DEDUCTIVE SYSTEMS AND CATEGORIES

II. STANDARD CONSTRUCTIONS AND CLOSED CATEGORIES*

by

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0. INTRODUCTION

We wish to explore the connection between

(1) pre-ordered sets with structure,

(2) deductive systems,

(3) categories with structure.

By a pre-ordered set we mean a set with a transitive and reflexive, but not necessarily anti-symmetric, relation $\leq$. It would be premature to give a rigorous definition of what we have in mind by "structure" and "deductive system". Instead we shall illustrate with an example.

(1) A semilattice (with largest element) is a triple $(A,T,\wedge)$, where A is a pre-ordered set, T an

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element of $\underline{A}$, and $\wedge$ a binary operation on $\underline{A}$ such that

$$A \leq T,$$

$$C \leq A \wedge B \Longleftrightarrow C \leq A \;\&\; C \leq B.$$

(2) For any discrete set $\underline{X}$ there is a deductive system $D(\underline{X})$ described as follows:

Its terms are all elements X of $\underline{X}$, also T is a term; finally if A and B are terms, so is $A \wedge B$. (It is understood that nothing else is a term.)

Its formulas are all expressions of the form $A \longrightarrow B$ where A and B are terms.

Its axioms (or axiom schemes) and rules of inference are listed as follows:

$$1:\ A \longrightarrow A\,, \qquad 2:\ \frac{A \longrightarrow B \quad B \longrightarrow C}{A \longrightarrow C}\,, \qquad 3:\ A \longrightarrow T\,,$$

$$4:\ \frac{C \longrightarrow A \wedge B}{C \longrightarrow A}\,, \qquad 5:\ \frac{C \longrightarrow A \wedge B}{C \longrightarrow B}\,, \qquad 6:\ \frac{C \longrightarrow A \quad C \longrightarrow B}{C \longrightarrow A \wedge B}\,.$$

Proofs are written in tree-form. For example, the following is a proof:

$$\dfrac{\dfrac{\dfrac{A \wedge B \xrightarrow{1} A \wedge B}{A \wedge B \longrightarrow B}\,5 \qquad \dfrac{A \wedge B \xrightarrow{1} A \wedge B}{A \wedge B \longrightarrow A}\,4}{A \wedge B \longrightarrow B \wedge A}\,6$$

Any formula appearing as a proof, in particular the last

formula, is called provable or a theorem. Thus we see from the above that $A \wedge B \longrightarrow A \wedge B$, $A \wedge B \longrightarrow B$, $A \wedge B \longrightarrow A$, and $A \wedge B \longrightarrow B \wedge A$ are all theorems, but only the first is an axiom.

(3) A cartesian category is a quintuple $(\underline{A},T,\wedge,\alpha,\beta)$, where $\underline{A}$ is a category, T is an object of $\underline{A}$, $\wedge: \underline{A} \times \underline{A} \longrightarrow \underline{A}$ is a bifunctor, and α and β are natural isomorphisms as follows:

$$\alpha(A): \underline{1} \longrightarrow [A,T] ,$$

$$\beta(A,B,C): [C,A] \times [C,B] \longrightarrow [C,A \wedge B] .$$

Here $\underline{1}$ is "the" one-element set, and $[A,B]$ is short for $\mathrm{Hom}_{\underline{A}}(A,B)$. Thus T is "the" terminal object of $\underline{A}$ and $\wedge$ is the usual categorical product.

We shall see later that the deductive system $D(\underline{X})$ can be used to construct not only the free semilattice, but also the free cartesian category generated by $\underline{X}$. In the first case we are only interested in the theorems of the deductive system, but in the second case the proofs will play an important role.

More generally one can form a deductive system $D(\underline{X})$ when $\underline{X}$ is a pre-ordered set by adjoining the axiom

$$0: X \longrightarrow Y$$

whenever $X \leq Y$ in $\underline{X}$; also when $\underline{X}$ is a category by adjoining a family of axioms

$$D_f\colon X \longrightarrow Y \ ,$$

one for each map $f\colon X \longrightarrow Y$ is $\underline{X}$. For technical reasons it is important to regard D_f and D_g as distinct axioms when $f \neq g$, even though they assert the same formula.

We shall lean heavily on concepts and techniques developed by logicians. However one question they seem to have ignored; namely, when are two proofs of the same formula equivalent? For example, consider the proof:

$$\dfrac{\dfrac{\dfrac{}{A \wedge B \xrightarrow{1} A \wedge B}\,4}{A \wedge B \longrightarrow A} \qquad \dfrac{\dfrac{}{A \wedge B \xrightarrow{1} A \wedge B}\,5}{A \wedge B \longrightarrow B}}{A \wedge B \longrightarrow A \wedge B}\,6 \ .$$

Nobody would deny that this proof might be regarded as, in some sense, equivalent to the shorter proof:

$$A \wedge B \xrightarrow{1} A \wedge B \ .$$

On the other hand, there is a good case for considering the following proofs as inequivalent:

$$\dfrac{\dfrac{}{A \wedge A \xrightarrow{1} A \wedge A}\,4}{A \wedge A \longrightarrow A} \ , \qquad \dfrac{\dfrac{}{A \wedge A \xrightarrow{1} A \wedge A}\,5}{A \wedge A \longrightarrow A} \ .$$

In fact, they admit different "generalizations".

Our program will be to look at two kinds of structured categories in some detail: standard constructions and closed categories. It had been planned to discuss also cartesian closed categories and combinators, but time did not permit inclusion in this volume. A treatment of this topic and also the development of a general theory of structured categories must be deferred.

1. STANDARD CONSTRUCTIONS

By a closure system we shall understand a pair $(\underline{A},T)$ where $\underline{A}$ is a pre-ordered set with an operation T satisfying the following rules:

$$A \leq T(A) ,$$

$$T(T(A)) \leq T(A) ,$$

$$A \leq B \longrightarrow T(A) \leq T(B) ,$$

for all elements A and B of $\underline{A}$. It will be convenient to write $T^2(A)$ for $T(T(A))$, etc., also $T^0(A)$ for A.

As a device for constructing free closure systems it is useful to look at a deductive system $D(\underline{X})$ generated by a pre-ordered set $\underline{X}$. Its terms are elements of $\underline{X}$ and expressions of the form $T(A)$ when A is a term. Of course, instead of describing the set of terms recursively, we can, in this case, list all terms explicitely as being expressions of the form $T^n(X)$, where X is an element of

$\underline{X}$ and n is a non-negative integer. The formulas of $D(\underline{X})$ are expressions of the form $A \longrightarrow B$ where A and B are terms. They also may be listed explicitely as being expressions of the form $T^m(X) \longrightarrow T^n(Y)$. It remains to present the axioms and rules of inference:

$$A \xrightarrow{1_A} A \ ,$$

$$\frac{A \xrightarrow{Q} B \qquad B \xrightarrow{P} C}{A \xrightarrow{PQ} C} \ ,$$

$$X \xrightarrow{D_{X,Y}} Y \ , \text{ whenever } X \leq Y \text{ in } \underline{X},$$

$$A \xrightarrow{h(A)} T(A) \ , \qquad T^2(A) \xrightarrow{m(A)} T(A) \ ,$$

$$\frac{A \xrightarrow{P} B}{T(A) \xrightarrow{T(P)} T(B)} \ .$$

The last rule of inference, for example, must be interpreted as a rule for generating a proof $T(P)\colon T(A) \longrightarrow T(B)$ from a given proof $P\colon A \longrightarrow B$.

An example of a proof in $D(\underline{X})$ is the following:

$$\cfrac{\cfrac{\cfrac{A \xrightarrow{1_A} A}{T(A) \longrightarrow T(A)} \qquad T^2(A) \xrightarrow{m(A)} T(A)}{T^2(A) \longrightarrow T(A)}}{}$$

denoted unambiguously by $m(A)T^2(1_A)$.

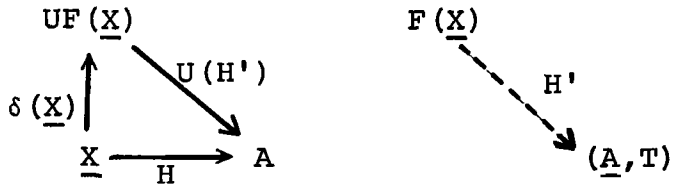
We now construct the free closure system $F(\underline{X})$ generated by the pre-ordered set $\underline{X}$. Its elements are the terms of $D(\underline{X})$, and $A \leq B$ in $F(\underline{X})$ means that $A \longrightarrow B$ is provable in $D(\underline{X})$. $T(A)$ is the same in $F(\underline{X})$ as in $D(\underline{X})$. It is easily seen that $F(\underline{X})$ is a closure system. In what sense is it free?

By Ord we shall mean the category of pre-ordered sets and order preserving mappings. By Clo we shall mean the category of closure systems and those order preserving mappings which preserve T exactly, let us call them strict maps. Thus a strict map $H: (\underline{A},T) \longrightarrow (\underline{B},T)$ is a map $H: \underline{A} \longrightarrow \underline{B}$ in Ord such that

$$H(T(A)) = T(H(A))$$

for all elements A of $\underline{A}$. It is a sore point of our theory that "=" in the above cannot be replaced by "$\leq$ and $\geq$", otherwise we could not make the following statements without first generalizing the concept of "adjoint". There is an obvious forgetful functor $U: \text{Clo} \longrightarrow \text{Ord}$. As is well known, a left adjoint F of U is determined by a solution to the universal mapping problem: Given any pre-ordered set $\underline{X}$, find a closure system $F(\underline{X})$ and an order preserving mapping $\delta(\underline{X}): \underline{X} \longrightarrow U(F(\underline{X}))$ so that for every closure system $(\underline{A},T)$ and every order preserving mapping $H: \underline{X} \longrightarrow \underline{A}$ there exists a unique strict map $H': F(\underline{X}) \longrightarrow \underline{A}$ for which

$U(H')\ \delta(X) = H.$



It remains to define $\delta(\underline{X})$. We stipulate that $\delta(\underline{X})(X) = X$ for each element X of $\underline{X}$ and verify that $\delta(\underline{X})$ preserves order. Indeed, if $X \leq Y$ in $\underline{X}$, then $X \longrightarrow Y$ is an axiom $D_{X,Y}$ in $D(\underline{X})$, hence provable and, therefore, $X \leq Y$ in $F(\underline{X})$.

Now, assume that $(\underline{A},T)$ is any closure system and $H: \underline{X} \longrightarrow \underline{A}$ any order preserving mapping. Define $H': F(\underline{X}) \longrightarrow (\underline{A},T)$ recursively by $H'(X) = X$ and $H'(T(A)) = T(H'(A))$, or all at once by $H'(T^n(X)) = T^n(H(X))$. It is easily verified that H' is a strict map in Clo, in fact the only strict map for which $U(H')\delta(\underline{X}) = H$.

Here it should be pointed out that it is in fact easy to list all theorems of $D(\underline{X})$. They are precisely the formulas $T^m(X) \longrightarrow T^n(Y)$ where $X \leq Y$ in $\underline{X}$ and $m = 0$ or $m = n$ or $n > 1$. Thus $F(\underline{X})$ is isomorphic to the closure system whose elements are the non-negative integers, with the pre-order relation $m \leq n$ whenever $m = 0$ or $m = n$ or $n > 1$, and for which T is the successor function.

We shall see later, in the context of standard constructions, that $\delta(\underline{X})$ is <u>full</u> in the sense that $X \leq Y$ in $F(\underline{X})$ only when $X \leq Y$ in $\underline{X}$. It will also follow from what will be said later that Clo is equational (tripleable) over Ord.

A <u>standard construction</u> is a quadruple $(\underline{A},T,\eta,\mu)$ where $\underline{A}$ is a category, $T\colon \underline{A} \longrightarrow \underline{A}$ is a functor, and $\eta\colon 1_{\underline{A}} \longrightarrow T$ and $\mu\colon T^2 \longrightarrow T$ are natural transformations satisfying $\mu\cdot(\eta T) = 1_T = \mu\cdot(T\eta)$ and $\mu\cdot(\mu T) = \mu\cdot(T\mu)$. By Trip we shall mean the category whose objects are small standard constructions and whose maps, called <u>strict</u> maps, say from $(\underline{A},T,\eta,\mu)$ to $(\underline{B},T,\eta,\mu)$, are those functors $H\colon \underline{A} \longrightarrow \underline{B}$ which preserve the structure exactly, in the sense that

$$HT = TH,\ H\eta = \eta H,\ H\mu = \mu H\ .$$

What has been said in apology for using strict maps in Clo also applies here.

We wish to construct a left adjoint to the obvious forgetful functor $U\colon$ Trip $\longrightarrow$ Cat. To this purpose we slightly modify the deductive system $D(\underline{X})$ used earlier. Now $\underline{X}$ may be any small category, not just a pre-ordered set, and we must speak of the objects rather than of the elements of $\underline{X}$.

Also the axiom $D_{X,Y}$ is replaced by a family of axioms

$$D_f\colon X \longrightarrow Y\ ,$$

one for each map $f\colon X \longrightarrow Y$ in $\underline{X}$. It is understood that $D_f \neq D_g$ whenever $f \neq g$.

We now construct the free standard construction $F(\underline{X})$ together with a functor $\delta(\underline{X})\colon \underline{X} \longrightarrow U(F(\underline{X}))$. The objects of $F(\underline{X})$ are the terms of $D(\underline{X})$. The maps of $F(\underline{X})$ are equivalence classes of proofs of $D(\underline{X})$, according to a certain equivalence relation $\equiv$ between proofs, to be defined presently. We write $[P]$ for the set of all proofs $Q \equiv P\colon A \longrightarrow B$. Identity maps, composition of maps, T, η, μ, and $\delta(\underline{X})$ are defined thus:

$$1_A = [1_A],\ [P][Q] = [PQ],\ T([P]) = [T(P)],$$
$$\eta(A) = [h(A)],\ \mu(A) = [m(A)],$$
$$\delta(\underline{X})(X) = X,\ \delta(\underline{X})(f) = [D_f].$$

It remains to specify the equivalence relation $\equiv$. We take it to be the smallest transitive, symmetric, and reflexive relation on the set of proofs of $D(\underline{X})$ which satisfies the following rules:

(1) The substitution rules

$$\frac{P \equiv Q \quad R \equiv S}{PR \equiv QS}\ ,\qquad \frac{P \equiv Q}{T(P) \equiv T(Q)}\ ,$$

because only by virtue of these rules are the above

definitions of $[P][Q]$ and $T([P])$ justified.

(2) Rules which allow us to conclude that $UF(\underline{X})$ is a category:

$$1_B P \equiv P, \quad P 1_A \equiv P, \quad (PQ)R \equiv P(QR) .$$

The last rule, for example, asserts the equivalence of the following two proofs:

$$\frac{\dfrac{B \xrightarrow{Q} C \quad C \xrightarrow{P} A}{A \xrightarrow{R} B \quad B \longrightarrow D}}{A \longrightarrow D} \quad , \quad \frac{\dfrac{A \xrightarrow{R} B \quad B \xrightarrow{Q} C}{A \longrightarrow C \quad C \xrightarrow{P} D}}{A \longrightarrow D} .$$

(3) Rules which assure that T is a functor:

$$T(1_A) \equiv 1_{T(A)} , \quad T(PQ) \equiv T(P)T(Q) .$$

(4) Rules which assure that η and μ are natural:

$$T(P)h(A) \equiv h(B)P , \quad T(P)m(A) \equiv m(B)T^2(P) .$$

(5) Rules which assure the commutativity conditions for η and μ:

$$m(A)h(T(A)) \equiv 1_{T(A)} \equiv m(A)T(h(A)) ,$$

$$m(A)T(m(A)) \equiv m(A)m(T(A)) .$$

(6) Rules which assure that $\delta(\underline{X})$ is a functor:

$$D_{1_X} \equiv 1_X , \quad D_{fg} \equiv D_f D_g .$$

PROPOSITION 1. (F,δ) is a solution of the universal mapping problem for U; that is, for each small

category $\underline{X}$, each small standard constructions $(\underline{A},T,\eta,\mu)$, and each functor $H: \underline{X} \longrightarrow \underline{A}$, there exists a unique strict map $H': F(\underline{X}) \longrightarrow (\underline{A},T,\eta,\mu)$ such that $U(H')\ \delta(\underline{X}) = H$.

Proof. Given $H: \underline{X} \longrightarrow \underline{A}$, we construct H' as follows:

$$H'(X) = H(X),\ H'(T(A)) = T(H'(A)) .$$

Before defining $H'([P])$, we define $H'(P)$ for each proof P in $D(\underline{X})$ recursively thus:

$$H'(1_A) = 1_{H'(A)},\ H'(PQ) = H'(P)\ H'(Q),\ H'(T(P)) = T(H'(P)),$$

$$H'(h(A)) = \eta(H'(A)),\ H'(m(A)) = \mu(H'(A)),\ H'(D_f) = H(f) .$$

It is easily verified that

$$P \equiv Q \Longrightarrow H'(P) = H'(Q) ,$$

by induction on the length of the proof of equivalence. Consequently, we are justified in defining

$$H'([P]) = H'(P) .$$

Moreover, one quickly verifies that

$$H'(T(A)) = T(H'(A)),\ H'(T([P]) = T(H'([P]),$$
$$H'(\eta(A)) = \eta(H'(A)),\ H'(\mu(A)) = \mu(H'(A)),$$

hence H' is indeed a strict map in Trip.

Finally

$$U(H')\ \delta(\underline{X})(X) = H(X),\ U(H')\ \delta(\underline{X})(f) = H(f)\ ,$$

and these equations force the definition of H'.

REMARK 1. As is well known, F can be made into a functor and δ into a natural transformation such that F becomes the left adjoint of U with adjunction $\delta\colon 1 \longrightarrow UF$. In the present situation one can easily extend the isomorphism $[F(\underline{X}), (\underline{A},T,\eta,\mu)] \cong [\underline{X},\underline{A}]$ in Ens to one in Cat. However this is not typical for structured categories and depends on the fact that T is a covariant functor.

REMARK 2. It may easily be shown at this stage that Trip is equational (tripleable) over Cat by verifying Beck's conditions. However the verification is too easy in this example to be typical, and we shall defer it to another structured category.

PROPOSITION 2. $\delta(\underline{X})$ is full and faithful.

Proof. (a) Given a proof $P\colon X \longrightarrow Y$ in $D(\underline{X})$, we claim that $P \equiv D_f$ for some map $f\colon X \longrightarrow Y$ in $\underline{X}$. Actually we shall show more generally that any proof $P\colon A \longrightarrow Y$ has this form. We proceed by induction on the length of P.

If $P = 1_Y$, then surely $P \equiv D_{1_Y}$. If $P = D_f$, there is nothing to prove. $P = T(A)$, $h(A)$, or $m(A)$ are all ruled out. There only remains $P = QR$, where $R: A \longrightarrow B$ and $Q: B \longrightarrow Y$. By inductional assumption we have $Q \equiv D_f$ for some $f: Z \longrightarrow Y$. Hence, again by inductional assumption, $R \equiv D_g$ for some $g: X \longrightarrow Z$. Therefore $P \equiv D_f D_g \equiv D_{fg}$.

An easier proof of (a) will be given later.

(b) Suppose $D_f \equiv D_g$, we wish to prove that $f = g$. We need a definition and a lemma. $\underline{2}$ is the category with two objects 0 and 1, and one non-identity map $0 \longrightarrow 1$.

DEFINITION. The generality of a proof P in $D(\underline{X})$ is the set of all functors $H: \underline{2} \longrightarrow \underline{X}$ for which there exists a proof P' in $D(\underline{2})$ such that $P = H(P')$. By the last equation we mean that P results from P' if all occurrences of 0, 1, and $\longrightarrow$ in P' are replaced by $H(0)$, $H(1)$, and $H(\longrightarrow)$ respectively.

LEMMA 1. Equivalent proofs have the same generality.

Assuming this for the moment, we shall continue with (b). Let $H: \underline{2} \longrightarrow \underline{X}$ be defined by

$$H(0) = X,\ H(1) = Y,\ H(\longrightarrow) = f\ .$$

Then $H(D_{\longrightarrow}) = D_f$. Therefore, by the lemma, there exists a map $x: 0 \longrightarrow 1$ in $\underline{2}$ such that $H(D_x) = D_g$. Clearly, $x = \longrightarrow$, hence $f = g$. It remains to prove the lemma.

Proof sketched. We proceed by induction on the length of the proof that the given proofs are equivalent. For example, assume that the last step in the proof of equivalence was a substitution rule, say

$$\frac{P \equiv Q}{T(P) \equiv T(Q)}$$

By inductional assumption, P and Q have the same generality. Let $H: \underline{2} \longrightarrow \underline{X}$ and $H(P^*) = P$, where $P^*: T(A') \longrightarrow T(B')$ in $D(\underline{2})$. Then $P^* = T(P')$, where $P': A' \longrightarrow B'$ and $H(P') = P$. Therefore, there exists $Q': A' \longrightarrow B'$ with $H(Q') = Q$, hence $H(T(Q')) = T(H(Q')) = T(Q)$.

We hasten to point out that the converse of the above lemma does not hold. For example, the proof

$$\frac{T^2(A) \xrightarrow{m(A)} T(A) \qquad T(A) \xrightarrow{h(T(A))} T^2(A)}{T^2(A) \longrightarrow T^2(A)}$$

is not equivalent to the axiom $T^2(A) \longrightarrow T^2(A)$, although it is easily seen to have the same generality. Indeed, supposing

$$h(T(A))\ m(A) \equiv 1_{T^2(A)} \qquad \ldots\ldots (7)$$

we should have

$$\eta(T(A))\ \mu(A) = 1_{T^2(A)}$$

in $F(\underline{X})$. But then it would easily follow that $T^2(A) \cong T(A)$ not only in $F(\underline{X})$, but in every small standard construction. To see that this is not so, take for example $\underline{A}$ = the category of countable sets, $T(A)$ = the underlying set of the free group generated by A, etc.

A standard construction is called <u>idempotent</u> if $\eta T \cdot \mu = 1_{T^2}$. For idempotent constructions we have the same deductive system as for standard constructions, but one additional rule to be satisfied by the equivalence relation on proofs. We shall see later that the converse of the above lemma does hold for idempotent constructions.

We are interested in obtaining a decision procedure for the deductive system $D(\underline{X})$: Given terms A and B, can we find all proofs of the formula $A \longrightarrow B$ at least up to equivalence? Of course this presumes that all maps $f: X \longrightarrow Y$ in $\underline{X}$ are known. Note that a decision procedure for the mere existence of proofs is easy: $T^m(X) \longrightarrow T^n(Y)$ is a theorem if and only if there exists a map $f: X \longrightarrow Y$ and $m = 0$ or $m = n$ or $n > 1$.

First we shall replace $D(\underline{X})$ by a Gentzen-type deductive system $G(\underline{X})$ with the same terms and formulas.

However $G(\underline{X})$ has the following axioms and rules of inference:

$$D_f\colon X \longrightarrow Y \quad \text{when} \quad f\colon X \longrightarrow Y \quad \text{in} \quad \underline{X}\ ,$$

$$\frac{A \xrightarrow{P} B}{A \xrightarrow{r(P)} T(B)} \quad , \quad \frac{A \xrightarrow{P} T(B)}{T(A) \xrightarrow{l(P)} T(B)}\ .$$

These are derived rules in $D(\underline{X})$. In fact the proofs

$$\frac{A \longrightarrow B \quad B \longrightarrow T(B)}{A \longrightarrow T(B)} \quad , \quad \frac{\dfrac{A \longrightarrow T(B)}{T(A) \longrightarrow T^2(B)} \quad T^2(B) \longrightarrow T(B)}{T(A) \longrightarrow T(B)}$$

show that we may define

$$r(P) = h(B)P \quad , \quad l(P) = m(B)T(P)\ .$$

In view of these definitions, each proof of $G(\underline{X})$ may be expanded to a uniquely determined proof of $D(\underline{X})$. In the converse direction we have the following.

PROPOSITION 3. Each proof in $D(\underline{X})$ is equivalent to (the expansion of) a proof in $G(\underline{X})$.

Proof. For any proof $P\colon A \longrightarrow B$ in $G(\underline{X})$, we have

$$T(P) \equiv 1_{T(B)}T(P) \equiv m(B)T(h(B))T(P) \equiv m(B)T(h(B)P) = l(r(P))\ .$$

Next we shall prove by induction on the number of occurrences of T in A that 1_A is equivalent to a proof in $G(\underline{X})$.

Surely $1_X \equiv D_{1_X}$. Now assume the result for A, say $1_A \equiv P_A$ in $G(\underline{X})$, then

$$1_{T(A)} \equiv T(1_A) \equiv T(P_A) \equiv 1(r(P_A))$$

by the above. Again, we have

$$h(A) \equiv h(A)1_A = r(1_A) \equiv r(P_A) ,$$
$$m(A) \equiv m(A)1_{T^2(A)} \equiv m(A)T(1_{T(A)}) = 1(1_{T(A)}) \equiv 1(P_{T(A)}) .$$

There only remains transitivity. This is handled by the following:

CUT ELIMINATION THEOREM. If $Q: A \longrightarrow B$ and $P: B \longrightarrow C$ are proofs in $G(\underline{X})$, then PQ is equivalent to a proof in $G(\underline{X})$.

Proof. We proceed by induction on the number of occurrences of T in A, B, C.

Case 1. Both premises are instances of D_f. Then $D_f D_g \equiv D_{fg}$.

Case 2. The last step in the proof of the second premise uses r. Then we have

$$\dfrac{A \xrightarrow{Q} B \qquad \dfrac{B \xrightarrow{R} C}{B \longrightarrow T(C)}}{A \longrightarrow T(C)} \equiv \dfrac{\dfrac{A \xrightarrow{Q} B \qquad B \xrightarrow{R} C}{A \longrightarrow C}}{A \longrightarrow T(C)}$$

since

$$r(R)Q = (h(C)R)Q \equiv h(C)(RQ) = r(RQ) \ .$$

By inductional assumption RQ is equivalent to some proof in $G(\underline{X})$.

Case 3. The last steps in the proofs of both premises use 1. Then we have

$$\dfrac{\dfrac{A \xrightarrow{S} T(B)}{T(A) \longrightarrow T(B)} \quad \dfrac{B \xrightarrow{R} T(C)}{T(B) \longrightarrow T(C)}}{T(A) \longrightarrow T(C)} \equiv \dfrac{A \xrightarrow{S} T(B) \quad \dfrac{B \xrightarrow{R} T(C)}{T(B) \longrightarrow T(C)}}{\dfrac{A \longrightarrow T(C)}{T(A) \longrightarrow T(C)}}$$

since

$$\begin{aligned} 1(R)1(S) &= m(D)T(R)\ m(B)T(S) \\ &\equiv m(D)\ m(T(D))T^2(R)T(S) \\ &\equiv m(D)\ T(m(D)T(R)S) \\ &= 1(1(R)S) \ . \end{aligned}$$

By inductional assumption $1(R)S$ is equivalent to some proof in $G(\underline{X})$.

Case 4. The last step in the proof of the first premise uses r and the last step in the proof of the second premise uses 1. Then we have

$$\dfrac{A \longrightarrow T(B) \quad \dfrac{A \xrightarrow{S} B \quad B \xrightarrow{R} T(C)}{T(B) \longrightarrow T(C)}}{A \longrightarrow T(C)} \equiv \dfrac{A \xrightarrow{S} B \quad B \xrightarrow{R} T(C)}{A \longrightarrow T(C)}$$

since

$$1(R)r(S) = m(C)T(R)h(B)S \equiv m(C)h(T(C)RS \equiv RS \ .$$

By inductional assumption RS is equivalent to a proof in $G(\underline{X})$.

The proof of the theorem is now complete.

The advantage of the system $G(\underline{X})$ over the system $D(\underline{X})$ is this: given a formula $A \longrightarrow B$, we can find all possible proofs of this formula in a finite number of steps, by working backwards.

To give one application let us present a new proof of Proposition 2(b); that is, that $\delta(\underline{X})$ is full. Assume that $P: X \longrightarrow Y$ in $D(\underline{X})$. By Proposition 3, P is equivalent to some proof Q in $G(\underline{X})$. Since T does not occur in the formula $X \longrightarrow Y$, Q must be D_f for some $f: X \longrightarrow Y$.

As another application, one may translate the recursive description of the proofs in $G(\underline{X})$ into a recursive description of the maps in $F(\underline{X})$. It then easily follows that $F(\underline{X}) = F(\underline{1}) \times X$. This result is due to Bill Lawvere, who also observed that $F(\underline{1})$ is isomorphic with the category of finite ordinals and order preserving maps.

We are now in a position to obtain the converse of Lemma 1 for idempotent constructions.

COHERENCE THEOREM FOR IDEMPOTENT CONSTRUCTIONS. Two proofs in $D(\underline{X})$ with the same generality are equivalent.

Proof. Since each proof in $D(\underline{X})$ is equivalent to one in $G(\underline{X})$ and therefore has the same generality, we need only consider proofs in $G(\underline{X})$.

Let $P: A \longrightarrow B$ be a proof in $G(\underline{X})$. It contains exactly one axiom $D_f: X \longrightarrow Y$. Define $H: \underline{2} \longrightarrow \underline{X}$ by

$$H(0) = X,\ H(1) = Y,\ H(\longrightarrow) = f\ .$$

Let P' be the proof in $G(\underline{2})$ obtained from P by replacing f by $\longrightarrow$. Then $H(P') = P$. Now, assume that $Q: A \longrightarrow B$ has the same generality as P. Then there exists a proof Q^* in $D(\underline{2})$ such that $H(Q^*) = Q$. But, by Proposition 3, Q^* is equivalent to some proof Q' in $G(\underline{2})$, hence, also $H(Q^*) \equiv H(Q')$. It follows from the lemma below that $P' \equiv Q'$, hence

$$P = H(P') \equiv H(Q') \equiv H(Q^*) = Q\ .$$

LEMMA 2. For idempotent constructions, any two proofs of $T^m(0) \longrightarrow T^n(1)$ in $G(\underline{2})$ are equivalent.

Proof. We proceed by induction on $m + n$. Aside from symmetry, we need only consider the following cases:

$$(P,Q) = (D_f,D_g),\ (1(P'),1(Q')),\ (r(P'),r(Q')),\ (1(P'),r(Q'))\ .$$

In the first case f and g are maps $0 \longrightarrow 1$, hence $f = g$.

In the second case we have $P' \equiv Q'$ by inductional assumption, hence $P \equiv Q$.

The third case is similar to the second.

We shall consider the fourth case after the following:

SUBLEMMA. If $T(A) \longrightarrow B$ is provable in $G(\underline{X})$ then $B = T(C)$ and $A \longrightarrow B$ is also provable.

Proof. This is shown by induction on the length of the proof of $T(A) \longrightarrow B$. If the last step in this proof is 1, there is nothing to show, so we may assume it is r. Then $B = T(C)$ and $T(A) \longrightarrow C$ is a theorem. By inductional assumption, $C = T(D)$ and $A \longrightarrow C$ is provable, hence so is $A \longrightarrow T(C) = B$.

We now return to the fourth case in the proof of Lemma 2.

We have proofs $P = 1(P')$ and $Q = r(Q')$ of $T(A) \longrightarrow T(B)$ in $G(\underline{2})$, where $P'\colon A \longrightarrow T(B)$ and $Q'\colon T(A) \longrightarrow B$. In view of the sublemma, $B = T(C)$ and there is a proof $R\colon A \longrightarrow B$. By inductional assumption, $P' \equiv r(R)$ and $Q' \equiv 1(R)$,

hence,

$$P \equiv l(r(R)) = m(B)T(T(R)h(A)) \equiv m(B)T^2(R)T(h(A)) \equiv T(R)m(A)T(h(A)) \equiv T(R)$$

and

$$Q \equiv r(l(R)) = h(B)m(C)T(R) = h(T(C))m(C)T(R) \equiv T(R)$$

by the idempotent rule (7). This is the only place where the idempotent rule comes in.

2. CLOSED CATEGORIES

We might as well start with a bang and define a biclosed monoidal category as a 10-tuple $(\underline{A}, I, \cdot, /, \backslash, \rho, \lambda, \alpha, \beta, \gamma)$ where

$\underline{A}$ is a category,

I is an object of $\underline{A}$,

$\cdot : \underline{A} \times \underline{A} \longrightarrow \underline{A}$, $/ : \underline{A} \times \underline{A}^{opp} \longrightarrow \underline{A}$, and $\backslash : \underline{A}^{opp} \times \underline{A} \longrightarrow \underline{A}$ are bifunctors,

$\rho, \lambda, \alpha, \beta, \gamma$ are natural isomorphisms:

$$\rho(A) : A \cdot I \longrightarrow A, \quad \lambda(A) : I \cdot A \longrightarrow A ,$$

$$\alpha(A,B,C) : (A \cdot B) \cdot C \longrightarrow A \cdot (B \cdot C) ,$$

$$\beta(A,B,C) : [A \cdot B, C] \longrightarrow [A, C/B] ,$$

$$\gamma(A,B,C) : [A \cdot B, C] \longrightarrow [B, A \backslash C] ,$$

such that the following composite maps are identities:

$$A\cdot B \longrightarrow (A\cdot I)\cdot B \longrightarrow A\cdot(I\cdot B) \longrightarrow A\cdot B\ ,$$

$$((A\cdot B)\cdot C)\cdot D \longrightarrow (A\cdot B)\cdot(C\cdot D) \longrightarrow A\cdot(B\cdot(C\cdot D)) \longrightarrow$$

$$A\cdot((B\cdot C)\cdot D) \longrightarrow (A\cdot(B\cdot C))\cdot D \longrightarrow ((A\cdot B)\cdot C)\cdot D\ .$$

Some historical remarks are in order. Monoidal categories $(\underline{A}, I, \cdot, \rho, \lambda, \alpha)$ were studied by Benabou (1963). The last two "coherence" conditions were discussed by MacLane (1963). Closed monoidal categories $(\underline{A}, I, \cdot, \backslash, \rho, \lambda, \alpha, \gamma)$ were studied by Eilenberg and Kelly (1966). Residuated categories $(\underline{A}, \cdot, /, \backslash, \alpha, \beta, \gamma)$ were studied in Deductive Systems and Categories (I). However one cannot strip off $\cdot$ and α without making some other changes, as is clear from the paper by Eilenberg and Kelly. The deductive system corresponding to residuated categories had already been studied in 1958 with a view of applications to mathematical linguistics.

With minor changes the proofs in Deductive Systems and Categories (I) go through to establish the following results:

(1) A free biclosed monoidal category $F(\underline{X})$ together with a functor $\delta(\underline{X}): \underline{X} \longrightarrow U(F(\underline{X}))$ can be constructed for any small category $\underline{X}$ using a suitable deductive system $D(\underline{X})$.

(2) $\delta(\underline{X})$ is a full embedding.

(3) $D(\underline{X})$ may be replaced by a Gentzen-type system $G(\underline{X})$, allowing one to find all proofs of a formula $A \longrightarrow B$ in $D(\underline{X})$ up to equivalence.

(4) Two proofs of $A \longrightarrow B$ in $D(\underline{X})$ are equivalent if and only if they have the same generality.

To explain the concept of generality (called "scope" in the paper referred to) let us consider two possible definitions.

DEFINITION A. The Generality of a proof $P: A \longrightarrow B$ in $D(\underline{X})$ is the class of all pairs $(\underline{X}', H)$ where $\underline{X}'$ is any small category and $H: \underline{X}' \longrightarrow \underline{X}$ is a functor such that there exists a proof P' in $D(\underline{X}')$ with $H(P') = P$.

Here $H(P')$ is the proof obtained from P by replacing each object and each map of $\underline{X}'$ occurring in P' by its image under H.

It turns out that when testing whether two proofs have the same Generality we need not look at all small categories $\underline{X}'$ but only at categories of the form $n\underline{2}$ (sum of n disjoint copies of $\underline{2}$), where n depends on the width of the given proofs. We make the following alternative definition.

DEFINITION B. The generality of a proof $P: A \longrightarrow B$ in $D(\underline{X})$ is the set of all pairs $(\underline{X}', H)$ where $\underline{X}' = n\underline{2}$ for some finite n and . . . . . .

We note that (3) and (4) together allow us to compute $[A,B]$ in $F(\underline{X})$, assuming of course that $[X,Y]$ in $\underline{X}$ is known.

A functor H between biclosed monoidal categories is called a strict map if it preserves the structure exactly, that is,

$$H(I) = I, H(\cdot) = H(\ ) \cdot H(\ ), \text{ etc.}$$

$$H\rho = \rho H, \text{ etc, } H(\alpha(A,B,C)) = \alpha(H(A), H(B), H(C)),$$

$$H(\beta(A,B,C)(f)) = \beta(H(A), H(B), H(C))(H(f)), \text{ etc.}$$

Let BMC denote the category of small biclosed monoidal categories, then the forgetful functor $U: BMC \longrightarrow Cat$ has a left adjoint F. We may now ask whether BMC consists of exactly the algebras of the standard construction $(UF, \delta, -)$ in the sense of Beck's Precise Tripleableness Theorem. Beck's conditions, as modified by Paré, demand this:

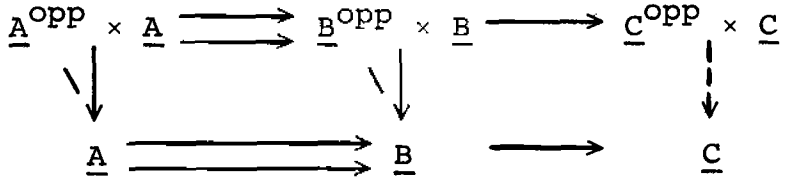
(1) U reflects isomorphisms.

(2) Given two strict maps $H_1, H_2: (\underline{A}, \ldots) \longrightarrow (\underline{B}, \ldots)$ in BMC and a functor $K: \underline{B} \longrightarrow \underline{C}$ such that $\underline{A} \rightrightarrows \underline{B} \longrightarrow \underline{C}$ is an absolute coequalizer diagram (that is, $L(\underline{A}) \rightrightarrows L(\underline{B}) \longrightarrow L(\underline{C})$ is a coequalizer diagram for every functor L with domain Cat), then the pair H_1, H_2 has a

coequalizer K' and $U(K') \cong K$ canonically.

We shall not give here the complete proof that BMC does indeed satisfy the Beck-Paré conditions, but we shall look at a crucial part of the argument for condition (2).

Assume that H_1, H_2, and K are given and consider the following diagram:



Since H_1 and H_2 are strict maps, the two squares on the left commute. Since the bottom is an absolute coequalizer diagram, the top is a coequalizer diagram. It follows that there is a unique functor $\underline{C}^{opp} \times \underline{C} \longrightarrow \underline{C}$ to make the right square commute, let us also denote it by $\backslash$. Then K preserves $\backslash$ exactly.

We may deal similarly with the functors $\cdot$ and $/$, but what about the natural transformations ρ to γ? The trick is to view these also as functors, thus for example $\beta: \underline{A}^{opp} \times \underline{A}^{opp} \times \underline{A} \longrightarrow Ens^2$ is the functor which, when followed by the canonical functor $Ens^2 \longrightarrow Ens \times Ens$ assigning to each map $f: X \longrightarrow Y$ the pair (X,Y) yields the functor $\underline{A}^{opp} \times \underline{A}^{opp} \times \underline{A} \longrightarrow Ens \times Ens$ whose value at

(A,B,C) is $([A\cdot B,C],[A,C/B])$.

We shall now turn to another way of studying closed categories by viewing them as multicategories. The connection between Gentzen's method in logic and multilinear mappings in algebra has been pointed out before (1961). I am told that multicategories have also been studied by Benabou and Cartier.

A multicategory consists of a class of objects together with a class of multimaps

$$g\colon A_1, A_2, \ldots, A_n \longrightarrow B\ ,$$

n being any non-negative integer. Among the multimaps are the identity maps $1_A\colon A \longrightarrow A$. Multimaps may be composed by "substitution" as follows: Given multimaps

$$g\colon A_1, \ldots, A_n \longrightarrow B,\quad f\colon \ldots, A, \ldots \longrightarrow B\ ,$$

there is a multimap

$$f(\ldots,g,\ldots)\colon \ldots, A_1, \ldots, A_n, \ldots \longrightarrow B\ .$$

Substitution, also called cut, must satisfy four conditions. These will be stated as soon as we have introduced a convenient notation.

We shall use capital Greek letters to stand for finite sequences of objects.

Thus a multimap may be denoted by $g: \Lambda \longrightarrow B$. We write $\emptyset$ for the empty sequence, for which $n = 0$. A multimap $b: \emptyset \longrightarrow B$ may also be regarded as an element of B. If Φ and Ψ are finite sequences of objects, their juxtaposition is denoted by Φ,Ψ. The reader should ignore the comma when Φ or Ψ is the empty sequence. The cut now takes the form

$$\frac{\Lambda \xrightarrow{g} A \quad \Phi,A,\Psi \xrightarrow{f} B}{\Phi,\Lambda,\Psi \longrightarrow B} \ .$$

We shall avoid the notation $f(\Phi,g,\Psi)$ for the resulting multimap as it tends to become troublesome. We now state the four conditions.

(1) The following multimap is equal to f:

$$\frac{A \xrightarrow{1_A} A \quad \Phi,A,\Psi \xrightarrow{f} B}{\Phi,A,\Psi \longrightarrow B} \ .$$

(2) The following multimap is equal to g:

$$\frac{\Lambda \xrightarrow{g} A \quad A \xrightarrow{1_A} A}{\Phi,\Lambda,\Psi \longrightarrow B} \ .$$

(3) The following multimaps are equal:

$$\frac{\Lambda \longrightarrow A \quad \Phi,A,\Psi \longrightarrow B}{\dfrac{\Phi,\Lambda,\Psi \longrightarrow B \quad \Gamma,B,\Delta \longrightarrow C}{\Gamma,\Phi,\Lambda,\Psi,\Delta \longrightarrow C}} \ , \quad \frac{\dfrac{\Phi,A,\Psi \longrightarrow B \quad \Gamma,B,\Delta \longrightarrow C}{\Lambda \longrightarrow A \quad \Gamma,\Phi,A,\Psi,\Delta \longrightarrow B}}{\Gamma,\Phi,\Lambda,\Psi,\Delta \longrightarrow C} \ .$$

(4) The following multimaps are equal:

$$\dfrac{\dfrac{\Delta \longrightarrow D \quad \Phi,C,\Theta,D,\Psi \longrightarrow B}{\Gamma \longrightarrow C \quad \Phi,C,\Theta,\Delta,\Psi \longrightarrow B}}{\Phi,\Gamma,\Theta,\Delta,\Psi \longrightarrow C} \quad , \quad \dfrac{\dfrac{\Gamma \longrightarrow C \quad \Phi,C,\Theta,D,\Psi \longrightarrow B}{\Delta \longrightarrow D \quad \Phi,\Gamma,\Theta,D,\Psi \longrightarrow B}}{\Phi,\Gamma,\Theta,\Delta,\Psi \longrightarrow C} \quad .$$

We may call (1) and (2) identity laws, (3) the associative law, and (4) the commutative law for multicategories.

We denote the class of all multimaps $\Lambda \longrightarrow B$ by $[\Lambda;B]$. The reader will appreciate the reason for use of the semicolon.

EXAMPLES

1. Any category becomes a multicat if we put $[A_1, \ldots, A_n; B] = \emptyset$ for $n \neq 1$.

2. Any monoidal category becomes a multicat if we define recursively

$$[\emptyset;B] = [I;B] ,$$

$$[\Gamma;B] = [C;B] \Longrightarrow [\Gamma,A;B] = [C\cdot A;B] .$$

Thus, for example,

$$[A,B,C;D] = [(A\cdot B)\cdot C;D] .$$

3. Any (left) closed category becomes a multicat if we define recursively

$$[\emptyset;B] = [I;B], \quad [C,\Gamma;B] = [\Gamma;C\backslash B] .$$

Thus, for example,

$$[A,B,C;D] = [C;B\backslash(A\backslash D)]$$
$$= [I;C\backslash(B\backslash(A\backslash D))] .$$

We also mention one application. If $\underline{M}$ is any given multicategory, we may define $\underline{M}$-categories, that is categories for which $[A;B]$ is in some sense an object of $\underline{M}$. This may be done in the same way as it was done by Benabou (1965) for monoidal categories.

Certain structures are more easily introduced into multicats than into categories. Thus we may readily define monoidal multicats, left closed multicats, right closed multicats, and biclosed multicats by leaving out unwanted data from the following.

A <u>biclosed monoidal multicategory</u> is a 9-tuple $(\underline{M}, I, \cdot, /, \backslash, i, m, r, l)$ where $\underline{M}$ is a multicat, $\cdot$, $/$, and $\backslash$ are binary operations on the set of objects of $\underline{M}$, and $i: \emptyset \longrightarrow I$, $m: A,B \longrightarrow A\cdot B$, $r: A/B,B \longrightarrow A$, and $l: A,A\backslash B \longrightarrow B$ are multimaps such that the following induced mappings are one-one and onto:

$$[\Phi, I, \Psi; C] \longrightarrow [\Phi, \Psi; C] ,$$
$$[\Phi, A\cdot B, \Psi; C] \longrightarrow [\Phi, A, B, \Psi; C] ,$$
$$[\Gamma; A/B] \longrightarrow [\Gamma, B; A] ,$$
$$[\Gamma; B\backslash A] \longrightarrow [B, \Gamma; A] .$$

The following cuts show how, for example, the first and third of the above mappings are induced:

$$\frac{\emptyset \xrightarrow{i} I \quad \Phi,\ I,\ \Psi \longrightarrow C}{\Phi,\ \Psi \longrightarrow C} \ , \quad \frac{\Gamma \longrightarrow A/B \quad A/B,\ B \xrightarrow{r} A}{\Gamma,\ B \longrightarrow A} \ .$$

A <u>functor</u> $E\colon \underline{M} \longrightarrow \underline{N}$ between multicategories associates with each object A of $\underline{M}$ an object $E(A)$ of $\underline{N}$ and with each multimap $f\colon A_1, \ldots, A_n \longrightarrow B$ a multimap $E(f)\colon E(A_1), \ldots, E(A_n) \longrightarrow B$, and it preserves identities and cuts. Let Mult be the category of small multicategories and functors.

By a <u>strict</u> map E between biclosed monoidal multicategories, we mean a functor which preserves the structure exactly. Thus

$$E(I) = I,\ E(A \cdot B) = E(A) \cdot E(B),\ \text{etc.},\ E(i) = i,$$
$$E(m) = m,\ \text{etc.}$$

Let BMM be the category of small biclosed monoidal multicategories and strict maps. There are obvious functors BMM $\longrightarrow$ BMC and BMC $\longrightarrow$ BMM. They establish an equivalence of categories, but it is rather tedious to show this in detail. Here is a description of these functors, at least on objects.

BMM $\longrightarrow$ BMC. Given any biclosed monoidal multicategory $(\underline{M}, I, \cdot, /, \backslash, i, m, r, l)$, form $(\underline{A}, I, \cdot, /, \backslash, \rho, \lambda, \alpha, \beta, \gamma)$ as follows. Suppress all multimaps which are not maps and define ρ, α, β, β^{-1}, for example, as follows:

$$\dfrac{\dfrac{A \longrightarrow A}{A,I \longrightarrow A}}{A \cdot I \longrightarrow A} \quad , \qquad \dfrac{\dfrac{\dfrac{B,C \longrightarrow B \cdot C \quad A, B \cdot C \longrightarrow A \cdot (B \cdot C)}{A,B,C \longrightarrow A \cdot (B \cdot C)}}{A \cdot B, C \longrightarrow A \cdot (B \cdot C)}}{(A \cdot B) \cdot C \longrightarrow A \cdot (B \cdot C)} \quad ,$$

$$\dfrac{\dfrac{A,B \longrightarrow A \cdot B \quad A \cdot B \longrightarrow C}{A,B \longrightarrow C}}{A \longrightarrow C/B} \quad , \qquad \dfrac{\dfrac{A \longrightarrow C/B \quad C/B, B \longrightarrow C}{A,B \longrightarrow C}}{A \cdot B \longrightarrow C} \quad .$$

BMC $\longrightarrow$ BMM. Given any biclosed monoidal category $(\underline{A}, I, \cdot, /, \backslash, \rho, \lambda, \alpha, \beta, \gamma)$, form $(\underline{M}, I, \cdot, /, \backslash, m, r, l)$ as follows. Define $A_1, \ldots, A_n \longrightarrow B$ as in Example 2 above. Let

$$\begin{aligned} i\colon \emptyset \longrightarrow I \quad &\text{mean} \quad 1_I\colon I \longrightarrow I \ , \\ m\colon A,B \longrightarrow A \cdot B \quad &\text{mean} \quad 1_{A \cdot B}\colon A \cdot B \longrightarrow A \cdot B \ , \\ r\colon A/B,\ B \longrightarrow A \quad &\text{mean} \quad \beta^{-1}(1_{A/B})\colon A/B \cdot B \longrightarrow A \end{aligned}$$

and similarly for l.

We shall now turn our attention to BMM. Given any small multicategory $\underline{X}$, we can form a deductive system $M(\underline{X})$ whose terms are the objects of $\underline{X}$; I, and $A \cdot B$, A/B, $A\backslash B$ if A and B are terms; and whose formulas have the form

$A_1, A_2, \ldots, A_n \longrightarrow B$ where the A_i and B are terms. Its axioms and rules of inference are as follows:

$$1_A\colon A \longrightarrow A \ ; \quad \text{the cut;} \quad M_f\colon X_1, \ldots, X_n \longrightarrow Y \ ,$$
$$\text{whenever} \quad f\colon X, \ldots, X_n \longrightarrow Y \quad \text{in} \quad \underline{X} \ ;$$
$$\emptyset \longrightarrow I; \ A,B \longrightarrow A\cdot B; \ A/B,B \longrightarrow A; \ A, A\backslash B \longrightarrow B \ ;$$

$$\frac{\Phi,\Psi \longrightarrow C}{\Phi,I,\Psi \longrightarrow C} \ ; \quad \frac{\Phi,A,B,\Psi \longrightarrow C}{\Phi,A\cdot B,\Psi \longrightarrow C} \ ; \quad \frac{\Lambda,B \longrightarrow A}{\Lambda \longrightarrow A/B} \ ; \quad \frac{B,\Lambda \longrightarrow A}{\Lambda \longrightarrow B\backslash A} \ .$$

It is clear that we can introduce a suitable equivalence relation on $M(\underline{X})$ and use it to construct the free bi-closed monoidal multicategory $F(\underline{X})$ generated by $\underline{X}$ and a functor $\delta(\underline{X})\colon \underline{X} \longrightarrow U(F(\underline{X}))$ showing that F is the left adjoint of U. Again, BMM will be equational over Mult.

We shall now replace $M(\underline{X})$ by a Gentzen-type deductive system $G(\underline{X})$. Its terms and formulas are the same as those of $M(\underline{X})$. Here are its axioms and rules of inference:

$$M_f\colon X_1, \ldots, X_n \longrightarrow Y$$
$$\text{whenever} \quad f\colon X_1, \ldots, X_n \longrightarrow Y \quad \text{in} \quad \underline{X} \ ,$$

$$\longrightarrow I \qquad \emptyset \longrightarrow I \qquad \frac{\Phi,\Psi \longrightarrow C}{\Phi,I,\Psi \longrightarrow C} \qquad I \longrightarrow$$

$$\longrightarrow \cdot \qquad \frac{\Gamma \longrightarrow C \quad \Delta \longrightarrow D}{\Gamma,\Delta \longrightarrow C\cdot D} \qquad \frac{\Phi,A,B,\Psi \longrightarrow C}{\Phi,A\cdot B,\Psi \longrightarrow C} \qquad \cdot \longrightarrow$$

$$\longrightarrow / \quad \frac{\Lambda, B \longrightarrow A}{\Lambda \longrightarrow A/B} \qquad \frac{\Lambda \longrightarrow B \quad \Phi, A, \Psi \longrightarrow C}{\Phi, A/B, \Lambda, \Psi \longrightarrow C} \qquad / \longrightarrow$$

$$\longrightarrow \backslash \quad \frac{B, \Lambda \longrightarrow A}{\Lambda \longrightarrow B \backslash A} \qquad \frac{\Lambda \longrightarrow B \quad \Phi, A, \Psi \longrightarrow C}{\Phi, \Lambda, B \backslash A, \Psi \longrightarrow C} \qquad \backslash \longrightarrow$$

Note that, aside from the axiom M_f which involves none of the symbols $\cdot$, /, \, half the rules serve to introduce one of these symbols on the right and the other half serve to introduce one on the left. The notation $\longrightarrow /$ and $/ \longrightarrow$ is found in Kleene's book. Surely, given a formula in $G(\underline{X})$, we can find all possible proofs of it by working backwards, eliminating one symbol at each step.

PROPOSITION 1. Every rule of $G(\underline{X})$ is a derived rule in $M(\underline{X})$.

Proof. The only new rules are $\longrightarrow \cdot$, $/ \longrightarrow$, and $\backslash \longrightarrow$. Here is how the first two are shown in $M(\underline{X})$:

$$\frac{\Gamma \longrightarrow C \qquad \dfrac{\Delta \longrightarrow D \quad C, D \longrightarrow C \cdot D}{C, \Delta \longrightarrow C \cdot D}}{\Gamma, \Delta \longrightarrow C \cdot D}, \qquad \frac{\Lambda \longrightarrow B \quad \dfrac{A/B, B \longrightarrow A \quad \Phi, A, \Psi \longrightarrow C}{\Phi, A/B, B, \Psi \longrightarrow C}}{\Phi, A/B, \Lambda, \Psi \longrightarrow C}.$$

PROPOSITION 2. Each proof in $M(\underline{X})$ is equivalent to the expansion of a proof in $G(\underline{X})$.

Proof. First we show, by induction on the number of symbols in A, that $A \longrightarrow A$ is a theorem in $G(\underline{X})$ whose proof, when expanded in $M(\underline{X})$, is equivalent to $1_A: A \longrightarrow A$.

Clearly $1_X: X \longrightarrow X$ is equivalent to $M_{1_X}: X \longrightarrow X$, since $\delta(\underline{X})$ is a functor.

$C \cdot D \longrightarrow C \cdot D$ may be proved thus in $G(\underline{X})$:

$$\dfrac{\dfrac{C \xrightarrow{P} C \qquad D \xrightarrow{Q} D}{C,D \longrightarrow C \cdot D}}{C \cdot D \longrightarrow C \cdot D} \quad ,$$

and we may assume, by inductional assumption, that $P \equiv 1_C$ and $Q \equiv 1_D$. Expanding in $M(\underline{X})$, we see that the above is equivalent to

$$\dfrac{\dfrac{C \xrightarrow{1_C} C \qquad \dfrac{D \xrightarrow{1_D} D \qquad C,D \longrightarrow C \cdot D}{C,D \longrightarrow C \cdot D}}{C \cdot D \longrightarrow C \cdot D} \equiv \dfrac{C,D \longrightarrow C \cdot D}{C \cdot D \longrightarrow C \cdot D} \quad ,$$

since $F(\underline{X})$ is a multicategory. Finally, the last proof is equivalent to $1_{C \cdot D}: C \cdot D \longrightarrow C \cdot D$, since $[C \cdot D;\ C \cdot D] \longrightarrow [C,D;\ C \cdot D]$ is one-one and onto.

$C/D \longrightarrow C/D$ may be proved thus in $G(\underline{X})$:

$$\dfrac{\dfrac{D \xrightarrow{Q} D \qquad C \xrightarrow{P} C}{C/D,D \longrightarrow C}}{C/D \longrightarrow C/D} \quad ,$$

and we may assume, by inductional assumption, that $P \equiv 1_C$ and $Q \equiv 1_D$.

Expanding in $M(\underline{X})$, we see that the above is equivalent to

$$\dfrac{\dfrac{1_D}{D \longrightarrow D} \qquad \dfrac{\dfrac{C/D,D \longrightarrow C \qquad \dfrac{C \overset{1_C}{\longrightarrow} C}{}}{C/D,D \longrightarrow C}}{C/D \longrightarrow C/D}}{} \quad \equiv \quad \dfrac{C/D,D \longrightarrow C}{C/D \longrightarrow C/D} \ ,$$

since $F(\underline{X})$ is a multicategory. Finally, the last proof is equivalent to $1_{C/D}: C/D \longrightarrow C/D$, since $[C/D;\ C/D] \longrightarrow [C/D,D;\ C]$ is one-one and onto.

$C\backslash D \longrightarrow C\backslash D$ is treated similarly, thus completing the induction proof that $1_A: A \longrightarrow A$ is equivalent to a proof in $G(\underline{X})$.

$A,B \longrightarrow A\cdot B$ may be proved thus in $G(\underline{X})$:

$$\dfrac{A \longrightarrow A \qquad B \longrightarrow B}{A,B \longrightarrow A\cdot B} \ .$$

Expanding in $M(\underline{X})$ and using the above result, we see that this is equivalent to

$$\dfrac{\dfrac{1_A}{A \longrightarrow A} \qquad \dfrac{\dfrac{1_B}{B \longrightarrow B} \qquad A,B \longrightarrow A\cdot B}{A,B \longrightarrow A\cdot B}}{A,B \longrightarrow A\cdot B} \quad \equiv \quad A,B \longrightarrow A\cdot B \ ,$$

since $F(\underline{X})$ is a multicategory.

$A/B,B \longrightarrow A$ and $B,B\backslash A \longrightarrow A$ are treated similarly. There remains the cut, which will now be dealt with. Let us, for the moment, assume that the cut has been added to the rules of $G(\underline{X})$.

CUT ELIMINATION THEOREM. Every proof in $G(\underline{X})$ with cut is equivalent to a proof without cut.

Proof. We assume that P and Q are proofs of $G(\underline{X})$, expanded into proofs of $M(\underline{X})$, and want to show that the cut

$$\dfrac{\Lambda \xrightarrow{Q} A \qquad \Phi, A, \Psi \xrightarrow{P} B}{\Phi, \Lambda, \Psi \longrightarrow B}$$

is equivalent to a proof with cuts of smaller degree. By the <u>degree</u> of the exhibited cut we mean the total number of occurrences of the symbols I, $\cdot$, $/$, and $\backslash$ in Φ, Λ, Ψ, A, B.

Proof. We distinguish four cases.

Case 1. $P = M_f$ and $Q = M_g$. Then

$$\dfrac{\Lambda \xrightarrow{M_g} X \qquad \Phi, X, \Psi \xrightarrow{M_f} Y}{\Phi, \Lambda, \Psi \longrightarrow B} \quad \equiv \quad \Phi, \Lambda, \Psi \xrightarrow{M_h} B \quad ,$$

where h is the multimap in $\underline{X}$ obtained by substituting g into f, since $\delta(\underline{X})$ is a functor.

Case 2. The last step of Q does not introduce the main operation symbol of A. For example, assume that it is $I \longrightarrow$, then we claim that

$$\dfrac{\dfrac{\Gamma,\Delta \xrightarrow{R} A}{\Gamma,I,\Delta \longrightarrow A} \quad \Phi,A,\Psi \xrightarrow{P} B}{\Phi,\Gamma,I,\Delta,\Psi \longrightarrow B} \;\equiv\; \dfrac{\Gamma,\Delta \xrightarrow{R} A \quad \dfrac{\Phi,A,\Psi \xrightarrow{P} B}{\Phi,\Gamma,\Delta,\Psi \longrightarrow B}}{\Phi,\Gamma,I,\Delta,\Psi \longrightarrow B} \,,$$

where the cut on the right has smaller degree. To verify this, observe that

$$\dfrac{\Gamma,\Delta \longrightarrow A}{\Gamma,I,\Delta \longrightarrow A} \,, \qquad \dfrac{\emptyset \longrightarrow I \quad \Gamma,I,\Delta \longrightarrow A}{\Gamma,\Delta \longrightarrow A}$$

must by "inverses" in $M(\underline{X})$, in view of the isomorphism $[\Gamma,I,\Delta;A] \longrightarrow [\Gamma,\Delta;A]$. Hence, our task is equivalent to showing that

$$\dfrac{\emptyset \longrightarrow I \quad \dfrac{\Gamma,I,\Delta \longrightarrow A \quad \Phi,A,\Psi \longrightarrow B}{\Phi,\Gamma,I,\Delta,\Psi \longrightarrow B}}{\Phi,\Gamma,\Delta,\Psi \longrightarrow B} \;\equiv\; \dfrac{\dfrac{\emptyset \longrightarrow I \quad \Gamma,I,\Delta \longrightarrow A}{\Gamma,\Delta \longrightarrow A} \quad \Phi,A,\Psi \longrightarrow B}{\Phi,\Gamma,\Delta,\Psi \longrightarrow B} \,.$$

But this follows from the associative law for multicategories.

Case 3. The last step of P does not introduce the main operation symbol of A. For example, assume that it is $I \longrightarrow$, then we claim that

$$\dfrac{\Lambda \xrightarrow{Q} A \quad \dfrac{\Gamma,\Delta,A,\Psi \xrightarrow{R} B}{\Gamma,I,\Delta,A,\Psi \longrightarrow B}}{\Gamma,I,\Delta,\Lambda,\Psi \longrightarrow B} \;\equiv\; \dfrac{\Lambda \xrightarrow{Q} A \quad \Gamma,\Delta,A,\Psi \xrightarrow{R} B}{\dfrac{\Gamma,\Delta,\Lambda,\Psi \longrightarrow B}{\Gamma,I,\Delta,\Lambda,\Psi \longrightarrow B}} \,,$$

where the cut on the right has smaller degree. This is treated similarly to Case 2 above, using the commutative law for multicategories.

Case 4. The last steps of both P and Q introduce the main operation symbols of A. For example, assume they are $\longrightarrow I$ and $I \longrightarrow$, then we claim that

$$\dfrac{\emptyset \longrightarrow I \qquad \dfrac{\Phi,\Psi \longrightarrow B}{\Phi,I,\Psi \longrightarrow B}}{\Phi,\Psi \longrightarrow B} \quad \equiv \quad \Phi,\Psi \longrightarrow B \ .$$

This is an immediate consequence of the isomorphism $[\Phi,I,\Psi;B] \longrightarrow [\Phi,\Psi;B]$.

The reader may object that the examples chosen under cases 2 to 3 are hardly typical. However, all other subcases have already been treated in Deductive Systems and Categories (I).

By a <u>testing</u> multicategory we shall mean a category with $n + 1$ objects $1, \ldots, n + 1$ and only one non-identity multimap $1, \ldots, n \longrightarrow n + 1$. By the <u>generality</u> of a proof in $M(\underline{X})$ we mean the set of all pairs $(\underline{X}', H)$ where $\underline{X}'$ is a finite direct sum of testing multicategories and $H\colon \underline{X}' \longrightarrow \underline{X}$ is a functor such that there exists a proof P' in $M(\underline{X}')$ with $H(P') = P$.

PROPOSITION 3. Equivalent proofs in $M(\underline{X})$ have the same generality.

This may easily be shown by induction on the length of the proof that the given proofs are equivalent.

PROPOSITION 4. The functor $\delta(\underline{X}): \underline{X} \longrightarrow UF(\underline{X})$ defined by

$$\delta(\underline{X})(X) = X \ , \quad \delta(\underline{X})(F) = [M_f] \ ,$$

is full and faithful.

The proof is similar to that of the corresponding result for standard constructions and will be omitted.

COHERENCE THEOREM FOR BMM. Two proofs in $M(\underline{X})$ are equivalent only if they have the same generality.

Proof. In view of Proposition 2, it suffices to show that two proofs $P,Q: A \longrightarrow B$ in $G(\underline{X})$ with the same generality are equivalent. Look at all the axioms appearing in P from left to right. Suppose the i-th axiom involves the multimap

$$f_i: X_{i,1}, \ldots, X_{i,n(i)} \longrightarrow X_{i,n(i)+1} \ .$$

Let $\underline{X}'$ be the direct sum of the testing multicategories whose non-identity multimaps are

$$1, \ldots, n(i) \longrightarrow n(i) + 1 \ .$$

If K_i is the injection of the i-th testing multicategory into $\underline{X}'$, let $H: \underline{X}' \longrightarrow \underline{X}$ be determined by

$$HK_i(j) = X_{i,j}, \ HK_i(\longrightarrow) = f_i \ .$$

Now generalize P to a proof P' in $G(\underline{X}')$, replacing f_i by the i-th testing multimap. Then $H(P') = P$. Since P is assumed to have the same generality as Q, we have a proof Q^* in $M(\underline{X})$ such that $H(Q^*) = Q$. Now, by Proposition 2, $Q^* \equiv Q'$ for some proof Q' in $G(\underline{X})$. Therefore, also $H(Q') \equiv Q$, and $P \equiv Q$ will follow from $P' \equiv Q'$. That this is so will be shown in a lemma.

LEMMA. Let $\underline{X}$ be a finite direct sum of testing multicategories and assume that $A \longrightarrow B$ is a formula in $G(\underline{X})$ in which no object of $\underline{X}$ occurs more than once. Then any two proofs P and Q of $A \longrightarrow B$ in $G(\underline{X})$ are equivalent.

Proof. We roll up our sleeves and proceed by induction on the number of operation symbols in $A \longrightarrow B$. After disposing of the preliminary cases $P = M_f$, $Q = M_g$, and $P = \longrightarrow I$, $Q = \longrightarrow I$, we consider the following.

Case 1. Suppose the last step of P is $I \longrightarrow$ and the last step of Q is $\longrightarrow \cdot$. Then, say

$$P = \frac{\Gamma,\Theta,\Delta \xrightarrow{P'} C\cdot D}{\Gamma,I,\Theta,\Delta \longrightarrow C\cdot D} \quad , \quad Q = \frac{\Gamma,I,\Theta \xrightarrow{Q'} C \qquad \Delta \xrightarrow{Q''} D}{\Gamma,I,\Theta,\Delta \longrightarrow C\cdot D} \quad .$$

Now we have in $M(\underline{X})$

$$\frac{\emptyset \longrightarrow I \qquad \Gamma,I,\Theta \xrightarrow{Q'} C}{\Gamma,\Theta \longrightarrow C} \quad ,$$

hence $\Gamma,\Theta \longrightarrow C$ also has a proof Q^* in $G(\underline{X})$.

By inductional assumption we may put

$$P' \equiv \dfrac{\Gamma,\Theta \overset{Q^*}{\longrightarrow} C \quad \Delta \overset{Q''}{\longrightarrow} D}{\Gamma,\Theta,\Delta \longrightarrow C\cdot D}\ , \qquad Q' \equiv \dfrac{\Gamma,\Theta \overset{Q^*}{\longrightarrow} C}{\Gamma,I,\Theta \longrightarrow C}\ .$$

Thus, we wish to show that

$$\dfrac{\dfrac{\Gamma,\Theta \longrightarrow C \quad \Delta \longrightarrow D}{\Gamma,\Theta,\Delta \longrightarrow C\cdot D}}{\Gamma,I,\Theta,\Delta \longrightarrow C\cdot D} \equiv \dfrac{\dfrac{\Gamma,\Theta \longrightarrow C}{\Gamma,I,\Theta \longrightarrow C} \quad \Delta \longrightarrow D}{\Gamma,I,\Theta,\Delta \longrightarrow C\cdot D}\ .$$

Expanding these proofs in $M(\underline{X})$, we claim that

$$\dfrac{\Gamma,\Theta \longrightarrow C \quad \dfrac{\Delta \longrightarrow D \quad C,D \longrightarrow C\cdot D}{C,\Delta \longrightarrow C\cdot D}}{\dfrac{\Gamma,\Theta,\Delta \longrightarrow C\cdot D}{\Gamma,I,\Theta,\Delta \longrightarrow C\cdot D}} \equiv \dfrac{\dfrac{\Gamma,\Theta \longrightarrow C}{\Gamma,I,\Theta \longrightarrow C} \quad \dfrac{\Delta \longrightarrow D \quad C,D \longrightarrow C\cdot D}{C,\Delta \longrightarrow C\cdot D}}{\Gamma,I,\Theta,\Delta \longrightarrow C\cdot D}\ .$$

In view of the isomorphisms $[\Gamma,I,\Theta;C] \longrightarrow [\Gamma,\Theta;C]$ and $[\Gamma,I,\Theta,\Delta;C] \longrightarrow [\Gamma,\Theta,\Delta;C]$ this is tantamount to showing that

$$\dfrac{\dfrac{\emptyset \longrightarrow I \quad \Gamma,I,\Theta \longrightarrow C}{\Gamma,\Theta \longrightarrow C} \quad \dfrac{\cdots\cdots}{C,\Delta \longrightarrow C\cdot D}}{\Gamma,\Theta,\Delta \longrightarrow C\cdot D} \equiv \dfrac{\emptyset \longrightarrow I \quad \dfrac{\Gamma,I,\Theta \longrightarrow C \quad \dfrac{\cdots\cdots}{C,\Delta \longrightarrow C\cdot D}}{\Gamma,I,\Theta,\Delta \longrightarrow C\cdot D}}{\Gamma,\Theta,\Delta \longrightarrow C\cdot D}\ ,$$

and this holds by associativity.

Case 2. Suppose the last step of P is $I \longrightarrow$ and the last step of Q is $\longrightarrow /$. Then say

$$P \equiv \dfrac{\Gamma,\Delta \overset{P'}{\longrightarrow} A/B}{\Gamma,I,\Delta \longrightarrow A/B}\ , \qquad Q \equiv \dfrac{\Gamma,I,\Delta,B \overset{Q'}{\longrightarrow} A}{\Gamma,I,\Delta \longrightarrow A/B}\ .$$

As before, we use the inductional assumption to show that both premises may equivalently be inferred from $\Gamma,\Delta,B \longrightarrow A$.

In view of the fundamental isomorphisms of a biclosed monoidal multicategory, one wants to verify that

$$\dfrac{\dfrac{\emptyset \longrightarrow I \quad \dfrac{\Gamma,I,\Delta \longrightarrow A/B \quad A/B,B \longrightarrow A}{\Gamma,I,\Delta,B \longrightarrow A}}{\Gamma,\Delta,B \longrightarrow A}}{} \equiv \dfrac{\dfrac{\emptyset \longrightarrow I \quad \Gamma,I,\Delta \longrightarrow A/B}{\Gamma,\Delta \longrightarrow A/B} \quad A/B,B \longrightarrow A}{\Gamma,\Delta,B \longrightarrow A} \ ,$$

and this holds by associativity.

Cases 3 and 4. Suppose the last step of P is $I \longrightarrow$ and the last step of Q is $I \longrightarrow$ or $\cdot \longrightarrow$. These cases are treated similarly.

Case 5. Suppose the last step of P is $I \longrightarrow$ and the last step of Q is $/ \longrightarrow$. There are three subcases, only one of which we shall consider as an example.

$$P = \dfrac{\Phi,\Theta,A/B,\Lambda,\Psi \longrightarrow C}{\Phi,I,\Theta,A/B,\Lambda,\Psi \longrightarrow C} \ , \quad Q = \dfrac{\Lambda \longrightarrow B \quad \Phi,I,\Theta,A,\Psi \longrightarrow C}{\Phi,I,\Theta,A/B,\Lambda,\Psi \longrightarrow C} \ .$$

Using the inductional assumption as above, one wants to show that

$$\dfrac{\Lambda \longrightarrow B \quad \dfrac{\Phi,\Theta,A,\Psi \longrightarrow C}{\Phi,\Theta,A/B,\Lambda,\Psi \longrightarrow C}}{\Phi,I,\Theta,A/B,\Lambda,\Psi \longrightarrow C} \equiv \dfrac{\Lambda \longrightarrow B \quad \dfrac{\Phi,\Theta,A,\Psi \longrightarrow C}{\Phi,I,\Theta,A,\Psi \longrightarrow C}}{\Phi,I,\Theta,A/B,\Lambda,\Psi \longrightarrow C} \ .$$

We omit the easy verification.

There are essentially nine other cases. These have already been discussed in Deductive Systems and Categories (I). Unfortunately the five cases considered here are not at all typical of the difficulties that may arise. For

example, the special assumptions about the multicategory $\underline{X}$ have not yet been used. To deal with the remaining cases, three lemmas were required. The reader who wishes to take the trouble to look into these matters should be warned that one of these lemmas (Lemma 1), must be modified in the present context as follows:

If $\Phi,\Lambda,\Psi \longrightarrow B$ is provable in $G(\underline{X})$ and Λ is prime to B, then $\Lambda \longrightarrow I$ and $\Phi,\Psi \longrightarrow B$ are theorems.

The definition of "prime" must also be modified: We say that Λ is <u>prime</u> to Δ in $G(\underline{X})$, provided $\underline{X}$ contains no multimap $f: X_1, \ldots, X_n \longrightarrow Y$ with some X_i occurring in a term of Λ and Y occurring in a term of Δ.

Just to give the reader an idea, let us discuss one further case. Suppose the last steps of P and Q are $\longrightarrow \cdot$, then we may put

$$P = \frac{\Gamma \xrightarrow{P'} C \qquad \Theta,\Delta \xrightarrow{P''} D}{\Gamma,\Theta,\Delta \longrightarrow C \cdot D}, \quad Q = \frac{\Gamma,\Theta \xrightarrow{Q'} C \qquad \Delta \xrightarrow{Q''} D}{\Gamma,\Theta,\Delta \longrightarrow C \cdot D}.$$

We assert that Θ is prime to D. For, otherwise, there is a multimap $f: X_1, \ldots, X_n \longrightarrow Y$ in $\underline{X}$ with X_i in Θ and Y in D. Now Y must be introduced into the proof of $\Delta \longrightarrow D$ through some axiom $Y_1, \ldots, Y_p \longrightarrow Z$ $(Y = Y_j)$ or $Z_1, \ldots, Z_q \longrightarrow Y$.

The first possibility is excluded by the definition of $\underline{X}$ as a direct sum: the components can have nothing in common. In the second case all Z_j occur in Δ,D. As we have assumed that no object of $\underline{X}$ occurs more than once, X_i is not among the Z_j, and, therefore, Y would be in two disjoint components of $\underline{X}$, again a contradiction.

Now we may invoke the above mentioned lemma to infer that both $\Theta \longrightarrow I$ and $\Delta \longrightarrow D$ are theorems in $G(\underline{X})$. By inductional assumption,

$$P'' \equiv \dfrac{\Theta \longrightarrow I \quad \dfrac{\Delta \longrightarrow D}{I,\Delta \longrightarrow D}}{\Theta,\Delta \longrightarrow D} \quad , \quad Q' \equiv \dfrac{\Theta \longrightarrow I \quad \dfrac{\Gamma \longrightarrow C}{\Gamma,I \longrightarrow C}}{\Gamma,\Theta \longrightarrow C}$$

Substituting these proofs into P and Q, we may show that both P and Q are equivalent to

$$\dfrac{\Theta \longrightarrow I \quad \dfrac{\Gamma \longrightarrow C \quad \dfrac{\Delta \longrightarrow D \quad C,D \longrightarrow C\cdot D}{C,\Delta \longrightarrow D}}{\Gamma,\Delta \longrightarrow C\cdot D}}{\Gamma,\Theta,\Delta \longrightarrow C\cdot D}$$

in $M(\underline{X})$. We omit the details.

One final word about the usefulness of the above results. Proposition 2 tells us how to find all proofs of $A \longrightarrow B$ in $M(\underline{X})$ up to equivalence. Proposition 3 and the coherence theorem tell us exactly when two proofs of $A \longrightarrow B$ are equivalent.

Since multimaps in $F(\underline{X})$ are equivalence classes of proofs in $M(\underline{X})$, we thus have a method for computing $[\Lambda;B]$ in $F(\underline{X})$. This assumes, of course, that the sets $[X_1, \ldots, X_n; Y]$ in $\underline{X}$ are known.

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For additional items see the references in 4.

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