FROM λ-CALCULUS TO CARTESIAN CLOSED CATEGORIES

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Dedicated to H.B. Curry on the occasion of his 80th Birthday.

Haskell Curry may be surprised to hear that he has spent a lifetime doing fundamental work in category theory. The purpose of this account is to convince categorists that Cartesian closed categories (Eilenberg and Kelly, 1966) have been anticipated by logicians (Curry, 1930) by many years and, conversely, to persuade logicians that combinatory logic may benefit from being phrased in categorical language.

I have attempted to tell this story twice before (1972, 1974), but am not entirely satisfied with these earlier accounts. The present exposition is essentially my unscheduled talk at the 1977 Durham Symposium on applications of sheaf theory to logic, algebra and analysis.

I regret that limitations of space do not permit a discussion of illative combinatory logic (Curry and Feys, 1958) or combinatory type theory (Church, 1940) and applications thereof to the construction of free toposes.

Let me confess at once that I am not a historical scholar

and that I have taken some liberties with the original material. Thus, I have taken the opportunity to present the early discoveries of combinatory logic in the language of universal algebra.

Our story begins in 1924, when Schönfinkel studied what would now be called an albegra $A = (|A|, {}^{f}, I, S, K)$ consisting of a set $|A|$ equipped with a binary operation ^{f} and constants I, S and K. These were to satisfy the following identities:

(1) $I{}^{f}a = a$,

(2) $(K{}^{f}a){}^{f}b = a$,

(3) $((S{}^{f}f){}^{f}g){}^{f}c = (f{}^{f}c){}^{f}(g{}^{f}c)$,

for all elements a, b, c, f and g of $|A|$. Actually, Schönfinkel did not employ the language of universal algebra, and he defined I in terms of K and S, but of this we shall speak later. His main result would now be stated as follows.

PROPOSITION 1. Every polynomial $\varphi(x)$ over a Schönfinkel algebra A can be written in the form $f{}^{f}x$, where $f \in |A|$.

Polynomials are of course formed as words in an indeterminate x and are subject to the same three identities. More precisely, equality $\underset{x}{=}$ between polynomials is the smallest equivalence relation $\equiv$ between words in x which has the substitution property

$$(0_x) \qquad \frac{\varphi(x) \equiv \psi(x) \quad \alpha(x) \equiv \beta(x)}{\varphi(x){}^{f}\alpha(x) \equiv \psi(x){}^{f}\beta(x)}$$

and which satisfies

$$(1_x) \qquad I{}^{f}\alpha(x) \equiv \alpha(x),$$

$$(2_x) \qquad (K{}^{f}\alpha(x)){}^{f}\beta(x) \equiv \alpha(x),$$

$$(3_x) \qquad ((S{}^{f}\varphi(x)){}^{f}\psi(x)){}^{f}\gamma(x) \equiv (\varphi(x){}^{f}\gamma(x)){}^{f}(\psi(x){}^{f}\gamma(x)).$$

Alternatively, one may regard the polynomial $\varphi(x)$ as an element of the Schönfinkel algebra $A[x]$, which comes equipped with an element x and with a homomorphism $h_x: A \to A[x]$ with the usual universal property: for every algebra B, every homomorphism $f: A \to B$ and every element $b \in |B|$, there exists a unique homomorphism $f': A[x] \to B$ such that $f'h_x = f$ and $f'(x) = b$. In the special case when $B = A$ and f is the identity homomorphism $A \to A$, f' is the substitution homomorphism which allows us to replace x in any polynomial by b. Similarly, when $B = A[x]$ and $f = h_x$, f' allows us to replace x by a polynomial $\beta(x)$.

The proof of Proposition 1 is remarkably simple. It proceeds by induction on the length of the word $\varphi(x)$, which must be either x, or a constant or of the form $\psi(x){}^{f}\chi(x)$ not constant, in which last case we may assume by inductional assumption that $\psi(x) \underset{x}{=} g{}^{f}x$ and $\chi(x) \underset{x}{=} h{}^{f}x$. In the three cases we have

$$\varphi(x) \underset{x}{=} I{}^{f}x,$$

$$\varphi(x) \underset{x}{=} (K{}^{f}a){}^{f}x,$$

$$\varphi(x) \underset{x}{=} (g{}^{f}x){}^{f}(h{}^{f}x) \underset{x}{=} ((S{}^{f}g){}^{f}h){}^{f}x,$$

respectively.

This proof also yields an algorithm for converting every polynomial into the form $f\,x$. For example, one easily calculates

$$x\,x \underset{x}{=} ((S\,I)\,I)\,x.$$

In 1930, Schönfinkel's result was rediscovered by Curry. However, Curry was interested in imposing an additional requirement:

(4) If $f\,x \underset{x}{=} g\,x$ in $A[x]$ then $f = g$ in A.

For example, from

$$((S\,K)\,I)\,x \underset{x}{=} (K\,x)\,(I\,x) \underset{x}{=} x \underset{x}{=} I\,x$$

one could use (4) to obtain $(S\,K)\,I = I$, which equation cannot be derived from (1) to (3) alone. In the same way, one could deduce that $(S\,K)\,K = I$. In fact, Schönfinkel originally defined I by this equation. For reasons that will become clear later, we shall not follow him in this application of Occam's razor.

While (4) does not have the form of an identity, by which I mean an equation prefixed by universal quantifiers, Curry discovered that it could be replaced by a finite number of equations. These were later simplified, and Rosenbloom lists four, one of which reads

$$(S\,((S\,(K\,S))\,K))\,(K\,I) = I.$$

The reader will forgive us for not copying out the other three!

By a Curry algebra we shall mean a Schönfinkel algebra subject to certain additional equations or identities whose conjunction is equivalent to (4). Curry's result may then be formulated thus:

PROPOSITION 2. Over a Curry algebra A every polynomial $\varphi(x)$ may be uniquely written in the form $f\,x$ with $f \in |A|$.

This property of Curry algebras is called functional completeness. It is an immediate consequence of the equivalence of (4) with the conjunction of the above mentioned four equations. For a proof we could refer the reader to the book by Rosenbloom. However, we prefer to give another proof, in the course of which we shall discover five equations whose conjunction is equivalent to (4).

Proof. Let $\lambda_x \varphi(x)$ be defined by induction on the length of the word $\varphi(x)$ thus:

(i) $\lambda_x x = I$;

(ii) $\lambda_x a = K\,a$, when a is a constant;

(iii) $\lambda_x(\psi(x)\,\chi(x)) = (S\,\lambda_x \psi(x))\,\lambda_x \chi(x)$, when $\psi(x)$ and $\chi(x)$ are not both constant.

We shall prove below that the restriction on (iii) is not necessary. In view of the above proof of Proposition 1, we have

$$\varphi(x) \underset{x}{=} \lambda_x \varphi(x)\,x,$$

so the existence of f with $\varphi(x) \underset{x}{=} f\,x$ is assured. It remains to prove its uniqueness. First, we claim that

(*) $\varphi(x) \underset{x}{=} \psi(x)$ implies $\lambda_x \varphi(x) = \lambda_x \psi(x)$,

so that $\lambda_x \varphi(x)$ depends not just on the word $\varphi(x)$ but on the polynomial $\varphi(x)$, that is, the word modulo the equivalence

relation $\underset{x}{=}$.

To prove (*), we write $\varphi(x) \equiv \psi(x)$ for $\lambda_x\varphi(x) = \lambda_x\psi(x)$. It is easily seen that $\equiv$ is an equivalence relation between words which satisfies (0_x), that is, a congruence relation. If we make sure that $\equiv$ also satisfies (1_x) to (3_x), it will follow that $\equiv$ contains $\underset{x}{=}$, and this is what (*) asserts.

To say that $\equiv$ satisfies (1_x) means that

$$\lambda_x(I\,{}^f\alpha(x)) = \lambda_x\alpha(x).$$

Writing a for $\lambda_x\alpha(x)$, we may rewrite this, in view of the unrestricted (iii), as

$$(S\,{}^f(K\,{}^fI))\,{}^fa = a,$$

an easy consequence of

$$S\,{}^f(K\,{}^fI) = I, \tag{4.1}$$

which itself has already been derived from (4).

In the same manner we may obtain consequences (4.2) and (4.3) of (4) which imply (2_x) and (3_x) respectively. We shall not bother to spell them out.

We now turn to the uniqueness of f in Proposition 2. Suppose also $g\,{}^fx \underset{x}{=} \varphi(x)$, we claim that $g = \lambda_x\varphi(x)$. Now, by (*), $\lambda_x(g\,{}^fx) = \lambda_x\varphi(x)$, so it suffices to prove that $\lambda_x(g\,{}^fx) = g$.

Now a small calculation shows that $\lambda_x(g\,{}^fx) = (S\,{}^f(K\,{}^fg))\,{}^fI$, so we require the identity

$$(S\,{}^f(K\,{}^fg))\,{}^fI = g$$

for all g, which is an easy consequence of (4).

This identity must remain valid if we adjoin an indeterminate y to the algebra, so we have

$$(S\,{}^f(K\,{}^fy))\,{}^fI \underset{y}{=} y.$$

We may therefore replace the required identity by the equation

$$\lambda_y((S\,{}^f(K\,{}^fy))\,{}^fI = I. \tag{4.4}$$

Of course the λ may be eliminated from this using (i) to (iii).

It remains to show the validity of (iii) when $\psi(x)\,{}^f\chi(x)$ is constant, say $b\,{}^fc$. So we want to show that

$$\lambda_x(b\,{}^fc) = (S\,{}^f(K\,{}^fb))\,{}^f(K\,{}^fc).$$

But, by (ii), $\lambda_x(b\,{}^fc) = K\,{}^f(b\,{}^fc)$, so we are led to stipulate the identity

$$(S\,{}^f(K\,{}^fb))\,{}^f(K\,{}^fc) = K\,{}^f(b\,{}^fc)$$

for all b and c. Again, this is an easy consequence of (4). By the same argument as above, we may replace the stipulated identity by the equation

$$\lambda_x\lambda_y((S\,{}^f(K\,{}^fx))\,{}^f(K\,{}^fy)) = \lambda_x\lambda_y(K\,{}^f(x\,{}^fy)). \tag{4.5}$$

The proof of Proposition 2 is now complete, provided we adopt (4.1) to (4.5) as the five equations which a Curry algebra must satisfy in addition to the identities (1) to (3).

From now on we shall write $\lambda_x\varphi(x)$ for the unique f corresponding to $\varphi(x)$ by Proposition 2, as we did in the proof. The properties of the new symbol λ_x are embodied in the

λ-calculus of Church (1932). The equivalence of the systems of Curry and Church (λK-calculus, 1941) are summed up as follows.

PROPOSITION 3. The identities and equations of Curry algebras imply and are implied by the following:[1)]

(1) $I = \lambda_x x$,

(2) $K = \lambda_x \lambda_y x$,

(3) $S = \lambda_u \lambda_v \lambda_z ((u\,{}^{\backprime}z)\,{}^{\backprime}(v\,{}^{\backprime}z))$,

(4) $\lambda_x (f\,{}^{\backprime}x) = f$,

(5) $(\lambda_x \varphi(x))\,{}^{\backprime}a = \varphi(a)$.

The proof is almost straightforward. We shall only explain why (5) holds for Curry algebras. By Proposition 2, we have $f\,{}^{\backprime}x \underset{x}{=} \varphi(x)$, and we want to deduce from this that $f\,{}^{\backprime}a = \varphi(a)$. By the universal property of $A[x]$, there exists a unique homomorphism $h' : A[x] \to A$ such that $h'h_x = h_x$ and $h'(x) = a$. This is of course the substitution homomorphism which replaces x by a, hence yields the required equation from $f\,{}^{\backprime}x \underset{x}{=} \varphi(x)$.

To recapture the traditional terminology, let us mention that the theory of Curry algebras is called <u>combinatory logic</u>. Proposition 2 may then be compressed into the slogan:

combinatory logic = λ-calculus.

Incidentally, <u>combinators</u> are the canonical elements of Curry (or Schönfinkel) algebras, that is, the elements of the free Curry algebra generated by the empty set.

Both Schönfinkel and Curry had intended to use combinatory logic for the foundations of mathematics. An obstacle arose in the following result.

PROPOSITION 4. In a Curry algebra every element f has a "fixpoint" a such that $f\,{}^{\backprime}a = a$.

Proof. Let $b = \lambda_x (f\,{}^{\backprime}(x\,{}^{\backprime}x))$ and put $a = b\,{}^{\backprime}b$. Then

$$f\,{}^{\backprime}a = f\,{}^{\backprime}(b\,{}^{\backprime}b) = b\,{}^{\backprime}b = a.$$

If f is negation, usually denoted by $\neg$, we have $\neg a = a$, so a cannot be a proposition. We must therefore distinguish between propositions and other entities; but even this distinction does not prevent Russell's paradox from raising its head (e.g. Curry and Feys, 1958).

It is of course well-known that Russell's paradox may be avoided by introducing types. In the following exposition of typed combinatory logic I shall follow Curry and Feys in principle, even though I shall reject their permissiveness in allowing symbols with ambiguous types.

First of all we replace a Curry algebra by a many-sorted algebra, whose elements, which we call <u>entities</u>, may belong to different sorts, which we call <u>types</u>. If A and B are types, so is B^A, the type of all "functions" from B to A. For typographical reasons, we write $B \leftarrow A$ in place of B^A. The binary operation symbol ${}^{\backprime}$ may not be placed between entities indiscriminately, but is subject to the following rule:

(0) $$\frac{a \in A \qquad f \in B \leftarrow A}{f\,{}^{\ulcorner}a \in B},$$

meaning that, if a is of type A and f of type $B \leftarrow A$, then $f\,{}^{\ulcorner}a$ is of type B.[2)]

In place of the three constants I, K and S we adopt three families of constants:

(1) $I_A \in A \leftarrow A$ such that $I_A\,{}^{\ulcorner}a = a$;

(2) $K_{A,B} \in (A \leftarrow B) \leftarrow A$ such that $(K_{A,B}\,{}^{\ulcorner}a)\,{}^{\ulcorner}b = a$;

(3) $S_{A,B,C} \in ((A \leftarrow C) \leftarrow (B \leftarrow C)) \leftarrow ((A \leftarrow B) \leftarrow C)$

such that $((S_{A,B,C}\,{}^{\ulcorner}f)\,{}^{\ulcorner}g)\,{}^{\ulcorner}c = (f\,{}^{\ulcorner}c)\,{}^{\ulcorner}(g\,{}^{\ulcorner}c)$.

It is assumed here that $a \in A$, $b \in B$, $c \in C$, $f \in (A \leftarrow B) \leftarrow C$ and $g \in B \leftarrow C$.

A many-sorted algebra of the kind constructed above may be called a typed Schönfinkel algebra, or a typed Curry algebra if the appropriate equations are postulated.

Curry and Feys have pointed out that, if $\leftarrow$ is read as "if", the types of I_A, $K_{A,B}$ and $S_{A,B,C}$ are precisely the axioms of the intuitionist implicational calculus, while (0) bears an obvious relation to the usual rule of modus ponens.

Two remarks are in order. The same axioms also appear in the classical propositional calculus accompanied by an additional axiom involving negation. Nonetheless, the negationless formula $A \leftarrow (A \leftarrow (B \leftarrow A))$ is a theorem classically but not in the system without negation. This is why we call the present system intuitionistic.

Secondly, it should be pointed out that $A \leftarrow A$ is usually not taken as an axiom, but is deduced from the other two axioms. Nevertheless, we prefer to regard it as an axiom here.

Implicit in the the definition of a typed Schönfinkel algebra is a way of regarding each entity of type A as a proof of the formula A.

For example, $S_{A,B,A}$ is by definition a proof of the axiom $((A \leftarrow A) \leftarrow (B \leftarrow A)) \leftarrow ((A \leftarrow B) \leftarrow A)$ and $K_{A,B}$ is by definition a proof of the axiom $(A \leftarrow B) \leftarrow A$. Hence $S_{A,B,A}\,{}^{\ulcorner}K_{A,B}$ is a proof of the theorem $(A \leftarrow A) \leftarrow (B \leftarrow A)$ by modus ponens. Again, $K_{A,C}$ is a proof of the axiom $(A \leftarrow C) \leftarrow A$. Take $B = A \leftarrow C$, then $(S_{A,B,A}\,{}^{\ulcorner}K_{A,B})\,{}^{\ulcorner}K_{A,C}$ is a proof of the theorem $A \leftarrow A$. Since we regard $A \leftarrow A$ as an axiom, another proof is I_A. Incidentally, we see here that the derivation of $A \leftarrow A$ from the other axioms of the implicational calculus is nothing else than Schönfinkel's definition of I as $(S\,{}^{\ulcorner}K)\,{}^{\ulcorner}K$.

The association of entities with proofs becomes even more striking when we compare the free typed Schönfinkel algebra (generated by a set of letters) with pure intuitionistic implicational logic. Then

combinators = proofs.

The reader should note that throughout we distinguish between an axiom such as $(A \leftarrow B) \leftarrow A$ and its proof $K_{A,B}$, a pedantic but necessary distinction.

Let us now look at Proposition 1 for typed Schönfinkel algebras. The reader will easily convince himself that the proposition and its proof remain valid, provided x is an indeterminate of type A, where A is any type of $\mathcal{A}$. It asserts that, if $\varphi(x)$ is a polynomial of type B in the indeterminate x of type A, then there is a constant f of type $B \leftarrow A$ such that $\varphi(x) \underset{x}{=} f^{\frown}x$.

In the proof-theoretic interpretation we should regard x as an assumption that A holds. Proposition 1 then becomes the usual deduction theorem with an extra punch at the end:

if $\varphi(x)$ is a proof of B from the assumption x that A holds, then there is a proof of $B \leftarrow A$ without any assumption such that $\varphi(x) \underset{x}{=} f^{\frown}x$.

The extra bit at the end asserts the "equality" of two proofs. Perhaps it would have been better to speak only of "equivalence" of proofs.

As far as I know, this association between combinators and proofs is due to Curry and Feys (1958). It was developed further by Howard (see Stenlund, 1972) and the author (1969, 1972).[3)]

Proposition 2 will also remain valid for typed Curry algebras. It asserts that the proof f of $B \leftarrow A$, whose existence has been established in Proposition 1, is unique up to equivalence of proofs (which we called equality here).

Proposition 3 will remain valid for typed Curry algebras provided we use the typed λ-calculus and write $f = \lambda_{x<A}\varphi(x)$.

Proposition 4 will not remain valid for typed Curry algebras without very strong restrictions. After all, the whole purpose of introducing types was to render expressions such as $b^{\frown}b$ meaningless in general. If $b \in B$, it does have a meaning only if $B = B \leftarrow B$. Proposition 4 remains valid for typed algebras in the following sense:

if A is a type such that $A \leftarrow A = A$, then every entity f of type A has a fixpoint a such that $f^{\frown}a = a$.

The fragment of propositional logic investigated up to now studiously avoids conjunction, which classically was usually defined in terms of implication and negation. Let us now turn to a fragment of logic, the positive intuitionist propositional calculus, which deals with implication $\leftarrow$, conjunction $\wedge$ and truth T. We shall present this as a deductive system, making use of the symbol $\rightarrow$ for entailment, in the spirit of Gentzen. Here are our axioms and rules of inference, suitably labelled:

(1) $A \xrightarrow{1_A} A$, $\dfrac{A \xrightarrow{f} B \quad B \xrightarrow{g} C}{A \xrightarrow{gf} C}$;

(2) $A \xrightarrow{0_A} T$;

(3) $A \wedge B \xrightarrow{\pi_{A,B}} A$, $A \wedge B \xrightarrow{\pi'_{A,B}} B$, $\dfrac{C \xrightarrow{f} A \quad C \xrightarrow{g} B}{C \xrightarrow{\langle f,g\rangle} A \wedge B}$;

(4) $(A \leftarrow B) \wedge B \xrightarrow{e_{A,B}} A$, $\dfrac{C \wedge B \xrightarrow{h} A}{C \xrightarrow{h^*} A \leftarrow B}$

The labels are useful in naming proofs. For example, the commutative law is proved thus:

$$\frac{A \wedge B \xrightarrow{\pi'_{A,B}} B \quad A \wedge B \xrightarrow{\pi_{A,B}} A}{A \wedge B \xrightarrow{\langle \pi'_{A,B}, \pi_{A,B} \rangle} B \wedge A}$$

The label $\langle \pi'_{A,B}, \pi_{A,B} \rangle$ appearing in the last line may be used to denote the whole proof.

We are also interested in an equivalence relation between proofs, which we may as well denote by the equality symbol $=$. We do not bother to write down the usual reflexive, symmetric, transitive and substitution laws for equality. However, we list the following equations, where in $f = g$ it is assumed that f and g have the same source and target.

(1') $f 1_A = f$, $1_B f = f$, $(hg)f = h(gf)$, for all $f: A \to B$, $g: B \to C$ and $h: C \to D$;

(2') $f = 0_A$, for all $f: A \to 1$;

(3') $\pi_{A,B}\langle f,g \rangle = f$, $\pi'_{A,B}\langle f,g \rangle = g$, $\langle \pi_{A,B} h, \pi'_{A,B} h \rangle = h$, for all $f: C \to A$, $g: C \to B$ and $h: C \to A \wedge B$;

(4') $\varepsilon_{A,B}\langle h^* \pi_{C,B}, \pi'_{C,B} \rangle = h$, $(\varepsilon_{A,B}\langle k\pi_{C,B}, \pi'_{C,B} \rangle)^* = k$, for all $h: C \wedge B \to A$ and $k: C \to A \Leftarrow B$.

The reader will recognize that (1) and (1') define a category. Moreover, (2) and (2') assert that T is a terminal object, usually denoted by 1. Furthermore, (3) and (3') assert that $A \wedge B$ is the Cartesian product of A and B, usually written $A \times B$. Finally, (4) and (4') express the fact that $(-) \Leftarrow B$ is right adjoint to $(-) \wedge B: A \to A$, which makes $\Leftarrow$ an internal hom-functor, the usual notation for $A \Leftarrow B$ being A^B. (1) to (3) and (1') to (3') define a Cartesian category, (1) to (4) and (1') to (4') define a Cartesian closed category.

Cartesian closed categories were introduced under this name by Eilenberg and Kelly (1966). Lawvere (1969) also pointed out that $\times$ is right adjoint to the diagonal $A \to A \times A$ and emphasized the analogy with propositional logic. Our definition of Cartesian closed categories is slightly different, in as much as 1, $\times$ and exponentiation are not only said to exist, but are regarded as part of the structure. In fact, our notion of Cartesian closed category is algebraic over the category of categories, or the category of graphs for that matter, in much the same way that the notion of group is algebraic over the category of sets.

For future reference, we also note the following consequence of (1') to (4'):

(5') $\langle f,g \rangle h = \langle fh, gh \rangle$.

This is proved thus: let $k = \langle f,g \rangle h$, then, omitting subscripts, we have

$$\pi k = fh, \quad \pi' k = gh$$

by (3'), hence

$$\langle fh, gh \rangle = \langle \pi k, \pi' k \rangle = k,$$

by (3') again.

In the author's opinion, it is a tour de force to present propositional logic without conjunction. Curiously, the same tour de force is found in the paper by Eilenberg and Kelly, who

went to some trouble to eliminate the Cartesian product from Cartesian closed categories. One could argue that closed categories without products are essentially the same as typed Curry algebras. It therefore seems reasonable to proceed from the study of typed Curry algebras to the study of Cartesian closed categories.

I had proved in 1972 that functional completeness holds for Cartesian closed categories, provided that these satisfied a finite set of equations, like those due to Curry, and conjectured that these equations are already a consequence of (1') to (4') above, that is, they hold in any Cartesian closed category. I proved this later (1974) and shall give another exposition of the proof here. The new versions of Propositions 1 and 2 will be called Theorems 1 and 2.

First, we must explain what it means to adjoin an indeterminate morphism $x: 1 \to A$ to a Cartesian closed category $\mathbf{A}$, A being an object of $\mathbf{A}$, which we also regard as a formula. In the same spirit, we may regard x as an assumption that T entails A. The objects of $\mathbf{A}[x]$ are the same as those of $\mathbf{A}$, but the morphisms $\varphi(x): B \to C$ in $\mathbf{A}[x]$ may be regarded as proofs that B entails C on the assumption x. Equality between proofs or polynomials is determined by (1') to (4') and is denoted by $\underset{x}{=}$. One must check of course that $\mathbf{A}[x]$ thus constructed has the expected universal property.

THEOREM 1. (Deduction theorem). With every proof $\varphi(x): B \to C$ from the assumption $x: 1 \to A$ there is associated a proof $f: A \wedge B \to C$ not depending on this assumption. Moreover $f\langle x0_B, 1_B\rangle \underset{x}{=} \varphi(x)$.

It should be pointed out that we have presented the positive intuitionist propositional calculus as a deductive system, so that the usual deduction theorem becomes absorbed in the rules governing the deduction symbol $\to$, thus:

$$\frac{A \wedge B \to C}{A \to C \leftarrow B} \ .$$

However, at a higher level the horizontal bar functions as a deduction symbol and Theorem 1 plays the role of a new deduction theorem.

Proof. Clearly, every polynomial $\varphi(x)$ must have one of the five forms: k, x, $\langle\psi(x), \chi(x)\rangle$, $\chi(x)\psi(x)$, $\psi(x)^*$, where k is a constant and where $\psi(x)$ and $\chi(x)$ are shorter polynomials. By inductional assumption we have, omitting subscripts:

$$\psi(x) \underset{x}{=} g\langle x0, 1\rangle, \ \chi(x) \underset{x}{=} h\langle x0, 1\rangle.$$

The result now follows by verifying the following equations:

$$k\pi'_{A,B}\langle x0_B, 1_B\rangle \underset{x}{=} k \ ,$$

$$\pi_{A,1}\langle x0_1, 1_1\rangle \underset{x}{=} x \ ,$$

$$\langle g,h\rangle\langle x0_B, 1_B\rangle \underset{x}{=} \langle\psi(x), \chi(x)\rangle \ ,$$

$$h\langle \pi_{A,B}, g\rangle\langle x0_B, 1_B\rangle \underset{x}{=} \chi(x)\psi(x) ,$$

$$(g\alpha_{A,B,D})^{*}\langle x0_B, 1_B\rangle \underset{x}{=} \psi(x)^{*} ,$$

where $\alpha_{A,B,D}\colon (A\wedge B)\wedge D \to A\wedge(B\wedge D)$ is given by

$$\alpha_{A,B,D} = \langle \pi_{A,B}\pi_{A\wedge B,D}, \langle \pi' A,B \pi_{A\wedge B,D}, \pi' A\wedge B,D\rangle\rangle .$$

The last equation is proved by showing (omitting subscripts) that

$$\varepsilon\langle (g\alpha)^{*}\langle x0,1\rangle\pi,\pi'\rangle \underset{x}{=} g\langle x0,1\rangle ,$$

which follows from a routine calculation, as in (Lambek 1974, p. 272).

THEOREM 2. (Functional completeness of Cartesian closed categories). For every polynomial $\varphi(x)\colon B \to C$ in an indeterminate $x\colon 1 \to A$ there is a unique constant $f\colon A\times B \to C$ such that $f\langle x0_B, 1_B\rangle \underset{x}{=} \varphi(x)$.

Proof. Let us write $\kappa_{x\in A}\varphi(x)$ for the constant f which is assigned to the proof $\varphi(x)$ in the proof of Theorem 1. Thus, when k is a constant, we have:

(i) $\kappa_{x\in A} k = k\,\pi' A,B$;

(ii) $\kappa_{x\in A} x = \pi_{A,1}$;

(iii) $\kappa_{x\in A}\langle \psi(x), \chi(x)\rangle = \langle \kappa_{x\in A}\psi(x), \kappa_{x\in A}\chi(x)\rangle$;

(iv) $\kappa_{x\in A}(\chi(x)\psi(x)) = \kappa_{x\in A}\chi(x)\langle \pi_{A,B}, \kappa_{x\in A}\psi(x)\rangle$;

(v) $\kappa_{x\in A}(\psi(x)^{*}) = (\kappa_{x\in A}\psi(x)\alpha_{A,B,D})^{*}$,

where (iii), (iv) and (v) are subject to the restriction that $\varphi(x)$ is not constant.

Note, in particular, the following special case of (iv), in view of (i):

(vi) $\kappa_{x\in A}(k\ \psi(x)) = k\ \kappa_{x\in A}\psi(x)$,

where again it is assumed that $\psi(x)$ is not constant.

We first show that the above restrictions on (iii) to (v), and therefore on (vi), may be removed. For example, to take the most difficult case, let us check that (v) holds when $\psi(x)$ is a constant g. Note that $\pi'\alpha = \langle \pi'\pi, \pi'\rangle$ by (3'). Then

$$\begin{aligned}
(\kappa_{x\in A}\, g\, \alpha)^{*} &= (g\,\pi'\alpha)^{*} && \text{by (i)}\\
&= (g\,\beta)^{*} && \text{if } \beta = \langle \pi'\pi, \pi'\rangle\\
&= (\varepsilon\langle g^{*}\pi, \pi'\rangle\beta)^{*} && \text{by (4')}\\
&= (\varepsilon\langle g^{*}\pi\beta, \pi'\beta\rangle)^{*} && \text{by (5')}\\
&= (\varepsilon\langle g^{*}\pi'\pi, \pi'\rangle)^{*} && \text{by (3')}\\
&= g^{*}\pi' && \text{by (4')}\\
&= \kappa_{x\in A}(g^{*}) && \text{by (i) .}
\end{aligned}$$

We next show that $\kappa_{x\in A}\varphi(x)$ depends only on the polynomial $\varphi(x)$, which may be regarded as the proof $\varphi(x)$ modulo the equivalence relation $\underset{x}{=}$. Let us write $\varphi(x) \equiv \psi(x)$ for $\kappa_{x\in A}\varphi(x) = \kappa_{x\in A}\psi(x)$. Then it is easily checked that $\equiv$ has the substitution property and satisfies all the conditions which

equality in $A[x]$ must satisfy (see the sample calculation below). Since $\underset{x}{=}$ is by definition the smallest such equivalence relation, it follows that $\underset{x}{=}$ is contained in $\equiv$, that is,

(*) $\quad \varphi(x) \underset{x}{=} \psi(x)$ implies $\kappa_{x\in A}\varphi(x) = \kappa_{x\in A}\psi(x)$,

as claimed.

As promised, we shall prove, for example, that $\varepsilon < \chi(x)^{*}\pi,\pi'> \equiv \chi(x)$, to take the worst case. Indeed, writing $\kappa_{x\in A}\chi(x) = h$, we have

$$\kappa_{x\in A}(\varepsilon<\chi(x)^{*}\pi,\pi'>) = \varepsilon\,\kappa_{x\in A}<\chi(x)^{*}\pi,\pi'> \quad \text{by (vi)}$$

$$= \varepsilon<\kappa_{x\in A}(\chi(x)^{*}\pi),\kappa_{x\in A}\pi'> \quad \text{by (iii)}$$

$$= \varepsilon<\kappa_{x\in A}\chi(x)^{*}<\pi,\kappa_{x\in A}\pi>,\pi'\pi'> \quad \text{by (iv) and (i)}$$

$$= \varepsilon<(h\alpha)^{*}<\pi,\pi\pi'>,\pi'\pi'> \quad \text{by (v) and (i).}$$

Now the associativity morphism $\alpha = <\pi\pi,<\pi'\pi,\pi'>>$ clearly has an inverse α^{-1}, so the above

$$= \varepsilon<(h\alpha)^{*}<\pi\alpha,\pi\pi'\alpha>,\pi'\pi'\alpha>\alpha^{-1} \quad \text{by (5')}$$

$$= \varepsilon<(h\alpha)^{*}<\pi\pi,\pi'\pi>,\pi'>\alpha^{-1} \quad \text{by (3')}$$

$$= \varepsilon<(h\alpha)^{*}\pi,\pi'>\alpha^{-1} \quad \text{by (3')}$$

$$= h\alpha\alpha^{-1} = h \quad \text{by (4').}$$

We are finally ready to prove the uniqueness of f. Suppose $g<x0_B,1_B> \underset{x}{=} \varphi(x)$, then

$$\kappa_{x\in A}\varphi(x) = \kappa_{x\in A}(g<x0_B,1_B>) \quad \text{by (*)}$$

$$= g\,\kappa_{x\in A}<x0_B,1_B> \quad \text{by (vi)}$$

$$= g\,<\kappa_{x\in A}(x0_B),\kappa_{x\in A}1_B> \quad \text{by (iii)}$$

$$= g\,<\kappa_{x\in A}x<\pi_{A,B},\kappa_{x\in A}0_B>,1_B\pi'_{A,B}> \quad \text{by (iv) and (i)}$$

$$= g\,<\pi_{A,}1<\pi_{A,B},0_B\pi'_{A,B}>,\pi'_{A,B}> \quad \text{by (ii) and (i)}$$

$$= g\,<\pi_{A,B},\pi'_{A,B}> \quad \text{by (3')}$$

$$= g\,1_{A\times B} \quad \text{by (3')}$$

$$= g \quad \text{by (1') .}$$

REMARK 1. Theorems 1 and 2 remain valid for Cartesian categories, that is, if exponentiation is absent.

REMARK 2 (for categorists). Theorem 2 may be interpreted as saying that $A[x]$ is the Kleisli category of the obvious cotriple on A associated with the function $A \times (-)\colon A \to A$, or of the triple on A associated with the functor $(-)^{A}$, see (Lambek 1974) for details. It is also not difficult to prove directly that the Kleisli category has the universal property of $A[x]$. This alternative approach (loc. cit. § 5) bypasses the historical development discussed above, but also loses the algorithm for calculating $\kappa_{x\in A}\varphi(x)$. Such a direct approach, without mention of Kleisli categories by that name, was also used in (Volger, 1975).

As a special case of Theorem 2 we obtain the following.

COROLLARY. For every polynomial $\varphi(x)\colon 1 \to C$ in an indeterminate $x\colon 1 \to A$ there is a unique constant $g\colon A \to C$ such that $gx \underset{x}{=} \varphi(x)$ and, equivalently, a unique constant $h\colon 1 \to C^A$ such that $\epsilon_{C,A}\langle h,x\rangle \underset{x}{=} \varphi(x)$.

Proof. To obtain this from Theorem 2, merely put

$$g = \kappa_{x\in A}\varphi(x)\ \langle 1_A, 0_A\rangle ,$$

$$h = (g\pi'_{1,A})^* = (\kappa_{x\in A}\varphi(x)\ \langle \pi'_{1,A}, \pi_{1,A}\rangle)^* .$$

Actually, the corollary is no weaker than the theorem, since polynomials $B \to C$ are in one-to-one correspondence with polynomials $1 \to C^B$. To compare the corollary with the λ-calculus, we write

$$h^{\ulcorner} = \epsilon_{C,A}\ \langle h0_A, 1_A\rangle ,$$

so that

$$h^{\ulcorner}x \underset{x}{=} \epsilon_{C,A}\ \langle h,x\rangle \underset{x}{=} \varphi(x) ,$$

and define

$$\lambda_{x\in A}\varphi(x) = (\kappa_{x\in A}\varphi(x)\langle \pi'_{1,A}, \pi_{1,A}\rangle)^* = h.$$

There are two traditional applications of the λ-calculus. The first is to arithmetic, in particular, the theory of recursive functions. The second is to the foundations of mathematics via Curry'a illative combinatory logic or the type theory of Church. We shall briefly discuss the first application and regret that space does not permit discussion of the second and the use which has recently been made of it in the construction

of free toposes.

Writing

$$f\circ g = \lambda_x(f^{\ulcorner}(g^{\ulcorner}x)) ,$$

one may introduce natural numbers

$$0 = \lambda_x\, I,\ 1 = \lambda_x\, x = I,\ 2 = \lambda_x\, (x\circ x),\ \ldots\ldots$$

and successor, addition, multiplication and exponentiation by

$$S^{\ulcorner}n = \lambda_y(y\circ(n^{\ulcorner}y)) ,$$

$$m + n = \lambda_y\, ((m^{\ulcorner}y)\circ(n^{\ulcorner}y)) ,$$

$$m\ n = m\circ n ,$$

$$m^n = n^{\ulcorner}m .$$

These are recursive functions, and Kleene proved in 1936 that recursive (or Turing computable) functions are precisely those recursive functions which are definable in the type-free λ-calculus.

Unfortunately there are difficulties when one introduces types. If a has type A, then f and g in $(f\circ g)^{\ulcorner}a$ have types $A^A = B$, say. For $n^{\ulcorner}f$ to make sense, n will then have to be of type B^B, and for $n^{\ulcorner}m$ to make sense, m will have to be of type B. If m and n are both natural numbers, we are led to $B^B = B$, and this would be a consequence of $A^A = A$.

Now, it is certainly possible to postulate a type A such that $A^A = A$ (Scott, 1972). However, it seems more natural to postulate a type N with entities $0 \in N$, $S \in N \leftarrow N$ and

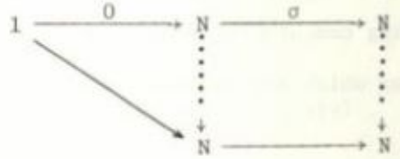
$R_A \in ((A \Leftarrow A) \Leftarrow (A \Leftarrow A)) \Leftarrow N$ satisfying certain equations. In the language of Cartesian closed categories, we thus require an object N and morphisms $0\colon 1 \to N$, $\sigma\colon N \to N$ and $\rho_A\colon N \to (A^A)^{(A^A)}$ satisfying certain identities, to wit:

$$(\rho_A\ 0)\,{}^{\ulcorner}f = I_A\ ,$$

$$(\rho_A(\sigma n))\,{}^{\ulcorner}f = f \circ ((\rho_A\ n)\,{}^{\ulcorner}f),$$

for all $n\colon 1 \to N$ and $f\colon 1 \to A^A$.

These identities are related to the Peano-Lawvere axiom, which deals with a morphism $A \times A^A \to A^N$ instead. They imply the existence (but not the uniqueness) of a morphism $N \to A$ such that the following diagram commutes for given morphisms $1 \to A$ and $A \to A$:



The Peano-Lawvere axiom also requires uniqueness.

Marie-France Thibault called a Cartesian closed category with N, 0, σ, ρ satisfying the above identities a <u>prerecursive category</u>. She proved the following:

(a) The set of primitive recursive functions is properly contained in the set F of functions represented by morphisms $N^k \to N$ in the free prerecursive category generated by the empty category ;

(b) F is properly contained in the set of all recursive functions ;

(c) F coincides with the set of type-recursive functions discussed by Grzegorczyk.

It seems clear from (Hindley, Lercher and Seldin, 1972, Chapter 11) that F is also essentially the same as the set of Gödel's functions of finite type.

FOOTNOTES

1) Here $=$ denotes equality in A, $A[x]$, $A[x,y]$, etc. simultaneously, subscripts having been suppressed. It is an equivalence relation subject to the following rules:

$$\frac{f = g \quad a = b}{f\,{}^{\ulcorner}a = g\,{}^{\ulcorner}b}\ ,\quad \frac{\varphi(x) = \psi(x)}{\lambda_x \varphi(x) = \lambda_x \psi(x)}\ .$$

2) Actually, the application symbol ${}^{\ulcorner}$ should carry subscripts A and B; but we omit these whenever they are clear from the context, as they are throughout this paper.

3) For related ideas and further references see also the thought provoking paper by Scott (1970).

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RELATING THEORIES OF THE λ-CALCULUS

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Dedicated to Professor H. B. Curry on the occasion of his 80th Birthday

Mathematical theories arise for many different reasons, sometimes in connection with specific applications and often owing to accidental inspiration. From time to time we ought to ask ourselves concerning our theories where *should* they have come from; usually the answer will have little to do with the exact historical development. The λ-calculus is, I feel, a case in point. In Scott (1980), in the Kleene Festschrift, I made up a story of where the theory of type-free λ-calculus *could* have come from. Any number of people who heard my lecture and read the manuscript were cross with me. They said "But it didn't develop that way! And besides we doubt it ever would have." But this reaction misses the point of my story. I shall not, however, repeat the earlier story here, for the point of the present paper is different. For those people who do not like to discuss philosophy - even Philosophy of Mathematics - my remarks here can be taken as a suggestion of how to group diverse models of λ-calculus rather uniformly under a general scheme. The scheme is by now rather well known and not at all original with me. What I hope can be regarded as a useful contribution is my putting of the ideas in a certain order. As I consider the order to be a natural one, I feel there is philo-