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Up-to-Homotopy Monoids

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Abstract

Informally, a homotopy monoid is a monoid-like structure in which properties such as associativity only hold 'up to homotopy' in some consistent way. This short paper comprises a rigorous definition of homotopy monoid and a brief analysis of some examples. It is a much-abbreviated version of the paper 'Homotopy Algebras for Operads' (`math.QA/0002180`), and does not assume any knowledge of operads.

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Introduction

This paper is a taster for the longer paper [Lei], 'Homotopy algebras for operads'. It is written for both the reader unfamiliar with the language of operads, and the reader who would like to get the essence of [Lei] without taking too much time. The purpose here is to define what a homotopy monoid is and to take a look at some examples.

I should first of all issue a warning that the term 'homotopy monoid' is not to be read in the same way as 'homotopy group'. Informally, a homotopy monoid is a monoid-like structure in which the associativity and unit laws don't hold

strictly, but only 'weakly' or 'up to homotopy' in some consistent way. Two examples are well-known. In a (non-strict) monoidal category, one has a tensor product $\otimes$ which is not (for instance) strictly associative, but is associative up to isomorphism; moreover, these isomorphisms obey rules of their own. The second example is that of a loop space, that is, the space of all based loops in a fixed topological space with basepoint. A loop space is not strictly speaking a topological monoid, because of the fact that if one chooses a rule for composing any pair of loops (e.g. travel each one at double speed) then that composition does not obey unit or associative laws. However, these laws are obeyed up to homotopy. Again, if one chooses particular homotopies to do this job then these homotopies almost obey certain laws of their own—that is, they obey them up to homotopy—and so on.

Those who do know about operads will know that monoids are the algebras for a certain operad. In the full-length version [Lei], homotopy P-algebras are defined for an arbitrary operad P, and the content of this paper is a special case. Another special case is the operad P whose algebras are *commutative* monoids; a homotopy P-algebra in the category of topological spaces is then exactly what Segal defined as a Γ-space in [Seg] (or 'special Γ-space', in more recent terminology). It is from Segal's paper that this work is descended.

Acknowledgements

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1 The Environment

Let $\mathcal{M}$ be a monoidal category. Our aim is to define what a homotopy monoid in $\mathcal{M}$ is, and so we need some notion of 'homotopy' or 'weak equivalence' in $\mathcal{M}$. This will be achieved by a very simple device: we mark out certain of the morphisms in $\mathcal{M}$ and decide to call them homotopy equivalences, or just 'equivalences'.

So: from now on suppose that $\mathcal{M}$ is equipped with a distinguished subcollection of its morphisms, and call these morphisms the *(homotopy) equivalences* in $\mathcal{M}$. For reasons which are more apparent in [Lei], we will only consider examples having the following properties:

- any isomorphism is an equivalence

- if $h = g \circ f$ is a composite of morphisms in $\mathcal{M}$, and if any two of f, g, h are equivalences, then so is the third
- if $A \xrightarrow{f} B$ and $A' \xrightarrow{f'} B'$ are equivalences then so is $A \otimes A' \xrightarrow{f \otimes f'} B \otimes B'$.

Examples

a. $\mathcal{M}$ is the monoidal category $(\mathbf{Top}, \times)$ of topological spaces with the usual (cartesian) product; homotopy equivalences are homotopy equivalences.

b. $\mathcal{M}$ is the monoidal category $(\mathbf{Cat}, \times)$ of (small) categories and functors, with cartesian product; equivalences are equivalences of categories—that is, those functors G for which there exists some functor F with $G \circ F \cong 1$ and $F \circ G \cong 1$. (F will be called a *pseudo-inverse* to G.)

c. $\mathcal{M}$ is any monoidal category, and the equivalences are just the isomorphisms. Given the first of the three conditions above, this is the smallest possible class of equivalences in $\mathcal{M}$.

d. $\mathcal{M}$ is $(\mathbf{ChCx}, \otimes)$, the monoidal category of chain complexes (of modules over a fixed commutative ring) with the usual tensor product; equivalences are chain homotopy equivalences.

e. $\mathcal{M}$ is $(\mathbf{Top}_*, \vee)$, where $\mathbf{Top}_*$ is the category of spaces with basepoint and continuous basepoint-preserving maps, and $\vee$ is the wedge product (join two spaces together by their basepoints); equivalences are homotopy equivalences which respect basepoints.

2 The Definition of Homotopy Monoid

We are nearly ready to give the definition of homotopy monoid, but first need a preliminary definition; it is one of the various notions of a map between monoidal categories.

Definition 2.1 *Let $\mathcal{N}'$ and $\mathcal{N}$ be monoidal categories. A* colax monoidal functor $\mathcal{N}' \longrightarrow \mathcal{N}$ *consists of a functor $X : \mathcal{N}' \longrightarrow \mathcal{N}$ together with maps*

$$\begin{array}{rccc} \xi_{AB}: & X(A \otimes B) & \longrightarrow & X(A) \otimes X(B) \\ \xi_0: & X(I) & \longrightarrow & I \end{array}$$

($A, B \in \mathcal{N}'$) satisfying naturality and coherence axioms.

Here $\otimes$ and I denote the tensor operation and unit object in both monoidal categories $\mathcal{N}'$ and $\mathcal{N}$. The coherence axioms can be found in [Lei, 1.1.1] or in [Bor, 6.4.1]. If using the latter source one must reverse all the arrows labelled by a τ, since what is being defined there is a *lax* monoidal functor.

Definition 2.2 *Let $\mathcal{M}$ be a monoidal category equipped with a class of equivalences, as in Section 1. A* homotopy monoid *in $\mathcal{M}$ is a colax monoidal functor*

$$(X, \xi) : \Delta \longrightarrow \mathcal{M}$$

for which the maps ξ_0 and ξ_{mn} are each homotopy equivalences ($m, n \geq 0$).

Here Δ is the category of finite ordinals, equivalent to the category of totally ordered finite sets (including $\emptyset$). The monoidal product on Δ is addition, and the unit object is the empty set. The objects of Δ are written $0, 1, \ldots$, so that n denotes an n-element totally ordered set.

The rest of this paper is devoted to examples of homotopy monoids. The non-trivial examples take a while to explain, so this section only covers them in a very sketchy way; we come back to two of them in Sections 3 and 4. Further details on all can be found in [Lei].

Examples

a. Suppose that the equivalences in $\mathcal{M}$ are just the isomorphisms, as in Example 1(c). Then a homotopy monoid in $\mathcal{M}$ is just a monoidal functor $\Delta \longrightarrow \mathcal{M}$. By a monoidal functor I mean what is sometimes called a strong monoidal functor, that is, a map of monoidal categories which preserves the tensor and unit up to coherent isomorphism. But it is well-known that monoidal functors $\Delta \longrightarrow \mathcal{M}$ are essentially just monoids in $\mathcal{M}$. Thus when $\mathcal{M}$ has no 'interesting' homotopy equivalences, a homotopy monoid in $\mathcal{M}$ is merely a monoid in $\mathcal{M}$.

   To be a little more precise, it is proved (essentially) in [Mac, VII.5.1] that the category of monoidal functors and monoidal transformations from Δ to $\mathcal{M}$ is equivalent to the category of monoids in $\mathcal{M}$. A monoidal functor (X, ξ) corresponds to a certain monoid structure on the object $X(1)$ of $\mathcal{M}$, and in general we will regard $X(1)$ as being in some sense the underlying space of a homotopy monoid (X, ξ).

b. A genuine monoid in **Cat** is a strict monoidal category; one would therefore expect a homotopy monoid in **Cat** to be something comparable to a (non-strict) monoidal category. This is discussed further in Section 3.

c. Any loop space is a homotopy monoid in **Top**: see Section 4.

d. A monoid in **ChCx** is a differential graded algebra, so a homotopy monoid in **ChCx** might be called a homotopy differential graded algebra. But there is already a commonly-used formulation of the concept of homotopy differential graded algebra, namely A_∞-algebras. So homotopy monoids in **ChCx** ought to be related somehow to A_∞-algebras; this is not discussed further here, but is in [Lei, 3.5]. Similar remarks apply to A_∞-spaces and homotopy monoids in **Top**.

3 Monoidal Categories

Let us now look more closely at homotopy monoids in **Cat**. Such a structure consists of a functor $C : \Delta \longrightarrow \mathbf{Cat}$ (previously called X) together with equivalences of categories

$$\begin{array}{rccc} \xi_{mn} : & C(m+n) & \longrightarrow & C(m) \times C(n) \\ \xi_0 : & C(0) & \longrightarrow & \mathbf{1} \end{array}$$

$(m, n \geq 0)$ fitting together nicely. Note that by assembling the components of ξ we obtain a (canonical) equivalence $C(n) \xrightarrow{\simeq} C(1)^n$ for each n; we regard $C(1)$ as the 'base category' of the homotopy monoid (C, ξ).

There now follows a comparison (in one direction) between homotopy monoids in **Cat** and monoidal categories in the usual sense.

Proposition 3.1 *A homotopy monoid in* **Cat** *gives rise to a monoidal category.*

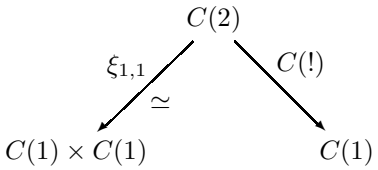
Sketch Proof Take a homotopy monoid (C, ξ) in **Cat**, and construct from it a monoidal category as follows.

Underlying category: $C(1)$.

Tensor: What we want to define is a functor

$$\otimes : C(1) \times C(1) \longrightarrow C(1);$$

what we actually have are functors



where ! is the unique map $2 \longrightarrow 1$ in Δ. So for each m and n, choose (arbitrarily) a pseudo-inverse ψ_{mn} to ξ_{mn}, and define $\otimes$ as the composite

$$C(1) \times C(1) \xrightarrow{\psi_{1,1}} C(2) \xrightarrow{C(!)} C(1).$$

Associativity isomorphisms: The next piece of data we need is a natural isomorphism between $\otimes \circ (\otimes \times 1)$ and $\otimes \circ (1 \times \otimes)$. To see why such an isomorphism should exist, consider what would happen if the ψ_{mn}'s were *genuine* inverses to the ξ_{mn}'s. Then the ψ_{mn}'s would satisfy the same coherence and naturality axioms as the ξ_{mn}'s (with the arrows reversed), and this would guarantee that all sensible diagrams built up out of ψ_{mn}'s commuted. Hence $\otimes$ would be strictly associative. As it is, ψ_{mn} is only inverse to ξ_{mn} up to isomorphism, and correspondingly $\otimes$ is associative up to isomorphism.

In practice, choose (at random) natural isomorphisms

$$\eta_{mn} : 1 \xrightarrow{\sim} \xi_{mn} \circ \psi_{mn}, \qquad \varepsilon_{mn} : \psi_{mn} \circ \xi_{mn} \xrightarrow{\sim} 1$$

for each m and n. Then a natural isomorphism

$$\alpha : \otimes \circ (\otimes \times 1) \xrightarrow{\sim} \otimes \circ (1 \times \otimes)$$

can be built up from the η_{mn}'s and ε_{mn}'s.

Pentagon: We must now check that the associativity isomorphism just defined satisfies the famous pentagon coherence axiom. This asserts the commutativity of a certain diagram built up from components of α, that is, built up from η_{mn}'s and ε_{mn}'s. However, this diagram does *not* commute, which is perhaps unsurprising since η_{mn} and ε_{mn} were chosen independently.

But all is not lost: for recall the result that if

$$F : A \longrightarrow B, \qquad G : B \longrightarrow A,$$
$$\sigma : 1 \xrightarrow{\sim} G \circ F, \quad \tau : F \circ G \xrightarrow{\sim} 1$$

is an equivalence of categories, then τ can be modified to another natural isomorphism τ' so that (F, G, σ, τ') is both an adjunction and an equivalence (see [Mac, IV.4.1]). So when we chose the natural isomorphisms η_{mn} and ε_{mn} above, we could have done it so that $(\psi_{mn}, \xi_{mn}, \eta_{mn}, \varepsilon_{mn})$ was an adjunction. Assume that we did so. Then this being an adjunction says that certain basic diagrams involving η_{mn} and ε_{mn} commute (namely, the diagrams for the triangle identities [Mac, IV.1]): and that is enough to ensure that the pentagon commutes.

Units: We also need to define the unit object for the tensor and the left and right unit isomorphisms, and to prove those coherence conditions which involve units. This is done by the same methods as above. □

So starting from a homotopy monoid in **Cat**, we have constructed a monoidal category. The construction involves arbitrary choices, but these choices only affect the resulting monoidal category up to isomorphism (that is, isomorphism in the category of monoidal categories and monoidal functors).

4 Loop Spaces

So far we have not actually seen any examples of homotopy monoids beyond the trivial. We remedy this now by sketching an argument that any loop space is a homotopy monoid in **Top**. Liberal use will be made of function spaces (exponentials); the conscientious reader should therefore regard 'space' as meaning 'compactly generated Hausdorff space', or take some other convenient cartesian closed substitute for the category of topological spaces.

Fix a space B with basepoint and form the loop space of B, that is, the space $\mathbf{Top}_*(S^1, B)$ of continuous basepoint-preserving maps from the circle S^1 to B.

Proposition 4.1 $\mathbf{Top}_*(S^1, B)$ *is a homotopy monoid in* **Top***, in a canonical way.*

Sketch Proof The object is to find a homotopy monoid (X, ξ) in **Top** with $X(1) = \mathbf{Top}_*(S^1, B)$. (To see why this is a reasonable interpretation of the statement of the Proposition, compare the role of $X(1)$ in Example 2(a) and $C(1)$ in Section 3.) The proof is in two steps.

a. S^1 is a homotopy comonoid, in the sense that there is a colax monoidal functor

$$(W, \omega) : (\Delta, +) \longrightarrow (\mathbf{Top}_*^{\mathrm{op}}, \vee)$$

with $W(1) = S^1$ and with the components ω_0, ω_{mn} of ω all homotopy equivalences. In other words, S^1 is a homotopy monoid in $(\mathbf{Top}_*^{\mathrm{op}}, \vee)$. We define (W, ω) as follows:

- $W(n)$ is the standard n-simplex Δ^n with its $(n+1)$ vertices collapsed to a single point, and this declared the basepoint, e.g.

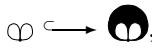
$$\begin{aligned} W(0) &= \bullet\,, \\ W(1) &= \bigcirc = S^1, \\ W(2) &= \end{aligned}$$



- W is defined on morphisms using the standard face and degeneracy maps of simplices
- ω is defined by face maps: for instance,

$$\omega_{1,1} : W(1) \vee W(1) \longrightarrow W(2)$$

is the inclusion



which is evidently a homotopy equivalence. ($W(2)$ can be thought of as a thickened wedge of two circles.)

b. There's a functor

$$\mathbf{Top}_*(—, B) : (\mathbf{Top}_*^{\mathrm{op}}, \vee) \longrightarrow (\mathbf{Top}, \times),$$

which is a (strong) monoidal functor (in the terminology of Example 2(a)) simply because $\vee$ is the coproduct in $\mathbf{Top}_*$. Note also that this functor preserves homotopy equivalences.

Composing the functors of (a) and (b) yields a homotopy monoid $(X, \xi) : (\Delta, +) \longrightarrow (\mathbf{Top}, \times)$ with $X(1) = \mathbf{Top}_*(S^1, B)$, as required. □

Let us finish by examining the homotopy monoid structure we have just put on the loop space, and in particular at how the composition of two loops is handled.

We have

$$X(2) = \mathbf{Top}_*(\text{⚇}, B),$$

and the pieces of (X, ξ) relevant to binary composition are the maps

$$\begin{array}{ccc} & X(2) & \\ {\scriptstyle \xi_{1,1}} \swarrow & \simeq & \searrow {\scriptstyle X(!)} \\ X(1)^2 & & X(1). \end{array}$$

This diagram is

$$\begin{array}{ccc} & \mathbf{Top}_*(\text{⚇}, B) & \\ \swarrow & \simeq & \searrow \\ \mathbf{Top}_*(\text{○○}, B) = \mathbf{Top}_*(S^1, B)^2 & & \mathbf{Top}_*(S^1, B), \end{array} \qquad (*)$$

where the map on the left is restriction to the two inner circles and the map on the right is restriction to the outer circle. Note that all of the data making up (X, ξ), and in particular the maps in $(*)$, is constructed canonically from B: no arbitrary choices have been made. In contrast, there is no canonical map

$$\mathbf{Top}_*(S^1, B)^2 \longrightarrow \mathbf{Top}_*(S^1, B)$$

defining 'composition': although the obvious and customary choice is to use the map described by the instruction 'travel each loop at double speed', this appears to have no particular advantage or special algebraic status compared to any other choice. Since the usual formulation of the idea of homotopy topological monoid, A_∞-spaces, does entail this arbitrary choice of a composition law for loops, one might regard this as a virtue of the definition presented here.

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