Lawvere Archive Posthumous Publication – August 27, 2025

The Variety of Mathematical Space*

F. William Lawvere

July 13, 2013

Abstract June 18, 2010: What is a space? A space is just an object in a category of spaces! The implied further question is given a very general answer involving lextensivity, as well as a much more structured answer involving dialectically coupled (cohesive/discrete) pairs of toposes. Examples can be analyzed and constructed using the simple geometric paradigm of figures and incidence relations, by which any lextensive category can be embedded in a Grothendieck topos; more refined subtoposes of the latter are specified by Grothendieck coverings that embody the geometrical equivalent of existential/disjunctive conditions on these extended spaces; a specific example involves a generalization of Maschke means. The extended spaces always include the Hurewicz exponential spaces, for example, spaces of functions and distributions equipped automatically with the ambient sort of cohesion. Examples important for smooth, analytic, and algebraic geometry are infinitesimally generated, pursuing an observation that goes back to Euler. A smooth account of points and components for algebraic geometry over a non-algebraically closed field is achieved by replacing Cantorian abstract sets with a Galois-Barr topos as the discrete aspect. The basic goal is to help make the advances in Algebraic Geometry during the past 50 years more accessible to students and to colleagues in related fields by utilizing the simplifying advances in categorical foundations during the same 50 years, especially guided by proposals made by Grothendieck's colloquium talk in 1973 at Buffalo [8].

*We do not address here the various metrics that a space might or might not carry, but rather the nature of the domains on which such metrics or other quantities might be defined.

What is a Space?

Mathematics requires the general notion of a 'cohesively extended object'. The inside of such a cohesively extended object may encompass figures of various shapes. It can serve as domain of variation for intensively variable quantities, such as functions, differential forms, cohomology classes, and also as domain of variation for extensively variable quantities, such as measures, distributions, and homology classes. Moreover, structures like groups are typically not purely abstract, but rather move in such a space as cohesive background. I chose 'cohesive' as the general adjective because the historically and philosophically natural term 'continuous' has for over a century been restricted to one specific mathematical determination.

However, many various determinations are needed in mathematical practice, so that a more concrete description involves several particular categories such as smooth, analytic, algebraic, simplicial, cubical, et cetera. Actually, all those categories do have non-trivial features in common, as partly described in my article 'Axiomatic Cohesion' [19] (2007). The axioms given there apply to many of the categories just mentioned; but not included is the case of 'classical continuity' that had become the default model of cohesion. That default model, the category of 'topological spaces', has severe drawbacks. For example, Peano's space-filling curves should have been the signal to develop something like Grothendieck's 'tame topology', but instead, the philosophical climate of the time led to their enshrinement as proof of the unfathomable nature of space (that students of analysis would have to face by abandoning their geometric intuition, as Peano urged). In the 1940's Hurewicz made explicit a second defect and boldly replaced the default category with the k-spaces (k = kompakt erzeugte [9] (1940's), later studied by Kelley, Brown, and Steenrod). His modification explicitly incorporated the cohesion that also infinite-dimensional spaces of variable quantities enjoy. Such infinite-dimensional cohesion had also been achieved in Bernoulli's calculus of variations and in Volterra's analytic functionals by a covariant determination [18] (2000) (Palermo).

For many reasons, the default model is difficult to shake off. Thus, for example, we find

today in Wikipedia the claims that both schemes and toposes are 'generalized topological spaces'; neither of those claims stands up under serious scrutiny. Such imprecise analogies do not honor the many insights discovered by Hausdorff and his successors using the contravariant topological space framework.

1 Objective distributions in lextensive categories; the dialectical opposition to non-cohesion

Because mathematically a space is internally characterized and transformed through closely related spaces, the best modern answer to the question 'What is a space?' is: 'A space is an object in a category of spaces'. Of course, that leads straight to another question, to which my axiomatic cohesion is one detailed answer. However, a very general answer that includes indeed classical topology, schemes, and toposes as well, yet still has nontrivial consequences, is the notion of extensive category: that is a category $\mathcal{C}$ with coproducts, denoted $+$, such that for any two objects X, Y, the extended sum functor

$$\mathcal{C}/X \times \mathcal{C}/Y \to \mathcal{C}/(X+Y)$$

is an equivalence. In other words, any space E spread over a sum $X+Y$ splits itself uniquely as a sum $E = E(X) + E(Y)$ of two pieces spread over the summands X and Y, respectively. Intuitively, $E \to S$ is a kind of objective distribution over S, and weighing or counting E would give rise to a quantitative distribution or extensive quantity. (For example, we could consider a gas E in a box S, as studied by Avogadro, or a tribe E dispersed over a continent S). The covariant functoriality, that is typical of extensive quantities, is achieved in this objective case by merely composing, and this process is additive (for example, the total or integral of a distribution over S is obtained by applying this functoriality to $S \to 1$). The pieces of E that result from distributing E in a sum $S = X + Y$ are obtained by the special pullbacks along the injections. If an extensive category has general fibered products, or just

products, these products provably distribute over sums; for short, we refer to such categories as lextensive or prextensive, respectively.

The prextensive condition on a category is a rudimentary general property of cohesion because, dialectically, it provides a context in which it is possible to detect non-cohesion (for example, by maps $S \to 2$; by the above distributivity, $2 = 1 + 1$ is a Boolean algebra.) Roughly, we can consider that the connected objects are the special ones that have no non-trivial maps to 'discrete' objects.

2 The geometry of figures and incidence relations

The more subtle cohesion in the inside of any object X is typically detected by its geometry of figures and incidence relations. In interesting cases this analysis of the inside of X is adequately done with the help of a smaller category $\mathcal{B} \hookrightarrow \mathcal{C}$; this analysis is embodied in the slice category $\mathcal{B}/X$; the objects of that category are the $\mathcal{B}$-shaped figures in X and the maps between these (commutative triangles in $\mathcal{C}$) determine the incidence relations (such as belonging, intersecting, et cetera). A (representable) subspace of X is an object of $\mathcal{C}/X$ such that any figure can belong to it in at most one way. If an inclusion $\mathcal{B} \hookrightarrow \mathcal{C}$ is adequate [10] (1960), then for any inclusion $U \hookrightarrow V$ between subspaces of any space X, the condition

every $\mathcal{B}$-figure in X that belongs to V also belongs to U

is sufficient for equivalence $U = V$ in $\mathcal{C}/X$. The shape-labeling functor $\mathcal{B}/X \to \mathcal{B}$ is a discrete fibration, and the adequacy is equivalent to the requirement:

any functor $\mathcal{B}/X \to \mathcal{B}/Y$ must be induced by a unique actual map $X \to Y$ in $\mathcal{C}$ provided that it preserves the $\mathcal{B}$-labeling.

(Such a functor is intuitively a 'continuous' process because it preserves the incidence relations of figures without tearing them.) Discrete fibrations are of course equivalent to contravariant set-valued functors; the simple fact that the objects of 'any' category can be analyzed as functors should become familiar to every student. The structuralist thesis that spaces 'are' structures is thus tempered by experience: each choice of an adequate category $\mathcal{B} \hookrightarrow \mathcal{C}$ yields a covariant analysis of the spaces in $\mathcal{C}$ as geometric structures of shape $\mathcal{B}$. (Dually, a choice of $\mathcal{T} \to \mathcal{C}$ yields a contravariant analysis of $\mathcal{C}$ in terms of algebraic structures of kind $\mathcal{T}$; a $\mathcal{T}$-algebraic structure is a discrete op-fibration over $\mathcal{T}$, whose maps act as algebraic operations, and functors between such op-fibrations are homomorphisms, because they preserve the operations labeled by the maps in $\mathcal{T}$). Considering the figures of an object in $\mathcal{C}$ to form a presheaf, rather than a discrete fibration, lead to the functor

$$\mathcal{C} \to \mathbf{Sets}^{\mathcal{B}^{\mathrm{op}}}$$

that assigns to every space X the restriction of the Yoneda representation to $\mathcal{B}$. Adequacy of $\mathcal{B}$ means in these terms that this functor is full and faithful. (Dually, algebraic structures of kind $\mathcal{T}$ can be considered as covariant functors from $\mathcal{T}$ to sets, and the assignment of the $\mathcal{T}$-algebra of intensively-variable quantities to each space X leads to a functor $\mathcal{C} \to (\mathbf{Sets}^{\mathcal{T}})^{\mathrm{op}}$ that is also a restriction of a Yoneda representation.)

Volterra, in the 1880's used the term 'elements' for what we have called 'figures'. Later the term 'element' came to suggest some special properties that a general figure does not possess. Grothendieck, around 1960 popularized the term 'point' for general figures. That terminology had a sort of brave new science-fiction air, but still is actually rather counter-intuitive; everybody knows that points are only a very special kind of figure. How special they are, becomes an important foundational question; Leibniz and Galois had already contributed some insights, to which we refer below.

3 Functions may determine figure types, if not general spaces

In analyzing or synthesizing a particular category $\mathcal{C}$ of spaces, the choices $\mathcal{B} \hookrightarrow \mathcal{C}$ and $\mathcal{T} \to \mathcal{C}$ of basic figure-shapes and basic function types, gives a particular view of the dialectics connecting, in each particular space, incidence relations of figures with algebraic operations on functions. Experience with projective space X suggests the necessity of a two-step analysis. More precisely, Liouville's theorem states that in analytic and algebraic contexts, all globally defined functions on X are constant, yet the space X is determined by the 'global' functions on each piece of a covering. This suggests that often a rather special choice $\mathcal{T} \to \mathcal{C}$ of 'algebraic theory' (although not co-adequate in all of $\mathcal{C}$) may be co-adequate for a small $\mathcal{B}$ that is in turn adequate in all of $\mathcal{C}$. Even fixing a 'small' such choice of the set-valued bi-module $\mathcal{T} \to \mathcal{B}$, may permit very large compatible categories $\mathcal{C}$: not only quotients of the 'affine' $\mathcal{B}$-spaces may be important, but especially the function spaces of Bernoulli-Volterra-Hurewicz are host to the functional analysis over any $\mathcal{T} \to \mathcal{B}$, and should be incorporated in $\mathcal{C}$, in order to reveal and utilize the intrinsic cohesiveness of the variable intensive and extensive quantities. Besides adequacy, a further constraint on enlarging $\mathcal{C}$ is governed by Grothendieck topology (which, for example, may strive to guarantee that the gluing of spaces in $\mathcal{C}$ agrees as much as possible with what can be calculated in terms of $\mathcal{T} \to \mathcal{B}$ itself.) In case $\mathcal{B}$ is itself extensive, that agreement is easily achieved for the initial case of disjoint gluing, so that less tautological Grothendieck topologies involve mainly constraining $\mathcal{C}$, in order to make some of the epimorphisms (existential statements from a logical point of view) agree with those in $\mathcal{B}$. The bi-module $\mathcal{T} \to \mathcal{B}$ is often just a full inclusion, for example, (the opposite) of polynomial k-algebras among finitely-generated k-algebras; then the elements of the bi-module, specifying the functions on each figure shape, and at the same time the figures in each function type, are identified with homomorphisms. Together with a choice of Grothendieck coverings, such a bi-module is data for an analysis/synthesis of a topos $\mathcal{C}$. A topos is a category with subspace representability, which implies both extensivity and Hurewicz exponentiation. The extent to which subspace

representors can be analyzed in terms of unions of 'computability degrees' or of 'Hilbert schemes' still needs to be clarified.

It should be emphasized that even without co-adequacy, the functor $\mathcal{C} \to (\mathbf{Sets}^{\mathcal{T}})^{\mathrm{op}}$, for a given choice $\mathcal{T} \to \mathcal{C}$, gives useful partial information about the spaces in $\mathcal{C}$; for example, if $\mathcal{T}$ consists of the finite powers of the truth-value object of a topos $\mathcal{C}$, this contravariant algebraic invariant contains all of the logic in a narrow sense (i.e. the full relational systems of the spaces, considered as Heyting algebras.) Full duality is not needed for a useful invariant, witness cohomology.

4 From algebraic geometry to smooth geometry and Grothendieck's newer proposals

Godement's book was a fundamental reference for me in 1960, but Gabriel's 1966 lectures in Oberwolfach revealed that the typical topos is not a generalized topological space, but rather a category of spaces; that had already been demonstrated implicitly by Grothendieck in his 1960 seminar on constructing analytic spaces, and even already by Eilenberg & Zilber in their 1950 introduction of simplicial sets [7] (1950). Insights from Gabriel were incorporated into my Chicago lectures on Categorical Dynamics [16] (1979). The axiomatic method in the 1967 Chicago lectures had later led to the elementary theory of toposes with Myles Tierney [15] (1970). The intended dynamical applications have been slow to materialize (as outlined in my Categories in Continuum Physics, Introduction to Springer Lecture Notes 1174 [17], (1986)). The axiomatic theory of synthetic differential geometry that emerged [14] (1981) by Anders Kock, includes algebraic geometry; it further reveals that smooth topology can also be treated by analogous methods, including nilpotent infinitesimals.

The treatment of smooth spaces emphasizes the elementary role of a concrete algebraic theory that is not given by a syntactical presentation, namely, the theory of C^∞ rings.

The developments of 1966-1973 indicated to several people that the foundations of algebraic geometry, in particular, could be considerably simplified relative to the 1957-1962 version. For example, in a 1973 Buffalo colloquium talk, Grothendieck himself urged abandonment of his previous definition of scheme (with its awkward dependence on the notion of locally ringed space). Though the category of schemes as such plays a key role, more streamlined and conceptual constructions of it can be found; axiomatic descriptions of the category itself can support directly the geometrical intuition, without necessarily referring to any particular construction. Grothendieck's 1973 Buffalo colloquium talk [8] was never published, but fragments of his proposal circulated in the community, usually under the slogan 'spaces are functors'. That is only half the story, because even having analyzed the domain and codomain categories for these functors, there remains the fact that neither the minimal (or affine) category of representable functors, nor the maximal category of all functors, has the geometrically correct colimits. A wise choice of Grothendieck topology is needed to specify an appropriate intermediate category whose quotients and sums are good for a specific geometrical purpose. Such an intermediate category is a Grothendieck topos; we need to utilize the functional analysis (!) that flows from that.

5 Infinitesimally generated toposes and 'il punto nella punta'*

Indeed, even finitary functional analysis reveals a fact implicitly known to Euler: the full category of spaces is actually infinitesimally generated. There are several senses of 'generated' for such toposes. Every object is a colimit of the objects in a given site, but the affine site in turn consists of finite limits of lines; the lines therefore generate the whole topos by means of the two Giraud operations: small colimits of finite limits. However, another operation is the

*The point in the tip.

Hurewicz exponentiation that exists in any topos, and the affine site results from applying that operation to still tinier spaces, as I will explain below.

The paradigm of functors and figures applies even to the classical topology (at least in its Fréchet-Fox-Johnstone [12] (1979) covariant form where the basic figures are convergent sequences). However, the more refined axiomatic cohesion does not apply to that example because, unlike the case of simplicial topology, the notion of connected component is not the discrete reflection.

There are many examples (especially those involving differential calculus) that not only satisfy the axiomatic cohesion, but are infinitesimally generated in the sense: every space is a colimit of finite limits of exponentials of infinitesimal spaces. More precisely, many toposes of spaces have a special object D, which could be referred to as 'il punto nella punta' (somewhat like the tip of a pencil point). It has only one rational point and yet its self-exponential D^D has two distinct points (showing that D is itself not reduced to the bare point but has non-vanishing extent), and yet D^D is connected in a suitable sense. That monoid object contains (as a retract) the space of those endomorphisms of D that preserve the point, and that space is just the 'line' as a multiplicative monoid (that contains the one-dimensional multiplicative group as the automorphisms of D). Since all finitary affine schemes are finite limits of copies of this line, it follows that the object D infinitesimally generates the whole topos. The objects D (and their closest relatives, such as $(D^2)/2!$) are roughly the same in the various toposes of interest; only with a second iteration of the exponentiation do the distinctions between smooth, analytic, and algebraic, become evident!

6 Leibnizian skeleton of any space

The special properties of the infinitesimal objects like D, in relation to the ambient topos that they generate, are very striking. They are typically part of a larger class of objects, namely all the spaces that (suggestive of the monads of Leibniz) have the property that every

connected component contains exactly one point. But even more potentially powerful is the existence of further right adjoints to the exponential tangent bundle functors $(\ )^D$; these in turn permit construction of Eilenberg-Mac Lane spaces for deRham cohomology. These Leibnizian objects form a quotient topos that is quintessential in the sense of [13] (2011), and thus, in particular, are co-reflective by a functor that we may call the infinitesimal skeleton.

In the case of algebraic geometry over a field k, the infinitesimal spaces will typically have finite-dimensional function algebras. A crude consequence of more precise statements by Hyman Bass [3] (1963), concerning the ubiquity of Gorenstein rings, or by Garrett Birkhoff [4] (1935) in the case of finitely-generated sub-directly irreducible algebras, is the principle that, in order to verify a consequence of a system of polynomial equations, it suffices to consider all their interpretations by linear transformations of finite-dimensional vector spaces over k. In other words, the infinitesimal skeleton $D(X) \to X$ of any space X is perceived as epimorphic by all affine spaces. (This is a qualitative strengthening of the Nullstellensatz.)

7 Relativized discreteness and the logic of Galois existence

In accord with Grothendieck's philosophy of relativization, the treatment of cohesion is a contrast with 'non-cohesion' via a pair of toposes connected by four adjoint functors. The lower (or relatively discrete) topos may have very special properties, such as Boolean logic, without reducing fully to the Cantorian totally abstract sets. (Such a reduction would in fact tend to destroy the good properties of the four connecting functors.) That lower topos helps to specify the distinction between points and the more general figures; in fact, a composite of two of the four functors, defines a discrete skeleton of any space.

Since Galois, we have been accustomed to the idea that a solution to a polynomial equation over k 'exists' if it really exists in a finite extension of k. The significance of the

principle 'Quantifiers and Sheaves' [15] (1970), derived from the elementary observation that the subjective rule of inference describing the existential quantification of a property along a map expresses precisely the objective adjointness of the operation of direct image along a map applied to a subobject of its domain. The slightly less elementary observation was the answer to the question: 'To what extent do figures belonging to the image of a map actually come from the domain of the map?' The answer was in principle already supplied by Volterra, Poincaré, and deRham, with their theory of exactness, and is now seen to reside in the sheaf-theoretic employment of coverings, which is a main example of Barr's theory of exactness [2]. As a byproduct, this observation brought out more sharply the actual content of the existential quantifier of logic: even in simple cases it only signifies actual existence, when epimorphisms split (i.e., when local sections can be glued) in the background category where theories are modeled.

From such considerations we see that, for algebraic geometry over a field k that is not algebraically closed, the category of abstract sets is not a good base topos. A topos defined by a site that involves finitely generated algebras over k, is cohesive over the topos of sheaves on a site dual to a category of finite field extensions of k. If we take all maps of fields as coverings, then this base topos is Boolean (as shown by Barr and Diaconescu [1]) and, moreover, its internal logic expresses precisely the traditional usage concerning the 'existence' of solutions to equations. Among the notions of covering (for the general algebras) whose trace on the fields is that of Barr as above, there is the topology that we could call 'amenable' that not only takes account of finite disjoint partitions of unity (to preserve the extensivity) but, more specifically, generalizes Maschke's considerations of finite group actions to take as covering any map $\mathbf{spec}(B) \to \mathbf{spec}(A)$ that admits an A-linear retraction $B \to A$ (which would be geometrically a distribution-valued section). Probably, this amenable topology has already occurred in practice, though I do not know a reference.

Concerning 'points' and the 'underlying topological space', all information in them can be extracted from the cohesive/discrete pair described, in a way that avoids the awkward behavior of the traditional formulation. That awkward behavior resulted from taking the

non-filtered colimit of the points defined over the specific extensions (thus, in effect, lumping all prime ideals together irrespective of what field of definition they come from); the products of the resulting external topological spaces do not agree with the geometrical products, hence the embarrassment that the underlying space of an algebraic group is not a topological group. A better-behaved partial invariant is the 'single' topological space object internal to the Galois-Boole-Barr base topos.

Acknowledgements

Bill prepared an abstract and paper titled What is a Space? for a talk in Pisa, Italy on June 18, 2010. A slightly updated version was presented at the Symposium on Sets in Geometry, July 26-29, 2011 in Nancy, France, for which a video is available on the internet (thanks to Michael Wright and the Archive Trust). Bill greatly appreciated the efforts made for both occasions. We would join him in thanking his colleagues and the organizers. Several updates were made for the 2013 version, including changing the title, and adding subtitles. We know Bill appreciated comments and suggestions from Matías Menni (October 20, 2010), and it is possible that Bill had planned further improvements. We are very thankful for Francisco Marmolejo's thoughtful and efficient work in reformatting the paper, and finishing the incomplete bibliography.

The Lawvere Archives Family

Bibliography

[1] Michael Barr & Radu Diaconescu. Atomic Toposes, *Journal of Pure and Applied Algebra* 17, 1980, 1-24.

[2] Michael Barr Non-Abelian Full Embedding: Outline, *Proceedings of the International Congress of Mathematicians,* Nice, France, 1970.

[3] Hyman Bass. On the Ubiquity of Gorenstein Rings, *Mathematische Zeitschrift* vol. 82, 1963, 8-28.

[4] Garrett Birkhoff. On the Structure of Abstract Algebras, *Mathematical Proceedings of the Cambridge Philosophical Society* vol. 31, 1935, 433-454.

[5] Ronnie Brown. Ten topologies for $X \times Y$. Quart. J. Math. Oxford Series (2) (1963). 303-319.

[6] Samuel Eilenberg & Norman Steenrod. *Foundations of Algebraic Topology*, 1952

[7] Samuel Eilenberg & Joseph A. Zilber. Semi-Simplicial Complexes and Singular Homology. *Annals of Mathematics* (2), Vol. 51, 1950, 499-513.

[8] Alexander Grothendieck. Unpublished Colloquium Talk, Buffalo, NY, 1973.

[9] Witold Hurewicz & Henry Wallman. *Dimension Theory.* Princeton University Press, 1941.

[10] John R. Isbell. Adequate Subcategories, *Illinois J. Math*, 4, 1960, 541-552.

[11] John R. Isbell. Small Adequate Subcategories, *Journal of the London Mathematical Society*, 43, 1968, 242-246.

[12] Peter Johnstone. On a Topological Topos, *Proceedings of the London Mathematical Society* (3), Vol. 38, 1979, 237-271.

[13] Peter Johnstone. Remarks on Punctual Local Connectednes, *Theory and Applications of Categories*, Vol. 25, 2011, 51-63.

[14] Anders Kock. *Synthetic Differential Geometry*, Cambridge University Press, 1981, second edition 2006.

[15] F. William Lawvere. Quantifiers and Sheaves. *Proceedings of the International Congress of Mathematicians*, Nice, France, 1970.

[16] F. William Lawvere. Categorical Dynamics, *Open House on Topos-theoretic Methods in Geometry*, Aarhus, 1979.

[17] F. William Lawvere. Introduction to Categories in Continuum Physics, *Springer Lecture Notes in Mathematics* 1174, 1986.

[18] F. William Lawvere. Volterra's Functionals and Covariant Cohesion of Space, *Rendiconti del Circolo Matematico di Palermo, Serie II* , 2000.

[19] F. William Lawvere. Axiomatic Cohesion, *Theory and Applications of Categories*, Vol. 19, 2007, 41-49.