Does Hegel have anything to say to modern mathematical philosophy?

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Abstract

This paper argues that Hegel has much to say to modern mathematical philosophy, although the Hegel-ian perspective needs to be substantially developed to incorporate within it the extensive advances in post-Hegelian mathematics and its logic. Key to that perspective is the *self-referential* character of the fundamental concepts of philosophy. The Hegelian approach provides a framework for answering the philosophical problems, discussed by Kurt Gödel in his paper on Bertrand Russell, which arise out of the existence in mathematics of self-referential, non-constructive concepts (such as *class*).

1. *Introduction*

Hegel's great work on Logic, the *Wissenschaft der Logik*[1], which we will abbreviate to *WL,* appeared in three volumes in 1812, 1813 and 1816. The work contains long discussions of mathematics. In particular, it develops philosophies of the *natural numbers*, the *negative numbers* and also of the *rational numbers*. There is an extensive and erudite account (with the real numbers presupposed for the sake of the argument) of how this philosophical understanding of mathematics ``makes sense'' of the (informal) infinitesmimal manipulations of the differential and integral calculus. (Hegel, of course, lived before the time of Weierstrass (late 19th century) when the modern, rigorous treatment of mathematical limits and the calculus was developed.)

The importance of mathematics in Hegelian logic is further emphasized by the fact that near the very end of *WL*, Hegel's discussion of cognition is primarily concerned with mathematics. In *analytic cognition*, he returns to number and the problems of formalization, while in *synthetic cognition*, he (like Kant) is considers Euclidean geometry.

Despite Hegel's lengthy and detailed treatment of mathematics, his influence on modern mathematical philosophy is almost negligible (implicit at best)[2]. Indeed, the analytic philosophy of the last century, which pretty much took mathematical philosophy as its source of inspiration, was very hostile to Hegel. Much of this can be attributed to the discomfort over Hegelian argument which does not seem formalizable in principle. Indeed, Hegel dismisses formalization as

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[1]

Notes

The edition of the *Wissenschaft* (*WL*) used in the paper is that of G. Lasson: G. W. Hegel, *Wissenschaft der Logik*, I, II, Vol. 56, 57, Der Philosophischen Bibliothek Band, Verlag von Felix Meiner, Hamburg, 1967. The English translation, *SL*, of *WL* is: *Hegel's Science of Logic*, translated by A. V. Miller, Humanities Press International, Inc., Atlantic Highlands, N.J., 1989.

[2] The relevance of Hegelian logic to themes in mathematical logic has, however, often been overlooked. I would argue, for example, that Russell's *no class theory* is anticipated and superseded in Hegel's treatment of reciprocity where *thinghood* passes over into *property* (*SL*, p.491).

``mechanical'' and emphasizes that philosophy is a matter of thinking and concepts, not a methodology whose objective is to reduce such thinking to algor-ithmic procedures. (An example of such an algorithmic procedure[3] in modern logic is the truth table method for determining which well-formed formulas in the *propositional calculus* are theorems - these are the *tautologies*.)

Of course to say this is not to say that analytical and algorithmic procedures are unnecessary - rather they are indispensable for a proper understanding of mathematical concepts (and Hegel is wrong to belittle them). His point is the Fregean one that formalism in mathematics is subordinate to concepts, and serves merely as a vehicle for the articulation of these concepts. (There is always a *point* to formalization, and this point is not given in the formalism but rather belongs to its conceptual basis.)

The person who finds much of value in the thought of both Hegelian and mathematical logics is faced with the daunting task of trying to show how the two logics relate to one another. The present paper attempts to show how this can be done. We first indicate the main thesis of the paper.

Mathematics as a subject has to be distinguished from its philosophy. Mathematical thought proceeds intuitively on the basis of certain assumptions. (The assumptions need not all be explicitly given as axioms - they can operate implicitly through intuitive insight.) Philosophy comes in when we question this procedure and try to justify the assumptions. However, philosophy and mathematics cannot be completely separated, since foundational (and therefore philo-sophical) difficulties arise *out of mathematics itself*. Examples of these difficulties in the history of mathematics are the existence of irrational numbers (Greek mathematics), infinitesimals and limits (Calculus) and the set theoretic paradoxes (late 19th century). Mathematics then cannot avoid philosophical problems, although mathematical creative thought, like scientific creative thought, continues unabated whatever these problems are.

Motivated by the work of Gödel, we will argue in this paper that the fundamental philosophical problems of mathematics all arise from the self-referential character of its concepts. Here, a concept A can be defined to be *self-refer-ential* if any conceptual analysis of it uses the original, unanalysed concept A itself. Equivalently, A is self-referential if it cannot be constructed out of simpler concepts. *Class* is an example of a mathematical self-referential concept. The logic needed to understand and work with self-referential concepts is not part of mathematics or of modern mathematical logic. In particular, the foundational systems of modern mathematical logic do not involve the self-referential, being constructive in character. Hence the only way that such systems have been able to treat the philosophical problems is in an *ad hoc* way so that they do not interfere with mathematical progress. Zermelo-Fraenkel set theory (**ZF**), for example, organizes and presents, precisely and brilliantly, what mathematicians feel intuitively that they should be able to do when working with sets. It gives, as it were, a procedural hand-book for such work. But the actual concept of *set*, which is non-constructive and self-referential, is left by **ZF** untreated.

The philosophical problems of mathematics are then to be understood as outward expressions, occurring historically (and unpredictably) in one form or another, of the conflict between the reliance of mathematics on self-referential concepts and yet the absence in mathematics of a logic of such concepts.

[3] See, for example, Chapter 1 of: W. S. Hatcher, *Foundations of Mathematics,* W. B. Saunders Company, Philadelphia, 1968.

These problems can, then, only be dealt with by developing a logic of self-referential concepts. Next we claim that the logic of Hegel is a logic of self-referential concepts, in which concepts such as *being*, when logically developed, refer *back to themselves.* So the simple answer to the question of how Hegelian logic relates to modern mathematical logic is that *the former presents (at least in principle) the missing logic of the self-referential concepts needed in the foundations of mathematics.* This Hegelian self-referential logic then completes the *non*-self-referential logic of present day mathematical philosophy. That, I believe, is what Hegel has to say to modern mathematical philosophy.

We now briefly indicate the argument of the paper. We begin (in § 2) by discussing Gödel's well-known paper on the mathematical logic of Bertrand Russell, in which he argues for the validity of self-referential concepts. In this paper, Gödel points out the inadequacy of the *vicious circle principle* to deal with fundamental concepts of mathematics, such as *natural number, class* and *concept*. These are self-referential in the sense defined above. They are therefore not *constructible.* Their validity (for Gödel) depends on their objectivity (Gödel's realism). The foundational problems of set theory - in particular, the question of what the correct axioms for set theory should be - are then (for Gödel) due to our inadequate insight into the nature of these ``objects''.

Absent from Gödel's account of self-referential concepts is any investigation into their *logic*. Perhaps Gödel felt that this missing logic is also to be determined by the supposed ``objects'' to which these concepts apply, similar to the way in which our understanding of a concept (such as *force*) and the logic that governs its use in science is determined by objective experience and scientific experiment. The problem with such an answer is that the ``objects'' to which self-referential concepts refer are mysterious at best. They are not given through sense perception. Further, the concepts of the objects of science and daily life (e.g. tables, planets, electrons), unlike the concept of *set* in mathematics, are not conceived of as self-referential in character.

The need for an investigation into the logic of self-referential concepts is therefore inescapable, and as mentioned above, we will argue that Hegelian logic provides (in principle) this self-referential logic which is missing in Gödel's account.

Before discussing Hegelian logic, however, we examine first (in § 3) the attempt by Plato in the *Republic* to solve, within the context of Greek mathematics, Gödel's problem of what the ``correct'' axioms for mathematics should be. Plato's account is an early version of the self-referential logic which we are seeking, and is discussed in the paper in order to provide helpful background and motivation for Hegel's much more sophisticated version.

For Plato, in order to obtain canonical axioms for mathematics, one needs to specify first a *starting point* requiring no assumptions. Once this starting point has been obtained, then on that basis, one works backwards to mathematics through a sequence of (what Plato calls) *ideas*. This sequence provides a grounding for mathematics, and in particular, will produce *canonical* axioms for the subject. The problem of the ``correct'' axioms for mathematics will then have been solved. The conceptual process which gives the starting point and the working back to mathematics is called by Plato *dialectic.* (Platonic dialectic, however, is not restricted to mathematics.) We argue that Platonic dialectic gives effectively a logic of self-referential concepts.

However, nowhere in the Platonic corpus do we see Plato working out in detail his program for the grounding of mathematics. Indeed, the program can only be judged a failure. The self-referential character, that is essential (as we saw above) for the philosophy of mathematics, is only implicit in Plato's work. Further, the detailed arguments of Platonic dialectic, presented, for example, in *Parmenides II*, come across as inconclusive. Hegel (I think justifiably) criticized them for being *negative* in character.

So we move on, in §4 of the paper, to the more satisfactory form of self-refer-ential logic in the work of Hegel. Here, the sequence of ideas of Platonic dialectic is replaced by that of a *conceptual* development. Further, in the mathematical context, Hegelian logic no longer aims to produce axioms for mathematics but instead gives a philosophical basis for the concepts operating already in existing mathematics. Hegelian dialectic, unlike Platonic dialectic, is pos-itive as well as negative, and is explicitly self-referential, the development being one through which what is initially given in immediate form is mediated, the concepts involved referring back to themselves.

To illustrate how Hegelian logic can provide a philosophical basis for key concepts in mathematics, we sketch in the final section of the paper a modern Heg-elian validation of the fundamental mathematical notions of *infinite set* and *natural number*. *WL* starts with the concept of *being* (*Sein)* in its complete simplicity and abstraction. This is reasonable since any thinking at all is about something or other, and so presupposes *being.* (The concept of *being* plays in Hegelian logic the role of Plato's *starting point* (above).) This initial concept of being in *WL* is developed conceptually through the famous triple of *being-nothing-becoming*, which results in the concept of qualitatively determined being (*Dasein)*. The logic of *Dasein* in turn gives the concepts of the *finite* and the *infinite*. The relation between the finite and the infinite in this development yields, we will argue, primitive forms of the concepts of *infinite set* and *natural number*. In fact, Hegel in *WL* gives a much more complete development of the concept *natural number* in the post-*Dasein* logic of *Fürsich-sein* and the quantitative, but this is beyond the scope of the paper[4].

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2. Gödel and self-referential concepts

In his well-known paper[5] ``Russell's Mathematical Logic'', Gödel discusses constructivism and self-referential concepts in mathematical logic. (Numbers in parentheses in this section refer to page numbers of that paper.)

Gödel conceived of classes and concepts as ``real objects'', ``existing independently of our definitions and constructions'' (220). One reason, he points out, why we need the concepts of mathematics to be objective is that if we take the contrary view - so that concepts are built up (constructed) by us out of simpler notions, then different constructions of what is intuitively the *same* concept would have to be regarded as *different*, since the content of each construction determines the constructed concept. This is not acceptable since the construction has to make sense of the intuitive concept, the conforming or otherwise of the construction to the primitive concept being the criterion of

[4] This is discussed in the author's paper: *The Successor Function and Induction Principle in a Hegelian Philosophy of Mathematics*, Idealistic Studies 30(2000), 25-60.
[5] pp. 211-232 of: *Philosophy of Mathematics, Selected Readings*, edited by P. Benacerraf and H. Putnam, Prentice-Hall, Inc. Englewood Cliffs, New Jersey, 1964.

success or failure of the enterprise. A construction cannot replace the original intuitive concept.

A good illustration of such a construction and its failure to account for the intuitive concept is that of *number* in Russell's ramified type theory[6]. Each type has its own definition of the natural numbers and a number such as 2 in one type is different (as a class) from 2 in another. So the definitions of 2 in the types are different, and none of them is the genuine 2 of intuition! Ironically, numbers (in the intuitive sense) are used to label the types in sequence, and so the constructed 2's depend implicitly for their meaningfulness on the use of the intuitive 2 *anyway*.

Gödel points out (220) that the *vicious circle principle* (in its primary form), to whose violation Russell, following Poincaré, attributed the set-theoretic paradoxes, applies *only to constructed entities*. Any translation of a sentence containing an impredicatively defined (and therefore non-constructed) concept (such as *class*) will itself contain the symbol for that impredicative concept, so that nothing has been achieved in the attempted translation. The articulation of the concept yields itself again, it is ``self-referential''.

From a realistic standpoint, such self-referential concepts are not, for Gödel, a problem. It just means that you can't *construct* the concept, but since the latter is conceived of objectively, it is ``there'' anyway, and you just have to deal with it *non-constructively*, as we do with most concepts in normal life, science, mathematics and philosophy. He goes on to say (221):

Nor is it self-contradictory that a proper part should be identical (not merely equal) to the whole, as is seen in the case of structures in the abstract sense.

Such a whole *cannot* be constructible since if it was, the construction of the part would give the whole. Yet the whole contains more than the part, and we have a contradiction. The concept of the whole is then not constructible, and so has to be *self-referential* (in the sense (earlier) that any analysis of it refers back to itself).

Gödel goes on to illustrate this identity of whole and part by pointing out that

The structure of the series of integers, e.g. contains itself as a proper part

In more detail, let **N** denote the natural numbers with the order relation R on $\mathbf{N}x\mathbf{N}$. (So $m \leq n$ if and only if $(m,n) \varepsilon R$.) Contained in **N** as a set is the proper subset $2\mathbf{N}$ of *even* natural numbers, and the restriction of R to $2\mathbf{N}x2\mathbf{N}$ is, qua structure, *identical* to R.

With reference to the concepts of *concept* and *class*, he writes (230)

The difficulty is only that we don't perceive the concepts of ``concept'' and of ``class'' with sufficient distinctness.

[6] For a short, helpful, though not entirely precise, description of the theory of types, see Chapter 7 of: Mary Tiles, *The Philosophy of Set Theory*, Basil Blackwell, New York, 1989. A precise discussion, in which the technical inadequacies in Russell's account are resolved, is given in Chapter 4 of: W. S. Hatcher, *Foundations of Mathematics*, ibid. (See also: Hao Wang, *From Mathematics to Philosophy*, Routledge and Kegan Paul, London, 1974, Chapter 3.)

In the case of the class concept, the suggestion is - and this is supported by comments in his *continuum hypothesis* paper[7] - that we need new axioms that will express a clearer understanding of it. Since the concept is conceived of objectively, a better understanding of the objects involved will lead to better axiomatization for the class concept, just as the principles of quantum mechanics express a better understanding of the physical world than those of Newton-ian mechanics.

Unfortunately, since the time of Gödel, the desired new axioms have not been been discovered. A possible exception to this claim is *Freiling's Axiom of Symmetry*[8]. This axiom, whose plausibility comes from intuitive probabilistic notions, says that for any function *f* from the real numbers into the set of countable subsets of real numbers, there exist *x* and *y* such that *x* is not in *f(y)* and *y* is not in *f(x)*. The axiom easily implies that the Continuum Hypo-thesis fails. (The Continuum Hypothesis is well known to be undecidable in standard set theory.)

In the sought for clarification of the class concept, the philosophy is not, for Gödel, an end in itself, since once the desired axioms have been obtained, then one proceeds with the development of set theory in the usual formal way. All that is required is that the new axioms will strike us as intuitively convincing. The adoption of axioms for foundations is analogous to the adoption of hypotheses in science.

However, for Gödel, philosophy is an aid for making progress with foundational problems. A good example of this is the use by Gödel himself of the *liar paradox* to motivate the incompleteness theorems of arithmetic[9]. Philosophical thought might help in the finding of new axioms.

While Hegel would have agreed with Gödel on the self-referentiality of fundamental concepts, he would, I believe, have criticized the latter's objectivist views as follows. These views depend essentially on *intuition*, but the nature of the intuition is philosophically undeveloped. In particular the ``objective'' awareness associated with sets and numbers is far from clear - they do not seem *at all* like the objects of sense perception. In Hegelian parlance, such intuition is ``immediate'' (*unmittelbar*).

Plato and Hegel, in their philosophies of mathematics, can be regarded as trying to give the missing philosophical *mediation* or justification underlying such mathematical intuition. We turn first to Plato's attempt to give this philosophical mediation. This will be helpful for two reasons. First, Plato's thought is closer to that of Gödel's considered above - indeed Gödel's mathematical philosophy is even called ``Platonism''. Second, as we will argue, the inadequacies in the Platonic position will motivate Hegel's much more sophisticated account.

3. *Plato and his dialectic*

In the *Republic*, VI, Plato describes the levels of apprehension of reality and truth, for both the visible and intelligible orders of things, in the geometrical terms of the *divided line*, the ratios of the lengths of the line segments produced by the division representing the comparative clarity or obscurity of the

[7] *What is Cantor's Continuum Problem?*, pp.258-273 in: *Philosophy of Mathematics*, *Selected Readings*, ibid.
[8] C. Freiling, *Axioms of symmetry: throwing darts at the real number line,* Journal of Symbolic Logic 51(1986), 190-200.
[9] See Gödel's comments on the role that his philosophical objectivism played in the discovery of the incompleteness theorems in: Hao Wang, ibid, Introduction.

insight associated with each level. The lower level of the intelligible order concerns primarily mathematical thought, which in the Greek context, had two areas, *geometry* and *arithmetic*.

Plato[10] points out that the students of geometry and arithmetic

first postulate the odd and the even and the various figures and three kinds of angles and other things akin to these in each branch of science, regard them as known, and, treating them as absolute assumptions, do not deign to render any further account of them to themselves or others, taking it for granted that they are obvious to everybody. They take their start from these, and pursuing the inquiry from this point on consistently, conclude with that for the investigation of which they set out.

Plato mentions two criticisms of this procedure.

(1) First, when the geometer proves a theorem e.g. about triangles, he does so on the basis of a drawn, physical, figure. However this is not the *real* triangle but just a resemblance of the original. The genuine, real triangle cannot be drawn but, according to Plato, can ``be seen only by the mind''. So the first criticism of the procedure of the geometer is that he is claiming to say things about the real triangle, when in fact he is really dealing with an imperfect ``image'' of that.

(2) The second criticism made by Plato is a consequence of the first. Since the geometer is unable to extricate himself from the derivative, seen triangle, he does not have access to the *real* triangle, and so has to *make assumptions* about it. Geometry does not proceed to ``a first principle''.

The same objections apply in the modern context to the other branches of mathematics, in particular, to the mathematics of *set*. Indeed, Gödel's view (§ 2) that we don't perceive the concept of *class* with sufficient distinctness is a variant of objection (1), the lack of clarity being due to us not ``seeing'' the *real* concept of class. It also seems to me likely that he would have ag-reed that this lack of clarity is a consequence of approaching the concept of *class* through its ``likeness'' in particular classes (such as the set of real numbers). We also saw objection (2) in §2 in our discussion of what axioms (assumptions) should be made for set theory.

These two criticisms of mathematical procedure and Plato's explanation for them lead to two other related criticisms that we now discuss in Platonic terms.

(3) The first of these concerns a very simple problem with the evaluation of mathematical work, one that, I believe, Hegel tries to answer in detail in his philosophy of mathematics. It is that there is no criterion given in the math-ematician's approach to his subject of *why certain mathematical concepts, definitions, etc. are important*. Why, in mathematics, should a certain result be seen as of fundamental importance while another is not. What makes one mathematical result (such as that *there are infinitely many primes*) interesting and even thrilling, while another - e.g. a huge tautology - is boring?

Presumably for Plato, the answer to this would lie in the clarity with which the mathematician sees the original reality. A great theorem (such as the *Pythagoras theorem*) is an expression of the essential nature of the *real* triangle, whereas

[10] p.745-746 in: *The Collected Dialogues of Plato*, edited by E. Hamilton and H. Cairns, Bollingen Series LXXI, Princeton University Press, 1996.

the uninteresting theorem does not penetrate that far and so reflects weakly (and superficially), in the world of imperfect ``images'', the properties of that triangle. In the imperfect image, we have not got (what Hegel called) the *concept* of the triangle. Genuine mathematical insight is value laden.

(4) Criticism (2) leads to the issue of the *arbitrariness* of axiomatizing. Different axiomatic schemas can be given for the *same* mathematical system. As Quine points out[11], *in the beginning there was Euclidean geometry*, and the selection of postulates (axioms) out of the totality of truths of Euclidean geometry is *conventional*, being

the separation [of this totality] into those to be taken as starting point (for the purpose of the exposition at hand) and those to be deduced from them.

This lack of canonicity in the axiomatizing is disturbing; for while we readily admit that the subject of Euclidean geometry is the same whichever axiomatic set up we choose to adopt, yet this well-defined, unique ``body of truth'' is being presented from an arbitrary starting point. We can say that once the axiomatic framework has been chosen, then mathematical deduction and proof takes over. But we are aware of geometry and arithmetic *before* the axiomatizing begins.

How, then, is mathematics to be talked about *before* the axiomatizing? Plato's answer to this (in the *Republic*) goes as follows. All thinking proceeds on the basis of certain presupposed truths or *assumptions*. So pre-axiomatic mathematical thought requires its own assumptions. However, these assumptions do not function in the same way as the axioms of mathematics. Rather they are only ``hypotheses, underpinnings, footings and springboards'' serving purely as an aid to reach a starting point that requires *no assumption*.

Once this starting point X has been determined, its content is then to be unfolded in a philosophical development of ``pure ideas'' or ``forms'' (ειδοι), which ``proceeds downwards'' back to mathematics. For Plato, this development will result in a *canonical* axiomatic basis for mathematics proper, and when we have that basis, the philosophy leaves off and the mathematics starts up again, but this time as properly founded. The reasoning through which X is specified out of pre-axiomatic mathematics and is then developed in reverse to ground mathematics is called[12] *dialectic*. We stress that for Plato, this philosophical development is one of *pure ideas*, not that of describing quasi-objects accurately. In this respect, Plato is closer to Kant, Hegel, Frege and Husserl - who emphasize *concepts* in their philosophy - than he is to Gödel.

Without the dialectical basis, geometry for Plato belongs to the *understanding*, not to *reason*, the latter being the highest level of clarity of science obtainable. Unfortunately, Plato does not give, in his middle period dialogues, extended examples of dialectical argument. (The long and difficult argument of *Parmenides II* belongs to Plato's late period when his views on the forms and dialectical methodology had changed somewhat from those of the *Republic*[13].)

We now need to connect the Platonic notions of *starting point* and *dialectic* to the views of Gödel's on self-referential concepts discussed in §2. Recall that for Gödel, the vicious circle principle applies only to *constructed* entities.

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[11] p. 392 of: W. V. Quine, *Carnap and Logical Truth*, pp. 385-406 in: *The Philosophy of Rudolph Carnap*, The Library of Living Philosophers, ed. P. A. Schilpp, The Open Court Publishing Co., La Salle, Illinois, 1963.
[12] *Republic,* 511b.
[13] See, for example, §2, Chapter 1 of: K. M. Sayre, *Plato's Late Ontology*, Princeton University Press, Princeton, New Jersey, 1983.

Further, impredicative concepts are allowable, even though we can't build them up in a constructive way (for such concepts refer to themselves when analysed). The ``starting point'' X for Plato (above) was *also* not constructed since it does not depend on ``assumptions'', and hence the building blocks for any proposed construction for it, external to it, are not available. So allowing, for argument's sake, that there is such an X, we are faced with the following two natural questions: *is X a self-referential concept (in the sense of Gödel) and is Plato's dialectical logic effectively a logic of self-referential concepts?*

The answer to both of these questions is surely ``yes''. In more detail, for Plato, as we have seen, the content of X articulates through dialectic a philosophical development of concepts entirely out of its own resources. If we then ask how this is possible, the only answer is that X itself must be a concept which *explains itself* in that development. So X is a self-referential concept, and our first question is answered affirmatively.

For the second question, although Plato does not explicitly say so, his dialectic is *de facto* a logic of self-referential concepts. To see this, we recall that dialectical argument moves out of mathematics up to X and then returns back down to mathematics again. This gives a logical circle (though not a vicious circle), and each concept of the circle relates *back to itself* through the argument. So dialectic is a self-referential logic.

The concept X and the logical circle[14] was developed for general philosophy in great detail in the Neo-Platonic philosophies of Plotinus and Proclus. There (as anticipated in the work of Plato), X is the *One* (*το εν*) which is also the *summum bonum*. However, the hypostatic, henadic character of Neo-Platonic philosophy is not appropriate to modern mathematics.

Plato's own program for justifying mathematics dialectically seems to have remained only a program. Perhaps the reason for the failure of the program lies in its inadequate grasp of dialectical (i.e. self-referential) logic. This inadequacy was made clear in modern times by Hegel after the notion of dialectic resurfaced out of the philosophy of Kant[15]. Hegel criticized Platonic dialectic (in particular, that of the *Parmenides*) on the grounds that it is inconclusive and negative[16], essentially destructive.

Hegel does, however, concede that there is a negative side to dialectic. For him, a philosophical starting point in a dialectical argument is, even before that argument begins, *defective*, in that it is an *abstraction,* and this defect in the starting point is exposed through the argument. However, Hegelian dial-ectic does not stop at this negative conclusion. Rather driven by the conceptual content of the argument, it returns back to the starting point and corrects the one-sidedness of the abstraction. So dialectic for Hegel also has a *positive* result[17].

This intrinsic ``return'' in Hegelian dialectic is of great interest from the perspective of our earlier discussion on the need for a logic of

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[14] cf. *In any divine procession the end is assimilated to the beginning, maintaining by its reversion thither a circle without beginning and without end.* (Proclus, *Elements of Theology,* Prop. 146, trans. E. R. Dodds, Clarendon Press, Oxford University Press, 2000). See also § 147 of: Proclus, *A Commentary on the first book of Euclid's Elements*, trans. by G. R. Morrow, Princeton University Press, 1970.

[15] Dialectic is considered by Kant in his *Transcendental Dialectic*. However, as Hegel points out (*SL*, p.831), ``the result to be drawn from it [dialectic] must be the opposite of that arrived at by Kant''.

[16] *SL*, p.55.

[17] *SL*, p.834.

self-referen-tial concepts. For the return is readily interpreted as just the expression of self-referentiality. So, for example, the mediation of the concept of *being* (that we will look at more closely in the next section) is nothing other than the articulation of the self-referential character of that concept. Hegelian logic is explicitly a logic of the self-referential whereas Platonic dialectic was only that implicitly. The main thesis of the paper is that Hegelian logic is (in principle) the answer to the search for the philosophy of self-referen-tial concepts that, as we have seen, is required for foundational philosophy. We will now illustrate this thesis by sketching how, within the Hegelian framework, we can obtain a (primitive) philosophical foundation for the key concepts, *infinite set* and *natural number*, of mathematics.

4. Infinite set and natural number from a Hegelian perspective

Hegel's treatment of number in the *Wissenschaft* is long and difficult and we will be content to indicate briefly how part of the argument goes and how this part can be used to give us a foundation for the concepts of *infinite set* and *natural number*. In the interests of clarity, we will indicate the main points of the argument by (a), (b), etc. For the purposes of the paper, we will focus the discussion on the self-referential concepts required in mathematics, even though Hegelian concepts such as *Dasein,* used in the argument, have a much wider philosophical range[18] than just the mathematical. Of necessity, much will be omitted from Hegel's argument, and we will feel free to relate his ideas to post-Hegelian mathematical philosophy.

(a) *The abstract concept of being (Sein)*

We start (as does *WL*) with the concept of *Being*. We argue first that such a concept is implicitly presupposed in modern approaches to mathematical logic, where one starts with immediately given *entities* (particular *beings)*.

There are two main such approaches. One either starts with entities taken as *individuals* in the world (Russell-Whitehead) (so that mathematics applies to the real world from the outset) or with the *natural numbers* themselves taken non-formally (Gödel) (making them available to label constants, variables, functions and predicates in formal mathematics). Being also returns in mathematical logic when one moves from formal systems *per se* to their interpretations, the latter giving a set theoretic basis for the former. Such an ontological presence in foundations ensures that mathematics is *about something* (a conviction felt by most mathematicians) and also provides partial comfort that the subject will not give rise to a contradiction.

Yet there are problems with building a foundational philosophy on immediately given entities. If one starts with individuals, it is not clear what is to count *as* an individual. This lack of specificity leads to problems such as whether there are infinitely many individuals (Russell's *axiom of infinity*). As for starting with numbers, it is not clear what kind of ``entities'' these are supposed to be. We could base them (as in the previous paragraph) on some set-theoretic intepretation, but then the choice of interpretation is arbitrary (not canonical) and relies on the unclarified notion of *set*.

These problems disappear when we switch from entity talk to the abstract, completely general, *concept* of being presupposed in such talk. This is exactly the starting point for Hegelian logic, and it effectively plays the role for that

[18] For a comprehensive discussion of the logic of *Sein* and *Dasein*, the reader is referred to Chapter 2 of the book by John Burbidge: *On Hegel's Logic,* Humanities Press, New Jersey, 1995.

logic that the starting point X of the last section does in Platonic dial-ectic. Starting with the concept of *being* is of fundamental importance for a philosophy of mathematics, since on the one hand, as a concept, it provides the basis for a logic of mathematical *concepts*, and on the other hand, it enables mathematics to apply to the world simply because the objects in the world *exist* and so fall under the being concept.

(b) *Being, nothing and becoming*

The starting concept of *being* (*Sein*) is completely abstract and general. As such it has no ``reference outwards'' (*noch nach aussen*). So being refers to itself (and so is a *self-referential* concept.) This concept of being has no content and is therefore the concept *of nothing at all*.

This result that the concepts of being and nothing coincide gives a simple illustration of *negative* dialectic (discussed in the previous section). Clearly, the concepts of being and nothing are *not* identical. Hegelian logic moves on to the *positive* by observing that what is left out of this apparently incoherent identity of being and nothing is the *becoming* through which the identity is established. One can see this by noticing how the concept of being arises through abstraction. The concept of being ``becomes'' as it runs through all the forms of concrete being in its effort to isolate what they all have in common. But what they all have in common is *nothing*, and this identity of being with nothing is realized only through becoming.

The intuitionistic view of the continuum gives a good example in mathematical philosophy of this becoming of being. In this connection, Brouwer and Weyl[19] recognize the continuum as containing individual numbers but not as a set of real numbers with ``finished'' being (*fertig seiende*). For them, it is rather to be thought of as a medium of free becoming (*Das Kontinuum als Medium freien Werdens*).

(c) *Dasein*

Being is then not to be thought of in terms of simple fixity but rather it is being-through-becoming, what Hegel calls *Dasein*[20] (there-being). *Dasein* can be helpfully illustrated as something *there* in front of us in a process of change. At the same time, as being, it has a constant qualitative character which determines its identity through the change. This character is to be thought of in terms of ``surface'' qualities. (The (essential) concept of an *object* with its *properties* is more advanced logically and so appears much later in *WL*.) Howev-er, *Dasein* is not restricted to changing things of sense perception - for example, the continuum as described in (b) is *Dasein*, and the *Dasein* concept keeps reappearing and further developing throughout *WL*[21]. As we will see, the concepts of the finite and the infinite in mathematics, in their simplest forms, are determined in Hegelian logic by *Dasein*.

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[19] p.94 of: H. Weyl, *On the New Foundational Crisis of Mathematics*, pp.86-118 in: P. Mancosu (editor), *From Brouwer to Hilbert: the debate on the foundations of mathematics in the 1920's*, Oxford University Press, New York, 1998.

[20] For a discussion of suggested translations of the term, see the Appendix to the author's paper: *Self-reference and the natural numbers as the logic of Dasei*n, Hegel Studien, 32(1997), 93-121.

[21] For example, in *Essence* logic, *Dasein* (*there-being* in Being logic) advances to *Gesetzsein* (*posited being*) (*WLII*, p.20), while in *Concept* logic, this essence form in turn advances to *An-und Fürsichsein* (*in-and-for-itself being*) (*WLII*, p.262).

From the point of view of our earlier discussion, the positing of being in being-as-*Dasein* is to be seen as an expression of the *self-referential* nature of the being concept. That is, the concept of being, through its intrinsic development, *produces itself again*, not in its original, empty, simplicity, but in *positive* mediated form.

(d) The finite and infinite

Dasein is the becoming of being through *nothing* and hence is intrinsically unstable. *Dasein* does not stand on its own feet (whereas *Sein* at first apparently did). When this is developed further, *Dasein* logic yields two self-refer-ential concepts of fundamental importance in mathematics, that of the *finite* (*Endlichkeit*) and the *infinite* (*Unendlichkeit)*.

For the first of these, when the non-being that is at the heart of *Dasein* is made explicit, it undermines the qualitative determinations that gave *Dasein* its stability, so that *Dasein ceases to be*. *Dasein* in this ceasing to be is defined by Hegel to be the *finite*. Hegel makes explicit the self-referential character of the finite when he says that finite things ``are negatively self-related''[22].

This leads (in *WL*) to the first of many ``infinite progresses'' (*die unendlich-en Progressen)* as follows. The finite has collapsed into nothingness through its ceasing to be, but nothing, the second term of (b), logically develops, through becoming and *Dasein,* to give the finite again. This in turn, as finite, ceases to be and the unstable process repeats *ad infinitum*. This is the *infinite progress* of the finite.

However, while the finite itself alternates between disappearance and emergence in the progress, the latter's *pattern* of *positing-disappearing-positing* is constant throughout, and it is this second order conceptual pattern which is defined by Hegel to be the *infinite*. The finite exhibits its *implicit* infinitude through its process of dissolution and return, while the infinite expresses the structure of the finite.

The concept of the infinite, implicit in that of abstract being, has thus become explicit. In contrast to Fregean thought, where the infinite comes about through the extension of classes and depends on what there is available in the world to fall under a concept, *the infinite in Hegel is intrinsic to the very nature of thought itself*. Its concept does not depend on what there is empirically in the world, and Russell's unsatisfactory *axiom of infinity* is not required. No future scientific discoveries (e.g. the existence of previously unknown stars) will change the basic concept of the infinite (or of number).

The concept of the infinite is not to be thought of as a potential infinite, intrinsically incomplete in its determination. (For more discussion of this, see (f).) It is rather a concept which is complete in itself, pictured as a circle closed up in itself as opposed to a line of infinite length that goes on and on for ever[23]. The infinite and the finite are no longer opposed to one another, but each is only through the other: the thought of one immediately contains the thought of the other. The concept of the infinite-through-finite and finite-through-infinite is explicitly the *self-referential*. The infinite leads back to itself through the finite whose pattern it is, while the finite gives rise to itself as a coherent concept only through its pattern (the infinite).

[22] *SL*, p.129.
[23] *SL*, p.149. See also Footnote 14.

We now show how the concept of the infinite that has emerged out of the logic of *Dasein* gives a foundation for two important, self-referential, mathematical concepts, namely, *infinite set* and *natural number*.

(e) *The primitive notion of infinite set*

While Hegel, who lived before the time of Cantor, has, of course, no explicit set theory, I want to claim that Hegel's concept of the infinite in (d) specializes to that of *infinite set* in modern mathematics. In this specialization, the set is determined by its *elements* (the form of the *finite* in that case) through their intrinsic reference-to-one-another so that one does not come to a stop at an element but is forced forwards to another in the infinite progress. The elements of the set are not *neutral* with respect to one another but rather are mutually determinative. The set itself is characterized not by a property (as in Frege's *axiom of abstraction*) but rather by its conceptual *pattern* through which an element refers to the rest. In this sense it is an *internally determined* set rather than a property functioning as a ``sieve'' through which everything in the universe is filtered, leaving behind as residue the elements of the set. This internal determination of *set* is expressed math-ematically in the *well-ordering theorem* (which is equivalent to the *axiom of choice*), so that the latter depends for its philosophical justification on (d).

(f) *The primitive notion of natural number*

Intuitively, the natural numbers provide an example of such an internally determined set. Given any number, e.g. 2, for it to be meaningful, it refers implicitly to all of the others - these are required for 2 to have its *place*. (Hegel discusses this in more detail, later in *WL,* in his treatment of *extensive and intensive quantum*).

This reference of numbers to each other is just the *ordinal* character of the *natural number*. The infinite progress of the finite exhibits this ordinal character in the forcing on of its *positing-disappearing-positing* pattern. The Heg-elian approach to number gives neither the *Platonic* nor the *Gödelian* view of number. There is no stripping away of shifting sense perceptions to achieve an intuition of pure numerical essences. Nor are there absolute, objective ``nat-ural numbers'' out there that we have an intuition of in a manner analogous to our awareness of the objects of sense perception. In fact we are *not* aware of any such ``objects'', but rather number thought articulates itself in its nec-essity in thought. The explanation for this lies in the *conceptual* character of number, which, as we have seen, is determined ultimately through the concept of being itself.

Nor is number to be thought of in Kantian terms as an *a priori* determination of the *temporal* form of inner intuition. Rather the *natural number* is given conceptually, *completely*, through the relations of finite-through-infinite and infinite-through-finite discussed in (d). It is only when the infinite progress of the finite is thought of in terms of everyday objects in time that it gives a *succession* of finites. This succession is the source of our way of listing the natural numbers in order. This listing is an *incomplete* infinity, and we only obtain in that way the natural numbers as a *potential infinite*, since wherever we stop, there is a beyond.

(g) *The quantitative character of number*

A more extensive development is needed before we arrive at the natural numbers in a form appropriate for mathematics and indeed for everyday life. Material for

such a development is given by Hegel, and goes through the self-referential proper (*Fürsichsein*) and the shift from the qualitative to the quantitative. The fundamental insight of this shift is the very simple one that number is *qualitatively neutral*. The number 3, for example, applies to *any* triple of things, whatever their qualitative character.

This insight is also of great importance for Fregean philosophy[24]. Indeed, it is the reason why, in the Fregean approach, a number *n* is defined intuitively as the class of *all n*-sets of things. The crucial difference between the Heg-elian and Fregean approaches to number is, as we saw in (d), that Hegel's is through the concept of being, whereas Frege's presupposes a world of things which can be clumped together into classes. Further, the concept for Frege lacks the self-referential logic needed to derive number from the concept of being.

(h) *Conclusion*

The Hegelian, self-referential basis for mathematical concepts that we have briefly discussed does not undermine mathematics. It does not interfere with, or even intrude into, mathematical reasoning. It does not imply that contradictions in mathematics are acceptable. Mathematical argument is precise and non-self-referential in character.

The self-referential philosophy lies rather in the *source* of such argument, operating implicitly in subjective, though not arbitrary, form through the immediate, creative thinking of the mathematician. The philosophy only emerges *after* the creative mathematics, whose philosophical sense it explicitly exhibits[25]. It thus lies at the level of the *meta-theory* of mathematics, though the latter is not to be understood as an abstract theory of mathematics but rather as the expression of the activity of the self-referential concepts that result-ed in the mathematical creation, but which has retreated into implicitness behind the hardened precision of its exposition in a mathematical paper. Phil-osophy brings this conceptual activity out into the open in explicit, mediated form.

The creative mathematician can, and indeed, should, for the most part, avoid philosophy in his or her mathematical work, since the philosophical thought will usually interfere with the intuitive, mathematical thought flow which produces the theorems. This limiting of the role of philosophy in mathematics is illustrated by the way in which mathematicians over the centuries have dealt with the crises that have threatened the subject. What happens *de facto* is not primarily a philosophical investigation at all, but rather an axiomatic redef-ining of the boundaries within which they work, a determining of procedural rules, what you can and can't do. This, as it were, ``rights the ship'' and mathematics gets started up again, leaving philosophy behind.

But what the mathematician leaves as *immediate* and resolves in a purely pragmatic way is exactly the self-referential philosophy of concepts sought by Gödel, and which Plato and Hegel strived to articulate, and the open minded thinker who is willing to struggle with the initially strange world of Hegelian logic will find in its conceptual developments an insightful, if incomplete, mediation of this immediacy.

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[24] A good discussion of this is given in Chapter 7 of: Mary Tiles, ibid.

[25] This view of mathematics is a special case of Hegel's well-known ``owl of Minerva'' comment in the Preface to Hegel's *Philosophy of Right* (pp.12-13 of: The Philosophy of Right, trans. by T. M. Knox, Oxford University Press, Oxford, 1952). As part of the history of the world (with which Hegel is concerned in that comment), philosophy as the thought of mathematics only appears when the mathematical creation is already there, actual and ``cut and dried''.

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