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THE INFLUENCE OF HERACLITUS ON MODERN MATHEMATICS

Among the pre-Socratic philosophers we know that Thales, Pythagoras, Zeno and Democritus were involved in mathematics in one way or another. In fact, Pythagoras coined the very word "mathematics." Heraclitus does not appear to have occupied himself with mathematical questions; so how can he be said to have influenced modern mathematics? The answer to this question will take us on a small detour.

What has survived of Heraclitus' thought is contained in a number of pithy statements. I shall only mention two:

> "You cannot step into the same river twice",

and

> "Men do not know how what is at variance agrees with itself; it is an attunement of opposite tensions, as in the bow or in the lyre".

Of all the Greek philosophers, Heraclitus has perhaps been the most influential in the twentieth century. When Mao Tse-tung refers to the "law of contradiction in things" as "the unity of opposites," he is quoting indirectly from Heraclitus. How did this come about?

The nineteenth century German philosopher Hegel developed the ideas of Heraclitus into his theory of dialectics, viewing the universe as a sort of divine debating society in which opposing ideas are forever struggling to produce a synthesis. Marx and Engels took up Hegel's dialectics and gave it a materialistic basis. Marx himself had studied Democritus for his doctoral dissertation, comparing him with Epicurus, and mentioned Heraclitus only marginally. However, both Engels and Lenin acknowledged that dialectic materialism originated in the thought of Heraclitus, as has been eloquently argued and amply documanted by Kessidi.

At first it seems reasonable to expect to find the dialectical process in history, economics, biology and perhaps other sciences. But mathematics is supposed to deal with eternal and unchanging objects such as the number 5. Indeed, when later dialecticians rephrased the Heraclitian doctrine of continual change by saying that "a is not always equal to a,"

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this appears to contradict a fundamental axiom of mathematics and is, in fact, not taken seriously by practicing mathematicians in communist countries.

However, Heraclitus' doctrine of the unity of opposites has occasionally been illustrated by examples from mathematics. Lenin asserted somewhere – and I quote from memory – that subtraction is the antithesis of addition, with arithmetic as the synthesis, and that integration is the antithesis of differentiation, with calculus as the synthesis. I did find a relevant quotation by Mao Tse-tung:

> The classification of scientific studies is based precisely upon the particular contradictions inherent in their objects .... For example, positive numbers and negative numbers in mathematics, action and reaction is mechanics,....

Other mathematical illustrations were not so fortunate. I recall one Marxist illustrating the change of quantity into quality by pointing out that the number 2 is even, instead of being twice as odd as the number 1.

I spent my sabbatical 1965–6 in Zürich, where I had many conversations with the young American mathematician, Bill Lawvere. We kicked around the idea that an interesting illustration of dialectic contradictions could be found in the "adjoint functors" of modern mathematics, which had recently been popularized by Peter Freyd in his book on Abelian categories. While the basic idea behind adjoint functors is fully contained in the "universal mapping problem" of Nicolas Bourbaki, their more symmetric presentation by Kan in 1958 relied on the notion of a category invented by Eilenberg and MacLane.

To me the connection between adjoint functors and dialectical contradictions was not much more than an amusing analogy, but to Bill Lawvere it was of profound significance and pervaded the whole structure and development of mathematics. Having been influenced by General Semantics in his youth, he was now becoming attracted to Marxism-Leninism and came to see the dialectical process everywhere in mathematics. When asked point blank years later he would not admit that adjoint functors were the only manifestations of contradictions in mathematics. However, in this essay, I shall confine myself mainly to this narrower view.

In his 1970 address on 'Quantifiers and Sheaves' to the International Congress of Mathematicians in Nice, Lawvere said the following:

> The unity of opposites... is essentially that between logic and geometry, ... We first sum up the principal contradictions of the Grothendieck-Giraud-Verdier theory of

topos in terms of four or five adjoint functors... When the main contradictions of a theory have been found, the scientific procedure is to summarize them in slogans which one then constantly uses as an ideological weapon for the further development and transformation of the thing. Doing this for "set theory" requires taking into account that the main pairs of opposing tendencies in mathematics take the form of adjoint functors....

Lawvere's ideas influenced a host of mathematicians working in the area of category and topos theory. Some of his students also adopted his ideological orientation. To show how even the philosophical vocabulary caught on, let me only quote a sentence in a review by Myles Tierney of an article by Anders Kock, which appeared in *Mathematical Reviews* in 1972: "No doubt, however, the reviewer has again failed in the struggle to grasp firmly the essential nature of the contradiction between "abstract" and "general"."

Again, in a recent paper, distributed in 1977, Henry Crapo writes: "The essential notion is that of a unity (of opposites) in a binary relation...." However, his notion of "unity of opposites" is quite different from the interpretations that will be discussed in the present essay. Let these examples suffice and let us turn to explain what the mathematical illustrations are all about.

I shall begin by explaining the notion of adjoint functors in a setting that does not presuppose an understanding of categories. Certain categories have always been known, namely preordered sets. A *preordered set* $\mathscr{A} = (A, \leq)$ is a set A together with a reflexive and transitive relation $\leq$ on A, that is, a binary relation satisfying

$$a \leq a$$

and

$$(a \leq a' \wedge a' \leq a'') \Rightarrow a \leq a''$$

for all elements a, a', a'' of A. We shall not assume the usual antisymmetry rule, although many of our examples do satisfy this law. We may write $a \cong a'$ to mean that $a \leq a'$ and $a' \leq a$.

Given two preordered sets $\mathscr{A} = (A, \leq)$ and $\mathscr{B} = (B, \leq)$, by a *functor* $F: \mathscr{A} \to \mathscr{B}$ we understand an order preserving function from A to B so that

$$a \leq a' \Rightarrow F(a) \leq F(a')$$

for all $a, a' \in A$.

Given two functors $F\colon \mathscr{A} \to \mathscr{B}$ and $G\colon \mathscr{B} \to \mathscr{A}$, we say G is *right adjoint* to F if

$$F(a) \leq b \Leftrightarrow a \leq G(b)$$

for all $a \in A$ and $b \in B$. This notion is not new; in fact, such a pair (F, G) used to be called a *Galois correspondence.*

Let us draw some obvious conclusions from this setup. Consider the functors $GF\colon \mathscr{A} \to \mathscr{A}$ and $FG\colon \mathscr{B} \to \mathscr{B}$. Of these the former is a *closure operation*, that is,

$$a \leq GF(a)$$
$$GFGF(a) \leq GF(a),$$
$$a \leq a' \Rightarrow GF(a) \leq GF(a')$$

for all $a, a' \in A$. The latter satisfies the dual properties:

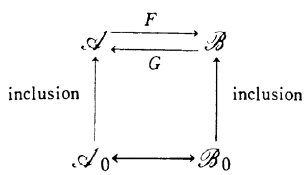
$$FG(b) \leq b,$$
$$FG(b) \leq FGFG(b),$$
$$b \leq b' \Rightarrow FG(b) \leq FG(b')$$

for all $b, b' \in B$; we might call it an *interior operation.* Moreover, the functions F and G set up a biunique (up to $\cong$) correspondence between the *closed* elements of A, that is, the elements of A for which $GF(a) \leq a$, and the *open* elements of B, that is, the elements b of B for which $b \leq FG(b)$. Thus, the closed elements of A form a preordered subset $\mathscr{A}_0 = (A_0, \leq)$ of $\mathscr{A}$ and the open elements of B form a preordered subset $\mathscr{B}_0 = (B_0, \leq)$ of $\mathscr{B}$, and $\mathscr{A}_0$ is equivalent with $\mathscr{B}_0$ in the sense that

$$GF(a) \cong a,\ FG(b) \cong b$$

for all $a \in A_0$ and $b \in B_0$.

The following picture should throw light on the situation:



The same picture will apply to categories in place of preordered sets later in this essay.

There are at least two ways in which we can interpret adjoint functors as exemplifying the dialectical process.

Interpretation I. We may think of the pair (F, G) as establishing a contradiction between $\mathscr{A}$ and $\mathscr{B}$. The equivalence between $\mathscr{A}_0$ and $\mathscr{B}_0$ is then the unity of opposites. (This interpretation becomes even more convincing if we replace $\mathscr{B}$ by $\mathscr{B}^{\mathrm{op}} = (B, \geq)$.)

Interpretation II. We may think of F as the thesis; its right adjoint G then becomes the antithesis. In this interpretation we should perhaps think of the adjunction itself as the synthesis. (This appears to be Lawvere's favorite interpretation.)

We shall illustrate these interpretations with a number of examples, dealing first with preordered sets, later with other categories.

EXAMPLE 1. Take $\mathscr{A}$ and $\mathscr{B}$ to be the ordered set of nonnegative integers, both with the natural order. Let

$$F(a) = \text{the } a\text{-th prime number when } a > 0,\ F(0) = 0;$$

$$G(b) = \text{the number of primes} \leq b.$$

Then $GF(a) \leq a$ holds for all positive integers a, while $b \leq FG(b)$ holds then and only then when b is a prime number. The "unity of opposites" here describes the biunique correspondence between positive integers and prime numbers.

Many examples arise from a binary relation $R \subseteq X \times Y$ between two sets X and Y. Take $\mathscr{A} = (\mathscr{P}(X), \subseteq)$ as the ordered set of subsets of X, ordered by inclusion, and take $\mathscr{B} = (\mathscr{P}(Y), \supseteq)$ as the ordered set of subsets of Y, ordered by the converse of inclusion. Put

$$F(a) = \{y \in Y \mid \forall_{x \in a}\, (x, y) \in R\},$$

$$G(b) = \{x \in X \mid \forall_{y \in b}\, (x, y) \in R\}$$

for any $a \subseteq X$ and $b \subseteq Y$. This situation is called a *polarity*. It gives rise to a lattice isomorphism between the lattice $\mathscr{A}_0$ of "closed" subsets of X and the lattice $\mathscr{B}_0$ of "closed" subsets of Y. (Note that open elements of $\mathscr{B}$ are closed subsets of Y, because $b \leq b'$ here means $b \supseteq b'$.)

Examples of polarities abound in mathematics; let me only mention two.

EXAMPLE 2. Take X to be the set of points of a plane, Y the set of half-planes, and write $(x, y) \in R$ for $x \in y$. Then, for any set a of points,

$UF(a)$ is the intersection of all half-planes containing a, in other words, the *convex hull* of a. Hence $GF(a) \leq a$ if and only if a is convex. For any set b of half-planes, $FG(b)$ is the set of all half-planes which contain the intersection of all planes in the set b. The "unity of opposites" here asserts that there are two equivalent ways of describing a convex set: by the points on it or by the half-planes containing it.

Our next example is a bit more technical and should be skipped by readers not familiar with the theory of rings.

EXAMPLE 3. Given a commutative ring C, take X as the set of elements of C, Y as the set of prime ideals of C. Again we write $(x, y) \in R$ for $x \in y$. Then $GF(a)$ is the intersection of all prime ideals containing the set a. In commutative algebra this is shown to be the set of all those $x \in X$ for which $x^n \in a$, for some natural number n, and is called the *radical* of a. On the other hand, the closure operation FG on the set of all subsets of Y makes Y into a topological space, in fact a compact space, called the *spectrum* of C. The "unity of opposites" here describes a biunique correspondence between "radical" ideals of the ring (ideals which coincide with their radical) and closed subspaces of its spectrum.

So far we have followed the first interpretation; but, according to the second interpretation, we could say that the above resolves a contradiction between a commutative ring and its spectrum.

Another example of a Galois correspondence is taken from elementary logic.

EXAMPLE 4. Take $\mathscr{A}$ and $\mathscr{B}$ both as the preordered sets of formulas of the propositional calculus, where $\leq$ is understood to mean entailment and is usually written $\vdash$. Given any formula p, we define $F\colon \mathscr{A} \to \mathscr{B}$ and $G\colon \mathscr{B} \to \mathscr{A}$ by

$$F(a) = p \wedge a,\ G(b) = p \Rightarrow b.$$

They form a pair of adjoint functors, because $a \vdash p \Rightarrow b$ if and only if $p \wedge a \vdash b$. It is easily seen that $GF(a) \leq a$ if and only if $p \Rightarrow a \vdash a$ and $FG(b) \leq b$ if and only if $b \vdash p$. (Here $p \Rightarrow a \vdash a$ means the same as $\vdash p \vee a$ classically, but not intuitionistically.) According to the first interpretation, we would thus be interested in the correspondence between formulas which entail p and formulas a for which $\vdash (p \Rightarrow a) \Rightarrow a$, but I don't know the significance of this correspondence. In fact, I believe that

Lawvere, who first studied this pair of adjoint functors, would see it as exemplifying the dialectical process according to the second interpretation as follows.

Proceeding from the functor F to its right adjoint G, we progress from the basic notion of conjunction to the more sophisticated antithetical notion of implication, thus introducing a new idea, albeit implicit in the old one. The synthesis of these two ideas is the system of logic embodying conjunction and implication, essentially the positive intuitionist propositional calculus, also known as Heyting algebra. (I am here neglecting the symbols T, $\perp$ and $\vee$.)

From now on we shall assume that $\mathscr{A}$ and $\mathscr{B}$ are categories and that (F, G) is a pair of adjoint functors between them. As in the special case of preordered sets, we obtain an equivalence between full subcategories $\mathscr{A}_0$ and $\mathscr{B}_0$, where $\mathscr{A}_0$ consists of all objects A of $\mathscr{A}$ for which the canonical arrow $A \to GF(A)$ is an isomorphism, while $\mathscr{B}_0$ consists of all objects B of $\mathscr{B}$ for which the canonical arrow $FG(B) \to B$ is an isomorphism. The meaning of all these terms will be explained in the appendix. Again we have two dialectical interpretations, as in the special case of preordered sets.

EXAMPLE 5. $\mathscr{A}$ is the category of Abelian groups and $\mathscr{B}$ is the opposite of the category of topological Abelian groups. Let K be the compact group of the reals modulo the integers. For any abstract Abelian group A, $F(A)$ is the group of all homomorphisms of A into K, with the topology induced by K. For any topological Abelian group B, $G(B)$ is the group of all continuous homomorphisms of B into K. Then $\mathscr{A}_0$ is $\mathscr{A}$, while $\mathscr{B}_0$ is the opposite of the category of compact Abelian groups. According to our first interpretation, we have here a contradiction between abstract and topological Abelian groups, the "unity of opposites" being the well-known Pontrjagin duality between abstract and compact Abelian groups.

EXAMPLE 6. $\mathscr{A}$ is the category of rings and $\mathscr{B}$ is the opposite of the category of topological spaces. For any ring A, $F(A)$ is the topological space of homomorphisms of A into the ring of integers modulo 2. For any topological space B, $G(B)$ is the ring of continuous functions of B into $\{0, 1\}$, with the ring structure inherited by that of $\{0, 1\}$. Then $\mathscr{A}_0$ is the category of Boolean rings and $\mathscr{B}_0$ is the opposite of the category of zero-dimensional compact Hansdorff spaces. According to our first interpreta-

tion, we have here a contradiction between algebra (the category of rings) and topology (the opposite of the category of topological spaces), the "unity of opposites" being the well-known Stone duality.

EXAMPLE 7. In a manner similar to Example 6, we study the contradiction between (the category of) Banach algebras and (the opposite of the category of) topological spaces and obtain the well-known Gelfand duality as the "unity of opposites."

The details of Examples 6 and 7 have been worked out in collaboration with Basil Rattray. There are numerous other examples of the same nature, let me just mention one more.

EXAMPLE 8. $\mathscr{A}$ is the category of presheaves on a topological space X. $\mathscr{B}$ is the category of spaces over X. There is a pair of adjoint functors between these categories which induces an equivalence between sheaves and local homomorphisms.

In Examples 5 to 8, the first interpretation was useful. In the following example, which is analogous to Example 4, the second interpretation is more interesting.

EXAMPLE 9. $\mathscr{A}$ and $\mathscr{B}$ are both the category of sets (or any Cartesian closed category). Given any object C, we consider the functors F and G determined by $F(A) = C \times A$ and $G(B) = B^C$. These are a pair of adjoint functors in view of the correspondence between arrows $C \times A \to B$ and $A \to B^C$. Here exponentiation should be viewed as the antithesis of Cartesian multiplication, and the struggle between the two produces essentially a Cartesian closed category. (We have ignored the terminal object.)

Up to this point we have considered mathematical illustrations of the Heraclitian principle of the unity of opposites. His doctrine of continual change has been less influential, but it too has recently entered the thinking of Bill Lawvere. In his talk at the Bristol Logic Colloquium of 1973, he advocated the view that "the theory of topoi ..... is a basis for the study of continuously variable structures, as classical set theory is a basis for a study of constant structures." The idea is that a sheaf of rings, for example, may be looked upon as a variable ring, or perhaps as a ring of variable quantities. He quotes Engels as saying that "the advance from constant quantities to variable quantities is a mathematical expression of the advance from metaphysics to dialectics". The surprising thing is that

results about constant sets when proved intuitionistically carry over to variable sets, that is to say, to sheaves.

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APPENDIX

A *category* consists of two classes, the class of *arrows* and the class of *objects*, together with two mappings between them, called *source* and *target*. We write $f: A \to B$ as shorthand for: source$(f) = A$ and target$(f) = B$. Moreover, there is given for each object A an arrow $1_A: A \to A$, called the *identity*, and a *composition* of arrows thus:

$$\frac{f: A \to B \quad g: B \to C}{gf: A \to C}.$$

Finally, the following equations are postulated:

$$1_B f = f, f 1_A = f, (hg)f = h(gf),$$

for all $f: A \to B$, $g: B \to C$ and $h: C \to D$.

A *functor* $F: \mathscr{A} \to \mathscr{B}$ between categories sends the objects of $\mathscr{A}$ to objects of $\mathscr{B}$ and the arrows of $\mathscr{A}$ to arrows of $\mathscr{B}$ so that

$$\frac{f: A \to B}{F(f): F(A) \to F(B)}$$

and

$$F(1_A) = 1_{F(A)}, \quad F(gf) = F(g)F(f).$$

A *natural transformation* $t: F \to G$ between two functors from $\mathscr{A}$ to $\mathscr{B}$ sends the objects of $\mathscr{A}$ to arrows of $\mathscr{B}$ so that $t(A) : F(A) \to G(A)$, for all objects A of $\mathscr{A}$, and so that

$$G(f)t(A) = t(A')F(f),$$

for all arrows $f: A \to A'$ in $\mathscr{A}$, as is illustrated by the diagram:

$$\begin{array}{ccc} F(A) & \xrightarrow{t(A)} & G(A) \\ {\scriptstyle F(f)}\uparrow & & \downarrow{\scriptstyle G(f)} \\ F(A') & \xrightarrow[t(A')]{} & G(A') \end{array}$$

We say that $G\colon \mathscr{B} \to \mathscr{A}$ is *right adjoint* to $F\colon \mathscr{A} \to \mathscr{B}$ if there are given natural transformations $\eta\colon 1_{\mathscr{A}} \to GF$ and $\varepsilon\colon FG \to 1_{\mathscr{B}}$, where $1_{\mathscr{A}}$ and $1_{\mathscr{B}}$ are the identity functors on $\mathscr{A}$ and $\mathscr{B}$ respectively, such that, for all objects A of $\mathscr{A}$ and B of $\mathscr{B}$,

$$G(\varepsilon(B))\eta(G(B)) = 1_{G(B)}, \varepsilon(F(A))F(\eta(A)) = 1_{F(A)},$$

that is, the following composite natural transformations are both identities:

$$G \xrightarrow{\eta G} GFG \xrightarrow{G\varepsilon} G, \quad F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon F} F.$$

We say that the adjoint pair (F, G) determines an *equivalence* between $\mathscr{A}_0$ and $\mathscr{B}_0$ if $\eta(A)$ and $\varepsilon(B)$ are isomorphisms (that is, have inverses) for all A in $\mathscr{A}_0$ and all B in $\mathscr{B}_0$.

The following categories are mentioned in the text.

When $\mathscr{A} = (A, \leq)$ is a preordered set, we may regard it as a category whose objects are the elements of A and whose arrows $a \to b$ are the pairs (a, b) for which $a \leq b$. In particular, a topological space gives rise to the preordered set of open subsets of X, which is then viewed as a category.

Other categories mentioned in the text are displayed in this table:

Category of	Objects	Arrows
sets	sets	functions
topological spaces	topological spaces	continuous functions
Abelian groups	Abelian groups	homomorphisms
topological Abelian groups	topological Abelian groups	continuous homomorphisms
rings	rings	homomorphisms
Banach algebras	Banach algebras	norm decreasing homomorphisms
presheaves on X	functors $X^{\mathrm{op}} \to$ Sets	natural transformations
spaces over X	continuous functions $f\colon Y \to Z$	continuous functions $h\colon Y \to Y'$ so that $f'h = f$

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