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# HEGEL'S EARLY GEOMETRY

## ABBREVIATIONS

The following abbreviations are used throughout.

*Diss.* *Dissertatio philosophica de Orbitis Planetarum*, pp.1-29 in: Georg Wilhelm Friedrich Hegel, Sämtliche Werke, Erster Band, Krit. Journal u. a. Schriften, Fr. Frommanns Verlag, Günther Holzboog, Stuttgart, 1958.

*Dok.* *Dokumente zu Hegels Entwicklung*, herausgegeben von Johannes Hoffmeister, Fr. Frommanns Verlag, Stuttgart, 1936.

*GS* *Geometrische Studien*, by G. W. F. Hegel, pp. 288-300 of *Dok.*.

*Hea.* *The thirteen books of Euclid's Elements, Vol. 1*, translated with introduction and commentary by T. L. Heath, Dover Publications, New York, 1956.

*JS* *Georg Wilhelm Friedrich Hegel Jenaer Systementwürfe II*, herausgegeben von R-P. Horstmann und J. H. Trede, Deutsche Forschungsgemeinschaft, Bonn-Bad Godesberg, 1971. (*JL* stands for the English translation of the *Logik* and *Metaphysik* of *JS*: *G. W. F. Hegel The Jena System, 1804-5: Logic and Metaphysics*, translation edited by J. W. Burbidge and G. di Giovanni, Introduction and explanatory notes by H. S. Harris, McGill-Queen's University Press, Kingston and Montreal, 1986.)

*Lor.* *Euklids Elemente, funfzehn Bücher*, aus Griechischen übersetzt von Johann Friedrich Lorenz, Halle, 1781.

*NP* *Georg Wilhelm Friedrich Hegel Enzyklopädie der philosophischen Wissenschaften, zweiter Teil, Naturphilosophie (1830)*, herausgegeben von F. Nicolin und O. Pöggeler, Verlag von Felix Meiner, Hamburg, 1959.

*P* *Proclus: A Commentary on the first book of Euclid's Elements*, translated with introduction and notes by G. R. Morrow, Princeton University Press, Princeton, N. J., 1970.

*TS* *Hegel's Development*, by H. S. Harris, two volumes: *TS I, Toward the Sunlight (1770-1801)*; *TS II, Night Thoughts (Jena 1801-1806)*, Oxford University Press, Oxford, 1972, 1983.

*WLg* *Georg Wilhelm Friedrich Hegel, Gesammelte Werke, Wissenschaft der Logik*, herausgegeben von F. Hogemann und W. Jaeschke, two volumes: *WLg I (1812-1813)*, *WLg II (1816)*, Felix Meiner Verlag Hamburg, 1978, 1981.

*WL* *Georg Wilhelm Friedrich Hegel, Wissenschaft der Logik (1812 ff.)*, herausgegeben von G. Lasson, two volumes: *WL I, WL II*, Verlag von Felix Meiner, Hamburg, 1934, 1966, 1967. (*SL* stands for the English translation of *WL*: *Hegel's Science of Logic*, translation by A. V. Miller, Humanities Press International, Inc., Atlantic Highlands, NJ, 1989.)

*Euclid 1* stands for the first book of Euclid's *Elements*. All of the English translations of Hegel texts in the paper are my own unless indicated otherwise. (For English-speaking readers, I have given page references to the standard translations *JL*, *SL* of *JS* and *WL*.) In translating, I have followed the practice of H. S. Harris and others by

inserting, in some places, words not in the original text to clarify the meaning. These insertions are surrounded by brackets [...]. Reference within the paper to Chapter a, section (b) is abbreviated to: a(b).

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## 1. *Introduction*

Throughout his life, Hegel maintained a deep and sustained interest in geometry and its philosophical basis. Geometry is investigated in his early work, the *Geometrische Studien* (*GS*), which was completed in September, 1800, during his time at Frankfurt. Although there are substantial new developments in Hegel's geometrical thought in his later work – especially in *JS*, *WLg*, *WL*, and *NP* - the discussion of GS is much more detailed than the rest, and is fundamental for all of his later thought on the subject.

At the invitation of Schelling, Hegel left Frankfurt for Jena in January of 1801. His thought developed rapidly, as he actively worked with Schelling within the framework of Schelling's *Identity philosophy*, the two of them collaborating on the journal *Kritische Journal der Philosophie*. The change in philosophical direction appeared quickly in his dissertation (*Diss.*) on the orbits of the planets, written within a year of the completion of GS. In *Diss.*, the influences of Plato, Kepler and Schelling are very clear, and it has an unrestrained speculative character. In relation to *GS*, the geometry of *Diss.* can be regarded as providing a philosophical rationale for *applied mathematics*.

Tensions built up between Schelling and Hegel over their collaboration, and after Schelling left Jena in 1803, Hegel worked on his first sustained treatment of Logic, Metaphysics and Natural Philosophy in the 1804/05 sketch of his system, *JS*. The calm, intense, rational thought of this work indicates, I believe, the beginning of Hegel's mature period, that was to culminate in *WL*. As far as geometry (and indeed mathematics in general) is concerned, *JS* represents a continuation of *GS* and *Diss.* but set within a much broader range of ideas. Since the objective of the present paper is to discuss Hegel's *early* geometry, the paper gives a detailed discussion of *GS*, and concludes with a brief description of the applied geometry of *Diss*..

In particular, the fundamental notion of mature Hegelian logic, the *Idea*, that links logic to nature, will not be considered here. Also not considered here are Hegel's later investigations into the logic of construction and proof, the role of the *axioms*, the nature of arithmetic, the infinite in mathematics, the *Calculus* and the significance of the Pythagoras theorem. I have, however, not hesitated to refer to Hegel's later discussions of mathematics when they help to clarify his earlier treatment of geometry.

Sadly, much of Hegel's original discussion in *GS* has been lost. For example, it seems likely that, in the original *GS*, Hegel treated all 48 of the Propositions in *Euclid 1* (ending in the Pythagoras theorem), but of this material, the discussions of only 14 of these propositions have survived. Also surviving is a valuable discussion of mathematical objects, a treatment of parallel lines and a puzzling section on geometrical algebra.

What survives of *GS* poses serious difficulties for the interpreter, being both fragmentary and its assertions cryptic. Some of it is not even in the state of a fair copy (*reine Schrift*). To help deal with these difficulties, we start in Chapter 2 by developing a background for the study of *GS*. In particular, we will investigate the basic themes of *GS* by understanding it, not only in terms of Kantian and post-Kantian philosophy, so influential at Hegel's time of writing, but primarily in terms of the great commentary *P* on Book 1 of the Elements of Euclid by the classical NeoPlatonic philosopher *Proclus*. The importance of *P* for Euclidean scholarship has long been recognized.[1] (Indeed, I do not think that it is in principle possible today to understand the philosophical basis for *Euclid 1* without the help of *P*.) In particular, *P* strongly influenced Kepler, who in turn influenced Hegel. It is not clear to me to what extent Proclus had a *direct* influence on Hegel, but that is not so important for our purposes: *P* provides a comprehensive, NeoPlatonic framework which provides a basis for understanding the terse account of geometry in GS. To this end, Chapter 2 is a study of a number of common themes of *GS* and *P*. Hegel's detailed treatment of geometry in *GS* is described in Chapter 3.

The key to Hegel's account of geometry in *GS* is his philosophical understanding of geometrical objects and their relationships, expressed in terms of the *limitation of concepts*. This account is inspired by the classical Pythagorean-Platonic theory of limitation, which is applied in

[1] The reader need only consider the prominent role of *P* in the 151 page introduction to the *Elements* in *Hea*..

detail by Proclus in *P* to geometry. However, Hegel's philosophy of limitation is distinctively *modern* in its *conceptual* character (consonant with the central role of the concept (*Begriff*) in the philosophy of Kant).[2] Hegel uses this philosophy to interpret and also to criticize Euclid's treatment of the foundations of geometry in *Euclid 1*. In particular, as we will see, he uses it to give a philosophical justification of point, line, plane and solid, the three dimensionality of space, the nature of angles and the triangle congruence theorems. He also gives a philosophical explanation, with particular stress on equality and inequality, of the order of the geometrical figures, and in particular, of the role of the paradigm mathematical objects in geometry. These paradigms include the circle, the straight line, the equilateral triangle and the square. Following Proclus, Hegel sees the paradigms as standing out from the other figures because of their self-identification through the *equality* of their parts. Hegel also analyses in detail the conceptual structure of -two of the paradigm objects in three dimensional space, the *cube* and the *sphere*.

Hegel's discussion of Propositions 1-12 and Propositions 26 and 29 of *Euclid 1* have survived in *GS*. He is concerned to understand these propositions within his philosophy of limits, and to correct those proofs of Euclid that are inadequate. For Hegel, the fundamental weaknesses in some of these proofs stems from the inadequacy of Euclid's (implicit) philosophy of geometry. In particular, for Hegel, this results in unsatisfactory and unnecessarily lengthy proofs of certain propositions: commenting on one of these, he says that *Euklid's Umweg dient allein, diesen Mangel an Verschiedenheit zu decken*[3] (``Euclid's detour serves only to cover this deficiency in [his grasp of] difference'').

For Hegel, this deficiency of Euclid's approach to geometry results in Euclid *not* seeing the common conceptual basis for identical triangles in different spatial locations, and instead, Euclid has to resort to the controversial ``method of superposition'' in which one lifts up a triangle (as if it were a piece of paper) and puts it on top of another to check for congruence. For Hegel, superposition is a fundamental error, based on a wrong philosophy. In *GS*, he corrects Euclidean proofs that use superposition, replacing them by philosophically adequate, conceptual proofs. However, *GS* is not just concerned with the philosophical basis of geometry: Hegel also comes up with some Euclidean proofs of his own, and was clearly well-informed about geometry.

The proper setting for Hegelian mathematics is within his Logic. The importance of mathematics in Hegel's logical investigations[4] prefigures the dominant role that mathematics plays in present day logic, so that the latter is often called *mathematical* logic. While I think that modern logic sometimes takes this mathematical dominance too far, Hegel does not make this error, and his logic spans the full range of concepts required

[2] Hegel's primary criticism of Proclus is exactly that the ``self-development'' of the unity (i.e the *One*) which is the basis of the Proclus philosophy ``is not made with the necessity of the concept'' (p.435 in: G. W. F. Hegel, *Hegel's Lectures on the History of Philosophy*, trans. E. S. Haldane and F. H. Simson, Vol. 2, The Humanities Press Inc., New York, 1955.).

[3] *GS*, p.290.

[4] In *WL* I, 84 pages (pp. 239-322) are devoted to the mathematical infinite and the calculus alone, more than is given to the discussion of topics concerning religion and the state in all of *WL*! This is not, of course, to deny Hegel's profound interest in the philosophy of religion and of the state – he investigated these in great detail in his other writings.

for thought in general: in particular, the scientific, religious, political, aesthetic and ethical, and of course the mathematical. The need to consider the somewhat difficult issues of logic in Hegelian philosophy is inescapable. Through such consideration, the structure of thought in general becomes explicit, or equivalently, in Hegelian terms, the concept becomes *für sich*. Without the discipline of the logical foundation, Hegelian thought is in danger of becoming ``soft'' and imprecise, at the mercy of thought content in which what is essential and what is on the surface are not distinguished.

The full scope of Hegel's geometrical philosophy presents an astonishingly erudite account of Euclidean geometry, in which the great insights of the classical Greek philosophers and geometers are integrated into a modern, post-Kantian perspective. It's ideas, and especially those of *GS*, are at first sight strange since most of the classical geometrical philosophy that inspired it has been forgotten over the last 100 years through the influence of the dominant, set theoretically based, mathematical philosophy (in which Cantor and Frege played such formative roles). Present day mathematics has, however, in topology and differential geometry, developed its own geometrical intuition and insight, based on, but going far beyond, Euclidean geometry, while at the same time, formulating its mathematics within a precise set-theoretic framework.[5]

Given that almost all mathematicians are, and always have been, (mildly) Platonic in their view of mathematics – mathematics is about *something* – the need for a philosophy of mathematical objects is inescapable. Rather than just being content with producing axioms which will allow mathematicians to do what they would do anyway and hoping (*pace* Gödel) that some inspiration will ennable the ``correct'' axioms for set theory to ``force'' themselves on us, we need to confront the nature of these objects and elucidate the philosophical principles through which they are to be understood. Particularly pressing is the need to understand the most visually obvious mathematical objects, those of geometry, which express our intuition of space and which are fundamental for physical science – not just the objects of classical geometry: triangles, circles, planes, etc., but also of their profoundly developed, modern counterparts: Riemannian manifolds, geodesics, the curvature tensor, etc.. I believe that Hegel provides a stimulating perspective on the nature of mathematics and its objects, and clarification of that perspective was the main motivation for writing this paper. For these reasons, I have not hesitated to refer, usually in footnotes, to issues in modern mathematics and its philosophy when I have felt that their interaction with Hegelian thought would prove helpful for both. But, of course, the primary emphasis of the paper is that of expounding Hegel's early philosophy of geometry.

My hope is that the tantalizing, and sometimes fragmentary, discussions of the philosophy of geometry, written by arguably the greatest (and most misunderstood) of all philosophers, will prove instructive in the search for a present day philosophy of mathematics.

## 2. *The NeoPlatonic basis for Hegel's early work on geometry*

[5] This is usually that of Zermelo-Fraenkel set theory, which is more satisfactory and easier to work with than Frege's system.

(a) *Background to Geometrische Studien*

Hegel's early thought on mathematics and its philosophy was shaped by his study of the *Elements* of Euclid. The results of that study appear in what survives of his manuscript *Geometrische Studien* (which we abbreviate to *GS*). Hegel completed his work on *GS* in the Frankfurt period. Indeed, the part of *GS* dealing with the first twelve propositions of *Euclid 1* (the first book of the *Elements)* is dated 23 September, 1800, when Hegel was visiting the nearby city of Mainz. At that time, Hegel was based in Frankfurt, employed as a *Hofmeister* (private tutor). However some of the material of *GS* goes back much earlier than 1800.[6]

*GS* also contains discussions of two other later propositions of *Euclid 1* (Propositions 26 and 29), an account of mathematical objects in general, an account of parallel lines and lastly some algebraic calculations with line lengths. It seems that much of Hegel's original manuscript has been lost. In particular, there is no account of the Pythagoras theorem of *Euclid 1*, a theorem that features prominently in Hegel's later geometrical thought. It would also seem that Hegel's geometrical investigations in *GS* went beyond *Book 1* of the *Elements*.[7]

The manuscript of *GS* is at Harvard University. We will use the edition of Hoffmeister in *Dok*..[8] Helpful comments on *GS* are given in the latter's *Anmerkungen*.[9]

In his study of *Euclid 1*, Hegel used its German edition *Lor.* by Johann Friedrich Lorenz. The Lorenz versions of the first 12 propositions of *Euclid 1* discussed by Hegel are given in the *Anmerkungen* of Hoffmeister. As early as 1785, Hegel made excerpts from *Lor.* and also from mathematical works of Kästner.[10] In particular, Hegel records in his *Tagebuch* (diary) that he studied excerpts on spherical trigonometry that he copied out of Lorenz's work.[11] (This is a puzzling statement since there is no trigonometry in *Lor.*.)

According to the authoritative English translator of the *Elements*, T. L. Heath, Lorenz's work is ``the first attempt to reproduce Euclid in German word for word''[12] and Lorenz himself in the title page to his book says that his work is a translation out of the Greek. However, this is not quite accurate. The definitions, postulates and axioms of Lorenz given in his version of *Euclid 1* are not exactly the same as those of Euclid – for example, Lorenz has 12 axioms compared with Euclid's 5. The statements of the propositions given by Lorenz are translations of Euclid's statements. The proofs that Lorenz gives are (usually) much shorter and more readable versions of those of Euclid, and he sometimes adds additional proofs of his own: for example, he gives two proofs for Proposition 31 of Book 6. *Lor.* (in my opinion) is an excellent, lucid,

[6] At least one part of *GS* goes back to 1785 when Hegel was attending the Stuttgart Gymnasium. Indeed, in his *Tagebuch* for that year (*Dok.*, p.17) he says, concerning the equilateral triangle, ``*Die simpelste Figur ist hier das gleichseitige Dreieck*'' which is echoed in *GS*: ``*Das gleichseitige Dreieck ist die einfachste Figur*''.
[7] Hegel refers to propositions from Book 2 of the *Elements* in *WLg II*, p.223; *WL II*, p.469 (*SL*, p.810).
[8] *Dok.* pp. 288-300.
[9] *Dok.*, pp. 470-473.
[10] See the discussion in: *TS I*, pp.50, 54, 55.
[11] *Dok.*, p.39.
[12] *Hea.*, p.107.

though also uncompromising, treatment of the *Elements*, that manages to get the essential content of all of the 15[13] books of the *Elements* in a one volume work of 367 pages. Clearly Hegel liked the book, for he used the 1793 edition for his own school teaching of the ``Elements of Mathematics'' at the Nuremberg Gymnasium later in his career.

While, no doubt, Hegel first learned geometry from studying *Lor.*, he went far beyond it, in particular studying in depth for himself the works of the great Greek geometers and their geometrical philosophies. (There is no *philosophy* in *Lor.*) Hoffmeister[14] points out that the unsatisfactory nature of the Euclidean approach to geometry was recognized and worked on in the second half of the 18$^{th}$ century by mathematicians such as Karsten (1778), Hindenburg (1781), Schulz (1783) and Lambert (1786), but it is not clear to what extent their work influenced Hegel's treatment of geometry in *GS*.

The philosophy of geometry that Hegel presents in *GS* is expressed in a very condensed, elliptic way. To make sense of it, I believe that we need to understand it against the background of the *NeoPlatonic* philosophy. It is in NeoPlatonism that we have the most systematic presentation of classical Greek geometry and its philosophy. In this context, we must not think of the NeoPlatonic philosophy of geometry in a narrow sense. Rather, it is to be regarded as the culmination of the complete Greek geometrical tradition, and it embraces not only the thought of the great philosophers, such as Plato and Aristotle, but also of the great geometers, such as Pythagoras, Euclid, Archimedes and Apollonius.

Of course the Kantian and post-Kantian idealist philosophy, dominant in Germany in Hegel's early years, is also fundamental in understanding *GS* (especially with its emphasis on the concept).[15] However, post-Kantian philosophy, as briefly discussed below, was itself influenced by the NeoPlatonic philosophy, and so the influences of these two philosophies on Hegel's geometrical thought are not incompatible.

It is natural to ask how much direct influence the philosophy of Hegel's friend Schelling had on the thought of *GS*. I think that the answer is *very little*. It is true that in his Frankfurt period, Hegel bought, and presumably studied in detail, the writings of Schelling.[16] However, I have been unable to find any detailed investigation by Schelling on geometry in his early (or indeed in his complete) *Werke*. Further, the Fichtean themes of Schelling's early work (as in the *Ideen zu einer Philosophie der Natur* (1797)) are absent in *GS*. As we would expect, Schelling's identity philosophy clearly influenced Hegel's thought after Hegel left Frankfurt and the two collaborated at Jena. In particular, Schelling's influence is very strong in Hegel's dissertation *Diss.*, with, for example, its understanding of the spatial *point* in terms of *mens*. (We will discuss this further in Chapter 4.)

We now come to what, I believe, is a central question for the student of *GS*: *what are the main sources, in classical Greek geometrical*

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[13] Books 14 and 15 of the *Elements* were not written by Euclid.
[14] *Dok.*, p.473.
[15] For a brief discussion of how Kant's view of geometrical objects may have influenced *GS*, see Footnote 90.
[16] p.100 of: *Georg Wilhelm Friedrich Hegels Leben*, K. Rosenkranz, Wissenschaftliche Buchgesellschaft, Darmstadt, 1969.

*philosophy, used by Hegel when writing GS?* I will argue in (e) that the main philosophical source for Hegel's philosophy of geometry in *GS* was (directly or indirectly) the commentary *P* on *Euclid 1* by the last of the great classical NeoPlatonic philosophers, Proclus. We start our justification of this claim by discussing briefly the Proclus commentary.

(b) *NeoPlatonism and geometry*

Proclus lived in the 5th century and in his later years, was head of the Academy in Athens. (As a result of the edict of Justinian, the Academy was closed down and its property confiscated in 529, not long after the death of Proclus.) The commentary *P* of Proclus on the first book of Euclid's *Elements* is one of the most valuable documents in ancient philosophy, not only for its information about the history of geometry before the time of Proclus but also for its insights into the foundations of mathematics.[17]

The Proclian account of geometry (and more generally, of mathematics) is highly Pythagorean in character as well as Platonic. We can summarize it as follows in terms of the *Prologue* to *P*. The quest for the first principles for mathematics is (for Proclus) the search for ``those all-pervading principles that generate everything from themselves''.[18] There is thus no clear distinction between pure and applied mathematics: the principles of mathematics go back to the principles of everything. There are two such principles: they are l*imit* (or end (πέρας)) and the *unlimited* (or boundless, endless[19] (απειρος)), that head the two columns of the Pythagorean categories. The *unlimited* is generative in character while *limit* acts as a bound for the unlimited and so gives rise to determinate things. According to Aristotle,[20] under *limit*, the Pythagoreans listed categories such as *odd*, *one*, *male*, *straight*, *good* and *square*, while under the *unlimited*, they listed the corresponding categories such as *even*, *many*, *female*, *curved*, *evil*, and *oblong*. The pairings of corresponding geometrical categories in the two columns appear in the thought of Proclus, Euclid and Hegel. For example, for Proclus, the unlimited and limited for *line* are respectively the *straight line* and the *curved line* (especially the *circle*).[21] Also the reduction by Euclid of the (unlimited) *oblong* (rectangle) to its limited counterpart, the *square*, is considered philosophically by Hegel in his mature logic.

As in Plato's Philebus[22], everything (for Proclus) is a ``conjunction'' of the unlimited and limit, though some things lean more to one than the other. So for example, the point, as completely simple, fixed in its position, is all limit. But as we will see later, the point, for Proclus, does have an unlimited side. This side is implicit, ``secret'', but is made explicit in its expansion, expressed in the point-line-surface-solid development. On the other hand, the idea of ``body'' has more of the unlimited in it, and so is bounded from without (rather than intrinsically in itself) and is ``divisible to infinity in

[17] *P*, p.xxxii; *Hea.*, p.29.
[18] *P*, p.4.
[19] Note the privative character of απειρος: the infinite as απειρος is a potential, not an actual, infinite. The infinite in Euclidean geometry is entirely of this kind.
[20] *Metaphysics*, 986a22-26.
[21] *P*, pp.88, 120.
[22] *Philebus*, 16d.

all directions''.[23] Above limit and the unlimited lies the *One*, the ineffable, incomprehensible, source of all that is, including the mathematical beings. Reality flows in its orders out from the One (procession), and reverts back to it (recession).

Beneath the two principles of *limit* and *the unlimited*, Proclus,[24] following Plato's *divided line* imagery of the *Republic*, distinguishes three levels of reality: (1) the ``highest intelligence'' *noesis* (νόησις) or *Nous*, (2) the level of discursive thought or understanding, *dianoia* (διάνοια), and (3) the world of sense perception and its thought in the form of *opinion*, *doxa* (δόξα). The three levels decrease in order of clarity and comprehension, the top level (noesis) being, as Morrow expresses it, ``immediate, total and sure comprehension'', to which, I suppose, the Kantian intuitive intellect, that was such strong motivation for Schelling and Hegel, corresponds. At the next level, that of the understanding (which corresponds, at a fundamental level, to the Kantian *Verstand*[25]), is the world of *mathematics*, of geometry in particular.

These three levels are cognitive in character, and a natural, pressing philosophical problem for mathematics (even today) is: *where does the geometer get his objects - triangles, circles, squares etc. - and how do these relate to their sensible counterparts?* So, for example, how does *the* mathematical triangle relate to one drawn with a pencil and straight edge (or in the sand). The first triangle is exact and unchanging, having infinitely thin, perfectly straight sides, while the second is drawn through a temporal process, can be erased, is ``rough'', and the lines are not exactly straight and have varying positive breadths. The solution to this problem for Proclus is as follows. The objects of mathematics come[26] by imaginative projection of mind (soul) from the vast store of forms implicit within the noetic world and of which the world of sense perception only provides, in drawn diagrams, a reminder (ανάμνησις). (As Morrow comments in his introduction to *P*,[27] this theory of projection anticipates the Kantian schematism of the categories.)

So for Proclus, there are *three* levels at which the concept *triangle* can be taken. There is first the *exact* triangle which is at the level of forms. Since a form is completely simple, this formal triangle is *completely simple*. Then there is the *mathematical* triangle which is a projection of the formal triangle in imagination (with some kind of extendedness and matter). Lastly, there is the *sensible* triangle which is a projection of the mathematical triangle into the world of sense perception, with the rough and ready nature of the drawn figure. Through the medium of the sensible triangle, we can ``see'' its originals, the *genuine* mathematical triangle and the simple formal triangle behind that, and this explains why, in geometry, the proofs are accompanied by drawn figures. Against the background of this projection theory of *P*, Proclus works out in great detail his interpretation of *Euclid 1*.

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[23] *P*, p.72.
[24] *P*, p.xxxiv.
[25] But Kant's *Verstand* is not as ``*pure*'' as the *dianoia* of Plato and Proclus. For Kant, Plato leaves the sense world and ventures *in den leeren Raum des reines Verstandes* (*Kritik der reinen Vernunft*, Einleitung III.)
[26] *P*, p.11 and pp.41-45.
[27] *P*, p.xxxv.

(c) *NeoPlatonism in Hegel's Frankfurt period*

*Neoplatonism* was, we may say, ``in the air'' at the end of the 18th and the beginning of the 19th centuries in German thought. In particular, in its aesthetic form, it features strongly in the work of Hölderlin (for example, in his *Hyperion* of 1797-1799) and Schelling (in his *Bruno* of 1802). Hegel came to Frankfurt in 1797 at the invitation of his friend Hölderlin, who was also a *Hofmeister* there. Hölderlin greatly influenced Hegel[28] in their time together at Frankfurt, and so it is reasonable to attribute to Hölderlin a NeoPlatonic influence on *GS*.

In this connection, Klaus Düsing[29] discusses the influence of the aesthetic Platonism of Hölderlin on Hegel at that time. Aesthetic Platonism, in which the beautiful is given the highest ontological status, was developed in the NeoPlatonic Florentine academy by Marsilio Ficino and Leone Ebreo. As Düsing points out, Hegel used an edition of Plato that contains Ficino's translations, and also showed an interest in Ficino's *Platonic Theology*. Further, earlier in his life at the Tübingen *Stift*, Hegel, with Hölderlin and other friends, studied the writings of Plato (as well as those of Kant, Jacobi and Spinoza), and even made his own attempts to translate some of Plato's work.[30] So the sustained common interest of Hölderlin and Hegel in the work of Plato over a number of years is well documented.

However, the ecstatic aesthetic idealism of Hölderlin's *Hyperion* seems almost completely absent from Hegel's work during the Frankfurt (or at any other) period.[31] In fact the bulk of his writings in the Frankfurt period are on religious and political themes rather than on aesthetic ones. The striking exception is the *ältestes Systemprogramm des deutschen Idealismus* (The earliest system-program of German idealism), with its enthusiastic endorsement of the aesthetic, in particular, of poetry (*Dichtung*). However, the authorship of the *ältestes Systemprogramm* is controversial – Hegel, Hölderlin and Schelling have been suggested, with the consensus of scholars, including O. Pöggeler and H. S. Harris, settling for Hegel.[32]

Since also, as far as I can see, Hölderlin had little or no interest in Euclidean geometry, it seems to me that the NeoPlatonic background to *GS* has to come from elsewhere. In our discussion of this, we will first describe how the thought world to which Hegel belonged can be regarded as

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[28] See, for example, p.80 of: T. Pinkard, *Hegel: A Biography*, Cambridge University Press, 2000, Cambridge.

[29] ``*Ästhetischer Platonismus bei Hölderlin und Hegel*'', by K. Dusing, pp.101-117 of: *Homburg vor der Höhe in der deutschen Geistesgeschichte Studien zum Freundkreis um Hegel und Hölderlin*, ed. C. Jamme und O. Pöggeler, Klett-Cotta, Zweite Auflage, Stuttgart, 1987.

[30] p.40 in: Rosenkranz, *ibid.*.

[31] There is no doubt, though, about the profound friendship of Hegel and Hölderlin in the Frankfurt period and of Hegel's knowledge of and interest in *Hyperion*. In a luncheon conversation in Berlin, 1830, Hegel talked ``with great spiritedness'' about his Frankfurt memories of Hölderlin and of Hyperion (pp.626-627, Pinkard, *ibid.*).

[32] But Pinkard (p.136, *ibid.*) says that ``the author, however, is most likely Hölderlin''. The issues involved in the authorship of the manuscript are discussed in: *Mythologie der Vernunft: Hegels ältestes Systemprogramm des deutschen Idealismus*, edited by C. Jamme and H. Schneider, Suhrkamp Verlag, Frankfurt am Main, 1984. This book contains a discussion of the authorship of the manuscript by O. Pöggeler. Harris's discussion of the authorship is given in *TS I*, pp. 249f.. Harris also gives an English translation of the manuscript in *TS I*, pp.510-512.

a transformed NeoPlatonism,[33] and then second, will discuss the possible influence of Proclus on Hegelian geometry.

(d) *Post-Kantian philosophy as a transformed NeoPlatonism*

The title of this section must not be taken to imply that post-Kantian philosophy in Germany is just a *repetition* of NeoPlatonic thought. The former is far more wide-ranging with many new ideas. What we are saying is that Neo-Platonic thought greatly influenced post-Kantian philosophy, and hence there is a basic common pattern of thought in the two philosophies.

The role of the NeoPlatonic One is taken over in the Kantian philosophy by the transcendental unity of pure apperception, but with the crucial difference that the manifold content to which this unity applies is given in external intuition, and the unity is empty without that manifold content which it unifies in concepts, judgements and syllogisms. (There is nothing of this dependence on sensible intuition in classical NeoPlatonic thought - rather, as for Plato, sense perception is a transitory, derivative form of knowledge, whose primary function is to point back to the genuine world of pure forms on which any validity that sense perception has is ultimately based.)

The Kantian philosophy gave rise to antinomies through the role played in it by the *Ding-an-sich*, the reality, independent of the observer, that is the source *somehow* of the manifold content on which the unity of pure apperception goes to work and of the distinction between the *noumenal* and the *phenomenal*. Fichte's modification of the transcendental philosophy and the influence of Kant's third Kritik led, in post-Kantian thought, to the abolishment of the Ding-an-sich, and with the return of dialectical thought, a view of the world emerged that was closer to that of the NeoPlatonic. Concept and intuition were now no longer completely separated, and instead, in Schelling's form of the Fichtean system, cognition and being are integrated in the *absolute* which is ``*the point of indifference*'' (using the language of magnetism) or *absolute point of identity*.[34] Thought and being are then not to be conceived of as radically different but rather each is through the other and they are just different forms of the same ultimate reality or absolute. Schelling says that ``we meet with nothing in philosophy but the Absolute – always just strictly the One, and just this unique One in particular forms''.[35] Recalling that the source of the world in NeoPlatonic thought is also the

[33] Neo-Platonic interpretations of Hegel's thought have been subject to criticism. For example, Clark Butler (*Hegel's Logic*, Northwestern University Press, 1996) criticizes ``J. N. Findlay's NeoPlatonic interpretation of the Hegelian system as the endless, asymptotic approach to the absolute as a transcendent one which can only be intuited'' (p.6, *ibid*.). There is a danger in oversimplifying Neo-Platonism so that it becomes the One processing into the world and reverting back to itself, giving the impression that that is all there really is to it. In fact classical Neo-Platonism is remarkably complex and detailed. One need only think of the intricate levels of reality expounded in Proclus's *Elements of Theology*. For our present purposes, the long, Neo-Platonic explanations of geometry, given by Proclus in his Commentary on the *Elements*, provide a remarkably coherent philosophy of geometry, and deserve the most serious consideration.

[34] A brief discussion of this is given in Chapter 4.

[35] Translation by H. S. Harris from the introduction to Schelling's lectures on the Philosophy of Art - p.54 of: G. W. F. Hegel, *Differenz des Fichte'schen und Schelling'schen Systems der Philosophie (The Difference between Fichte's and Schelling's system of philosophy)*, English translation by H. S. Harris and W. Cerf, State University of New York Press, Albany, New York, 1977.

*One*, we see that Schelling's identity philosophy and the NeoPlatonic philosophy have a common basis.

And it is, I believe, this fundamental identity of basis for both the NeoPlatonic and the identity philosophies that ennables Hegelian geometry to use both. The Platonic and NeoPlatonic investigations into geometry, that depended on the method of division, the Pythagorean categories and classical dialectic, in Hegel's treatment of geometry, became a post-Kantian philosophy of concepts functioning within an identity philosophy. In the very early *GS*, the NeoPlatonic framework dominates, but four years later, in *JS*, the NeoPlatonic philosophy and the identity philosophy are integrated into a philosophy of the *Concept*.

We now turn to the source of the NeoPlatonic influence on *GS*. In particular, what can we say about the specific influence on it of the Proclus commentary *P*? This question is discussed in (e) below. A difficulty for establishing *directly* a connection between *GS* and *P* is that nowhere in *GS* is there any explicit reference to the NeoPlatonic tradition in mathematics. (Even in his later logical works, *JS*, *WLg*, *WL*, there are very few explicit references to NeoPlatonism.) However, I am certain that the philosophy of *GS* is to be understood in terms of *P*. Summarizing the discussion of (e), I do not have *conclusive* evidence that Hegel studied *P* when thinking about *GS*, and it may be the case that the influence of *P* on Hegel was only indirect (in particular through Kepler). But I think it *likely* that Hegel was influenced directly by *P*.

(e) *The possible influence of Proclus on Hegel's geometrical thought*

Hegel may have been first introduced to the Proclus commentary in his time (1788-1793) at the Tübingen Stift. There, Hegel was taught mathematics and physics by C. F. Pfleiderer who was apparently a good mathematician and according to Betzendörfer[36] an admirer of the Greek mathematicians. It seems certain that a serious scholar of Greek mathematics, such as Pfleiderer, would have studied the Proclus commentary and may well have brought it to the attention of his students, one of whom was Hegel.

I think that it is almost certain that Hegel became aware of the commentary *P* for himself in the work of Kepler while he was at Tübingen.[37] Kepler had been, like Hegel, a scholar at Tübingen, and both were Swabian and (somewhat unorthodox) Lutherans. Kepler was strongly influenced by *P* and makes frequent references to it. He even starts the 1st book of his famous work *Harmonices Mundi* with a quotation in the original Greek from *P*, and says in the introduction to that book that he would not have needed to write the work if Proclus had written a commentary on the 10$^{th}$

[36] See *TS I*, p.81.

[37] Rosenkranz (*ibid.*, p.152) says, in the context of his discussion of *Diss.* (presented within a year of the completion of *GS*): ``Von Kepler's Harmonia mundi war Hegel tief durchdrungen.'' But it seems very likely that Hegel's reading of Kepler goes back to his Tübingen days. In this connection, Rosenkranz (*ibid.*, p.151) says, again in his discussion of *Diss.*: ``Auszüge aus Kant's Schriften zur Mechanik und Astronomie, aus Kepler, Newton u. A. finden sich bei ihm schon viel früher''. The influence on Hegel of Kepler's treatment of the classical Greek geometers is exemplified in *WL I*, p.311 (*SL*, p.303) where Hegel discusses Kepler's views on the work of Archimedes on measuring the circle (*Kreismessung*).

book of Euclid's *Elements*.[38] Hegel's admiration for and loyalty to Kepler is well-known, and it resulted in polemic against Newton. Although there is no mention of Kepler in *GS*, there is enthusiastic praise for him in *Diss.*.[39]

There is no doubt that in later life at least, Hegel studied and thought highly of the philosophy of Proclus. In his correspondence with the symbolist Creuzer,[40] he praises the work of Proclus, in particular, the *Platonic Theology*, as ``the true turning point or transition from ancient to modern times, from ancient philosophy to Christianity''. In Hegel's *History of Philosophy* lectures,[41] the work of Proclus is approvingly discussed, again with particular reference to the *Platonic Theology* as well as to the commentaries of Proclus on the dialogues of Plato. Hegel also refers in passing to the ``several mathematical works'' of Proclus. Hegel rated Proclus even higher than the great founder of NeoPlatonism, Plotinus.

As commented earlier, I think it likely that Hegel read and was directly influenced by *P* in his thinking on geometry. However, whether the influence was direct or indirect, *P* provides, I think, an indispensable framework for discussing *GS*. With the objective of using that framework to help understand *GS*, we now briefly sketch some of the themes of *P*, and describe the conceptual forms that these take in Hegel's own philosophy of geometry. We shall also point out some of the significant *differences* between the geometrical philosophies of Proclus and Hegel.

(f) *Proclus and Hegel: the philosophical basis for mathematics*

Mathematics for Proclus proceeds on the basis of certain assumptions (or hypotheses) - e.g. axioms - which are unjustified within the subject. The justification for these assumptions lies in the ``unhypothetical science'' of Plato's *Republic*[42] which Proclus (following Aristotle) identifies with metaphysics (first philosophy). Like Plato and Proclus, Hegel also sees mathematics as based on assumptions which are justified through philosophy: he says in *NP*

> Man könnte noch weiter den Gedanken einer *philosophischen Mathematick* fassen, welche dasjenige aus Begriffen erkännte, was die gewöhnliche mathematische Wissenschaft aus vorausgesetzten Bestimmungen nach der Methode des Verstandes ableitet.[43]

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[38] The influence of Proclus on Kepler is discussed in detail on p.99 and pp.167-170 of: J. V. Field, *Kepler's Geometrical Cosmology*, The University of Chicago Press, Chicago, 1988.

[39] In the dissertation, Hegel refers to the *sublimem Kepleri legem*.

[40] For Hegel's correspondence with Creuzer, with reference to Proclus, see p.368f. (letter 400) and p.466 (letter 389) in: G. W. F. Hegel, *Hegel: The Letters*, trans. by C. Butler and C. Seiler with commentary by C. Butler, Indiana University Press, Bloomington, 1984. Another friend of Hegel's, Victor Cousin, also translated works of Proclus (C. Butler and C. Seiler, *ibid.*, pp.635, 637).

[41] pp.434-435 in: E. S. Haldane and F. H. Simson, *ibid.*.

[42] Plato, *The Republic*, 510b-511d. The form of the unhypothetical science in Hegelian mathematics is discussed in more detail in the paper: Alan L. T. Paterson, *Does Hegel have anything to say to modern mathematical philosophy*, Idealistic Studies 32:2, 2002, 143-158.

[43] *NP*, § 259, Zusatz. *trans.* ``One could also formulate the thought of a philosophical mathematics, which cognizes out of concepts the same as what the usual mathematical science derives from presupposed determinations according to the method of the Understanding.''. Hegel expresses similar views in *Diss.*, p.25: ``*lemmata matheseos philosophica mutuanda sunt, atque inde theorematum universam fere mathesin adplicatam fundantium, et ad nostram usque aetatem veris demonstrationibus, quae mathematicae fieri*

While Hegel thinks that mathematics should not be mixed with the concept (*Begriff*) - mathematics is a science of the understanding (*Verstand*) - he also says that the concept can clarify the leading principles of *Verstand* itself, and of order and its necessity in arithmetic and in the theorems of geometry. Further, for Hegel, philosophy is definitely needed for a true account of mathematical determinations such as the infinite, infinitesimals and powers. In *GS*, conceptual (not mathematical) explanations are needed for the fundamental concepts of geometry, such as the basic figures of geometry and the dialectical development of point, line, surface, solid. For both Proclus and Hegel, philosophy (the unhypothetical science) also provides justification for the explicitly stated presuppositions on the basis of which geometry proceeds. These presuppositions, stated at the start of Euclidean geometry, are the *definitions*, *postulates* and *axioms* (common notions).[44]

But philosophy is not restricted only to the *beginning* of geometry, and in Hegel's geometrical philosophy, there are later mathematical claims of *Euclid 1* whose proofs need the insights of philosophy. Such an insight is based on the conceptual interplay of *limit/the unlimited*; the latter two are fundamental determinations of the geometry of both Proclus and Hegel. (The Proclus philosophy of limit was briefly discussed in 2(b).) Examples in *GS* of philosophical proofs occur in its *theory of parallels* (3(f)) and its discussions of the *congruence theorems* (3(d)). Another example of special importance, not in *GS* but in *JS* and *WL*, is that of the *Pythagoras theorem*.

Hegel strictly limits such ``philosophical'' proofs in *GS*. They are used only when the geometric proofs given by Euclid are unsatisfactory, and can only be made satisfactory by making explicit the underlying philosophical principles. For the rest of the time, proof in Hegelian geometry is the same as for Euclid, and (as we will see) Hegel even produces some geometrical proofs of his own.

What determines philosophical significance in mathematics for Hegel - and this is true in his thinking from *GS* onwards - is *conceptual identity* in general, and its particular mathematical, quantitative form as *equality*. The identity and equality deriving from the concept is realized through the *limits* which geometrical figures inherit through their determination as *limit/the unlimited*. For example, the triangle is determined through its *limits,* i.e. its sides and angles.

This is the same for Proclus, except that in his case, the *identity* and *equality* are derived through the forms from the *One* (rather than *conceptually*) and that he would also have understood this significance ethically and aesthetically. A good illustration of the role of identity and equality in the geometrical thought of Proclus is his philosophical justification in *P* of the *order* and *paradigms* of the main figures of geometry as they are presented at the beginning of *Euclid 1*. This order

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*nequeunt, carentium demonstrationes deducendae sunt: cujus rei viam notionibus quas exhibuimus tentare voluimus.''*. An example of the application of such a philosophical mathematics of concepts to applied mathematics - not, I fear, a convincing one - occurs before the above sentence in *Diss.*, where Hegel ``derives'' the cubic power occuring in Kepler's third law. For more discussion of this, see Chapter 4 below.

[44] Hegel in *WLg II*, p.222; *WL II*, p.466 (*SL*, p.808) says that the axioms of geometry are theorems taken mostly from logic. (In *WL*, logic, as *Wissenschaft*, is the exposition of the concept: for *all* knowledge, the *Wissenschaft* is Hegel's developed form of the unhypothetical science.)

is derived from the ``perfection'' of the One. The One is self-equal (corresponding to the formal ``A=A'' of the Fichtean philosophy) and so the simplest, highest, geometrical forms of the One are those that share that oneness of character and also (from the identity of the Good, the One and the Beautiful in the Platonic philosophy) are the most beautiful. As one might expect, the *circle* is high in the order of figures. It is for Proclus the paradigm closed curve. The paradigm triangle, for Proclus, is the *equilateral triangle* which is ``the most beautiful of triangles and most akin to the circle''[45] – the use of the *circle* in the construction of the *equilateral triangle* in the first proposition of *Euclid 1* ``seems to indicate in a likeness how the things that proceed from first principles receive perfection, identity, and equality from these principles''.[46]

In more modern language, we may say that the *oneness* of a figure – indicating its closeness to the NeoPlatonic *One* – is expressed through the symmetries of the figure, where a *symmetry* of a figure is a rigid motion (such as a rotation or a reflection) that preserves the figure.[47] For example, rotation through a fixed angle of a circle about its center and in its plane is a symmetry of the circle, while reflecting through a diagonal is a symmetry of the square – each leaves the figure as a whole where it was before. (Of course, a symmetry usually moves around points of the figure – for example, in the square case, two vertices are interchanged by the reflection while the other two are left fixed.) The existence of symmetries entails *equalities* between the parts (limits) of the figure, and the more symmetries a figure has, the higher it ranks in the order of *P*. So, for example, the square is ``superior'' to the rectangle (i.e. precedes it in the order) since the former has more symmetry than the latter: the rotation of the square about its center through a right angle preserves the square, but this is not true for the rectangle in general. Yet every symmetry of the general rectangle is a symmetry for the square since the square is a special kind of rectangle. This superiority can also be seen in terms of equality of the parts: the sides of a square have more equality (all four equal) than do the sides of the rectangle (which are only equal in pairs).

As we will discuss in more detail below, there is, I believe, a similar order of figures in Hegelian geometry, but justified in terms of conceptual unity rather than in terms of order of closeness to an originating One. This order, as for Proclus, is also based on identity and equality. While the order is not given in detail in *GS*, an instructive example of it, as it applies to triangles, is given by Hegel

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[45] *P*, p.167.

[46] *P*, p.167. Under the influence of Neo-Platonism, the ``perfection'' of the equilateral triangle is expressed artistically in the pyramidal schemes of Renaissance Art, e.g. in the equilateral triangle shape of the Virgin's form in Raphael's *Madonna of the Meadows.*

[47] These rigid motions – the *symmetries* of the figure - are dialectical (or Hegelian) in character. Indeed, the concept of the figure is realized through the action on it of its symmetries; through them, the figure identifies itself with itself through its ``difference''. This self-identity or invariance is intrinsic to the figural concept: it is not imposed by us from without. In contrast, in classical geometry, rigid motions were treated in an *external* way, such as in lifting up one figure and putting it on top of another, as in the method of superposition which Hegel criticized (3(d)). Technically, the *orbit* of a figure under the whole group of rigid motions expresses the full concept of the figure. (More generally in modern mathematics, this approach to geometry has achieved precise formulation in *Klein's Erlangen program*, for which geometry is defined in terms of the properties invariant under a group action.)

(after his discussion of Proposition 5 of *Euclid 1*) in the following comment:

> Die Gleichschenklichkeit ist ein viel geringerer Grad der Einheit in der Figur, als die Gleichseitigkeit.[48]

In explanation, we can say that the equilateral triangle is at a deeper (more conceptual) level of unity than is the isosceles triangle since the former has equality of all three sides while for the latter, only two sides are determined as equal, and the third side is arbitrary (unrelated to the other two). The equilateral triangle has more *symmetry* than the isosceles triangle, just as (in the above discussion of the order of figures in *P*) the square has more symmetry than the rectangle. We could also say that the equilateral triangle possesses more *determination* in the relations of its limits (e.g. equality of its sides) than does the isosceles triangle. So, consistent with the fundamental roles of the *unlimited/limit* in the thought of Pythagoras, Plato and Proclus, mathematical significance is, for Hegel a consequence of conceptual *determination* in terms of *equality.*

Determination of the circle is through the *equidistance* of its points from the center. Determination for the straight line is through *constancy* of direction. For both Proclus and Hegel (and inspired by Plato), the circle[49] is primary among closed curves, while the constancy of direction of the straight line makes it primary among the non-closed curves. Both philosophers attach ``deeper'' (and related) significance to these two curves, beyond their foundational role in geometry. Proclus expatiates enthusiastically on the simplicity and perfection of the circle and its character as *Nous*, while Hegel in *JS* sees in circular terms, the *One* as the *soul*, and in *WL*, sees in the circle the image (*Bild*) of the genuine infinite (*wahrhafte Unendlichkeit*).[50] (There is, however, nothing of this in what has survived of *GS*.)

In mathematical conceptual determination lies, I think, Hegel's answer to the important question, largely ignored in present day mathematical philosophy: *what is the source of mathematical significance: why should one theorem be regarded as ``great'' while another is boringly ``routine''?* The answer to this question, i.e. the source of *significance* in mathematics, for both Proclus and Hegel, comes from philosophy (in the form of the unhypothetical science).

We must emphasize again that the concept is not, for Hegel, a Proclian One (though it fulfills a similar unifying role). Nor, in terms of the identity philosophy, is it an ``absolute indifference'' point, sheer blank identity, *die Nacht auszugeben, worin, wie man zu sagen pflegt, alle Kühe schwarz sind* [51], but the concept is as much *difference* as it is *unity*, and in its articulation, is a whole of parts. It has also to be

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[48] t*rans.* ``the property of being isosceles is [at] a much more superficial level of unity for a figure than is [the property of being] equilateral.''

[49] e.g. *P*, p.104.

[50] *P*, p.117f.; *JS*, p.140 (*JL*, p.147); *WL* I, p.138 (*SL*, p.149). In the latter reference, Hegel's thought of the infinite as a circle ``*ohne Anfangspunkt und Ende*'' echoes the thought of Proclus, in which the``divine'' procession and reversion is presented as maintaining a circle ``without beginning and without end''. (Proposition 146 of: Proclus, *The Elements of Theology*, trans. E. R. Dodds, (Clarendon Press, Oxford, 2000).)

[51] This is a famous, critical comment about the identity philosophy made by Hegel in the preface to his *Phenomenology*.

stressed that the crux of NeoPlatonism, viz. how a perfectly simple, apart, *One* can give rise to *difference* (e.g. in the variety of forms) is not a problem for the Hegelian concept. For difference is intrinsic to that concept. The inadequate approach to difference in NeoPlatonic thought is, as we shall see, paralleled, for Hegel, in Euclid's own imperfect view of difference (3(d)).

(g) *Proclus and Hegel: point, line, surface and solid*

To illustrate the role of philosophy (the unhypothetical science) in the foundations of geometry for Proclus and Hegel, we now discuss their views, in *P* and *GS*, on the basic geometrical concepts, *point, line, surface and solid.* (Note here that the German word *Fläche* that Hegel uses can mean either plane or surface – the plane is just the *flat* surface, the paradigm surface. We will use either word to translate *Fläche* depending on the context.)

Appropriately, Euclid starts his *Elements* with the definition of a point. His definition is: *a point is that which has no part*. The explanation of this, for Proclus, is as follows. All levels of reality originate in the *One*. The One is completely simple, and reality is derived from the potentiality of the One through the determinations of the categories of *limit* and *the unlimited* (as we saw above). So mathematical objects, intermediate between sensible and intellectual objects, announce ``their unitary and undivided and generative reality''[52], the One. ``Enmattered'' reality in the world is the *body* or *solid*, e.g. tables and trees, and to specify, for the solid, the originating principle of the One in its geometric form, we have to go back through the sequence of *limitings* that gave rise to it.[53] The limit of the solid is the (curved) surface, as it is the *surface* that bounds the solid. Similarly, the bounding limit of the surface is the *line* (curved), e.g. the semicircle that bounds a hemisphere, while that of the line is the (end-)*point* (of which there are two), and there we come to a stop. The point is completely simple, determined by limit[54], *without parts*, and it is there that we have the One, or the simplicity of *Nous*, in geometric form.

For Proclus, the point is the ultimately simple element of geometry though not of mathematics as a whole. The number of arithmetic is ``purer'' and more fundamental than that of the point, since it has its existence in thought alone, and is ``free of any extraneous figure''. The point, on the other hand, is projected through imagination so that it has *position* in addition to its unitary character.[55]

As completely determinate and simple, the point is (in terms of the basic Pythagorean categories) given entirely as *limit*. But as everything is, as we saw earlier, a combination of limit and the unlimited (for, to Proclus, all forms partake of both limit and the unlimited), the point does have an unlimited side. However this side is initially only *potential*, becoming explicit in the process through which point determines other figures. For Proclus[56], the point ``secretly possesses the nature of the unlimited and strives to be everywhere in the things

[52] *P*, p.4.
[53] *P*, p.70.
[54] *P*, p.72.
[55] *P*, p.78.
[56] *P*, p.72.

that it bounds.'' This is why ``it is present in them an infinite number of times''.

So just as the One generates everything ``below it'', so its geometrical form as the point ``generates'' all of the geometrical figures, their bounding being in terms of its primary productions, *points*, *lines*, and *planes*. Proclus describes this generation in Pythagorean[57] terms, as realized through movement: through that, the point generates the line (the ``flowing of a point''), and in turn, the line through its movement generates the plane, and the plane through its movement generates the solid. We can think, for example, of a line rotating about a point on it (like the spokes of a wheel about its center) and sweeping out a plane. This generation reverses the direction of the *limiting* process above, where we went from solid to surface to line to point by taking boundaries. The generation process procedes through *unlimiting*, in which, for example, the point breaks out of its limitation and is no longer completely simple. (Proclus also discusses an alternative Pythagorean view of point-line-plane-solid in terms of the ``extendedness'' of the numbers 1,2,3,4.)

Hegel, in *GS* and indeed in his mature logic, argues for a dialectical version of this Pythagorean generation of point, line, plane and solid.[58] For the present, we will discuss his version briefly in relation to that of Proclus, and will examine it in more detail later (3(a)).

Starting with the *point*, Hegel says in *GS*:[59] *Der Punkt ist die einfachste Begrenzung der Unendlichkeit* (``The point is the simplest limitation of the infinite''). We can see this as a formulation of the view of Proclus that the point is the ultimately simple in geometry and contains within it the potentiality of the unlimited. In the Jena Logic, Hegel develops this further: he says of the world process (*Der Proceß der Welt*) that it is:[60]

> ein einfaches in seiner unendlich vielfachen Bestimmtheit; welche schlechte Unendlichkeit, unmittelbar als absolute Unendlichkeit, als Eins, als Punkt gesetz ist.

This ``bad infinity'' (*schlechte Unendlichkeit*) is the Hegelian version of the unlimited of Proclus which is ``secretly'' within the point, and becomes explicit through the generation of line, surface and solid. Note also in the above passage the ``multiple'' (*vielfachen*) characterization of the point (the simple). This corresponds, in the thought of Proclus (above), to the point being ``present in them [things] an infinite number of times''.

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[57] *P*, p.79-80. Walter Burkett (p.68f. in : *Lore and Science in Ancient Pythagoreanism*, trans. E. L. Minar, Jr., Harvard University Press, Cambridge, 1972) says that the derivation point-line-plane-solid through movement is probably due to the Pythagorean geometer, Archytas. Aristotle, however, attributes it to Platonism. (The doctrine may have entered Platonism through the influence of Archytas, since Plato and Archytas were friends (Plato's *Letters* 7, 9 and 12).)

[58] In his *Hegels Leben*, *ibid.*, p.100, Rosenkranz says that while at Frankfurt, Hegel studied intensely (*viel*) the writings of Sextus Empiricus. Hegel may well have first met the Pythagorean theory of point, line, plane, and solid in these writings. Sextus discusses the theory in: *Adversus mathematicos*, 10.281f..

[59] *GS*, p.294.

[60] *JS*, p.145; *JL*, p.151. *trans.* ``a simple in its infinite multiple determination; which bad infinity is posited immediately as absolute infinity, as one, as a point.''

Hegel gives us an explanation in *GS* of why we must stop at the solid[61] in this development. The reason is that *the solid is not limit*, while point, line and surface are. It is *limit* that drives on the development, since the limiting ``traps'' an unrealized infinite ``within'' it. It is this infinite which is expressed in the next stage of the development. Where there is no limit, there is nothing more to be expressed. For this reason, the limitless solid does not push forward to a fourth dimension, and space (for Hegel) is three-dimensional.

(h) *Proclus and Hegel: the nature of geometrical figures*

We saw above that for Proclus, the point as the *One* in geometry is given as limit, but also has the unlimited potentially within it. The point is the source of geometrical limiting and the infinite in the objects that it determines. At a deeper philosophical level, the *originating* One – not just its geometrical form as the point – is expressed at the highest level of *Nous* through the *forms*, and the mathematical objects (``mathematicals'') produced in turn from the forms by imaginative projection and studied through the understanding. In two dimensions, these mathematical objects are the figures; drawn geometrical figures are only reminders in sense perception of these objects and are indispensable helps for understanding them.

Each figure is a specification of limit and the unlimited, the bounding lines determining the figure as *limiting*, but also determining the spatial region outside as the *unlimited*. The order of figures then follows the number series, based on the number *n* of bounding curves, and for each such *n*, the figures are listed in order according to the quality of their self-identity and equality. The first of these figures (for a given *n*) is the paradigm. The paradigm figures are those mathematicals which are ``closest'' to the form of the circle and hence also closest to the One – they are the most ``One-like'', the most symmetrical (2(f)). If *n=1*, then the paradigm figure is the circle, if *n=2*, it is the semicircle, if *n=3*, then it is the equilateral triangle, and if *n=4*, then it is the square.

Turning to Hegel's view of geometrical figures in *GS*, we find similar themes, with paradigm figures determined by self-identity and equality within a theory of limit and the unlimited. In 2(i) we will compare in detail the views of the two philosophers on four of the canonical figures. For the present, however, we discuss a fundamental *difference*, alluded to in our earlier discussion, between Hegel and Proclus, viz. Hegel's treatment of the figures within a *conceptual*, rather than *NeoPlatonic*, framework.

The difference between the Proclus and Hegel views of figures can be helpfully indicated with reference to *WL*.[62] There, the concept of space is abstract, and conceptually posited in it are the *Gestalten und Figurationen* (``shapes and figures'') of geometry. Ontologically, these resemble, for Hegel, the purposive *ends* of teleology and are to be distinguished from the *concrete* things of Nature and Spirit. In

[61] Hegel follows Aristotle here in the latter's view of the spatial dimensions in *De Caelo*, 268a8 (in: *The Complete Works of Aristotle*, Vol. 1, ed. J. Barnes, Princeton University Press, Princeton, New Jersey, 1984): *A magnitude if divisible one way is a line, if two ways a surface, and if three, a body. Beyond these there is no other magnitude, because the three dimensions are all that there are ... .*

[62] *WLg II*, p.211; *WL* II, p.453 (*SL*, pp.796-797).

discussing the philosophical character of a geometrical figure, we have to distinguish clearly between what its defining *limits* are and what is conceptually *posited* (*gesetzt*) in it. What is given as limit (such as the vertices and sides in the triangle case) determines the figure. Further, it is the philosophical development of point, line, surface, solid that makes available these limits.

But more is posited in the figure than just the limits. For example, consider the surface of a (hollow) sphere. (We will look at this more closely in 2(i)(α) and 3(c).) The surface is limit in this case and there is no other limit. Now by definition – at the conceptual level – a sphere is a surface whose points lie at a fixed distance R from a given point, the *center*. The center does not lie on the sphere - instead, in Hegelian language, the center is *posited* in space (although it is not *limiting* for the sphere). Another example would be that of the triangle, where points (such as the *centroid*) and lines (such as the medians) are logically determined, even though they are not explicitly given for the triangle in its definition as a figure with three limiting sides.

So while the limits of a figure do determine it in space, its *conceptual basis* involves much more. And it is this basis for figures that gives to geometry the interesting insights formulated in its theorems. It is also this basis that gives ``permanence'' to the figures posited. For example, in geometrical construction, it is this that ennables a triangle to retain its integrity despite being broken up by lines drawn through it.[63]

Proclus, following Plato, interposes, as we have seen, between the form and the sensible figure the *mathematical* produced through imagination. But Hegel's conceptual view of figures does not require mathematicals. Indeed, his sympathy with Aristotle and the latter's antipathy to the mathematicals of the early Academy suggest that the thought of a separately existing mathematical with its strange kind of matter would have been unacceptable to him.

For Proclus, the ``genuine'' triangle is the pure form. It is completely simple, and separated out into parts only at the level of the imaginatively projected mathematical triangle, the latter then being represented as a drawn triangle of sense perception. But in the Hegelian position of *GS*, the triangle concept is *from the start* a whole of interrelated parts. For Hegel, the concept of a figure is the holding together of its parts (=limits), or the *necessity* of the figure, what gives it its coherence. Since whole and parts are together initially, Hegel's philosophy involves, as was discussed above, *difference*. I think that Hegel would have criticized Proclus as well as Euclid for his inadequate view of difference .

We now discuss more closely how the concept of a figure in *GS* relates to that figure through its limits. According to Hegel,[64] d*as Gesetz, die Einheit der Figur, ist das Verhaltnis der Grenzen.* This relation (*Verhaltnis*) is the determining of the figural concept in the figure through its limits. The relation is called by Hegel *die räumliche*

[63] Construction is discussed in *JS*, pp.113f. (*JL*, pp.118f.). There Hegel points out that construction is the presentation of a something (*Subjects*) divided in itself *das in der Vielheit es selbst bleibt*.

[64] *GS*, p.294. *trans.* ``the principle, the unity of the figure, is the relation of its limits''.

*Beziehung*[65] (the spatial relation). The concept *makes determinate* the figure by the way that it *relates* the boundaries of the figure. Since the figure is given by a *limitation* of the spatial relation, the figure is posited ``outside itself'' in its *exterior*, just as the point posits itself outside itself in the *line*.

It is important not to see the determining of geometrical figures for Hegel (or for that matter, Proclus) in modern, Cantorian, set theoretic, terms. For Hegel, the spatial relation for the triangle is given *conceptually*, not extensionally as a set of points.[66] The points that *are* determined are so, either through an initial positing (e.g. the center of a circle referred to in the latter's definition) or by explicit construction (e.g. determining the center of a rectangle by first joining opposite vertices to construct the diagonals and then specifying the center as their point of intersection).

The Hegelian *spatial relation of limits*, translated into a *relation* in the modern, set theoretic sense, gives each kind of figure as an *equivalence relation* on the plane. Let us illustrate this for the case of a circle. We say that two points in the plane are *equivalent* if they are equidistant from the origin. This is an *equivalence relation* on the plane in the modern sense, and its equivalence classes (*sets*) are just the circles in the plane centered at the origin. The difference of the modern equivalence relation from that of the Hegelian spatial relation is obvious: in the modern approach, circles (and figures in general) are regarded as highly infinite *sets* of points, rather than, as in the Hegelian approach, given as conceptual wholes specified through a philosophy of limit and the unlimited, whose conceptual structure, suppressed through the defining limitation, is brought to light through geometrical theorems and constructions.

(i) *Proclus and Hegel: the geometrical paradigms – (α) the circle and straight line, (β) the equilateral triangle and (γ) the square*

(α) *The circle and straight line*

The One in geometry, for Proclus, is, as we have seen, the point. Like the One, the point, while primarily limit, is also potentially the unlimited, and this potential is actualized in the ``flowing of the point'', through which it takes linear form. This form assumes two primary determinations, according to the principles of limit and the unlimited. As *limit*, the point is the *circle*, and as *unlimited*, it is *the straight line*. These are the canonical figures with one boundary. As appropriate for paradigms (cf. 2(f)) each of these is characterized by

[65] *GS*, p.295.

[66] The relation between extension and concept is posited in the Fregean philosophy through the *extension* and *abstraction* axioms, by which two sets are the same if they have the same elements and a concept (property) determines a unique set. The contradictory character of the axiom of abstraction (as in Russell's paradox) led to the downfall of Frege's foundation for mathematics. Hegel would, I think, have criticized modern set theory for separating a set from its elements in an external way. We can, I think, compare set-elements to the moment pair continuity-discreteness of quantity which Hegel discusses in *WL*. Hegel says (*WL I*, p.194 (*SL*, p.200)) that each moment contains the other: *jede dieser Größen beide Momente, sowohl die Kontinuität als die Diskretion an ihr hat und ihr Underschied nur dadurch konstituiert wird...* Hegel (and most of the mathematicians of his time) overemphasized magnitude in their approach to mathematics, and his treatment of the quantitative needs to be extended to incorporate concepts such as *sets*. For example, one could argue that the set is the continuity of its elements and its elements the discreteness of the set.

simple *equality*: the circle with equality of radii and the straight line with equality of direction at every point on it. The straight line is unlimited in the obvious sense: it ``goes on for ever'' and is never completed. In Euclidean geometry, we never deal with the complete line but just a line drawn ``long enough''. The line in Euclidean geometry is only potentially infinite.

The circle is, I suppose, *limit* for Proclus in that it bounds two regions of space, the inside and the outside and does not go off to infinity in any direction. More important for Proclus is the way that the circle reflects the character of the One. For him, the circle ``returns to itself'' and ``of its own accord, wraps itself back into unity''.[67] One way to interpret these comments is as follows. As we go round the circle we come back to where we started: through its own nature it identifies itself with itself again, imitating the One whose nature it expresses. This is illustrated, for Proclus, by the ``circular revolution of the heavens''. However, at a deeper level, this ``return'' of the circle to itself follows from the dialectical character of its *center* in linear expansion along the radii and its reflection back to itself.

This radial dialectic can be seen as following from the *definition* of the circle: the circle is (*Euclid 1*, Definition 15) a plane figure bounded by a closed curve such that there is a point (the *center*) within it with the property that distances R from the point to the points on the curve are the same. We note that in this definition, there is reference to a point (the center O) and the straight lines (the *radii*) that measure the distance R from O to the points on the curve. This requires philosophical explanation, and for Proclus, it is in terms of figures being a ``mixture'' of limit and the unlimited. Precisely, the point O is the geometrical ``One'' of the circle, and by the dialectic of point-line-plane-solid, it must generate the *unlimited*, i.e. *straight lines*. These straight lines are just the radii (or diameters) of the circle, ``flowing out'' from the center till they meet the boundary, the limit, of the circle. As limit, the circle bounds this outwards flow, and the center as One identifies with itself again through returning from the boundary back to O again along the radii. In this way the circle and its center realize in geometrical form the *One* in its activities of progression and reversion.[68]

However, as commented earlier, for Proclus, the form has to be distinguished from its mathematical projection. So, for example, the geometric circle, which is the *figure* of a circle, has to be distinguished from its noetic paradigm (its form). In the figure, the parts - the center, radii and circumference are all in different *places*. But they are all present at the *same* time. In the paradigm, however, not only are the parts present at the same time but also they are not distinguished spatially either - ``up there they [the parts] are all in one''.[69] (Spatial difference is not preserved in the ``genuine'', the ``truly real'', circle.)

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[67] *P*, p.122.

[68] This philosophical interpretation of the circle is remarkably close to the Fichtean image of the *Anstoss*, in which the self – thought of as a self-constituting point - posits itself absolutely in both centripetal and centrifugal directions. (See Fichte's *Wissenschaftslehre*, Part III, Second Discourse.)

[69] *P*, p.122.

Turning to Hegel, there is no doubt that for him, as for Proclus, the circle is of great philosophical (as well as geometrical) importance, and for similar reasons. It seems to me almost certain that Hegel discussed the circle in the original version of *GS*, but that discussion has not survived. However, in his discussion of the sphere in *GS*, Hegel refers to the role of the ``great circle'', which he says:[70]

> in sich zurückkehrende Linien, in sich zurückkehrende Fläche hat, wie die Kreise keine Punkte, so sie keine Linie zur Begrenzung; im Kreise liegt die Grenze (der Mittelpunkt, Diameter) außer dem, welche sie begrenzt.

Hegel notes that the choice of any other point in the sphere is arbitrary (*willkürlich*) and so is also not a limit of the sphere.

We can clarify the *GS* view of the circle ``as the relation of its limits'' by saying that it is through its limits (center and diameter) that the symmetry, which is the self-equality of the circle, is realized. The center, a *point*, generates (2(g)) the *line.* In the circle case, this line is the *radius* or, if we think of the generation in both directions, it is the *diameter*; the latter, in its turn, through rotation about the center sweeps out a circular region with its boundary the circle. The circle is then the relation of its limits in the sense that by *relating* the center limit and the (generic) diameter limit through rotating the latter about the former, we get the circle.

The expansion process from the center along the radii, that gives the circle, is reversed through contraction of the circle back along the radii. This is, I think, how we should understand *zurückkehrende Linien* in the above quotation from *GS*. The line through which the circle ``returns'' to the point is, of course, the radius. With this interpretation, Hegel's view of the circle is then seen to be a development of the Proclus view. But whereas, in the Proclus view, the circle is seen in terms of the procession from and reversion back to the One (as represented in the center), the Hegelian view is conceptual in character and relies on his theory of limit and negative being.

Hegel's view that the center and diameters together give the limit of the circle is puzzling. Obviously, the end-points of a (finite) line are the limits of the line. But in the case of a circle, the center, one of Hegel's proposed limits, is at a radius's distance away from the points on the circle – so how can the center possibly be a limit of the circle? (I have not been able to find in *P* the Hegelian view that the center is a limit of the circle.) Further, for the sphere itself, Hegel says that center is *not* limit: *der Mittelpunkt nicht zur Kugel, und ist nicht Grenze*. Why is the center a limit for the circle case but not for the sphere case?

We can understand this (somewhat speculatively) in terms of the difference between the circle and sphere for the philosophical foundations of geometry? The circle arises at the *beginning* of geometry *directly* out of the point (as the line in its limited form), whereas the sphere arises much later in the conceptual development of geometry: the

[70] *GS,* p.295. *trans.* `` has, in itself the returning line [a diameter], [which has] in itself the returning plane [the plane of the circle;] as the circle has no [posited] point [on it], so it has no [straight] line on its boundary; within the circle lies its limit (the center, diameter[s]), external to it, which it [the circle] bounds.''

geometry of points, lines and distances are already established in the case of the sphere but not in the case of the circle. In the definition of the sphere, its center is only *posited* (*gesetzt*) and is not limit. As we saw in 2(h), geometrical *positing* is, for Hegel, to be distinguished from geometrical *limiting*. We can regard the positing as what is *conceptually* required for a geometrical object, but what is posited is not necessarily given as limit of the object for sense perception. So the center of the sphere is not given when we look at a spherical surface: we just know that it has to be somewhere inside the sphere. The same applies to the circle, but the difference is that the circle is *by definition*[71] the dialectical expansion of the center point determined as limit, whereas the sphere is not determined as such.

The simplicity of the form for Proclus corresponds in the later thought of Hegel to the simplicity of the concept over its reality: geometrical reality has a ``superfluity'' over its determining concept.[72]

After his treatment of the canonical geometrical figures with *one* boundary (the circle and the straight line), Proclus moves on to the *semicircle*, the canonical figure with *two* boundaries. The semicircle is bounded, part by the circle and part by the straight line, and so has ``mixed'' boundary, part *limit* and part *the unlimited*. As far as I know, the semicircle does not feature in Hegelian philosophy and so we move on to the figures with three boundaries.

(β) *The equilateral triangle*

For Proclus, the (canonical) figures with three or more boundaries are rectlinear, the closed figures bounded by straight lines. Proclus, commenting on Definition 19 of *Euclid 1*, says[73] that ``the rectilinear figures come forth in orderly fashion according to the series of numbers from three to infinity''. So the rectilinear figures are determined by the *number series* rather than directly from the One through the form (as were the straight line and circle). The canonical figures with three or more sides have straight sides. Indeed, while, of course, there are closed figures (such as the *arch*) with some of its sides straight and others circular, the simplest ones have straight edges. Proclus does not say why this is the case, but perhaps it is because *the unlimited*, in the form of the straight line, dominates at the stage of three or more edges (the combination of circle/straight line occurring, as we saw in (α), at the semicircle stage). Indeed, the number series that determines the rectilinear figures is itself *unlimited*.

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[71] In his later logic, Hegel effectively says (*WLg II, p.224; WL II*, p.469 (*SL*, p.810)) that the circle, at the conceptual level, *is* the Pythagoras theorem expressed as the equation of the circle in analytic geometry: $x^2 + y^2 = r^2$.

[72] In *WLg II, p.223; WL II*, p.468 (*SL*, p.810)), Hegel distinguishes the concept of a triangle from its reality as the Idea and applies it to Proposition 4 of *Euclid 1*. At the level of the Idea, there is a redundancy: you don't really need 3 sides and 3 angles to determine the triangle – two sides and the angle between them is sufficient, and the remaining side and the other two angles are then determined automatically by the triangle concept. So there is an *Überfluß der Realität über die Bestimmtheit des Begriffs*. Thus as in *P* (where ``all is one'' at the form level) the Hegelian mathematical concept has a *simplicity* over its realization in sense perception. However, as commented earlier, the Hegelian concept involves *difference*, while the form of Proclus does not.

[73] *P*, p.128f..

The first of the figures with straight edges is, of course, the *triangle*.[74] Proclus follows the classification of triangles given in Definition 20 of *Euclid 1*, and understands the order of this classification as determined by decreasing identity (=equality). The order is: (1) the paradigm, the *equilateral triangle* (all sides equal, the most equality), (2) two sides only equal (*isosceles*) and (3) none equal (*scalene*). As discussed in 2(f), Hegel, like Proclus, evaluated figures in accordance with equality and identity, and in particular, agreed with Proclus on the priority of the equilateral triangle before the other triangles, in particular, before the isosceles triangle.

The concept of *angle* for Hegel naturally comes into the discussion of triangles (since each triangle comes with three angles, one at each corner). The above classification of triangles can, of course, also be thought of in terms of equality of *angles* rather than of sides. (Hegel's understanding of angles is discussed in 2(j) and 3(a).) The right angle is of particular significance for both Proclus and Hegel. It is the angle made by the vertical (*gnomon*) to the horizontal. It is an axiom of Euclid that there is only *one* right angle.

The right angle is for Proclus the ``measure of angles other than right'' in the sense that it determines the other two basic kinds of angle, *acute* (smaller than right) and *obtuse* (greater than right) – these other two angles are ``indefinite more-and-less''. The right angle, then, procedes from *limit*, while the acute and obtuse procede from the *unlimited*, ``receptive of the more-and-less''. It has the remarkable property that it equals its complementary angle, i.e. that the adjacent angle made by the vertical is also right. Using the identity (equality) criterion for significance (in terms of the *One* for Proclus and the *concept* for Hegel), the right angle is special since it is a ``self-identity'', *equal* with itself through its complement.

For these reasons, as is clear in Hegel's later thought, the right-angled triangle, as well as the equilateral triangle, is also paradigmatic for triangles, and through the Pythagoras theorem, even determines the concept of the circle. The right angle also contributes, as we will now see, to the paradigmatic role of the *square* among the four-sided (quadrilateral) figures.

(γ) *The square*

As with triangles, Euclid (Definition 22 of *Euclid 1*) classifies the quadrilateral figures in accordance with decreasing order of identity. Proclus follows this classification though augmenting it. Each class of quadrilaterals is assumed to exclude the others. So, for example, the oblong excludes the square. The Euclidean classification is as follows: the *square* (equal sides, equal angles all right angles), *oblong* (all right angles, opposite sides equal), *rhombus* (equal sides, no right angles), parallelogram or *rhomboid* (opposite sides equal, opposite angles equal), *trapezoid* (the rest). Proclus says[75] that the square ``is the only area whose very nature exhibits an ideal with respect both to its sides and to its angles''. Equality is, of course, the key here: all of

[74] Hegel agrees; he says that ``*das Dreieck die erste geradlinige Figur ist*'' (*NP*, §256, Zusatz).
[75] *P*, p.135.

its sides are *equal*, all of its angles are *equal* and the angle is the *right* angle, which, as we have seen, is for him the ``self-identical''.

*GS* discusses, for the most part, the early part of *Euclid 1* and it contains no discussion of the square. However, it seems almost certain that Hegel's view of the square is very similar to that of Proclus. This is clear from Hegel's later logic, where he interprets the priority or ``superiority'' of the square among quadrilaterals (in particular when compared with the rectangle=oblong) in terms of equality. For him, the square (like the right angle) is the *self-equal* (*dem sich selbst Gleichen*), while the rectangle is the *self-unequal* (*dem sich selbst Ungleichen*).[76]

(j) *Proclus and Hegel: angles*

The topic of the nature of an angle was much debated by the classical geometers. A central issue in this debate was how to understand *angle* in terms of the Aristotelian categories, in particular, the categories of *relation*, *quality* and *quantity*. We now summarize the discussion by Proclus of this issue.[77]

In terms of relation, angle is the inclination of one line to another in a plane, and so can be regarded as a *relation* between two lines. This is Euclid's point of view (Definition 8 of *Euclid 1*). However, angles can be straight or curved and this is a *qualitative* distinction. (For example, the angles made by two intersecting circles are curved.) Also, angles can be measured to give *numbers* (e.g. the ``45'' in a *45° angle*), and, of course, number is *quantitative*. Further, an angle can be divided (e.g. bisected) by a line, and surely only what is *quantitative* can be divided. Proclus discusses the difficulties in each of these three views of angle, and, following his predecessor Syrianus at the Academy, says that ``the angle as such is none of these things mentioned but exists as a combination of all these categories''.

In *GS*, Hegel, like Euclid, sees the (rectilinear) angle as a determination of the two lines that meet to form the angle. But consistent with the general approach of *GS*, he sees angle in terms of *limit*. While the two lines bound (limit) a region ``from the outside'', the determination of the angle is not limit in that sense but is a determination of *the limiting itself* of the two lines. Hegel says:[78]

> Der Winkel ist nicht eine Grenze (außer als Fläche, dann Grenze eines Raumes), aber die Bestimmtheit der Begrenzung zweier sich begrenzenden Linien, also als Bestimmtheit doch eine Grenze.

So two lines, that intersect, limit a (wedge shaped) region, the ``limitation'' of the two lines. But we have to limit that region *again* to get a single angle. The ``wedge'' shape between the two lines is lost when we limit to the angle. We may say that the angle is what is unchanged in the contraction of the wedges to the vertex. (This limiting of the limitation is, I think, approximated in classical geometry by the

[76] *WLg II*, p.224; *WL II*, p.469 (*SL*, p.810).
[77] *P*, p.98f..
[78] *GS*, p.293; *trans.* ``The angle is not a limit (like a surface [limiting] from the outside, in that case, the limit of a [three-dimensional region] of space), but rather [the angle] is the determination of the limitation of the two lines bounding it, [and] therefore as a determination, still [counts as] a limit.''

view of Apollonius, discussed by Proclus, that the angle is the *contracting* of a surface at a point under a broken line.)

Angle is, then, to be understood in terms of *limit* and *the unlimited*, the two concepts that, as we have seen, are foundational in Hegelian (as well as in Proclian) geometry. Angle is a limit in a *mediate* or *second-order* sense, being determined by the two lines enclosing the angle. For two reasons, I think, the *first* or *primary* limits of the triangle are its *sides*. First, the sides immediately specify the triangle as *limited*, cut out against the backdrop of the *unlimited* plane, in a way that angles do not. Second, geometry starts with the point which then develops into the *line*; the angle comes later and presupposes lines already ``there''. So the linear (the sides) has logical priority over the angular. We will discuss further Hegel's view of the angle in 3(a).

(k) *Proclus and Hegel: problems and theorems*

The propositions of the *Elements* fall into two kinds: *problems* and *theorems*. It is clear that Hegel was aware of the distinction in *GS* since he explicitly refers to Proposition 9 of *Euclid 1* (``*to bisect a rectilinear angle*'') as a problem (*Aufgabe)* and formulates its theorem (*Lehrsatz*) version. Hegel does not discuss the distinction in *GS*, but he does in his later work.[79] Since Hegel may well have learned about it from the Proclus commentary, we now briefly summarize the views of Proclus on the matter.

Proclus says: ``Problems require us to construct a figure, or set it at a place, or apply it to another, or inscribe it in or circumscribe it about another, or fit it upon or bring it into contact with another, and the like; theorems endeavor to grasp firmly and bind fast by demonstration the attributes and inherent properties belonging to the objects that are the subject-matter of geometry.''[80] The essential difference between a problem and a theorem is, then, that a problem primarily involves the *construction* of something while a theorem deals with understanding ``the objects of the theoretical sciences''. In *Euclid 1*, Proposition 1 asks for the construction of an equilateral triangle and so is obviously a problem. On the other hand, Proposition 5, which asserts that the base angles of an isosceles triangle are equal to one another, is a theorem. Of course, proofs of theorems often involve constructions (problems) and conversely, establishing the validity of a construction can require the use of theorems. Proclus indicates the difference between Euclid's

[79] Hegel says in *WLg* II, pp.206-207, *WL* II, pp.446-447 (*SL*, pp.790-791) that the difference between the theorem and the problem has to do with the difference between *synthetic cognition* and *analytic cognition*. Now the problems of geometry are *constructive* in character – e.g. to *construct* an equilateral triangle – and in the Jena Logic (*JS*, p.113 (*JL*, pp.118-119*)*), construction is the division (*Eintheilung*) of something which, when brought back to the unity of the definition, gives the proof. (Indeed, in terms of the identity philosophy, construction is, for Hegel, the *second potency*.) Presumably this ``bringing back'' is the synthetic cognition, referred to above in *WL*, that transforms a construction problem into a theorem. The NeoPlatonic version of this is that the ennunciation of the end of the proof of a theorem joins ``the end to the beginning in imitation of the *Nous* that unfolds itself and then returns to its starting point'' (*P*, p.164) – Hegel expresses a similar view, though in terms of conceptual mediation, in *WLg II*, p.225, *WL II*, pp.470-471, (*SL*, p.812). The distinction between construction and theorem is not a rigid one. Indeed, in *GS* (with reference to Proposition 9 of *Euclid 1*), Hegel points out that a problem can be reformulated as a theorem. Also, as pointed out above, theorems often involve constructions for their justification.

[80] *P*, pp.157-158.

treatments of problems and theorems: he points out that ``at the end of his demonstrations'' Euclid places ``*This is what was to be done*'' (in the case of problems) and ``*This is what was to be proved*'' (in the case of theorems).[81] (Traditionally, these endings are abbreviated to *Q.E.F.* and *Q.E.D.*.)

## 3. *Hegel's Geometrische Studien*

### (a) *Mathematical objects and the philosophy of limitation*

In this chapter, we will examine in more detail some of themes in *GS*, using the background provided by Chapter 2. We begin the discussion with Hegel's philosophy of mathematical objects. Of special importance for this philosophy is the following passage:[82]

> An jedem mathematischen Object ist zu begrenzen (a) sein positives, insofern es eine Begrenzung aufhebt (Linie die räumliche Beziehung von Punkten); als aufgehobnes selbst bleibt nur die Menge (der Punkte);(b) sein negatives α) insofern es als negiertes eines andern ist, Grenze ist, Linie der Fläche (nicht insofern es) negiert, begrenzt; denn insofern kommt nur dem andern ein Charakter zu, die negierende Linie, einer Linie, Fläche ist darum nichts mehr und nicht weniger; es entsteht nichts anderes in ihr – als erst durch die andere –, das β) isofern sie negiert wird, nur zum Teil begrenzt wird.

Here, there does not seem to be a noun to which the adjectives *positives*, *negatives* refer, and I understand them as referring to the two ``sides'' or ``beings'' of the object in relation to its limit. As we would expect, these are understood by Hegel in terms of his philosophy of limits. (They are also to be understood in terms of the finite and the infinite as we will see later.)

Precisely, each object becomes determinate through the intrinsic development (dialectic) of its *limitation*. In many cases, this limitation is the geometrical *boundary* – or part of that boundary - of the object.[83] So a (finite) line is limited by its end-points while a surface (e.g. a hemisphere) is limited by the curve bounding it (e.g. a circle). As we saw earlier (2(g)), the point is the limitation of the (finite) line. We can, I think, further say that the line only *is* through the development of the point through which one end-point of the line identifies with itself *qua* point with the other end-point. This identification (*Aufhebung*) stabilizes the fixed line with its two end-points. (The identification does not mean that the initial and final

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[81] *P*, p.66.

[82] *GS*, p.293; *trans.* `` In each mathematical object, there is, qua limiting: a) its positive [being], insofar as it [the object] sublates a limitation ([e.g. the] line as the spatial relation [or connection] of the point); as self-sublated, [the object] remains only quantitative [*die Menge*] (the point [remains *one* point]); b) its negative [being], α) in so far as it is an other as negated, [it] is limit[; so] the line negates the surface (not inasmuch as it [is a line but]) by bounding it; because [it has the] character only of approaching the other, [i.e.] the negating line, that's why the surface is nothing more nor less than a line; it [i.e. the surface] develops from nothing other in itself – rather it [is] first through the other – , β) it is negated only [in that] it is limited by the part.''

[83] But there are cases where the limitation at the conceptual level does not appear as a boundary at the level of sense perception. We saw this earlier in our discussion (2(i)) of the circle in relation to its center.

points coincide *geometrically* – for if they did, we would have a closed curve (paradigmatically, the circle) – but rather in the straight line case, the identification of the points is conceptual (or in the language of *Diss.*, *ideal*). Both are *point*, and this is expressed geometrically in the line determined as *one thing* in its beginning and end.

There are two sides to this process: the first is the completed (finite) line with the generating limitation of the point that gave rise to it suppressed. This is the *positive being* of the line. The other side is that generating limitation: this is the *negative being* of the line. As Hegel puts it in the above quotation, an object is positive being *insofern es eine Begrenzung aufhebt*, while it is negative being *insofern es als negiertes eines andern ist, Grenze ist*. The positive being of a geometrical object is that object fixed and stable, as it is in normal geometrical work, boundary included; for example, in the case of the surface, it is the *surface with its bounding curve*. In the object as positive, the limitation is present but with its generating dialectic suppressed. The *negative being* of the geometrical object, on the other hand, is the object's generating limitation *on its own*, the rest of the object, that ``fills out'' the limitation, being ``wiped out''. So the (curved) line *on its own* is the negative being of the surface that it limits, and an end-point *on its own* the negative being of the line of which it is an end-point. (In the latter case, the point in which the two end-points identify loses the sublated line that it generated and returns to being just a *one*, *nur die Menge (der Punkte)* (``only the quantitative (the point [remains a point])'').)

In the above quotation from *GS*, Hegel says that the line is *die räumliche Beziehung*[84] (``the spatial relation'') of a point. As we saw above, the point relates itself to itself through the end-points of the line, so that the line is the medium for the self-relating of the point. It is in this sense that the line is the spatial relation of the point. The spatial relation is also described by Hegel as *die objective Beziehung* (``the objective relation''). The difference between the two relations, is, I think, only one of emphasis, the relation in its spatial form referring back to the *negative* being of the object while the relation in its objective form points forward to the *positive* being of the object. (The spatial relation applies to geometrical objects in general, and will be considered below for the triangle.)

One might wonder how the above theory of limitation applies to a *curved* line. For given any two distinct points, there are infinitely many curved lines with these points as end-points. So how can the end-points – the limits of a (curved) line – determine the line? This is not a problem for the Hegelian position. As in the thought of Proclus, the *straight* line for Hegel is the simplest form of the (unlimited) line, and that kind of line *is* completely determined by the end-points (given the geodesic character of the straight line for Euclidean space). For curved lines, other conditions, notably curvature, come into play, e.g. the line might be required to be *circular*. That is, the limitation of the curve

[84] Hegel, it seems, began *JS* with *Einfache Beziehung* (which is translated in *JL* as *Simple Connection* (or *Simple Relation*)). The geometrical form of this in *GS*, applying to the point as the beginning of geometry, is *die räumliche Beziehung*. The treatment of the spatial relation of *JS* (p.199f.) is different from that of *GS*. The spatial relation of *JS* is *die Beziehung zweyer Dimensionen* of the plane, and the negative and positive associated with it are not discussed in terms of limit but rather in terms of the plane as the *Andersseyn* of space.

requires *additional* determinations apart from the end-points. (In the case of the circle, as we saw in 2(i),(α), these extra determinations are its center and diameters.)

We also saw in 2(g) that Euclid, Proclus and Hegel all agree that the point has primacy in geometry. Hegel says in *GS*:

> Der Punkt ist die einfachste Begrenzung der Unendlichkeit, die darum nichts absolutes ist, weil sie begrenzt werden kann; wäre sie absolut, so wäre keine Mathematik möglich.[85]

Here, the point as the limitation of the *Unendlichkeit* is, as explained in 2(g), to be understood in the Proclian terms of *limit* and *the unlimited*. The point *is*, for Hegel, only as self-sublating: an *Außersichkommen*[86] or ``coming out of itself''. The point is a first for geometry, but is not *absolute*: for if it were, then it would be complete in itself, and have no unfulfilled potential for producing line, plane and body. In that case, geometry would not be possible (*möglich*) since these other geometrical objects would not be ``there'' to be studied![87] The point has, we may say, only *negative* being:[88] it has no positive being. Because of this, it is the potentiality for being *only* the limit of something that *has* genuine positive being. Another way of saying this is that the point is the only mathematical object that is not the result of a sublation, or alternatively, in later Hegelian terminology, is the only mathematical object that is not a *second negation*.

Hegel then uses the concept of *Grund* (``ground'') to elucidate mathematical objects:[89]

> Der Grund (Veranlassung überhaupt und hier in der Geometrie) ist Vermannigfaltigung der Einheit der Grenze; die neuen mannigfaltigen müssen aber selbst wieder Begrenzte sein. Die Vermannigfaltigung der Einheit der Grenzen gibt aber eine neue Gestalt der Figur, d.h. eine neue Einheit des Ganzen.

The ground of an object as the *Vermannigfaltigung der Einheit der Grenze* is to be understood, as in the Kantian philosophy, in terms of an object as a conceptually determined manifold, but with this crucial difference,

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[85] *GS*, p.294; *trans.* ``*The point is the simplest limitation of the infinite, [but] it is nothing absolute, because it can become bounding [as the end of a line]; if it were absolute, then no mathematics would be possible.*''

[86] In his later logic (*WL* I, p.313; *SL*, p.304) Hegel describes the line as the *Außersichkommen des Punktes*.

[87] Hegel (above) says that *mathematics* would not be possible if the point were absolute. But presumably arithmetic and number theory, which do not depend on geometry, could still be possible even if the point *were* absolute.

[88] This *negative* being of the point is reflected in Euclid's *negative* definition of the point as having *no part*.

[89] *GS*, p.294; *trans.* ``The ground ([i.e.] the reason [for something] in general and [in particular] here in geometry) is the manifoldness of the unity of the limit[s]; but the [resulting] new manifoldness must itself be limited again. The manifoldness of the unity of the limits gives a new form of the figure, i.e. a new unity of the whole.'' (Hegel's view of ground is developed further in *JS*, where the ground is *die Reflexion des Erkennens selbst* (*JS*, p.135; *JL*, p.141).)

that *the determination is through the limits.*[90] Indeed the unity of the limits *is* the concept that determines the geometrical manifold of the figure. In the case of geometrical objects, the ground, I think, can be regarded as the manifoldness of the *negative being* of the figure, not completely determined into positive being. There are therefore further determinations of the manifold which posit it as *eine neue Gestalt der Figur,* determined through a *second order* limitation. The secondary determinations are, for Hegel, *magnitude determinations* and the *angles*, which we now discuss in more detail.

These secondary determinations are derived from relations between the primary limits: Hegel says (as we saw in 2(h)): *Das Gesetz, die Einheit der Figure, ist das Verhältnis der Grenzen*. For Hegel, such a *Verhältnis* (relation) has two forms. The first is given by a *quantitative* relation between the limits, and since it is determined conceptually, it has to involve *equality*, quantitative *identity* (2(f)). So the relation is one of *numerical equality between limits of the same kind*: *Größe ihrer selbst, also Gleichheit der Art, derselbe Maßstab*. The equality can be between limits that are *lines* or limits that are *angles*. An example of the first case is the *equilateral triangle*. Initially, it is only a triangle in general, but there is *secondarily* imposed an additional relation of *equality* between the *lengths* of its line limits (=sides). The quantitative relation between limits is, Hegel says, *positive.* I think that the reason for this is that number is *fixed*, indifferent,[91] and so its negativity (dialectic) is completely external to it.

The other kind of secondary determination is that of angle. Its secondary character is, I think, confirmed in geometry by the fact that the sides of a triangle (the primary limits) determine the angles *but the converse is not true*. (If two triangles have the same angles then the triangles are only similar, not necessarily congruent.)

The angle in a triangle is determined for Hegel through the *spatial relation* (*räumliche Beziehung*) of the triangle. This relates the sides (limits) to one another through the triangle concept, and when ``filled out'' into positive being, gives a completely determinate spatial region (the interior of the triangle with its edges). Clarifying this *Beziehung*, Hegel says: *der Winkel ist als begrenzt α) die Begrenzung der Unbestimmtheit der räumlichen Beziehung der sich begrenzenden Linien; er ist als Begrenzung durch sie angedeutet*[92] (`` the angle is, as limited,

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[90] However, Kant does view the geometrical figures in terms of the *limitation* of space, and perhaps his brief treatment of this, in the *transzendentalen Ideal* in the *Kritik der reine Vernunft*, influenced Hegel. There, Kant says: *Alle Mannigfaltigkeit der Dinge ist nur eine eben so vielfältige Art, den Begriff der höchsten Realität, der ihr gemeinschaftliches Substratum ist, einzuschränken, so wie alle Figuren nur als verschiedene Arten, den unendlichen Raum einzuschränken, möglich sind.* (Kant, however, cautions the reader that this way of talking is not *strict* (*genau*).) So for Kant, as for Hegel, the geometrical figures arise as ways of *limiting* (*einzuschränken*) infinite space (cf. also Hegel's discussion of the positing of figures in absolute space (2(h))). Kant, again like Hegel in *GS*, relates this limitation to the concept of *ground*: indeed, he says that *der Möglichkeit allen Dingen die höchste Realität als ein Grund ... liegen*. However, unlike Hegel, Kant does not seem to have developed a philosophy of geometry in terms of a theory of limits.

[91] In connection with arithmetic, Hegel says (*WL I,* p.200 (*SL*, p.205)): *Denn die Zahl ist die gleichgültige Bestimmtheit, träge; sie muß von außen betätigt und in Beziehung gebracht werden. trans.* ``For number is the indifferent determinateness, listless; it must be operated on from outside and be brought into relation.''

[92] The β) that followed this is lost.

α) the limitation of the indeterminacy of the spatial relation of its bounding lines; as limitation, it is indicated through them.'').

I interpret this as follows. As discussed in 2(j), the spatial region is further limited through the contraction of the ``wedge'' shapes formed by the sides at each vertex. The resultant limitation produces the *angles* of the triangle. The principle of this secondary determination is *die Beziehung selbst der Grenzen, als Raum (eingeschlossne Winkel)* (``the self relation of the limits as space ([the] enclosed angle)''). The secondary determination is then *equality of angles*.

Since, as we saw, the first form of the relation (equality of sides) is *positive*, it is reasonable to regard its second form (equality of angles) as *negative* (although Hegel does not explicitly say so). This negative character is clear when we consider how the angle at a vertex is determined through the *vanishing* of the spatial ``wedge'' at the vertex. The angle itself becomes *positive* in its measurement through the lengths of the primary limits, i.e. the sides of the triangle that bound it: the angle *als Grenze, und dasjenige, das sie begrenzen und bestimmen, wird gemessen* (``as limit, is measured [by the limits] that bound and determine it''). So for example, in a triangle $\Delta ABC$, $\angle A$ is greater than $\angle B$ if and only if the side *a* opposite $\angle A$ is *longer* than the side *b* opposite $\angle B$ (Proposition 18 of *Euclid 1*.) The quantitative determination of angle in terms of sides achieves its complete expression in the *trigonometry* of elementary mathematics, where the trigonometric functions of an angle - such as its *sine* and *cosine* - are just *ratios* of the lengths of the *sides* of a right-angled triangle containing it.

(b) *The three dimensionality of space*

As we saw in 2(g), the point, for Hegel, is the *simplest limitation of the infinite*. It was also commented there that Hegel's view of the point corresponds to that of Proclus, where the point is the ultimately simple geometrical object, entirely limit but with the potential infinite implicit within it. Hegel also views, as does Proclus, the fundamental geometrical objects as arising through a process of limiting, first the point and then line, plane and solid, each limiting what comes after it. We also saw in 3(a) that a geometrical object has both positive and negative being, the latter being the generating limitation that determines the former, and the former determining the latter as its boundary. The two kinds of being are linked through the *spatial* or *objective relation.*

So the line is the objective relation of its limiting end-points. Similar considerations apply to the surface as the objective relation of its bounding (curved) line (e.g. a circular region bounded by a circle), and of the solid (body) as the objective relation of its bounding surface (e.g. a solid ball bounded by a spherical surface). At this stage, for Hegel, the pattern breaks off, thus forcing space to be three-dimensional. The justification of this ``breaking off'' lies, again, in his theory of limits.

For unlike the point, line and surface, t*he solid is not limit*. Hegel says:[93]

[93] *GS*, p.295; *trans.* ``The principal proof that there are only three dimensions must be that the spatial relation of the point, line and plane, posited outside themselves, is of a different nature than they are themselves; but the solid is not; that means that these [point, line and plane] are limits, [while it, the body] is not. Its limit [if

> Der Hauptbeweis, daß nur drei Dimensionen sind, müßte sein, daß die räumliche Beziehung des außer sich gesetzten, Punkte, Linie, Fläche, von andrer Art als sie selbst sind; beim Körper nicht; das hieße, daß jene Grenze sind, dieser nicht. Grenze sein ist Aufhörung der Beziehung; also darum, weil Körper nicht Grenze ist

What replaces the limiting role of point, line and plane in the case of the solid is, then, *Aufhörung der Beziehung,* the *ceasing to be of the [spatial] relation*. It is true, Hegel says in *GS*, that we can think of the solid, as for the point, line and surface, as being *außer sich gesetz* (``posited outside itself''), but this positing is fundamentally different from the way that point, line and plane are posited outside themselves. In the latter cases, the geometrical object is the *limit* of a different, higher dimensional object: the point of a line, the line of a surface and the surface of a solid. But while, it is true, we can conceive an *outside* of a solid, that outside is *another solid* with a space in between. That is, we have *two* bodies, the one external to the other. The spatial relation of the solid has ``ceased'' in the spatial gap. We can also conceive of the solid as going outside itself in an other, more intrinsic, way. In that way, the solid is, Hegel says, *durchaus selbst ein bezognes* (``absolutely a drawing out of itself''). I take it that an example of this would be a body expanding or contracting. But in either case, it is solid going to solid, the spatial relation, that determined the original boundary, ceasing and being reestablished in another solid. Solid is not limit, and we cannot go up to a fourth dimension. Space is only three dimensional.

There is, however, present a deeper relation between the solid and the point that brings the solid back to the point and so closes up the conceptually related point, line, plane and solid in a logical circle. In a cryptic comment of *GS*, Hegel says that there are only three dimensions[94]

> denn der Körper ist selbst wieder Punkt, d.h. allen entgegengesetzt außer sich; der sich selbst entgegengesetzte Körper als Einheit wäre Linie.

I understand this (somewhat speculatively) as follows. Recall that in the preceding paragraph, we saw that, for Hegel, there are two ways in which the solid is external to itself: in the first, it is external to itself as another body, and in the second, it changes its boundary in its expansion and contraction. We can understand Hegel's comment above about the solid ``being'' the point in either of the two ways. In the first way, the point also is outside itself since it posits other points external to it as the end-points of the lines that are potential within it and through which it achieves its self-identification. Further, in *NP*,[95] Hegel says that the negation of space itself is immediately *unterschiedslose Außersichsein* (``differenceless being-outside-itself'') and this (he says) is the *point*. In that respect, the solid is the *same* as the point in that it is *solid again* as posited outside itself through

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you want to call it that] is the ceasing to be of the relation; that is why the solid is not limit''.

[94] *GS*, p.295; *trans.* ``for the solid is itself again [the] point, [i.e.] outside itself [in] complete contradiction. The self-opposed body as unity was the line.''

[95] *NP*, §256 .

the spatial gap. *Both point and solid are conceptually the same contradiction of being-outside-itself*.

Following the second way, we can regard the identity of point and solid as realized by the *contracting* inwards of the solid. Non-precisely, the solid is conceptually a ``black hole'', its ``contraction'' to a point, the ``undoing'' of the initial expansion of the point from its immediacy into line that started off the dialectic of space. The latter, then, takes us from point (zero-dimensional), to line (one-dimensional), to surface (two dimensional), to solid (three-dimensional) and then back to point, where we started from. Space has identified itself dialectically with its immediate starting place, and so is complete in its three dimensionality.[96]

A consequence of this is that space is initially posited on its own without time. Indeed, for Hegel and Newton, as well as for common sense, the time ``dimension'' is fundamentally different from the spatial dimensions. The science of space is geometry, but there is, as Hegel says in *NP*, §257-259, no science of time; rather, time is the positing of the self-externality of space; it is the negativity of space, and the ``dimensions'' of time - past, present and future - are *das Werden der Äußerlichkeit als solches*. It is this negativity that drives the point-line-plane-solid dialectic, but is not explicitly posited in space, where the geometrical figures lie inertly side by side; time, on the other hand, is just this negativity *explicitly posited*. In terms of the physics of matter, the *change* in real things associated with time – giving their past, present and future - is just the expression of the logic of the *finiteness* (Dasein) characterizing these things, and the negativity in the becoming of these things is just time.

In modern science, time is not derived philosophically from space (as Hegel did), and the time ``dimension'' is integrated with the spatial dimensions. Special relativity theory, for example, is four dimensional, with time giving the fourth dimension.[97] Further, in the first attempt to

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[96] It is possible that Schelling's philosophy of potencies influenced Hegel here. If so, presumably the development of the point into the line would correspond to the first potency (transition of unity into difference), of the line into plane to the second potency (transition of difference into unity) and of the plane into the body - which coincides with the starting point - to the third potency (absolute identity). As we will see in Chapter 4, the argument of *Diss.* understands the point-line-plane-body dialectic in nature in terms of Schelling's potencies, and this gives support, I think, to the interpretation above of the solid identifying with the point through contraction. Note, though, that in *Diss.* ``bodies'' (such as the solar system) can be *ideal*.

[97] It is sometimes argued that Hegel anticipated relativity. We need to be careful about such claims. What Hegel says (*NP*, §257, Zusatz) is that space is the immediate existence of Quantity, whose negation *in gleichgültiges Bestehen zerfällt* (``collapses in indifferent existence''). Time is the existence of this negation; it is space as *für sich in seiner ganzen Unruhe* (``for itself in its total restlessness'') through which the ``indifferent'' space is sublated, so that while the spatial point (as in geometry) is only an abstraction, the spatial point *in time* is *actual*.

So for the Hegelian position, time is implicit within space and is thus *not* to be thought of as a separate dimension, apart from and mathematically on a par with each of the three spatial dimensions, as it is thought of in the 4 dimensional world of *Special Relativity*. The three dimensionality of space appears in *NP before* time is considered, and the latter is derived from the former. Hegel (*ibid.*) explicitly warns about taking space and time to be quite separate: *In der Vorstellung ist Raum und Zeit weit auseinander, da haben wir Raum und dann auch Zeit; dieses "Auch" bekämpft die Philosophie.* (``For [popular] thought, space and time are far apart, so that we have space and then also time; philosophy fights against this ``also''.'') In relativity theory, space and

produce a unified field theory (unifying electromagnetic theory and the geometry of space-time), Kaluza and Klein[98] attached circles to the points of space-time thus giving a fifth dimension that (it is claimed) we don't notice because of the perfect homogeneity in the circular direction. The modern ``grand unified theories'', which attempt to unify the non-gravitational forces, use at least 24 dimensions. In view of this, and the ``bad press'' that Hegel's *Naturphilosophie* has got, it is easy to mock Hegel's insistence on the fundamental three dimensional nature of reality.

In defense of Hegel, however, scientific experiments are done in the familiar three dimensional world evolving in time, and any theory of the world involving more than three spatial dimensions is only speculative if it is separated from its verification when projected into three dimensional terms. So, for example, in special relativity, the verification of the contraction in length of a measuring rod on board a moving spaceship, when compared with the length of an identical measuring rod at rest on earth, involves measurements with clocks and observers in the *familiar* three dimensional world (which is taken for granted). We may say that the remarkable, higher dimensional, theories are *essential* – or even in Hegelian terms, *ideal* - in character, and become *real* only in so far as they refer to and explain what happens in the everyday world. It is in this sense that the real world is three dimensional. This

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time are, as I think Hegel would have said, *externally* combined, i.e. without their philosophical significance clarified. However, the formulae of special relativity are *symmetrical* in the space and time variables. Hegel's logical differentiation between space and time – space first and then time as its inner negation – is not reflected in special relativity, and so, I think, cannot be valid as it stands. However, Hegel's *internal* explanation of the relation between space and time can, I think, be readily modified to give a philosophical basis for their relation in relativity theory. One needs, I think, to start natural philosophy, not, as Hegel does, with nature immediately determined as indifferent, spatial side-by-sideness, but rather with nature as *becoming*, whose positing is a positing as negation (rather like *Schein* in Hegel's *Wesen* logic), with space and time posited through each other *together*, on the same logical footing. Such an approach would also unify the point of space with the instant of time in a space-time point, and the instants of time would be just as much an indifferent side-by-sideness in time as are the points in space, and space as much the negation of time as time is of space. Such a start to the Philosophy of Nature is consistent, I think, with Hegel's comments on Nature in the Jena System, e.g. (*JS*, p.177; *JL*, p.185) Nature as *den sich aus seinem Anders sich werdenden* (``the becoming itself through itself out of its otherness'). Such a symmetry between space and time is interestingly evoked by Hans Castorp in Thomas Mann's book (p.95): *Der Zauberberg* (S. Fischer Verlag, Frankfurt am Main, 1974) where he says that measuring time with space is the same as measuring space with time (*Wir messen also die Zeit mit dem Raume. Aber das ist doch ebenso, als wollten wir den Raum an der Zeit messen,*).

However, this is not the end of the story. For example, while there can be *many* spatial dimensions in modern Physics, there is only *one* time coordinate so that space and time *are* distinguished at a very basic level. Indeed, the symmetry between space and time in the special theory is only at the one dimensional level. Further, in relativistic cosmology, the differences between time and space reemerge and in certain respects, the theory ressembles the Newtonian. In particular there is a time coordinate, called *world time*, that reinstates in relativistic terms the old Newtonian idea of *absolute time (e.g.* p. 184, pp. 315f. of: R. D'Inverno, *Introducing Einstein's Relativity*, Clarendon Press, Oxford, 1995). This in turn gives the basis for *absolute distance* (*ibid.*, p.325). It is an interesting, and, I think, important, philosophical task to develop a modern Hegelian philosophy of space-time-matter, first by understanding special relativity in terms of Nature as becoming, with space and time playing symmetrical roles, immediately ``indifferent'', and then second by examining general relativity in terms of Hegel's treatment of space and time in *NP*, where time actualizes space.

[98] See, for example, p.311 of: B. Booss and D. D. Bleecker, ``*Topology and Analysis - The Atiyah-Singer Index Formula and Gauge-Theoretic Physics*'', trans. by D. D. Bleecker and A. Mader, Springer-Verlag, New York, 1985.

everyday, three dimensional character of the world was, I think, stressed too much by Hegel in his criticism of Newton, since the higher dimensional theories whose projections explain the ``real'' world *cannot* be created by staying only with three spatial dimensions. But surely Hegel is correct in seeing, as does common sense, the *solid* as ultimately the real thing in our world.

(c) *The cube and the sphere*

We saw (2(h), 3(a)) that for Hegel, the principle of a figure is the relation of its limits. In *GS*, Hegel discusses this figural principle for the cube and the sphere. Underlying this discussion, of course, are the four basic geometrical objects which structure the objects of geometry in general: *point, line, plane, solid*.

Starting with the cube, the presence of these four objects is obvious. The *point* is present in the 4 vertices, the *line* in the 12 edges, the *plane* in the four faces, and the *solid* as the whole cube itself. Clearly, in terms of Hegel's theory of limits, the cube is very simple and basic. The *sphere* is more interesting.

The sphere here is not to be thought of as a solid ball but rather as the *surface* of such a ball. Hegel says[99] *in der Kügel keines als Grenze, und zugleich gesetzte, beschränkende Grenze* (``in the sphere there is nothing but limit, and at the same time, it is a posited, bounding limit''). Hegel means here that the sphere *per se* is itself the limit that separates the region inside it from the region outside it. Hegel then goes on to analyse the sphere, using his philosophy of limit, in terms of *point, line and plane*.

There is no particular *point* specified *on* the sphere through the necessity of the sphere concept.[100] However, the sphere conceptually determines more than just the bounding surface of a ball. It determines a *point, the center*. Indeed, by definition, to belong to the sphere is to be at a certain fixed distance (the radius) from the center. Hegel says that *der Mittelpunkt gehört nicht zur Kugel, und ist nicht Grenze, auch wenn er gesetzt würde* (``the center does not belong to the sphere and is not a limit of the sphere even though it is posited [by it]''). We recall that Hegel also says that the center of a circle *is* a limit of the circle even though this is not the case for the sphere. We discussed a possible explanation for this difference between the circle and the sphere in 2(i).

The natural *lines* for the sphere are the diameters or radii. As in the case of the center, the diameters are not limits. Hegel says *die Linie nicht als Begrenzung* (``the line is not a limitation of the sphere''). It is true that the line in both its unlimited (straight) and limited (circular) forms is present in the sphere: in the first case, as diameters and radii, and in the second, as the *great circles*.[101] (Both

[99] *GS*, p.294.

[100] In his dissertation (*Diss.*,p.27) Hegel echoes the corresponding *arbitrariness* of the point on the *circle* (as well as that of the latter's diameters): *In circulo formali aequalis distantiae notio a puncto peripheriam efficit: et primitivus ejus character est, ut neque ulla diameter neque ullus peripheriae locus reliquis infinite multis excellat.*

[101] A circle on the sphere is a great circle if its radius is greatest possible, i.e. is that of the sphere itself. For example, if we regard the surface of the earth as a sphere, then the equator would be a great circle.

kinds pass through the posited point, the center, which distinguishes them from arbitrary lines passing through the sphere and arbitrary circles lying on the sphere.) Each diameter determines two bounding points on the sphere, and each great circle, of course, lies on the sphere. But the choices of diameter and great circle are *still* arbitrary and neither diameter nor great circle *bounds* the sphere: rather, they are given *through* the sphere. As Hegel says, they are limited by the sphere, not the other way round.

Turning to the *plane*, the latter is also *not* a limit of the sphere: there is no plane lying on the sphere. However, there are *intrinsic* planes connected with the sphere, viz. the planes of the *great circles*, but again, these are arbitrary, and are limited by, but do not themselves limit, the sphere. Hegel says that *der Punkt*[102] *die Umschließung, wie die größten Kreise, in sich zurückkehrende Linien,*
*in sich zurückkehrende Fläche hat* (``the point, as enclosing, [in the form of] the great circle, has in itself the returning line, [which has] in itself the returning plane''). This brings us back to the dialectic of point, line and plane for the circle (2(i)).

(d) *Congruence and superposition*

In the Introduction, we discussed briefly Hegel's criticism of the method of superposition that Euclid used to prove congruence for triangles. We now examine this in more detail.

In *Euclid 1*, two figures are called *congruent* if they are the same in every respect (apart from their spatial position). So, for example, for triangles ABC and DEF to be congruent, we require corresponding sides and corresponding angles to be exactly the same: AB=DE, BC=EF, CA=FD, $\angle$A=$\angle$D, $\angle$B=$\angle$E, $\angle$C=$\angle$F. In certain instances, Euclid proves that two triangles are congruent by *superposition*. For this, one of the triangles is regarded as being lifted up and *superimposed* on (laid on top of) the other. The two figures should then exactly coincide. (The fourth *common notion* (or axiom) of *Euclid 1* then asserts that the two figures are equal.) The problem is*: how can you lift up ``pure'' triangles?* You can certainly lift up paper triangles, but they are inexact: they are not the triangles of pure geometry, which is an exact science. Hegel points out[103] that you cannot handle geometrical figures or lines as one does material things: *auf Figuren kann sowenig die Behandlungsart materieller Dinge angewendet werden, als auf Linien* (``the way of handling material things can be applied as little to figures as to lines'').

As we noted in the *Introduction,* for Hegel, the fundamental problem behind Euclid's use of the method of superposition is the latter's inadequate philosophical grasp of *difference*. Euclid, for Hegel, takes difference as if it were a *thing* (*Verschiedenheit der Dreiecke als Dinge*). For Hegel, when difference is considered *conceptually*, then the problems with superposition do not arise. One rather has to consider what is required for a figure to be completely determined according to its concept. In this connection, a major problem with the geometer's approach – and one that Hegel criticizes Euclid for making – is in not

[102] This word is illegible in the manuscript and *Punkt* (``point'') is my guess. *Punkt* fits in well with the words *Linie* and *Fläche* immediately following: the theme of point, line, plane and solid is, as we have seen, fundamental in Hegelian philosophy.
[103] *GS*, p.289.

distinguishing the concept of a figure from the actual drawn figure, the latter being located in a particular place with its parts spread out separately.

It is the concept of a geometrical figure that differentiates it into a whole of interrelated parts (or limits). The relationship *figure-limits* is then a special case of the *whole-parts* relationship that Hegel investigated in his later logic,[104] and the form of congruence theorems should be stated as follows:[105] *ein Dreick ist durch soviel und soviel Bestimmungen ganz bestimmt* (``a triangle is through such and such determinations [of its parts] completely determined''). The concept is what preserves the integrity of the triangle throughout the geometrical processes of construction and proof, and the new knowledge gained about the triangle through these processes is an elaboration of, or the bringing out of what is implicit in, that same concept.

Since, for Hegel, congruence for triangles is about the *identity* of two distinct triangles, the concept realized in the two triangles *is the same* even though the two triangles are distinguished spatially. So in the ``real'' world of drawn figures, the concept is ``doubled''.[106] The congruent figures are just different realizations of the *same* concept. For Hegel, this has the remarkable consequence that at the conceptual level, *there is only one triangle in every congruence proof.* Since congruence proofs are *proofs from the concept*, they have a philosophical character, and illustrate the use of philosophy as the unhypothetical science to clarify mathematical arguments (2(f)).

This oneness of the triangle at the conceptual level is interestingly illustrated by Hegel's discussion in *GS* of Proposition 5, the proposition which shows that the angles at the base of an isosceles triangle coincide. He again criticizes the method of superposition used in the proof by Euclid, and approves of a very simple method of proof which, according to Proclus, was due to Pappus. Suppose that the triangle is $\Delta ABC$ with $AB=AC$. Then by Proposition 4,[107] $\Delta ABC$ is congruent to $\Delta ACB$ so that $\angle ABC=\angle ACB$, i.e. the base angles are equal as required. What is striking about this proof is that the two triangles $\Delta ABC$, $\Delta ACB$ are the *same triangle* – we have just gone round the vertices in different ways. For Hegel, this is confirmation in geometry of the *unity* of the triangle concept that is misunderstood in the method of superposition. The difference aspect of the proposition comes from *the triangle concept itself* and does not come through separate triangles (as it were) accidentally turning up in different spatial positions, and only coming to identity through human intervention by the ``laying on'' of superposition.

All of the congruence criteria for triangles have then to be reformulated in terms of the triangle concept. Consider, for example, the ``three sides'' congruence criterion of Proposition 8 of *Euclid 1*: *two triangles*

[104] *WLg I*, p.354; *WL II*, p.138ff..
[105] *GS*, p.294.
[106] For more details about the ``doubling'' of a triangle in GS, see 3(e) below. In Hegel's later writings, it is developed further in *JS* where it is compared with the vanishing of the differential. Hegel says (*JS*, p.20 (*JL*, p.22)) that *die Realität eine Verdopplung der Einheit* (``reality is a doubling of unity''), and in his discussion of the Idea in *WLg II*, p.223, *WL II*, p.468 (*SL*, p.810), the congruence theorems are said to ``contain two parts'', the concept and the reality.
[107] This proposition says that two triangles are congruent if two sides and the included angle are the same for the triangles – see 3(e).

*are congruent if their corresponding sides are equal*. In the philosophy of *GS*, this is to be reformulated in terms of *one* triangle: *if the sides of a triangle are determined, then the whole of the triangle is determined.* In particular, the three angles are determined. (The philosophical proof for this depends on the angle being the *relation* of the sides that enclose it (2(j), 3(a)), resulting in the angle being determined once these sides are.) The other congruence criteria discussed by Hegel for a triangle ABC are: *two sides and the included angle* (Proposition 4) and *two angles and a corresponding side* (Proposition 26). We will examine in detail Hegel's reformulation and proof of Proposition 4 in 3(e) below.

Hegel is not alone in his criticism of Euclid's use of the method of superposition (though I think that his explanation of why it is unsatisfactory is unique). Indeed, the method has been criticized by many post-Euclidean geometers. A detailed discussion of this is given by Heath.[108] Heath points out that Euclid himself uses the method with reluctance. Peletier (1557) observed that if the method of superposition was legitimate in geometry, then geometry would be full of such proofs. (The method could, for example, have been used to prove Propositions 2 and 3 of *Euclid 1* as Hegel himself points out.[109]) Bertrand Russell suggested that Euclid should have taken the first proposition in which the method is used, Proposition 4, as an axiom, and indeed it is effectively taken that way by Hilbert in his *Grundlagen der Geometrie*.

It seems to me that the difficulty with superposition cannot be solved by putting in an extra axiom (as Russell and Hilbert wanted to do): doing that just evades the problem of understanding the unarguable *necessity* that we feel with congruence, without any explanation of that necessity. The axiomatic solution just ennables us to ``get by''. Rather, I think that Hegel is correct in seeing the problem here as based on a philosophical error. For Hegel, there are not posited two triangles which are then to be compared as far as equality of sides and angles are concerned but rather t*he two triangles appear only on the sensuous level*, and congruence is a consequence of the concept of a triangle being completely determined through the positing of certain relationships between its limits.

(e) *Hegel's discussion of 14 propositions of Euclid 1*[110]

Hegel starts by commenting on the first four propositions. Proposition 1 constructs an equilateral triangle on a given straight line using two intersecting circles. Hegel gives a reason why Euclid starts his *Elements* with the construction of the equilateral triangle. It is that *für den Anfang wird die Gleichheit in einer Figur aufgegeben* (``at the beginning, equality is given in a figure''). This figure is the equilateral triangle, for *das gleichseitige Dreieck ist die einfachste Figur, sowohl als gleichseitig, als als Dreieck* (``The equilateral triangle is the simplest figure, both in terms of equality of sides as well as in terms of [a] triangle.''). Hegel's explanation of why the *Elements* starts with the construction of the equilateral triangle thus agrees substantially with that of Proclus, discussed in 2(f), 2(i)(β), with its emphasis on identity and equality. However, Hegel's version is

[108] *Hea.*, pp.249-250.
[109] *GS*, p.288.
[110] The present writer's translation of *GS* contains, together with their diagrams and proofs, statements of the Euclidean propositions discussed by Hegel.

totally without the theory of *Soul* which Proclus attaches to his own explanation. (For Proclus, but not for Hegel, the equilateral triangle is ``the likeness of the first soul''.[111])

Proposition 1 serves as the basis for Propositions 2 and 3, both of which involve *equal lines*. Proposition 2 asks for the construction, with a given point as an end-point, of a straight line equal in length to a given straight line, while Proposition 3 asks, given two unequal straight lines, to cut off from the greater a straight line equal in length to the lesser. The philosophical basis for all three propositions is then *equality*, in particular equality for lines and for the sides of a triangle. Hegel then moves on to Proposition 4. This proposition says that *two triangles are congruent if two sides and the included angle for each triangle are the s*ame, and Euclid proves it using the method of superposition. In 3(d), we discussed in some detail the method of superposition and Hegel's philosophical criticism of it.

We now describe Hegel's philosophical formulation and proof of Proposition 4. Proposition 4 is first reformulated conceptually: we are no longer given two separate triangles but only one. Hegel does not explicitly state his conceptual version of the proposition but I think that it can be expressed as follows: *for a $\Delta ABC$, if $\angle A$ and the lengths of the sides AB, AC are determined, then the remaining side BC and the remaining angles $\angle B$ and $\angle C$ are also determined, i.e. the whole triangle is determined.*

The proof goes as follows. Since AB and AC are determined, so are their limits, the end-points A,B, C. However these limits are posited only for their respective lines AB, AC and so are not *vollständig gesetzt* (``completely posited''), for *sie schließen nicht die Unendlichkeit des Raums aus* (``they do not exclude the infinite of space''). I think that what Hegel means is that if we only know the lengths of AB, AC, then there are infinitely many possible positions for the points B and C: for example, if r is the given length of AB, B can be any point on the circle center A and radius r. For the *complete* positing of B and C, we also have to include the determination of $\angle$A: the lengths of AB, AC *together with $\angle A$* determine B and C uniquely for the triangle. But B, C are the limits for the side BC. So the straight line BC is also posited since the limits of a straight line determine the line. (In ordinary geometrical terms, we get the line BC by joining up B and C using a straight edge.) Once we have *all* of the sides of the triangle determined, then so are all of the angles, since, as we saw earlier, for Hegel, an angle is the determination of the *limitation* of the two lines bounding it: since the pairs of lines that bound the angles of the triangle are now determined, their limitations in the angles are also determined. So the whole triangle, i.e. all of its sides and all of its angles, are determined. This completes Hegel's proof of Proposition 4.

Another way of expressing this proposition – and this is, I think, philosophically better – is to say that a triangle is determined *in concept* by two sides and their included angle. The other parts of the triangle are determined through its conceptual necessity of whole and parts – the spatial location of a drawn realization of the triangle is irrelevant. So in the proof, we can vary the position of A and rotate

[111] *P*, p.167. However, in *JS*, p.140 (*JL*, p.147) Hegel does conceive of the circle in terms of soul (*Seele*).

the triangle about A in any way that we like. The variations in the position of the triangle in space are, in the language of Hegel's later logic (2(i),(α)), *der Überfluß der Realität über die Bestimmtheit des Begriffs*.[112]

It remains to show how the original *Euclidean* Proposition 4 follows from Hegel's version of the proposition above. In the Euclidean Proposition 4, we are given two *separate* triangles ΔABC, ΔDEF with ∠A=∠D, AB=DE, AC=DF. We have to show that the remaining corresponding parts of the two triangles are the same. This follows immediately from Hegel's proposition: for once you have two sides and the included angle determined in a triangle, then everything else in that triangle is completely determined by them. Since the determinations of the two sides and included angle are exactly the same for the two triangles, ΔABC, ΔDEF, it follows by Hegel's proposition that their corresponding parts are exactly the same. So the triangles are congruent (and conceptually the same).

For Hegel, philosophical (and not just geometrical) insight is also required for a proper understanding of the relationship between Propositions 5 and 6. Proposition 5 says that in a triangle, if two sides are equal, then the angles opposite them are equal; the converse is given in Proposition 6. We can express these propositions in Hegelian form as asserting that equality of the two angles for a triangle completely determines the *same* triangle concept as does equality of the two sides. As in the case of Proposition 4 (and congruence in general), the real issue is:[113] *wie viel ist notwendig zur Bestimmung des Ganzen, oder wenn das Ganze gesetzt ist, wie viel muß ich von den Teilen erkennen, um daraus allein die Erkenntnis des Ganzen, das aber schon gesetzt ist, erkennen zu können?* (``how much is necessary for the determination of the whole, or, when the whole is posited, how much must I know about the parts in order to cognize, solely from that knowledge, the whole that already is posited?'').

Euclid's proof of Proposition 5 is involved, and as discussed in 3(d), Hegel approves of a simple, alternative proof due to Pappus, which uses Proposition 4 and *only one* triangle which is regarded as two different triangles (by going round the vertices both ways). In this connection, Proclus[114] says that we can ``see two triangles in this single one'', and this is echoed by Hegel who says that in Proposition 5, the triangle *zweierlei wären* (``was double[d]''). I think that this indicates for Hegel that the difference involved here belongs to the *single* triangle concept rather than to the two distinguished triangles that appear in the geometrical proof. In the case of Proposition 6, Hegel says that it is the *angle* that is doubled, and sketches a proof of the proposition that uses Proposition 26.[115]

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[112] As observed in Footnote 72, this remark of Hegel is applied by him to Proposition 4, the ``superfluity'' being with reference to the other side BC and the other two angles ∠B and ∠C of the triangle. But the ``superfluity'' applies, I think, even more to the location of a triangle in space.

[113] *GS*, p.292.

[114] *P*, p.195.

[115] Proposition 6 is not used by Euclid till Book 2 of the *Elements*, so that it is legitimate to prove it using Proposition 26 – asserting congruence for two triangles given two angles and a corresponding side – which Hegel discusses later in *GS*. Elegant proofs of Proposition 6 using Proposition 26 were given by the Arabian geometer an Nairīzī (who died c. 922) and independently much later by Todhunter in his 1862 edition of Euclid – these are given in *Hea.*, p.258. Hegel may well have worked out for himself

Both propositions depend on a ``higher thing'' (*von einem höhern abhangen*): this higher thing is, I believe, the *concept of the triangle* whose necessity, expressed through the interrelationships of its parts, gives the equivalence of the two conditions: *equal sides* and *equal angles*. But this ``higher thing'' *nicht fürs Bewußtsein gesetzt war* (``was not posited for [our] consciousness''). Rather, I think, we need to move from geometry to philosophy to make it explicit.

An interesting criticism by Hegel of Euclid's proof of Proposition 6 (and also that of Proposition 7) concerns the use of *argument by contradiction*. (Hegel calls such a proof *apagogic* (*apagogisch*).) His objection is not so much a criticism of proof by contradiction *in principle* but rather that in each of these propositions, the given proof by contradiction *requires a construction that cannot be done*. Indeed, as the figures for these propositions show, these proofs by contradiction are only possible by *drawing impossible diagrams*. But in the Euclidean framework, we can only work with *constructed* figures. The proof that Hegel gives of Proposition 6 does not rely on contradiction.

As indicated in the preceding paragraph, I do not think that Hegel regards as unacceptable, in general, *proof by contradiction* (as did Brouwer in modern mathematics). To reject *all* such proofs has grave consequences for mathematics. Indeed, as we will see, Hegel himself gives an apagogic proof in *GS* of his version of the congruence result Proposition 26: *in $\Delta ABC$, the point C is determined uniquely by AB, $\angle B$, and $\angle C$.* This proof, however, is conceptual in nature and does not involve an impossible geometric construction. It is only the Euclidean proofs by contradiction that depend on an *impossible* construction that Hegel finds unacceptable.

It is the *constructive* character of Euclidean geometry that rules out impossible constructions. Any construction that you consider has actually to be done. So in the context of Euclid's proof of Proposition 6, Hegel says: *Wäre AC, AB ungleich, so mache BD=AC; diese Konstruktion, zeigt es sich, ist nicht möglich* (``If it were the case that AC [is less than] AB, then BD=AC; this construction shows itself to be not possible''). Once you consider ``possible'' constructions, then you are effectively dealing with the *set* of all constructions and getting involved in infinite set theory (with its possible paradoxes) thus leaving Euclidean geometry. It shows great insight on Hegel's part to have noticed this essential, constructive character of the *Elements* – it is exactly in this respect that Euclidean geometry differs from modern axiomatic geometries (such as that of Hilbert).[116]

It is undeniable, however, that Hegel regarded apagogic proofs with some suspicion throughout his life. For example, in his discussion of the second Kantian antinomy in *WL*,[117] he rejects the apagogic elements in Kant's arguments. (The source of the second antinomy for Hegel lies in

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that Proposition 26 can be used to prove Proposition 6. At any rate, it illustrates impressively how knowledgable he was about geometry.

[116] Bernays (*On Platonism in Mathematics*, p.275 of: *Philosophy of Mathematics*, Editors P. Benacerraf and H. Putnam, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1964) espresses the difference between the geometries of Euclid and Hilbert well: *Euclid speaks of figures to be constructed, whereas, for Hilbert, system[s] of points, straight lines, and planes exist from the outset.*

[117] *WLg I*, p.113f., *WL I*, p.183f. (*SL*, p.190f.).

the character of the concept of quantity, whose determinations, *continuity* and *discreteness*, are null in isolation and consist only in their transition from the one into the other.[118])

Returning to GS, Propositions 7 and 8, that give the ``three sides'' criterion for congruent triangles, are criticized similarly. In *Euclid 1*, the first has an apagogic proof and the second relies on superposition. Hegel gives conceptual proofs for both, avoiding apagogy and superposition. He regards Proposition 7 as *durchaus analytisch* (``completely analytic'').

Propositions 9 and 10 are problems rather than theorems. They show respectively how to bisect a rectilinear angle and how to bisect a finite straight line. Hegel comments that the propositions are (effectively) *ganz dasselbe* ``completely the same'', both using the construction of an equilateral triangle on a given line (Proposition 1). However, while it is true that the proofs of both propositions construct an equilateral triangle, it is not correct to say that the propositions are the same. The reason is as follows. Both proofs use a construction in which two triangles have a common base. In the proof of Proposition 9, one of the triangles is isosceles and the other is equilateral. But in the proof of Proposition 10, *both* triangles are equilateral. I think it likely that the reason for Hegel's misunderstanding is the poor figure given by Lorenz[119] where the isosceles triangle used in the proof of Proposition 9 *looks like* an equilateral one.

Propositions 11 and 12 deal with one problem: how to construct a perpendicular line from a given point to a given line. Proposition 11 is the case where the point is on the line, while Proposition 12 is the case where the point is not on the line. Hegel points out that the solution to both problems lies in constructing an isosceles triangle on the given line. (His comments are purely geometrical, with no philosophy involved.)

In a different part of *GS* (as it appears in *Dok.)*, there is a collection of comments that look like conceptual proofs. These are analogous to the proofs that Hegel gives for congruence theorems. *GS* does not say what Proposition is being proved here, but inspection shows that it is Proposition 26 of *Euclid 1*. The proposition gives the congruence theorem: *two triangles are congruent when two angles and a corresponding side are the same*. The proof falls into two cases, the first case where the specified side joins the equal angles, and the second case, where the side subtends one of the equal angles.

As usual, Hegel's discussion of the proposition is in terms of the determination of the whole of $\Delta ABC$ by some of the parts: two angles and a side determine everything in the triangle. In his proof of case 1 of the proposition, discussed in α), Hegel shows that given BC, $\angle B$ and $\angle C$, then the point A is uniquely determined, and hence also each of the remaining sides AB, AC. β) continues the discussion of α).

At the end of β), he raises a question which is equivalent to proving case 2 of the proposition. In that case, one is given AB, $\angle B$ and $\angle C$, and has to show that $\Delta ABC$ is completely determined. This will be

[118] See Footnote 66.
[119] *Lor.*, p.8.

achieved if it can be shown that the point C is *uniquely* determined by the given data. For then AC and BC are determined, and hence also the angle of their inclination ∠A. So all of the sides and angles of the triangle are determined. Hegel then gives two proofs of the unique determination of C.

The first proof is *apagogic* (i.e. by contradiction), showing that if the point C is not uniquely determined, then the length of either of the lines BC, AC is not determined, and ∠C (or, I think, ∠B) would be changed. But ∠C, ∠B are determined and the contradiction follows. The second proof is direct and, I think, essentially goes as follows. The length AB is given and the directions of AC, BC relative to AB are given. Since *sich gerade Linien nur in Einem Punkte schneiden* (``straight lines intersect only in one point''), it follows that C, the point of intersection of AC, BC is uniquely determined.

(f) *The philosophy of parallel lines*

*Euclid 1* treats the basic geometry of parallel (straight) lines in Propositions 27-31. Euclid defines[120] two straight lines to be *parallel* if they are ``in the same plane and being produced indefinitely in both directions, do not meet one another in either direction''. This is what Hegel calls the *negative* definition of parallel lines (presumably because there is a *not* in the definition). Hegel also gives a *positive* definition: two lines (in a plane) are called *parallel* if they[121] *überall gleich weit entfernt sind* (``everywhere are equally far apart''). This positive definition[122] is specified by Hegel in two ways, a) and b).

The first way, a), was, according to Proclus[123], given by Posidonius: parallel lines are lines in a single plane that neither converge nor diverge, but have all the perpendiculars equal that are drawn to one of them from the points on the other. The second way, b), in which the positive definition can be determined is through the interior angle that a third line (a transversal, or, as Hegel expresses it, a *distance line* (*Entfernungslinie*)) makes with one of the lines. I think that what he means is the following. Fix an angle θ and one of the parallel lines, *l*. For any point A on *l*, consider the line (the *distance line*) through A that makes (interior) angle θ with *l*. The point at which this distance line cuts the *other* parallel line *m* is determined, and hence so also is the distance between the two points. The two lines *l*, *m* are then called *parallel* when this distance is the same wherever the initial point A is taken on *l*. If the interior angle is a right angle, then of course this distance will be the perpendicular distance as in the first way.

Hegel notes, with reference to the second way that *die Entfernungslinien im Fall b sind gleich mit dem Begriffe der Parallellinien* (``the distance lines in case b are equal in concept with that of the parallel lines''). Indeed the distance lines, because they make the same angle with the given lines, themselves form a family of parallel lines, and

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[120] *Hea.*, p.154.
[121] *GS*, p.298.
[122] Heath (*Hea.*, p.249) classifies into three groups all of the definitions of parallel lines that have been given. Hegel's positive definition is a variant of the third group (which says that two lines are parallel when the distances between them are constant).
[123] *P*, p.138.

Hegel's interesting comment is an example of how two things (distance lines, parallel lines) which, when they appear in a geometrical diagram are completely different, nevertheless at the conceptual level are the same. (Earlier (3(d),(e)) we had other examples of this: the congruence conditions are the same conceptually as the concept of the whole triangle, and the equality of the base angles in a triangle is conceptually the same as the equality of the opposite sides.)

Hegel's philosophical account of parallelism is not entirely clear. For example, the existence of just two equal distance lines making the same angle $\theta$ with $l$ is sufficient to ensure that $l$ and $m$ are parallel, but Hegel does not discuss this. It is tempting to define the lines $l$ and $m$ to be parallel if a distance line makes the same angle $\theta$ with $m$ as it does with $l$, but Hegel expressly rules this out: this equality *wäre noch zu erweisen* (``is still to be proved''). This is a corollary of Euclid's Proposition 27.

Part of Hegel's discussion of Proposition 29 of *Euclid 1* has survived. As in the case of Proposition 26, there is no explicit mention of the proposition in Hegel's account. Proposition 29 asserts that *a straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the interior angles on the same side equal to two right angles.* Euclid uses in his proof of this proposition his famous fifth postulate: *if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.*

Many (failed) attempts were made since the time of Euclid to prove the fifth postulate from the other assumptions of Euclidean geometry. (Proclus, in particular, had a wrong ``proof'' of the fifth postulate.) Indeed, it is more like a proposition than a postulate and it is understandable that geometers would want to prove it. The postulate is equivalent to the intuitively obvious fact that there is no upper limit to the area of a triangle, and is also equivalent to the Pythagoras theorem. We now know that the fifth postulate does not hold in non-Euclidean geometry and is independent of the other assumptions of Euclidean geometry, so that attempts to ``prove'' it are doomed to failure. Unfortunately, there is no discussion of the fifth postulate in *GS*.[124]

Using the positive definition of parallel lines Hegel gives two proofs of the first part of Proposition 29. The proofs seem to be original. The precise assumptions that Hegel is making here are not clear to me, and the text is difficult to follow. The proofs, as far as I can see, are purely geometrical in nature with no philosophical justification required.

[124] Hegel does discuss briefly the fifth postulate in *WLg, II*, p. 221, *WL II*, pp. 465-466 (*SL*, pp.807-808). There, he says that Euclid showed good sense (*richtige Sinn*) in not trying to prove the fifth postulate since such a proof can only be obtained from the Concept (*solche Deduktion nur aus dem Begriffe geführt werden kann*), i.e. from the unhypothetical science. Unfortunately, Hegel's philosophical proof of the fifth postulate has not survived. But it would have fitted in well with his philosophical interpretation of the Pythagoras theorem, since the latter is equivalent to the fifth postulate.

This confirms, if such evidence be needed, that Hegel, while not a creative mathematician, fully accepted the essential validity of the Euclidean approach. Hegel took geometry very seriously and did his own research in it. For a non-professional mathematician, he knew the subject very well, though there is an occasional lapse in his geometrical insight. (One such lapse is in his misunderstanding of the ``ambiguous case'' for congruent triangles,[125] a case that (strangely) is not in the *Elements* and which, I believe, Hegel uses wrongly twice in this section and also once wrongly in *JS*[126].)

In geometry, philosophy only comes in at certain places, viz. where mathematical resources fail in principle, and the subject requires the philosophical concepts of the ``unhypothetical science''. Only then does the indispensable, philosophical basis for geometry require to become explicit, and Euclidean methods need ``correcting''.

(g) *Algebra and geometry*

There is a geometrical algebra in the *Elements* of Euclid: indeed, Book 2 of Euclid can be regarded as algebraic in the sense that problems that nowadays would be solved by (elementary) algebra are solved by Euclid geometrically. For example, where we talk of a product $xy$ of positive numbers $x$, $y$, Euclid talked about the area of a rectangle ABCD (with sides AB of length $x$ and BC of length $y$). The modern algebraic approach to elementary geometry goes back at least as far as Heron (c. 3rd century AD). As Heath points out[127], ``Book II gives the geometrical proofs of a number of algebraical formulae''.

However, the algebra that Hegel is involved with in this part of *GS* is really just the adding and subtracting of line intervals occurring within a certain geometrical figure. In this figure, A, B are the centers of two distinct circles, AB intersects the circles, between A and B, in the points F,G, while E is the point of intersection of AB with the line CD joining the points of intersection of the circles. The radii of the circles are the same in length. In his calculations, he uses, without proof, the equality FE=EG, which easily gives AE=BE, AF=GB. (This equality FE=EG is not difficult to prove.)

I confess that I do not know what Hegel is trying to do here. He starts with a long algebraic calculation dealing with the lengths of line segments on AB. The calculation begins with the equality AF=GB, and ends with the equality AF+FE-FE=BG+EG-EG, which, upon cancelling the FE's and the EG's, is trivially the same as what he started with, AF=GB. Maybe this calculation is very early (schoolboy?) work of Hegel in which he was doing a mathematical experiment, not designed for publication. A little later, Hegel gives an argument to show that three pairs of triangles in a closely related figure are congruent. It ends with the equality: GC-GD-DE=CE, which is trivially true from the diagram.

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[125] The ambiguous case is discussed in detail by Heath (*Hea.*, pp.306-307). This case says that *if two triangles have two sides equal to two sides respectively, and if the angles opposite to one pair of equal sides are also equal, then the angles opposite the other pair of equal sides are either equal or complementary, and in the former case, the triangles are congruent.*

[126] *JS*, p.19; *JL*, p.21.

[127] *Hea.*, p.372.

In between these two algebraic calculations, Hegel has an interesting short philosophical discussion about the relation between geometry and algebra. He says:[128]

> In der Algebra ist das x bestimmt; aber seine Bestimmtheit ist nicht gesetzt; in der Geometrie gibt es kein x, es ist alles bestimmt, und als so[l]ches gesetzt; sein Bestimmtwerden ist nicht gesetzt; seine Abhängigkeit ist =x.

Since its determination is not posited, the *x* of algebra *is a variable, universal in character*. The Euclidean line, as indicated in modern mathematics by the variable *x* which is supposed to range over the set *R* of real numbers, is not geometrical in the sense that a drawn line or triangle are geometrical. *Euclidean geometry has no* x's. In geometry, we are presented with a definite, drawn triangle with no variables; everything in the triangle is fixed. In geometry, *es ist alles bestimmt*. The *x* of algebra is to be compared with the *possibility* of a triangle, as when I ask: *draw me a triangle*! There are infinitely many possibilities.

This is further clarified by Hegel in his specification of the *conceptual* difference between algebra and geometry: he says that in algebra, *sind die Bedingungen gegeben und das bedingte nicht* (``the conditions are given and the conditioned [is] not''), while in geometry it is the other way round; in geometry *ist das Bedingte gegeben, und die Bedingungen sind als solche zu setzen, zu zeigen* (``the conditioned is given, and the conditions as such are to be posited, to be shown''), and a geometrical problem *gibt bestimmt den Begriff des aufgegebenen* (``makes determinate the concept of the concealed [conditions]''). I think that we can understand this as follows.

Starting with geometry, the drawn triangle is initially given as *conditioned* in the obvious way: its sides are straight and there are three of them. But the conditions that relate the angles and sides, at that initial stage, are ``concealed''. Geometrical argument is required to uncover these concealed conditions, to ``posit'' them. An example of such a concealed condition would be the congruence criterion of Proposition 4: two sides and an included angle *condition* (determine) the triangle completely, even although we can clearly understand what a triangle is without knowing the proposition.

With algebra, on the other hand, the conditions are given. So, for example, if we wish to prove a geometrical theorem by algebraic methods (``analytic geometry'') we write down *equations* for lines, circles, conics etc. and work with them algebraically. I think that these equations are the ``conditions'' that (Hegel says) are ``given'': so, for example, the *condition* for a line is given algebraically by: *ax+by+c=0*, and this determines the line *completely*. Further, these equations involve variables such as *x*. But as a variable, *x* is not determined as a *specific* number and what the conditions condition, e.g. an actual triangle, is not given in the algebra. It is true that, from a modern point of view, we could define the triangle as a *set* of pairs of numbers *(x,y)*, but that is *still* at the level of conditions: that is, unless we

[128] *GS*, p.297; *trans*. `` In algebra, *x* is fixed; but its determination [exact value] is not posited; in geometry, no *x* is given, everything is determined, and as such[,] posited; the becoming of its determination is not posited; its dependence is =x.''

regard this highly infinite set as a given thing – and we have no direct, non-geometrical intuition for that – the definition just says that a point *(x,y)* is in the triangle if and only if the coordinates *x,y* satisfy certain inequalities (conditions). If we rely on algebra *alone*, the actual triangle is not given. Unlike geometry, there is no spatial intuition in algebra. So, as Hegel says, in algebra, exactly *what* is conditioned (i.e. the geometrical triangle) is not given.

In fact, analytic geometry is a powerful tool for proving geometrical results, but only when the geometrical figure and the algebra *are present together* – the given geometrical object together with its formal algebraic determinations. Real number and algebra become involved in a geometrical problem when we explicitly assign a number pair (x,y) to a point. The geometrical interpretation of x,y are as *coordinates* of the point, i.e. the distances of the point as measured from two fixed perpendicular lines (the x- and the y-axes). (This idea was due to Descartes and Fermat.) The calculations of analytic geometry depend crucially on the Pythagoras theorem which ennables one to calculate the distance between two points (x,y), (x',y'). So in that respect as well, analytic geometry still depends on Euclidean geometry.

These observations of Hegel on the difference between algebra and geometry are, I think, very valuable, and a corrective to the commonly held, simplistic, view that elementary geometry can be ``reduced'' to the algebra of numbers and hence to set theory.

## 4. *Hegel's Dissertatio*

Hegel's early philosophy of geometry, discussed above, is incomplete in a number of ways. In his mature philosophy of geometry, especially as presented in *JS*, *NP*, *WLg*, *WL* Hegel addressed this incompleteness and developed further some of the themes of *GS*. A detailed discussion of this mature philosophy is beyond the scope of this paper. Instead we will briefly describe the geometrical philosophy of Hegel's dissertation, composed within a year of *GS*,[129] and which, as discussed in the introduction, can be regarded as providing the transition between the early and mature geometrical philosophies of Hegel.

Hegel moved to Jena to join Schelling soon after the completion of *GS*, and this resulted, at least from a geometrical perspective, in a drastic change in Hegel's thought. This is very clear in Hegel's dissertation *Diss*. It comes as a shock to the reader to go from the careful, detailed geometrical perspective of *GS* to the ``applied mathematics'' of the dissertation, formulated in terms of Schelling's identity philosophy (using categories such as potencies, cohesion and polarities such as

[129] Hegel submitted his dissertation in October, 1801, the year that he moved to Jena. Opinions differ as to whether Hegel prepared some of *Diss.* when he was at Frankfurt or whether it was completely composed when Hegel was in Jena. For the different positions, see the discussions of Harris and Hoffheimer (Ch. 2 of *TS II*; Ch. 4 of Hoffheimer's dissertation). If Hegel worked on *Diss.* when he was at Frankfurt, he would surely have been thinking about it and *GS* at around the same time. The different treatments of geometry in the two works makes this unlikely, and so I am inclined to think that *Diss.* was composed entirely at Jena when Hegel was directly under Schelling's influence. This is confirmed by Rosenkranz's statement (*ibid.*, p.154) that *Diss.* was written in the spring and summer of 1801. (See also p.72 of the paper: T. G. Bucher, *Wissenschaftstheoretische Überlegungen zu Hegels Planetenschrift,* Hegel Studien 18 (1983), 65-137.)

subject/object and ideal/real). Discussion of *Diss.* has, for the most part, centered round Hegel's speculations, influenced by the *Timaeus*, on a ``rationally'' based sequence of numbers which give the distances of the planets from the sun, and the counterexample to that proposed sequence resulting from the discovery of the asteroid *Ceres*.[130] We will not be concerned with these speculations in this paper. Indeed, Hegel's discussion of the sequence is a short addendum to the main text of *Diss.* and is not essential for the rest.

In *Diss.*, Hegel gives a philosophical interpretation of the geometrical structure of reality, in particular, of space, time, matter and the solar system. Unlike *GS*, the dissertation has almost no discussion of geometry as a science but it shows that Hegel's mathematical interests had extended to the *Calculus* (a theme that Hegel investigates in great detail in *WL I*). The dissertation does address two fundamental problems not dealt with in *GS*. These are:

1) what is the philosophical basis for the *One* of geometry, i.e. the *point*?
2) how does applied mathematics relate to pure mathematics?

Starting with 1), we saw earlier that in GS, the point generates the basic objects of geometry – line, plane and solid – but Hegel does not say very much about the philosophical basis of the point.[131] In the identity philosophy of Schelling, the point is presented within a much more detailed theory. Indeed, the NeoPlatonic *One* appears as the *point* of indifference between objective and subjective, the real and the ideal. (Subjectively the point is conceived of as *mind.*) This One itself is absolute, indivisible, undifferentiated. The form-based Platonic philosophy through which the classical NeoPlatonic philosophers understood procession and reversion, is more muted in Schelling's view of the absolute, and influences of Spinoza, Kant and Fichte are clear. We can, I think, say that the NeoPlatonic procession and reversion assumes in Schelling's work the more modern form of the *philosophy of nature* and so provides a philosophical basis for understanding how mathematics applies in nature and so for answering 2). We now describe very briefly the background in the thought of Schelling to *Diss.*.

For Schelling[132] opposition and duality arise within this absolute One itself[133] as the ``self-differentiation of the undivided absoluteness into

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[130] A detailed discussion of the philosophical background to Hegel's sequence is given by T. G. Bucher (*ibid.*). A clear, short account of the sequence is given by Adler in his introduction to the English translation of *Diss.*: G. W. F. Hegel: *Philosophical Dissertation on the Orbits of the Planets (1801), Preceded by the 12 Theses defended on August 27, 1801*, translated with Foreword and Notes by P. Adler, Graduate Faculty Philosophy Journal, Vol. 12, Number 1&2, pp.269-309, 1987.

[131] As we saw in 2(g), he does however say in *GS* that *der Punkt ist die einfachste Begrenzung der Unendlichkeit*.

[132] Cf. R. Stern's discusssion on p.xxi of: *Ideen zu einer Philosophie der Natur (Ideas for a Philosophy of Nature)*, trans. E. E. Harris and P. Heath, Cambridge University Press, Cambridge, 1988. (This translation will be abbreviated to *Ideas* below.)

[133] H. S. Harris points out (pp.23, 57 of Harris and Cerf, translation of the *Differenz* essay, *ibid.*) that Hegel and Schelling disagree in their understanding of *philosophical difference*. In Schelling's identity theory, difference arises within the absolute itself only to be reconciled in the identity of identity and difference, whereas for Hegel in the *Differenz* essay, as for Fichte (and Kant), difference (reflection) is ``antinomic'' in character. This disagreement can, I think, be at least partially explained in terms of the shift in Schelling's thought between the first and second editions of his *Ideen*. When writing the *Differenz* essay, Hegel had only the first edition available. This appeared in 1797, and is Fichtean in character, expressed in

subject and object''.[134] Subject and object are also expressed in terms of the *ideal* and the *real*, the worlds of *mind* and *nature*. However, the absolute-real is the same as the absolute ideal, and at the level of that identity - that of the *Absolute* or the *One* - the two are integrated indifferently in the *point of identity*, conceived of as an *absolute knowing*, an eternal act of cognition.[135] In this act, essence and form are the same; essence (the subjective) resolved into form (the objective) is the absoluteness become objective (in Nature) and conversely form resolved into essence gives the absolute as subjective. (This can be regarded as a development of the procession and reversion in NeoPlatonic thought.)

In the Jena period, Schelling formulated his philosophy in terms of *Potenzen* (*potencies*)[136] through which the absolute ``is posited as the undivided whole under distinct determinations''.[137] There are three potencies:[138] the transition of unity into difference, the transition of difference into unity and third, absolute identity, the identity of identity and difference. These potencies are the principles of both the real and ideal worlds. In the *real* series, the potencies appear as the three dimensions of space – line, plane and depth - while in the *ideal* series, they appear as self-consciousness, sensation and intuition. Hegel understands change in the real series in terms of these potencies: *nihil enim est mutatio aliud, quam aeterna identitatis ex differentia restitutio et nova differentiae productio, contractio et expansio.*[139]

The resolution of essence into form (above) is conceived by Schelling in terms of the absolute as mind (the subjective) becoming explicit in Nature (the formed objective). According to Schelling,[140] ``Nature should be mind made visible, mind the invisible nature''. This, and other Schellingian themes, are present in Hegel's dissertation.[141] However Hegel brings a number of his own ideas to the discussion.

In *Diss.*, the *making visible* of mind is to be understood as the One (or absolute) as the geometrical point – the geometrical form of *the point of indifference* - developing through the dialectic of line, plane and solid to provide the three dimensional world for Nature, matter and the solar

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terms of dualism and polarity. In the second edition of 1803, Schelling works within the framework of the identity philosophy.

[134] p.47 of *Ideas*, *ibid.*.

[135] pp.44-47 of *Ideas*, *ibid.*.

[136] I am inclined to think that Schelling's theory of *Potenz* is a development of the NeoPlatonic view of Proclus on the nature of procession and reversion. In the Proclus theory (e.g. Proposition 21 of *The Elements of Theology*, *ibid.*), reality falls into *orders* (e.g. the soul order) and each order has its beginning in a *monad*, the ``single originative principle'' of the order. The *monad* of Proclus seems very similar to the *Potenz* of Schelling, and indeed Schelling discusses monads and potencies together (pp.48-49 of *Ideas, ibid.*).

[137] This is taken from the introduction to Schelling's lectures on the Philosophy of Art, in the translation by H. S. Harris (p.54 of: Harris and Cerf, translation of the *Differenz* essay, *ibid.*).

[138] pp.xxi, 180 of *Ideas*, *ibid.*.

[139] *Diss.*, p.23.

[140] p.42 of *Ideas*, *ibid.*. In particular, the third potency in Nature corresponds to the third dimension, *depth*.

[141] e.g. *Diss.*, p.23: *Differentia reali polorum et linea cohaesionis ita intellecta, ad alteram differentiam, idealem sive potentiarum, subjecti et objecti transeamus*. Of course, Baader influenced both Hegel and Schelling strongly at this time. His influence on *Diss.* is illustrated by Hegel's use of the analogy of the lever (*vectis*). See also Hegel's fifth dissertation thesis.

system. (As we noted above, Schelling provides a similar basis for Nature using his theory of potencies.) Hegel says:[142]

> Ut realem materiam intelligamus, spatii abstractae notioni contraria sive subjectivitatis forma addenda est, quam voce magis latina mentem, et si ad spatium referatur punctum appellemus. Quo modo punctum, aut sub differentiae ipsi propria forma, tempus, atque spatium elementa constituunt materiae, quae quidem non ex iis conflata, sed eorum principium est.

So for Hegel with connotations of the activity of the Platonic demiurge, the Kantian understanding of time as the form of *inner* intuition, the Fichtean *Self* and Schelling's understanding of nature as *mens*, *mind in the form of point is the subjective principle behind space and time*. For Hegel, the elements of space constitute matter *(spatium elementa constituunt materiae*) but not in the sense of being actually built up from spatial ``atoms'' but rather in the sense that the point is the *principium* of these elements. Matter is, Hegel says, ``objective gravity'' (*materia sit objectiva gravitas*). Matter is to be understood in terms of the *real* series, and corresponds to time and space in the *ideal* series.[143] The line of pure geometrical space (the *ideal* line) has its real counterpart in matter as the *line of cohesion* (below) (the *real* line).

The real line in matter is expressed in terms of *forces* (as in Kant's theory), and of course gravity, objective matter, is a kind of force: *nihil enim aliud est corpus quam vis physicae*. The fact that a force acts along a straight line, i.e. is *vectorial* in character, is, I think, also understood by Hegel as the real counterpart of the ideal line in *GS*. Indeed, in *GS*, the point expressed its content in the form of the *line* with its limiting end-points. This is also the case for the real series in *Diss.* but there is an indeterminacy for end-points or *poles* of the line in that series which, as we will see, results in *motion*. (This is not present in pure geometry.)

The line in the *real* series, i.e. the linear holding together of matter through forces, is conceived by Hegel in terms of *cohesion*. Cohesion was an important category in Schelling's philosophy of nature, being close to Schelling's third potency (above): cohesion is defined by Schelling as ``synthesis of identity and difference'', associated with clinging together.[144] The real line is then for Hegel the ``line of cohesion'' (*linea cohaesionis*). This line ``constitutes gravity'' (*quam gravitas constituens*). He discusses examples of this line in nature, in particular the case of the solar system which we look at below. The point of indifference can be identified for the lines of cohesion as they appear in nature as physical systems. In general, this point is the *center of mass* (= center of gravity) of the system – indeed, in the absence of external forces, the center of mass is *at rest* (indifferent). This point is a *mathematical* point, a point of ideal geometry as it arises out of *mens*.

As we saw earlier in our discussion of *GS*, the point identifies itself with itself through the end-points of the line that it expressed. So in

[142] *Diss.*, p.23.
[143] *Diss.*, p.20.
[144] pp.65, 116 of *Ideas*, *ibid.*.

the ``real series'', the line of cohesion, originating in the point of indifference, has its ``end-points'' or nodes lying on it where the line is ``disrupted'' (*disrupta*). (This is reminiscent of the *Knotenlinie von Maßverhältnissen* (nodal line of measure relations) treated much later by Hegel in *WLg I*, *WL I*.) The nodes divide the line up into ratios that Hegel finds particularly significant in the case of the *solar system*. We now briefly describe Hegel's geometrical understanding of physics and in particular, of the solar system. Hegel's argument in *Diss.* is difficult to follow, and I have been free in my interpretation below.

For the real series, the line of cohesion can be complete with its two end-points (poles) fixed – which is how it's ideal version is treated in *GS* – or just one end-point fixed or even none fixed. A good example where the two poles of the line of cohesion are fixed occurs for the bar magnet where the line *joins the two poles*: in this case, the point of indifference is the ``neutral'' point at the middle of the line. Another example is that of the *lever* with its two end-points, one of which supports the weight to be moved and the other on which a force is exerted, such as in a dolly. The fulcrum is the indifference point. In the real series, the end-points, while fixed relative to the line, may not be fixed relative to space. When we lift a weight using a dolly both end points (and the line of cohesion) *move* although the point of indifference (the center of gravity) is unchanged. An example where one end-point is fixed and the other is not is that of the *pendulum* where, of course, the end with the bob swings back and forwards. The pendulum itself is an incomplete lever (*vectem incompletum*) – it has ``lost'' its other pole. Hegel sees relations between these examples: in the fifth thesis for *Diss*. he claims that the magnet is the natural lever, and the gravitation of the planets toward the sun is the pendulum of nature.

More precisely, I think, the solar system has to be thought of as a *system of pendula*. It is fixed at the point of indifference and each of the planets moves like the bob of a pendulum (except that its orbit is a closed curve). How are we to understand the line of cohesion in this case? Regarding the planetary orbits as (roughly) circular and centered at the sun, the natural lines will be those associated with the circle, and these, as we saw in our discussion of *GS* (2(i)(α)) are the radial lines or diameters. We take the line of cohesion to be one of these radial lines. As Hegel pointed out in *GS*, there is no canonical radius and in this sense, the line of cohesion is ideal – but the arbitrariness will not matter. There is, however, a canonical *plane*. This is the *ecliptic plane*, the plane that contains most of the planetary orbits including that of the earth, in particular, the radial line. The orbit of each planet intersects that line at exactly one point, and so the planets produce a sequence of ``nodes'' (*nodi*) on it. Note also that the radial ``line of cohesion'' sweeps out the ecliptic plane as it rotates about the point of indifference; this is the physical counterpart of the dialectic of *GS* where the line gives the plane as its ``positive being'' (3(a)).

In the language of the *Timaeus*, we may say that the solar system is the ``moving image'' of the eternal geometrico-philosophical logic of point, line and plane. Next, how does the third dimension of space enter into the solar system? The solar system is not a *real* solid body. Instead, for Hegel, the three dimensional character of the system is to be

understood in terms of the *ideal* body:[145] *in motu autem circulari, quo corpus ideale producitur.* One can interpret this in terms of Hegel's understanding of a geometrical figure in *GS*: there, the conceptual whole determined itself through its limiting parts. Here, the ``whole'' of the solar system is ideal but its ``parts'' or ``limits'' are real bodies – we do not see the ideal body as real but rather as the system articulated in a collection of bodies whirling in space. In a similar way (3(d)), we do not see in the real world the conceptual *necessity* through which the sides of a triangle determine the whole triangle in pure geometry.[146]

In *Diss.*, there are two primary cases to be considered for bodies in motion. First, the body may be *real*, in which case its motion is free fall, or the body is *ideal* (such as the solar system above) in which case it is determined as a system of real bodies in circular motion. In both cases, Hegel has to relate classical Euclidean geometry to the more modern mathematics of *analytic geometry* and *the calculus*. Both of these require *algebra*, and it is natural to look for the earlier philosophy of point, line, plane and solid appearing in algebraic form.

For this Hegel falls back on the mathematical origin of the theory of potencies (powers) from the work of Eschenmayer. The line, dimension 1,[147] is represented by power one, $x^1$; the plane, dimension 2, by power 2, $x^2$; and finally the body, dimension 3, by power 3, $x^3$. Also x represents the length of a line interval, $x^2$ the area of a square (in the plane) and $x^3$ the volume of a cube (in three dimensional space). Hegel interprets physical laws in terms of powers and the philosophy of point-line-plane-body. So the motion of a *real* body falling freely to the earth – described by $s=\frac{1}{2}gt^2$ – involves first power of *s* and second power of *t*, and is interpreted by Hegel in terms of the transition of the line into the plane. What, then, is the algebraic equation for the circular motion of the *ideal* body. Hegel's answer is: *Kepler's third law*.

This law says that for any planet in a (roughly) circular orbit, we have $T^2=kr^3$, where T is the period of its orbit, r is the radius of the orbit and k is a constant depending only on the mass of the central body (the sun). The problem is how to understand this in terms of an *ideal* body, and Hegel does so by uniting *linea distantiae et motus quadratum* in the

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[145] *Diss.*, p.25.

[146] This ideal body can also, I think, be understood in terms of Schelling's third potency, absolute identity, which gives the third, ``body'' dimension of *depth* (Footnote 140) and reestabilishes the original ideal unity of the point as *mens* with which the process begun, cf. Footnote 96.

[147] In this discussion, it is natural to ask: how does the *point* fit in with this – does it correspond to some mathematical power? By analogy, it is reasonable to associate it with the $0^{th}$ power of algebra, $x^0$. In algebra, $x^0$ is defined to be 1 except when x=0 when it is not defined. Clearly the $0^{th}$ power is more subtle than the other powers, and indeed is defined in algebra the way that it is for formal reasons (so that the laws of exponents remain valid for all integer values). A likely answer, I think, is that the point, as representing the absolute One, *mens*, is at a different philosophical level from line and plane, the latter arising through the *difference* of the One. This is, perhaps, related to the argument of *WL I*, p.253, where Hegel claims that $y=ax$ is only *formally* a function of *x*. For otherwise, if we treat $y=ax$ as function of *x* like the other powers, then the rule of differentiation would give $dy/dx=a\cdot x^0$ thus involving the $0^{th}$ power. As a result, the first power is regarded as having a different philosophical status from that of the higher powers. Hegel understands $y=ax$ in *WL I* not as a *function* of *x* but rather as a *relation* between *y* and *x* given by the ratio *a*: $y/x=a$, a fixed quantum. As Hegel puts it (*WL I*, p.289): D*ie gerade Linie hat ein empirisches Quantum*.

cube. The line is present in the *radial line* (the line of cohesion) whose length is r, the square as the *second* power of the *period* T and also as referring to the ecliptic *plane* in which the planet orbits, and the cube (expressing the ideal body) as the *third* power of the radius.

From a present day perspective, it is hard to be sympathetic to this kind of explanation of why a cube power comes into Kepler's third law. One observation, though, is important. Hegel's approach is not one of ``*a priorism*'': for both Kepler and Hegel, the third law, like all scientific laws, is established on the grounds of *empirical evidence* (in Kepler's case, on the basis of the data left to him by Tycho Brahe). There is, of course, no particular reason why one would *expect* the cube of *r* to be connected with the square of T as in the Kepler law: other powers, *a priori*, could be involved, or indeed a general mathematical function of *r,T*. Only *after* the law has been grounded empirically, do Kepler and Hegel look for a philosophical explanation – basically, for an explanation in terms of the unhypothetical science.

Hegel himself, later in his life, expressed dissatisfaction with his treatment of the planets in *Diss.*.[148] Yet he continued to believe in the special philosophical significance of mathematical powers even in his later logic.[149] There, as far as applicability (*Anwendbarkheit*) is concerned, we are dealing with mathematics *in Natur* and as we have seen, for Hegel and Schelling, this mathematics *must* involve power functions since Nature is philosophically determined by the *Potenzen* operative in the fundamental structures of space and time. The unhypothetical science in this case is a philosophy of powers, based ultimately on the dialectic of point-line-plane-body. That dialectic involves moving from a lower power to a higher power (e.g. from a line into a plane) and also in the reverse direction (as we saw in 2(g)). These are reflected in *calculus*, the first direction involving *integration*, where $\int x^n dx = x^{n+1}/(n+1)$ (one power *higher*), and the second direction in *differentiation*, where $d/dx(x^n) = nx^{n-1}$ (one power *lower*).

I think that Hegel's fundamental error with respect to the work of Newton lies in his lack of appreciation for the importance of differential equations. A *differential equation* in Physics is an equation relating the derivatives (including usually those of order > 1) of the functions involved in the physical process, and that specifies the evolution of the process. Differential equations embody the laws operative in Nature and

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[148] See the discussion in T. G. Bucher, *ibid.*.

[149] The theory there (e.g. *WL I*, p.308f.; *SL*, p.300f.) assumes a more sophisticated form in which the (Taylor) expansion of functions in terms of powers is allowed, the powers counting as *moments* (*Potenzenmomente*). This ennables Hegel to include higher transcendental functions such as *sin x* and $e^x$ in his theory of powers; so, for example $e^x = \sum x^n/n!$, the summation being over all integers $n \geq 0$. Of course an *infinite* sum is involved here, and issues of convergence are involved. Indeed, as Fourier discovered in his investigations into heat, functions that are the sums of ``Fourier series'' cannot usually be expressed in terms of power series, and most continuous functions are not differentiable (and so *cannot* in principle be given by a power series). The *measurable* functions, required in modern integration theory, are at a much higher level of generality than even the continuous functions.

At the general philosophical level, Hegel in *WLg* I and *WL* I actually rejected the three *Potenzen* on the grounds that their use is an excuse for evading the task of grasping conceptual determinations, e.g. (*WL* I, p.335 (*SL*, p.325)): *Der Gebrauch jener Formen ist darum weiter nichts als ein bequemes Mittel, es zu ersparen, die Begriffsbestimmungen zu fassen, anzugeben und zu rechtfertigen.* (``The use of those forms is therefore irrelevant other than as a lazy way of saving oneself [the trouble] of grasping concept determinations, of pointing out and justifying [them].'')

the information that they give is much more far-reaching and insightful than what can be obtained from simple algebraic equations (such as $s=\frac{1}{2}gt^2$). In particular, all of the information concerning the movement of a planet round the sun – including that expressed in Kepler's laws – comes from studying the appropriate differential equation (Newton's universal law of gravitation). This equation gives a lot more information than just Kepler's third law – for example, it applies to electrical phenomena as well. I think that differential equations can be made to fit in quite naturally with Hegelian principles.

We can illustrate the difference between the Hegel-Kepler approach to science and the differential equation approach of Newton (and present day science) as follows. In *Harmonices Mundi*, Kepler[150] established that musical ratios (such as 4:5) correspond remarkably closely to the ratios between the extreme velocities of the planets and explained it platonically in terms of an archetypical harmony (``the music of the spheres''). From a modern perspective, however, this impressive correspondence between the musical and velocity ratios is rather to be understood ``scientifically'' in terms of the stability of the planetary orbits, established using the methods of differential equations.

Hegel has had no lack of critics for his support of Kepler against Newton. But so-called ``scientific'' methods are hopelessly limited for the understanding of the fundamental problems of humanity – those concerning the natures of ethics, religion, mysticism and aesthetics, and even of the natures of mathematics and science themselves. These problems do not only involve *abstraction* from the individual, as when someone peers through a telescope and records observations, and what only survives is the results, the observer qua truth being only an x, indifferently replaceable by another:[151] rather, human individuality is central to them, and one requires for their study a notion of truth which is closer to the notion of ``genuineness'' in normal life.[152] This notion of truth projects down to that which (legitimately) informs the abstractions of ``scientific'' methods, but at the price of restricted content, in much the same way as the qualitative is lost when we evaluate something *only* quantitatively. In both cases, what is abstracted from in the projecting is still present implicitly, waiting to reemerge.[153] Plato, Kepler and Hegel tried to formulate this notion of truth. In the present day world, its clarification sadly seems further away from resolution than ever.

Tensions[154] built up between Hegel and Schelling in their collaboration on the *Kritische Journal der Philosophie*, and Hegel gained his philosophical freedom after Schelling left Jena in 1803. Hegel's 1804/05 account (*JS*) of Logic, Metaphysics and Natural Philosophy establishes the continuity of the thought of *GS* as far as geometry, and more generally, mathematics,

[150] See J. V. Field, *ibid.*, p. 97, 150.
[151] Hegel specifies the fundamental failing of this kind of abstraction (*WLg* II, p.40; *WL* II, p.249 (*SL*, p.609)): *Dies macht also diese Allgemeinheit zur abstrakten, daß die Vermittlung nur Bedingung ist oder nicht an ihr selbst gesetzt ist.* (``[What] therefore makes this universality into the abstract is that the mediation is only condition, or is not posited in itself.'')
[152] This notion of truth also operates in creative mathematics and science.
[153] In Hegelian terms, the reemergence is exemplified by the transition from quantity to quality, through which Hegel understands the nodal line of measure relations and also the change in the qualitative character of a state resulting from the expansion of its population (*WLg 1*, pp. 219-220; *WL 1*, p.384 (*SL*, p.371)).
[154] p.110f. of: T. Pinkard, *ibid.*.

is concerned. In *JS*, a transition from logic/metaphysics to natural philosophy is established, geometrical themes such as the philosophical significance of the circle and of the congruence theorems, are revisited, and the crucial role of the Pythagoras theorem for geometry, missing in *GS*, is discussed. Further, fundamental philosophical issues for geometry, not discussed in *GS*, are addressed: in particular how *construction* and *proof* are to understood. Also appearing in *JS* are discussions of *calculus* which, of course, Hegel would have studied in the work of Newton while preparing *Diss.*. With *JS*, then, we enter the period of Hegel's mature thought on mathematics. But this is another story.