4 Bolzano on Logic in Mathematics and Beyond[1]

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Bolzano's Logical and Mathematical Work at a Glance

According to standard narratives, the origins of formal logic as we know it are to be found within the push toward logicism, axiomatisation and the foundations of set theory for which Frege's foundational project in mathematics often serves as muster. Frege, however, was by no means the first logician of the 19th century to seek to provide a new logical foundation to mathematical knowledge. At least one other author was driven by concerns, insights, ambitions and philosophical acumen that were as remarkable as Frege's. This author's efforts too resulted in a full-scale logical system whose conceptual resources, while they do not have the elegance and simplicity of Frege's "concept-script", are nonetheless as rich as those of first-order predicate calculus and powerful enough to generate Russell's paradox.[2] This author is Bernard Bolzano.

Bernard Placidus Johann Nepomuk Bolzano was born on 5 October 1781 in Prague. He was the son of an Italian father and of a German-Speaking Czech mother. His early schooling was unexceptional: private tutors, Lyceum. In the second half of the 1790s, he studied philosophy and mathematics at the Charles-Ferdinand University. He began his theology studies in the fall of 1800 and simultaneously wrote his first mathematical treatise, the *Considerations on Some Objects of Elementary Geometry*. When he was completing his doctorate in 1804, he applied for two positions at Charles University in Prague: one in Mathematics, the other one in the "Sciences of the Catholic Religion". He obtained the second. He was hastily ordained and took up his function in 1805. His professional career would be punctuated by sickness – he suffered from respiratory illness – and controversy: Bolzano's liberal views on public matters and politics would serve him ill in a context dominated by conservatism in Austria. In 1819, he was accused of "heresy" and subjected

1 This chapter is based on material that was first published as part of an entry in the *Internet Encyclopedia of Philosophy*. It has been revised and enhanced. I'd like to thank Sean Morris for a careful reading and helpful comments.

2 Cf. Simons (1997) and Lapointe (2011, Chapter 3).

to an investigation that would last five years after which he was forced to retire and banned from publication and public office. From then on, he devoted himself entirely to his philosophical work. The result is monumental.[3]

Bolzano is known to have been an important contributor to early modern mathematics. However, his *Considerations on Some Objects of Elementary Geometry* (1804) received virtually no attention at the time they were published and the commentators who have appraised his early mathematical work concur to say that Bolzano's positions may not have more than historical interest (cf. Russ 2004 and Sebestik 1992[4]; see also Waldegg 2001). Part of the reason for this lukewarm appraisal rests in the fact that Bolzano's investigations into geometry fail to anticipate modern axiomatic approaches to the discipline and did not belong to the trend that would culminate with the birth of non-Euclidean geometries, the existence of which Bolzano's contemporary Johann Carl Friedrich Gauss (1777–1855) claimed to have discovered and whose first samples were found in the works of Nikolai Lobatchevski (1792–1856) and Janos Bolyai (1802–60), whom Bolzano did not read.[5]

By contrast, Bolzano is renowned for his anticipation of significant results in the arithmetisation of analysis. This is important: Bolzano's advances here cannot be dissociated from his conclusions regarding the need for a logical reform. Three booklets that appeared in 1816–17 have drawn the attention of historians of mathematics and logic, one of which, the *Pure Analytic Proof…* was re-edited twice at the turn of the 20th century, in 1894 and 1905 (Rusnock 2000, 56–86; 158–98). At the time of their publication, however, they attracted hardly any notice. Only one review is known (see Schubring 1993, 43–53). And even though, according to Grattan-Guiness (1970), there is a possibility that Cauchy plagiarised the *Pure Analytical Proof…* in his *Cours d'Analyse*, this hypothesis is disputed by Freudenthal (1971) and Sebestik (1992, 107ff).

Over the course of the 1820s, Bolzano returned to the philosophical and methodological investigations he had initiated earlier in the *Contributions to a Better Founded Exposition of Mathematics* (1810), a book in two volumes on logical methodology which seems to have received little attention. At the end of the 1830s, after he had worked out the basis for the logical system he deploys in full in his main logical treatise, i.e. the *Theory of Science* (1837), Bolzano returned once more to the philosophy of mathematics and spent the last years of his life working on the

3 Cf. Lapointe (2014a).

4 As Sebestik explains (1992, 35 note), Bolzano, however, never put into question the results to which he had come (Bolzano 1804) in his attempt to prove Euclid's parallel postulate.

5 See Sebestik (1992, 33–72) for a discussion of Bolzano's contribution to geometry; see also Russ (2004, 13–23).

Theory of Magnitudes (*Grössenlehre*). The latter remained unpublished until after his death, and only excerpts appeared in print in the 19th century, most notably the *Paradoxes of the Infinite* (1851). The *Theory of Function* (1930) and the *Pure Theory of Numbers* (1931) were edited by the Czech mathematician Karel Rychlik as part of a commission for the Royal Bohemian Academy of Science. All of these works have now been translated into English by Steve Russ (2004).

By current analytical standards, the scope of Bolzano's philosophical project is considerable. Bolzano delivered a fully worked-out alternative to the logic of his time. Bolzano understood the main obstacle to the development of mathematics in his time to be the lack of proper logical resources – post-Cartesian theories of conceptual analysis and syllogistic inference being unfit for the purpose of modelling deductive reasoning in arithmetic and geometry. Bolzano correctly saw the logic of his contemporaries in the first decades of the 19th century as solidly anchored in the new Kantian epistemology and not, as many have assumed, in some stale modern version of traditional Aristotelian syllogistic.[6] While Bolzano's efforts toward the reform of logic were by no means solely motivated by the needs of mathematical practice – he considered the need to provide ethics and the philosophy of religion with a solid logical and epistemological foundation to be prevalent – it is nonetheless anchored in a general concern for methodology and the theory of knowledge in the context of a theory of deductive systems, i.e. the "purely conceptual disciplines". In such disciplines – mathematics, ethics, pure physics, etc. – truths are related as "grounds to consequences, a notion that can in turn be modelled as part of a theory of deductive theories.

Bolzano's efforts toward developing this new logic fell into two phases: the first extends throughout the 1800s and 1810s and includes the period over the course of which Bolzano produced and published the *Contributions*… (1810) as well as the bulk of his strictly mathematical work (1816, 1817a, 1817b). The second period covers the 1820s and culminated in the publication of the *Theory of Science* (1837), a four-volume treatise extending over 2400 pages that includes everything from a calculus of classes, a theory of conceptual analysis and a semi-formal account of logical consequence to a theory of linguistic interpretation and an account epistemic justification and objective proof. In the *Contributions*, Bolzano's undertaking remained largely programmatic. By the time he had started writing the *Theory of Science*, he had abandoned the main doctrines at play in his earlier work. Between the *Elements of Logic* (1812) and *On the Mathematical Method* (1841), Bolzano rejected his theory of the "multiple copula", his views on the content and structure of concepts, as well as the detail of his positions on modality and

6 See my Introduction, infra.

inference.[7] Nonetheless, it is fair to say that the leitmotiv of Bolzano's mature epistemology already came through in the early 1810s. Bolzano's foremost motivations to call for a logical reform was his fundamental disagreement with the idea that the proper foundation of mathematical knowledge rests in the "Kantian Theory of Construction of Concepts through Intuitions", a criticism to which he devoted an "Appendix" to the *Contributions…* of 1810.[8] As Bolzano saw it, an adequate account of deductive reasoning – in mathematics or anywhere else – must exclude appeal to non-conceptual inferential steps, be they putative "pure intuitions" or any other proxy for logic. In this, Bolzano can be seen to have anticipated an important aspect of later criticisms of Kant, including Bertrand Russell's.[9]

Bolzano's Logical Reform

Even while accounting for the fact that he is credited with anticipating some of 20th century's major philosophical-logical innovations, e.g. antipsychologistic semantic realism, the definition of analyticity and logical truth as invariance under substitution and the Tarskian definition of logical consequence using similar substitutional resources, Bolzano's reform of logic was considerably more radical than what it would be natural to assume in light of the standard narrative. There are two reasons for this. First, standard narratives in the history of logic are almost always based on an approach to historiography that involves deliberately projecting contemporary norms and standards onto past theoretical achievement and to assess philosophical view in light of these. However, Bolzano's main theoretical innovations – I discuss his views on analyticity, deducibility and ground-consequence below – draw on conceptual resources that are exceptionally germane to contemporary analytical philosophers. Because the standard approaches tend to track past anticipations of contemporary ideas and to emphasise theoretical features that speak directly to analytical readers' own current interests, it is bound to picture Bolzano as a kindred spirit whose views do not need to be contextualised to be properly understood.

Second, standard narratives in the history of logic and analytical philosophy are eminently deficient when it comes to understanding the context in which logic developed in the century before and after the publication of the *Critique of Pure Reason*. In particular, they tend to overlook the importance of Kant and the work that was done within his "school" in shaping the discipline over the course of the 19th century.

7 Cf. Lapointe (2011, 18–90).

8 The text has been translated into English by Rusnock (2000, 198–204) and Russ (2004, 132–7).

9 Cf. Russell (1903, §§ 4, 5, 423, 433–4).

As a result, they fail to do justice to what turned out to be the context of Bolzano's logical reform and Bolzano's actual philosophical reasons to effect the shifts in question. The resources Bolzano had to deploy to achieve his results were specifically designed to address problems he found with the doctrines of his contemporaries. In order to appraise the magnitude of Bolzano's proposed change, and do justice to his contribution to the discipline, his views also need to be understood as arising from this context.

One important feature of Bolzano's theory that has been almost constantly overlooked is the significance of his rejection of his predecessors' account of conceptual relations in terms of inclusion in setting the stage for his own views on conceptual and propositional content. The standard account of conceptual relations was couched on a number of doctrines inherited at various stages, for instance, (i) the idea that definition is a matter of (Porphyrian) division and results in conceptual hierarchies in which (ii) the canon of inverse proportion between comprehension (*Inhalt*) and extension (*Umfang*) holds.[10] What is especially relevant here is how (i) and (ii) tie into theories of deduction at the time. The doctrine that concepts' structure is decompositional is the basis for the doctrine according to which logical deduction proceeds on the basis of the principles of identity and/or contradiction and/or sufficient reason and/or excluded middle, i.e. the principles that were meant to do the heavy deductive lifting in the theories of Leibniz, Wolff and even the early Kant.[11]

The connection between decompositional analysis and the theory of deduction in post-Cartesian theories was such that it would have been impossible to dismiss the first without creating deep conceptual difficulties for the second (and vice versa). At least part of the reason why Bolzano was led to develop a new account of analyticity, inference and validity (more on this below) was precisely his rejection of the mainstream account of conceptual analysis in terms of inclusion and decomposition – a criticism that extends over dozens of more or less contiguous pages in the *Theory of Science*. The significance of Bolzano's rejection of decompositional analysis – this included an explicit rejection of both (i) and (ii)[12] – is inescapable and it should be properly contextualised.

10 See Lapointe (2011, Chapter 2).

11 Cf. Heis (*infra*) for a description of various ways of conceiving of the relation between these principles in and after Kant. As Heis explains, we must be cautious not to infer from the fact that Kant thinks that all *analytic judgements* derive from the principle of contradiction that all *logical knowledge* derives from the principle of contradiction. But this is precisely what is at stake here: given that there are, in Kant and in Bolzano's views, non-empirical (or, as Bolzano sees it, purely conceptual) truths that are not analytic, which principles need to be at work to justify them? And are these principles themselves logical (Bolzano says yes) or otherwise (i.e. "transcendental," in Kant's view)?

12 Cf. Lapointe (2011, Chapter 2).

In the *Theory of Science*, Bolzano's theory of propositions and ideas provides the footing for a series of investigations into the epistemology and pragmatics of the deductive disciplines – including a theory of scientific demonstration. This theory is based on two crucial innovations. First, Bolzano commits to a form of antipsychologistic, non-hypostatic realism about propositions of which we find another version in Lotze.[13] On Bolzano's account, properties such as analyticity and deducibility (*Ableitbarkeit*) are defined not for subjective thoughts or sentences but for what Bolzano conceives to be the "objective content" (*objektiver Inhalt*) of the former and the "meaning" (*Sinn*, *Bedeutung*) of the latter. He calls these entities, whose role is to be the primitive bearers of semantic properties 'propositions in themselves' (*Sätze and sich*). Second, properties such as analyticity and deducibility are defined on the basis of a substitutional method. Bolzano's strategy is to define "classes" of propositions on the basis of a fixed vocabulary that they share or, put in his own terms, to use substitution as a procedure to "generate" classes of propositions that differ only by virtue of the fact that certain (non-logical) parts are exchangeable (*veränderlich*).

This approach allows Bolzano to define some interesting types of semantic regularities. For instance, propositions have a certain "degree of validity" (*Grad der Gültigkeit*). This property comes to the fore when we consider the set of all propositions that differ only by virtue of some determinate arbitrarily exchangeaboue components, and ponder the ratio of true to false propositions in that set.[14] Bolzano would agree to say, e.g. that, if we take 'Caius' to be arbitrarily exchangeable in the proposition 'Caius, who lives in Saguenay is a descendent of Nicolas Audet', the degree of validity of the latter the is higher than that of 'Caius, who lives in Saguenay is a descendent of the current Belgian King'. This is documented by a quick survey of genealogical data for the region where I was born.

As Bolzano sees it, analyticity is just like validity a property defined on the basis of the substitutional method. Of course, Bolzano's theory of analyticity is an eminently favoured topic in the literature (cf. Bar-Hillel 1950; Etchemendy 1988, 2008; Künne 2006; Lapointe 2000, 2008, 2014; Morscher 2003; Neeman 1970; Proust 1981, 1989; Textor 2000, 2001). This should be no surprise. Not only is analyticity a topic that has been central to analytical philosophy in the 20th century, but Bolzano's theory speaks directly to what analytical philosophers identified to be the main problems of previous conceptions. On the one hand, Bolzano's definition applies to all grammatical constructions, including those that are not of the subject-predicate form. This was notoriously not the

13 Although it's unclear that Lotze was aware of Bolzano's work in any detail, the conceptual points of convergence are striking. Stang's presentation of Lotze (*infra*) is eminently propitious to a comparison with Bolzano.

14 Cf. Bolzano (1837, §147); see Lapointe (2011, Chapters 4 and 5) for a discussion.

case for Leibniz's and Kant's, for instance, who both defined analyticity precisely in terms of the inclusion of the predicate-concept in the subject-concept in categorical judgements. A judgement or proposition is analytic on their account if the analysis or decomposition of the subject-concept into its parts reveals the predicate-concept as one of them. In this respect, the advantage of Bolzano's approach is that it affords for a maximally broad range of applications: 'analytical' can be predicated or denied of conditionals, disjunctions, conjunctions, etc., and, in general, of propositions that present a syntactic complexity that is foreign to post-Cartesian categorial logics.

Bolzanian Analyticity

As Bolzano sees it, a proposition is analytic when its truth remains constant under substitution of some of its component parts. To be precise, on Bolzano's account, a proposition S is analytic with respect to some variable components i, j… if and only if

i i, j, … can be varied so as to yield at least one "objectual" substitution instance of S
ii All substitution instances of S have the same truth-value as S

A substitution instance is "objectual", on Bolzano's account, if the concept designated by the subject "has an object" (at least on), i.e. if it has an extension (*Umfang*).[15] On this account, propositions can be analytically true or analytically false. When they are analytical, they are not absolutely so either: propositions are always analytic relative to some exchangeable parts.

Admittedly, there are "problems" with Bolzano's conception of analyticity, at least if we assume that the notion is supposed to provide an insight into the property, e.g. Carnap and Quine sought to define under the same name.[16] For Bolzanian analyticity does not provide an account of what it means for a proposition to be true *by virtue of meaning* alone and to be knowable as such. Arguably, however, Bolzano did not think that his definition of what he calls 'analyticity' was meant to do that. The problem Bolzano understood Kant to be grappling with was not the problem of truth by virtue of meaning but a different one: to provide a systematic account of quantificational statements that express generality. There is ground to debate whether this constitutes a gross misunderstanding of Kant's views, or whether Bolzano is here picking up on the fact that Kant himself does not offer a treatment of generality that

15 In Bolzano's view, there are no empty extensions.
16 Cf. Lapointe (2014b).

is both (i) adequate in that it does more than append quantificational particles to categorical statements and (ii) clearly separate from that of accounting for analyticity.[17] Strictly speaking, however, a statement ascribing Bolzanian analyticity to a given propositional form, say,

'X who is a man is mortal' is analytic with respect to X

if it is true, is true because every objectual substitution instance of 'X who is a man is mortal' is also true.

Bolzano's definition of what he calls 'analytic' propositions offers a fairly clear description of the property we associate today with substitutional quantification. If a proposition, say, 'Caius who is man is a mortal', is Bolzano-analytically true with respect to 'Caius', then all substitution instances of the corresponding propositional form 'X who is a man is mortal' are true.[18] The main conclusion to be drawn at this point is the following: a proper contextualisation of Bolzano's views on analyticity should show that there is no simple way to project back contemporary logical and semantic concerns and concepts onto Bolzano's theories. The question Bolzano sought to answer when he developed his views on what he calls 'analyticity' was something like "What does it mean to say that a statement, say,

Caius who is a man, is mortal

is generally true?" This question he answers with roughly the following: it means that all instances of the corresponding propositional form

X who is a man, is mortal

are true. Given the centrality of quantificational issues and the limitation of syllogistic theories in this respect, a reform of logic that would include a systematic treatment of generality was undeniably urgent and crucial

17 Kant says little about the relation between the two notions of universality and necessity, what he does say can at best allow us to infer that he did not distinguish the two notions properly (cf. Lapointe 2000, Chapter 1). While this will sound odd to contemporary readers, the project of testing the career of the two notions in the 17th and 18th centuries would throw light on a neglected topic that is taken for granted.

18 Bolzano primarily deals not with sentences and words but with their objective "meaning" (*Sinn*), i.e. objective ideas and propositions in themselves. Yet in Bolzano's theory, there is (at least) one objective idea for every object, and in this sense, Bolzano assumes that there is in principle a "name" for every object. For this reason, though Bolzano's approach to quantification is substitutional, he is not liable to the reproach following which his interpretation of the universal quantifier cannot account for every state of the world: the resources he assumes he has at his disposal are in principle as rich as necessary to provide a complete description of the domain the theory is about.

in context. Bolzano's analysis of generality – even under a retrospective misnomer – was crucial to articulating the logical syntax that underpins his entire logic. Bolzano not only saw a great deal of significant, articulate structure within concepts and propositions and offered an elaborate theory of such structure; he also had an account of quantifiers and logical connectives – including a non-extensional definition of deducibility, which is meant to account for conditionality – that goes much beyond anything that was available at the time.[19]

Importantly, Bolzano did not fail to distinguish the problem of generality and that of truth by virtue of meaning, and he does have a remarkably sophisticated and systematic account of the latter as well. Bolzano deals with truth by virtue of meaning as part of his theory of "a priori" or "conceptual" knowledge. The most important feature of Bolzano's theory in this respect is the fact that he understood conceptual, i.e. a priori knowledge to be invariably deductive. Strictly speaking, in Bolzano's view, there is no such thing as truth by virtue of meaning outside of a deductive order, a point which was quite unclear before him but which will be echoed in a number of his successors.[20]

Deducibility

Given the last remark, then, it would be right to conclude that Bolzano's account of deduction is key to his epistemology. Bolzano, however, has not one but two at once competing and complementary accounts of logical deduction: "deducibility" (*Ableitbarkeit*) and ground-consequence (*Abfolge*). Just like Bolzanian analyticity, and in spite of its great similitude with Tarskian logical consequence in the resources it deploys to model entailment, Bolzanian deducibility is not meant to define a relation by virtue of which truth is preserved a priori and as a matter of necessity from premises to conclusion. The reason why Bolzanian deducibility does not, on its own, provide such an account is that in Bolzano's theory, the task of providing an account of what it means to (i) preserve truth from premises to conclusion (ii) as a matter of necessity, is precisely "split" between deducibility and ground-consequence. The notion of deducibility presents a semantic account of (i), i.e. an account of truth-preservation that is neither trivial nor careless (see Siebel 1996, 2002, 2003; van Benthem 1985, 2003). On the other hand, Bolzano's attempt to do justice to (ii) results in a definition of objective "ground-consequence" that constitutes the basis of his account of a priori knowledge and mathematical explanation – a theory some have sought to vindicate[21]

19 Cf. Lapointe (2011, 54ff).

20 Cf. Lapointe (2009).

21 Many have translated Bolzano's term 'Abfolge' into English with 'grounding' – including me at various places. Terminological similarities with theories of "grounding" in the

(see Mancosu 1999; Tatzel 2002, 2003; Dubucs and Lapointe 2006). Both notions are immensely original and interesting. While the division of labour between them runs counter to what we assume makes the contemporary, post-Tarskian notion of logical consequence interesting in the first place, namely the fact that it provides a semantic account of what it means for the truth of the conclusion to be necessarily entailed by the truth of the premises, there is nonetheless great value in understanding Bolzano's strategy in context. If nothing else, it is an occasion to re-evaluate our own assumptions concerning the relationship between truth-preservation, entailment and logicality.[22]

Although Bolzano did not say this explicitly, Bolzano's definition of deducibility made it possible for the first time to systematically identify inference forms – propositions in which at least one part is considered to be exchangeable – that preserve truth from premises to conclusion without being constrained by a particular regimentation of grammar. Post-Cartesian syllogistic theory, by contrast, models inference within a language that is both considerably regimented and resource-poor – it does not add much to medieval Aristotelian syllogistic in this respect. Traditional syllogistic definitions of validity generally concur to suppose that there is only a finite number of possible forms of inference, namely 256. This number comes up as a result of various assumptions, including the supposition that:

- inferences ought to be modelled as having exactly two premises and one conclusion and – in the case of classical syllogistic ;
- the form of the latter is invariably *subject-copula*-predicat; and
- there are only four variants of the *subject-copula-predicate* form, namely the forms that are traditionally referred to as a, e, i and o.
- any given inference contains at most three different terms.

While later syllogistic theories were developed to account for inferences that contain hypothetical and disjunctive premises, definitions of validity for such inference forms were premised on the same type of syntactic limitations.

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context of contemporary work on fundamentality and dependence in metaphysics could lead to mistaken assumptions as to the nature of Bolzano's own concern. Bolzanian ground-consequence is not a metaphysical notion and should not to be confused with contemporary mereological and non-mereological accounts of "grounding". The purpose of Bolzanian ground-consequence is to define the order in which propositions are related in axiomatic systems, not to provide a model for relations of dependence between facts, events, properties and so on.

22 Etchemendy (1990), is an example of such questioning. See also Rusnock and Burke (2010) for a critical perspective.

From Bolzano's perspective, syllogistic theory marks out a syntax whose expressive resources are too limited to account for the richness of inferential practices in the sciences and beyond. Bolzano generalises the problem: as he sees it Aristotle's crucial intuition as regards the notion of validity is that it ought not bound to any particular syntax.[23] He writes:

> Aristotle began with such a broad definition of the word syllogism that one is astonished that he could have subsequently restricted the concept of this kind of inference so severely. He writes (in Anal Pr. I, 1) 'syllogism is a discourse in which, certain things being stated, something other than what is stated follows of necessity from their being so.' This definition obviously fits every inference, not only with two, but also with three and more premises, and not only simple inferences but complex ones as well.
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> (1837 §262: 535)

As Bolzano sees it, then, in order to do justice to Aristotle's insight, we need to first take stock of the fact that inference forms come in all sizes and packages. If that is the case, however, syllogistic theory's approach – which consists in identifying which inferential forms among those that are syntactically canonical always preserve truth from premise to conclusion – becomes intractable.

Whether or not Bolzano was aware of it, he has at least one other reason to be dissatisfied with classical syllogistic definitions of validity and it is connected with his views on the nature and role of definition in theorising. Knowing which inferential forms among a predefined set satisfy the intuitive Aristotelian definition of deduction does not, by Bolzano's account, provide a genuine "determination" (*Bestimmung*) of the notion of good or valid deduction, that is, a definition that allows us to properly operationalise the concept within the logical theory. By Bolzano's account, merely knowing what falls in the extension of a concept – in this case the class of all putatively valid syllogistic inferential forms – does not amount to having a proper understanding or determination of that concept. Although Bolzano does not say so himself, determination is a form of conceptual exercise whose end-purpose is to serve the explanatory goal of a theory. Since definitions are partial grounds for conclusion we draw about the corresponding objects, when we define concepts, i.e. when we identify the components and structure of what ultimately counts as a principle or axiom within a deductive order, we

23 This would have been even more obvious to Bolzano in light of the beginning of the second book of the *Prior Analytics*, which is devoted to the relationship between premises and conclusion as regards their truth-value.

are in fact pinning down its role in that theory. I refrain from using the term 'inferential role' in this context in reference to Bolzano's notion of definition so as to avoid anachronism and hasty associations. But given Bolzano's views on a priori deductive knowledge, he has something of the sort in mind: concepts (and by extension their determination) are always part of deductive structures, and we use them to draw conclusions within a theory.[24]

Bolzano's definition of "deducibility" is based on the same conceptual resources as his definition of analyticity. As Bolzano presents it, deducibility is a property defined for sets of propositional forms, namely sets of propositional forms in which some play the role of premises and others the role of conclusion.

I describe deducibility as a property of sets of "propositional forms", where Bolzano describes deducibility as a property of classes or "species" of propositions *with respect to some exchangeable components*. Here again, the relativisation of deducibility to some variable component is what allows it to mark generality, i.e. semantic invariance under substitution. As Bolzano puts it, the set of propositions T is deducible from the set of propositions S with respect to constituent-ideas i, j,… if and only if

i i, j, … can be varied so as to yield at least one true substitution instance of S and T (compatibility)
ii whenever S is true, T is also true

Bolzano's discussion of deducibility is quite detailed – it extends over three dozen sections – and insightful. In particular, his definition yields a series of theorems that contribute to further delineating the notion he has in mind, the most significant of which are the following:

- It is not the case that if T are deducible from S, then S are deductible from T… (asymmetry).[25]
- If T are deducible from S and R are deducible from T, then R are deducible from S (transitivity).
- In addition, assuming that the propositions included in the set S are compatible in the sense specified by the aforementioned condition (i), i.e. they share at least one variable that make them all true at the same time, then S is deducible from S (reflexivity).

24 I explain this in more detail in Lapointe (2009).

25 This is different from anti-symmetry. Deducibility is not anti-symmetric, since, in the case in which T and S are logically analytic, if T are deducible from S, then S are deducible from T.

There are striking similarities between Bolzano's notion of deducibility and post-Tarskian model theoretic accounts of logical consequence. But one obvious difference between Bolzanian deducibility and post-Tarskian logical consequence is the fact that Bolzanian deducibility indiscriminately includes *formally* valid arguments as well as materially valid arguments. In other terms, all arguments in which the conclusion is a logical consequence of the premises in the contemporary, post-Tarskian sense are instances of Bolzanian deducibility. For instance,

Caius is rational

is Bolzano-deducible with respect to 'Caius', 'man' and 'rational' from

Caius is a man
Men are mortal

But Bolzanian deducibility also allows for cases of merely "materially" valid consequence. For instance,

Caius is mortal

is Bolzano-deducible with respect to 'Caius' from

Caius is a man

Assuredly, the distinction between arguments of the former kind and arguments of the latter is an important one, and providing an account of this distinction is at the core of logical theory. Part of the reason for this is that it is assumed that logic distinctively seeks to account for "formal" and "a priori" features of languages and deductive systems. Provided one has a satisfactory account of logical form, in order to know that the conclusion follows from the premises of a formally valid argument one only needs to consider its structure or form, the presumption being that one can do so *a priori*: no other kind of knowledge is required than logical knowledge to recognise logical validity. In the materially valie argument, however, in order to infer from the premise to the conclusion, one needs more than logical knowledge: one also needs to understand the signification of the terms, e.g. 'man' and 'mortal' in the previous example, since in order to know that Caius is mortal one also needs to know, in addition to the fact that Caius is a man, that all men are mortal. But this is not something one knows a priori.

One consequence of the failure to provide an account of the difference between logical and material consequence is that we are left unable to account for the fact that we seem able to extend our knowledge through

putatively pure logical cognitions. If we know, for instance, that all instances of *modus ponens* are logically valid, we can infer from two propositions whose truth we've recognised:

> If Caius is a man, then he is mortal
> Caius is a man

A new proposition:

> Caius is mortal

whose truth we might not have previously known. But Bolzano's account of deducibility does not allow one to extend one's knowledge in this way. In order to know that an argument is an instance of Bolzano-deducibility, one already needs to know for every substitution, both that the premises are true *and* that the conclusion is true. What we have, then, is a systematic way to model the formal features – variant and invariant – of arguments that preserve truth "universally", i.e. for every substitution instance. But the substitutional method does not give us the means to know a priori which arguments preserve truth from premises to conclusion and to leverage this knowledge to acquire new knowledge.

Most arguments that count as logically valid in the post-Tarskian sense would also count as instances of Bolzanian deducibility. Most, but not all: Bolzano's account of the conditions under which an argument counts as an instance of deducibility is in fact both broader – as we've seen it includes instances of merely material consequence – and stricter than what we find in standard model-theoretic practice. In order for a conclusion to be Bolzano-deducible from a given set of premises, there must be at least one substitution that makes both the premises and the conclusion true at once – what Bolzano calls the "compatibility" (*Verträglichkeit*) condition – a requirement that is not reflected in classical conceptions of logical consequence. As a result, Bolzano's programme converges with some contemporary attempts at a definition of non-classical notions of logical consequence in at least two different respects.

First, given the compatibility condition, while a logical truth may follow from any (set of) true premises, *nothing* as opposed to everything is deducible from a contradiction. In other terms, the compatibility condition invalidates the *ex contradictio quod libet* or explosion principle: no substitution of 'p' in ' 'q' is deducible from 'p and non-p' ' can fulfil the compatibility constraint. In other words, no interpretation of 'p' in 'p and non-p' can yield a true variant since there is therefore no idea in a proposition of this form – a logical falsehood – that can be varied so as to make it true. If that is the case, trivially, the premise and the conclusion can never all be true at once. This has at least two further upshots. First, the compatibility constraint invalidates the law of contraposition.

Whenever at least one of the propositions in the antecedent is analytically true – when all its substitution instances are true – we cannot infer from:

T are deducible from S

to

¬S are deducible from ¬T

since ¬S entails a contradiction. For instance, while

Caius is a physician who specialises in the eyes.

is deducible with respect to 'ophthalmologist' from

Every ophthalmologist is an ophthalmologist

and

Caius is an ophthalmologist

It is not the case that

It is not the case that every ophthalmologist is an ophthalmologist
It is not the case that Caius is an ophthalmologist

is deducible with respect to the same component from:

It is not the case that Caius is a physician who specialises in the eyes

Second, the compatibility condition makes Bolzano's logic non-monotonic. Classical logical consequence is monotonic: if a conclusion follows from some premises, then it follows from these premises and any other premise. However, by Bolzano's account, whenever the premise added contains a contradiction or contains information that contradicts the information already at hand, the conclusion no longer follows.

Ground-Consequence

The contemporary reader of Bolzano is likely to expect an account of validity and inference to do more than what Bolzanian deducibility does, namely to identify inference forms such that true substitutional variants of the premises co-occur with true substitutional variants of the conclusion. Bolzanian deducibility does not, however, epitomise our

intuitions as to what is presumably distinctive of knowledge we acquire in "formal" or "conceptual" disciplines such as mathematics and pure physics, and this is a problem Bolzano was well aware of. Bolzano considered mathematical knowledge to be "purely conceptual", and in his eye, this implied that it is both "necessary" and "a priori", two qualities for whose explanation substitutionally defined notions are not equipped. Hence for Bolzano the problem of providing an epistemological foundation to mathematics is ultimately that of defining the logical relations by virtue of which true propositions are ordered within an axiomatic structure. To provide an account of mathematical knowledge and do justice to what makes it a priori and necessary one must therefore, by Bolzano's account, be in a position to offer a theory of deductive theories, that is, a theory of system in which true propositions relate as "objective grounds" to "objective consequence". Bolzano called "ground-consequence" (*Abfolge*) the relation he took to define this relation.

As Bolzano conceived of it, to say that mathematical knowledge is a matter of ground-consequence is to say that mathematical proposition are justified by establishing their status as "objective consequences" of "objective grounds". Ultimately, then, propositions be it in mathematics, in ethics or in logic are a priori and known as such to the extent that they acquire their justification objectively from "primitive truths" (*Grundsätze*), which by their very nature do not themselves require a justification. Objective, and a priori justification is thus a defined only for truth that relate as objective ground-consequence.[26]

One important feature of Bolzano's account of objective justification is that it is meant to do justice to the idea that in order to know that a proposition is true necessarily we need to know *why* it is true. In this sense at least, Bolzanian ground-consequence is meant to have explanatory import. My knowing that 'p' grounds 'q' has explanatory value: q, because p. Bolzanian ground-consequence is designed so as to capture roughly what, according to Bolzano, the truly scientific mind ought to mean when in the conduct of a scientific inquiry, she uses the phrase '…, because….' in response to the question 'why …?' What's special about arithmetic and geometry is the fact that since the propositions they contain are invariably true – Bolzano was a child of his time in assuming that the "fundamental propositions" in deductive orders need to be true – and "purely conceptual", grasping any such proposition as the conclusion of a ground-consequence inference invariably warrants knowledge that is a priori in the sense that it does not involve "extra-logical" resources. The challenge for Bolzano was thus to define the relation of ground-consequence in such a way as to have the resources to model every inferential step along the way.

26 I discuss this and what follows in more details in Lapointe (2011, 72–90).

When Bolzano speaks of ground-consequence, what he has in mind is invariably *immediate* ground-consequence. Mediate ground-consequence thus is a derivative notion. We can define mediate ground-consequence as the transitive closure of the more primitive notion of immediate ground-consequence: p$\phi$is the mediate consequence of ϕ_1, …, ϕ_n, if and only if there is a chain of immediate consequences starting with ϕ_1, …, ϕ_n and ending with p. p$\phi$is the immediate consequence of ϕ_1, …, ϕ_n if there are no intermediate logical steps between ϕ_1, …, ϕ_n and p. Importantly, ground-consequence is anti-reflexive. p cannot be its own ground, whether mediate or immediate. The anti-reflexive character of grounding can be inferred from its asymmetry. If grounding were reflexive, then p could be grounded in itself. But given that if p grounds q, it is not the case that q grounds p, this would imply a contradiction: by substitution p could at once ground itself and not ground itself. Finally, Bolzanian ground-consequence is unique: for every true proposition that is not primitive, there is a unique tree structure that relates it to the primitive truths from which it can be deduced. Deducibility, by contrast, is not unique. Likewise, unlike ground-consequence, which holds only between truths Bolzanian-deducibility can hold between false substitutional instances as well as true ones.

What precedes is sufficient to establish that ground-consequence is not a special case of deducibility in Bolzano's account, and vice versa. On the one hand, not all cases of deducibility are cases of ground-consequence. For instance, while

> (T1) It is warmer in the summer than in the winter

is deducible (with respect to 'summer' and 'winter') from

> (T2) Thermometers, if they function properly, are higher in the summer than in the winter

(T1) is not an objective consequence of (T2) in Bolzano's sense. Bolzano assumes that the objective ground of a proposition needs to play a role in explaining its truth. But the reason for (T2), i.e. the reason why thermometers are higher in the summer, is precisely (T1), i.e. that it is warmer then, so that, in the previous example, the order of ground-consequence is the reverse of the deducibility relation.

Logical consequence as we conceive of it today is meant to combine the main features of Bolzanian deducibility and ground-consequence. Bolzano thought that instances of deducibility that are also instances of ground-consequence were interesting enough to deserve their own name: 'formal ground-consequence'. In an inference that fits both deducibility and ground-consequence, true interpretations of the premises always coincide with true interpretation of the conclusion AND the conclusion

follows *necessarily* from the premises: its truth is objectively justified by the truth of the primitive propositions from which it is derived. But the notion of formal ground-consequence is theoretically inert: it is not an additional resource of Bolzano's logic but a designation for types of inferences that happen to fall under two different rubrics.

Conclusion

When past philosophers put forward doctrines and theories that resonate with those that make for the matter of contemporary philosophical discussion, the temptation is great to ignore the context in which they emerged and to emphasise similarities, possibly more in an effort to validate recent developments than in order to do justice to the development of the discipline. Given the extent of Bolzano's anticipatory spirit, his theories offer an eminently fertile ground for such an approach. Bolzano's contribution to the philosophy of logic and semantics bears on problems prima facie similar to those that later defined early analytical philosophy, such as for instance the pursuit of logical foundations for axiomatic disciplines and deductive theories. The resources he deploys, while they can't be said to have the same elegance and simplicity as those that have shaped the fields after the turn of the 20th century, are complex and sophisticated enough to be contemplated with awe. The same holds for his contribution to the pragmatic and cognitive basis of mathematical practice. Far from ignoring epistemic and practical constraints, Bolzano discusses them in detail. As a mathematician, Bolzano was attuned to philosophical concerns that escaped the attention of most of his contemporaries and many of his successors.

Much can be learned by comparing our theories with those of Bolzano, of course. But this does not exhaust the range of purpose history of philosophy may serve. One important task the historian of philosophy may set for herself is to track the factors that have contributed to doctrinal and theoretical transformation *in context* so as to understand the dynamics of disciplinary development. This requires that past doctrines and theories, whether or not they appear to present similarities with those to which contemporary philosophers subscribe and which they value be contextualised and understood as the result of agreement, criticism, reinterpretation, and/or rejection of other doctrines and theories that shaped the author's effective background.

What's relevant with respect to our discussion of Bolzano is the fact that the context in which his views on logic emerged and developed is prone to being mischaracterised or oversimplified. On the one hand, the idea that Bolzano's logic should be understood as a "development" of Leibniz's has the same sort of import as the claim that Kripke's views were shaped by Russell's and Wittgenstein's. It feeds into a superficially

traditionalist narrative whose purpose is not to do justice to the way in which doctrines and theories effectively transformed as a result of philosophical activity. The purpose of such claims is often broadly genealogical, part of an attempt to position oneself in some putatively worthy lineage. The same holds for the claim that Bolzano's philosophy is "anti-Kantian".[27]

In order to appreciate the connection between Bolzano's views and the post-Kantian context, a rather substantial overhaul of our assumptions concerning the development of logic over the 17th and 18th centuries would be required. For instance, the view that logical theories, including Kant's, were then still "broadly Aristotelian" and that logic was the more or less stagnant repository of doctrines that had not evolved since the middle ages does not survive proper contextualisation. The benefit of doing justice to logical theories in view of their context however is not merely historical. What's interesting in observing that Bolzano's views on conceptual analysis are the result of his rejection of the Leibnizo-Kantian decompositional approach is that it emphasises the fact that philosophical concepts and doctrines are not abstract and immutable but rather historical constructs that cannot be dissociated from the broader theoretical and metaepistemological frame to which they are attached. The same holds for the observation that the doctrines Bolzano associates with "analyticity" and "deducibility", rather than providing an account of truth by virtue of meaning, seek to offer a substitutional account of generality and truth-preservation in a context where such an account was still at best pedestrian.

The lesson here is not trivial. Each time we identify the way in which critical engagement with specific texts and corpora in a given context leads to doctrinal and theoretical transformations, we acquire metacognitive insight into our own practice. Disciplinary history teaches us as much about the past as it teaches about the mechanisms that underpin our own philosophies.

Cited work

Bar-Hillel, Yehoshua (1950) "Bolzano's definition of analytic propositions", *Methodos* 32–55 [republished *Theoria* 16, 1950, 91–117; reprinted in *Aspects of language. Essays and lectures on philosophy of language, linguistic philosophy and methodology of linguistics*, Jerusalem, The Magnes Press 1970, 3–28].

Benthem, Johan van (2003) "Is there still logic in Bolzano's key?" in *Bernard Bolzanos Leistungen in Logik, Mathematik und Physik*, Edgar Morscher (ed.), Sankt Augustin, Academia, 11–34.

27 See Lapointe and Armstrong (2014).

Benthem, Johan van (1985) "The variety of consequence, according to Bolzano," *Studia Logica* 44/4, 389–403.

Bolzano, Bernard (1851) *Paradoxien des Unendlichen*, (reprint) Wissenschaftliche Buchgesellschaft, 1964. [Dr Bernard Bolzano's Paradoxien des Unendlichen herausgegeben aus dem schriftlichem Nachlasse des Verfassers von Dr Fr. Příhonský, Leipzig, Reclam. (Höfler et Hahn (eds.), Leipzig, Meiner, 1920)].

Bolzano, Bernard (1837) *Wissenschaftslehre*, Sulzbach, Seidel.

Bolzano, Bernard (1931) *Reine Zahlenlehre* [*Pure Theory of Numbers*], *Spisy Bernada Bolzana*, Prague, Royal Bohemian Academy of Science.

Bolzano, Bernard (1930) *Funktionenlehre* [*Theory of Function*] Spisy Bernada Bolzana, Prague, Royal Bohemian Academy of Science.

Bolzano, Bernard (1817a) *Rein analytischer Beweis des Lehrsatzes, dass zwischen je zwey Werthe, die ein entgegengesetzes Resultat gewähren, wenigstens eine reelle Wurzel der Gleichung liege*, Prague, Haase. [2nd edition, Leipzig, Engelmann, 1905; Facsimile, Berlin, Mayer & Mueller, 1894].

Bolzano, Bernard (1817b) *Die drey Probleme der Rectification, der Complanation und der Cubirung, ohne Betrachtung des unendlich Kleinen*, Leipzig, Kummer.

Bolzano, Bernard (1816) *Der binomische Lehrsatz und als Folgerung aus ihm der polynomische, und die Reihen, die zur Berechnung der Logarithmen une Exponentialgrösse dienen*, Prague, Enders.

Bolzano, Bernard (1812) *Etwas aus der Logik*, Bolzano *Gesamtausgabe*, Gesamtausgabe, Stuttgart, Frohmann-Holzboog, vol. 2 B 5, 140ff.

Bolzano, Bernard (1810) *Beyträge zu einer begründeteren Darstellung der Mathematik*, Prague, Widtmann. [Darmstadt, Wissenschaftliche Buchgesellschaft, 1974].

Coffa, Alberto (1991) *The semantic tradition from Kant to Carnap*, Cambridge, Cambridge University Press.

Dubucs, Jacques and Lapointe, Sandra (2006) "On Bolzano's alleged explicativism," *Synthese* 150/2, 229–46.

Etchemendy, John (2008) "Reflections on consequence," in Patterson, Douglas (ed.), *New Essays on Tarski and Philosophy*, Oxford University Press, 263–99.

Etchemendy, John (1990) *The Concept of Logical Consequence*, Cambridge, MA, Harvard University Press.

Etchemendy, John (1988) "Models, semantics, and logical truth," *Linguistics and Philosophy* 11, 91–106.

Freudenthal, H. (1971) "Did Cauchy plagiarize Bolzano?" *Archives for the History of Exact Sciences* 7, 375–92.

Grattan-Guiness, Ivan (1970) "Bolzano, Cauchy and the "new analysis" of the early nineteenth century," *Archives for the History of Exact Sciences* 6, 372–400.

Künne, Wolfgang (2006) "Analyticity and logical truth: from Bolzano to Quine", in *The Austrian Contribution to Analytic Philosophy*, Mark Textor (ed.), New York, Routledge, 184–249.

Lapointe, Sandra (2014a) "Bernard Bolzano," in *Oxford Bibliographies in Philosophy*, Duncan Pritchard (ed.), New York, Oxford University Press.

Lapointe, Sandra (2014b) "Bolzano, Quine and logical truth," in *A Companion to Quine*, Gilbert Harman and Ernest Lepore (eds.), London, Wiley-Blackwell, 296–312.
Lapointe, Sandra (2014c) "Bolzano and the analytical tradition," *Philosophy Compass* 9/2, 96–111.
Lapointe, Sandra (2011) *Bolzano's Theoretical Philosophy*, Houndmills, Palgrave.
Lapointe, Sandra (2009) "A priori knowledge and Bolzano's Classical Model of Science," *Synthese* 174/2, 263–81.
Lapointe, Sandra (2008) *Qu'est-ce que l'analyse?* Paris, Vrin.
Lapointe, Sandra (2000) "Analyticité, universalité et quantification chez Bolzano," *Les Études Philosophiques* 2000/4, 455–70.
Lapointe, Sandra and Chloe Armstrong (2014) "Bolzano, Kant and Leibniz," in *New Anti-Kant*, Sandra Lapointe and Clinton Tolled (eds.), Houndmills, Palgrave, 272–90.
Lapointe, Sandra and Chris Pincock (eds.) (2017) *Innovations in the History of Analytical Philosophy*, Houndmills, Palgrave.
Lapointe, Sandra and Clinton Tolley (eds.) (2014) *New Anti-Kant*, Houndmills, Palgrave.
Morscher, Edgar (2003) "La définition bolzanienne de l'analyticité logique," *Philosophiques* 30/1, 149–69.
Neeman, Ursula (1970) "Analytic and synthetic propositions in Kant and Bolzano," *Ratio* 12, 1–25.
Příhonský, František (1850) *Neuer Anti-Kant: oder Prüfung der Kritik der reinen Vernunft nach den in Bolzanos Wissenschaftslehre niedergelegten Begriffen*, Bautzen, Hiecke.
Proust, Joëlle (1989) *Questions of Form. Logic and the Analytic Proposition from Kant to Carnap*, Minneapolis, University of Minnesota Press.
Proust, Joëlle (1981) "Bolzano's analytic revisited," *Monist* 64/2, 214–30.
Rusnock, Paul (2000) *Bolzano's Philosophy and the Emergence of Modern Mathematics*, Amsterdam, Rodopi.
Rusnock, Paul and Mark Burke (2010) "Etchemendy and Bolzano on logical consequence," *History and Philosophy of Logic* 31/3, 3–29.
Russ, Steve (2004) *The Mathematical Works of Bernard Bolzano*, Oxford, Oxford University Press.
Russell, Bertrand (1903) *The Principles of Mathematics*, Cambridge, Cambridge University Press.
Sebestik, Jan (1992) *Logique et mathématique chez Bernard Bolzano*, Paris, Vrin.
Schubring, Gert (1993) "Bernard Bolzano. Not as unknown to his contemporaries as is commonly believed?" *Historia Mathematica* 20, 43–53.
Siebel, Mark (2003) "La notion bolzanienne de déductibilité," *Philosophiques* 30/1, 171–89.
Siebel, Mark (2002) "Bolzano's concept of consequence," *Monist* 85, 580–99.
Siebel, Mark (1996) *Der Begriff der Ableitbarkeit bei Bolzano*, Sankt Augustin, Academia Verlag.
Simons, Peter (1997) "Bolzano on Collections," *Grazer Philosophische Studien* 53, 87–108.

Tatzel, Armin (2003) "La théorie bolzanienne du fondement et de la consequence," *Philosophiques* 30/1, 191–217.
Tatzel, Armin (2002) "Bolzano's theory of ground and consequence," *Notre Dame Journal of Formal Logic* 43, 1–25.
Textor, Mark, (2001) "Logically analytic propositions 'a posteriori'?" *History of Philosophy Quarterly* 18, 91–113.
Textor, Mark (2000) "Bolzano et Husserl sur l'analyticité," *Les Études Philosophiques* 2000/4 435–54.