be their ``true'' philosophical significance for Hegel. This theory, expounded briefly in *WL* and *JL*, may well have been developed in the missing parts of *GS.*

Problems and theorems for Proclus and Hegel

The propositions of the *Elements* fall into two kinds: *problems* and *theorems*. It is clear that Hegel was aware of the distinction since he explicitly refers to Proposition 9 (``*to bisect a rectilinear angle''*) as a problem and formulates its theorem version. As far as I know, there is no surviving discussion by Hegel of the distinction. Since Hegel may well have learned about it from the Proclus commentary[69], we now briefly summarize the views of Proclus on the matter.

Proclus says: ``Problems require us to construct a figure, or set it at a place, or apply it to another, or inscribe it in or circumscribe it about another, or fit it upon or bring it into contact with another, and the like; theorems endeavor to grasp firmly and bind fast by demonstration the attributes and inherent properties belonging to the objects that are the subject-matter of geometry.''[70] The essential difference between a problem and a theorem is, then, that a problem primarily involves the *construction* of something while a theorem deals with understanding ``the objects of the theoretical sciences''. Proposition 1, which asks for the construction of an equilateral triangle, is obviously a problem. On the other hand, Proposition 5, which asserts that the base angles of an isosceles triangle are equal, is a theorem. Of course, proofs of theorems often involve constructions (problems) and conversely, the validity of a construction can require the use of theorems. Proclus indicates the difference between Euclid's treatments of problems and theorems: he points out that ``at the end of his demonstrations'' Euclid places ``*This is what was to be done''* (in the case of problems) and ``*This is what was to be proved''* (in the case of theorems).[71] (Traditionally, these endings are abbreviated to *Q.E.F.* and *Q.E.D.*.)

For Proclus, interpreting the views of earlier thinkers in terms of his own projection theory of geometry, construction is the result of the shaping in imagination[72] by our ideas of the ``intelligible matter'' of geometry, ``taking eternal things as if they were in the process of coming to be''. I think that Hegel may well have followed a Kantian version of this imaginative account of geometrical construction. He talks in *WL* of how geometrical objects are abstract determinations of

space; these ``shapes and configurations'' are posited in it, and ``resemble the products of external purposiveness''.[73]

With this general background to Hegel's philosophy of geometry (interpreted against the background of the philosophy of Proclus) we now discuss in detail the themes of *GS*.

B Hegel's Geometrical Studies

GS, as edited by Hoffmeister, is treated continuously although, as Hoffmeister comments in his *Anmerkungen*, it contains material composed by Hegel at different times. For reasons of coherence and readability, it seemed to me important to organize *GS* into its major parts. There are four major parts, which I have titled: *The nature of mathematical objects*, *The first twelve propositions of Euclid 1*, *The philosophy of parallel lines* and *On the algebra of a geometrical figure*. Of course, these titles are entirely mine, and do not appear in the original manuscript. This presentation of *GS* has also involved minor rearrangement of material, and the order in which the four parts appear has been changed from that in Hoffmeister's edition. Explanations for the changes in order are given below, which discusses the main themes of the four parts.

The nature of mathematical objects

This is the most philosophical of the four parts. On linguistic grounds, Hoffmeister dates it earlier than the discussion of *Euclid 1* that it immediately follows in *Dok.*. For this and other reasons (e.g. the general nature of the treatment of mathematical objects which naturally should come before the geometrically technical treatment of the propositions of *Euclid 1*), I have changed Hoffmeister's order of presentation and put this section first. I have also made a small number of minor changes with regards to the *order* of the material in the section. By doing this, fragments on the same theme are combined to give a more coherent, readable treatment. For example, the first paragraph, dealing with figures, sides and angles has been combined with the similar material later in the section, so that the section starts (more appropriately) with a discussion of mathematical objects in general (rather than just the special case of rectilinear figures).

There are, I think, five major themes in this part. These are:
(a) *mathematical objects and the theory of limitation,*
(b) *point, line, plane and solid, and the three dimensionality of space*,
(c) *the cube and the sphere*,
(d) *congruent triangles and the method of superposition*,
(e) *the nature of an angle*.
We discuss these briefly in turn.

(a) *Mathematical objects and the theory of limitation*

Hegel starts by distinguishing the *positive* and *negative* being of a mathematical object in terms of his theory of limits. (These kinds of being are understood in terms of the finite and the infinite later.) Each object as determinate comes with a limitation whose development results in the object. So, for example, as we saw earlier, the point

logically contains the line which it ``generates''. The point is the limitation whose development is the line. In many cases, the limitation of a geometrical object is its *boundary*.[74] So a (finite) line is limited by its end-points while a surface is limited by the curve bounding it. The *positive being* of the object is the object as sublating the limitation, the limitation being present but its generating dialectic suppressed. The positive being of a geometrical object is that object in the usual sense with its boundary included. So, for example, the positive being of a line is that line with its two end-points, while the positive being of the surface is the surface with its bounding curve.

The *negative being* of the geometrical object is the boundary itself. So the (curved) line is the negative being of the surface at its edge, and the end-points the negative being of the line. Hegel says that the line is *the spatial relation of a point*. One can interpret this, as was done earlier, by saying that the point is potentially a line – in Pythagorean terms, the line is generated through the *movement* of the point and, with a Hegelian addition, comes back to itself *qua* point as the *other* point at the end of the line. The line then is, we may say, the development of the point, and more generally, the positive being of a geometrical object is the development or filling in of its negative being (its limitation).

One might wonder how a curved line can have only its end-points as limitation. Surely the curvature of the line needs to be taken into account, since it is not completely determined by these points. This is not a problem for the Hegelian position. As in the thought of Proclus, what is significant (at least initially) is the *simplest* form of the line, i.e. the *straight* line, and this *is* determined by the end-points (given the geodesic character of the straight line for Euclidean space). For curved lines, other conditions come into play, e.g. the line might be required to be *circular*. That is, the limitation of the curve has extra determinations apart from the end-points. (As in our earlier discussion of the circle, these extra determinations, for Hegel, are its center and diameters.)

In Euclidean geometry, the *point* is defined as ``that which has no part'' (*Euclid 1*, Definition 1). This is reiterated by Proclus, for whom the point is the first principle of geometry.[75] Hegel expresses a similar view in *GS*: *The point is the simplest limitation of the infinite*. Hegel is then saying that the point is the simplest limited form in mathematics of the infinite. The point is not, however, absolute, in the sense of being completely separate from other objects, like the NeoPlatonic One. Rather, for Hegel, the point ``is nothing absolute because it can become bounding''. The meaning of this, and its form in the thought of Proclus, was discussed earlier in this introduction: the point is potentially the end-point of a line, and so is other than itself, not an absolute self-dependence.

Hegel then uses the concept of ``ground'' to elucidate mathematical objects. The ground is ``the reason [for something] in general''. (I think that this is different from his view of ground in *JL*[76] where the ground is ``the reflection of cognition itself''.) In the case of geometrical objects, the ground is ``the manifoldness of the unity of the limit''. I think that this means that geometrical objects are *spatial* in character and hence, as at the end of *WL* and the beginning

of *NP*, have the character of being external to themselves; the limit of the object refers intrinsically to what is outside it and this outside, incorporated into the object, makes the object manifold in character and gives it its *positive being* – space is quantitative, rather than qualitative, and the quantitative has multiplicity. The unity of the object is located in its concept: the object is determined by the concept through the latter's limitation. The concept or principle of the figure is the ``unity of the whole'', ``the relation of its limits''. So for example the concept of a triangle has as its limitation, the three sides, and the concept as the relation of the three sides determines the interior of the triangle (which is away from the limits) and the angles.

In the case of the triangle, this relation of limits, according to Hegel, assumes two specific, additional, forms. The first is a relation of *magnitude*, as when in an isosceles triangle, we require two sides to be *equal* in length: so two of the limits (the sides) are related through magnitude. The second is ``the self-relation of the limits as space, (e.g. the enclosed angle)''.

We discussed briefly the second form, the angle, earlier in this introduction. (We will return to the angle a third time below.) Recall that the sides are the first or primary limits of the triangular figure while the angles are secondary. More precisely, the sides related to one another through the triangle concept determine a completely determinate spatial region (the interior of the triangle). At each vertex, this region determines a ``wedge'' with bounding sides, and these determine the enclosed angle. So the angles of a triangle are also limits but of a derivative kind. The relation of the sides determine the angles and their magnitudes. Indeed, as limit, the angle is *measured*; it is measured in terms of the lines that bound and determine them. (For example, the trigonometric angle functions are just ratios of the bounding *sides* of a right-angled triangle). So we have the limits of the triangles (the sides) initially in relation through the triangle concept, and that relation is then determined as angle. This philosophical relationship between sides and angles is well illustrated by Proposition 5 of *Euclid 1*: *the equality of two sides of a triangle entails equal opposite angles.* (Note that in this Proposition, the magnitude relation is present here in the equality of sides, while the self-relation of the sides is present in the angles. There is also a magnitude relation present in the equality of the angles.)

The genesis of geometrical objects, for Hegel, lies in the dialectical development of point, line, plane and solid. We discussed in **A** this dialectic as it occurs in the thought of Proclus and Hegel. We now look at Hegel's account of this topic, as it appears in *GS*, in more detail.

(b) *Point, line, plane and solid, and the three dimensionality of space*

As observed in **A**, Hegel says in *GS* that the point is the *simplest limitation of the infinite*. It was also commented that this is to be regarded as a modern formulation of Proclus's view of the point as the ``offspring'' of the One, the simplicity of *Nous* in its geometrical form, the first principle of geometry. (Of course, as Proclus points out, different sciences have different first principles – arithmetic has the unit, and physics has the elements.) Hegel in *GS* also views,

as does Proclus, the fundamental geometrical objects arising through a process of limiting, first the point and then line, plane and solid, each as limiting what comes after it. What comes after is potential in what comes before – the object as negative being precedes the object as positive being. So, for example, the point is the limit of the line since the boundary of the line segment consists of points. We recall also that for Hegel in *GS*, the line is the ``spatial relation'' of the point in the sense that the point relates itself to itself through the expression of the line as the potential infinite implicit within it, the expression starting at a point and ending at a point.

Alternatively expressed, he says that ``the objective relation of a point posited outside itself is the line''. Similarly, the ``objective relations'' of the line and the surface posited outside themselves are respectively the surface and the solid. The objectivity here just refers to the implicit line within the point becoming objective in an actual line, and this line, of course, is outside the ``generating'' point. As in Hegel's logic of *Dasein* in *WL*, the line, potential within the point which ``limits'' that potential, ``traps'' it in as it were, is posited ``on the other side'' of the limit. The transition from point to line is, for Hegel, to be thought of dialectically, according to the concept, and only *exemplified* in the production of a line through the motion of a particle (``point'') in space.

The German noun, that we have translated as ``relation'', is *Beziehung*. In *GS*, the simplest geometrical object, the *point*, is thought of as a relation. *Beziehung* is an important word in the Jena Logic. Indeed, the title of the first section of *JS* is *Einfache Beziehung*. (This title is conjectural since most of the first section of *JS* is missing.) The word *Beziehung*, which is translated in *JL* as *connection*, can also be translated as *relation*, and it seemed to me preferable to do so in the mathematical context of *GS*. However, the reader should also bear in mind that, compatible with *JL*, ``spatial relation'', ``objective relation'' can also be rendered ``spatial connection'', ``objective connection''. *GS* is consistent with *JS* in that the *first* of the geometrical objects, the point, and the *starting point* of general logic is conceived of in *relational* terms.[77] (*Relation* is also one of the four fundamental Kantian category groups with which a post-Kantian logic might reasonably start, although Kant uses *der Relation* instead of *die Beziehung* as the name for that category group.)

To reiterate, in the point as the limit of the line we can see the point *relating* itself to itself. For a given line *relates* or *connects* the two end-points, so that the point as one end-point relates itself to itself as the other end-point through the line. Similar considerations apply to the surface as the objective relation of its bounding (curved) line (e.g. a circular region bounded by a circle), and of the solid (body) as the objective relation of its bounding surface (e.g. a solid ball bounded by a spherical surface). At this stage, the pattern breaks off. For Hegel, the justification of this ``breaking off'' lies in his theory of limits.

For unlike the point, line and surface, t*he solid is not limit*. What corresponds to the limiting role for point, line and plane in the case of the solid is the *ceasing to be of the relation*. It is true, Hegel says in *GS*, that we can think of the solid in limiting terms as being ``posited outside itself'', but this positing is fundamentally

different from the way that point, line and plane are posited outside themselves. In the latter cases, the geometrical object is the boundary of a different, higher dimensional object: the point of a line, the line of a surface and the surface of a solid. But while, it is true, we can conceive an outside of a solid, that outside is *another solid* with a space in between. That is, we have *two* bodies, the one external to the other. The spatial relation of the solid has ``ceased'' in the spatial gap. We can also conceive of the solid as going outside itself in an other, more intrinsic, way. In that way, the solid is ``absolutely a drawing out of itself''. I take it that an example of this would be a body expanding or contracting. But in either case, it is solid going to solid, the spatial relation, that determined the boundary, ceasing and being reestablished in the solid. Solid is not limit, and we have not gone up a dimension.

There is, however, a relation between the solid and the point. In a puzzling comment of *GS*, Hegel says that ``the solid is itself again [the] point, [i.e.] outside itself [in] complete contradiction''. I understand this (somewhat speculatively) as follows. Recall that in the preceding paragraph, there were two ways in which the solid was external to itself: in the first, it is external to itself as another body, and in the second, it changes its boundary in its expansion and contraction. We can understand Hegel's comment in either of the two ways. For first, the point is outside itself since it posits other points external to it as the end-points of the lines that are potential within it. Further, this self-externality is the conceptual character of the point – spatiality is immediately the point as ``differenceless self-externality''.[78] In this respect, the solid is the same as the point in that it is *solid again* as posited outside itself through the spatial gap. *Both are conceptually the same*. Following the second way, we can regard this identity of point and solid as realized by the contracting inwards of the solid. In non-precise terms, solid is conceptually a ``black hole'', its ``contraction'' in a point. The dialectic of space then takes us from point (zero-dimensional), to line (one-dimensional), to surface (two dimensional), to solid (three-dimensional) and then back to point, where we started from. There is no fourth dimension, and this completes Hegel's philosophical proof of the three dimensionality of space.

Ultimately any validity to Hegel's argument relies on the fact that the real world is for us three dimensional. In modern science[79], of course, the mathematical models of the universe use more than three dimensions. Special relativity, for example, is four dimensional, with time giving the fourth dimension[80]. (In the later Hegelian position, time is the appearing of the ``restlessness'' of space, the ``restlessness'' being the same as the dialectic of the point. One can see this in the way that the point gives the line in the Pythagorean account through *movement*, which of course is time dependent.) Further, in the first attempt to produce a unified field theory (unifying electromagnetic theory and the geometry of space-time), Kaluza and Klein attached circles to the points of space-time thus giving a fifth (unobservable) dimension. The modern ``grand unified theories'', which attempt to unify the non-gravitational forces, use at least 24 dimensions. In view of this, and the ``bad press'' that Hegel's *Naturphilosophie* has got, it is easy to mock Hegel's insistence on the fundamental three dimensional nature of spatial reality.

In defense of Hegel, however, scientific experiments are done in the familiar three dimensional world with time treated ``normally'' and any theory of the world is only speculative if it is separated from its verification when projected into three dimensional terms. So, for example, in special relativity, the verification of the contraction in length of a measuring rod on board a moving spaceship, when compared with the length of an identical measuring rod at rest on earth, involves measurements with clocks and observers in the *familiar* world (which is taken for granted). We may say that the remarkable, higher dimensional, theories are *essential* in character, and become *real* only in so far as they refer to and explain what happens in the everyday world. It is in this sense that the real world is three dimensional. This everyday, three dimensional character of the world was, I think, stressed too much by Hegel in his criticism of Newton, since the higher dimensional theories whose projections explain the ``real'' world *cannot* be created by staying only with three spatial dimensions. But surely Hegel is correct in seeing, as does common sense, the *solid* as ultimately the real thing in our world, as when we lift and lay a book and kick a soccer ball.

(c) *The cube and the sphere*

We saw that for Hegel, the principle of a figure is the relation of its limits. In *GS*, Hegel discusses this figural principle in the (three-dimensional) cases of the cube and the sphere. Underlying this discussion, of course, are the four basic geometrical objects which form the constituting limits of geometry: *point, line, plane, solid*. Indeed, everything important in Hegelian geometry reduces to these four fundamental geometrical objects, justified, as we have seen, on metaphysical grounds through the unhypothetical science.

Starting with the cube, the presence of these four objects is obvious. The point is present in the 4 vertices (corners), the line in the 12 edges, the plane in the four faces, and the solid as the whole cube itself.

But in terms of the point-line-plane-solid dialectic, the *sphere* is more interesting than the cube. The sphere here is not to be thought of as a solid ball but rather as the *surface* of a ball. Hegel says ``in the sphere there is nothing but limit'', by which, I think, he means, that the sphere *per se* is itself the limit that separates the region inside it from the region outside it. Hegel then goes on to analyse the sphere in terms of *point, line and plane*. (The fourth fundamental geometric object, the *solid*, is not relevant here since the sphere is a surface.)

Beginning with the *point*, there is no particular point specified *on* the sphere through the necessity of the sphere concept. However, the sphere conceptually determines more than just the bounding surface of a ball. It determines a *point*, *the center*. Indeed, the center is involved in the very definition of a sphere: to belong to the sphere is to be at a certain fixed distance (the radius) from the center. Hegel says that the center does not belong to the sphere and is not a limit of the sphere. Rather, the center is *posited* by the sphere. A little later in *GS*, and as discussed in **A**, Hegel says that the center of a circle *is* the limit of the circle. I do not know why this is not the same for the sphere. (Perhaps the difference between circle and sphere

in this regard lies in the fact that the circle arises at the *beginning* of geometry out of the point as the line in its limited form, whereas the sphere arises later in the conceptual development of geometry after the third dimension has been established.)

Turning next to the *line*, one might see the line in the sphere in the form of its diameters and radii. (In the two dimensional version of the sphere, i.e. the circle, Proclus and (I think) Hegel see it that way as was discussed earlier.) However, as Hegel points out, the diameters and radii do not belong to the sphere. So rather than looking for line in its *unlimited* form (straight) we look for line in its *limited* form, i.e. as a *circle*. The natural class of circles that lie on the sphere are the *great circles:* a circle on the sphere is a great circle if its radius is greatest possible, i.e. is that of the sphere itself. The great circles of the sphere *do* lie on the sphere and their planes pass through the specified point of the sphere, the center. But there are infinitely many great circles and none of them is specified through logical necessity. The great circles do not bound the sphere (though, we can add, if you cut along one, it will bound the resulting hemispheres). They are posited by the sphere and ``it bounds them as circles''. I think that this means that the sphere bounds the circle on either side, that, in modern terms, the circle is a *submanifold* of the sphere - the latter provides the circle with its ``world''.

Turning to the *plane*, the latter is also not a limit of the sphere. However, there are, as we have seen, intrinsic planes connected with the sphere, viz. the planes of the *great circles*. Hegel says that ``the point, as enclosing, [in the form of] the great circle, has in itself the returning line, [which has] in itself the returning plane''. This brings us back to the case of the circle which was discussed above.

(d) *Congruent triangles and the method of superposition*

The conceptual explanation of geometrical figures discussed earlier leads to an important philosophical interpretation of certain kinds of propositions. These are propositions of the form: *when are two triangles congruent*, i.e. *are identical*? (Identity here means that the corresponding parts (i.e. the limits, the sides and angles) are the same, and this entails that the two triangles are the same in all respects.) The answers to this question in *Euclid 1*, are of the following form: *if some of the corresponding parts are the same then the triangles are congruent.* To establish congruence, Euclid uses the *method of superposition*. What this amounts to is that one of the two triangles is ``lifted up'' and placed (superimposed) on top of the other, and the two identified if they exactly match

Hegel's approach to congruence is interesting, and one that differs fundamentally from the Euclidean. The crucial point is that one has to understand the triangle (and the other geometrical figures, of course) as conceptually determined. That we have a concept of a triangle is undeniable: we have no difficulty talking about and recognizing triangles as having three sides and three angles, and in their different kinds (acute, right angled, obtuse). The drawn triangle is a spatial representation of the triangle concept. (As we saw earlier, the concept takes over the role in Hegelian philosophy that the form of

a triangle plays in NeoPlatonic thought.) The concept differentiates the whole into its interrelated parts (or limits) and these in turn determine the complete triangle. The relationship triangle-limits is then a special case of the whole-parts relationship that Hegel investigated in his later logic, and the form of congruence theorems should be stated as follows: *a triangle is through such and such determinations of its parts completely determined*. This was discussed in **A** above, and will now be examined in more detail below.

The conceptual relationship of the parts gives the *necessity* operating within the geometrical processes of construction and proof. It is in these processes that the *concept* of triangularity expresses itself and it is the reason for the *constancy* of the triangle throughout: the triangle that we start with at the beginning of a proof is the same as that with which we finish (even though as a figure it may have been cut up in the construction). The new knowledge gained about the triangle through proof is *conceptual* in character.

Since congruence (for Hegel) is about the *identity* of two given, distinct triangles, the concept realized in the two triangles *is the same* even though the two triangles are distinguished spatially. They are just different realizations of the *same* concept. It follows that correct congruence (=identity) proofs have to be formulated *conceptually*, i.e., in terms of our earlier discussion, the ``correct'' proofs for congruence must come from philosophy, *the unhypothetical science*. It cannot be dealt with by looking at a figure and imagining triangles in the figure being moved around and superimposed upon one another for comparison purposes. Congruence is saying something about the *concept* of a triangle and how this concept relates the whole to the parts and parts to whole. So in *GS*, the equality or congruence of two triangles, spatially distinct in general, just affirms that the triangle concept is determined in the two triangles in the same way. More precisely, the equality conditions on the parts identify conceptually *the same triangle*. There is only *one* triangle.

Consider, for example, the ``three sides'' congruence criterion of Proposition 8 of *Euclid 1* that ``two triangles are congruent if they have all three sides equal''. In the philosophy of *GS*, this is really just saying: *if the (limiting) sides of a triangle are determined, then the whole of the triangle is determined.* In particular, the three angles are determined. The philosophical proof for this depends on the angle as the *relation* of the limiting sides (see (e) below), so that the angles are determined once the sides are. The other congruence criteria discussed by Hegel for a triangle ABC are: *two sides and the included angle* (Proposition 4) and *two angles and a corresponding side* (Proposition 26).

The account of congruence given by Hegel does not, then, use superposition; indeed, that method is, for Hegel, a ``superfluous and lengthy'' way of showing identity. The superposition method is, I think, to be regarded as an external reflection of the *conceptual* determination of the triangle concept by the conditions, the determination which is the *genuine* reason for congruence. Hegel devotes a comparatively large amount of space in *GS* (e.g. in his discussions of Propositions 4, 5, 8 and 26) to the understanding of congruence and his criticism of the method of superposition.

Hegel is not alone in his criticism of the method of superposition (though I think that his explanation of why it is unsatisfactory is unique). Indeed, the method has been criticized by many post-Euclidean geometers. A detailed discussion of this is given by Heath.[81] Heath points out that Euclid himself uses the method with reluctance. Peletier (1557) observed that if the method of superposition was legitimate in geometry, then geometry would be full of such proofs. (The method could, for example, have been used to prove Propositions 1,2 and 3.) Bertrand Russell suggested that Euclid should have taken the first proposition in which it is used, Proposition 4, as an axiom, and indeed it is effectively taken that way by Hilbert in his *Grundlagen der Geometrie*.

Hegel's conceptual approach to congruence is based, as we have seen, on his version of the unhypothetical science. Mathematical objects have a *conceptual* basis, and Hegel points out that the ``way of handling material things'' cannot be applied to mathematical things like figures and lines. Further (it seems to me) that the difficulty cannot be solved by putting in an extra axiom (as Russell and Hilbert wanted to do): doing that just evades the problem of understanding the unarguable *necessity* that we feel with congruence, without any explanation of that necessity. The axiomatic solution just ennables us to ``get by''.

Rather, the problem of superposition cannot be solved by an *ad hoc* mathematical assumption, for it is based on a philosophical error. For Hegel, there are not posited two triangles which are then to be compared as far as equality of sides and angles are concerned. *The two triangles appear only on the sensuous level*. Rather, the correct understanding of congruence is, as we have seen above, the one that sees it as the the concept of a triangle completely established by certain of its parts through their interrelationships as triangle limits.

This theory of congruence is developed further in Hegel's mature logic. In the later account, the Neo-Platonic theory of limits, on which the *GS* approach relies, is implicit and instead, the explanation of congruence is in terms of the *Idea*. So[82] he says that ``these [congruence] theorems themselves contain two parts, one of which may be regarded as the concept, and the other as the reality, as the element that completes the former into reality.''. In spatial reality, the conceptual unity of the *triangle* is dispersed into the sensuous triangle and has redundancy in its parts. A congruence theorem, such as the one which states that two sides and the included angle determine the whole triangle, effectively strips away the redundancy of the sensuous triangle so that ``the remaining two angles and the third side are the superfluity of reality over the determinateness of the concept''. Thus the sensuous (real) triangle contains 6 parts, three sides and three angles, but for the triangle concept, three of them suffice. The equality of the two triangles then follows, as in the *limit theory* account of *GS*, not by superimposing the triangles but because conceptually, the triangle concept is determined, e.g., by two sides and the included angle.

A line on its own is, as we have seen, determined by its limits, the end-points. However, in the *triangle* context, a side can be determined in other ways. The side, we may say, is determined by the triangle *as a whole*, not isolated on its own. For example, to determine a side L

of a triangle, it would suffice to be given the length of L, another side M, a common end-point where L and M intersect and the angle that the two sides make with one another. Then the end-points of L, and therefore L itself, are completely determined, as are all the other sides and angles of the triangle.

This oneness of the triangle at the conceptual level is interestingly illustrated by Hegel's discussion in *GS* of Proposition 5, the proposition that shows that the angles at the base of an isosceles triangle coincide. He again criticizes Euclid's method of superposition and approves of a very simple method of proof which, according to Proclus, was due to Pappus. Suppose that the triangle is ?ABC with AB=AC. Then ?ABC is congruent to ?ACB (two sides and the included angle) so that ?ABC=?ACB, i.e. the base angles are equal. Here we have, I think, for Hegel, confirmation in geometry of the *unity* of the triangle concept that is misunderstood in the method of superposition. The proof uses only *one* triangle - ?ABC is the same triangle as ?ACB. The difference aspect of the proposition comes from *the triangle concept itself* and does not come through separate triangles (as it were) accidentally turning up in different spatial positions, and only coming to identity through human intervention by superposition.

(e) *The nature of an angle*

The topic of the nature of an angle was much debated by the classical geometers. An account of the different views is given by Proclus.[83] He mentions three (Aristotelian) categories under which ``angle'' had been put: that of *relation*, *quality* and *quantity*. From the relation point of view, angle is the inclination of two distinct, intersecting lines in a plane. As such, it is a *relation* between two lines. This is Euclid's point of view (Definition 8 of *Euclid 1*). However, angles can be straight or curved and this is a *qualitative* distinction. (For example, the angles made by two intersecting circles is curved.) From the quantity point of view, angles can be measured to give *numbers* (e.g. the ``45'' in a *45º angle*), and, of course, number is *quantitative*. Proclus discusses the difficulties in each of the three views of angle, and, following his predecessor Syrianus at the Academy, says that ``the angle as such is none of these things mentioned but exists as a combination of all these categories''.

In *GS*, Hegel, like Euclid, sees the (rectilinear) angle as a determination of the two lines that meet to form the angle. But consistent with the general approach of *GS*, he sees angle in terms of *limit*. While the two lines bound (limit) a region ``from the outside'', the determination of the angle is not limit in that sense but is a determination of *the limiting itself* of the two lines. Hegel says: ``rather [the angle] is the determination of the limitation of the two lines bounding it, [and] therefore as a determination, still [counts as] a limit.'' So two lines, that intersect, limit an angular region, the ``limitation'' of the two lines. But we have to limit that *again* to get a single angle. The ``wedge'' shape between the two lines is lost when we limit to the angle, and angle is a ``second order'' limit. (This limiting of the limitation is, I think, approximated in classical geometry by the view of Apollonius, discussed by Proclus, that the angle is ``the contracting of a surface at a point under a broken line''.)

The first 12 propositions of *Euclid 1*

Hegel's objective is to give a philosophical interpretation of *Euclid 1* and using this interpretation, to correct what he sees as unsatisfactory proofs in *Euclid 1*. It seems likely that Hegel wrote an account of all of the propositions in *Euclid 1* but only made a ``fair copy'' for the first twelve propositions. If so, his account of Propositions 13-25, 27-28, 30-47 have been lost. His treatment of Propositions 26 and 29 have survived in *GS*, together with a theory of parallel lines that seems to have arisen out of his study of Propositions 27-31 of *Euclid 1*. There are also philosophical interpretations of other theorems of *Euclid 1* (such as the Pythagoras theorem (Proposition 47)) in Hegel's later logical works.

Hegel starts by commenting on the first four propositions. Proposition 1 constructs an equilateral triangle on a given straight line using two intersecting circles. Hegel gives a reason why Euclid starts his *Elements* with the construction of the equilateral triangle. It is that ``at the beginning, equality is given in [the form of] a figure''. This figure is the equilateral triangle, ``the simplest figure, both in terms of equality of sides as well as in terms of [a] triangle''. Hegel's explanation of why the *Elements* starts with the construction of the equilateral triangle agrees substantially with that of Proclus discussed in **A**, with its emphasis on identity and equality. However, Hegel's version is totally without the theory of *Soul* which Proclus attaches to his own explanation. (For Proclus, but not for Hegel, the triangle is ``the likeness of the first soul''.)

Proposition 1 serves as the basis for Propositions 2 and 3, both of which involve *equal lines*. Proposition 2 asks for the construction, with a given point as an end-point, of a straight line equal in length to a given straight line, while Proposition 3 asks, given two unequal straight lines, to cut off from the greater a straight line equal in length to the lesser. The philosophical basis for all three propositions is then *equality*, in particular equality for triangles, at the level of the triangle as a whole and at the level of the parts (sides). Hegel then moves on to Proposition 4. This proposition says that two triangles are congruent if two sides and the included angle for each triangle are the same, and Euclid proves it using the method of superposition. We discussed, in some detail above, the method of superposition and Hegel's philosophical criticism of it.

As we have seen, the relation between the concept and the drawn triangle lies in the ``positing'' of the latter through its limits. To illustrate this, we now describe the philosophical proof by Hegel of Proposition 4. Proposition 4 is first reformulated conceptually: we are no longer given two separate triangles. Hegel does not explicitly state his conceptual version of the proposition but I think that it can be expressed as follows: *for a ?ABC, if ? A and the lengths of the sides AB, AC are determined, then they determine the remaining side BC and the angles ? B and ? C, i.e. the whole triangle is determined.* The proof goes as follows. Since AB and AC are determined, so are their limits, the end-points A,B, C. However these limits are posited only for their respective lines AB, AC and so are not ``completely posited'', for ``they do not exclude the infinite of space''. I think that what Hegel

means by this is that if we only know the lengths of AB, AC, then there are infinitely many possible positions for the points B and C: B can be any point on the circle center A and radius the given length of AB, and similarly for C. For the *complete* positing of B and C, we also have to include the determination of ?A: the lengths of AB, AC *together with* ?*A* determine B and C uniquely for the triangle. But B, C are the limits for the side BC. So the straight line BC is also posited (since the limits of a geometrical figure determine the figure). (In plain geometrical terms, we get the line BC by joining up B and C using a ruler.) Once we have *all* of the sides determined then so are all of the angles, since, as we saw earlier, for Hegel, an angle is the determination of the *limitation* of the two lines bounding it: since the pairs of lines that bound the angles of the triangle are now determined, their limitations in the angles are also determined. So the whole triangle, i.e. all of its sides and all of its angles, are determined. This completes Hegel's proof of Proposition 4.

We note that in this proof, the complete positings are *for the triangle as a whole*, i.e. the triangle *according to its concept* and not with respect to position in space. So in the proof, we can vary the position of A and rotate the triangle about A in any way that we like. Such variations in the spatial position of the triangle do not affect the proof: the determinations of the triangle parts are relative to the triangle as a whole. The variations in the position of the triangle in space are, in the language of Hegel's later logic, the ``superfluity of reality over the determinateness of the concept''.[84]

It remains to show how the original *Euclidean* Proposition 4 follows from Hegel's proposition above. In the Euclidean Proposition 4, we are given two *separate* triangles ?ABC, ?DEF with ?A=?D, AB=DE, AC=DF. We have to show that the remaining corresponding parts of the two triangles are the same. This follows immediately from Hegel's proposition: for once you have two sides and the included angle determined, then everything in the triangle is determined by them. Since the determinations of the two sides and included angle are exactly the same for the two triangles, the two triangles are the same in concept, and so their corresponding parts are the same. No superposition is required. (The coincidence of the triangles through laying one on top of the other in the Euclidean proof is now seen to have its valid explanation through the triangle as a conceptual relation of whole and parts.)

For Hegel, philosophical (and not just geometrical) insight is also required for a proper understanding of the relationship between Propositions 5 and 6. These two propositions are converses to one another: together they say that *in a triangle, the two base angles are the same if and only if the two opposite sides are the same in length.* Euclid's proof of Proposition 5 is involved. It also uses Proposition 4 whose proof, as we have seen, requires superposition and so is unsatisfactory. The ``deficient grasp of difference'', which Hegel sees as a fundamental defect in the Euclidean approach, is present in the proof of Euclid's Proposition 5. As in the case of Proposition 4 (and congruence in general), the real issue is: *how much is needed in the parts to determine the whole*. In the present case of Propositions 5 and 6, equality of the two angles determines the *same* triangle concept as does equality of the two sides. As discussed above in (d),

Hegel approves of a simple classical proof of these propositions, due to Pappus, which uses *only one* triangle. The use of only *one* triangle in that proof, or, as Hegel puts it, the fact that the triangle has been *doubled*[85], indicates, for Hegel, awareness that difference in geometry is *conceptual*, not spatial, in character. The Pappus proof does not use separate triangles in different spatial locations, but rather *the difference is in the same triangle*.

An interesting criticism by Hegel of Euclid's proofs of Proposition 6 and 7 concerns their use of *argument by contradiction*. (Hegel calls such a proof *apagogic*.) His objection is not so much a criticism of proof by contradiction *in principle* but rather that in each of these propositions, the proof requires a construction that cannot be done. His point is that we cannot perform a ruler and compass construction if it is impossible to do it. Indeed, as the figures for these propositions show, these proofs by contradiction are only possible by *drawing impossible diagrams*. But in the Euclidean framework, we can only work with *constructed* figures. (The same problem effectively arises in the mathematics of number, and is emphasized in Brouwer's intuitionistic foundation for mathematics.) Hegel resolves this problem with Euclid's proof of Proposition 6 by indicating an alternative argument not relying on contradiction.

As indicated in the preceding paragraph, I do not think that Hegel regards as unacceptable in general *proof by contradiction* (as did Brouwer in modern mathematics). (To reject *all* such proofs has grave consequences for mathematics.) Indeed, he himself gives an apagogic proof in *GS* of Proposition 26: *in a triangle ABC, the point C is determined uniquely by AB, ? B, and ? C.* This proof, however, is conceptual in nature and does not involve an impossible geometric construction. It is only the Euclidean proofs by contradiction that depend on an *impossible* construction that Hegel finds unacceptable.

It is the constructive character of Euclidean geometry that rules out impossible constructions. Any construction that you consider has actually to be done. Once you consider ``possible'' constructions, then you are effectively dealing with the *set* of all constructions and one gets involved in infinite set theory (with its possible paradoxes) and leaves Euclidean geometry. It shows great insight on Hegel's part to have noticed this essential, constructive character of the *Elements* – it is exactly in this respect that Euclidean geometry differs from modern axiomatic geometries (such as that of Hilbert).[86]

But this does not rule out apagogic proofs in mathematics. Here is a simple apagogic proof in algebra that, I think, Hegel would have accepted. One wants to show that if $x>a>0$ then $x^2>a^2$. Suppose not. Then $x^2=a^2$, and we derive a contradiction by dividing that inequality through by x and using $a/x<1$. This is acceptable since there *are* x's for which x^2 is less than or equal to a^2: the x with which we started just does not happen to be one of them. But in the geometrical case, it is different: *real things cannot be contradictory* and a geometrical argument relying on a construction has to use a *genuine* construction. It is not permitted to consider hypothetical constructions. So in the context of Proposition 6, Hegel says: ``If it were the case that AC [is less than] AB, then BD=AC; this construction shows itself to be not possible.''

It is undeniable, however, that Hegel regarded apagogic proofs with some suspicion throughout his life. For example, in his discussion of the second Kantian antinomy in *WL*[87], he rejects the apagogic elements in Kant's arguments. (The source of the second antinomy for Hegel lies in the character of the concept of quantity, whose determinations, *continuity* and *discreteness*, are null in isolation and consist only in their transition from the one into the other.)

Proposition 7 and Proposition 8 (that give the ``three sides'' criterion for congruent triangles) are criticized similarly. In *Euclid 1*, the first has an apagogic proof and the second relies on superposition. Hegel gives conceptual proofs for both, avoiding apagogy and superposition.

Propositions 9 and 10 are problems rather than theorems. They show respectively how to bisect a rectilinear angle and how to bisect a finite straight line. Hegel comments that the propositions are (effectively) ``completely the same'', both using the construction of an equilateral triangle on a given line (Proposition 1). However, while it is true that the proofs of both propositions construct an equilateral triangle, it is not correct to say that the propositions are the same. The reason is as follows. Both proofs use a construction in which two triangles have a common base. In the proof of Proposition 9, one of the triangles is isosceles and the other is equilateral. But in the proof of Proposition 10, both triangles are equilateral. I think it likely that the reason for Hegel's misunderstanding is the poor figure given by Lorenz[88] where the isosceles triangle *looks like* an equilateral one. (This is discussed in more detail in an end-note to the present translation.)

Propositions 11 and 12 deal with one problem: how to construct a perpendicular line from a given point to a given line. Proposition 11 is the case where the point is on the line, while Proposition 12 is the case where the point is not on the line. Hegel points out that the solution to both problems lies in constructing an isosceles triangle on the given line. (His comments are purely geometrical, with no philosophy involved.)

In a different part of *GS* (as it appears in *Dok.)*, there is a collection of comments that look like conceptual proofs. These are analogous to the proofs that Hegel gives of congruence theorems (such as Proposition 4). *GS* does not say what Proposition is being proved here, but inspection shows that it is Proposition 26 of *Euclid 1*. It seemed to me appropriate to include this proposition with the others in the second part of the translation. The proposition gives the congruence theorem: *two triangles are congruent when two angles and a corresponding side are the same*. The proof falls into two cases, the first case where the specified side joins the equal angles, and the second case, where the side subtends one of the angles.

As usual, Hegel's discussion of the proposition is in terms of the determination of the whole of the triangle ABC by some of the parts: two angles and a side determine everything in the triangle. In his proof of case 1 of the proposition, discussed in a), Hegel shows that given BC, ?B and ?C, then the point A is uniquely determined, and hence

also each of the remaining sides AB, AC. ß) continues the discussion of a).

At the end of ß), he raises a question which is equivalent to proving case 2 of the proposition. In that case, one is given AB, ?B and ?C, and has to show that the triangle ABC is completely determined. This will be achieved if it can be shown that the point C is *uniquely* determined by the given data. For then AC and BC are determined, and hence also the angle of their inclination ?A. So all of the sides and angles of the triangle are determined. Hegel then gives two proofs of the unique determination of C.

The first proof is by contradiction (or in Hegel's language, it is *apagogic*), showing that if the point C is not uniquely determined, than the length of either of the lines BC, AC is not determined, and ?C would be changed. But ?C is determined and the contradiction follows. The second proof is direct and, I think, essentially goes as follows. Since the sum of the angles of a triangle add up to two right angles (Proposition 32) and two of the angles, ?B and ?C, are determined, then the third angle ?A is also determined. So ``the [relative] direction[s] of AC and BC and AB are (*ist*) determined''. Now AB is given and the directions of AC, BC relative to AB are given. Since ``straight lines intersect only in one point'', it follows that C, the point of intersection of AC, BC is uniquely determined.

The philosophy of parallel lines

In *GS*, this is the last section, preceded by the material of *On the algebra of a geometrical figure* below. I have changed this order since this section on parallel lines deals with a topic in *Euclid 1* that follows the propositions discussed in the previous section, and indeed contains a discussion of Euclid's Proposition 29.

Euclid 1 treats the basic geometry of parallel lines in Propositions 29-31. Euclid defines[89] two straight lines to be *parallel* if they are ``in the same plane and being produced indefinitely in both directions, do not meet one another in either direction''. This is what Hegel calls the *negative* definition of parallel lines (presumably because there is a *not* in the definition). He also gives the *positive* definition: two lines (in a plane) are called *parallel* if they ``everywhere are equally far apart''. This positive definition can be specified in two ways.

The first way was, according to Proclus[90], given by Posidonius: *parallel lines are those which, (being) in one plane, neither converge nor diverge, but have all the perpendiculars equal which are drawn from the points of one line to the other*. This means that the distance from any point on one line to the other line is the same for all of the points on the lines.

The second way[91] in which the positive definition can be determined is through the interior angle that a third line (a transversal, or, as Hegel expresses it, a *distance line* (*Entfernungslinie*)) makes with one of the lines. Hegel's exposition here is rather confusing. I think that there are two possible interpretations of the second way. The simplest is this. Fix an angle ? and one of the parallel lines. For

any point A on that line, consider the line (the *distance line*) through A that makes (interior) angle ? with the parallel line. The point at which the distance line cuts the *other* parallel line is determined, and hence also the distance between the two points. The two lines are then called *parallel* when this distance is the same wherever the initial point A is taken on the first parallel line. (If the interior angle is a right angle, then of course this distance will be the perpendicular distance as in the first way.) A disadvantage of this interpretation is that the distance for the distance line is determined *after* the angle, rather, as one might expect, coming *before* the angle. The second interpretation, though more involved than the first, does specify distance first, and so I think is preferable. It is discussed in an end-note to the translation of this part of *GS*.

Hegel notes, with reference to the second way, that ``the distance lines are equal in concept with that of the parallel lines''. Indeed the distance lines, because they make the same angle with the given lines, themselves form a family of parallel lines, and Hegel's interesting comment is an example of how two things (distance lines, parallel lines) which, when they appear in a geometrical diagram are completely different, nevertheless at the conceptual level are the same. (Earlier, we had other examples of this: for example the congruence conditions are the same conceptually as the concept of the whole triangle, and the equality of the base angles in a triangle is conceptually the same as the equality of the opposite sides.)

Part of Hegel's discussion of Proposition 29 of *Euclid 1* has survived. As in the case of Proposition 26 above, there is no explicit mention of the proposition in Hegel's account. Proposition 29 asserts that *a straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the interior angles on the same side equal to two right angles.* Using the positive definition of parallel lines Hegel gives two proofs of the first part of Proposition 29. The proofs seem to be original. The precise assumptions that Hegel is making here are not clear to me, and the text is hard to follow. The proofs, as far as I can see, are purely geometrical in nature with no philosophical argumentation required.

This confirms, if such evidence be needed, that Hegel, while not a creative mathematician, fully accepted the essential validity of the Euclidean approach, and that he spent the time making himself skillful in its methods. Hegel took geometry very seriously and did his own research in it. For a non-professional mathematician, he knew the subject very well, though there is an occasional lapse in his geometrical insight. (One such lapse is in his misunderstanding of the ``ambiguous case'' for congruent triangles, a case that (strangely) is not in the *Elements* and which, I believe, Hegel uses wrongly twice in this section.)

In geometry, as discussed in **A**, philosophy only comes in at certain places, viz. where mathematical resources fail in principle, and the subject requires the philosophical concepts of the ``unhypothetical science''. Only then does its indispensable, philosophical basis need to become explicit, and Euclidean methods need ``correcting''.

On the algebra of a geometrical figure

There is a geometrical algebra in the *Elements* of Euclid: indeed, Book 2 of Euclid can be regarded as algebraic in the sense that problems that nowadays would be solved by (elementary) algebra are solved by Euclid geometrically. For example, where we talk of a product xy of positive numbers x, y, Euclid talked about the area of a rectangle ABCD (with sides AB of length x and AC of length y). The modern algebraic approach to elementary geometry goes back at least as far as Heron (c. 3rd century AD). As Heath points out[92], ``Book II gives the geometrical proofs of a number of algebraical formulae''.

However, the algebra that Hegel is involved with in this last part of *GS* is really just the adding and subtracting of line intervals arising from a figure in which two circles intersect. (See Figure 3 of the translation.) In this figure, A, B are the centers of the circles, AB intersects the circles, between A and B, in the points F,G, while E is the point of intersection of AB with the line CD joining the points of intersection of the circles. The radii of the circles are the same in length. In his calculations, he uses, without proof, the equality FE=EG, which easily gives AE=BE, AF=GB. The simple proof of these equalities is given in the translation.

I confess that I do not know what Hegel is trying to do in this part of *GS*. He starts with a long algebraic calculation dealing with the lengths of line segments on AB. The calculation begins with the equality AF=GB, and ends with the equality AF+FE-FE=BG+EG-EG, which, upon cancelling the FE's and the EG's, is trivially the same as what he started with, AF=GB. Maybe this calculation is very early (schoolboy?) work of Hegel in which he was doing a mathematical experiment, not designed for publication. The part finishes with an argument in which Hegel shows that three pairs of triangles in Fig. 4 (which is close to Fig. 3) are congruent. It ends with the equality: GC-GD-DE=CE, which is trivially true from the diagram.

In between these two algebraic calculations, Hegel has an interesting philosophical discussion on the relation between geometry and algebra. He says:

> In algebra, x is fixed; but its determination [exact value] is not posited; in geometry, no x is given, everything is determined, and as such[,] posited; the becoming of its determination is not posited; its dependence is =x.

In this ``not fixing'' of the x in algebra, reality as being, or as the Idea, as it is immediately determined in space, disappears; *x is a variable, universal in character*. The Euclidean line, as indicated by the variable x which is supposed to range over the set R of real numbers, is not geometrical in the sense that a drawn line or triangle are geometrical. Euclidean geometry has *no x's*. In geometry, we are presented with a definite, drawn triangle with no variables; everything in the triangle is fixed. But the x of algebra is the *possibility* of a triangle, as when I ask: *draw me a triangle*! There are infinitely many possibilities.

This is further clarified by Hegel in his specification of the *conceptual* difference between algebra and geometry: he says that in

algebra, *the conditions are given and the conditioned [is] not* , while in geometry it is the other way round; in geometry, *the conditioned is given, and the conditions as such are to be posited, to be shown,* and a geometrical problem *makes determinate the concept of the concealed [conditions]*. I think that we can understand this as follows.

Starting with geometry, the drawn triangle is initially given as *conditioned* in the obvious way: its sides are straight and there are three of them. But the conditions that relate the angles and sides, at that initial stage, are ``concealed''. Geometrical argument is required to uncover these concealed conditions, to ``posit'' them. An example of such a concealed condition would be the congruence criterion of Proposition 4: two sides and an included angle *condition* (determine) the triangle completely.

With algebra, on the other hand, the conditions are given. So, for example, if we wish to prove a geometrical theorem by algebraic methods (``analytic geometry'') we write down *equations* for lines, circles, conics etc. and work with them algebraically. These equations are, I think Hegel is saying, the ``conditions'' that are ``given'': so, for example, the *condition* for a line is given algebraically by: $ax+by+c=0$, and this determines the line *completely*. Further, these equations involve variables such as x. But as a variable, x is not determined as a *specific* number and what the conditions actually condition, e.g. an actual drawn triangle, is not given in the algebra. It is true that, from a modern point of view, we could define the triangle as a *set* of pairs of numbers (x,y), but that is *still* at the level of conditions: that is, unless we regard the highly infinite set as a given thing – and we have no direct, non-geometrical intuition for that – the definition just says that a point (x,y) is in the triangle if and only if the coordinates x,y satisfy certain inequalities (conditions). If we rely on algebra *alone*, the actual triangle is not given. Unlike geometry, there is no spatial intuition in algebra. So, as Hegel says, in algebra, exactly *what* is conditioned (i.e. the geometrical triangle) is not given. The relation betweeen algebra and geometry is similar to that between pure logic and the philosophy of nature in the Hegelian philosophy.

In fact, analytic geometry is a powerful tool for proving geometrical results, but only when the geometrical figure and the algebra are present together – the given geometrical object together with its formal algebraic determinations. Real number and algebra become involved in a geometrical problem when we explicitly assign a number pair (x,y) to a point. The geometrical interpretation of x, y are as *coordinates* of the point, i.e. the distances of the point as measured from two fixed perpendicular lines (the x- and the y-axes). (This idea was due to Descartes and Fermat.) The calculations of analytic geometry depend crucially on the Pythagoras theorem which ennables one to calculate the distance between two points (x,y), (x',y'). So in that respect as well, analytic geometry still depends on Euclidean geometry.

These observations of Hegel on the difference between algebra and geometry are, I think, very valuable, and a corrective to the commonly held, simplistic, view that elementary geometry can be ``reduced'' to the elementary algebra of numbers and hence to set theory.

On the translation

As discussed above, I have organized the material into four main parts. The headings for these parts are, of course, due to the translator. They are given to help orient the reader to the main themes of *GS*. Without such an organization of the material, *GS* becomes very fragmentary. As background for Hegel's discussion of the first 12 propositions of *Euclid 1* and also of Propositions 26, 29, I have given their Euclidean statements and a sketch of their proofs. This background is necessary in order to make sense of Hegel's discussion.

Following the practice of H. S. Harris and others, I have inserted in some places words not in the original text to clarify the meaning of that text. These insertions are surrounded by brackets [...]. Comments made by the translator on the text are also surrounded by brackets but are in italic font.

Translation

Geometrical Studies

THE NATURE OF MATHEMATICAL OBJECTS

In each mathematical object, there is, *qua* limiting:
  a) its positive [being], insofar as it [the object] sublates a limitation ([e.g. the] line as the spatial relation [or *connection*[93]] of the point); as self-sublated, [the object] remains only quantitative [*die Menge*] (the point [remains a point])[94];
  b) its negative [being],
    a) in so far as it is an other as negated, [it] is limit[; so] the line negates the surface (though the latter is still a surface in the sense of positive being)[95] by bounding it; because [it has the] character only of approaching the other, [i.e.] the negating line, that's why the surface is nothing more nor less than a line; it [i.e. the surface] develops from nothing other in itself – rather it [is] first through the other - ,
    ß) it is negated only [in that] it is limited by the part.

The point is the simplest limitation of the infinite, [but] it is nothing absolute, because it can become bounding [as the end of a line]; if it were absolute, then no mathematics would be possible. The positive of each mathematical object is its sublation of the infinite.

[*The nature of figures*]

The ground ([i.e.] the reason [for something] in general and [in particular] here in geometry) is the manifoldness of the unity of the

limit (*Vermannigfaltigung der Einheit der Grenze*)[96]; but the [resulting] new manifoldness must itself be limited again. The [new] manifoldness of the unity of the limits gives a new form of figure, i.e. a new unity of the whole. The principle (*Gesetz*), the unity of the figure, is the relation of [its] limits; this relation can be[97]:

1. its magnitude, therefore equality of kind (*Art*)[, qualitatively indifferent], [referring to] the same standard, [or]
2. the self-relation of the limits as space ([e.g. the] enclosed angle).

In 1., it [the principle] is positive; in 2., it is[,] as limit[, negative], and is measured as that which they [the limits] bound and determine.

The propositions [of the form]: under which conditions are [two] triangles equal to one another, are therefore [to be] expressed: *a triangle is through such and such determinations completely determined.*

[*Cube and Sphere*]

[The] objective relation of a point posited outside itself is the line; [the] objective relation of a line posited outside itself is the surface; the objective relation of a surface posited outside itself is the solid. In the cube [for example] there is in existence

a) the point,
b) the line,
c) the surface,
d) the solid;

in the sphere there is nothing but limit, and at the same time, it is a posited, bounding limit that bounds [a region in space][98][; in the case of the sphere]:

a) *the point* as such is not [a limit of the sphere] – [for:]
   a) the center, [the point determined by the sphere], does not belong to the sphere and is not a limit of the sphere even though it is posited [by it],
   ß) if any other [point than the center] is posited [in the sphere], [then that choice] is arbitrary, and [it is] therefore not a limitation;
b) *the line* is not such a limitation [of the sphere: for,]
   a) the diameters [and] radii do not belong to the sphere, and, [depending on] where they are posited, they are arbitrarily bounding [where they meet the sphere] (though they are not bounded themselves);
   ß) the great circles[99] do not bound the sphere, [but] are given through it and arbitrarily, [depending on] where they are posited, and it bounds them as circles;
c) *the plane* [is likewise not a limit of the sphere]:
   a) the plane of a great circle does not belong to the sphere; its place is arbitrary, and it is limited by, but does not limit, [the sphere];
   ß) the point[100] as enclosing, [in the form of] the great circle[101], has, in itself the returning line [a radius], [which has] in itself the returning plane [the plane of the circle;] as the circle has no [posited] point [on it], so it has no [straight] line [for such a line would have to have a bounding point posited on the circle]; within the circle lies its limit (the center, diameter[s]), external to it [the circle], which it [the circle] bounds.

[*The three dimensionality of space*]

There are only three dimensions, for the solid is itself again [the] point, [i.e.] outside itself [in] complete contradiction. The self-opposed body as unity was the line.

The principal proof that there are only three dimensions must be that the spatial relation of the point, line and plane, posited outside themselves, is of a different nature than they are themselves; but the solid is not; that means that these [point, line and plane] are limits, [while it, the body] is not. Its limit [if you want to call it that] is the ceasing to be of the relation; that is why the solid is not limit – the points (limits of lines) are such that they exclude another space and so their relation with another space; the solid[,] posited outside itself, is either two solids, with a space in between, the ``not'' of their relation, but [is, rather, the latter's] sundering – or [it] is one solid, when it is not the cessation of the relation in some regard or other, but [rather] absolutely a drawing out of itself.

[*Congruence and angles*]

[That two rectilinear] figures are congruent means: their [corresponding] sides and angles are equal. The angles [of such a figure] are determined through its enclosing sides, are comparable[102] through the opposite [sides] and come about[,because of that,] through the limitation of the sides. The angle[103] is not a limit ([in the way that] a surface limits from the outside a [three-dimensional region] in space), but rather [the angle] is the determination of the limitation of the two lines bounding it, [and] therefore as a determination, still [counts as] a limit.

The angle is, as limited,
a) the limitation of the indeterminacy of the spatial relation of its bounding lines; as limitation, it is indicated through them.
[*The ß) that presumably followed this is lost*.]

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23. Sept. 1800
Mayence/Vend. l'an IX

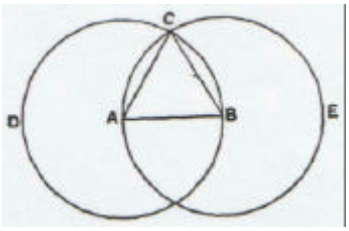
THE FIRST TWELVE PROPOSITIONS OF *EUCLID 1*

[*This part of Hegel's geometrical studies is a discussion of the first 12 propositions of Euclid 1. (We have also included here Hegel's discussion of Proposition 26. His discussion of Proposition 29, which deals with parallel lines, is included in the next part, which deals with the philosophy of parallel lines.) As shown above, its date is 23 September, 1800. This must have been when Hegel completed it, for on the next day[104], he went to work on: Der Begriff der Positivität (a revised introduction for his earlier essay on the Positivity of the Christian Religion). In September of 1800, Hegel visited[105] the city of Mainz which had been in French hands since 1798. Hegel also uses the French name Mayence for Mainz and the French revolutionary calendar to date the period of his visit. (``Vend.'' stands for the Autumn month*

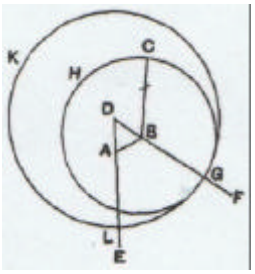
Vendemiaire (''Vintage''), which was the first month of that calendar. 1800 was the ninth year of that calendar.) To help with the reading c Hegel's text, the original propositions of Euclid 1, discussed by Hegel, will be stated following the translation of Hea., and a brief description of their Euclidean proofs will also be given. (Hegel himself does not state the propositions nor give the proofs.)]

[Propositions 1-3 of Euclid 1]

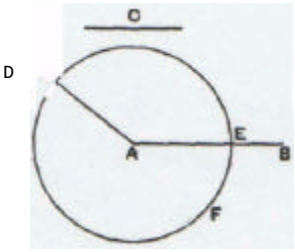
Proposition 1. *On a given finite straight line to construct an equilateral triangle. (Sketch of proof:*
let AB be the line. The proof constructs the circle center A and radius AB and the circle center B radius AB. Then AABC is equilateral (since the radii AC, BC equal AB) , where C is a point of intersection of the circles.)



Proposition 2. *To place at a given point (as an extremity) a straight line equal to a given straight line. (Sketch of proof:*
let A be the point and BC the given straight line. Construct (Proposition 1) an equilateral triangle ABD. Extend the line DB to intersect the circle center B and radius BC at the point G. Extend the line DA to intersect the circle center D and radius DG at the point L. Then AL is the line placed *at* A that we want. For AL=DL-DA=DG-DB=BG=BC using equality of radii and the equilateral condition.)



Proposition 3. *Given two unequal straight lines, to cut off from the greater a straight line equal to the less. (Sketch of proof:*
let AB, c be the lines with AB the longer. Let (Proposition 2) AD be a line of length c, and let the circle center A and radius AD intersect AB at E. Then AE has length c and we cut at E.)



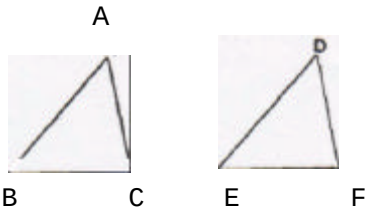
[Hegel's comments on Propositions 1-3]

Proposition 1 directly serves *[dient]* as a basis for [Propositions] 2 and 3, where [in Proposition] 2 the problem[106] is to draw a line equal in length to a given line, and [in Proposition] 3, [the problem is] to take away from a larger line a given smaller one; for at the beginning, equality is given in [the form of] a figure, and this [figure] precedes even the construction[107] of two equal lines [in Proposition 2]. The equilateral triangle is the simplest figure, both in terms of equality of sides as well as in terms of [a] triangle.

[Proposition *4 of Euclid 1]*

Proposition 4. *If two triangles have the two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, they will also have the base equal to the base, the triangle will be equal to the triangle, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend.*
[*More briefly expressed, this says that two triangles are congruent if two sides and the included angle are the same in each* .]

Sketch of proof*:*
the proof is by superposition of one triangle on top of the other so that corresponding sides lie on top of one another. Euclid describes this as one triangle ``being applied'' (efa?µ???µ????) to the other. The third sides and then both triangles coincide.)



[*Hegel's comments on Proposition 4*]

The laying of points on top of each other and the putting of triangles on top of each other is not an acceptable, appropriate requirement. [If we allow the use of superposition] of figures [to prove Proposition 4] why should [it not have been] used on lines to prove Propositions 2 and 3? Then[108] such problems can be solved because of this in the simplest [way]. The way of handling material things can be applied as little to figures as to lines. - This method [of superposition] is, in addition, superfluous and lengthy. Its conclusion is that [since] the end-points of the equal sides and the sides [themselves] fall on top of one another, therefore so also do the third sides; and moreover, the [corresponding] angles are equal [since the triangles now coincide]. [Instead], the argument given below [is] that if something is given in one triangle and it is also given in the other [triangle], then this means [that both triangles] are determined in the same way. With the concept of the determined being [*Bestimmtseins*] comes also the positing of both [triangles], and comparison is superfluous; for it is unnecessary to posit a whole as [actually] given; for the question is purely one of [logical] consequence[s]; no comparison is needed. When in a triangle two sides and the included angle are determined, [then] so also is [determined] the third side. [For], if ?A and [the sides] AB, AC are given, then the end-points B and C [of these sides] are posited.

[*The Hegelian version of Proposition 4 can, I think, be stated as follows: for a ?ABC, if ?A and the lengths of the sides AB, AC are determined, then the remaining side BC and the angles ?B and ?C are also determined, i.e. the whole triangle is determined.*]

Proof.
B and C are posited, [and] so therefore also is [the line] BC, for B and C are the two points which bound BC.

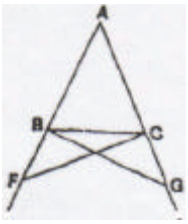
B and C are limits for AB and AC but only in that respect[;] they are not completely posited, that is, they do not exclude the infinite of space, what they as points ought to be[.] As [these points are only given as] limit points of a line BC, that is why the condition, that ?A be determined, is necessary, i.e. that AC, AB have only one slope

relative to each another, and as a result, B and C as points are together completely posited; earlier on, [B and C] were posited only through consideration of A, which is the other end-point of AB and AC; but when the slope of these lines [relative to one another] is [also] determined, it is the same for each point[109] [i.e. each of the points, B, C is also determined].

That the angles ? B and ? C are determined follows from the determination of the slope[s] of the lines AB, BC for ? B; [and of] AC, BC for ? C - exactly as in the Euclidean proof.[110]

[Proposition 5 of Euclid 1]

Proposition 5. *In isosceles triangles the angles at the base are equal to one another, and if the equal straight lines be produced further, the angles under the base will be equal to one another.*
(*Sketch of proof*:
extend AB to D, AC to E. Take F on BD and then G on CE such that AG=AF (Proposition 3). Then AC=AB, AF=AG and ? FAC=? GAB (same angle). So ?AFC=?AGB (two sides and the included angle (Proposition 4)). Next BF=CG (since AF-AB=AG-AC), ? BFC=? CGB, BG=FC (these two equalities coming from ?AFC= ?AGB). So ?BFC=?CGB (two sides and the included angle). So the angles ? FBC, ? GCB under the base are equal. Further, ? ABC=? ABG - ? CBG=? ACF - ? BCF=? ACB (using ?AFC=?AGB, ?BFC=?CGB).)



[Hegel's comments on Proposition 5]

The property of being isosceles is [at] a much more superficial level of unity for a triangle *(in der Figur)* than is [the property of being] equilateral.

The proof of the proposition follows from the concept of distance and that through which it [the isosceles triangle] is determined.

In the Euclidean proof, the triangles whose equality is proved using Proposition 4 can not be layed on one another. It [the Euclidean proof] necessarily has to use this detour through [these] triangles because it must compare [them]. Because of this, [viz.] that the triangles do not at all lie on top of one another, but only [have] parts in common, the same deficiency that was pointed out *(erfordert)* in the proof of the fourth Proposition - [concerning] the difference of the triangle - is not so conspicuous.[111] Indeed, when [the] difference of the triangle [is taken as conceptual and] not necessarily as if it were a thing in the manner of Euclid, then the proof [of Proposition 5] can be made shorter: we are given:

AB=AB
AC=AC (presupposition [Common notion 4])
? A=? A.

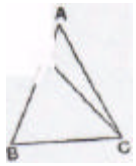
Therefore ?ABC=?ABC; therefore ?B=?C.[112] Better, that is where ?ABC is treated as two different triangles, [we have]:

AB=AC
AC=AB
?A=?A.

[So] ?ABC=?ACB; and ?B=?C.[113] Euclid's detour serves only to cover the deficiency in [his grasp of] difference.

[*Proposition 6 of Euclid 1*]

Proposition 6. *If in a triangle two angles be equal to one another, the sides which subtend the equal angles will also be equal to one another.* (*This is the converse to the first part of Proposition 5.*
Sketch of proof:
suppose that ?ABC=?ACB. Suppose that AC?AB. Without loss of generality, we can take AB to be longer than AC, and cut off (Proposition 3) from AB a line segment DB equal in length to AC. Then ?DBC=?ACB (Proposition 4) which is absurd and we have a contradiction. (As Heath points out, Euclid needs a postulate to justify this conclusion – for our purpose, we can argue that congruent triangles have the same area, but the area of ?ABC=area of ?DBC + area of ?ADC> area of ?DBC, contradiction.))



[*Hegel's comments on Proposition 6*]

This is the converse to the previous proposition. It's proof is apagogic.[114] It appears nevertheless strange that what it demands to be done is impossible. If it were the case that AC [was less than][115] AB, then BD[116]=AC; this construction[117] [as shown in the proof of Proposition 6] shows itself to be not possible.

Otherwise, the proposition can be proved in a way similar to the previous one [Proposition 5]; in the [discussion of the][118] previous [proposition], the equality of the sides was used to construct the same triangle, [so that] the triangle is doubled, while here [it is] the angle [that is to be doubled]. One extends BC to D and E so that BE=CD; and this makes ?E=?D; (to be sure, it has not been [proved] that this construction gives two equal angles [?E, ?D], but on this point, it is [enough] to say that the equality of ?E and ?D can just as well be proved [*gesetz*] in the same way as the [equality of the] enclosed angles were proved in the [Hegelian version of] Proposition 4.[119])



Then in ?ACE and ?ABD[120], [we have]

?E=?D
?C=?B
EC=BD

[, so that] ?ACE=?ABD. Therefore AC=AB using Proposition 26, Case 1, which case depends only on Proposition 4[121] - also it then [follows] that ?AEB=?ADC[122], and ? ABE=? ACD.

[We have in Propositions 5 and 6] a pair of converse propositions[123][;] this leads one even to conjecture that the two determinations, which in the different propositions occur as conditions and consequences, are both dependent on something higher, and it is mere arbitrariness that [in each of these propositions,] the one [determination] is posited [as condition], and the other as consequence, but yet [this higher thing] is considered not posited; while it has been proved, as has been shown, that each is determined together with the other, yet still this [mutual determination] was not posited for [our] consciousness. This [criticism] is valid in general for all [of the] propositions, and the essential question for every theorem, with regard to the whole of the posited, is this [*heißt*]: *how much is necessary for the determination of the whole*, or, *when the whole is posited, how much must I know about the parts in order to cognize, solely from that knowledge, cognition of the whole that already is posited.*

From the concept of the distance of a point from a line there results that in the isosceles triangle, [two of the] lines and [two of the] angles are determined together.

[*Proposition 7 of Euclid 1*]

Proposition 7. *Given two straight lines constructed on a straight line (from its extremities) and meeting in a point, there cannot be constructed on the same straight line (from its extremities), and on the same side of it, two other straight lines meeting in another point and equal to the former two respectively, namely each to that which has the same extremity with it.*
(Euclid's statement of the proposition is (notoriously) periphrastic - more simply, the proposition says that a triangle with a given base line and required to lie on one side of that base line is determined uniquely by its other two sides. (The requirement that the triangle lie on one side of the base line is to rule out its mirror reflection in the base line.) *Sketch of proof*: suppose not. Then there are lines AC, BC, AD, BD, on the same side of the base line AB, with AC=AD, BC=BD and C different from D. Then ?ACD and ?BCD are isosceles, and by Proposition 5, ? ACD=? ADC and ? BCD=? BDC. Then ? BDC=? ADC+? BDA>? ADC=? ACD=? ACB+? BCD>? BCD, contradicting, ? BCD=? BDC.)



[*Hegel's comments on Proposition 7*]

The proof again is apagogic, and that's why the construction demanded is impossible.

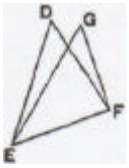
The point [C], in which the lines [AC, BC] are supposed to meet, is unique (*ist nur Einer*); for the lines are given, that is, determined, that is, the distance[s] of C from A and B are determined absolutely; therefore the [common] limit [C] of the lines [AC, BC] is posited; at the same time, the lines are supposed to intersect, that is, the point

C (the [common] limit of both) has to be unique; this proposition is completely analytic.[124]

[*Proposition 8 of Euclid 1*]

Proposition 8. *If two triangles have the two sides equal to two sides respectively, and have also the base equal to the base, they will also have the angles equal which are contained by the equal straight lines.* (This says that two triangles ?ABC, ?DEF are congruent if their corresponding sides are all equal, i.e. AB=DE, BC=EF and CA=FD.
Sketch of proof:
suppose that the corresponding sides for ?ABC, ?DEF are equal as above. Lift (``apply'') ?ABC onto ?DEF with B coinciding with E, BC lying along EF and both triangles on the same side of EF. Then since BC=EF, C coincides with F. The point A is then at a position G. Also ED=BA=EG, FD=CA=FG, and by Proposition 7, D=G. So ? BAC=? EGF=? EDF. Similarly, ? ABC=? DEF and ? ACB=? DFE. So the triangles ABC, DEF are congruent.)



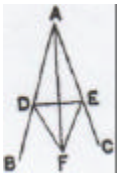


[*Hegel's comments on Proposition 8*]

The proof begins with one side [the base BC of the triangle] thought of as posited; therefore its posited limits [B, C] are determined, and as a result, through the determination of the other sides, the third point [A of the triangle] is determined by Proposition 7. The positing of two triangles is completely superfluous. For this [positing of two triangles] itself is referred back to the fundamental proposition: [two things] that coincide with one another are equal to one another[125] – to be precise, from the fixing of the point [A], the equality of the angles is to be derived. Through the fixing of the point [A,] the directions [or] slopes of the sides [AB, AC meeting at A] are immediately determined; this follows [since] the end-points of the two sides [AB, AC] come together [at A], i.e. the lines have to incline against one another.

[*Proposition 9 of Euclid 1*]

Proposition 9. *To bisect a rectilinear angle.*
(*Sketch of proof*:
let ? BAC be the angle to be bisected. Take D any point on AB different from A with AD less than both AB and AC. Cut off AE from AC with AE=AD (Proposition 3). Join DE and construct the equilateral triangle ?DEF on the opposite side of DE from A (Proposition 1). Then AF bisects ? BAC. For (Proposition 8) the triangles ?ADF, ?AEF are congruent since they have equal sides, so that ? DAF=? FAE.)



[*Hegel's comments on Proposition 9*]

[This is a] problem.[126] This proposition can also be expressed as the theorem[127]: *when two [distinct] isosceles triangles [gleichen Dreiecken] have a common base [Seite], then the angle [of any one of the triangles] opposite the common side is bisected [by the line joining the vertices of the triangles not on the common side]*. The solution to the problem [differs only from] Proposition 9 [in] the construction of the equilateral triangle.

[*An alternative proof by Hegel using Propositions 4, 5*][128]

On the face of it [in Proposition 9], only two sides [AD=AE, DF=EF] of each of [the] triangles [ADF, AEF] are constructed equal to the [corresponding] two sides of the other; the third [common side AF] is, because of that, already determined, and that is why it is not taken into account. But the angles ADE and AED [are] determined [as equal], when AD=AE [Proposition 5]; and DF=EF by construction [of the equilateral triangle DEF]. For from AD=AE, [we get by Proposition 5]

? ADE=? AED - and from DF=FE, [we get]

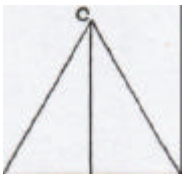
? FDE=? FED. [So by adding angles]

? FDA=? FEA.

[It follows from Proposition 4 that the triangles ?ADF, ?AEF are congruent, and so ? DAF=? FAE.]

[*Proposition 10 of Euclid 1*]

Proposition 10. *To bisect a finite straight line*.
(*Sketch of proof*:
let AB be the line to be bisected and ?ABC an equilateral triangle constructed on it (Proposition 1). Let ? ACB be bisected by CD where D lies on AB (Proposition 9). Then ?ACD=?BCD (two sides and the included angle (Proposition 4)) and AD=BD.)



[*Hegel's comments on Proposition 10*]

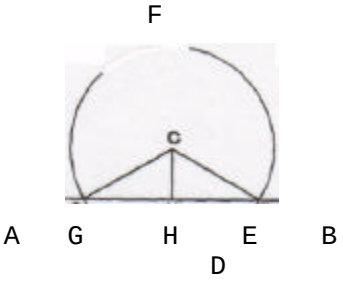
The proposition is completely the same [as Proposition 9].[129]

[*Propositions 11 and 12 of Euclid 1*]

Proposition 11. *To draw a straight line at right angles to a given straight line from a given point on it*.
(*Sketch of proof*:
let AB be the given line and C the given point on it. Take D to be any point on AC not equal to C. We can suppose that DC<CB. Then cut off CE equal to DC on the other side of C from D (Proposition 3) so that C is the mid-point of DE. Construct an equilateral triangle FDE (Proposition 1). Then ?FDC=?FEC (Proposition 8) so that ? FCD=? FCE. By definition of ``right angle''[130], FC is at right angles to AB.)



Proposition 12. *To a given infinite straight line, from a given point which is not on it, to draw a perpendicular straight line .*
(*Sketch of proof*:
let AB be the (potentially) infinite straight line and C the given point not on it. Let D be a point on the other side of AB from C and let the circle center C, radius CD, cut AB at G and E. Let H be the mid-point of GE (Proposition 10). Then ?CHG=?CHE (Proposition 8) and ? CHG and ? CHE are equal adjacent angles. So CH is perpendicular to AB.)

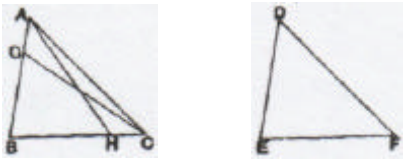


[*Hegel's comments on Propositions 11 and 12*]

[These are one] problem. As theorem[s], both belong together. In the isosceles triangle the line determined by the mid-point of the base line and the opposite point is perpendicular [to the base line]. The solution to the problem in both cases exists in this, that the infinite line AB gives the base of the isosceles triangle, so that in the first case [Proposition 11] the given [point] C is the same as the mid-point [of the base], [while] in the second case, equal lines are drawn to the point C located outside the base line.

[*At the end of the first part of* GS *(as it appears in Dok.), there is a discussion by Hegel that looks like a conceptual proof of some geometrical result. Closer inspection shows this to be a proof of Proposition 26 of Euclid 1, though no explicit mention of that proposition is made. This proposition is another congruence result for triangles, and it seemed appropriate to include Hegel's discussion of it at this place in the translation along with the other congruence results. As we saw above, Hegel suggests using Case 1 of Proposition 26 to prove Proposition 6.*]

Proposition 26. *If two triangles have the two angles equal to two angles respectively, and one side equal to one side, namely, either the side adjoining the equal angles, or that subtending one of the equal angles, they will also have the remaining sides equal to the remaining sides and the remaining angle to the remaining angle.*
(*Sketch of proof*:
Let ?ABC, ?DEF be the triangles.
There are two cases to be considered: For case 1, we are given BC=EF, ? ABC=? DEF and ? ACB=? DFE. For case 2, we are given AB=DE, ? ABC=? DEF and ? ACB=? DFE. In each case, we have to show that the triangles ?ABC, ?DEF are congruent.



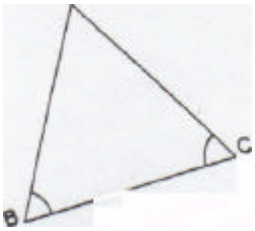
Case 1. Suppose that AB does not equal DE. We can suppose AB>DE. Let G be the point between B and A with BG=ED (Proposition 3). Then BG=ED, BC=EF, ? GBC=? DEF. So the triangles ?GBC, ?DEF are congruent (Proposition 4). But then ? EFD=? BCG<? BCA=? EFD and we have a contradiction. So AB=DE. Since also BC=EF, ? ABC=? DEF, the triangles ?ABC, ?DEF are congruent, again by Proposition 4.

Case 2. Suppose that BC does not equal EF. We can suppose that BC>EF, and construct H on the line BC so that EF=BH<BC. As in case 1, the triangles ?ABH, ?DEF are congruent, and so ? AHB=? DFE=? ACB. By Proposition 16 (which Hegel does not discuss in what survives of *GS*), ? DFE=? AHB=? ACB + ? HAC>? ACB which gives a contradiction. So BC=EF and the triangles ?ABC, ?DEF are congruent, again by Proposition 4.)

[*Hegel's comments on Proposition 26*[131] *follow. As we would expect, Hegel interprets the proposition in terms of the determination of the whole triangle by some of its parts.]*

[*Hegel gives a proof of Case 1 of Proposition 26: BC,* ? *B,* ? *C determine AB and AC. So ?ABC is determined (since all of its sides are, and Proposition 8 applies).*]

A



a) BC determines, posits [both] B and C, [and, since ? B, ? C are given], therefore also AB and AC, [and hence] the point [A], where AB and AC intersect, bound each other[; there is] only one [such point A], because straight lines which incline to each other intersect only in one point; this point is the limit of both lines, [and] so [there is] only one [such] limit, and [*oder*] its limitation that posits [completely each] existing line [AB, AC]; i.e. determines its length.

A



ß) B and C [and ? B, ? C] determine AB as well [since they determine BC and a) applies]. [Further] A and B as limits of the line AB are given, as points, where they have to be cut from BC and AC, through which AC and BC are limited; is it because of that that C also as [a] point is determined, [and why] a multiplicity of points C is not[132] possible when ? B and ? C and AB are determined?

B C D

[*Hegel follows his discussion of Case 1 by giving a proof, by contradiction, of Case 2: AB,* ? *B,* ? *C determine C uniquely. Again, it follows by Proposition 8 that ?ABC is completely determined.*]

Another length for [either of] the lines BC or AC would give a different point for C, and then [with] AB remaining the same, the angle ? C is changed, as AB is [only] the limiting line of BC and AC[;] [but it is on these lines, not on AB] that the determination of the magnitude of ? C depends; [so] ? C [is determined], yet it changes with each change of AC and BC; [this gives an] apagogic proof.[133]

[*Hegel's direct proof of Case 2: AB,* ? *B,* ? *C determine C uniquely.*]

More directly: [we require to show] that C is the unique point in which BC and AC intersect when [AB is fixed and] ? B and ? C remain unchanged; the [relative] direction[s] of AC and BC and AB are (*ist*) determined by the angles ? B and ? C, because these lines bound the angles (*als begrenzende die Winkel*); if then AB is assumed to have a fixed length, forming a triangle with BC and AC – [then] the end-points [A,B] of AB

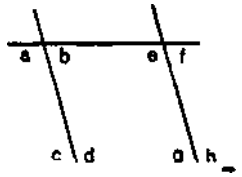
are given [(since AB is given;] so these points are also given for AC and BC[; the point of intersection C of AC, BC is also given] because straight lines intersect only in one point [and the directions of AC and BC relative to AB, and the points A, B, are determined].

THE PHILOSOPHY OF PARALLEL LINES

Lines which never meet when infinitely extended ([the] negative [definition]), or which everywhere are equally far apart ([the] positive [definition]), are called *parallel lines*. The distance measured [between two parallel lines] is [given by] a straight line; in order to measure the different distances [between the parallel lines], [one] must take a third [a *distance line*] and the equality or inequality between [the different choices of the third must] be examined ; here, [for parallel lines], [there] must be an equality in the *distance*, for otherwise, it would be completely undetermined which point on one line [to which] point on the other line are to be taken [in order to determine] the third [line] which measures the distance; [such] a determination is only brought about as follows [*hierin zu bringen*]: [either][134]

a) [the determination is] through [this], that always, two points are taken, one on each line, which lie at the shortest distance apart; or such lines, which give the shortest distance are [to be taken to be] the distance lines; the shortest are [the] perpendicular lines; so all perpendicular lines, which are drawn between parallel lines, are equal; or

b) that the angles, which [the] distance [lines] make with the parallel lines are equal, the a with the e, the b with the f – exactly that applies also to the perpendicular [case], and the universal, whose determination [as] measure in these distance lines is required, is this: *the angles, which the distance lines make with the parallel lines, are equal ; the b with the f [and] the a with the e*; either



a) these angles also [satisfy:] a [is] equal to b, [and] e [is] equal to f; [in that case], the [distance] lines are perpendicular [to the parallel lines], and [give] the shortest [distance]; [or]

b) [this is] not [the case]; [then] through the determination [of the angles] (equality in the general [case], the angles undetermined as far as their [exact] size), the size of the line becomes determined, because of the unity of the points of intersection, which [unity], and therefore the length of the line, is determined through the angle – also the distance lines in case b) are equal in concept with [that of] the parallel lines[135] – that, in addition, a also equals d and h, or that i and k are equal, is to be proved; (conversely, the lines as equal [in length] determine [the angles] so that the angles a and e are equal, for the size of the line determines the angle.[136])

[*Above, Hegel says that a=d is to be proved. Below he describes two proofs of this. We first give Proposition 29 of Euclid 1, part of which proves that a=d. The diagram in the proof is effectively included*

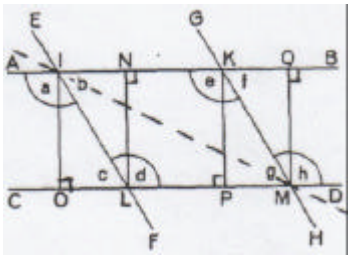
in Hegel's Fig. 1 below, with I in place of G and L in place of H . The proof uses Propositions 13 and 15 of Euclid 1 as well as Euclid's famous fifth postulate; none of these is discussed in what survives of GS. Proposition 13 says that given a straight line DBC and a point A not on the line then the sum of the angles ? ABC, ? ABD is equal to two right angles. Proposition 15 says that when two lines intersect, then opposite angles are equal. Postulate 5 says that if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles]

Proposition 29. *A straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the interior angles on the same side equal to two right angles (i.e. ? AGH=? GHD=? EGB and ? BGH+? GHD=two right angles)*.
(*Sketch of proof*:
suppose ? GHD<? AGH. Then
? GHD+? BGH <? AGH+? BGH=two right angles (Proposition 13). By Postulate 5, AB and CD will meet, contradicting the fact that AB and CD are parallel.
Similarly, we obtain a contradiction if we suppose that ? AGH<? GHD. So ? AGH=? GHD.
By Proposition 15, ? AGH=? EGB so that ? EGB=? GHD. Lastly ? BGH+? GHD=? BGH+? AGH=two right angles.)

[*Two proofs by Hegel of part of Proposition 29, viz. that a=d (which corresponds to ? AGH=? GHD above).*]

In order to determine the distance line, it is only necessary to posit a equal to e, and b equal to f; to this end, [it is sufficient] to prove that a=d determines both [equalities], or one of the determined part[s] (*Teil*) [determines] both opposite lines [IL, KM]; draw [Fig. 1] from I the perpendicular IO [to CD], [and] from L the perpendicular LN [to AB], so that

Fig. 1[137]



IL=IL; IO=LN [*since AB is parallel to CD*];
and ? IOL=? INL [*right angles*];
so ?ILN=?ILO[138]
and ? b=? c; (and ? OIL=? ILN, because [*also weil*] ? OIA=? NLM)[139] and because b and c are complementary angles for a and d, [we get]

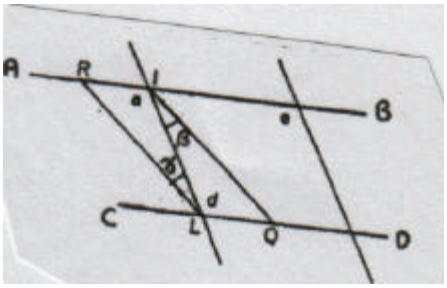


Fig. 2[140]

? a=? d. [*Q. E. D.*]

Or shorter [Fig. 2]: draw from I opposite e the line IQ and from L opposite a the line LR[141] so that ? ß=? ?; then IQ=RL; so ?IQL=?ILR [Proposition 4] and ? a=? d. [*Q. E. D.*]

Both proofs [are] the same, the first for b and c; the second for d and a; in the first, it is unimportant that LN and IO are perpendicular [to AB, CD respectively]; if not, ? ILN must be taken equal to ? LIO.

So because [Fig. 1] ?ILM=?IKM[142] (for ? e=? a=? d; IM=IM; MK=IL): [we have] IK=LM; and therefore (because ? a=? d=? EIK, and ? c=? b=? FLM) also EF and GH [are] parallel.

ON THE ALGEBRA OF GEOMETRICAL FIGURES

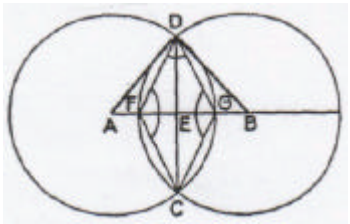
[*I do not know what Hegel's point was in these exercises of adding and subtracting lengths of lines. Explanations of the steps in the argument are given in italics. To clarify the explanations, I have also labelled the equations*.

The figure below used by Hegel is similar to one used by Apollonius to bisect a line (Hea, p.268), except that the radius of the circle there is AB. We first prove the following elementary proposition that is used in Hegel's calculations. I have not been able to find the proposition in Euclid's Elements. Part of its proof appears in Hegel's discussion of Fig. 4 below.]

Proposition. *If FB=AG [i.e. the circles have equal radii] then*
FE=EG (a) *AE=BE (b).*

(*Proof*. Since AD=DB, ?DAB is isosceles. So (Proposition 5) ? DAB=? DBA. Then ?DAG=?DFB (Proposition 4) so that DG=DF. Similarly, CF=CG. Next (Proposition 8) ?DFC=?DGC so that ? FDE=? GDE. Then ?DFE=?DGE (Proposition 4) so that FE=EG. Then AE=AG-EG=FB-FE=BE.)

Fig. 3[143]



If FB=AG, then FE=EG.

[*This is (a) of the above Proposition. The assumption that the radii FB=AG is made throughout*.]
[From Fig. 3,]
AG=AF+FE+EG
FB=FE+EG+GB

_____

[*Subtracting and cancelling the FE's and EG's gives AG-FB=AF-GB=0, so that*]
AF=GB. (c)[144]

[*Also, from (c),*
AG=AB-GB=AB-AF=FB. (d)
Next, from Fig. 3 and (d)]
AG[=AF+FE+EG=(AF+FE+EG)[145]-AG+FB]=(AF+FE+EG)-(AE+EG)+(FG+GB) (1)
[*Similarly*]
FB=(FE+EG+GB)-(EB+EF)+(FG+AF) (2)
[*So from (1) and (d)*]
AG-AF-FE[=EG-AG+FB]=EG (2)
[*Again using (d) and Fig. 3*]
AE+EG[=AG=FB]=BE+EF (4)
[*Trivially*]
BG+FE+EG=BG+EG+EF (5)
[*By (c)*]
FB-GB=FB-AF[146] (6)
[*From Fig. 3*]
AF+FG=AF+FE+EG (7)
GB+FG=FE+EG+GB (8)
[*From (7) and (8)*]
[0=] AF+FG-AF-FE-EG=GB+FG-FE-EG-GB (9)
[*Substituting FG=FE+EG in (9)*]
[0=]AF+FE+EG-AF-EF-EG=GB+GE+FE-GB-GE-FE
[*From (a) and Fig. 3*]
FG AE=AF+FG-EG
FG-EG=FG-EF EB=GB+FG-FE
AG=AF+FG
AE+EG[=AG]=AF+FE+EG (10)
[=BF *by (d)*]=BE+EF
[=EG+GB+EF *from Fig. 3*]
=AF+EG+FE (*by (c)*)
[*From Fig. 3,(a) and (b)*]
[0=]AE-AF-FE=BE-AF-EG
[*and using (c) and trivial cancellation*]
AF+FE-FE=BG+EG-EG

A___F___E__G___B [147]
A___F___E__G___
F___E__G___B

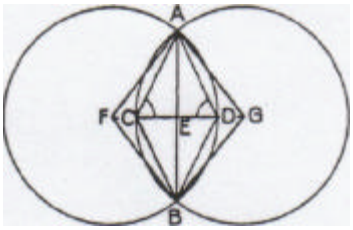
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In algebra, x is fixed; but its determination [exact value] is not posited[148]; in geometry, no x is given, everything is determined, and as such[,] posited; the becoming of its determination is not posited; its dependence is =x. In algebra[,] the conditions are given and the conditioned [is] not; in geometry, the conditioned is given, and the conditions as such are to be posited, to be shown [*zu setzen, zu zeigen*]; a geometrical problem makes determinate the concept of the concealed [*aufgegebenen*] [conditions].

[*The next figure (Fig. 4) that*
Hegel discusses is effectively

the same as Fig. 3 with the four radii AD, BD, AC, BC added, but the labelling of the points is changed. Justifications for the steps of Hegel's reasoning below are omitted – they are very similar to those used in the discussion of Fig. 3 above. The reader may find it helpful to change the labelling in Fig. 4 back to that of Fig. 3 for easier comparison.]

[Fig. 4]



[In Fig. 4] the points of intersection [*Grenzung[s]punkte*] A and B [of the circles] are given, [and] also C and D, so [there are given]: the line AB, the line CD [*jene der Linie AB; diese der Linie CD*], and each of the lines CE and ED; also [given are] the directions[149] [*die Richtung*] of the lines AB and CD, and also their intersection point E.

So in ?ACE and ?ADE
? EDA=? ECA
(AC=AD
AE=AE)
and ? CAE=? DAE
further ?AGB=?AFB
and when ?ACD is subtracted from [the equal] triangles ?FDA and ?GAC, we get
?DAG=?FCA
and ? FAC=? GAD
and ? CAE=? DAE
GC-GD-DE=CE [*obvious from the figure*]

FA=AG[150]
FD=GC
? AFD=? AGC[151]
[So] ?AFD=?AGC
and AD=AC
? FDA=? GCA

Notes

[1] *JL*, p.ix.
[2] At least one part of *GS* goes back to 1785 when Hegel was attending the Stuttgart Gymnasium. Indeed, in his *Tagebuch* for that year (*Dok.*, p.17) he says, concerning the equilateral triangle, ``*Die simpelste Figur ist hier das gleichseitige Dreieck*'' which is echoed in *GS*: ``*Das gleichseitige Dreieck ist die einfachste Figur*''.
[3] *Dok*. pp.288-300.
[4] *Dok.*, pp.470-473.
[5] ``*Die Euclid-Bearbeitung*'', 2 Auflage, Halle, 1781.
[6] *Dok.*, p.39; *TS*, p.55.
[7] *TS*, pp.50, 54.
[8] *Hea.*, p.107.
[9] *Dok.*, p.473.
[10] Ibid, p.xxxii.
[11] *Hea.*, p.29.
[12] Note the privative character of ape????: the infinite as ape???? is a potential, not an actual, infinite. The infinite in Euclidean geometry is entirely of this kind.
[13] *Metaphysics*, 986a22-26.
[14] *P*, p.120.
[15] *SL*, p.810.

[16] *Philebus*, 16d.
[17] *P,* p.72.
[18] *P,* p.11, pp.41-45.
[19] See, for example, p.80 of: T. Pinkard, *Hegel: A Biography*, Cambridge University Press, 2000, Cambridge.
[20] ``*Ästhetischer Platonismus bein Hölderlin und Hegel''*, by K. Dusing, pp.101-117 of: *Homburg vor der Höhe in der deu*tschen *Geistesgeschichte Studien zum Freundkreis um Hegel und Hölderlin*, ed. C. Jamme und O. Pöggeler, Klett-Cotta, Zweite Auflage, Stuttgart, 1987.
[21] p.40 in Rosenkrantz's *Life of Hegel*. (K. Rosenkranz, *Georg Wilhelm Friedrich Hegels Leben*, Wissenschaftliche Buchgesellschaft, Darmstadt, 1969.)
[22] There is no doubt, though, about the profound friendship of Hegel and Hölderlin in the Frankfurt period and of Hegel's knowledge of and interest in *Hyperion*. In a luncheon conversation in Berlin, 1830, Hegel talked ``with great spiritedness'' about his Frankfurt memories of Hölderlin and of Hyperion (pp.626-627, Pinkard, ibid.).
[23] But Pinkard (ibid.) says (p.136) that ``the author, however, is most likely Hölderlin''. The issues involved in the authorship of the *ältestes Systemprogramm* manuscript are discussed in: *Mythologie der Vernunft: Hegels ältestes Systemprogramm des deu*tschen *Idealismus*, edited by C. Jamme and H. Schneider, Suhrkamp Verlag, Frankfurt am Main, 1984. This book contains a discussion of the authorship of the manuscript by O. Pöggeler. Harris's discussion of the authorship is given in *TS*, pp.249f.. Harris also gives an English translation of the manuscript in *TS*, pp.510-512.
[24] Neo-Platonic interpretations of Hegel's thought have been subject to criticism. For example, Clark Butler (*Hegel's Logic*, Northwestern University Press, 1996) dismisses, I think unfairly, ``J. N. Findlay's Neoplatonic interpretation of the Hegelian system as the endless, asymptotic approach to the absolute as a transcendent one which can only be intuited'' (p.6, *ibid.*). There is a danger in oversimplifying Neo-Platonism so that it becomes the One processing into the world and reverting back to itself, giving the impression that that is all there really is to it. In fact classical Neo-Platonism is remarkably complex and detailed. One need only think of the intricate levels of reality expounded in Proclus's *Elements of Theology.* For our present purposes, the long, Neo-Platonic explanations of geometry, given by Proclus in his Commentary on the *Elements,* provide a remarkably coherent philosophy of geometry, and deserve the most serious consideration. This philosophy underpins the full scope of Greek geometry (and the *One* is not mentioned very much)!
[25] A helpful account of this is given by H. S. Harris in his introduction to the English translation of Hegel's Difference essay. See p. 57 of: G. W. F. Hegel, *The Difference between Fichte's and Schelling's System of Philosophy*, trans. H. S. Harris and W. Cerf, State University of New York, Albany, 1977. Harris also discusses how Hegel's understanding of the ``absolute'' in the Difference essay differed from Schelling's Spinozistic view of it.
[26] Translation by H. S. Harris from the introduction to the lectures of Schelling on the Philosophy of Art: p.54 of Harris and Cerf, ibid..
[27] *JL*, p.117-118.
[28] *TS*, p.81.
[29] The influence of Proclus on Kepler is discussed in detail on pp.167-171 of: J. V. Field, *Kepler's Geometrical Cosmology*, The University of Chicago Press, Chicago, 1988.
[30] Rosenkrantz (ibid, p.152) says: ``Von Kepler's Harmonia mundi war Hegel tief durchdrungen.'' This statement by Rosenkrantz is with reference to Hegel's dissertation on the planetary orbits, submitted in 1801 within a year of the completion of *GS*. As commented above, it seems very likely that Hegel's reading of Kepler goes back to his Tübingen days. The influence on Hegel of Kepler's treatment of the classical Greek geometers is exemplified in *SL*, p.303, where Hegel discusses Kepler's views on the cyclometry of Archimedes.
[31] For Hegel's correspondence with Creuzer, with reference to Proclus, see p.368f. (letter 400) and p.466 (letter 389) in: G. W. F. Hegel, *Hegel: The Letters*, trans. by C. Butler and C. Seiler, Indiana University Press, Bloomington, 1984. Another friend of Hegel's, Victor Cousin, also translated works of Proclus (C. Butler and C. Seiler, *ibid,* pp.635, 637).
[32] pp.434-435 in: G. W. F. Hegel, *The History of Philosophy*, trans. E. S. Haldane and F. H. Simson, Vol. 2, The Humanities Press Inc., New York, 1955.
[33] Plato, *The Republic*, 511. The form of the unhypothetical science in Hegelian mathematics is discussed in more detail in the paper: Alan L. T. Paterson, *Does Hegel have anything to say to modern mathematics*, Idealistic Studies 32:2, 2002, 143-158.
[34] ``One could also formulate the thought of a philosophical mathematics, which cognizes out of concepts the same as what the usual mathematical science derives from presupposed determinations according to the method of the Understanding.'' *NP*, § 259, Zusatz (my translation).

[35] Hegel in *WL* I, p.466, says that the axioms of geometry are logical propositions. (Logic, as *Wissenschaft*, is the exposition of the concept, and as absolute knowledge, is Hegel's developed form of the unhypothetical science.)
[36] Some of the details of the Proclian order are considered later in the Introduction, but for the convenience of the reader, we give here a brief summary. It follows Plato (*P*, p.84) by starting with the point, the ``likeness of the One'' . Next comes the line, whose two fundamental forms are the circle (corresponding to limit) and the straight line (corresponding to the unlimited). The circle is ``the first and simplest and most perfect of the figures'' (*P*, p.117). It is the primary figure with one limiting boundary. The primary figure with two boundaries is the semicircle, intermediate between the circle and the rectilinear figures (*P*, p.126). Then (*P*, p.128) ``the rectilinear figures come forth in orderly fashion according to the series of numbers from three to infinity'': triangle, quadrilateral, pentagon, etc.. Triangles and quadrilaterals are then classified according to equality and inequality with respect to sides and angles (*P*, pp. 130, 133). The primacy of the equilateral triangle among triangles is illustrated by the way that Euclid defines triangle (*Euclid 1*, Definition 20). Similiarly, Euclid's definition of quadrilateral figures begins with the square (*Euclid 1*, Definition 22) and its special place among the quadrilaterals is due to its exhibiting ``an ideal with respect both to its sides and to its angles'' (*P*, p.172).
[37] *P*, p.167.
[38] *P*, p.167. Under the influence of Neo-Platonism, the ``perfection'' of the equilateral triangle is expressed artistically in the pyramidal schemes of Renaissance Art, e.g. in the equilateral triangle shape of the Virgin's form in Raphael's *Madonna of the Meadows.*
[39] *P*, p.104.
[40] *P*, p.147f,; *WL* I, p.138. In the latter reference, Hegel's thought of the infinite as a circle ``ohne Anfangspunkt und Ende'' echoes the thought of Proclus, in which the``divine'' procession and reversion is presented as maintaining a circle ``without beginning and without end''. (Proposition 146 of: Proclus, *Elements* of Theology, trans. E. R. Dodds, (Clarendon Press, Oxford, 2000).)
[41] This is a famous, sarcastic comment on the identity philosophy made by Hegel in the preface to his *Phenomenology*.
[42] *P*, p.4.
[43] *P*, p.70.
[44] *P*, p.72.
[45] *P*, p.72.
[46] Hegel reformulates this view of the point in *JS*: the world process posits itself as ``something simple in its infinitely complex determinacy''. This simple as a ``bad infinity is posited immediately as absolute infinity, as one, as a point'' (*JL*, p.151). This ``bad infinity'' is the Hegelian version of the unlimited of Proclus which is ``secretly'' within the point, and becomes explicit through the generation of line, surface and solid.
[47] *P*, p.79-80. Walter Burkett (p.68f. in : *Lore and Science in Ancient Pythagoreanism*, trans. E. L. Minar, Jr., Harvard University Press, Cambridge, 1972) says that the derivation point-line-plane-solid through movement is probably due to the Pythagorean geometer, Archytas. Aristotle, however, attributes it to Platonism. (The doctrine may have entered Platonism through the influence of Archytas, since Plato and Archytas were friends (Plato's *Letters* 7, 9 and 12).)
[48] In his *Hegels Leben*, *ibid*, p.100, Rosenkrantz says that Hegel studied the writings of Sextus Empiricus. Sextus discusses the Pythagorean views on point, line, plane and solid in: *Adversus mathematicos*, 10.281f..
[49] *JL*, p.151.
[50] *WL* I, p.115: ``so the point is its own dialectic to become within the line, [and] the line is [its own] dialectic to become within the plane, [and] the plane its own dialectic to become within the totality of space'' (my translation). Hegel goes on, however, to say that the ``movement'' view of the generation of point, line, plane, solid in the Pythagorean position is contingent and imaginative. Rather it is the conceptual contradiction of a thing in its *limit* which impels the dialectic and gives point, line and plane no stability. So siding with Aristotle against Proclus, it follows for Hegel that ``there is no such thing as a point, line or plane'' (*SL*, p.129).
[51] Rosenkrantz (*ibid*, p.100) says that Hegel studied Schelling's writings during the Frankfurt period.
[52] ``Ut realem materiam intelligamus, spatii abstractae notioni contraria sive subjectivitatis forma addenda est, quam voce magis latin mentem, et se ad spatium referatur punctum appellemus.''
[53] *SL*, p.796-797.
[54] The relation between concept and extension is posited in the Fregean philosophy in the *extension* and *abstraction* axioms, by which two sets are the same if they have the same elements and a concept (property) determines a unique set. The contradictory character of the axiom of abstraction (as in Russell's paradox) led to the downfall of Frege's foundation for mathematics.

[55] *P*, p.122.
[56] *P*, p.122.
[57] *P*, p.167.
[58] *P*, p.122.
[59] Note here that, as boundary, the circle itself falls into the Pythagorean limit category, which as we saw earlier, is how Proclus sees the circle (in contrast to the straight line as the unlimited). As discussed below, the Hegelian transformation of this takes the circle as representing the genuine infinite and the straight line as the bad infinite.
[60] The diameter is generated as a line by the center. But it is still a limit of the circle even though it is dependent on the circle's primary limit, viz. the *center*. In this respect, the limiting role of the diameter is to be compared with that of the *angles* of a triangle. As we will see below, the angles of a triangle are limits of the triangle, but they also depend on the primary limits, viz. the sides of the triangle.
[61] *WL I*, p.138.
[62] *P*, p.128f..
[63] *P*, p.106-108.
[64] ``die dem rechten Winkel, dem sich selbst Gleichen enspricht,'', *WL* II, p.469.
[65] *JL*, p.121.
[66] *SL*, p. 810; *JL*, p.122.
[67] *P*, p.135.
[68] *WL II*, p.469.
[69] *P*, p.63f.; *Hea.*, p.124f..
[70] *P*, p.157.
[71] *P*, p.66.
[72] *P*, p.64.
[73] *SL*, p.796.
[74] But there are cases where the limitation at the conceptual level does not appear as a boundary at the level of sense perception. We saw this earlier in our discussion of the circle.
[75] *P*, p.76.
[76] p.141.
[77] It is also noteworthy that a major theme of *GS*, viz. Hegel's theory of limits, is also prominent in what survives of the first section of *JS.*
[78] *PN*, p.31.
[79] See, for example, p.311 of: B. Boss and D. D. Bleecker, ``*Topology and Analysis - The Atiyah-Singer Index Formula and Gauge-Theoretic Physics*'', trans. by D. D. Bleecker and A. Mader, Springer-Verlag, New York, 1985.
[80] It is sometimes argued that Hegel anticipated relativity. We need to be careful about such claims. What Hegel says (*PN*, p.34) is that space is the immediate existence of Quantity, whose negation ``falls apart into indifferent subsistence''. Time is the existence of the ``inner negation'' of space, the ``pushing apart'' of the spatial elements in their indifferent separation, the explicit bringing out of the truth of space underlying the superficial, passive apartness of space. As such, time is the ``unrest'' of space in which the latter is sublated, so that while the spatial point is an abstraction, the spatial point in time is *actual*, and as we have seen, generates the line through its motion.

So in the Hegelian position, time is implicit within space, is its dialectic, and is thus not to be thought of as a separate dimension, apart from and mathematically on a par with each of the three spatial dimensions, as it is thought of in the 4 dimensional space of *Special Relativity*. (The three dimensionality of space appears in *PN* before time is discussed.) In Relativity theory, space and time are, as I think Hegel would have said, *externally* combined, i.e. without their philosophical significance clarified. But the formulae of special relativity are *symmetrical* in the space and time variables, so that Hegel's logical differentiation between space and time – space first and then time as its inner negation – is not present in special relativity. However, Hegel's philosophical, *internal*, explanation of the relation between space and time can, I think, be modified to give a philosophical basis for their relation in relativity theory. One needs, I think, to start natural philosophy, not, as Hegel does, with nature immediately determined as indifferent, spatial side-by-sideness, but rather with nature as *becoming*, whose positing is a positing as negation (rather as *Schein* in Hegel's essence logic), with everyday space and time posited through each other *together*, on the same logical footing. Such an approach would also unify the point of space with the instant of time in a space-time point, and the instants of time would be just as much an indifferent side-by-sideness in time as the spatial points in space, and space as much the negation of time as time is of space. Such a start to the Philosophy of Nature is consistent, I think, with Hegel's comments on Nature in the Jena Logic, e.g. Nature as *the becoming itself out of its otherness* (*JL*, p.185). Such a symmetry between space and time is interestingly evoked by Hans Castorp in Thomas Mann's book: *The Magic Mountain (*trans. J. E. Woods, Vintage

International, New York, 1995), where (p.64) he says that measuring time with space is the same as measuring space with time.

[81] *Hea.*, pp.249-250.

[82] *SL*, p.810.

[83] *P*, p.98f..

[84] *SL*, p.810. This remark of Hegel is applied by him to Proposition 4, the ``superfluity'' being with reference to the other side BC and the other two angles ?B and ?C of the triangle. But the ``superfluity'' applies, I think, even more to the location of the triangle in space. This location of the triangle is *irrelevant* for the congruence theorems. The necessity of the concept only applies ``internally'' to the triangle as a whole of parts. For Hegel, Euclid's ``deficient grasp of difference'' is exactly the error of seeing difference in terms of spatial location rather than in terms of the *conceptual* difference of whole and parts.

[85] This duplication feature of Euclid's congruence theorems for triangles is repeated by Hegel on *JL*, p.21. See also *JL*, p.22 where Hegel says that ``reality is a doubling of unity'', and *SL*, p.810, where the congruence theorems are said to ``contain two parts'', the concept and the reality.

[86] Bernays (*On Platonism in Mathematics*, p.275 of: *Philosophy of Mathematics*, Editors P. Benacerraf and H. Putnam, Prentice-Hall, Inc., Englewood Cliffs, New Jersey) puts the difference between the geometries of Euclid and Hilbert well: *Euclid speaks of figures to be constructed, whereas, for Hilbert, system[s] of points, straight lines, and planes exist from the outset*.

[87] *SL*, pp.190ff..

[88] *Lor.*, p.8.

[89] *Hea.*, p.154.

[90] *P*, p.138; *Hea.*, p.190f..

[91] This second way is a variant of the *direction theory* approach to parallel lines that seems to have originated with Leibniz. A discussion of the difficulties with this approach is given in *Hea.*, p.194.

[92] *Hea.*, p.372.

[93] Hegel, it seems, began *JL* with *Simple Connection* (*Einfache Beziehung*). So in the logical development of geometry, it is reasonable for Hegel to begin with space (the simple, undifferentiated medium for geometrical determination) as *connection*. In this spatial context, we have preferred to translate *Beziehung* as *relation* rather than *connection*.

[94] The view expressed here is one that is discussed a number of times by Hegel (e.g. in *JL*, *WL*, *NP*). It is, as we described in the introduction, that the concept of a point dialectically refers to the *line* whose concept, in turn, involves as limits (the end-) points. The last assertion of a) is, I think, saying that the object (e.g. the point), positively defined as a limit of something else, can be regarded as something in its own right, and then it loses its qualitative, limiting character, and becomes purely quantitative. For a discussion of positive and negative in early Hegelian logic, see *JL*, p.133ff..

[95] This is nothing more than my speculation about what Hegel really means here. The German is: *(nicht insofern es)*. Hoffmeister changes *er* to *es* presumably because it refers to the neuter noun *Object*.

[96] I take this to means that when we look for the ground of a phenomenon, then the latter is treated as a ``limit'' and the ground is the explanation for it, involving a determination of one or more interrelated concepts. Conversely, the interrelated concepts find their reality in the phenomenon. A very simple example of this would be the sides of a triangle as limit and the triangle concept determined in the triangle. In the later logics of *JL, WL, ground* is essential in character, coming after the determinations of reflection. But in *GS* we have not reached that stage of development. I think that Hegel's view of *ground* is a development of the NeoPlatonic procession of objects from and their reversion back to the noetic (conceptal) world. In the more developed account of *JL*, p.140: *[the being of the determinacy] displays itself as that which it is, as a many; it returns into itself; and it is thus the unity of itself, and its opposed determinacy. This unity is its ground; the ground is the unity itself as self-equivalent determinacy.* Ground is (*JL,* p.141) *the reflection of cognition itself, as the self-enclosed simple.*

[97] An example of 1) would be the determination of a triangle through the magnitudes of its limits (its sides). The relation between these magnitudes is the principle of the triangular figure. In 2), I think that Hegel has in mind a V-shaped figure with its angle determined by the two bounding legs of the V. Why does Hegel talk here about the principle of the shape as the ``self-relation'' of the limits? I think that he means that the primary limit of the V-shape is the line, and the line relates itself to itself through its two realizations as the bounding legs, and that this relation is the angle. So the enclosed angle is the determining principle of the shape.

[98] This refers, of course, to the sphere as a *surface* - not a solid ball.

[99] A great circle is the circle in which a plane through the center cuts the sphere; in the case of the surface of the earth (regarded as a sphere) the equator would be an example of a great circle.
[100] This word is illegible in the manuscript and the translation given (``point'') is a guess. It does fit in naturally with the following mention of line and plane: the theme of point, line, plane and solid is, as we have seen, fundamental in Hegelian philosophy. (The fourth of these, the solid, does not apply here since the sphere is a surface.)
[101] This is very close to the NeoPlatonic view of the circle, especially that of Proclus, discussed in the introduction. Hegel takes the metaphysics of the circle much further in his discussion of cognition and the soul in *JS*. He describes cognition in *JS* in terms of the geometry of the circle. Cognition is (*JL*, p.141) ``the circular movement of the return into itself'' and comes ``to a point as reflection while it is circumference as movement''. Also (JS, p.140; JL, p.147) the soul is thought of as the ``whole'' circle and its ``peripheral'' movement. Also, *die sich auf sie zugleich also Mittelpunkt bezieht, und zugleich als gerade Linie ins unbestimmte ausdehnt* : ``it [the soul] at the same time relates itself to itself as middle point [the center], and at the same time extends as [a] straight line [the radius] indefinitely'' (my translation).
[102] In a triangle, one angle is less than another if and only if the side opposite the first is smaller in length than that opposite the second (Proposition 18, *Euclid 1*).
[103] The problem of the nature of an angle is discussed in the Introduction.
[104] Dok., p.470; TS, pp. 523-524.
[105] T. Pinkard, *ibid*, p.746.
[106] Hegel follows the distinction between *problem* and *proposition* (theorem) made in Greek geometry. This is discussed in the introduction.
[107] lit. *distortion*. *GS* starts, as does *WL*, with what is most simple and immediate for the matter under investigation. Since *Euclid 1* is concerned primarily with triangles, and the equilateral triangle is, as we saw, the triangle in its simplest form, we have, for Hegel, an explanation of why that triangle is constructed in the first proposition of the *Elements*. The equality of its sides is the source of the equality of line lengths in the next two propositions. One can even regard *Euclid 1* as the mediation of the equilateral triangle. (However, as Proclus noted, the proof of Proposition 1 also uses the *circle* which is for geometry of even more importance than the triangle. See the Introduction.)
[108] ``Warum soll es bei Figuren statthaben, da es bei Linien 2. und 3. Satz nicht gebraucht wird?'' The point that Hegel is making is the same as that made by Peletier (referred to in the Introduction) that if the method of superposition is legitimate in geometry, then the subject would be full of such proofs.
[109] What Hegel is saying is that in a triangle ?ABC, given the point A fixed and the lengths of the lines AB and AC, the triangle is not determined. For that, we also need to determine ?A.
[110] I do not know what Hegel means by this. Euclid derives the equality of the corresponding angles in the two triangles from the fact that the two triangles, when superimposed, coincide.
[111] I think that what Hegel means is this. The problem with the proof of Proposition 4 is that we have to lay one triangle on top of another triangle exterior to it, but one cannot do this in principal since triangles are not the kinds of things that one *can* lift and lay. The problem with superposition is, however, even more serious for the proof of Proposition 5 which uses Proposition 4 to show that ?AFC=?AGB and ?BFC=?CGB. Superposing as in Proposition 4 would require ?AFC placed on ?AGB and ?BFC placed on ?CGB. But ?AFC cannot be placed on ?AGB without having to double parts (since, for example, the side AC in ?AFC is part of the side AG of ?AGB). When we ``lift'' ?AFC to place it on ?AGB, we remove parts of ?AGB so that that triangle is dismembered and no longer ``there'' to have something laid on it. Of course, this is not a problem for Hegel with his view of the triangle as determined by a concept. Proposition 4 is used in the proof of Proposition 5, but there, the difficulty is not so obvious, since in that proof, we do not explicitly place one triangle on top of another. Rather, the triangles being compared just have certain parts in common, and the corresponding other parts of the triangles are not brought together.
[112] What is puzzling in this argument is that Hegel does not say where the isosceles hypothesis AB=AC is used. The best that I can come up with is this. Since AB=AC, the first two of the three equalities can be interchanged without changing their content: the AB that we put in place of AC has exactly the same length as AC. The *same* triangle ?ABC is then determined by two sets of conditions in which the roles of B and C are interchanged, and it follows by symmetry that ?B=?C. Note that we are dealing with only *one* triangle here. The proof for ?ABC in its ``difference'' (as ?ACB) follows.
[113] As described in the Introduction, this ``shorter'' proof is classical, being due to Pappus. Echoing the Hegelian theme of identity and difference for the triangle, Proclus says of the Pappus proof that ``it is possible for us, by assumption, to see two triangles in this single one and so prove the equality of the angles at the base''.

[114] This is the first apagogic proof in *Euclid 1*. See the Introduction for a discussion of apagogic proofs (proofs from contradiction).
[115] Hegel has: ``Wäre AC, AB ungleich'', but in the preceding proof to which Hegel is referring, AC is less than AB (rather than just *unequal* to AB), and this makes sense of the rest of the sentence.
[116] Hoffmeister changes the BD to BC, but the BD that Hegel has in the manuscript is correct.
[117] In mathematics construction and proof are distinguished, but, for Hegel (*JL*, p.181) ``in absolute spirit construction and proof are absolutely one''. We can see, then, why, for Hegel, a proof that uses an ``impossible'' construction is a problem for philosophy (absolute truth): not only are the construction and proof different (as they are in all mathematical proofs) but worse, the construction is *impossible* and so the very *opposite* of truth. As discussed in the Introduction, the problem with proofs by contradiction is a serious one in modern mathematical philosophy.
[118] Hegel has ``um durch einen Zusatz'': the ``Zusatz'' (addition) that he is referring to is the Pappus proof that was given in his discussion of Proposition 5.
[119] I have been unable to prove this without proving first that AB=AC.
[120] Hegel has ?AED. (He corrects it to ?ABD below.)
[121] Proposition 26 of *Euclid 1*, and Hegel's discussion of it, are given below. Proposition 6 is not used by Euclid till Book 2 of the *Elements*, so that it is legitimate to prove it using Proposition 26. Elegant proofs of Proposition 6 using Proposition 26 were given by the Arabian geometer an Nairizi (who died c. 922) and independently later by Todhunter in his 1862 edition of Euclid (*Hea.*, p.258). Hegel may well have worked out for himself that Proposition 26 can be used to prove Proposition 6. At any rate, it illustrates how knowledgable he was about geometry.
[122] *GS* has ?ACD.
[123] Strictly Proposition 6 is the converse to the first part of Proposition 5 as was observed by Proclus.
[124] This argument that Hegel gives to prove Proposition 7 is hard to clarify. What has to be shown is that, with AB given, and the distances AC, BC fixed (and large enough and lying on one side of AB) then the point C is uniquely determined. There are subtleties with Proposition 7. Indeed, as Heath points out (*Hea.,* p.258), Euclid's argument needs a postulate to ensure that ?BDC>?ADC. Yet Hegel claims to explain Propostion 7 in one sentence! Hegel does not even refer to the fact that AC, BC lie on one side of the base line. And just why the point C is determined by the distances AC, BC is not explained. Then he claims, without justification, that two non-parallel lines intersect in exactly one point. (However, he does return to this after Proposition 26.) Perhaps his argument for Proposition 7 is ultimately based on his philosophical understanding of the *straight line* as discussed in *WL I*, p.204. There, Hegel dismisses Kant's claim that the proposition: ``the shortest distance between two points is given by the straight line'' is synthetic *a priori*. Hegel claims, on the contrary, that it is ``wholly analytic'' (*SL*, p. 209). He justifies this on the grounds that the simplest line is the straight line, and simplicity in the world of Quantum, as applied to the line, gives the shortest distance property. (In modern mathematics, this property of the straight line assumes the much more general form that the geodesics in a Riemannian manifold give the shortest distance between points – this however, is a highly non-trivial theorem.) Hegel's ``wholly analytic'' comment above echoes the *GS* comment by Hegel that Proposition 7 is ``completely analytic''.
[125] This is Euclid's Common Notion 4 (*Hea.,* p.155).
[126] A problem can however be made into a theorem by including in the latter's statement the construction required for the solution of the problem. The theorem then is the statement that the construction solves the problem. Hegel does this with Euclid's Proposition 9 below.
[127] The German reads: *Wenn von gleichen Dreiecken eine gleiche Seite auf die gleiche Seite fällt, so ist der Winkel, [der] jetzt entsteht, das Doppelte der noch getrennten* . In trying to make sense of this, I have given a very free translation. Hegel's expression *gleichen Dreiecken* seems most naturally to be understood as *congruent triangles*, since Euclid uses ``equal triangles'' to mean ``congruent triangles'' (e.g. in the statement of Proposition 4), and Hegel uses *Gleichheit* for *congruence* in his comments on Proposition 5. But then Hegel's version of Euclid's Proposition 9 is false. Further, in Euclid's construction of Proposition 9, one of the triangles is isosceles and the other is equilateral, and the two triangles are in general *not* equal. I think that the reason for this puzzle lies in the fact that Hegel used Lorenz's translation of the *Elements* in his study of *Euclid 1* and that Lorenz (*Lor.*, p.8; *Dok.,* p.472) has a poor figure for Proposition 9. In that figure, the isosceles triangle on top looks as if it is equilateral, congruent to the (genuinely) equilateral triangle underneath and with which it shares a common base. This suggests that one could interpret *gleichen Dreiecken* as ``equilateral triangles''. I think that Hegel did mean that, having been misled by the figure. Confirmation of this is given by his discussion of Proposition 10, where Hegel says that its construction is ``completely the same'' as that of Proposition 9. Yet the

proof of Proposition 10 uses two *equilateral* triangles. Taking *gleichen Dreiecken* (as I have done) in Hegel's theorem (rather than problem) version of Proposition 9 to mean ``isosceles triangles'' gives a correct result. The same proof works when both triangles are isosceles, and the result is more general.

[128] In *GS*, the following paragraph comes after Proposition 10. But it gives an alternative proof of Proposition 9 (not using Proposition 8), and so I have included it in Hegel's comments on Proposition 9.

[129] This is not correct. (See the note above.)

[130] Definition 10 of *Euclid 1* (*Hea.*, p.153) defines ``right angle'' as follows: *When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, and the straight line standing on the other is called a perpendicular to that on which it stands*.

[131] The diagram opposite a) is presumably for case 1 of Hegel's version of Proposition 26, where the indicated angles and the side joining the vertices of these angles are given determined. The diagram opposite ß) is a triangle ABD with C on BD and the line AC drawn. However, there is no mention of D in ß) and the diagram seems irrelevant to Proposition 26.

[132] The word ``not'' is missing from Hegel's manuscript. But it is required since ?B and ?C and AB do in fact determine the triangle ABC (and hence also C).

[133] This is a proof by contradiction: the assumption that C is not uniquely determined leads to ?C not being fixed; but the fixity of ?C was a hypothesis.

[134] The simplest explanation of what Hegel is saying was discussed in the Introduction, **B**. An alternative explanation is as follows. For two given lines to be parallel, we need to specify the existence of a certain family of distance lines (transversals) cutting the two given two lines, a distance line passing through each point of these parallel lines. Now given any point A on one of the parallel lines, to determine the distance line through A, we first have to specify a *distance number d*; this number determines (in general) two points B, C on the other parallel line for which AB=AC=d (B, C are the points at which the other parallel line and the circle center A and radius d intersect); there are then two distance lines AB, AC and they make complementary angles with the parallel lines. We need an extra condition to determine completely the family of distance lines. In a), the extra condition takes the form of the existence of a *shortest* distance, given by the perpendiculars between the lines. (Then B=C and no choice is necessary.) In b), we have d> the shortest distance, and the extra condition that distinguishes between AB, AC is that the distance lines of the family are those that make *the same fixed angle a* with the parallel lines. (This eliminates one of AB, AC for each A, and the family of distance lines is determined.) This extra angle condition on its own is actually all that is needed for parallelism.

[135] I think that this refers to the fact that EF, GH, like the original pair of lines, are *parallel,* and so the same concept applies to both pairs.

[136] ``denn die Größe der Linie ... durch die Winkel bestimmt''. Some words are missing here, so that the translation that I have given is a guess. However, I think what Hegel is saying here is clear. Earlier in the paragraph, Hegel said that the *length of the line is determined through the angle*. Here he gives the converse: *equal line lengths determine equal angles.* However, this requires qualification since, as we saw in Endnote 134, there are in general two families of distance lines for a given distance d, and the angles that the families make with each parallel line are not the same (though they are complementary).

[137] In Fig. 1, I have added perpendicular signs at O, N, P and Q. I have also added the line IM which is required in the last paragraph of *GS*.

[138] Perhaps, Hegel is appealing here to a congruence theorem which is (inexplicably) not in the *Elements*. This is the so-called *ambiguous case* (*Hea.,* pp. 306-307). This case says that *if two triangles have two sides equal to two sides respectively, and if the angles opposite to one pair of equal sides are also equal, then the angles opposite the other pair of equal sides are either equal or complementary, and in the former case, the triangles are congruent*. In Hegel's application, the triangles are ILN, ILO, and IL=IL, IO=LN and the angles ?IOL, ?INL opposite IL, IL are right angles and so equal. Hegel wants to conclude that the triangles ILN, ILO are congruent. Unfortunately, however, he can only conclude from the ambiguous case either that b=c or that they are complementary. (One could overcome this difficulty by saying that b and c cannot be complementary, since then one of the triangles ILN, ILO would have two angles each greater than or equal to a right angle and this contradicts the fact (Proposition 17) that the sum of any two angles of a triangle are less than two right angles.) I think it likely that Hegel misunderstood the ambiguous case – this point is discussed further in Endnote 142 below. Hegel, perhaps, shows awareness of the ambiguous case by his comment in the margin: ``*(a theorem which Euclid does not have – because IL determines the point L for OL, and the point I for IN).*'' On the other hand, perhaps Hegel is not appealing to the ambiguous case but rather using a conceptual argument of his own, missing from *GS*, to prove congruence. If so, I cannot construct the argument.

[139] That ?OIL=?ILN follows from the congruence of the triangles ?ILO, ?ILN. The angles ?OIA, ?NLM are equal since they are right angles. (Does this need proof? – it is not clear to me exactly what Hegel's assumptions are.) Lastly since b=c, their complementary angles a, d are equal (both a+b, d+c equal two right angles (Proposition 13)).

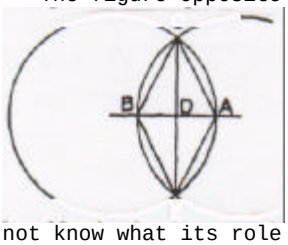
[140] In Fig. 2 in *Dok.*, ?ß is ?BIL and ?? is ?CLI. However the proof requires ?ß to be ?LIQ and ?? to be ?ILR, and I have changed Fig. 2 accordingly. I have also added the points A, B, C, D as in Fig. 1. Hegel is using the congruence theorem Proposition 4 to prove that the triangles IQL, ILR are congruent. I am not sure what Hegel's assumptions are here, and exactly why IQ=RL.

[141] Hegel has RL. I have omitted a semicolon that Hegel has following IQ. (For Hegel's argument, LR and OQ *have* to be drawn so that ?ß=??.)

[142] Hegel seems to be trying to apply to the triangles ?ILM, ?IKM the ambiguous congruence theorem discussed in Endnote 138 above. It seems likely that Hegel misunderstood this case. Not only does he seem unaware of the ambiguity in the two applications of this case in ***GS,*** but he says (incorrectly) in *JL*, p.21 that ``in a triangle the rest is fully determined by three elements (if there is a line among them ....''.) So Hegel's proof of the congruence of ?ILM, ?IKM is not valid as it stands.

[143] The lines AD, DB are not in Hegel's figure – it was convenient to include them for the purposes of the proof of the Proposition. Further, these lines also appear in the next figure. I have labelled the figures of this section Fig. 3 and Fig. 4. (Hegel does not label them.)

[144] The figure opposite appears in Hegel's manuscript. I do



not know what its role is in ***GS*** since it is not used in the rest of the surviving discussion. It is the same as Fig. 3 but with different letters for the points.

[145] Here and elsewhere in this algebraic section, I have used parentheses, not in Hegel's text, for clarity of reading.

[146] Hegel has ``(AF-)''. I take it that this means that subtracting GB is, by (c), the same as subtracting AF.

[147] Lorenz draws lines in this way throughout his *Elemente*, e.g. *Lor.*, p.221. The line drawn here is the one joining the centers of the two circles in Fig. 3, and the order of points from left to right on that line is given. This makes equalities such as (10) very easy to see.

[148] I have been unable to find any discussions by Hegel of the relation of algebra to geometry in his mature works. However the indeterminacy of *x* in algebra is reiterated in *SL*, p.246 where Hegel says: *The letters with which general arithmetic [i.e. algebra] operates, the next universality into which numbers are raised, do not possess the property of having a specific numerical value; they are only general symbols and indeterminate possibilities of any specific value.*

[149] Perhaps Hegel just means that AB and CD are perpendicular.

[150] This part of Hegel's argument is the same as part of the proof of the Proposition following Fig. 3.

[151] [In the margin]: ``for the point A cuts the same arc lengths AD and AC and the same for the straight lines [AD and AC] and the radii FA=AG.'' For Hegel, this marginal comment justifies why FA=AG in the proof of the congruence of triangles AFD, AGC, and why AD=AC in the argument immediately following.

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