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Binary Relations in the Social and Mathematical Sciences

This is an exposition of some applications of the calculus of binary relations to anthropology, linguistics, computer science and mathematics. Binary relations appear in the rewrite systems used by anthropologists to study kinship terminologies of primitive societies. They also serve to model the syntactic calculus, a form of categorial grammar once proposed by the author. Moreover, if we introduce an operation *perp* as the complement of the converse, we may take the calculus of binary relations as a model for cyclic linear logic, studied by Yetter in a first attempt to remove commutativity from linear logic, recently of interest in theoretical computer science. Partial recursive functions may best be viewed as recursively enumerable binary relations which happen to be single-valued; they are of the form f composed with g converse, where f and g are primitive recursive functions satisfying the condition: g converse composed with g is contained in f converse composed with f. In mathematics, binary relations were pushed into the background when it was decided that functions be single-valued and universally defined. However, they made a comeback in homological algebra, where they provide the easiest construction for the so-called connecting homomorphism. While such constructions were originally confined to abelian categories, it is now clear that they can be extended to arbitrary categories, provided one chooses appropriate generalizations of 'exactness' and other relevant concepts.

The insistence that a *function* has to be universally defined and single valued is a fairly recent innovation in mathematics. For instance, the square root, viewed as a 'function' from Z to Z is neither and would better now be called a *relation*. Relations have recently come into their own, see for instance the books by MacLane (1967), Freyd and Scedrov (1990) and Tarski and Givant (1987). But long before their acceptance in mathematics, relations played a role in the kinship systems of various cultures. As Chomsky (1979) remarked: "the Greeks made up number theory, others made up kinship systems."

1. Cousinhood in the Trobriands

In English, a (first) cousin may be described as a parent's sibling's child, in other words, to say that x's cousin is y means that, for some z and t, x's parent is z and z's sibling is t and t's child is y. To introduce our notation, we write this thus:

$$\exists_z \exists_t (xPz \wedge zSt \wedge tCy),$$

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that is,

$$xPSCy,$$

where juxtaposition denotes the relative product.

In French, one must distinguish between $PSCM$ and $PSCF$, where M and F are subrelations of the identity, restricted to males and females respectively.

In some languages, not only the gender of y, but also that of x and of the intermediate z and t may be relevant. Thus, in discussing Trobriand kinship terminology, one must consider

$$G_1PG_2SG_3CG_4,$$

where each $G_i = M$ or F.

According to Malinowski (1932), a person whom we would call a (first) cousin in English, may be referred to as *tuwa, bwada, luta, latu, tabu* or *tama*, depending on the values of the G_i, or he or she may not be viewed as a relative at all. No wonder, Malinowski's translation was challenged by Leach (1958), who asserted: "when terms are projected onto a genealogical diagram, the underlying logic is utterly incomprehensible."

To discover which kinship term corresponds to each of the sixteen kinship descriptions covered by the formula $G_1PG_2SG_3CG_4$ requires a fairly sophisticated calculation. Such a calculation may be based on a set of postulates, discovered by Lounsbury (1965) after analyzing the original data collected by Malinowski. This is not to say that Trobrianders are consciously aware of such postulates, just as we are not aware of the grammatical rules underlying spoken English.

According to Bhargava and Lambek (1995), Lounsbury's rules may be expressed in our notation as follows, where G may be M or F:

$$PFC \to S,$$
$$GSG \to G \text{ after } P \text{ or before } C,$$
$$MSF \to MPF \text{ after } P,$$
$$FSM \to FCM \text{ before } C.$$

Note the absence of $PMC \to S$, which has been interpreted as reflecting the fact that Trobrianders did not acknowledge the father's role in procreation. Note also that FSM is the converse (or reciprocal) relation of MSF, making the fourth rule a mirror image of the third.

We shall consider four cases, depending on whether G_2 and G_3 are F or M.

$$G_1PFSFCG_4 \to G_1PFCG_4 \to G_1SG_4, \tag{1}$$

replacing FSF by F and then PFC by S. If $G_1 = G_4$, the cousin is called *tuwa* or *bwada*, depending on whether he or she is older or younger than ego. If $G_1 \neq G_4$, the cousin is called *luta*.

$$G_1PFSMCG_4 \to G_1PFCMCG_4 \to G_1SMCG_4, \tag{2}$$

replacing FSM by FCM and then PFC by S. If $G_1 = M$, we replace MSM by M and obtain MCG_4, and the cousin is called *latu*, the primary meaning of which is 'child'. If $G_1 = F$, we replace FSM by FCM, and the cousin is called *tabu*, the primary meaning of which is 'grandchild' or 'grandparent'.

$$G_1PMSFCG_4 \to G_1PMPFCG_4 \to G_1PMSG_4, \tag{3}$$

replacing MSF by MPF and then PFC by S. If $G_4 = M$, we replace MSM by M and obtain G_1PM, and the cousin is called *tama*, the primary meaning of which is 'father'. If $G_4 = F$, we replace PMS by PMP, and the cousin is again called *tabu*.

$$G_1PMSMCG_4 \to G_1PMCG_4, \tag{4}$$

replacing MSM by M. No further rule applies and, indeed, as Malinowski insisted, this kind of cousin is not considered to be a relative.

2. Models of Bilinear Logic

If R, S and T are three binary relations on a set X, one may define $R \otimes S = RS$, T/S and $R\backslash T$ as follows:

$$x(R \otimes S)y \Leftrightarrow \exists_{z \in X}(xRz \wedge zSy),$$
$$x(T/S)y \Leftrightarrow \forall_{z \in X}(ySz \Rightarrow xTz),$$
$$x(R\backslash T)y \Leftrightarrow \forall_{z \in X}(zRx \Rightarrow zTy),$$

and verify that

$$R \otimes S \leq T \Leftrightarrow R \leq T/S \Leftrightarrow S \leq R\backslash T$$

and, of course

$$(R \otimes S) \otimes T = R \otimes (S \otimes T).$$

Thus, binary relations on a set provide a model of the so-called *syntactic calculus* (Lambek, 1958). While this calculus had originally been intended for application to natural languages, Andréka and Mikulás (1993) (following a suggestion by van Benthem (1991)) proved a completeness result for it in terms of relational models.

One may also consider the relations I (sometimes incorporated into the syntactic calculus) and its complement O, defined by

$$xIy \Leftrightarrow x = y, \qquad xOy \Leftrightarrow x \neq y,$$

and then introduce

$$R^{\perp} = O/R = R\backslash O.$$

One may verify that

$$R \otimes I = R = I \otimes R, \qquad R^{\perp\perp} = R,$$
$$R \otimes S \leq O \Leftrightarrow R \leq S^{\perp} \Leftrightarrow S \leq R^{\perp}.$$

Thus, one obtains a model of Yetter's (1990) *cyclic bilinear logic*, a noncommutative generalization of Girard's (1987) linear logic, of interest in theoretical computer science. Cyclic bilinear logic still retains an element of commutativity, in as much as

$$R \otimes S \leq O \Leftrightarrow S \otimes R \leq O.$$

It is possible to get rid of this too, so that O/R and $R\backslash O$ become two distinct 'negations' of R (Lambek, 1993); however, this would take us away from the calculus of relations.

Note also the *converse* $R^\vee$ of R, defined by

$$xR^\vee y \Leftrightarrow yRx,$$

which satisfies

$$(R \otimes S)^\vee = S^\vee \otimes R^\vee, \qquad R^{\vee\vee} = R, \qquad I^\vee = I$$

and

$$R^\perp = \neg(R^\vee) = (\neg R)^\vee,$$

where $\neg$ denotes the Boolean complement. According to Kneale and Kneale (1984), the last equation was already pointed out by De Morgan. An axiomatic description of the calculus of relations, incorporating the notion of converse and the usual Boolean operations, may be found in the book by Tarski and Givant (1987).

An interesting generalization of binary relations as subsets of $X \times X$ replaces them by downward closed subsets of $X^{\mathrm{op}} \times X$, where X is a partially ordered set and X^{op}, its opposite, has the reverse order.

3. Recursive Functions

As I have shown elsewhere (Lambek, 1992), primitive recursive functions $f\colon N^k \to N$ may be viewed as operations of an algebra. It seems that partial and general recursive functions are most easily introduced as relations.

We are interested in binary relations on N which are recursively enumerable in the sense that their graphs are. Now, it was shown by Rosser (1936) that, if a set is recursively enumerable, then it can already be enumerated by primitive recursive functions. Hence, to say that the binary relation R on N is *recursively enumerable* means that there exist two primitive recursive functions $f, g\colon N \to N$, such that

$$xRy \Leftrightarrow \exists_{z \in N}(x = fz \wedge y = gz).$$

We may regard f and g themselves as binary relations, so that, e.g.,

$$xfz \Leftrightarrow x = fz.$$

Then R is recursively enumerable if and only if $R = fg^\vee$ for some primitive recursive functions f and g.

A small argument is required to show that the set of recursively enumerable relations is closed under composition. Given primitive recursive functions f, g, h and k, we claim that $fg^{\vee}hk^{\vee}$ is recursively enumerable. Indeed,

$$x(g^{\vee}h)y \Leftrightarrow gx = hy.$$

Now, clearly, the set of such pairs (x, y) is recursive, hence recursively enumerable, hence

$$g^{\vee}h = pq^{\vee},$$

for some primitive recursive functions p and q. Thus,

$$fg^{\vee}hk^{\vee} = (fp)(kq)^{\vee}$$

is also recursively enumerable.

For $R = fg^{\vee}$ to be *single valued*, we require that $RR^{\vee} \leq I$, that is,

$$g^{\vee}g \leq f^{\vee}f.$$

Thus, a *partial recursive* function (in one variable) may be defined as a binary relation $fg^{\vee}$ on N where f and g are primitive recursive functions and $g^{\vee}g \leq f^{\vee}f$.

We define the *domain* and *image* of a binary relation as follows:

$$x \in \text{Dom } R \Leftrightarrow \exists_y yRx, \qquad y \in \text{Im } R \Leftrightarrow \exists_x yRx.$$

Now

$$\text{Dom}(fg^{\vee}) = \text{Dom}(g^{\vee}) = \text{Im } g,$$

hence $fg^{\vee}$ is *universally defined* if and only if g is surjective. Therefore, a *general recursive* function (in one variable) may be defined as a partial recursive function $fg^{\vee}$ such that g is surjective.

We also note that

$$\text{Im}(fg^{\vee}) = \text{Im } f,$$

from which it follows again that a set enumerable by a general recursive function $fg^{\vee}$ is already enumerable by the primitive recursive function f. Of course, this result depends on our new definition of 'general recursive', which was only justified by Rosser's (1936) original observation.

4. Relations in Algebra

There are at least two places were relations turned out to be significant in algebra: in group theory and in homological algebra.

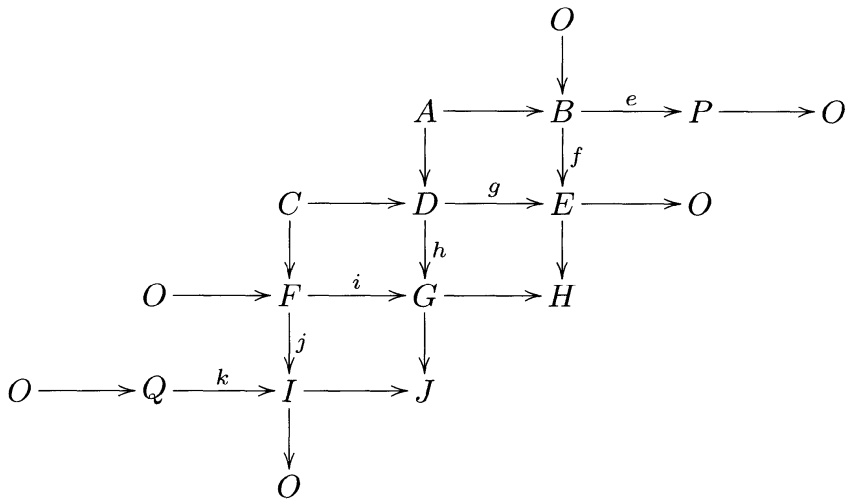
Already Goursat (1889) had made an observation which, in modern terminology, may be expressed as follows: given a *homomorphic relation* ρ between groups G and H, that is, a binary relation whose graph is a subgroup of $H \times G$, $\rho^{\vee}\rho$ and $\rho\rho^{\vee}$ are congruence relations on subgroups of G and H, respectively, and the factor groups of these subgroups modulo the respective congruence relations are isomorphic.

As I pointed out in (Lambek, 1957), Goursat's result may be exploited to give an alternative derivation of the lemma which Zassenhaus used to establish the Jordan–Hölder–Schreier Theorem. His lemma concerns two subgroups U and V of G with normal subgroups U' and V', respectively, and produces an isomorphism

$$\frac{U' \cdot (U \wedge V)}{U' \cdot (U \wedge V')} \cong \frac{V' \cdot (U \wedge V)}{V' \cdot (U' \wedge V)}.$$

As it turns out, this is an easy consequence of Goursat's result if we take $\rho = \kappa\lambda$, where κ is the congruence relation on U determined by U' and λ the congruence relation on V determined by V', both κ and λ being viewed as homomorphic relations on G.

There is a well-known example in homological algebra where binary relations are useful (see MacLane (1967)). Given a ring R with unity, consider the following diagram in the category of R-modules and R-homomorphisms:



Assume that all rows and columns are exact and that all squares commute. Then the binary relation

$$\varphi = kk^{\vee}ji^{\vee}hg^{\vee}fe^{\vee}e$$

is a homomorphism $\varphi\colon\ B \to I$, called the *connecting homomorphism*, and the sequence

$$A \to B \to I \to J$$

is exact.

Both the Zassenhaus Lemma and the connecting homomorphism have been generalized to arbitrary *Goursat varieties*, that is, varieties of algebras in which, for any two congruence relations θ and θ', one has

$$\theta\theta'\theta\theta' = \theta\theta',$$

to mention only one characterization (Lambek, 1995a, 1964).

In what follows, we shall leave varieties behind and proceed to arbitrary categories, or rather 'operational categories'.

5. Relations in Categories

We claim that the construction of the connecting homomorphism can be extended to a large class of categories, after appropriate definitions have been adopted to generalize the notions of 'exactness' and 'commutativity'.

To make a category $\mathcal{C}$ into an *operational* category $\mathcal{C}_\mathcal{G}$, we require an auxilary category $\mathcal{G}$ and a functor $[-,-]\colon \mathcal{G}\times\mathcal{C}\to$ Set into the category of sets. In the special case when $\mathcal{G}$ is a subcategory of $\mathrm{Set}^\mathcal{C}$, this functor may be taken to be 'application'. In many examples, $\mathcal{G}$ is a subcategory of $\mathcal{C}^{\mathrm{op}}$, $\mathcal{C}$ is locally small and $[-,-]$ is the usual Hom-functor.

A *potential relation* $\rho\colon A \not\to B$ in $\mathcal{C}_\mathcal{G}$, A and B being objects of $\mathcal{C}$, is a natural family of ordinary relations

$$\{\rho_G\colon [G,A] \not\to [G,B]\}_{G\in\mathcal{G}}.$$

We say that ρ is *universally defined* if $1_A \leq \rho^\vee\rho$ and *single-valued* if $\rho\rho^\vee \leq 1_B$. It is a morphism if it is both universally defined and single-valued; we then write $\rho\colon A \to B$. A morphism is thus a natural transformation $[-,A]\to[-,B]$. It is *injective* if $\rho^\vee\rho \leq 1_A$ and *surjective* if $1_B \leq \rho\rho^\vee$.

Special potential relations are the following:

(1) *representable* morphisms $[-,f]\colon [-,A]\to[-,B]$, where $f\colon A\to B$ in $\mathcal{C}$, we usually just write $[-,f]=f$;

(2) the *converse* $\rho^\vee\colon B\not\to A$ of a potential relation $\rho\colon A\not\to B$, where $(\rho^\vee)_G = (\rho_G)^\vee$ is the usual converse;

(3) the *relative product* $\sigma\rho\colon A\not\to C$ of two potential relations $\rho\colon A\not\to B$ and $\sigma\colon B\not\to C$, where $(\sigma\rho)_G=\sigma_G\rho_G$ is the usual relative product;

(4) a *zig-zag relation* has the form $fg^\vee h\cdots$;

(5) a *congruence relation* on B has the form $h^\vee h$, where $h\colon B\to C$;

(6) a *co-congruence relation* on B has the form $hh^\vee$, where $h\colon C\to B$;

(7) any intersection of potential relations $A\not\to B$.

It is often useful to assume that $\mathcal{G}$ *generates* $\mathcal{C}$ in the following sense: if $f,g\colon A\to B$ and $[G,f]=[G,g]$ for all G in $\mathcal{G}$, then $f=g$. Following Isbell (who considered the case $\mathcal{G}\subseteq\mathcal{C}$), we call $\mathcal{G}$ *adequate* if every morphism $A\to B$ is representable, that is, every natural transformation $[-,A]\to[-,B]$ has the form $[-,f]$ for some $f\colon A\to B$. If $\mathcal{G}$ generates and is adequate, this arrow $f\colon A\to B$ is uniquely determined and we have justified our identification of the morphism $[-,f]$ of $\mathcal{C}_\mathcal{G}$ with the arrow f of $\mathcal{C}$. In what follows we shall assume that all morphisms of $\mathcal{C}_\mathcal{G}$ are arrows of $\mathcal{C}$.

A *left fork*

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{h} C$$

is *exact* if $h^\vee h$ is the smallest congruence relation on B containing $fg^\vee$. In particular, this implies that the intersection $\mathrm{Im}(f,g)$ of all congruence relations on B containing $fg^\vee$ is also a congruence relation. (Note that the potential relations $B\not\to B$ form

a complete lattice.) Thus, exactness of the left fork asserts that $\mathrm{Im}(f,g) = \mathrm{Ker}\, h$, where $\mathrm{Ker}\, h = h^\vee h$.

A *right fork*

$$C \xrightarrow{h} B \underset{g}{\overset{f}{\rightrightarrows}} A$$

is *exact* if $hh^\vee$ is the largest co-congruence relation on B contained in $g^\vee f$. This implies that the join $\mathrm{Ker}(f,g)$ of all co-congruence relations on B contained in $g^\vee f$ is also a co-congruence relation, and exactness of the right fork asserts that $\mathrm{Im}\, h = \mathrm{Ker}(f,g)$, where $\mathrm{Im}\, h = hh^\vee$.

We shall distinguish two kinds of squares, depending on which arrows are doubled. We say that the *left square*

$$\begin{array}{ccc} A & \rightrightarrows & B \\ \downarrow\downarrow & & \downarrow f \\ C & \xrightarrow{g} & D \end{array}$$

quasi-commutes if

$$ff^\vee g\, \mathrm{Im}(A \rightrightarrows C)g^\vee ff^\vee = gg^\vee f\, \mathrm{Im}(A \rightrightarrows B)f^\vee gg^\vee.$$

We say that the *right square*

$$\begin{array}{ccc} D & \xrightarrow{g} & C \\ \downarrow f & & \downarrow\downarrow \\ B & \rightrightarrows & A \end{array}$$

quasi-commutes if

$$f^\vee fg^\vee\, \mathrm{Ker}(C \rightrightarrows A)gf^\vee f = g^\vee gf^\vee\, \mathrm{Ker}(B \rightrightarrows A)fg^\vee g.$$

We shall assume that the operational category $\mathcal{C}_\mathcal{G}$ has the *Goursat* property:

$$\rho\rho^\vee\rho\rho^\vee = \rho\rho^\vee$$

for every zig-zag relation ρ. Consider now the diagram

$$\begin{array}{ccccccc} & & A & \rightrightarrows & B & \xrightarrow{e} & P \\ & & \downarrow\downarrow & & \downarrow f & & \\ & C & \rightrightarrows & D & \xrightarrow{g} & E & \\ & \downarrow\downarrow & & \downarrow h & & \downarrow\downarrow & \\ & F & \xrightarrow{i} & G & \rightrightarrows & H & \\ & \downarrow j & & \downarrow\downarrow & & & \\ O & \xrightarrow{k} & I & \rightrightarrows & J & & \end{array}$$

and assume that all forks are exact, that all squares quasi-commute, that f and i are injections and that g and j are surjections. Then the zig-zag relation φ: $B \nrightarrow I$ defined by

$$\varphi = kk^\vee ji^\vee hg^\vee fe^\vee e$$

is a morphism and

$$A \rightrightarrows B \longrightarrow I \rightrightarrows J$$

is exact, meaning that both forks are.

So far, we have not committed ourselves as to which *potential* relations in $\mathcal{C}_G$ should be called 'relations'. For the present, we have in mind all zig-zag relations; but, for other purposes, we may also wish to admit intersections of such. For more details, see the author's 'Relations in categories' (Lambek, 1995b). For an earlier attempt to generalize the connecting homomorphism to Maltsev categories, see Carboni *et al.* (1990).

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