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ARE OPERADS ALGEBRAIC THEORIES?

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Abstract

I exhibit a pair of non-symmetric operads that, while not themselves isomorphic, induce isomorphic monads. The existence of such a pair implies that if 'algebraic theory' is understood as meaning 'monad' then operads cannot be regarded as algebraic theories of a special kind.

Introduction

Operads have become increasingly useful in many subjects, including algebraic geometry [**5**], algebra [**9**], theoretical physics [**6**, **9**], category theory [**7**], and of course algebraic topology [**1, 8**–**10**]. Correspondingly, there are many viewpoints on operads. Perhaps the most widely shared is that an operad is an algebraic theory of some kind: in other words, an operad consists of a collection of operations, each taking a given number of arguments, together with a rule for substitution.

The view that an operad is an algebraic theory would seem to be confirmed by the fact that any operad has a category of algebras. Nevertheless, the possibility remains that the passage from an operad to its category of algebras involves a loss of information. The purpose of this note is to show by example that such a loss can indeed occur, in the setting of operads without symmetric group action.

The answer to the question of the title is, therefore, 'no', if 'algebraic theory' is interpreted as 'monad' and 'operad' as 'non-symmetric operad'. It has been shown elsewhere that for symmetric operads the answer is 'yes': see Section 3.

The passage from operads to algebraic theories can be expressed more precisely as follows. I use 'operad' to mean 'non-symmetric operad of sets'. Any operad induces a monad on **Set**, the algebras for which are exactly the algebras for the operad. Any map of operads induces a map between the resulting monads, where by definition a map of monads is a natural transformation-preserving multiplication and units in an obvious sense made precise below. This defines a functor

$$(\text{operads}) \longrightarrow (\text{monads on } \mathbf{Set}).$$

However, this functor does not reflect isomorphism: in other words, there exist non-isomorphic operads P and P' whose associated monads *are* isomorphic. This implies, of course, that the categories of algebras for P and P' are isomorphic, so P and P' are 'Morita equivalent' in a strong sense. It also implies that an operad should not be regarded as merely a monad with certain properties: the canonical map from isomorphism classes of operads to isomorphism classes of monads is not injective.

Such a pair of operads is constructed as follows. Any operad P gives rise to a new operad P^{rev}, whose induced monad is isomorphic to that of P (Section 1). It

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is then just a matter of finding an operad P such that $P \ncong P^{\mathrm{rev}}$, which is done in Section 2. Further comments follow in Section 3.

1. *The reverse of an operad*

For each operad P, I define its 'reverse' P^{rev} and show that the monads induced by P and P^{rev} are isomorphic.

Let P be an operad; then P consists of a sequence $(P(n))_{n\in\mathbb{N}}$ of sets together with a unit element $1 \in P(1)$ and a composition operation

$$\begin{array}{rcl} P(n) \times P(k_1) \times \ldots \times P(k_n) & \longrightarrow & P(k_1 + \ldots + k_n) \\ (\theta, \theta_1, \ldots, \theta_n) & \longmapsto & \theta\circ(\theta_1, \ldots, \theta_n) \end{array}$$

for each $n, k_i \in \mathbb{N}$, satisfying axioms. Its *reverse* P^{rev} is defined as follows: $P^{\mathrm{rev}}(n) = P(n)$ for all $n \in \mathbb{N}$, the identity of P^{rev} is the same as that of P, and the composition $\circ_{\mathrm{rev}}$ is given by

$$\theta\circ_{\mathrm{rev}}(\theta_1, \ldots, \theta_n) = \theta\circ(\theta_n, \ldots, \theta_1)$$

($n, k_i \in \mathbb{N}$, $\theta \in P(n)$, $\theta_i \in P(k_i)$). This does define an operad P^{rev}: all that needs checking is associativity, which is straightforward.

Let (S, μ, η) be the monad on **Set** induced by P. Then for any set X, we have

$$SX = \sum_{n\in\mathbb{N}} P(n) \times X^n;$$

the unit map

$$\eta_X : X \longrightarrow SX$$

picks out the identity element of $P(1)$, and the multiplication map

$$\mu_X : S^2X \longrightarrow SX$$

is given by

$$\begin{array}{l} \Big(\theta, \big(\theta_1, x_1^1, \ldots, x_1^{k_1}\big), \ldots, \big(\theta_n, x_n^1, \ldots, x_n^{k_n}\big)\Big) \\ \qquad\qquad \longmapsto \Big(\theta\circ(\theta_1, \ldots, \theta_n), x_1^1, \ldots, x_1^{k_1}, \ldots, x_n^1, \ldots, x_n^{k_n}\Big) \end{array}$$

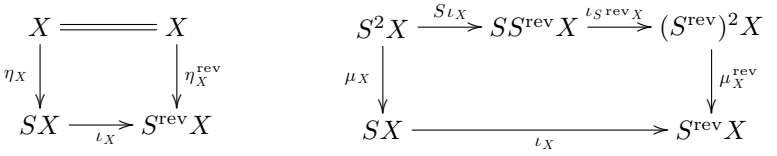
($n, k_i \in \mathbb{N}$, $\theta \in P(n)$, $\theta_i \in P(k_i)$, $x_i^j \in X$). The monad $(S^{\mathrm{rev}}, \mu^{\mathrm{rev}}, \eta^{\mathrm{rev}})$ induced by P^{rev} is given by $S^{\mathrm{rev}} = S$, $\eta^{\mathrm{rev}} = \eta$, and

$$\begin{array}{l} \mu_X^{\mathrm{rev}} : \Big(\theta, \big(\theta_1, x_1^1, \ldots, x_1^{k_1}\big), \ldots, \big(\theta_n, x_n^1, \ldots, x_n^{k_n}\big)\Big) \\ \qquad\qquad \longmapsto \Big(\theta\circ(\theta_n, \ldots, \theta_1), x_1^1, \ldots, x_1^{k_1}, \ldots, x_n^1, \ldots, x_n^{k_n}\Big). \end{array}$$

There is a natural isomorphism $\iota : S \xrightarrow{\sim} S^{\mathrm{rev}}$ whose component at a set X is

$$\begin{array}{rccc} \iota_X : & SX & \xrightarrow{\sim} & S^{\mathrm{rev}}X \\ & (\theta, x_1, \ldots, x_n) & \longmapsto & (\theta, x_n, \ldots, x_1) \end{array}$$

($n \in \mathbb{N}$, $\theta \in P(n)$, $x_i \in X$). Using the above descriptions of the monad structures, it is straightforward to check that ι is an isomorphism of monads; in other words, that for each X the following diagrams commute.



So, as promised, any operad P gives rise to a new operad P^{rev} inducing the same monad as P.

Observe also that reversal works for (non-symmetric) operads in any symmetric monoidal category $\mathcal{V}$. The definition of P^{rev} is an absolutely straightforward generalization of the case $\mathcal{V} = \mathbf{Set}$, using the symmetry of $\mathcal{V}$. If $\mathcal{V}$ has countable coproducts and tensor distributes over them, then any operad P in $\mathcal{V}$ induces a monad on $\mathcal{V}$, algebras for which are algebras for P; and just as above, the monads induced by P and P^{rev} are isomorphic.

2. *The counterexample*

To find a pair of non-isomorphic operads whose induced monads are isomorphic, it suffices to find an operad that is not isomorphic to its reverse.

This task is not completely straightforward, since many commonly encountered operads admit a symmetric structure and any such operad *is* isomorphic to its reverse. Indeed, let $\sigma_n \in S_n$ denote the permutation reversing the order of n letters: then for any symmetric operad P, there is an isomorphism $P \xrightarrow{\sim} P^{\mathrm{rev}}$ sending $\theta \in P(n)$ to $\theta \cdot \sigma_n \in P^{\mathrm{rev}}(n)$. Further, several well-known operads that do not admit a symmetric structure are, nevertheless, isomorphic to their reverse: this applies, for instance, to Stasheff's operad of associahedra [**9**, **11**].

Here is an operad P not isomorphic to its reverse. Let $P(n)$ be the set of all n-tuples $(f_1, \ldots, f_n)$ of order-preserving continuous maps $f_i : [0,1) \longrightarrow [0,1)$ of the half-open real interval such that if $i < j$ and $t_i, t_j \in [0,1)$ then $f_i(t_i) < f_j(t_j)$. The identity of P is the identity map $\mathrm{id}_{[0,1)} \in P(1)$, and composition

$$P(n) \times P(k_1) \times \ldots \times P(k_n) \longrightarrow P(k_1 + \ldots + k_n)$$

is

$$\Big((f_1, \ldots, f_n), (f_1^1, \ldots, f_1^{k_1}), \ldots, (f_n^1, \ldots, f_n^{k_n})\Big) \longmapsto \Big(f_1 f_1^1, \ldots, f_1 f_1^{k_1}, \ldots, f_n f_n^1, \ldots, f_n f_n^{k_n}\Big).$$

Seen another way, P is an endomorphism operad. For consider the (non-symmetric) monoidal category of ordered topological spaces, where the product $X \boxplus Y$ is defined by taking the disjoint union of X and Y and adjoining the relation $x < y$ for each $x \in X$ and $y \in Y$. (Compare the addition of ordinals.) Then $P(n)$ is the set of maps $[0,1)^{\boxplus n} \longrightarrow [0,1)$, with the usual endomorphism operad structure.

To prove that P is not isomorphic to P^{rev}, I introduce some temporary terminology. Let Q be an operad. An element $\gamma \in Q(1)$ is *constant* if

$$\text{for all } n \in \mathbb{N} \text{ and all } \phi, \phi' \in Q(n), \qquad \gamma \circ (\phi) = \gamma \circ (\phi').$$

An element $\phi \in Q(n)$ is *surjective* if

$$\text{for all } \theta, \theta' \in Q(1), \qquad \theta \circ (\phi) = \theta' \circ (\phi) \quad \Longrightarrow \quad \theta = \theta'.$$

The following lemma shows that these terms have the expected meanings when Q is P or P^{rev}. For convenience, I write an element $(g) \in P(1)$ as simply g.

LEMMA. (i) *An element $g \in P(1)$ is constant in the sense above if and only if the map $g : [0,1) \longrightarrow [0,1)$ is constant in the usual sense.*

(ii) *An element $(f_1, \ldots, f_n) \in P(n)$ is surjective in the sense above if and only if the union of the images of $f_1, \ldots, f_n$ is $[0,1)$.*

Moreover, both statements remain true when P is replaced by P^{rev}.

Proof. For (i), the 'if' is clear. Now suppose that g is not constant in the usual sense, so that there exist $t, t' \in [0,1)$ with $g(t) \neq g(t')$. If we take $f, f' : [0,1) \longrightarrow [0,1)$ to be the constant functions with respective values t and t' then $f, f' \in P(1)$ with $g \circ (f) \neq g \circ (f')$, so g is not constant in the sense above.

For (ii), the 'if' is also clear. Conversely, if the union of the images of $f_1, \ldots, f_n$ is not $[0,1)$ then, by continuity, at least one of the following holds:

- $n = 0$;
- $n \geqslant 1$ and $f_1(0) > 0$;
- $\sup f_{i-1} < f_i(0)$ for some $i \in \{2, \ldots, n\}$;
- $n \geqslant 1$ and $\sup f_n < 1$.

In all cases, there is some nonempty open interval $(a,b) \subseteq [0,1)$ that does not meet the union of the images of $f_1, \ldots, f_n$. We can construct a continuous order-preserving map $h : [0,1) \longrightarrow [0,1)$ that is not the identity but satisfies $h(t) = t$ for all $t \notin (a,b)$, and this gives distinct elements h, id of $P(1)$ satisfying $h \circ (f_1, \ldots, f_n) = \mathrm{id} \circ (f_1, \ldots, f_n)$. So $(f_1, \ldots, f_n)$ is not surjective in the sense above.

The final claim of the lemma follows immediately from the definition of P^{rev}. □

We can now show that the following isomorphism-invariant property of an operad Q holds when $Q = P$ but fails when $Q = P^{\mathrm{rev}}$:

- there exist $\phi \in Q(2)$ and a constant $\gamma \in Q(1)$ such that $\phi \circ (\gamma, \mathrm{id}) \in Q(2)$ is surjective.

It will follow that $P \ncong P^{\mathrm{rev}}$.

To see that the property holds for $Q = P$, let $g, f_1 : [0,1) \longrightarrow [0,1)$ both be the map with constant value 0, and let $f_2 : [0,1) \longrightarrow [0,1)$ be the identity. Then $g \in P(1)$ is a constant, $\phi = (f_1, f_2)$ is an element of $P(2)$, and if $e = \phi \circ (g, \mathrm{id})$ then

$$e_2 = f_2 \circ \mathrm{id} = \mathrm{id},$$

and so e is surjective.

To see that the property fails for $Q = P^{\mathrm{rev}}$, we have to see that given $\phi = (f_1, f_2) \in P(2)$ and a constant $g \in P(1)$, the composite $e = \phi \circ (\mathrm{id}, g)$ in P cannot be surjective. Indeed, let b be the constant value of g; then

$$e_2 = f_2 \circ g = (\text{constant map with value } f_2(b)),$$

so by order-preservation

$$\mathrm{image}(e_1) \cup \mathrm{image}(e_2) \subseteq [0, f_2(b)] \subsetneq [0,1),$$

as required.

3. *Further comments*

The properties of the functor

$$G : (\text{operads}) \longrightarrow (\text{monads on } \mathbf{Set})$$

can be analyzed more precisely. The monads in the essential image of G (that is, the monads isomorphic to $G(P)$ for some operad P) are the strongly regular finitary monads. By definition, this is the class of monads whose corresponding algebraic theory can be presented by finitary operations and equations in which the same variables appear on each side of the equals sign, in the same order and without repetition. (For instance, the theory of monoids is allowed, but the theories of commutative monoids and groups are not.) Another description is that they are the cartesian monads S such that there exists a cartesian natural transformation, respecting the monad structures, from S to the free monoid monad. The original source on strong regularity is [**3**]; proofs of the results just mentioned can be found in [**7**, C.1 and 6.2].

The functor G does not reflect isomorphism, as has been shown. It does reflect isomorphisms (plural): that is, if $f : P \longrightarrow P'$ is a map of operads and $G(f)$ is an isomorphism, then so too is f. This is easily shown, as is the fact that G is faithful. But since a full and faithful functor reflects isomorphism, G cannot be full. To prove this more directly, let P be the operad of the previous section and take the isomorphism $\iota : S \xrightarrow{\sim} S^{\mathrm{rev}}$ of Section 1, where S and S^{rev} are the monads induced by P and P^{rev} respectively. Then since G reflects isomorphisms, there is no map $f : P \longrightarrow P^{\mathrm{rev}}$ satisfying $G(f) = \iota$: so again, G is not full.

Here I have stuck to operads of sets; I know little about the situation for other types of operad. Trivially, taking discrete spaces on the set-theoretic example above yields a pair of non-symmetric topological operads that induce isomorphic monads but are not themselves isomorphic.

The situation for symmetric operads is completely different: symmetric operads of sets *can* be identified as monads of a special kind. Precisely, the canonical functor

$$(\text{symmetric operads}) \longrightarrow (\text{monads on } \mathbf{Set})$$

defines an equivalence between the category of symmetric operads and the category of analytic monads and weakly cartesian maps. This is a result of Weber [**12**], using Joyal's characterization of the endofunctors on **Set** induced by species, and of the natural transformations induced by maps between them [**4**].

There are at least two abstract perspectives on the construction of the reverse of an operad. First, write T for the free monoid monad on **Set**. Then a (non-symmetric) operad amounts to a cartesian monad $S = (S, \mu, \eta)$ on **Set** together with a cartesian natural transformation $\pi : S \longrightarrow T$ respecting the monad structures, and the monad induced by the operad is simply S. (For an explanation and proof, see for instance [**7**, Corollary 6.2.4].) Now, there is an involution ρ of the monad T given by reversing the order of finite lists, which implies that any operad P described by a pair (S, π) gives rise to a new operad described by the pair $(S, \rho \circ \pi)$; this operad is P^{rev}. From this point of view, the monad induced by P^{rev} is not just isomorphic but equal to that induced by P.

Second, given any cartesian monad T on a category $\mathcal{E}$ with pullbacks, there is a category of so-called *T-operads*. (See [**2**] or [**7**, Chapter 4].) Any T-operad P induces

a monad T_P on $\mathcal{E}$, and algebras for the operad are by definition algebras for this monad. When T is the free monoid monad on **Set**, these are the standard notions of 'non-symmetric operad', 'induced monad', and 'algebra'. Inevitably, if we have an isomorphism $(\mathcal{E}, T) \xrightarrow{\sim} (\mathcal{E}', T')$ between two different cartesian monads on two different categories then there is an induced isomorphism between the categories of T-operads and T'-operads, and if P' is the T'-operad corresponding to a T-operad P then the monad $T'_{P'}$ on $\mathcal{E}'$ is obtained by transporting the monad T_P on $\mathcal{E}$ across the isomorphism. In particular, this holds for the isomorphism

$$(\mathrm{id}, \rho) : (\mathbf{Set}, T) \xrightarrow{\sim} (\mathbf{Set}, T),$$

where T is the free monoid monad and ρ is as above; the resulting automorphism of the category of non-symmetric operads sends P to P^{rev}, and by the preceding comments the respective induced monads are isomorphic.

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