Frege and Hegel on concepts and number

Alan L. T. Paterson

1. Introduction

In 1812, the first volume of Hegel's great work on Logic, the *Wissenschaft der Logik*, appeared, followed up by the second volume in 1813. In this paper, we will abbreviate the *Wissenschaft* to *WL* and its English translation by A.V. Miller to *SL*.[1] In *WL*, Hegel develops his philosophy of *the concept (der Begriff)*. The concept, as distinguished from concepts in general, is articulated through the whole gamut of key concepts such as being, something, quantity, measure, ground, identity, difference, whole and parts, existence, appearance, the absolute, substance, the judgement and the syllogism. Hegel then moves on to the logic underlying the three main branches of science of his time, *mechanism* (physics), *chemism* (chemistry), and *teleology* (biology). The work culminates in the *Idea*, the source of objective truth as such, so that the logical structure of *WL* is the touchstone of truth in all areas of thought and experience. In particular, the logical system of *WL* is the source of validity for the other two areas of his philosophical system, the *philosophy of nature* and the *philosophy of spirit*.

In 1884, Frege published his remarkable little book *Die Grundlagen der Arithmetik*[2] on the philosophy of number. In the book, he develops his philosophy of *concepts* and their relation to the objects and concepts that fall under them. Frege's definition of number is in fact formulated in terms of concepts. The English translation of the *Grundlagen* by J.L. Austin[3] will be abbreviated to *FA*.

At first sight, there seems to be very little at all in common between Hegelian logic and Fregean logic. Hegel is not at all interested in Leibniz's program of *characteristica universalis*[4] which was so influential for motivating the work of

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Notes

The edition of *WL* used in the paper is that of G. Lasson: G. W. Hegel, *Wissenschaft der Logik*, I, II, (Vol. 56, 57, *Der Philosophischen Bibliothek Band*), Verlag von Felix Meiner, Hamburg, 1967. The English translation *SL* of *WL* is: *Hegel's Science of Logic*, translated by A. V. Miller, Humanities Press International, Inc., Atlantic Highlands, N.J., 1989. In quotations from *SL*, I have followed the present day practice of rendering *Begriff* by *concept* rather than by Notion.

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Die Grundlagen der Arithmetik, von G. Frege, Breslau, Verlag von Wiljhelm Koebner, 1884. Austin's translation of the *Grundlagen* (*infra*) contains also the German original.

3

The Foundations of Arithmetic, Philosophical Library, New York, 1950. I have made minor changes to Austin's translation where I have felt that it does not completely capture Frege's meaning.

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Hegel (p.685, *SL*) comments disparagingly that *this was a pet idea of Leibniz, embraced by him in his youth, and in spite of its immaturity and shallowness not relinquished by him even in later life .... .*

5 In this connection, the mathematical logician Hao Wang (p.51ff. of: *From Mathematics to Philosophy*, Routledge and Kegan Paul, London, 1974) lists the *pros* and *cons* of the influence of

Frege. Leibniz's program can be regarded as the philosophical forerunner to the modern formal system, in which philosophical issues are to be reduced to calculation (comparable with practical arithmetic).

Rather, for Hegel, the true conceptual *meaning* of the canons of logical inference (such as the syllogism) has to be given, and this conceptual meaning links up the aspects of the syllogism in a non-formal, logical development. This dismissal of formalism by Hegel is regrettable since in fact expressed in the formalism and emerging out of its interpretation, there are profound and important insights into logic and mathematics which are inaccessible without the formal clothing. Hegel's dismissal of the Leibniz program is perhaps understandable in view of the minimal achievements in advancing the aims of the program that had been made up to the time of Hegel.

On the other hand, the Hegelian scholar must be impressed at how *limited* the range of Fregean thought is, giving little or no help at all to traditional philosophical issues such as those concerning science, religion, ethics, aesthetics and metaphysics. In contrast, in the Hegelian philosophy, logic underpins the full range of philosophical thought, ``across the board''. To the Hegelian, the precision of the Fregean philosophy is artificial, and offers, instead of philosophical bread, only a stone.

What we will argue for in this paper is that the main problem with the Fregean philosophy is its *overly rigid view of concepts*. Reaction to such conceptual rigidity has resulted, in the 20th century, in Wittgenstein's *Philosophical Investigations*. We will develop the argument that when Frege's notion of concept is replaced by the Hegelian notion of concept, then Fregean and Hegelian logic shed light on each other, and suggest a modern Hegelian philosophy of number and indeed of mathematics in general in which Hegelian principles are seriously brought to bear on the issues of modern mathematical logic.

In view of the apparent differences between Fregean and Hegelian philosophy and (it would seem) the almost empty class of philosophers who have studied both, it is desirable to make this thesis of the paper initially plausible by indicating some common ground for the logics of Frege and Hegel. To this end, we now list five significant common features in these logics. We will, at the same time, indicate the different ways in which these features are understood by the two philosophers.

(a) *The primacy of concepts*

The first of these features is the fundamental emphasis on *concepts* in the work of both philosophers. This provides a common basis for both philosophies: *both are philosophies of conceptual development*. In the case of Frege, this development is expressed technically in his *Begriffschrift, concept* writing. Hegel's logical development in *WL* is expressed in the conceptual deepening and mediation of (what he calls) the concept (*Der Begriff*), which, as described in the first paragraph of the paper, is expressed through particular concepts, and in which the simple concept of *Being* is developed into the structured concept of the *Idea* in which concepts express their objective content.

However, the concepts relevant to logic are, for Hegel, at a much more general level than those of Frege. Hegel considers concepts such as *Being* and *Whole and*

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mathematical logic on philosophy this century, and doubts that the influences have been good ones. Wittgenstein in his later years thought that mathematical logic was a bad influence both in philosophy and in mathematics.

Parts as against the relatively concrete concepts of Frege (such as a *white cat* or a *black, silken cloth*).

(b) *The structure of concepts*

Both Frege and Hegel view concepts as *structured by other concepts*. Concepts, for Frege, are either simple or complex, the complex ones having other concepts as components. These components are called the *characteristics* of the concept in question, and (in a Fregean simile), the characteristics of a concept relate[5] to the concept as stones do to the house that they make up.

Frege uses other metaphors to express how concepts relate to their characteristics and the objects that fall under them: a concept is ``unsaturated'', has a ``logical place'' to be filled by objects or other concepts[6] and this filling is absolutely clear and precise. In particular, we must know *exactly* when an object falls under the concept or not. In fact, E. W. Kluge[7] has argued that for Frege, objects are nothing other than complete concepts, and are just property-complexes on the Leibnizian model.

Like Frege, Hegel regards the concept as structured by component concepts corresponding to Frege's characteristics. These are called the *moments* of the concept. The concept develops through its moments and their interaction, the latter being analogous to the interaction of distance and force in the *principle of moments* of elementary physics.

In the Hegelian philosophy, the thinking of the concept through its moments, through its own nature, leads one beyond the concept as given originally. And this is not to be thought of subjectively, as if it is only *our thinking about it* that brings out the ``leading on''. Rather it is intrinsic to the concept itself.

In contrast, the Fregean characteristics *statically* make up the concept, as it were, relating to the concept as independent, fixed parts to a whole. The Fregean concept is completely precise, its characteristics are given and it is known exactly when an object ``falls under'' the concept or not. But for Hegel, concepts are *dynamic* in character, compressing in tension within them conceptual relations which push and develop them. To take an analogy, the Fregean concept is to be compared to a point mass in Newtonian mechanics, constant, preserving itself in its interactions, whereas the Hegelian concept is a modern particle, an ``energy packet'', appearing, disappearing, in a flux of energy, breaking up into smaller particles, etc. (but nonetheless, conceptually coherent).

(c) *Criticism of abstraction*

In this paper, we will criticize this Fregean position as giving a passive, inert view of concepts in which a concept is fixed in abstraction with its constitutive characteristics. At the same time, while the paper argues that Frege's view of concepts is vitiated by such *abstraction*, it should be stressed that both Frege (in general) and Hegel were very much aware of the dangers of abstraction. In a

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[5] See p.4 of *Gottlob Frege: On the Foundations of Geometry and Formal Theories of Arithmetic*, trans. E. W. Kluge. As another example, the concept black, silken cloth has the characteristics of black, silken and cloth.

[6] p. xvii of E. W. Kluge, *supra*.

[7] p. xviii, *supra*.

letter to the formalist J. Thomae, Frege even goes into verse about these dangers:

> Abstraction's might a boon is found
> While man does keep it tamed and bound;
> Awful its heav'nly powers become
> When that its stops and stays are gone.

As for Hegel, in discussing the transition from the particular to the universal[8], he argues that this can happen in two ways. One is the way of *abstraction*, a *false path* in which one *rises to the higher and the highest genus* and stays on the *surface*, giving the *determinateness of indeterminateness*[9]. The other way is through the *individual*, the way of the *concept*.

It is perhaps the extreme abstraction involved in formalist views of mathematics, pointed out by Frege, that is the real reason for Hegel's antipathy to formal logic, an antipathy which he surely took too far.

(d) *The content neutrality of number*

A key feature of Frege's *Grundlagen*, which is the rationale for Frege's definition of number, is that *number is context independent*, so that a number such as 5 applies to collections of things whatever their *qualitative* character: shoes, ships, cabbages, kings, planets, angels, possible worlds, etc.. This remarkable feature of number distinguishes it from qualitative concepts, such as the color concept *white*. These concepts are limited in the things to which they apply.

This feature of number is, interestingly enough, also fundamental in Hegel's development of number in *WL*. In Hegel's logic, number comes in the *first* part, that dealing with *Sein* (being), at the end of a long development in which the concept of being is taken from its original, most abstract form, through its finite, relatively determinate form of *Dasein* and then through its infinite form of *Fürsichsein*. At that stage *the qualitative character of being becomes ``indifferent''*: it vanishes with the ``becoming'' of the *quantitative*. The latter determined in its finite form gives *quantum*, whose precise concept is number. Number as the result of the qualitative's becoming indifferent in Hegelian logic exactly corresponds to Frege's emphasis on the content-neutral character of number. The difference is that for Frege, this character is simply *noticed,* whereas for Hegel, it is *deduced.*

(e) *Number as based on the concept of set*

In the work of Frege, the context independent nature of number leads the way to his definition of number. For since
number has to apply to a collection of things without regard to the nature of the things collected, it follows the number of the collection gives the same thing for *all* collections equinumerous to the original one. So, for example, the number 6 must apply equally to 6 cats, 6 dogs, 6 anythings, and so is associated with *all* 6-collections, a very large, highly infinite, collection of *sets* (or *classes*). The precise definition that Frege gives of number is as follows[10]:

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[8]

[9] p.619, *SL*.

[10] p.609, *SL*.

> the number which belongs to the concept F is the extension of the concept ``equinumerous (*gleichzahlig*) to the concept F''.

Here, a collection is determined by a *concept* and two concepts are *equinumerous* if their extensions can be paired up in a one-to-one way. We are then confronted with Cantor's infinite set theory with its well-known difficulties, since infinite sets arise not only as the extensions of concepts but also as the set of concepts themselves which are equinumerous to the original one. (We will discuss Frege's definition of number in more detail later in the paper.)

Since the philosophies of number of both Frege and Hegel arise out of the qualitative neutrality of number, and since, for Frege, such neutrality is expressed in terms of *sets* (or *classes*), it is natural to look at the development of Hegelian logic leading up to number to see if it contains a corresponding latent set theory. Of course, we cannot expect explicit set theoretic concepts in the work of Hegel since set theory and the work of its founder Georg Cantor occurred well after Hegel's death. Frege was at an advantage in this respect.

We now want to argue that such a latent set theory *is* to be found in the Hegelian development of *Sein*, viz. in *Fürsichsein*, the treatment of the *one and the many*. Indeed, the connection between *the one and the many* and *set theory* is clear in a famous definition by Cantor of the concept of *set*. According to Cantor[11],

> Unter einer ``Mannigfaltigkeit'' oder ``Menge'' verstehe ich nämlich allgemein jedes Viele, welches sich als Eines denken lässt ...

or in English translation,

> By a manifold or set I understand precisely each general many, which can itself be thought of as one ...

With this interpretation of *Fürsichsein*, the concept of *set* (or *class*) can be regarded as implicit in *Fürsichsein,* and, as in Frege's approach, precedes that of number.[12]

While Hegel does not develop any set theory in the modern sense, the philosophical one-many basis for sets that he gives is important. A justifiable criticism of Frege is the absence of such a philosophical basis for sets (extensions of concepts) on which his definition of number depends. (Uncritical use of set theoretic concepts were the downfall of the Fregean system.

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pp.79-80, *FA*. Austin in his translation *FA*, translates *gleichzahlig* as equal. This is a poor translation of that word, since equal concepts should have equal content, and of course that is exactly what Frege does not want: the point about number is that it is content independent as discussed above. It is also a poor translation since it does not bring out the *zahl* aspect of *gleichzahlig*. In set theory, two eqequinumerous sets are often called *similar*.

[11] G. Cantor, *Gesammelte Abhandlungen mathematischen und philosophischen Inhalts*, ed. E. Zermelo, Springer, Berlin, 1932, p.204. See the discussion of Cantor's ideas on sets in Hao Wang, *supra*, p.188.

[12] Of course, the theme of the one and the many was extensively discussed much earlier in Greek philosophy, and one could equally argue that there is latent set theory also in Greek philosophy. I think (as Cantor surely did) that this is true, but Hegelian logic is far more developed than Greek philosophy, and the detailed logical development of *WL*, I would argue, gives considerable help with the kind of reasoning needed to develop set theory within the Hegelian framework of the one and the many.

Post-Fregean mathematical logic devoted a great deal of effort to clarify the concept of *set*.)

An explanation of these five parallels between Fregean and Hegelian logic is given in the present paper. It is that Fregean logic, when released from its rigid understanding of *concept*, as discussed earlier, reveals a Hegelian character which ennables it to fit into the Hegelian framework when the latter is developed in a modern formulation. Such a formulation not only enriches Fregean logic, but reciprocally, also Hegelian logic, in that there are serious shortcomings in the Hegelian view of number from a modern perspective. An example of this is the fundamental *induction principle* of number which receives no discussion by Hegel (no doubt because recognition of its importance came later in the development of mathematics). In contrast, the induction principle is part of the logical development of Frege's system.

The main themes of the paper have been discussed above. They will be examined in more detail in the rest of the paper. Frege's notion of concepts and *number* are discussed in § 2 and Hegel's logical development, which gives the basis of his view of number, is discussed in § 3. The fourth section of the paper discusses Hegel's account of concepts and number. The last section deals with the relation between these two views of number, and the paper closes with a few (somewhat speculative) comments on a modern Hegelian logical framework which provides a setting for the Fregean philosophy.

2. Frege and the conceptual character of number

In his *Grundlagen*, Frege discusses and rejects a number of plausible accounts of number and instead proposes and defends a conceptual view of number. For the purpose of presenting Frege's notion of *concept* and of how it applies to *number*, we will start with Frege's discussion of Mill's empiricist view of number.

According to Mill, a number names a property belonging to an aggregate (*Aggregat*) or collection of things, the property being the *characteristic manner* in which the collection is separated into parts[13].

Mill, then, thought that the number *five* is a property of, for example, the collection of fingers on my right hand, the ``parts'' being the fingers on that hand into which the collection divides ``characteristically''. But, Frege points out, there is no unique such ``characteristic manner''. There are *various* ways in which a collection can be separated out into parts for counting purposes - a town can be treated as one, but equally as many if we consider it to be made up of houses[14]. And what collection of things is the number *0* a property of? For in that case, there is nothing there to be separated into parts!

Another difficulty with Mill's view arises from his seeming presupposition that number only applies in a situation where we are tangibly presented with an empirical collection of things, before our very eyes as it were, and out of which the number arises as a property (determined precisely through counting). But in actual fact, we do not need to be presented with such a collection in order to know that it has a number. Frege illustrates this by pointing out[15]:

[13] p.30 of *FA*.

[14] p.33, *FA*.

[15] p.30, *FA*.

Must we literally hold a rally of all the blind in Germany before we can attach any sense to the expression ``the number of blind in Germany''?

This brings out the fact that number is applicable to a much wider range of collections than those of empirical things lying before our very eyes, and applies without change *to the collections defined by ideas and concepts* without any content restriction. In this connection, number differs radically from collection properties, such as ``hard'' or ``blue''. These properties do not apply to concepts such as angels, actions or thoughts, *but number does*. Motion, for example, is not blue (or salty or hard) but it is a ``1''. Frege cites Leibniz, who pointed out that God, an angel, a man, and motion together make 4! Number is thus ``quality'' or ``content'' neutral in the sense that it applies to concepts independently of the qualitative contents of the concepts[16].

All of this suggests that *whatever* number is, it applies primarily to *concepts*, not collections, and this indeed is Frege's position. The collection of things (objects) which satisfy a concept is called the *extension* of a concept. So for example, a concept such as ``citizen of New York'' has, as its extension, *all of the people living in New York*. The number of the concept is just the *population* of New York, the number of people living in that city.

Empirical collections to which we attach a number, such as the plates on a person's table at dinner time, are, for Frege, concealed concepts, given by description. An example would be the concept[17]: ``plate on my table at dinner time''.

Motivation for Frege's precise definition of *number* is given, in a very simple way, from the content neutrality of number. For the number 5, say, is not attached to any particular concept F under which 5 things fall, for that would make 5 depend on the qualitative content of F. Instead, 5 has to apply to *all* such concepts, with no favoritism[18], and this leads to Frege's famous definition of number which runs as follows: the *number which belongs to the concept F is the extension of the concept ``equinumerous to the concept F''.*

So what the definition is saying is that we start with a concept F and its extension (the things that fall under it). We then consider the concept of ``being a concept which is equinumerous to F''. This concept in turn has an extension: its members are just those concepts whose extensions are equinumerous to that of F, i.e. whose extensions can be put into one-to-one correspondence with the extension of F. The extension of the concept ``being a concept which is equinumerous to F'' is then the *number of F*[19]. So a number is a huge collection of concepts, and is parasitic on concepts.

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[16] Not every concept, however, has, for Frege, a number which applies to it. *Red* is an example of such a concept.

[17] It is arguable whether such a concept is a legitimate concept, but we will not discuss this in this paper.

[18] For the same reason, Frege would not have accepted more recent ways of defining number (such as those due to Zermelo and von Neumann) in which a number such as 5 is defined as a *particular* set of five elements (constructed from the empty set).

[19] Frege (p.80, *ibid*) comments that he believes that the number of the concept F can actually be identified with a concept rather than with an extension. Theconcept is: ``equinumerous to the concept F''.

We now look more closely at ow Frege's treatment of the number solved (for him) a variant of the problem of the *one and the many* of Greek philosophy. (The Hegelian answer to this problem will be considered in § 5.)

This arises as follows. While number is quality neutral, it is equally clear that for counting purposes, *we cannot dispense with qualities* for distinguishing the items or units involved. For example, if I am counting the number of chairs in this room, I need to distinguish the chairs from other items in the room. A chair is brown, hard, has a certain shape, and these qualities distinguish it from the carpet. So I do not include the carpet in the chair counting. Further, the chairs are distinguished among themselves by their spatial position in the room, for example, one being *to the right* of another. If we neutralize qualitative content to reach number, then it appears that the carpet is no longer distinguished from a chair and should be included in the counting, and the different chairs cannot be distinguished. Counting collapses!

So, as Frege expresses it[20]:

> We were thus forced, it seemed, to ascribe to units two contradictory qualities, namely identity and distinguishability.

Frege's solution to this follows naturally from his view on the relation between concepts and number described above. For a concept applies to a whole variety of things, but number does not apply to that variety. Rather *number applies to the concept which itself applies to the variety of things.* Frege's solution of how units can be both one and many is thus as follows. They are *one* in that they are covered by the *same* concept, and they are *many* in that each on its own satisfies the conditions required for it to fall under the concept. Of course, the units can satisfy these conditions without coinciding. The *identity* of the units then lies in the concept, not in the individual units *per se*, and number applies to these different items only through the mediation of the concept to which it belongs.

But this leaves a problem of how the concept relates to the objects that fall under it. To take a Fregean example, we obtain the concept of *cat* by considering white cats, black cats, any old cat and then ``disregard the properties that serve to distinguish them''[21]. The problem of the *one and the many* raises its head again. In the counting, we go *back* from the unitary concept to its extension, so that both processes of rising from the objects to their concept and back again are required to be completely determinate in the Fregean view. The ``many'' objects have to dissolve into the ``one'' concept, and equally, that concept has to yield up the objects again. The problems with this - arising out of the philosophies of both Hegel and Wittgenstein - will be considered in the next section.

3. The transition from the Fregean concept to the Hegelian concept

In the introduction, we pointed out that both Fregean and Hegelian philosophies are philosophies of the concept. In this section, we discuss briefly the Hegelian concept, and then use this Hegelian perspective (and the ideas of Wittgenstein's later philosophy) to specify what is wrong with the Fregean concept. Hegel's view of number will be discussed in more detail in the fourth section of the paper.

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[20] p.58, *FA*.

[21] p.45, *FA*.

Frege emphasized, as we have seen, the conceptual character of number and number as belonging to concepts. For Hegel, this conceptual character pertains to philosophy in general, not just to the logic of mathematics. Philosophy's concern is with *the* concept, *Der Begriff*, and logic is just the development of the concept. The development expresses the content of the concept through the specific concepts that structure it.

The Hegelian concept is at a much more general level than the Fregean. For Hegel, ``the collection of knives on the table at dinner time'' would not, I think, even have counted as a concept, as it did for Frege. Indeed, we would naturally think of *pointing out* the collection of knives rather than *contemplating* it. However, there is no doubt that there is conceptual content in the description of this collection (e.g. the concept of *knife*) and so for the rest of the paper, we will ignore this difererence between the Hegelian and Fregean concept.

The one concept runs through all of Hegelian logic[22], and it appears in three main forms. Initially, it is at the level of *Being* (*Sein*), of surface phenomena which relate to one another in a qualitative way. At that level, the concept only operates implicitly behind the scenes. Examples of Being concepts are (abstract) *being* itself, *something*, *limit*, *the infinite*, *the one and the many*, *quantity*, *number* and *measure*.

The next level is that of *Essence* (*Wesen*) in which the concept is explicitly present but only as mediated, appearing in the phenomena. Typically, essence concepts have a dual character, one side of the duality corresponding to what is ``behind the scenes'' and is appearing and the other side to the phenomenal aspect. Examples are the *law of contradiction*, *force and its expression*, *whole and parts*, *inner and outer*, *appearance and reality*, and *substance and accidents*. In each case, the concept is present but not ``for-itself''. It only appears and is inferred indirectly. (In the case of the law of contradiction, the duality is the contradictory form of appearance (e.g. in motion) as against the identity of essence which is negative in character, not positively abstract[23].)

The third level is that of the *concept* proper (*Begriff*), in which the concept is fully present in its moments of universality, particularity and individuality and which incorporates and preserves all that went before at the other two levels. The third level deals with the forms of rational inference, in particular, with the syllogism, and also of the transition from purely logical thought to its embodiment in scientific thought (Nature).

Hegelian logic is a *development,* in which the content of thought powers the exposition. It is not a formal system. (Hegel, like Frege, is not a formalist.)

In order to relate Hegelian logic to the Fregean arguments considered earlier, we focus on the *Being* level. There are two reasons for this. Firstly, properties such as *blue* and *hard,* that Frege distinguishes from and uses to motivate his definition of number, are *qualities* in the Hegelian sense. *Quality* is discussed by Hegel in *Being*. Secondly, Hegel's extensive discussion of *number* occurs in *Being*.

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[22] cf. p.39, *SL*: there, Hegel argues that the content of a specific concept is the determinateness which is the *specific form of this substantial oneness, a moment of the form as totality*, of that same concept *which is the foundation of the specific concepts.*

[23] pp.439-440, *SL.*

A brief discussion of the long development from the qualitative to the quantitative (number) in Hegel's treatment of *Being* will be given in the next section. For the present, however, we want to focus on how Hegel sees the relation between the qualitative and the quantitative and compare it with the Fregean perspective discussed in § 2.

The relation, for Hegel, is strikingly the same as that involved in the *universality of application of number* which we saw above was stressed by Frege, and which was the basis for his definition of *the number belonging to a concept F*. In that definition, *all* concepts (regardless of the character of the items in their extensions), whose extensions are equinumerous with that of F, are grouped together and the resultant class is defined to be the *number belonging to F*.

The terminology that Hegel uses is different from that of Frege and its origins are located in the philosophy of indifference (*Indifferenz*) of Schelling and himself. The argument is of great interest since it not only confirms the central ``quality neutral'' character that number has in Frege's philosophy, but gives a logical development showing how number arises out of quality, an issue that Frege leaves unanswered. In fact, such an answer cannot be given in the Fregean concept, since the latter is rigid. The main claim of the paper is that this is the fundamental problem with the Fregean concept, a problem that can be overcome when the Fregean concept is developed in the Hegelian direction.

To this end, we recall that (§ 3) how Frege dealt with the problem of *the one and the many* that arose from the counting of units. We also pointed out that there was a corresponding problem, not dealt with by Frege, associated with the concept in relation to the objects in its extension. We now examine this problem more carefully.

At the level of normal discourse, the Fregean view of objects and concepts is very plausible. For example, one of the concepts that Frege talks about is ``moon of Jupiter''. There seems initially to be no ambiguity in this concept - it is clear and fixed. equally, the objects to which it applies seem equally clear and fixed.

But more careful inspection, brings up doubts. What exactly is to count as a moon of Jupiter, in distinction from, say, a piece of debris rock orbiting round Jupiter? This is the point of Aristotle's well-known *elenchi: how many hairs have to be removed from the tail of a horse before it can be called bald*[24]? The concept *bald*, like the concept *moon of Jupiter*, is not precisely defined.

Another example of a concept considered by Frege is that[25] of ``inhabitant of Germany''. In answer to an objection that this concept varies from year to year so that the number belonging to it is not defined, he argues that this concept[26]

> contains, in fact, a time-reference as a variable element in it, or, to put it mathematically, is a function of time.

For Frege, we can remove this concept variability by giving the current date: so

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[24] See Hegel's discussion of the *elenchi* on p.335, *SL*.

[25] p.59, *FA*.

[26] This is an instance of Frege's view that concepts are functions. In this case, the concept is a two variable function, the variables being persons and time.

> the number belonging to the concept ``inhabitant of Germany at New Year, 1883, Berlin time'' is the same for all eternity.

But this time specification is itself not completely precise. By ``New Year, 1883'', do we mean New Year's day? When exactly does that start? - this is only determined up to an arbitrary small interval of time, e.g. a second, and hence the starting time for the New Year is not precisely specified. (The number 2,in contrast, *is* precisely specified: it is not determined up to a ``little more'' or a ``little less''.) And what is to count as an inhabitant of Germany? Is a man whose feet straddle the boundary line between France and Germany on New Year's day to be counted as an inhabitant or not? If he moves over completely to the German side later on New Year's day, does he suddenly become an inhabitant?

The same point is made by Wittgenstein in respect of his well-known discussion of the concept of ``game'' in the *Philosophical Investigations*[27]. At first sight, the game concept seems to be a perfectly clear concept - we have no trouble in using it. But:

> Consider, for example the proceedings that we call ``games''. I mean board-games, card-games, ball-games, Olympic games, and so on. What is common to them all? - Don't say: ``There *must* be something common, or they would not be called ``games'''' - but *look and see* whether there is anything in common to all. - For if you look at them you will not see something that is common to *all*, but similarities, relationships, and whole series of them at that.

In this connection, Wittgenstein explicitly refers to Frege's comparison of a concept with an area[28].

> Frege compares a concept to an area and says that an area with vague boundaries cannot be called an area at all. This presumably means that we cannot do anything with it. - But is it senseless to say: ``Stand roughly here''?

This illustrates the *rigidity* problem with the Fregean concept stressed in this paper. Against the Fregean view, concepts are dynamic. But this dynamic is (against the Wittgensteinian view) not that of a language game but rather *is part of the cognitive content of the concept itself*. That is, our thinking through the concept reveals its intrinsic movement: we cannot stay in one place, but the concept forces us on. It is not so much that concepts have, as Wittgenstein expresses it, ``blurred edges'' but rather that *they are on the move*. When we try to clarify the ``blurred edges'', the edges and the concept *shift*. This is a fundamental insight of Hegelian logic, and the origin of its principle of logical development (dialectic).

Since concepts are dynamic, one cannot precisely determine whether something falls under a concept or not. Indeed, the something itself is not precisely determined. But this undermines (even without having to consider set theoretic paradoxes) the Fregean view of number since the class of objects falling under the concept is not determined precisely. So if you have to approach number

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[27] § 66 of: L. Wittgenstein, *Philosophical Investigations*, translated by G. E. Anscombe, Basil Blackwell, Oxford, 1967.

[28] § 71, *ibid*.

through an abstracted concept (as Frege requires) we don't know what collection is to be counted. A different approach to number is required.

In the remainder of the paper, we will discuss the Hegelian understanding of concepts and number and claim that this gives us the basis for the approach that we are seeking. In fact, the Hegelian character of the problem is indicated by the presence of an ``infinite regress'' - in Hegelian terminology, an *infinite progress.* There are many infinite progresses (*die unendlichen Progressen*) in *WL*, and they arise in Hegelian logic when *finite*[29] concepts come up against their limitation and indicate the *infinite* concepts on which they implicitly depend. In such a progress, the *dynamic* character of concepts, that we have stressed above, becomes explicit and breaks out of the finite mold of thought.

The infinite progress associated the difficulty discussed above of defining exactly when an object falls under a concept is as follows. As we noted, the Fregean notion of a ``crystal clear'' concept is open to the Wittgensteinian criticism that what at first appears a clear, precise concept, such as game or ``inhabitant of Germany'', is in fact imprecise and not completely specified. It is true that we can still investigate the ``similarities, relationships and (the) whole series of them at that'' in games. But these similarities, etc. are *themselves* open to the same lack of specification (if we want to call it that) that the original game concept had. How are we then to understand this, for it would appear that, as we seek to refine concepts and make them more precise, we are involved in an endeavor that in principle never comes to an end? For we don't just want more and more precision in the concepts - we want precision absolutely, so that the philosophy of concepts will match the mathematical precision of 1+1=2 and Pythagoras Theorem. The imprecision is just not going away in the constant refinement of concepts, and the concepts are ``tainted'' with it at every level. We are trapped in an infinite progress - there is no solid conceptual footing but just a moving on to greater and greater levels of precision, a process that never stops in principle and becomes more and more pointless.

The emergence of this infinite progress out of the work of Frege then indicates that we are close to the themes of Hegelian logic. So we now turn to the Hegelian concept, and Hegel's view of number.

4. Hegel on concepts and number

We considered above the infinite progress arising from trying to make concepts completely ``precise''. There is no doubt that there is imprecision in concepts, and this is not just a regrettable inadequacy that we can ``fix'', but rather is ineluctable to concepts. Without the ``imprecision'', we would not be able to use concepts in the way that we do, and normal, ``everyday'', thought, which is the basis of all thought, would be impossible. To deal with this, we must no longer think of concepts as the same *`a la Frege* ``from all eternity'', in which case they would indeed deserve the epitaph: RIP! Rather, as we have argued in § 3, concepts are dynamic, not completely given, spread out in front of us, but determining themselves, arising from the implicit, inchoate to the explicit and articulated. Conceptual investigation in philosophy is then not reductionist in character, for then the primary ``atomic'' concepts in a reductionist philosophy would be simple and static in that philosophical system, not determining themselves.

[29] The precise Hegelian definition of *finite*, and also of *infinite*, as applied to concepts is given in § 4.

Nor is such a dynamic character to be thought in terms of subjectivity, in which *we* develop concepts. Rather, *they develop themselves* independently of our subjectivities - it is not we that wield concepts like tools for our ends, as if they were hammers and saws, but rather the other way round[30]. Truth is the objectivity of concepts, and this objectivity is intrinsic to concepts, the subjective processes of our thinking being only the means through which we become aware of that objectivity[31]. The two are not separated out, anymore than the objective and subjective are separated out for a small child crawling about on the carpet. The objective and the subjective only arise and are distinguished later through conceptual development and experience[32].

This gives the starting place for Hegel's logic. It is not, like Frege's, one in which we already presuppose a world of objects and clear-cut concepts under which an object either definitely falls or definitely does not fall. The simple starting place for a conceptual development is rather the most vague and nebulous that one can attach a name to, from which the blur and haze will clear only through the development intrinsic to it, and which will form the matrix of concepts and their relationships studied in Logic. This starting place is, for Hegel, that of *being* (*Sein*). After all, concepts are *about* something or other, and refer to *things*. We would say that such things *are*, have being, exist. Initially, the being with which logic must start is being at its most general and abstract since it has to apply at all conceivable levels of existents.

This initial concept of being is explicated in Hegel's famous starting triple of concepts, *being*, *nothing* and *becoming*. The ``nothing'' content of being is exactly the point made by Wittgenstein in his critique (discussed in § 3) of the claim that concepts have a completely precise character. In this case, the concept involved is the ``being'' concept. There is *nothing* that all forms of being have in common, and the absolutely precise, all-inclusive (and therefore absolutely abstract) concept of being is the concept of nothing at all.

What is left out of this identity of being and nothing is *becoming* (and it is this that is absent in Frege-inspired logic). The concept of being ``becomes'' as it runs through its indefinitely multifarious realizations to the nothing which is what they all have in common. The truth of being and nothing is the becoming process in which the concept see-saws from one extreme to the other and is meaningful only in this see-sawing.

Implicit in the becoming of being is one form of being set against the other as we move from one to the other and contrast one against the other, highlighting the difference between them, the difference that is stretching the being concept (until it snaps into nothing and incoherence).

This takes us to the next determination of being, that of *Dasein*. There is disagreement among scholars as to how this term should be translated into

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[30] Hegel (p.35, *SL*) expresses it: *it is our thinking that must accommodate itself to them [concepts] and our caprice and freedom ought not to want to mold them to suit itself.* For Hegel, the attempt to mold concepts for self-interest is doomed to failure anyway due to the *cunning of reason*.

[31] *the objective concept of things constitutes their essential import*, p.36, *SL*.

[32] Subjective, here, is not to be confused with *relative*, in which, for example, the answer to whether or not it is hot or cold in a room is relative to the person that you ask, and one man's meat is another man's poison. Rather, *subjective* in the Hegelian sense refers to thought content and is interpersonal.

English[33]. But the conceptual content of *Dasein* as against *Sein* is that the former is *limited* being, something as against its other, this opposition (to use an *Essence* term) being intrinsic to it. It arises out of the incoherent notion of Being with which we started through *Dasein*'s incorporating the *nothing* side within it, what is *not Dasein* being implicitly part *of Dasein*. (The *becoming* aspect of Being is left implicit initially in *Dasein* since *Dasein* is a form of Being, but as we will see, it will reemerge later in the *infinite progress* of *Dasein* as finite.)

The nothing incorporated into Being as *Dasein* gives it, then, a *limited* character, contrasting it in this respect with the Being with which we originally started and which covered everything (and hence said nothing about anything). The simplicity of this original Being is present in *Dasein*, but now in the form of *simple* characteristics of Being. Each of these characteristics does *not* apply to everything, so that the *nothing* aspect of Being is present in *Dasein* through them. These characteristics are *qualitative* in character. The qualitative covers concepts such as red, healing, good, these taken in their simplest, most immediate sense. In this sense, the qualitative *Dasein* concept is the *whole* of *Dasein*, there being at this level of logic no distinguishing between a thing as a substance or object which has the property or accident of being red but is still thought of as independently existing apart from this determination[34]. Rather at the primitive *Dasein* level of thought, the quality is everything. This is illustrated by empiricism taken to its extreme, where surface qualities are all that there is.

Dasein is then not to be thought of in terms of *a* thing but rather as thing in its most attentuated form, a *something* characterized qualitatively, and through its simple qualities, distinguished from *other* somethings. This is expressed in everyday life when we are in a situation where we can't remember a detail about which we are being asked, and say: *Oh! - it is something or other*. At any rate, *Dasein* as *something* intrinsically involves *Dasein* as an other which provides the limiting background against which it is defined. So *Dasein* as the quality *red* is *only* defined, only meaningful, as against *Dasein* as not red, e.g. blue or warm. *Dasein* explicitly posited as thus limited is (by Hegelian definition) *finite* being.

Dasein is conceptually contradictory, since no sooner is it thought about than it is replaced by what it isn't, and this leads to the first of many ``infinite progresses'' in *WL*, the expression of the contradiction in *Dasein* in which it is something in its own right and yet at the same time has its feet taken from under it by the other, only through whose determining it *is* in the first place[35].

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[33] For a discussion of suggested translations of the term, see the author's discussion in the Appendix to his paper: *Self-reference and the natural numbers as the logic of Dasein*, Hegel Studien, 32(1997), 93-121.

[34] For Hegel, things and their properties are *Essence* categories (p.484, *SL*).

[35] As an aside, it is to be observed that in 20th century philosophy, this *Dasein* argument has been expressed in its most notable form in Heidegger's use of it to describe the ``human condition'' in *Sein und Zeit*. (This has been translated as *Being and Time* by J. Macquarrie and E. Robinson, Oxford, Basil Blackwell, 1973.) The structure of human *Dasein* is that of *Care*, in which is expressed the conceptual contradictory character of the Hegelian *Dasein* in relation to an *other*. This *other* assumes, in particular, the form of *other human beings*, and the contradiction in *Dasein* shows itself in *anxiety* (for *Dasein* is *insecure*, resting on what is not solid) and expresses itself in idle talk, curiousity, ambiguity, falling and ``throwness''.

There is, however, an important distinction between Hegel's understanding of *Dasein* and Heidegger's. Heidegger requires effectively theological presuppositions in the form of *authentic being* which describes that from which *Dasein* has fallen. (So, e.g., (§ 436 of *Being and Time)*,

The infinite progress of *Dasein* is the endless repetition of *Dasein* in which it is posited only for it to disappear, its nothingness made explicit, when the other, on which its holding together depends, is made explicit. But this other itself is *Dasein* and so itself disappears and the process continues *ad infinitum*.

Note that the concept of the infinite, shrouded initially in the vagueness of abstract being, has now become explicit. In contrast to Fregean thought, where the infinite comes about through the extension of classes and depends on what there is available in the world to fall under a concept, *the infinite in Hegel is intrinsic to the concept itself*, to the very nature of thinking itself. The concept of the infinite does not depend on what there is empirically in the world, and Russell's unsatisfactory *axiom of infinity* is not required.

The *Dasein* infinite progress is conceptually incoherent and forces us on to a more satisfactory concept of the infinite, not a concept which in its thinking goes on and on for ever and is intrinsically incomplete, but rather a concept which is complete in itself, pictured as a circle closed up in itself as opposed to a line of infinite length that goes on for ever[36]. The infinite and the finite (*Dasein*) are no longer opposed to one another, but each is only through the other: the thought of either immediately gives rise to the thought of the other.

A good example of a concept that is infinite in the above sense is that of *number*. This is infinite since it applies to infinitely many numbers 1,2,3,..., but at the same time incorporates the finite (*Dasein*) since the items that it determines and without which it would not be number are *finite* numbers (such as 5). In the case of number, the infinite progress arising from its *Dasein* character and considered in the general case above, is very specific: it is the infinite progress of *enumerating* the nubers 1,2,3,4,..., 1000000,1000001,...., in which one number is posited and then disappears in the expressing of the next in a never-ending process.

This new kind of being conceived through an infinite concept is called by Hegel *Fürsichsein*, Being-for-self. It is a dynamic concept of being in which its completeness through its infinite character is established only through its incomplete, finite character and hence is thought of not as fixed and static (as in the Fregean concept) but rather as always establishing itself.

This notion is illustrated for Hegel in *self-consciousness*, which *is*, has actuality, only in consciousness of the finite (e.g. objects of perception or thought) but in that act transcends that consciousness and refers back to itself.

As will be briefly described below, *Fürsichsein*, expressed in the logic of the *one and the many*, gives the basis for the *quantitative* and *number*. The Hegelian approach, as we will argue, thus gives a logical explanation of the qualitative indifference of number rather than, as in the Fregean approach, just being observed as a ``matter of fact''. For Hegel, the ``indifference'' of number with respect to quality is the result of a conceptual development, while for Frege,

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Heidegger talks of the *distinction between the Being of existing Dasein and the Being of entities, such as Reality, which do not have the character of Dasein.* For Hegel, on the other hand, infinite being - which corresponds to Heidegger's authentic being - arises out of the thought of the *finite* and does not require the presupposition of some pellucid, primitive form of being against which *Dasein* is to be contrasted. Indeed, on Hegelian grounds, such a presupposed authentic being, as standing outside the finite, would itself be ``*Daseined*'' against that finite as its other and so itself be finite. (This is, in fact, the argument that Hegel uses against the first form of the infinite, the affirmative infinite, in the infinite progress of *Dasein*.)

[36] p.149, *SL*.

the qualitative neutrality of number is explained through *abstraction*, and indeed[37] the technique that Frege uses to define conceptual expressions such as ``the direction of a line a'', ``the shape of A'' and ``the number of F's'' by means of extensions of classes, is called in mathematical logic ``definition by abstraction''.

We will not go into most of the (difficult) details of just how, in Hegelian logic, quantity arises out of and expresses the truth of *Fürsichsein*. Rather, we will be content to sketch the main idea of how Hegel gets from *Fürsichsein* to number via the *one and the many*.

The *one and the many* arises out the infinite of *Fürsichsein* basically, the present author believes, through that infinite's incorporating within itself the infinite progress out of which *Fürsichsein* arose in the first place. (Hegel does not say this explicitly but I believe that it is implicit in his discussion.) In that progress, we have *Dasein* repeated and annulled indefinitely often, and the infinite of *Fürsichsein* holds it all together through this repeated, common (conceptual) *pattern* in which *Dasein* established itself through its disappearing. Seen backwards through the eyes of *Fürsichsein*, the qualitative character of *Dasein* disappears since the qualitative characterizes *Dasein* and *Dasein* is disappearing in the progress. The instances of *Dasein* in the progress are thus now viewed as all the same (since any qualitative distinguishing has been deleted through the disappearing), yet they have to be distinguished for there to be any *Progress*. They are then thought of as many ones or *units* held together in the infinite of *Fürsichsein* as a one. So we obtain from *Fürsichsein* the *one and the many*. This is reminiscent, as commented in § 1, of Cantor's definition of *set*, and we can say, from a modern perspective, that the *one and the many* is effectively the primitive notion of *set* in Hegelian logic.

As described above, the ones in the many are not qualitatively distinguished, yet they are different from one another. We can say that they are on the way to being identified in their conceptual holding together in the many as one. Hegel describes this conceptually dynamic situation using terminology derived from Physics. The concept of the one and the many is expressed in two ``moment'' concepts. Firstly, the many held together as a one gives the moment of *attraction*, and secondly, the many as scattered, each as a solitary one in its own right, preserving itself in its difference from the other ones, gives the moment of *repulsion*.

The one and the many as such is a contradictory concept since it is simultaneously a ``simple immediacy'' (all of the many sunk within the concept of the one) and an ``infinite mediation'' since the many is itself a one only by its bundling together all of the ones of the many, this bundling being both infinite (from the infinite character of *Fürsichsein)* and mediatory (in the collecting of the ones of the many). The ones limit one another through these moments, and this *Dasein* character, which has been transcended in *Fürsichsein,* leads to the resolution of the contradictory character of the *one and the many* in *quantity (Quantität).*

For Hegel, the shift from the one and the many to quantity takes place in the *neutralizing* of the interaction of the moments of attraction and repulsion. This neutralizing is implicit in the concept of *Fürsichsein* since the latter cuts through the limiting of one thing against another in the infinite, the conceptual basis for that limiting. The attraction-repulsion moment pairing of the *one and*

[37] e.g. p.112 of: M. Dummett, *Truth and other Enigmas,* Footnote 2.

the many slides over into the passive continuity-discreteness moment pairing of quantity. These moments are present in an obvious way in mathematics, the *continuity* moment being illustrated by the real numbers forming a *continuum*, while the *discrete* moment is illustrated by collections that we count or by the integers. (Hegel argues, however, that *both* moments are always present in quantity, though one may be more obvious than the other.)

Quantity as thus conceived is, like *Fürsichsein*, *infinite* in character, while numbers (like 5) are *finite*. As one might expect, since the *finite* for Hegel belongs to *Dasein,* the finite numbers arise out of quantity through a *Dasein* argument. More precisely[38], the two moments of continuity and discreteness *limit* each other to give what Hegel calls *quantum (number)* in which quantity is ``daseined'' against itself. Through this, quanitity is finitized. As such it is number. The continuity-discreteness moments of quantity slide over into the numerical moments of its finitized form. These are the *unit-amount* moments of number though of in terms of counting.

The *Dasein* in number as finitized quantity is not the earlier *Dasein* in which something and other limited each other through radical ``either-or'' qualitative difference, in which something *is* only when its other is *not*. Rather the numbers *themselves* are *daseined against each other*, but in a purely ``external'' way, each number being ``comfortable'' in itself. Number is not *daseined* within itself. The latter is illustrated by the counting process, in which *any* of the units can be the last one counted, so that the *Dasein* ``limit'' of the number of items in a collection, made explicit through the last one counted, where the counting comes to an *end*, is indifferently spread among the units. It does not matter which one comes last - you get the same number anyway.

So we arrive through a Hegelian route at the ``content neutral'' character of number. In the Hegelian context, this is just the ``passivity'' of number, its intrinsic qualitative ``indifference'' which it inherits as a determination of *quantity*. However, in the Hegelian context, this character of number is *derived*, ultimately from the concept of abstract being, unlike that of the Fregean context in which it is simply *observed*. The Hegelian account is conceptually much richer, having implicit within it the logic of *Sein*, *Dasein*, *Fürsichsein* and *Quantität*.

5. A Hegelian interpretation of Frege's definition of number

Number then, for Hegel (as for Frege), is conceptual in character. For Hegel it is the concept of *daseine*d quantity. Thus for Hegel, a number is quantity determined as *finite* through *Dasein*. The conceptual content of *number* is determined through its moments of *unit (continuity)* and *amount discretess)*. *Unit is* the common ``continuous'' character of the discretely distributed items making up a collection that is to be counted, and *amount* makes explicit the common unit character through of the items as ``discrete''. Each moment refers to the other. The *unit* refers to the items that *amount* counted as that common character which specified what was to be included in the tallying of the amount, while *amount* refers back to the items thus included and hence to the unit determination which made them eligible items in the first place.

We are, however, *fa*ced with the problem of how to relate Frege's equinumerous definition of number to Hegel's *Dasein* view of number. We now argue that in fact, regarding *set* as a determination of *Fürsichsein* preceding that of number, the

[38] p.201, *SL*.

Hegelian ``indifference'' understanding of quantity on which his view of number is based gives a natural explanation of the Fregean definition of *the number of a concept* which uses equinumerous sets. Indeed, when we talk about the *number* of a collection (defined by a concept) from a Hegelian perspective, then the items of the collection ``slide'' into indifference in the conceptual shift from *Dasein* to *Fürsichsein* to *Quantität*. The items then lose their qualitative character so that the particular characteristics of the items in the collection do not matter any more - equinumerous collections have their distinguishing characteristics blotted out in the indifference and so give the same outcome. This is how we can interpret Frege's position on number in the Hegelian context. This gives a defense of Hegel's ``indifference'' view of number and, I believe, avoids the set theoretic difficulties of Frege's approach[39].

Given two equinumerous collections, when (in the Hegelian approach), the qualitative is made ``indifferent'' in the quantitative, the two collections logically ``slide into one another'' and identify in number. We can illustrate this with images on a computer monitor screen. On one side of the screen, there is a collection of 5 squares, and on the other side, a collection of 5 circles. Using computer graphics, imagine the squares rounding out at the corners into circular shape, and simultaneously the circles ``squaring'' out. At the same time, the changing squares and circles are moving into the middle of the screen till they lie on top of one another. (This ``lying on top'' is the equinumerous aspect of this situation, where the squares and circles are paired up in a one-to-one correspondence.) There is then only one 5 collection and the shapes are neither squares or circles. Here, of course, there is not complete indifference since the five shapes that are left do have qualitative characteristics, but these characteristics that distinguished the two collections (e.g. squareness, circularity, position on the screen) have been removed in the ``making indifferent''.

For the concept of five, however, we need to go further. We need to consider the concept of the removal of *all* such qualitative marks. The new five collection itself, thought of numerically, is *completely* indifferent to the qualitative. It is this which makes the conceptual thinking of number in Hegelian logic difficult. It has to be thought of through its ``movement'', the numerical concept determining itself, and we must not allow ourselves to freeze the concept and focus on a fixed result. Rather, we need to see number as the *indifference process*. In this respect, the Hegelian view of number is to be compared with a video camera recording of an event, while the Fregean view is to be compared with taking a still photograph of the event.

Some of the well-known set-theoretic problems with the equinumerous concept used in Frege's definition of number are as follows. One needs to know, given two sets, if there is a one-to-one correspondence between them, i.e. if there is a function from one to the other which is one-to-one and onto. Regarding a function as a *set* (through its *graph*), it follows that whether there is or is not such a function depends on the set theory that one is using, and so is not canonical. One might say that the main difficulties lie with infinite sets, and this is the case. But even for finite sets, Frege is assuming in advance that of two finite sets, it is determined whether or not they are equinumerous, and he is also faced with the problem that there are infinitely many finite sets of arbitrarily large size to be considered (so that the problems with infinite sets have not gone away even if we restrict to the finite context).

[39] Of course, the technical questions of set theory in modern mathematics (e.g. the truth or otherwise of the continuum hypothesis) lie far beyond the philosophical genesis of set considered here.

At a more philosophical level, for two equinumerous sets, there are many one-to-one correspondences. Yet being equinumerous is intrinsic, canonical, to the sets involved, and should not depend on whether or not there exists (accidentally as it were) such a correspondence. Being equinumerous should not depend on the particular ``pairing up'' of elements given by a particular correspondence.

Examining this point from the Hegelian approach described above, the sets as defined conceptually are equinumerous through their concepts, through their intrinsic conceptual movement away from the qualitative content of the concepts and their conceptual merging into one another through the resulting conceptual indifference. The indifference applies even (in terms of the example above) to the question of which circles are identified with which squares, so that the necessity for a one-to-one correspondence falls away. This is what is operative in our knowing that the number of knives on the dinner table is the same as the number of forks. It is not, as noted in § 2, that this *particular* knife is paired up with that *particular* fork. Rather, for numbering purposes, the *set* of forks and the *set* of knives are merged together in no particular pairing.

And the number of a concept (such as that of the *dinner table knives*) does not depend on comparison with other concepts (such as the *dinner table forks*) since the transition from qualitative to quantitative in the concepts *leads to them coinciding anyway*. That is, the knives in their becoming from quality to quantity will become what the forks are becoming, *so that there is no need to involve the forks when considering the number of knives*. This, is, of course, obvious from an everyday standpoint: when we count the knives, 7 of them say, we look only at them, and ignore everything else (such as the forks). In particular, we do not consider other 7-element sets of things. In contrast, Frege's approach makes the number of knives *conceptually depend* on the (equinumerous) set of forks, and this does not conform with how we count. The Hegelian account that we have described above *does* not make the number of knives depend on the fork concept, and conforms to our normal way of thinking about number.

It will be seen from the above that in this Hegelian interpretation of Frege's definition of number, the concepts involved are dynamic in character. This is obvious even at a very simple level, e.g. if we think of knives as taken out of a drawer, and being moved about on the table and paired up with the forks. The concept of the table knives is not *itself* changed through such movement and pairing. It is the intrinsic activity that determines a concept to number. This conceptual activity is lacking in the Fregean concept, and its absence is the source of the abstract rigidity of Frege's concept. This rigidity is overcome when the Hegelian concept is allowed to develop within the Fregean philosophy.

Having discussed how the identity of the numbers of two equinumerous sets can be understand from a Hegelian perspective, we now turn to how Frege's problem, discussed in § 2, concerning how a unit can be both one and many, is to be understood in Hegelian logic. Frege's solution to this, we recall, was that when we say that the unit is a *one*, we means that the *unit* of a collection has been identified with the *concept* of the pertinent collection, but when we say that the units are *many*, we really mean that the *things falling under the concept* are different.

Let us turn to how Hegel's indifference view of number would deal with this same problem, viz. *how can units in counting be the same and yet there not just be one of them*. This question, at first sight, appears a problem for this Hegelian view since as the objects of a collection become qualitatively indifferent, will

they not become the *same*, as their distinguishing characteristics are stripped away, so that only one is left? And even then, the one left will itself disappear for there will be no distinguishing characteristics to pick it out from its background.

The crucial difference between Hegel and Frege in this matter is that for Hegel, the unit is *not* the whole concept but *only one of its moments*, and the other moment (amount) incorporates into the concept the character of the items covered by the concept. So unlike Frege's, Hegel's does not separate unit as concept from unit as an item to which the concept applies. The Hegelian concept incorporates both in the interaction of its moments. This also explains why, in normal language, both aspects of *unit* are used interchangeably. It is because they are just related sides of the same concept. For Hegel (as for Aristotle), form *cannot* be considered separate from content[40]. It also explains why, in the indifference process, the units do not merge into nothingness, since both unit (continuity) and amount (discreteness) are preserved in the process as the moments of number.

In more detail, as we saw in § 4, for Hegel, the conceptual moments of quantity are *continuity* and *discreteness*, and at the level of number, the unit is the *continuity moment* of number. So for Hegel, the term *unit* has a different, though related sense, to the term *unit* as used by Frege. For Frege, unit precedes the number stage, so that it still involves qualitative content such as *being a moon*. For Hegel, what corresponds to Frege's *unit* is *continuity*, which is given at the later stage of *quantity*, and Hegel's *unit* is what continuity goes into in the shift from quantity to number.

So Hegel's answer to the above objection of Frege about how we can ``ascribe to units two contradictory qualities, namely identity and distinguishability'' is that the continuity and discrete concepts involved are (in the form of unit and amount) *moments* of number, each logically presupposing the other. The unit moment of a concept intrinsically presupposes items distinguished through discreteness (amount) in the concept's extension, and conversely, the amount moment intrinsically presupposes a common character for the items in the concept's extension and this brings us back to the unit concept again. It is in the relationship of these conceptual moments that the Hegelian answer to the above question raised by Frege lies.

So (in Fregean terms) ``identity and distinguishability'' - in Hegelian terms, continuity and discreteness, - each logically involves the other, and we cannot form the abstractions and treat them, as Frege did, in isolation, purely separated out from one another.

6. Conclusion

In this paper, we have discussed the differing positions of Hegel and Frege with respect to concepts and number. These involve philosophical issues which, on the whole, have been submerged in the post-Fregean developments of mathematical logic. These developments, many based on ideas originally introduced by Frege himself, have led to the important present-day formal systems of mathematics, such as the *type* theories of Russell and Whitehead, Quine's *New Foundations*, Zermelo-Fraenkel set theory, category theory based systems and the system **S** for

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[40] As Hegel points out in his second Preface to *WL* (p.39, *SL*), *content in its own self possesses form, in fact it is through form alone that it has soul and meaning, and that it is form itself which is transformed only into the semblance of a content.*

elementary number theory. But Frege was opposed to the formal approach in mathematics without a philosophical basis.

The formalization of number does not capture the *essence* of number, but only gives a formal model[41] for it. While this is a profound and remarkable achievement, such a model does not give a conceptual understanding of number, but rather gives what is needed for mathematicians to get their number theoretic arguments to work. The philosophical interpretation of such a formal system is another matter and cannot be avoided. In this respect, formal systems themselves *presuppose* the informal concept of number. For example, the natural numbers, in the usual sense, are needed from the outset to provides labels for the variables, constants, functions and predicates of the system. Further, in metamathematics, intuitive number theory is explicitly required, and the Gödel incompleteness theorems indicate very clearly the inadequacy of the formal notion of number when compared with the intuitive notion. This brings us back to where we started before the launching of the formalization program and we cannot escape the philosophical issues which remain implicit, ``inner'', within, though motivating, the formal enterprize.

This ``inner'' is conceptual in character, as both Frege and Hegel saw. In particular, Hegel, in his logic, attempts to give the ``inner'' behind mathematical and logical concepts, their ``exegesis'' as it were. So, for example, for Hegel, the reason why we have numbers in mathematics and calculate with them the way that we do is rooted in the fundamental, primitive, concepts of being, *Sein*, *Dasein* and *Fürsichsein*. And the rationale for the logical formalism of the classical syllogism is through the logic of the concept and its moments.

The main philosophical defect of the Fregean logic, and why Fregean logic is ultimately not adequate for such a conceptual philosophy (while, I claim, Hegelian logic in principle is), lies in the *rigidity* of the Fregean concept. It is this that *allows* Frege to claim that his *Begriffsschrift* (concept-writing) *exactly* follows or expresses the concepts involved. But the ``inner'' conceptual development powering mathematics and its logic cannot be completely expressed in any external, frozen symbolism but rather can only be understood in its own terms.

This is not to say that Hegelian logic is the last word on the subject. As Hegel was well aware[42], there are serious inadequacies in his logic. (This is even clearer from a modern perspective.) But the method of his logic and the broad sweep of deeply related developing concepts that it gives, provides, I claim, a basis for a rethinking of modern logic.

Of the 20th century schools of mathematical logic, intuitionism comes closest to recognizing the concepts of mathematics in their development. In particular, Brouwer and Weyl[43] recognize the continuum as containing individual numbers but not as a *set* of real numbers with ``finished'' being (*fertig seiende*). For them, it is rather to be thought of as a *medium of free becoming* (*Das Kontinuum als Medium freien Werdens*). But the intuition underlying this is not traced back, as Hegel did, to more basic concepts (such as Being), and from a Hegelian

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[41] This is ``model'' in the intuitive sense of the word, not in the technical meaning that it has in metamathematics.

[42] e.g. p.31, *SL*.

[43] e.g. p.94 of: H. Weyl, *On the New Foundational Crisis of Mathematics*, in: P. Mancosu, *From Brouwer to Hilbert*, pp.86-118.

perspective, its constructivist character shows it to be *finite* (in the Hegelian sense that we described in § 4). The levels of construction (types) and even number itself as an ``incomplete infinite'' is vitiated by an infinite progress just as *Dasein* is in Hegelian logic.

Present day mathematical logic is, however, dominated by the formal approach to mathematics, in which the concept, both in its Fregean and Hegelian forms, has become suppressed, and mathematical logic largely reduced to a branch of mathematics such as Algebra and Topology.

The present situation of mathematical logic is curiously similar to that which occurred in classical Greek geometry. Euclidean geometry then played the role that formal systems do today (with geometrical diagrams and permissable constructions substituting respectively for well-formed formulas and rules of inference). Arithmetic and geometry at that time were the two main branches of ``pure'' mathematics, but it was noticed that there were common concepts and principles in these, such as the axioms concerning equals. A similar situation holds today where the different formal systems for mathematics exhibit common features.

However, the reaction of the Neoplatonist Proclus is very different from that of today. For Proclus, the explanation of the principles common to arithemetic and geometry was that there had to be a ``general mathematics'' underlying both. This general mathematics showed itself to be, in turn, dependent on a ``higher'' science. This higher science is described by Proclus in Aristotelian terms as the ``science of being as being'', i.e. *metaphysics*, the ``truly universal'', unhypothetical science, whose unifying principle is Plato's *dialectic,* the source of mathematical creativity and of logical development in mathematics.

In contrast, the dominant present-day formal approach to mathematics and its logic is content to compare the different formal systems with open-minded tolerance, each with its own merits and demerits, and to ``pass'' on the philosophical issues. It does not, as Proclus did, go further and investigate philosophically the ``general mathematics'' and the philosophy underlying that. But the philosophical issues do not go away.

What this paper has argued is that the philosophy underlying the formalizing of mathematical logic is rather to be based on the Fregean insight into the canonical, conceptual character of logic and mathematics, but developed within a modern Hegelian framework, in which the implicit infinite Hegelian character of the Fregean concept is allowed to break through and transform that concept. Such a philosophy would complete in Hegelian form, the program which Proclus, in the limited context of Greek mathematics, tried to realize, viz. the detailed articulation of the Platonic insight into *dialectic* as the capstone[44] of the mathematical sciences.

Department of Mathematics
University of Mississippi

[44] Republic, 534e.