Deductive Systems and Categories

I. Syntactic Calculus and Residuated Categories

by

JOACHIM LAMBEK*

McGill University, Montreal

Some years ago, G. D. Findlay and the author observed a connection between proofs in certain deductive systems and canonical mappings in certain categories [6, see also 8, Appendix II]. It is the object of the present paper to give a more formal expression to this connection.

Roughly speaking, the situation is this: While deductive systems may be viewed as free algebras on the category of pre-ordered sets, they may also be used to construct free algebras on the category of all categories in some universe. The crucial step in the argument is the construction of a category whose objects are the terms of a given deductive system and whose maps are equivalence classes of proofs. An equivalence relation is manufactured ad hoc to serve this purpose. However, in the pilot project under investigation, it turns out that two proofs are equivalent if and only if they have the same "scope", in the sense that any method of generalizing one proof (without changing the rules of inference) leads to a generalization of the other.

The present investigation is not carried out in all generality. Instead, we take as a pilot project a particular deductive system, which has been called "syntactic calculus" elsewhere [8, 9]. This system possesses a very simple decision procedure à la Gentzen, which has the pedagogical advantage of being free from so-called "structural" rules of inference [7]. As has already been pointed out [9], this decision procedure may be lifted to the level of categories, where it corresponds to Bourbaki's method of multilinear mappings.

The author plans to pursue this subject further, both by investigating standard constructions (= triples = monads) on the category of categories in general, and by looking at special deductive systems, in particular at the propositional calculus and its relation to Curry's theory of combinators (see [3, Chapter 9]).

*Part of this work was done while the author was a member of the Mathematical Research Institute of the E. T. H. in Zürich in 1966.

MATHEMATICAL SYSTEMS THEORY, Vol. 2, No. 4.
Published by Springer-Verlag New York Inc.

1. Residuated Ordered Sets

An *ordered* set $\underline{A} = (|\underline{A}|, \leqslant)$ consists of a set $|\underline{A}|$ and a binary relation $\leqslant$ on $|\underline{A}|$ which is reflexive and transitive. (The usual term for this is "quasi-ordered" or "pre-ordered". We omit the prefix, as we shall have no use for the anti-symmetry law.) If A and B are elements of $|\underline{A}|$, we write "$A \cong B$" for "$A \leqslant B$ and $B \leqslant A$". (The anti-symmetry law would demand that $A \cong B$ implies $A = B$.) The ordered sets $\underline{A}$ in a given universe are the object of a category Ord whose maps are order-preserving mappings.

A *residuated* ordered set $(\underline{A}, \cdot, /, \backslash)$ is an ordered set $\underline{A} = (|\underline{A}|, \leqslant)$ together with binary operations $\cdot, /, \backslash : |\underline{A}| \times |\underline{A}| \to |\underline{A}|$ such that, for all $A, B, C \in |\underline{A}|$,

$$(A \cdot B) \cdot C \cong A \cdot (B \cdot C)$$

$$A \cdot B \leqslant C \Leftrightarrow A \leqslant C / B$$

$$A \cdot B \leqslant C \Leftrightarrow B \leqslant A \backslash C.$$

For example, given a semigroup $(S, \cdot)$, the set of all subsets of S becomes a residuated ordered set if we define, for all subsets A, B of S,

$$A \cdot B = \{a \cdot b \mid a \in A \text{ and } b \in B\}$$

$$A / B = \{s \in S \mid \forall_{b \in B}\ s \cdot b \in A\}$$

$$A \backslash B = \{s \in S \mid \forall_{a \in A}\ a \cdot s \in B\}\ .$$

Similarly, the set of all additive subgroups of a ring may be turned into a residuated ordered set. (In these examples, the anti-symmetry law does hold.)

The residuated ordered sets $(\underline{A}, \cdot, /, \backslash)$ whose underlying sets $|\underline{A}|$ are in a given universe are the objects of a category. It is natural to allow as maps $f: (\underline{A}, \cdot, /, \backslash) \to (\underline{B}, \cdot, /, \backslash)$ of this category all mappings $f: |\underline{A}| \to |\underline{B}|$ which preserve order and the three operations, the latter up to isomorphism, to be precise:

$$A \leqslant B \Rightarrow f(A) \leqslant f(B)$$

$$f(A \cdot B) \cong f(A) \cdot f(B)$$

$$f(A / B) \cong f(A) / f(B)$$

$$f(A \backslash B) \cong f(A) \backslash f(B).$$

For technical reasons we shall also consider *strict* maps which preserve the operations up to equality, that is, with $\cong$ replaced by $=$.

There is an obvious "forgetful" functor U from the category of all residuated ordered sets with strict maps to the category Ord of all ordered sets in a given universe:

$$U(\underline{A}, \cdot, /, \backslash) = \underline{A}, \qquad U(f) = f.$$

We shall show that this functor U has a left adjoint F. To be precise, for any ordered set $\underline{X}$ we shall construct a "free" residuated ordered set $F(\underline{X})$ and a map $m(\underline{X})\colon \underline{X} \to UF(\underline{X})$ in Ord with the usual universal property. All this can be done even if maps are not restricted to be strict, but then the word "adjoint" would have to be redefined and would deviate from its usual meaning.

The construction of $F(\underline{X})$ will depend on a certain deductive system $D(\underline{X})$. This will be described, as is customary in logic, by laying down what are its terms, formulas, proofs, and theorems.

The *terms* of $D(\underline{X})$ are described inductively by stipulating that all elements of $|\underline{X}|$ are terms, and that if A and B are terms then so are $A \cdot B$, A / B, $A \backslash B$.

The *formulas* of $D(\underline{X})$ have the form $A \to B$, where A and B are terms.

The *theorems* of $D(\underline{X})$ are formulas which are provable from the following axiom schemes and rules of inference. It is assumed that A, B, C are arbitrary terms and that X and Y are arbitrary elements of $|\underline{X}|$.

$$D1\colon\quad A \to A$$

$$D2\colon\quad (A \cdot B) \cdot C \to A \cdot (B \cdot C) \qquad D20\colon\quad A \cdot (B \cdot C) \to (A \cdot B) \cdot C$$

$$D3\colon\quad \frac{A \cdot B \to C}{A \to C / B} \qquad D30\colon\quad \frac{A \cdot B \to C}{B \to A \backslash C}$$

$$D4\colon\quad \frac{A \to C / B}{A \cdot B \to C} \qquad D40\colon\quad \frac{B \to A \backslash C}{A \cdot B \to C}$$

$$D5\colon\quad \frac{A \to B \quad B \to C}{A \to C}$$

$$D6\colon\quad X \to Y \text{ when } X \leqslant Y \text{ in } \underline{X}.$$

Proofs are written in tree form, see for example the derived rule of inference $D8$ below. For typographical reasons it is often convenient to refer to a proof by its label.

Labels of proofs are defined inductively: $D1$, $D6$, $D2$, and $D20$ are labels of proofs. If a formula is derived by rule Dm, where $m = 3$, 30, 4, or 40, from a formula whose proof has been labelled Dn, the resulting proof is labelled Dmn. If a formula is derived by rule $D5$ from formulas whose proofs have been labelled Dm and Dn, the resulting proof is labelled $D5mn$.

In describing the label of a proof we have suppressed subscripts. Actually we should write more cumbersomely $D1_A$, $D6_{X,Y}$, $D2_{A,B,C}$, etc. A proof may be identified with its label with subscripts. Often one can reconstruct the proof of a given formula even from its label without subscripts.

Here are some theorem schemes with their proofs:

$$D31\colon\quad A \to (A \cdot B) / B$$

$$D41\colon\quad (C / B) \cdot B \to C$$

$$D3041\colon\quad B \to (C / B) \backslash C.$$

Other easily proved theorems are the following:

$$(C / B) \cdot (B / A) \to C / A$$

$$A / B \to (C / A) / (B / A)$$

$$(A \backslash B) / C \rightleftarrows A \backslash (B / C)$$

$$(A / B) / C \rightleftarrows A / (C \cdot B).$$

The following three *derived* rules of inference are sufficiently important to receive short labels:

$$D7 : \frac{A \to A' \quad B \to B'}{A / B' \to A' / B}$$

$$D70: \frac{A \to A' \quad B \to B'}{A' \backslash B \to B' \backslash A}$$

$$D8 : \frac{A \to A' \quad B \to B'}{A \cdot B \to A' \cdot B'} .$$

For example, a proof of $D8$ will be written out in detail:

$$\dfrac{\dfrac{\dfrac{A \overset{Dm}{\to} A' \quad A' \to (A' \cdot B) / B}{A \to (A' \cdot B) / B}\,D5 \qquad \dfrac{A' \cdot B \overset{D1}{\to} A' \cdot B}{A' \cdot B \to A' \cdot B}\,D3}{A \cdot B \to A' \cdot B}\,D4 \qquad \dfrac{\dfrac{B \overset{Dn}{\to} B' \quad B' \to A' \backslash (A' \cdot B')}{B \to A' \backslash (A' \cdot B')}\,D5 \qquad \dfrac{A' \cdot B' \overset{D1}{\to} A' \cdot B'}{A' \cdot B \to A' \cdot B'}\,D30}{A \cdot B \to A' \cdot B'}\,D5$$

Thus we have

$$D8mn = D545m31405n301,$$

and similarly

$$D7mn = D53405n30413541m$$

and

$$D70mn = D53045m3401305401n.$$

PROPOSITION 1. *The forgetful function U from the category of all residuated ordered sets with strict maps to the category of all ordered sets in a given universe has a left adjoint F.*

To be precise, given any ordered set $\underline{X}$ in the universe, there is a residuated ordered set $F(\underline{X})$ in the same universe and an order-preserving mapping $m(\underline{X})$: $\underline{X} \to UF(\underline{X})$ with this property: For any residuated ordered set $(\underline{A}, \cdot, /, \backslash)$ and any order-preserving mapping f: $\underline{X} \to \underline{A}$, there exists a unique strict map f': $F(\underline{X}) \to (\underline{A}, \cdot, /, \backslash)$ such that $U(f')m(\underline{X}) = f$. (We shall see later that $m(\underline{X})(X) \leqslant m(\underline{X})(Y)$ if and only if $X \leqslant Y$.)

Actually we are more interested in the construction of F than its existence, which might be proved more briefly otherwise.

Proof. We put $F(\underline{X}) = (|F(\underline{X})|, \leqslant, \cdot, /, \backslash)$, where $|F(\underline{X})|$ is the set of terms of $D(\underline{X})$, $A \leqslant B$ means that $A \to B$ is a theorem in $D(\underline{X})$, and $A \cdot B$, A / B, $A \backslash B$ are as in $D(\underline{X})$. We take $m(\underline{X})$ to be the inclusion mapping of $|\underline{X}|$ into $|F(\underline{X})|$, regarded as a map in Ord. Indeed, $m(\underline{X})$ preserves order, for if $X \leqslant Y$ in $\underline{X}$, then $X \to Y$ is a theorem in $D(\underline{X})$, in view of axiom $D6$, hence $X \leqslant Y$ in $(|F(\underline{X})|, \leqslant) = UF(\underline{X})$.

Now let $(A, \cdot, /, \backslash)$ be any residuated ordered set, and consider any map $f: \underline{X} \to \underline{A}$ in Ord. Define the mapping f': $|UF(\underline{X})| \to |\underline{A}|$ recursively by

$$f'(X) = f(X)$$
$$f'(A \cdot B) = f'(A) \cdot f'(B)$$
$$f'(A / B) = f'(A) / f'(B)$$
$$f'(A \backslash B) = f'(A) \backslash f'(B).$$

Here X is any element of $|\underline{X}|$ and A, B are any elements of $|\underline{A}|$.

We shall prove that f' preserves order, hence it is a map $UF(\underline{X}) \to \underline{A}$ in Ord. Since it clearly preserves the three operations (with $=$), it is then a strict map $F(\underline{X}) \to (\underline{A}, \cdot, /, \backslash)$ in the category of residuated ordered sets.

Suppose $A \leqslant B$, then $A \to B$ is a theorem in $D(\underline{X})$. We prove by induction on the length of the proof of $A \to B$ that $f'(A) \leqslant f'(B)$. When $A \to B$ is an instant of $D1$, then $B = A$ and the result is clear. When it is an instant of $D2$ or $D20$, the result follows from associativity in $\underline{A}$. When it is an instant of $D6$, then it has the form $X \to Y$, where $X \leqslant Y$ in $\underline{X}$, hence $f'(X) = f(X) \leqslant f(Y) = f'(Y)$. When it is an instant of $D3$, say $A \to C / D$ inferred from $A \cdot D \to C$, we have $f'(A) \cdot f'(D) = f'(A \cdot D) \leqslant f'(C)$, by inductional hypothesis, hence $f'(A) \leqslant f'(C) / f'(D) = f'(C / D)$. The remaining rules are treated similarly.

Finally we observe that $(U(f')m(\underline{X}))(X) = f'(X) = f(X)$. Moreover, if g is any strict map such that $U(g)m(\underline{X}) = f$, it is easily shown by induction that $g = f'$.

The proof is now complete. A few remarks are in order concerning the relaxation of "strictness".

Given two maps g, h: $(\underline{B}, \cdot, /, \backslash) \to (\underline{A}, \cdot, /, \backslash)$, we write $g \cong h$ to mean that $g(B) \cong h(B)$, for all B in $|\underline{B}|$. The free residuated ordered set $F(\underline{X})$ constructed above has the property that any map g: $F(\underline{X}) \to (\underline{A}, \cdot, /, \backslash)$ is isomorphic to a strict map g_0. We need only put $g_0(X) = g(X)$, for all X in $|\underline{X}|$, and then define inductively $g_0(A \cdot B) = g_0(A) \cdot g_0(B)$, etc. In particular, assume that f: $\underline{X} \to \underline{A}$ is any given map in Ord, and that $U(g)m(\underline{X}) \cong f$ in Ord. Then $g \cong f'$, where f' is defined as above.

If we redefine the word "adjoint" along these lines, we may omit the word "strict" from the statement of Proposition 1.

2. Residuated Categories

By a *residuated category* we shall mean a system $(\underline{A}, \cdot, /, \backslash, a, b, c)$ where

1. $\underline{A}$ is a category,
2. $\cdot : \underline{A} \times \underline{A} \to \underline{A}$, $/ : \underline{A} \times \underline{A}^{\mathrm{opp}} \to \underline{A}$, $\backslash : \underline{A}^{\mathrm{opp}} \times \underline{A} \to \underline{A}$ are functors,
3. a, b, c are natural isomorphisms

$$a(A, B, C): (A \cdot B) \cdot C \to A \cdot (B \cdot C)$$

$$b(A, B, C): [A \cdot B, C] \to [A, C / B]$$

$$c(A, B, C): [A \cdot B, C] \to [B, A \backslash C],$$

4. the composite map

$$A \cdot (B \cdot (C \cdot D)) \to (A \cdot B) \cdot (C \cdot D) \to ((A \cdot B) \cdot C) \cdot D$$
$$\to (A \cdot (B \cdot C)) \cdot D \to A \cdot ((B \cdot C) \cdot D) \to A \cdot (B \cdot (C \cdot D))$$

obtained by repeated application of a is the identity.

Note that we have written $[A, B]$ for Hom (A, B).

Residuated categories are closely related to the "closed categories" of [4]. Condition 4 is also called "coherence" and has been discussed at length by Mac Lane [10].

As an example, though by no means the simplest example, take $\underline{A}$ to be the category of all unitary $(R - R)$-bimodules, where R is an associative ring with 1, and define

$$A \cdot B = A \otimes_R B$$

$$A / B = \mathrm{Hom}_R(B_R, A_R)$$

$$A \backslash B = \mathrm{Hom}_R({}_R A, {}_R B).$$

As is well known, these are not only abelian groups but also $(R - R)$-bimodules.

Other examples are the residuated ordered sets considered above, if ordered sets are regarded as categories in the usual manner.

All residuated categories in a given universe form themselves a category if we give a suitable definition for a map $(\underline{A}, \cdots) \to (\underline{B}, \cdots)$. A naive and somewhat unnatural definition would say that such a map is a functor $T: \underline{A} \to \underline{B}$ satisfying

$$T(A \cdot B) = T(A) \cdot T(B),$$

$$T(A / B) = T(A) / T(B),$$

$$T(A \backslash B) = T(A) \backslash T(B),$$

and similar equations for maps, as well as

$$T(a(A, B, C)) = a(T(A), T(B), T(C)),$$

$$T(b(A, B, C)(f)) = b(T(A), T(B), T(C))(T(f)),$$

$$T(c(A, B, C)(f)) = c(T(A), T(B), T(C))(T(f)),$$

where f is any map in $[A \cdot B, C]$. We shall call such a functor T a *strict* map. Other maps will be discussed at the end of this chapter.

Again we have a "forgetful" functor U from the category of all residuated categories and strict maps to Cat, the category of all categories in a given universe:

$$U(\underline{A}, \cdot, /, \backslash, a, b, c,) = \underline{A}, \qquad U(T) = T.$$

PROPOSITION 2. *The forgetful functor U from the category of all residuated categories and strict maps to the category of all categories in a given universe has a left adjoint F.*

To be precise, given any category $\underline{X}$ in the universe, there is a residuated category $F(\underline{X})$ in the same universe and a functor $m(\underline{X})$: $\underline{X} \to UF(\underline{X})$ with this property: For any residuated category $(\underline{A}, \cdot, /, \backslash, a, b, c)$ and any functor T: $\underline{X} \to \underline{A}$, there exists a unique strict map T': $F(\underline{X}) \to (\underline{A}, \cdot \cdot \cdot)$ such that $U(T')m(\underline{X}) = T$. (We shall see by and by that $m(\underline{X})$ is full and faithful.)

Again we are more interested in the construction of F than in its existence.

Proof. First, we construct the deductive system $D(\underline{X})$ as in the case when $\underline{X}$ was an ordered set, with one difference: The old axiom scheme $D6$ is replaced by

$$D6_f\text{: } X \to Y \text{ when } f\text{: } X \to Y \text{ in } \underline{X}.$$

Thus, with each map f: $X \to Y$ in $\underline{X}$ there is associated an axiom $D6_f$: $X \to Y$. If f, g: $X \to Y$ are distinct maps, $D6_f$ and $D6_g$ are regarded as distinct axioms. Here we are taking the unorthodox position that the same assertion, to wit $X \to Y$, may be made by two distinct axioms.

Next, we describe the category $UF(\underline{X})$. Its maps are equivalence classes of proofs in $D(\underline{X})$ with respect to a certain equivalence relation $\equiv$, which will be defined presently. Composition will be by the rule of transivity $D5$. Thus, if Dm: $A \to Y$ and Dn: $B \to C$ are proofs, we define the composition of the equivalence classes $[Dm]$: $A \to B$ and $[Dn]$: $B \to C$ to be the equivalence class of the proof

$$D5\text{: } \frac{A \overset{Dm}{\to} B \qquad B \overset{Dn}{\to} C}{A \to C},$$

that is,

$$[Dn]\ [Dm] = [D5mn].$$

The identity maps in the category $UF(\underline{X})$ will be equivalence classes of instances of axiom scheme $D1$.

We complete the construction of $F(\underline{X})$ by defining

$$[Dm] \cdot [Dn] = [D8mn]$$
$$[Dm] \,/\, [Dn] = [D7mn]$$
$$[Dm] \setminus [Dn] = [D70mn]$$
$$a = [D2]$$
$$a^{-1} = [D20]$$
$$b([Dm]) = [D3m]$$
$$b^{-1}([Dm]) = [D4m]$$
$$c([Dm]) = [D30m]$$
$$c^{-1}([Dm]) = [D40m].$$

The functor $m(\underline{X})\colon \underline{X} \to UF(\underline{X})$ is determined by

$$m(\underline{X})(X) = X, \qquad m(\underline{X})(f) = [D6_f].$$

It remains to specify the equivalence relation $\equiv$ on the set of proofs. Our procedure will be to collect all properties that we want $\equiv$ to satisfy and then to take the smallest equivalence relation (intersection of all equivalence relations) with these properties.

We shall only write $Dm \equiv Dn$ when Dm and Dn are proofs of the same formula $A \to B$. Trivially, $\equiv$ must satisfy the reflexive, symmetric, and transitive properties, and, almost equally trivially, the substitution properties:

$$\frac{Dm \equiv Dm'}{Dkm \equiv Dkm'} \qquad (k = 3, 30, 4, 40)$$

$$\frac{Dm \equiv Dm' \quad Dn \equiv Dn'}{D5mn \equiv D5m'n'}.$$

It follows that also

$$\frac{Dm \equiv Dm' \qquad Dn \equiv Dn'}{Dkmn \equiv Dkm'n'} \qquad (k = 7, 70, 8).$$

$E1$. Composition in the category $UF(\underline{X})$ must have the usual properties. Thus we require

$$D51m \equiv Dm$$
$$D5m1 \equiv Dm$$
$$D55mnp \equiv D5m5np.$$

$E2$. $\cdot$, $/$, and $\setminus$ shall be bifunctors. Thus we require

$$Dk11 \equiv D1$$

$$Dk5mm'5nn' \equiv D5kmnkm'n'$$

for $k = 7, 70, 8$.

$E3$. a shall be a natural isomorphism with inverse a^{-1}. Thus we require

$$D588mnp2 \equiv D528m8np$$

$$D5220 \equiv D1$$

$$D5202 \equiv D1.$$

$E3'$. b and c shall be natural isomorphisms with inverses b^{-1} and c^{-1} respectively. For example, the naturality of b amounts to the commutativity of the following diagram:

$$\begin{array}{ccc} [A' \cdot B', C] & \xrightarrow{b(A', B', C)} & [A', C \,/\, B'] \\ \Big\downarrow {\scriptstyle [[D8mn], Dp]} & & \Big\downarrow {\scriptstyle [[Dm], [D7pn]]} \\ [A \cdot B, C'] & \xrightarrow[b(A, B, C')]{} & [A, C' \,/\, B] \end{array}$$

where $[Dm]: A \to A'$, $[Dn]: B \to B'$, $[Dp]: C \to C'$.

Assuming that $[Dk]: A' \cdot B' \to C$, this means that

$$[D7pn]b([Dk])[Dm] = b([Dp][Dk][D8mn]),$$

that is to say, $D55m3k7pn \equiv D3558mnkp$. Moreover we require $D34m \equiv Dm$ and $D43m \equiv Dm$.

The reader will have no difficulty in writing down the analogous properties concerning c.

$E4$. The composite map $A \cdot (B \cdot (C \cdot D)) \to \cdots \to A \cdot (B \cdot (C \cdot D))$ discussed above shall be the identity. Thus we require

$$D5555225212512 \equiv D1.$$

$E5$. $m(\underline{X})$ shall be a functor, that is

$$m(\underline{X})(1) = 1,\ m(\underline{X})(g)m(\underline{X})(f) = m(\underline{X})(gf),$$

where $f: X \to Y$ and $g: Y \to Z$ in $\underline{X}$. Thus we require

$$D6_1 \equiv D1$$

$$D56_f6_g \equiv D6_{gf}.$$

Now let $(\underline{A}, \cdot, /, \backslash, a, b, c)$ be any residuated category and consider any functor $T: \underline{X} \to \underline{A}$. We claim that there exists a unique strict map T': $F(\underline{X}) \to (\underline{A}, \cdots)$ such that $U(T')\, m(\underline{X}) = T$.

We begin by defining T' recursively as a function on objects:

$$T'(X) = T(X) \quad \text{when } X \text{ is any object of } \underline{X},$$
$$T'(A \cdot B) = T'(A) \cdot T'(B),$$
$$T'(A / B) = T'(A) / T'(B),$$
$$T'(A \backslash B) = T'(A) \backslash T'(B).$$

To define T' on maps, we first introduce $T'(Dm)$ for each proof, by induction on the length of the proof, and then show that this depends only on the equivalence class of the proof, so that we may write $T'([Dm]) = T'(Dm)$.

$$T'(D1) = 1$$
$$T'(D2) = a(T'(A), T'(B), T'(C))$$
$$T'(D20) = a(T'(A), T'(B), T'(C))^{-1}$$
$$T'(D3m) = b(T'(Dm))$$
$$T'(D30m) = c(T'(Dm))$$
$$T'(D4m) = b(T'(Dm))^{-1}$$
$$T'(D40m) = c(T'(Dm))^{-1}$$
$$T'(D5mn) = T'(Dn)\,T'(Dm)$$
$$T'(D6_f) = T(f).$$

Assuming that $Ds \equiv Dt$, we shall show that $T'(Ds) = T'(Dt)$. The proof of this will be by induction on the length of the proof of the fact that $Ds \equiv Dt$. Now this will have been established by invoking one of the rules $E0$ to $E5$.

$E0$. If $Ds \equiv Dt$ has been deduced by the reflexive, symmetric, or transitive property, the result is clear. If it has been deduced from one of the substitution properties, the following case will illustrate what happens in general. Suppose we have concluded that $D3m \equiv D3n$ because $Dm \equiv Dn$. Then, by inductional assumption, $T'(Dm) = T'(Dn)$, hence

$$T'(D3m) = b(T'(Dm)) = b(T'(Dn)) = T'(D3n).$$

$E1$. Suppose we have used $D51m \equiv Dm$, then

$$T'(D51m) = T'(Dm)\ T'(1) = T'(Dm).$$

Similarly $T'(D15m) = T'(Dm)$ and

$$\begin{aligned} T'(D55mnp) &= T'(Dp)(T'(Dn)T'(Dm)) \\ &= (T'(Dp)T'(Dn))T'(Dm) \\ &= T'(D5m5np) \end{aligned}$$

$E2$. For example

$$\begin{aligned} T'(D85mn'5nn') &= (T'(Dm')T'(Dm)) \cdot (T'(Dn')T'(Dn)) \\ &= (T'(Dm') \cdot T'(Dn'))(T'(Dm) \cdot T'(Dn)) \\ &= T'(D58mn8m'n'). \end{aligned}$$

This argument uses the fact that $T'(D8mn) = T'(Dm) \cdot T'(Dn)$, which may be verified by a straightforward computation.

$E3$. We have

$$\begin{aligned} T'(D588mnp2) &= T'(D2)T'(D88mnp) \\ &= a((T'(Dm) \cdot T'(Dn)) \cdot T'(Dp)). \end{aligned}$$

By naturality of a, this is

$$(T'(Dm) \cdot (T'(Dn) \cdot T'(Dp)))a = T'(D8m8np)T'(D2) = T'(D528m8np).$$

Moreover, $T'(D5220) = T'(D20)T'(D2) = a^{-1}a = T'(D1)$ etc.

$E3'$ is similar to $E3$.

$E4$. Applying T' to the composite proof of $A \cdot (B \cdot (C \cdot D)) \to A \cdot (B \cdot (C \cdot D))$, we obtain

$$T'(D5555225212512) = T'(D512)T'(D2)T'(D521)T'(D2)T'(D2).$$

To make this intelligible, one must insert subscripts. But then it is easily seen that this is the analogous map of $T'(A) \cdot (T'(B) \cdot (T'(C) \cdot T'(D)))$ into itself. Since $(\underline{A}, \cdots)$ satisfies the coherence condition 4, this must be the identity map, that is, $T'(D1)$.

$E5$. Finally we have

$$T'(D6_1) = T(1) = 1 = T'(D1)$$

and

$$T'(D56_f6_g) = T'(D6_g)T'(D6_f) = T(g)T(f) = T(gf) = T'(D6_{gf}).$$

This completes the proof that $T'(Dm)$ depends only on the equivalence class of Dm, and we shall put $T'([Dm]) = T'(Dm)$. T' is then a functor, for example,

$$\begin{aligned} T'([Dn][Dm]) &= T'([D5mn]) \\ &= T'(D5mn) \\ &= T'(Dn)T'(Dm) \\ &= T'([Dn])T'([Dm]). \end{aligned}$$

It is easily seen that it is a strict map in the category of residuated categories in the given universe. Thus, for example,

$$T'(A \cdot B) = T'(A) \cdot T'(B),$$

by construction, and

$$T'(a(A, B, C)) = T'([D2_{A,B,C}]) = a(T'(A), T'(B), T'(C)).$$

We compute

$$U(T')(m(\underline{X})(X)) = U(T')(X) = T'(X) = T(X),$$

$$U(T')(m(\underline{X})(f)) = U(T')([D6_f]) = T'(D6_f) = T(f).$$

Thus $U(T')m(\underline{X}) = T$. Is T' unique with this property? Suppose S is any strict map such that $U(S)m(\underline{X}) = T$. It is easily shown that S agrees with T' as an object function, by proving that $S(A) = T'(A)$ by induction on the length of the term A. But we can also show that $S([Dm]) = T'([Dm])$, by induction on the length of the proof Dm. This is an immediate consequence of the following observations:

$$S([D1_A]) = S(1_A) = 1_{S(A)},$$

$$S([D2_{A,B,C}]) = S(a(A, B, C)) = a(S(A), S(B), S(C)),$$

$$S([D3_{A,B,C}m]) = S(b(A, B, C)([Dm]) = b(S(A), S(B), S(C))(S([Dm])),$$

$$S([D5mn]) = S([Dn][Dm]) = S([Dn])S([Dm]),$$

$$S([D6_f]) = U(S)(m(\underline{X})(f)) = T(f),$$

with similar observations concerning $D20$, $D30m$, $D4m$, $D40m$.

We have seen in the above proof that with any functor $T: \underline{X} \to U(\underline{A}, \cdot\ \cdot\ \cdot)$ there is associated canonically a map $T': F(\underline{X}) \to (\underline{A}, \cdot\ \cdot\ \cdot)$ in the category of residuated categories. The maps in $\underline{A}$ of the form $T'([Dm])$, where Dm is a proof in $D(\underline{X})$, may be called *canonical* with respect to T. Canonical maps in A may also be defined recursively without reference to $F(\underline{X})$. The relationship between canonical maps defined recursively and proofs in $D(\underline{X})$ had already been pointed out in [9, Appendix II].

Let us return to the question of what maps to admit to the category of residuated categories. Presumably the strict maps considered hitherto are rather unnatural. A more sophisticated definition is the following. A map $(\underline{A}, \cdot\ \cdot\ \cdot) \to (\underline{B}, \cdot\ \cdot\ \cdot)$ is a quadruple (T, u, v, w) where $T: \underline{A} \to \underline{B}$ is a functor and u, v, w are natural isomorphims

$$u(A, B): T(A \cdot B) \to T(A) \cdot T(B)$$

$$v(A, B): T(A \,/\, B) \to T(A) \,/\, T(B)$$

$$w(A, B): T(A \setminus B) \to T(A) \setminus T(B)$$

satisfying three obvious commutativity conditions. Composition of such maps also has an obvious definition. If u, v, w are identity maps, (T, u, v, w) will be a strict map and may be identified with T.

We may remark that the category of residuated categories, like Cat, is "two-dimensional" in the sense of Benabou; one can speak of maps from (T, u, v, w) to (T', u', v', w'): $(\underline{A}, \cdot\ \cdot\ \cdot) \to (\underline{B}, \cdot\ \cdot\ \cdot)$. Such a map is a natural

transformation $t: T \to T'$ satisfying three obvious commutativity conditions. Of particular interest in the case when t is an isomorphism.

Any map $(S, u, v, w): F(\underline{X}) \to (\underline{A}, \cdot\ \cdot\ \cdot)$ is isomorphic to a strict map. If we assume that $U(S, u, v, w)m(\underline{X}) \cong T$, it is easily seen that $(S, u, v, w) \cong T'$. The word "strict" may be omitted from Proposition 2, provided the word "adjoint" is redefined according to this observation.

3. A Gentzen-Type Deductive System

It is known [8] that the deductive system $D(\underline{X})$ has a decision procedure, that is, an algorithm for deciding whether a given formula $A \to B$ is a theorem. (Actually it was assumed in [8] that $\underline{X}$ is a set without further structure, but essentially the same method works if $\underline{X}$ is an ordered set or a category.) This procedure was modelled on Gentzen's method for the intuitionist propositional calculus [7] and will now be sketched.

For any category $\underline{X}$ we construct a deductive system $G(\underline{X})$, which will ultimately turn out to be equivalent to $D(\underline{X})$.

The formulas of $G(\underline{X})$ have the form

$$A_1, A_2, \cdots, A_n \to B,$$

where n is a positive integer and $A_1, A_2, \cdots, A_n, B$ are terms of $D(\underline{X})$.

In stating the following axiom schemes and rules of inference for $G(\underline{X})$, we assume that Λ, Δ, Φ, and ψ are finite sequences of terms of $D(\underline{X})$. Moreover, Λ and Δ are assumed to be non-empty, while Φ and ψ may be empty sequences. "Φ, ψ" denotes the sequence obtained by juxtaposing Φ and ψ, if ψ is empty this is just Φ.

$$G1: \quad X \to Y \text{ when } f: X \to Y \text{ in } \underline{X}$$

$$G2: \quad \frac{\Lambda, B \to A}{\Lambda \to A / B} \qquad G20: \quad \frac{B, \Lambda \to A}{\Lambda \to B \setminus A}$$

$$G3: \quad \frac{\Lambda \to B \qquad \Phi, A, \psi \to C}{\Phi, A / B, \Lambda, \psi \to C} \qquad G30: \quad \frac{\Lambda \to B \qquad \Phi, A, \psi \to C}{\Phi, \Lambda, B \setminus A, \psi \to C}$$

$$G4: \quad \frac{\Phi, A, B, \psi \to C}{\Phi, A \cdot B, \psi \to C}$$

$$G5: \quad \frac{\Lambda \to A \qquad \Delta \to B}{\Lambda, \Delta \to A \cdot B}$$

We observe that the conclusion of each of rules $G2$ to $G5$ contains all the terms which occur in the premise or premises and one new occurrence of one of the operation symbols $\cdot$, $/$, or $\setminus$. Given any formula of $G(\underline{X})$, one can decide in a finite number of steps whether it is a theorem, by working backwards.

We shall establish one theorem scheme and a number of derived rules of inference.

$$G6: \quad A \to A$$

$$G7: \quad \frac{\Lambda \to A \qquad \Phi, A, \psi \to C}{\Phi, \Lambda, \psi \to C}$$

$$G8: \quad \frac{\Lambda \to A / B}{\Lambda, B \to A} \qquad\qquad G80: \quad \frac{\Lambda \to B \backslash A}{B, \Lambda \to A}$$

$$G9: \quad \frac{\Phi, A \cdot B, \psi \to C}{\Phi, A, B, \psi \to C}.$$

Proof of $G6$. Clearly $X \to X$ is a theorem for every object X of $\underline{X}$, in view of $G1$. Assuming that $B \to B$ and $C \to C$ are theorems, we prove $A \to A$ for $A = B \cdot C$, B / C, or $B \backslash C$, thus:

$$\frac{\dfrac{B \to B \quad C \to C}{B, C \to B \cdot C}}{B \cdot C \to B \cdot C} \qquad\qquad \frac{\dfrac{C \to C \quad B \to B}{B / C, C \to B}}{B / C \to B / C.}$$

Proof of $G7$. We distinguish four cases.

Case 1. Both premises are instances of $G1$:

$$\frac{X \overset{f}{\to} Y \qquad Y \overset{g}{\to} Z}{X \to Z}$$

Then the conclusion may also be obtained by rule $G1_{gf}$. Thus one replaces $G7 1_f 1_g$ by $G1_{gf}$.

Case 2. The last step in the proof of the first premise uses one of rules $G2$ to $G5$, but does not introduce the main operation symbol of A. Then $\Lambda \to A$ is inferred by rule $G3$, $G30$, or $G4$ from one or two formulas, one of which has the form $\Lambda' \to A$. One replaces

$$\frac{\dfrac{\Lambda' \to A \cdots}{\Lambda \to A} \qquad \Phi, A, \psi \to C}{\Phi, \Lambda, \psi \to C}$$

by

$$\frac{\Lambda' \to A \quad \Phi, A, \psi \to C}{\dfrac{\Phi, \Lambda', \psi \to C \cdots}{\Phi, \Lambda, \psi \to C}}$$

Thus one replaces $G7kmn$ by $Gk7mnp$ or $G7kmpn$ by $Gk7mnp$.

Case 3. The last step in the proof of the second premise uses one of rules $G2$ to $G5$, but does not introduce the main operation symbol of A. Then $\Phi, A, \psi \to C$ is inferred from one or two formulas, one of which has the form $\Phi', A, \psi' \to C'$. One replaces

$$\frac{\Lambda \to A \qquad \dfrac{\Phi', A, \psi' \to C' \cdots}{\Phi, A, \psi \to C}}{\Phi, \Lambda, \psi \to C}$$

by

$$\dfrac{\dfrac{\Lambda \to A \qquad \Phi', A, \psi' \to C'}{\Phi', \Lambda, \psi' \to C'} \quad \cdots}{\Phi, \Lambda, \psi \to C}.$$

Thus, for example, one replaces $G7m2n$ by $G27mn$.

Case 4. The last steps in the proofs of both premises introduce the main operation symbol of A. For example, if this operation is $\cdot$ and Φ and ψ are empty, one replaces

$$\dfrac{\dfrac{\Lambda \to A \qquad \Lambda' \to A'}{\Lambda, \Lambda' \to A \cdot A'} \qquad \dfrac{A, A' \to C}{A \cdot A' \to C}}{\Lambda, \Lambda' \to C}$$

by

$$\dfrac{\Lambda' \to A' \qquad \dfrac{\Lambda \to A \qquad A, A' \to C}{\Lambda, A' \to C}}{\Lambda, \Lambda' \to C}.$$

Thus one replaces $G75mn4p$ by $G7n7mp$.

If we measure the *degree* of an application of rule $G7$ by the total number of occurrences of the operation symbols $\cdot, /, \backslash$, in $\Phi, \Lambda, \psi, A, C$, it is easily verified that in each of cases 2 to 4 an application of $G7$ is replaced by one or two other applications of $G7$ with smaller degree, while in case 1 the degree is 0. $G7$ thus holds by induction on the degree.

Proof of $G9$. This is shown by induction on the number of occurrences of the operation symbols $\cdot, /, \backslash$ in the premise, that is, on the length of a proof of the premise.

If the premise has been obtained from $\Phi, A, B, \psi \to C$ by $G4$, there is nothing to prove. Otherwise it will have been obtained by Gk from one or two premises, one of which must have the form $\Phi', A \cdot B, \psi' \to C$. By inductional assumption $\Phi', A, B, \psi' \to C$ is a theorem in $G(\underline{X})$. It is easily verified in the different cases that the same rule Gk leads from $\Phi', A, B, \psi' \to C$ and the other premise, if any, to the conclusion $\Phi, A, B, \psi \to C$.

A shorter proof, using $G7$, is given by $G9n = G7566n$.

Proofs of $G8$ and $G80$ are similar to that of $G9$ and will be omitted.

$G7$ is Gentzen's "cut". $G8$, $G80$, and $G9$ are converses of $G2$, $G20$, and $G4$ respectively. We shall now investigate under what circumstances $G5$, $G3$, and $G30$ admit converses.

Suppose Λ and Δ are sequences of terms of $G(\underline{X})$. We say that Λ is *prime to* Δ provided $\underline{X}$ contains no map $f\colon X \to Y$ with X occurring in a term of Λ and Y occurring in a term of Δ.

LEMMA 1. *Suppose that* $\Phi, \Gamma, \psi \to C$ *is a theorem in* $G(\underline{X})$ *and* Γ *is prime to* Φ, C, ψ. *Then* Γ *is empty.*

Proof. We proceed by induction on the length of the proof of $\Phi, \Gamma,$

$\psi \to C$. If it is an instance of $G1$, there is nothing to prove. Suppose it has been established by rule $G2$, thus:

$$\frac{\Phi, \Lambda, \psi, B \to A}{\Phi, \Lambda, \psi \to A / B}.$$

Looking at the premise, we see that Λ is prime to Φ, A, ψ, B, and deduce that Λ is empty, by inductional hypothesis. Rules $G20$ and $G4$ are treated similarly.

Suppose the theorem has been established by rule $G3$, thus:

$$\frac{\Lambda \to B \qquad \Phi', A, \psi' \to C}{\Phi, \Gamma, \psi = \Phi', A / B, \Lambda, \psi' \to C}.$$

By inductional hypothesis applied to the second premise, we deduce that A is not prime to C, therefore A / B is not in Γ.

Case 1. A / B is in $\psi = \theta, A / B, \Lambda, \psi'$, then $\Phi' = \Phi, \Gamma, \theta$. Looking at the second premise $\Phi, \Gamma, \theta, A, \psi' \to C$, we observe that Γ is prime to $\Phi, C, \theta, A, \psi'$ (the last three bits being contained in ψ) and deduce that Γ is empty, by inductional assumption.

Case 2. A / B is in $\Phi = \Phi', A / B, \theta$, but Γ is neither in ψ' nor in Λ. Then $\Gamma = \Gamma_1, \Gamma_2$, where $\Lambda = \theta, \Gamma_1$ and $\psi' = \Gamma_2, \psi$. Looking at the first premise $\theta, \Gamma_1 \to B$, we observe that Γ_1 is prime to B and deduce that Γ_1 is empty, by inductional hypothesis. Looking at the second premise $\Phi', A, \Gamma_2, \psi \to C$, we observe that Γ_2 is prime to Φ', A, C, ψ and deduce that Γ_2 is empty, by inductional hypothesis. Thus $\Gamma = \Gamma_1, \Gamma_2$ is empty.

Case 3. A / B is in $\Phi = \Phi', A / B, \theta$, and Γ is in $\psi' = \theta', \Gamma, \psi$, where $\theta = \Lambda, \theta'$. Looking at the second premise $\Phi', A, \theta', \Gamma, \psi \to C$, we observe that Γ is prime to $\Phi', A, \theta', C, \psi$ and deduce that Γ is empty, by inductional hypothesis.

Case 4. A / B is in $\Phi = \Phi', A / B, \theta$, and Γ is in $\Lambda = \theta, \Gamma, \Lambda'$. Looking at the first premise $\theta, \Gamma, \Lambda' \to B$, we observe that Γ is prime to B and deduce that Γ is empty, by inductional hypothesis.

Rules $G30$ and $G5$ may be treated similarly, thus completing the proof of Lemma 1.

LEMMA 2. *Suppose that* $\Lambda, \Delta \to A \cdot B$ *is a theorem in* $G(\underline{X})$, Λ *is prime to* Δ, B, *and* Δ *is prime to* Λ, A. *Then* $\Lambda \to A$ *and* $\Delta \to B$ *are theorems.*

Proof. We proceed by induction on the length of the proof of $\Lambda, \Delta \to A \cdot B$. We may assume that the last step in the proof used one of rules $G3$ to $G5$.

Suppose the rule used was $G5$, say $\Delta = \Delta_1, \Delta_2$ and

$$\frac{\Lambda, \Delta_1 \to A \qquad \Delta_2 \to B}{\Lambda, \Delta_1, \Delta_2 \to A \cdot B}.$$

By assumption, Δ_1 is prime to Λ, A. Applying Lemma 1 to the first premise, we see that Δ_1 is empty, hence the premises are precisely $\Lambda \to A$ and $\Delta \to B$.

Suppose the rule used was $G4$, say $\Delta = \Delta_1, D \cdot E, \Delta_2$ and

$$\frac{\Lambda, \Delta_1, D, E, \Delta_2 \to A \cdot B}{\Lambda, \Delta_1, D \cdot E, \Delta_2 \to A \cdot B}.$$

By assumption, Λ is prime to $\Delta_1, D, E, \Delta_2, B$, and Δ_1, D, E, Δ_2 is prime to Λ, A. By inductional hypothesis applied to the premise, we deduce that $\Lambda \to A$ and $\Delta_1 D, E, \Delta_2 \to B$ are theorems. Then also $\Delta \to B$ is a theorem, by $G4$.

Suppose the rule used was $G3$, thus:

$$\frac{\Gamma \to E \qquad \Phi, D, \psi \to A \cdot B}{\Lambda, \Delta = \Phi, D \,/\, E, \Gamma, \psi \to A \cdot B}.$$

Case 1. $\Phi = \Lambda$ and $D \,/\, E$ is in Δ. Then we may apply the inductional hypothesis to the second premise and deduce that $\Lambda = \Phi \to A$ and $D, \psi \to B$ are theorems. Then also $\Delta = D \,/\, E, \Gamma, \psi \to B$ is a theorem, by $G3$.

Case 2. $\Lambda = \Phi, D \,/\, E, \Lambda'$ and ψ is contained in Δ, thus $\Delta = \theta, \psi$ and $\Gamma = \Lambda', \theta$. We look at the second premise and infer, by inductional hypothesis, that $\Phi, D \to A$ and $\psi \to B$ are theorems. Then also $\Phi, D \,/\, E, \Gamma \to A$ is a theorem, by $G3$. Applying Lemma 1 to the first premise, we see that θ is empty. Therefore $\Delta = \psi \to B$ and $\Lambda = \Phi, D \,/\, E, \Gamma \to A$ are theorems.

Case 3. $\Lambda = \Phi, D \,/\, E, \Lambda'$ and ψ is not contained in Δ, thus $\Gamma = \Lambda', \theta$ and $\Delta = \theta, \psi$. We look at the second premise and infer, by inductional assumption, that $\Phi, D \to A$ and $\psi \to B$ are theorems. Applying Lemma 1 to the first premise, we again find θ empty and deduce that $\Delta \to B$ and $\Lambda \to A$ are theorems.

Case 4. Λ is in $\Phi = \Lambda, \theta$, then $\Delta = \theta, D \,/\, E, \Gamma, \psi$. Looking at the second premise, we see, by inductional hypothesis, that $\Lambda \to A$ and $\theta, D, \psi \to B$ are theorems. Then also $\Delta \to B$ is a theorem, by $G3$.

Since $G30$ is symmetric to $G3$, this completes the proof of Lemma 2.

LEMMA 3. *Suppose that $\Phi, A \,/\, B, \Lambda, \psi \to C$ is a theorem in $G(\underline{X})$, Λ is prime to Φ, A, ψ, C, and Φ, A, ψ is prime to Λ, B. Then $\Lambda \to B$ and $\Phi, A, \psi \to C$ are theorems.*

The proof is similar to that of Lemma 2 and will be omitted.

4. Decision Procedures for $D(\underline{X})$ and $UF(\underline{X})$.

In order to compare $G(\underline{X})$ with $D(\underline{X})$ we shall associate with each non-empty sequence Λ of terms a single term $|\Lambda|$ of $D(\underline{X})$. This is done inductively by putting

$$|A| = A, \quad |\Lambda, A| = |\Lambda| \cdot A.$$

Thus, for example, $|A, B, C| = (A \cdot B) \cdot C$. The following metatheorem thus establishes a decision procedure for $D(\underline{X})$.

METATHEOREM. *$\Lambda \to B$ is a theorem in $G(\underline{X})$ if and only if $|\Lambda| \to B$ is a theorem in $D(\underline{X})$. In particular, $A \to B$ is a theorem in $G(\underline{X})$ if and only if it is a theorem in $D(\underline{X})$.*

Proof. First assume that $\Lambda \to B$ is a theorem in $G(\underline{X})$ with proof $P = Gm$. We will show that $|\Lambda| \to B$ is a theorem in $D(\underline{X})$ with proof $d(P) = Dm^+$. Dm^+ will be defined by induction.

If $\Lambda \to B$ is an instant of $G1_f$ then $|\Lambda| \to B$ is an instant of $D6_f$. Thus

$$D1_f^+ = D6_f .$$

If $\Lambda \to A \,/\, B$ is obtained by $G2$ from $\Lambda, B \to A$, then $|\Lambda| \to A \,/\, B$ is obtained by $D3$ from $|\Lambda, B| = |\Lambda| \cdot B \to A$. Thus

$$D(2m)^+ = D3m^+.$$

If $\Lambda \to B \setminus A$ is obtained by $G20$ from $B, \Lambda \to A$, then $|\Lambda| \to B \setminus A$ is obtained by $D30$ from $B \cdot |\Lambda| \to A$. This is not the same as $|B, \Lambda| \to A$, but follows from it and the associativity relation Da: $B \cdot |\Lambda| \to |B, \Lambda|$ by $D5$. Thus $D(20m)^+ = D305am^+$.

Similarly $|\Phi, A \cdot B, \psi| \to C$ is obtained from $|\Phi, A, B, \psi| \to C$ in $D(\underline{X})$, thus $D(4m)^+ = D5bm^+$, where Db: $|\Phi, A \cdot B, \psi| \to |\Phi, A, B, \psi|$, provided Φ is non-empty. If Φ is empty, a simplification is possible, and we may write $D(4m)^+ = Dm^+$.

Similarly we also have $D(5mn)^+ = D5c8m^+n^+$, where Dc: $|\Lambda, \Delta| \to |\Lambda| \cdot |\Delta|$ by associativity.

There remain only $G3$ and $G30$ to be considered. For reasons of symmetry, we only look at the former. First, assume that Φ and ψ in the conclusion of $G3$ are empty. Writing $L = |\Lambda|$, we then have:

$$\begin{array}{ll} D7: & \dfrac{A \to C \qquad L \to B}{A \,/\, B \to C \,/\, L} \\ D4: & \dfrac{A \,/\, B \to C \,/\, L}{(A \,/\, B) \cdot L \to C} \end{array}$$

Thus $D(3mn)^+ = D47n^+m^+$. Next, suppose Φ empty and $\psi = P$. Then we have

$$\begin{array}{ll} D3: & \dfrac{A \cdot P \to C}{A \to C \,/\, P} \\ D7: & \dfrac{A \to C \,/\, P \qquad L \to B}{A \,/\, B \to (C \,/\, P) \,/\, L} \\ D4: & \dfrac{A \,/\, B \to (C \,/\, P) \,/\, L}{(A \,/\, B) \cdot L \to C \,/\, P} \\ D4: & \dfrac{(A \,/\, B) \cdot L \to C \,/\, P}{((A \,/\, B) \cdot L) \cdot P \to C} \end{array}$$

Thus $D(3mn)^+ = D4473n^+m^+$. Similarly one may deal with the remaining cases in which Φ is not empty.

We have thus shown one of the two implications of the metatheorem. Conversely, we shall assume that $|\Lambda| \to B$ is a theorem in $D(\underline{X})$, we wish to show that $\Lambda \to B$ is a theorem in $G(\underline{X})$. It suffices to show that $|\Lambda| \to B$ is a theorem in $G(\underline{X})$, for then $\Lambda \to B$ can be derived by repeated applications

of $G9$. Thus we want to obtain $D1$ to $D6$ as derived rules of inference in $G(\underline{X})$. This will imply that each proof $P = Dn\colon A \to B$ in $D(X)$ can be expanded to a proof $g(P) = Gn^*\colon A \to B$ in $G(\underline{X})$. We shall define Gn^* by induction.

Indeed, we may put

$$G1^* = G6,\ G6_f^* = G1_f,$$

$$G2^* = G4456566.$$

The reader will easily reconstruct the last proof from its label. A similar expression is found for $G20^*$. Suppose we already know that Dn may be expanded to Gn^*; then we put

$$G(3n)^* = G27256636n^*,$$

$$G(4n)^* = G475n^*64366.$$

For example, the latter expansion looks like this:

$$G5:\ \dfrac{\dfrac{\overset{Gn^*}{A \to C \,/\, B} \qquad \overset{G6}{B \to B}}{A, B \to (C \,/\, B) \cdot B} \qquad G3:\ \dfrac{\overset{G6}{B \to B} \qquad \overset{G6}{C \to C}}{C \,/\, B, B \to C} \quad G4:}{G4:\ \dfrac{A, B \to C}{A \cdot B \to C}}\ (C \,/\, B) \cdot B \to C$$

$D30$ and $D40$ are treated similarly. Finally, $D5$ is just a case of $G7$, that is, we put $G(5mn)^* = G7m^*n^*$. The proof of the metatheorem is now complete.

In proving the above metatheorem we have translated each proof $Gm\colon \Lambda \to B$ in $G(\underline{X})$ into a proof $Dm^+\colon |\Lambda| \to B$ in $D(\underline{X})$. In particular, a proof of the form $G7mn$ may be thus translated, after first eliminating the derived rule of inference $G7$ by the method of the preceding chapter. On the other hand, $G7mn$ may be translated directly, for example, when Φ and ψ are empty:

$$D5:\ \frac{|\Lambda| \overset{Dm^+}{\to} A \qquad A \overset{Dn^+}{\to} C}{|\Lambda| \to C}.$$

It will be of considerable use to us to know that the roundabout translation of $G7mn$ is equivalent to the direct translation:

$$\begin{aligned}
E6.\ D(7mn)^+ &\equiv D5m^+n^+ && \text{if } \Phi \text{ and } \psi \text{ are empty}\\
&\equiv D58m^+1n^+ && \text{if } \Phi \text{ is empty and } \psi = P\\
&\equiv D5c581m^+n^+ && \text{if } \Phi \text{ is non-empty and } \psi \text{ is empty}\\
&\equiv D5c5881m^+1n^+ && \text{if } \Phi \text{ is non-empty and } \psi = P
\end{aligned}$$

with similar expressions when ψ consists of more than one term. Here

$$Dc\colon |\Phi, \Lambda| \to |\Phi| \cdot |\Lambda|$$

in the third equivalence and

$$Dc\colon |\Phi, \Lambda, P| \to (|\Phi| \cdot |\Lambda|) \cdot P$$

in the fourth, using associativity. (It will be observed that, in view of $E4$, Dc is uniquely determined up to equivalence.)

Proof. We examine in detail the four cases that arise in the proof of $G7$.

Case 1. $G71_f 1_g = G1_{gf}$, hence, by $E5$,

$$D(71_f 1_g)^+ = D1^+_{gf} = D6^+_{gf} \equiv D56_f 6_g = D51^+_f 1^+_g .$$

Case 2. Take for example $k=4$, $\Lambda' = D, E$, and $\Lambda = D \cdot E$. Then $G74mn = G47mn$, hence

$$D(74mn)^+ = D(47mn)^+ = D(7mn)^+.$$

By inductional hypothesis, we may put this

$$\equiv D5m^+n^+ = D5(4m)^+n^+.$$

Case 3. In the example considered, $G7m2n = G27mn$, hence

$$D(7m2n)^+ = D(27mn)^+ = D3(7mn)^+ \equiv D358m^+1n^+,$$

by inductional assumption. In view of $E1$, $E3'$ and $E2$, this may be written

$$\equiv D3558m^+n^+1 \equiv D55m^+3n^+711 \equiv D5m^+3n^+.$$

Case 4. In the example considered, $G75mn4p = G7n7mp$. Thus we have

$$D(75mn4p)^+ = D(7n7mp)^+ \equiv D5c581n^+(7mp)^+ \equiv D5c581n^+58m^+1p^+$$

by inductional assumption, where $Dc\colon |\Lambda| \cdot |\Lambda'| \to |\Lambda, \Lambda'|$. Now

$$\begin{aligned} D58m^+1p^+ &\equiv D43558m^+1p^+1 && \text{by } E3' \text{ and } E1 \\ &\equiv D455m^+3p^+711 && \text{by } E3' \\ &\equiv D455m^+3p^+1 && \text{by } E2 \\ &\equiv D45m^+3p^+ && \text{by } E1. \end{aligned}$$

Therefore

$$\begin{aligned} D581n^+58m^+1p^+ &\equiv D581n^+45m^+3p^+ \\ &\equiv D435581n^+(45m3p^+)1 && \text{by } E3' \text{ and } E1 \\ &\equiv D4551345m^+3p^+71n^+ && \text{by } E3' \\ &\equiv D455m^+3p^+71n^+ && \text{by } E1 \text{ and } E3' \\ &\equiv D43558m^+n^+p^+1 && \text{by } E3' \\ &\equiv D58m^+n^+p^+ && \text{by } E3' \text{ and } E1. \end{aligned}$$

Hence

$$\begin{aligned} D(75mn4p)^+ &\equiv D5c58m^+n^+p^+ \\ &\equiv D55c8m^+n^+p^+ \qquad \text{by } E1 \\ &\equiv D5(5mn)^+(4p)^+, \end{aligned}$$

and this is the result that was to be proved in the case under consideration.

The equivalence relation $\equiv$ between proofs in $D(\underline{X})$ may be transferred to $G(\underline{X})$. Recalling that each proof $Gm\colon \Lambda \to B$ in $G(\underline{X})$ gives rise to a proof $Dm^+\colon |\Lambda| \to B$ in $D(\underline{X})$, we define

$$Gm \equiv Gn \Leftrightarrow Dm^+ \equiv Dn^+.$$

We observe the following substitution properties:

*E*7.

$$\frac{Gm \equiv Gm'}{Gkm \equiv Gkm'} \qquad (k = 2, 20, 4, 5)$$

$$\frac{Gm \equiv Gm' \qquad Gn \equiv Gn'}{Gkmn \equiv Gkm'n'} \qquad (k = 3, 30, 7).$$

With the exception of $k = 7$, these are immediate consequences of the above definition and the proof of the metatheorem. Take, for example, the case $k = 2$. From $Gm \equiv Gn$, i.e. $Dm^+ \equiv Dn^+$, we deduce by *E*0 that $D3m^+ \equiv D3n^+$, i.e. $D(2m)^+ \equiv D(2n)^+$, i.e. $G2m \equiv G2n$. The case $k = 7$ is treated similarly, with the help of *E*6.

We assert that the transfer of the equivalence relation $\equiv$ from $D(\underline{X})$ to $G(\underline{X})$ is an "extension", in the sense that $Dm \equiv Dn \Leftrightarrow Gm^* \equiv Dn^*$, i.e. $Dm^{*+} \equiv Dn^{*+}$. This is an immediate consequence of the following:

*E*8. $Dm^{*+} \equiv Dm$.

Proof. Recall that $G1^* = G6$, hence $D1^{*+} = D6^+$. We will show that $D6_A^+ \equiv D1_A$, by induction on the length of A, consulting the proof of $G6_A$.

Indeed, we have $G6_X = G1_{1_X}$ hence, $D6_X^+ = D1_X^+ = D6_X \equiv D1_{1_X}$ by *E*5. Moreover, $G6_{B\cdot C} = G456_B6_C$, hence

$$D6_{B\cdot C}^+ = D(456_B6_C)^+ = D(56_B6_C)^+ = D86_B^+6_C^+ .$$

By inductional hypothesis, we may put this

$$\equiv D81_B1_C \equiv D1_{B\cdot C}$$

by *E*2.

The cases $A = B \,/\, C$ and $A = B \setminus C$ are handled similarly. Thus we have shown

$$D1^{*+} \equiv D1.$$

Recall that $D6_f^* = G1_f$, hence $D6_f^{*+} = D1_f^+ = D6_f$.

Recall that $G2^* = G4456566$, hence

$$D2^{*+} = D(4456566)^+ = D(56566)^+ = D5281811,$$

as is easily seen, by writing out the proofs in detail. Moreover, using $E2$ and $E1$, this is

$$\equiv D52811 \equiv D521 \equiv D2.$$

Thus we have shown the first of

$$D2^{*+} \equiv D2, \qquad D20^{*+} \equiv D20,$$

the second being similar.

Assuming that $Dn^{*+} \equiv Dn$, we claim also that

$$D(kn)^{*+} \equiv Dkn,$$

when $k = 3, 30, 4, 40$. For example, recalling that $G(4n)^* = G475n^*64366$, we have

$$\begin{aligned} D(4n)^{*+} &= D(475n^*64366)^+ \\ &\equiv D58n^{*+}14711. \end{aligned}$$

Using the inductional assumption and $E2$, this is

$$\begin{aligned} &\equiv D58n141 \\ &\equiv D4358n141 && \text{by } E3' \\ &\equiv D43558n1411 && \text{by } E1 \\ &\equiv D455n341711 && \text{by } E3' \\ &\equiv D455n11 && \text{by } E2 \\ &\equiv D4n && \text{by } E1. \end{aligned}$$

Finally, assuming $Dm^{*+} \equiv Dm$ and $Dn^{*+} \equiv Dn$, and recalling that $G(5mn)^* = D7m^*n^*$, we have

$$\begin{aligned} D(5mn)^{*+} &= G(7m^*n^*)^+ \\ &\equiv D5m^{*+}n^{*+} \\ &\equiv D5mn. \end{aligned}$$

This completes the proof of $E8$.

We are now in a position to establish a relative decision procedure for the underlying category $UF(\underline{X})$ of the free residuated category $F(\underline{X})$ generated by the category $\underline{X}$. Assuming that, for any two objects X and Y of $\underline{X}$, $[X, Y]$ is completely determined, we shall see how to compute $[A, B]$, for any two objects A and B of $UF(\underline{X})$. Now a map $A \to B$ is the equivalence class of a proof $Dm: A \to B$. The above metatheorem suggests that we look at the equivalence classes of proofs $Gn: A \to B$.

In general, let Mult (A, B) denote the set of all equivalence classes of proofs Gn: $A \to B$. We may think of these equivalence classes as *multilinear maps.* Now all proofs of $A \to B$ and therefore all multilinear maps $A \to B$ are easily found by working backwards. (See the comment after the introduction of $G(\underline{X})$.) A decision procedure for $UF(\underline{X})$ is therefore given by the following corollary to the metatheorem and $E8$:

PROPOSITION 3. *The assignments* * *and* $^+$ *establish a one-one correspondence between* $[A, B]$ *and* Mult (A, B).

Proof. In view of $E8$, $Dm \equiv Dm'$ implies $Dm^{*+} \equiv Dm'^{*+}$, that is, $Gm^* \equiv Gm'^*$. We thus have a mapping which assigns to the equivalence class $[Dm]$ the equivalence class $[Gm^*]$. It is one-one, because $Gm^* \equiv Gm'^*$ implies $Dm \equiv Dm^{*+} \equiv Dm'^{*+} \equiv Dm'$. It is onto, because $Gn \equiv Gn^{+*}$, since $Dn^+ \equiv Dn^{+*+}$ by $E8$.

Multilinear maps were used by Bourbaki [2] in resolving problems about the tensor product and by Findlay and the author [6, 9] in studying canonical maps in the category of bimodules.

As an application of our decision procedure we mention the following:

ILLUSTRATION. *The functor* $m(\underline{X})$: $X \to UF(\underline{X})$ *of Proposition 2 is full.*

Proof. Suppose $[Dm]$: $m(\underline{X})(X) \to m(\underline{X})(Y)$ in $UF(\underline{X})$, that is, $[Dm]: X \to Y$. Now it is clear that in $G(\underline{X})$ any proof of $X \to Y$ must have the form $G1_f$: $X \to Y$. Therefore $Gm^* \equiv G1_f$, hence $Dm \equiv Dm^{*+} \equiv D1_f^\dagger \equiv D6_f$, and so $[Dm] = [D6_f] = m(\underline{X})(f)$.

Similarly one may show that in Proposition 1, $m(\underline{X})(X) \leqslant m(\underline{X})(Y)$ implies $X \leqslant Y$.

5. When Are Two Proofs Equivalent?

We have shown that any category $\underline{X}$ generates a free residuated category $F(\underline{X})$. For the moment we shall assume that $\underline{X}$ is a discrete category, that is, just a set. Can we be sure that the underlying category $UF(\underline{X})$ is more than just an ordered set? That the answer to this question is "yes" is shown by the following argument [6].

Let X and Y be distinct objects of $\underline{X}$ and consider the maps

$$p = [D31]: X / Y \to ((X / Y) \cdot Y) / Y$$

$$q = [D7411]: ((X / Y) \cdot Y) / Y \to X / Y.$$

If $UF(\underline{X})$ were just a pre-ordered set, that is, a category which admits only one map between two given objects, the maps qp and pq would both be 1. Let us assume that this is the case, for argument's sake.

Now take $(\underline{A}, \cdot\ \cdot\ \cdot)$ to be the residuated category of all vector spaces ($(R - R)$-bimodules) over a field R, and introduce a functor T: $\underline{X} \to \underline{A}$ by putting $T(X) = A$, $T(Y) = B$, where A has dimension 1 and B has dimension 2. It does not matter what value T has for other objects (elements) of $\underline{X}$.

We then have canonical maps

$$T'(qp)\colon A \,/\, B \to A \,/\, B$$

$$T'(pq)\colon ((A \,/\, B) \cdot B) \,/\, B \to ((A \,/\, B) \cdot B) \,/\, B$$

both equal to 1. This establishes an isomorphism

$$A \,/\, B \cong ((A \,/\, B) \cdot B) \,/\, B.$$

Now $A \,/\, B = \mathrm{Hom}_R(B, A) \cong B \otimes {}_RA$ has dimension 2, while the $RHS \cong A \otimes {}_RB \otimes {}_RB \otimes {}_RB$ has dimension 8, a contradiction.

Actually, it is easily shown that

$$qp = [D5317411] = [D1] = 1.$$

Therefore

$$pq = [D5741131] \neq [D1] = 1.$$

The following consideration makes it plausible why $pq \neq 1$, that is $D5741131 \not\equiv D1$.

Consider the following proof in $D(\underline{X})$, where X, Y, Z, T are distinguished objects of $\underline{X}$.

$$\dfrac{\dfrac{\dfrac{X \,/\, Y \to X \,/\, Y}{(X \,/\, Y) \cdot Y \to X} \quad Z \to Z}{((X \,/\, Y) \cdot Y) \,/\, Z \to X \,/\, Z} \qquad \dfrac{(X \,/\, Z) \cdot T \to (X \,/\, Z) \cdot T}{X \,/\, Z \to ((X \,/\, Z) \cdot T) \,/\, T}}{((X \,/\, Y) \cdot Y) \,/\, Z \to ((X \,/\, Z) \cdot T) \,/\, T}$$

This also has label $D5741131$. If $Y = Z = T$, it specializes to the above proof of

$$((X \,/\, Y) \cdot Y \,/\, Y \to ((X \,/\, Y) \cdot Y) \,/\, Y.$$

On the other hand, the proof $D1$ of this formula yields a different generalization:

$$((X \,/\, Y) \cdot Z) \,/\, T \to ((X \,/\, Y) \cdot Z) \,/\, T.$$

We might say that the proofs $D5741131$ and $D1$ of the same formula have different "scope". We should not expect proofs of different scope to be equivalent.

Later we shall indeed give a rigorous definition of the word "scope" to be able to state a result of this nature. First we shall consider a special situation. From now on $\underline{X}$ may be any category, not necessarily discrete.

PROPOSITION 4. *Assume that no three objects of $\underline{X}$ are connected by maps and that no object of $\underline{X}$ occurs more than once in the formula $\Gamma \to C$. Then any two proofs $P, Q\colon \Gamma \to C$ in $G(\underline{X})$ are equivalent.*

Proof. The proof is by induction on the length of a proof of $\Gamma \to C$, that is, on the number of occurrences of the operation symbols $\cdot$, $/$, $\backslash$ in $\Gamma \to C$. A little reflection will show that, aside from arguments by sym-

metry, we need only consider the following eight cases: the last step in P involves rule Gi and the last step in Q involves rule Gj, where

$$(i, j) = (2, 4), (2, 3), (4, 3), (4, 4), (4, 5), (3, 30), (3, 3), (3, 5), (5, 5).$$

Case 1. $P = G2m$: $\Gamma \to A \,/\, B$; $Q = G4n$: $\Gamma = \Phi, D \cdot E, \psi \to A \,/\, B$. By inductional assumption, $\Gamma, B \to A$ can have only one proof up to equivalence. Now, by $G4$ and $G9$, $\Phi, D \cdot E, \psi, B \to A$ is a theorem if and only if $\Phi, D, E, \psi, B \to A$ is. Therefore P is equivalent to P':

$$\dfrac{\dfrac{\Phi, D, E, \psi, B \overset{Gk}{\to} A}{\Phi, D \cdot E, \psi, B \to A}}{\Phi, D \cdot E, \psi \to A \,/\, B}.$$

Again by inductional assumption, $\Phi, D, E, \psi \to A \,/\, B$ can have only one proof up to equivalence. But $G2k$ is one such proof. Therefore Q is equivalent to Q':

$$\dfrac{\dfrac{\Phi, D, E, \psi, B \overset{Gk}{\to} A}{\Phi, D, E, \psi \to A \,/\, B}}{\Phi, D \cdot E, \psi \to A \,/\, B}.$$

It is not difficult to verify that $P' \equiv Q'$, that is, $G24k \equiv G42k$. For example, taking $\Phi = F$ and ψ empty, P' gives rise to the following proof in $D(\underline{X})$:

$$\begin{array}{ll} D8: & \dfrac{F \cdot (D \cdot E) \overset{D20}{\to} (F \cdot D) \cdot E \qquad B \overset{D1}{\to} B}{(F \cdot (D \cdot E)) \cdot B \to ((F \cdot D) \cdot E) \cdot B \qquad ((F \cdot D) \cdot E) \cdot B \overset{Dk^+}{\to} A} \\ D5: & \\ D3: & \dfrac{(F \cdot (D \cdot E)) \cdot B \to A}{F \cdot (D \cdot E) \to A \,/\, B} \end{array}.$$

On the other hand, Q' gives rise to the following:

$$D5: \quad \dfrac{F \cdot (D \cdot E) \overset{D20}{\to} (F \cdot D) \cdot E \qquad D3: \dfrac{((F \cdot D) \cdot E) \cdot B \overset{Dk^+}{\to} A}{(F \cdot D) \cdot E \to A \,/\, B}}{F \cdot (D \cdot E) \to A \,/\, B}.$$

The equivalence of these two proofs is established thus:

$$\begin{aligned} d(P') &= D358201k^+ \\ &\equiv D3558201k^+1 && \text{by } E1 \\ &\equiv D55203k^+711 && \text{by } E3' \\ &\equiv D55203k^+1 && \text{by } E2 \\ &\equiv D5203k^+ && \text{by } E1 \\ &= d(Q'). \end{aligned}$$

Case 2. $P = G2m$: $\Gamma \to A \,/\, B$; $Q = G3kn$: $\Gamma = \Phi, D \,/\, E, \Lambda, \psi \to A \,/\, B$. By inductional assumption, $\Phi, D, \psi \to A \,/\, B$ can have only one proof up to equivalence. By $G2$ and $G8$, it is a theorem if and only if $\Phi, D, \psi, B \to A$ is.

Therefore it has a proof $G2k'$, and $Q \equiv Q'$:

$$\dfrac{\Lambda \overset{Gk}{\to} E \qquad \dfrac{\Phi, D, \psi, B \overset{Gk'}{\to} A}{\Phi, D, \psi \to A \,/\, B}}{\Phi, D \,/\, E, \Lambda, \psi \to A \,/\, B}.$$

Again by inductional assumption, $\Gamma, B \to A$ can have only one proof up to equivalence. But it has proof $G3kk'$, and therefore $P \equiv P'$:

$$\dfrac{\dfrac{\Lambda \overset{Gk}{\to} E \qquad \Phi, D, \psi, B \overset{Gk'}{\to} A}{\Phi, D \,/\, E, \Lambda, \psi, B \to A}}{\Phi, D \,/\, E, \Lambda, \psi \to A \,/\, B}.$$

One may easily show that $P' \equiv Q'$ by computing $G23kk' \equiv G3k2k'$.

Case 3. $P = G4m$: $\Gamma = \Phi, A \cdot B, \psi \to C$; $Q = G3kn$: $\Gamma = \Phi', D \,/\, E, \Lambda, \psi' \to C$. Thus $\Phi, A \cdot B, \psi = \Phi', D \,/\, E, \Lambda, \psi'$. There are three subcases, according to whether $A \cdot B$ is in Φ', Λ, or ψ'.

Subcase 1. $\Phi' = \Phi, A \cdot B, \theta$; $\psi = \theta, D \,/\, E, \Lambda, \psi'$. Arguing as above, we see that $P \equiv P'$:

$$\dfrac{\dfrac{\Lambda \overset{Gk}{\to} E \qquad \Phi, A, B, \theta, D, \psi' \overset{Gk'}{\to} C}{\Phi, A, B, \theta, D \,/\, E, \Lambda, \psi' \to C}}{\Phi, A \cdot B, \theta, D \,/\, E, \Lambda, \psi' \to C}$$

and that $Q \equiv Q'$:

$$\dfrac{\Lambda \overset{Gk}{\to} E \qquad \dfrac{\Phi, A, B, \theta, D, \psi' \overset{Gk'}{\to} C}{\Phi, A \cdot B, \theta, D, \psi' \to C}}{\Phi, A \cdot B, \theta, D \,/\, E, \Lambda, \psi' \to C}.$$

Again it is easily verified that

$$P' \equiv G43kk' \equiv G34kk' = Q'.$$

Subcases 2 and 3 are treated similarly. It ultimately amounts to showing that $G43kk' \equiv G34kk'$, $G34kk' \equiv G3k4k'$. The details will be omitted.

Case 4. $P = G4m$: $\Gamma = \Phi, A \cdot B, \psi \to C$; $Q = G4n$: $\Gamma = \Phi', D \cdot E, \psi' \to C$. Thus $\Phi, A \cdot B, \psi = \Phi', D \cdot E, \psi'$. Let us say that $\Phi = \Phi', D \cdot E, \theta$, then $\psi' = \theta, A \cdot B, \psi$. It is easily seen that $P \equiv P'$ and $Q \equiv Q'$:

$$\dfrac{\dfrac{\Phi', D, E, \theta, A, B \overset{Gk}{\to} C}{\Phi', D \cdot E, \theta, A, B \to C}}{\Phi', D \cdot E, \theta, A \cdot B \to C} \qquad \dfrac{\dfrac{\Phi', D, E, \theta, A, B \overset{Gk}{\to} C}{\Phi', D, E, \theta, A \cdot B \to C}}{\Phi', D \cdot E, \theta, A \cdot B \to C}.$$

Again it is easily verified that $P' \equiv Q'$. (In this case both have label $G44k$; the distinction lies in the subscripts which have been omitted.)

Case 5. $P = G4m$: $\Gamma = \Phi, A \cdot B, \psi \to C$; $Q = G5kn$: $\Gamma = \Lambda, \Delta \to C = D \cdot E$. Then $\Phi, A \cdot B, \psi = \Lambda, \Delta$. Let us say, $\Phi = \Lambda, \theta$ and $\Delta = \theta, A \cdot B, \psi$. Then $P \equiv P'$:

$$\dfrac{\dfrac{\Lambda \overset{Gk}{\to} D \qquad \theta, A, B, \psi \overset{Gk'}{\to} E}{\Lambda, \theta, A, B, \psi \to D \cdot E}}{\Lambda, \theta, A \cdot B, \psi \to D \cdot E}$$

and $Q \equiv Q'$:

$$\dfrac{\Lambda \overset{Gk}{\to} D \qquad \dfrac{\theta, A, B, \psi \overset{Gk'}{\to} E}{\theta, A \cdot B, \psi \to E}}{\Lambda, \theta, A \cdot B, \psi \to D \cdot E}.$$

Again it is easily verified that $P' = G45kk' \equiv G5k4k' = Q'$.

Up to now the special assumptions concerning the category $\underline{X}$ have not come into play. They will however in the following cases.

Case 6. $P = G3jm$, $Q = G30j'm'$:

$$\dfrac{\Lambda \overset{Gj}{\to} B \qquad \Phi, A, \psi \overset{Gm}{\to} C}{\Phi, A \,/\, B, \Lambda, \psi \to C} \qquad \dfrac{\Lambda' \overset{Gj'}{\to} B' \qquad \Phi', A', \psi' \overset{Gm'}{\to} C}{\Phi', \Lambda', B' \,\backslash\, A', \psi' \to C}$$

where $\Phi, A \,/\, B, \Lambda, \psi = \Phi', \Lambda', B' \,\backslash\, A', \psi'$.

Subcase 1. $B' \,\backslash\, A'$ is in $\Phi = \Phi', \Lambda', B' \,\backslash\, A', \theta$, then $\psi' = \theta, A \,/\, B, \Lambda, \psi$.

We claim that $\Phi', A', \theta, A, \psi$ is prime to Λ, B. For otherwise there would be a map $f\colon X \to Y$ in $\underline{X}$, X in $\Phi', A', \theta, A, \psi$ and Y in Λ, B. But then the proof of $\Lambda \to B$ must contain an axiom $G1_g$, where $g\colon Y \to Z$ or $g\colon Z \to Y$. Since X occurs at most once in the conclusion, $X \neq Z$, and so the three distinct objects X, Y, Z would be connected by maps f and g, contradicting the assumption.

Similarly one shows that Λ is prime to $\Phi', A', \theta, A, \psi, C$. It then follows from Lemma 3 that $\Phi', A', \theta, A, \psi \to C$ is also a theorem. But, by inductional assumption, $\Phi', A', \theta, A \,/\, B, \Lambda, \psi \to C$ has only one proof up to equivalence. Therefore Q is equivalent to Q':

$$\dfrac{\Lambda' \overset{Gj'}{\to} B' \qquad \dfrac{\Lambda \overset{Gj}{\to} B \qquad \Phi', A', \theta, A, \psi \overset{Gk}{\to} C}{\Phi', A', \theta, A \,/\, B, \Lambda, \psi \to C}}{\Phi', \Lambda', B' \,\backslash\, A', \theta, A \,/\, B, \Lambda, \psi \to C}.$$

As in earlier cases, it readily follows that $P \equiv P'$:

$$\dfrac{\Lambda \overset{Gj}{\to} B \qquad \dfrac{\Lambda' \overset{Gj'}{\to} B' \qquad \Phi', A', \theta, A, \psi \overset{Gk}{\to} C}{\Phi', \Lambda', B' \,\backslash\, A', \theta, A, \psi \to C}}{\Phi', \Lambda', B' \,\backslash\, A', \theta, A \,/\, B, \Lambda, \psi \to C}$$

One easily computes that $P' = G3j30j'k \equiv G30j'3jk = Q'$.

Subcase 2. $B' \,\backslash\, A'$ is in ψ, hence $A \,/\, B$ is in Φ'. An argument symmetrical to that for subcase 1 will apply.

Subcase 3. $B' \,\backslash\, A'$ is in $\Lambda = \theta, B' \,\backslash\, A', \Omega$ and $A \,/\, B$ is in $\Lambda' = \Sigma, A \,/\, B, \theta$. Looking at $\Phi', \Sigma, A, \psi \to C$ and applying Lemma 1, we see that Σ, A is not prime to Φ', C, ψ, say $f\colon X \to Y$ with X in Σ, A and Y in Φ', C, ψ. Looking at $\Sigma, A \,/\, B, \theta \to B'$, we see that its proof must contain an axiom $G1_g$ with $g\colon X \to Z$ or $g\colon Z \to X$. In any case, $Z \neq Y$, since Y may occur at most once in the conclusion. Thus X, Y, Z are three distinct objects of $\underline{X}$ connected by maps f and g, contradicting the assumption on $\underline{X}$, and so subcase 3 cannot arise.

Case 7. $P = G3jm$, $Q = G3jm'$. This is quite similar to case 6, but an additional subcase may arise:

$$\frac{\Lambda \to B \qquad \Phi, A, \psi \to C}{\Phi, A \,/\, B, \Lambda, \psi \to C} \qquad \frac{\Lambda' \to B \qquad \Phi, A, \psi' \to C}{\Phi, A \,/\, B, \Lambda', \psi' \to C}$$

where $\Lambda, \psi = \Lambda', \psi'$. For example, $\Lambda = \Lambda', \theta$ and $\psi' = \theta, \psi$. Looking at $\Lambda', \theta \to B$, we see that θ must be empty. For otherwise there would be a map $f\colon X \to Y$ with X in Λ', θ and Y in B. But the proof of $\Lambda' \to B$ must contain an axiom $G1_g$ with $g\colon Y \to Z$ or $g\colon Z \to Y$, where $Z \neq X$ since X can occur at most once in the conclusions. Thus three distinct objects X, Y, Z would be connected by the maps f and g in X, contradicting the assumption on $\underline{X}$. Therefore $\Lambda' = \Lambda$ and $\psi' = \psi$. Now, by inductional assumption $\Lambda \to B$ and $\Phi, A, \psi \to C$ have proofs which are unique up to equivalence, hence $P \equiv Q$.

Case 8. $P = G3im$, $Q = G5jn$:

$$\frac{\Lambda \overset{Gi}{\to} B \qquad \Phi, A, \psi \overset{Gm}{\to} D \cdot E}{\Phi, A \,/\, B, \Lambda, \psi \to D \cdot E} \qquad \frac{\Gamma \overset{Gj}{\to} D \qquad \Delta \overset{Gn}{\to} E}{\Gamma, \Delta \to D \cdot E}$$

hence $\Phi, A \,/\, B, \Lambda, \psi = \Gamma, \Delta$.

Subcase 1. $A \,/\, B$ is in $\Delta = \theta, A \,/\, B, \Lambda, \psi$. As in subcase 1 of case 6, we argue that $\theta, A, \psi \to E$ is a theorem. By inductional assumption, $\theta, A \,/\, B, \Lambda, \psi \to E$ has only one proof up to equivalence, and this proof must be $G3ik$. Therefore $Q \equiv Q' = G5j3ik$. As in earlier cases, we argue that $P \equiv P' = G3i5jk$, and verify that $P' \equiv Q'$.

Subcase 2. $A \,/\, B$ is in $\Gamma = \Phi, A \,/\, B, \theta$ and Δ is in $\psi = \Omega, \Delta$, hence $\theta = \Lambda, \Omega$. As before, we use Lemma 3 to show that $\Phi, A, \Omega \to D$ is a theorem. As in earlier cases, we argue that

$$P \equiv P' = G3i5kp \equiv G53ikp = Q' \equiv Q.$$

Subcase 3. $A \,/\, B$ is in $\Gamma = \Phi, A \,/\, B, \theta$ and ψ is in $\Delta = \Omega, \psi$, hence $\Lambda = \theta, \Omega$. Applying Lemma 1 to $\theta, \Omega \to B$, we see that Ω is empty (otherwise there is a map $f\colon X \to Y$ with X in Ω and Y in B, and looking at $\Lambda \to B$, there would be a map g connecting Y to another object Z in Λ, B). Therefore $\Delta = \psi$ and we are in subcase 2.

Case 9. $P = G5im$, $Q = G5jn$. The argument is similar to the above and will be omitted.

The proof of Proposition 4 is now complete. Before giving an application of this, we need some auxiliary results.

Given a functor $F\colon \underline{X} \to \underline{X}'$ and a term A of $G(\underline{X})$, we write $G(F)(A)$ for the term of $G(\underline{X}')$ obtained from A by replacing all occurrences of objects X of $\underline{X}$ in A by $F(X)$. Thus

$$G(F)(A(X, Y, \cdots)) = A(F(X), F(Y), \cdots).$$

Also, if $\Lambda = A_1, A_2, \cdots, A_n$, we write

$$G(F)(\Lambda) = G(P)(A_1), \cdots, G(F)(A_n).$$

Finally, if P is any proof in $G(\underline{X})$, we write $G(F)(P)$ for the proof in $G(\underline{X}')$ obtained from P by replacing all occurrences of objects X of $\underline{X}$ in P by $F(X)$ and all occurrences of maps f of $\underline{X}$ in P by $F(f)$. Thus

$$G(F)(P(X, \cdots, f, \cdots)) = P(F(X), \cdots, F(f), \cdots).$$

LEMMA 4. *Let P: $\Lambda \to B$ be a proof in $G(X)$. Then there exists a category X^Λ in which no three distinct objects are connected by maps, a functor F: $\underline{X}^\Lambda \to \underline{X}$, and a generalization P^Λ: $\Lambda^\Lambda \to B^\Lambda$ of P such that $G(F)(P^\Lambda) = P$ and such that no object of $\underline{X}^\Lambda$ occurs more than once in $\Lambda^\Lambda \to B^\Lambda$.*

Proof. Let all instances of the axiom scheme $G1$ in the proof P: $\Lambda \to B$ be enumerated by $G1_{f_i}$, $i = 1, 2, \cdots, n$ where f_i: $X_i \to Y_i$ in X. (We do not rule out that $f_i = f_j$ when $i \neq j$.) Let $\underline{X}^\Lambda$ be the sum (disjoint union) of n copies of the category 2, which has two objects 0 and 1 and one map a: $0 \to 1$ other than the identities, with injection functors K_i: $2 \to \underline{X}^\Lambda$. Now for each i there is a functor F_i: $2 \to \underline{X}$ such that $F_i(0) = X_i$, $F_i(1) = Y_i$, $F_i(a) = f_i$. Let F: $\underline{X}^\Lambda \to \underline{X}$ be the unique functor such that $FK_i = F_i$.

Let P^Λ: $\Lambda^\Lambda \to B^\Lambda$ be obtained from P by replacing each axiom $G1_{f_i}$ by its analogue $G1_{K_i(a)}$ in $G(\underline{X}^\Lambda)$. It is easily seen that P^Λ is a proof in $G(\underline{X}^\Lambda)$. Moreover, $F(K_i(a)) = F_i(a) = f_i$, hence $G(F)(P^\Lambda) = P$. It is easily seen that each object of $\underline{X}^\Lambda$ occurs exactly once in $\Lambda^\Lambda \to B^\Lambda$, by induction on the length of the proof. Finally, each object of $\underline{X}^\Lambda$ has the form $K_i(0)$ or $K_i(1)$, hence partakes in at most one map $K_i(a)$ of $\underline{X}^\Lambda$.

We define $D(F)$ quite similarly to $G(F)$ above. Thus, for a proof in $D(X)$, we put

$$D(F)(P(X, \cdots, f, \cdots)) = P(F(X), \cdots, F(f)),$$

and obtain a proof in $D(\underline{X}')$, where F: $\underline{X} \to \underline{X}'$.

We recall from the proof of the Metatheorem of Section 4 that each proof $P = Gm$ in $G(\underline{X})$ gives rise to a proof $d(P) = Dm^+$ in $D(\underline{X})$.

LEMMA 5. *If F: $\underline{X} \to \underline{X}'$ is any functor, and P any proof in $G(\underline{X})$,*

$$D(F)(d(P)) = d(G(F)(P)).$$

Proof. We wish to show that it does not matter whether the function d is applied to a proof P in $G(\underline{X})$ before or after specialization, that is, replacing X by $F(X)$ and f by $F(f)$. This is clear from the recursive construction of the function d, which depends only on the form of P, that is, only on the label Gm without subscripts. (For this reason, subscripts were in fact omitted.)

$E9$. Let F: $\underline{X} \to \underline{X}'$ be a given functor. If P, Q: $A \to B$ are proofs in $D(\underline{X})$, then

$$P \equiv Q \Rightarrow D(F)(P) \equiv D(F)(Q).$$

If P', Q': $\Lambda \to B$ are proofs in $G(\underline{X})$, then

$$P' \equiv Q' \Rightarrow G(F)(P) \equiv G(F)(Q).$$

Proof. The first assertion is proved by observing that equivalence of proofs in $D(\underline{X})$ depends only on the form of the proofs, hence is preserved upon replacing X by $F(X)$ and f by $F(f)$.

The second assertion is now shown as follows. Assume $P' \equiv Q'$ in $G(\underline{X})$, that is, $d(P') \equiv d(Q')$ in $D(\underline{X})$. In view of Lemma 5,

$$d(G(F)(P')) = D(F)d(P')) \equiv D(F)(d(Q')) = d(G(F)(Q')),$$

that is to say, $G(F)(P') \equiv G(F)(Q')$.

COROLLARY. *If X is a discrete category and $\Lambda \to B$ is a theorem in $G(X)$ in which every element of $\underline{X}$ occurs at most twice, then any two proofs of $\Lambda \to B$ are equivalent.*

Proof. Given proofs $P, Q\colon \Lambda \to B$ in $G(\underline{X})$, construct X^Λ, $F\colon \underline{X}^\Lambda \to \underline{X}$, and $P^\Lambda\colon \Lambda^\Lambda \to B^\Lambda$ as in Lemma 4. Suppose Q contains the axiom $Q1_X\colon X \to X$. Then X occurs in $\Lambda \to B$, hence $Gf_X\colon X^\Lambda \to X^\Lambda$ occurs in P^Λ. Let $Q^\Lambda\colon \Lambda^\Lambda \to B^\Lambda$ be formed from Q by replacing each axiom $Q1_X\colon X \to X$ in Q by $Gf_X\colon X^\Lambda \to X^\Lambda$. Then Q^Λ is a proof in $G(\underline{X}^\Lambda)$ and $G(F)(Q^\Lambda) = Q$. Now, by Proposition 4, $P^\Lambda \equiv Q^\Lambda$. Therefore, by $E9$,

$$P = G(F)(P^\Lambda) \equiv G(F)(Q^\Lambda) = Q.$$

By the *scope* of a proof $P\colon A \to B$ in $D(\underline{X})$ we mean the set of all $F\colon \underline{X}^0 \to \underline{X}$, with $\underline{X}^0$ in the given universe, for which there exists a generalization $P^0\colon A^0 \to B^0$ in $D(\underline{X}^0)$ such that $D(F)(P^0) = P$. (One would be tempted to speak of the "generality" of a proof rather than its "scope", were it not for the fact that a proof of $(A \cdot B) \cdot C \to (A \cdot B) \cdot C$ using the associative law has the same scope as one that does not.)

PROPOSITION 5. *The following statements concerning proofs $P, Q\colon A \to B$ in $D(\underline{X})$ are equivalent: (1) $P \equiv Q$; (2) P and Q have the same scope in $D(\underline{X})$; (3) There exists a category $\underline{X}^0$ in which no three objects are connected by maps, a functor $F\colon \underline{X}^0 \to \underline{X}$, and proofs $P^0, Q^0\colon A^0 \to B^0$ in $G(\underline{X}^0)$ such that $G(F)(P^0) = g(P)$ and $G(F)(Q^0) = g(Q)$ and no object of $\underline{X}^0$ occurs more than once in $A^0 \to B^0$.*

Proof. (1) $\Rightarrow$ (2). Suppose $P\colon A \to B$ and $Q\colon A \to B$ are equivalent proofs in $D(\underline{X})$. We shall show that they have the same scope, by induction on the length of the proof that $P \equiv Q$ using rules $E0$ to $E5$. (See the proof of Proposition 2 for these rules.)

$E0$. If the last step in the proof of $P \equiv Q$ uses reflexivity, symmetry, or transitivity, the result is obvious. Let us assume that it uses the substitution property, for example, concerning rule $G3$. Then P and Q have the form

$$\frac{A \cdot B \overset{P'}{\to} C}{A \to C \,/\, B} \qquad \frac{A \cdot B \overset{Q'}{\to} C}{A \to C \,/\, B}$$

where $P' \equiv Q'$. By inductional hypothesis, P' and Q' have the same scope. Assume $F\colon \underline{X}^0 \to \underline{X}$ and $D(F)(P^0) = P$, where $P^0\colon A^0 \to C^0 \,/\, B^0$ is a proof in $D(\underline{X}^0)$. Then surely there exists a proof $P'^0\colon A^0 \cdot B^0 \to C^0$ in $D(\underline{X}^0)$, where

$D(F)(P'^0) = P'$ and P^0 is obtained from P'^0 by following it up with rule $G3$. Since Q' has the same scope as P', Q' also has a generalization Q'^0: $A^0 \cdot B^0 \to C^0$ with $D(F)(Q'^0) = Q'$. Let Q^0 be obtained from Q'^0 by following it up with rule $G3$, then $D(F)(Q^0) = Q$. This shows the scope of P is contained in the scope of Q. The converse inclusion holds by symmetry.

$E1$. We will show that $D51m$ has the same scope as Dm and that $D55mnp$ has the same scope as $D5m5np$.

First, let P be a proof with label $D51m$, thus

$$\frac{A \xrightarrow{Q} B \qquad B \to B}{A \to B}$$

where Q has label Dm. Assume that P^0: $A^0 \to B^0$ is a generalization of P in $D(\underline{X}^0)$ with F: $\underline{X}^0 \to \underline{X}$ and $D(F)(P^0) = P$. Then P^0 has the form

$$\frac{A^0 \xrightarrow{Q^0} B^0 \qquad B^0 \to B^0}{A^0 \to B^0}$$

where $D(F)(Q^0) = Q$. This shows that the scope of P is contained in the scope of Q. The converse conclusion is even easier.

Next, let P and Q be proofs with labels $D55mnp$ and $D5m5np$ respectively:

$$\frac{A \xrightarrow{R} B \qquad \dfrac{B \xrightarrow{S} C \qquad C \xrightarrow{T} D}{B \to D}}{A \to D} \qquad \frac{\dfrac{A \xrightarrow{R} B \qquad B \xrightarrow{S} C}{A \to C} \qquad C \xrightarrow{T} D}{A \to D}.$$

Assume that F: $\underline{X}^0 \to \underline{X}$ and P has generalization P^0: $A^0 \to D^0$ in $D(\underline{X}^0)$ with $F(P^0) = P$. Then P^0 has the form

$$\frac{A^0 \xrightarrow{R^0} B^0 \qquad \dfrac{B^0 \xrightarrow{S^0} C^0 \qquad C^0 \xrightarrow{T^0} D^0}{B^0 \to D^0}}{A^0 \to D^0}$$

Here R^0, S^0, T^0 are generalizations of R, S, T respectively, with $D(F)(R^0) = R$, etc. Therefore the following is a generalization Q^0 of Q:

$$\frac{\dfrac{A^0 \xrightarrow{R^0} B^0 \qquad B^0 \xrightarrow{S^0} S^0}{A^0 \to C^0} \qquad C^0 \xrightarrow{T^0} D^0}{A^0 \to D^0}$$

and $D(F)(Q^0) = Q$. The converse inclusion is proved similarly.

Rules $E2$ to $E5$ are treated in a similar fashion. We shall omit the details. The inductional proof of the implication (0) $\Rightarrow$ (1) is thus complete.

(2) $\Rightarrow$ (3). Construct $\underline{X}^\Delta$, F: $\underline{X}^\Delta \to \underline{X}$, and P^Δ: $A^\Delta \to B^\Delta$ in $G(\underline{X}^\Delta)$ such that $G(F)(P^\Delta) = g(P)$, as in Lemma 4. Then, by Lemma 5,

$$d(g(P)) = d(G(F)(P^\Delta)) = D(F)(d(P^\Delta)).$$

Now $d(g(P)) \equiv P$, by $E8$, and, since (1) $\Rightarrow$ (2), $d(g(P))$ and P have the same scope. If we assume (2), this is also the scope of Q. Therefore there exist proofs $P', Q': A^{\wedge} \to B^{\wedge}$ in $D(\underline{X}^{\wedge})$ such that $D(F)(P') = P$ and $D(F)(Q') = Q$.

(3) $\Rightarrow$ (4). Assume (3). By Proposition 4, $g(P^0) \equiv g(Q^0)$ in $G(\underline{X}^0)$, that is, $d(g(P^0)) \equiv d(g(Q^0))$ in $D(\underline{X}^0)$. In view of $E8$, this implies $P^0 \equiv Q^0$. It then follows from $E9$ that $D(F)(P^0) \equiv D(F)(Q^0)$, that is, $P \equiv Q$.

This completes the proof of Proposition 5. We observe that, if the actual construction of $\underline{X}^{\wedge}$ and $F: \underline{X}^{\wedge} \to \underline{X}$ implied by the proof of Lemma 4 is borne in mind, condition (3) provides us with a decision procedure for determining when two proofs in $D(\underline{X})$ are equivalent.

ILLUSTRATION. *The functor $m(\underline{X})$: $\underline{X} \to UF(\underline{X})$ of Proposition 2 is faithful.*

Proof. Suppose $f, g: X \to Y$ in $\underline{X}$ and $m(\underline{X})(f) = m(\underline{X})(g)$, that is, $D6_f \equiv D6_g$. Then the proofs $D6_f$ and $D6_g$ have the same scope. Let 2 be the category with two objects 0 and 1 and one non-identity map $a: 0 \to 1$. Let $F: 2 \to \underline{X}$ be the functor defined by

$$F(0) = X, F(1) = Y, F(a) = f.$$

Then $D6_a$ is a proof in $D(2)$, and $D(F)(D6_a) = D6_f$. Therefore 2 must contain a map b such that $D(F)(D6_b) = D6_g$, that is $F(b) = g$. It follows from the paucity of maps in the category 2 that $a = b$, hence $f = g$.

REFERENCES

[1] J. BENABOU, Catégories avec multiplication, *C. R. Acad. Sci. Paris* **256** (1963), 1887–1890.
[2] N. BOURBAKI, *Algèbre Multilinéaire*, Hermann, Paris, 1948.
[3] H. B. CURRY and R. FEYS, *Combinatory Logic*, Vol. 1, Amsterdam, North-Holland Publ. Comp., 1958.
[4] S. EILENBERG and G. M. KELLY, Closed categories, Proc. Conference Categorical Algebra, LaJolla 1965, pp. 421–562, Springer-Verlag, New York, 1966.
[5] S. EILENBERG and J. C. MOORE, Adjoint functors and triples, *Illinois J. Math.* **9** (1966), 381–398.
[6] G. D. FINDLAY and J. LAMBEK, Calculus of Bimodules, unpublished manuscript, 1955.
[7] S. C. KLEENE, *Introduction to Metamathematics*, Van Nostrand, New York, 1952.
[8] J. LAMBEK, The mathematics of sentence structure, *Amer. Math. Monthly* **65** (1958), 154–169.
[9] J. LAMBEK, On the calculus of syntactic types, *Amer. Math. Soc., Proc. Symposia Appl. Math.*, **12** (1961), 166–178.
[10] S. MACLANE, Natural associativity and commutativity, *Rice University Studies* **49** (1963), No. 4., 28–46.

(Received 20 December 1967)