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# The algebra of distributive and extensive categories

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# Thesis Summary of Stephen Geoffrey Lack: The algebra of distributive and extensive categories

The aim of this thesis is to develop the algebra of distributive categories, guidance being drawn from the point of view that distributive categories are a categorical analogue to commutative rings. Many things done in the theory of commutative rings can be adapted to distributive categories; similarly much that is true of the *category* of commutative rings can be adapted to the case of the *bicategory* of distributive categories.

Corresponding to the category of abelian groups are the bicategory of categories with sums and the bicategory of extensive categories. The category $\mathbf{Ab}$ of abelian groups has a monoidal structure, and so it is possible to consider categories enriched in the monoidal category $\mathbf{Ab}$. In much the same way, the bicategories of categories with sums and of extensive categories both possess a monoidal structure in the appropriate bicategorical sense, and one can formulate the notion of a bicategory enriched in one of these monoidal bicategories.

The first three chapters develop the theory of monoidal bicategories, and of bicategories enriched in such. In the fourth chapter we turn to the distributive and extensive categories themselves; most of this chapter concerns actual facts about such categories rather than the bicategories to which they belong, the major exception being the construction of the extensive completion of a distributive category. In the fifth chapter some of the theory of coinverters is given. Coinverters are the bicategorical version of quotients, and as such of course crucial in any investigations in bicategorical algebra. We prove some results of general interest and some which are specifically geared towards the needs of later chapters. The sixth chapter presents the monoidal structure of the bicategories of categories with sums and of extensive categories. The final chapter presents the bicategory of distributive categories, and develops notions of simplicity, ideal, module, and so on, at the level of distributive categories.

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except where specifically indicated in the text.

*Στην υγειά μασ κι ασ πεθάνουν οι εχθροί μασ.*

# Acknowledgements

Over the three years and several months I have spent working towards a Ph. D there are many people who have made my life richer and more enjoyable and it is a great pleasure to be able to give them some small thanks here. Sean, Mike and I were at the University of Sydney together, became even closer friends here — Sean despite living with me for a year — and doubtless will remain so when I return to Sydney. Chris and Angus became good friends early on, and were always available to pop out for a pint if ever one felt such a need. Each has influenced the final appearance of the dissertation in a small but significant way. In my second year Lisa and Simon were constant companions, and I must have spent almost as much time at their house as at my own. While in Paris during my third year Duncan shared the joys of discovering a strange new city, and did his best to prevent my ever completing a thesis. Mary, Alain, Stéphane, et tous les autres du troisième étage à Montrouge, que puis-je vous dire mieux que "A la petite pause de 11 heures". In my final term at Cambridge I met a whole new circle of friends: Emma — who makes a cameo appearance a little later on, Loopy Lu the crazy rehead, the betoggled Yash, and Dan the venison king and fellow addict of Bogarde, Fellini and Pulp Fiction; they made what could have been a grim few months into a real joy.

My parents were always a reliable source of support throughout my time overseas and I am very grateful to them for that ... and much more.

I have indicated elsewhere specific intellectual debts I feel, but I should like here to acknowledge and thank the people who have shaped me in a more general way as a mathematician. In roughly chronological order of my coming into contact with them, they are Bob Walters, Aurelio Carboni, Max Kelly, Peter Johnstone, and Martin Hyland. Peter, as my supervisor, of course deserves extra thanks for all he has done for me in that respect over the three years.

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*Cambridge* Steve Lack

*26 November 1995*

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# Introduction

A distributive category is a category with finite products and finite coproducts, satisfying a distributive law expressing that

$$A \times (B + C) \cong A \times B + A \times C.$$

There is an obvious formal analogy here with commutative rings, and it is the pursuit and understanding of this analogy which is the subject of this dissertation. Before considering the similarities between these two situations, let us for a moment consider the differences.

1. Unlike the case of the elements of a ring, the objects of a category can, although not equal, still be isomorphic, in which case we should still like to consider them as being "the same". However in category theory it is never enough to know that two objects are isomorphic, one must always also know which is the isomorphism. This then raises the questions of *naturality* and *coherence* of these isomorphisms, with which we shall be occupied for much of this dissertation. Similarly when we consider not categories but bicategories, we shall replace isomorphism by equivalence as our notion of "sameness"; this continues on and there are biequivalences of bicategories, and triequivalences of tricategories.

2. In distributive categories there is nothing corresponding to subtraction or additive inverses. In fact we shall see in Chapter 4 that there can *never* be objects corresponding to additive inverses in a category with finite coproducts. Thus the analogy should be not so much with commutative rings, as

with the algebraic structures christened commutative *rigs* by Schanuel [37]; they are the "rings without negatives", which possess two binary associative unitary structures, one distributing over the other. Rigs are also sometimes known as semirings. In fact much of the theory of commutative rings carries over to the case of rigs. In particular the category of commutative monoids is symmetric monoidal closed, and so one can consider categories enriched in it; the category of commutative rigs admits such an enrichment, just as the category of commutative rings admits an enrichment in the category of abelian groups.

3. Analogous to the commutative monoids in the case of commutative rigs are the categories with finite coproducts. These form not just a category but a 2-category, and although they do have a monoidal structure, it is not the structure of a monoidal category on the underlying category of the 2-category, but rather the structure of a *monoidal bicategory*, or one-object tricategory. Similarly, the distributive categories and their maps form not just a category but a 2-category, and this admits an enrichment not in the ordinary sense of $\mathcal{V}$-categories [26], but rather of $\mathcal{W}$-bicategories, defined in Chapter 3, for $\mathcal{W}$ the monoidal bicategory of categories with finite coproducts.

4. There is a second class of categories which can be used to replace the commutative monoids as the recipients of the underlying additive structure; these are the categories with finite coproducts satisfying a certain additional property, called *extensivity*. This property has no real analogue among the commutative monoids; although it is true that the extensive categories enjoy a certain "cancellation" law (see Lemma 4.3.2), this does not serve to characterize them.[1] Once more the extensive categories and their maps form a 2-category which admits the structure of a monoidal bicategory; the 2-category of distributive categories which are moreover extensive admits an

[1] Schanuel [37] has a different notion of cancellativity, not even satisfied by all extensive categories.

enrichment (once more in the bicategorical sense) in this monoidal bicategory of extensive categories.

5. Although not actually a difference between the theories of distributive categories and commutative rings, it is perhaps apposite here to make the following word of caution. There is another "categorical version" of rings than the distributive categories, namely the one-object abelian-group-enriched categories. In fact such a category is precisely a (not necessarily commutative) ring: the arrows of the category are the elements of the ring, the additive structure is given by the abelian group structure on the hom-set coming from the enrichment, and the multiplicative structure by composition in the category. Needless to say this is quite distinct from the notion of ring we have in mind.

These differences notwithstanding, we are still able to adapt a large amount of the basic theory of abelian groups and commutative rings to the case of categories with finite coproducts (possibly extensive) and distributive categories. Thus we are able to obtain sensible notions, for example, of tensor product, quotient, ideal, module, and polynomial distributive category. Intuition is always sought by considering the *Burnside rig* [37], comprising the isomorphism classes of objects of the distributive category, with multiplication and addition formed by taking the categorical product and coproduct.

The first three chapters develop background material on bicategories, tricategories, and enriched bicategories. Chapter 1 gives basic definitions of bicategories and tricategories, almost all of which is well known. In the case of bicategories the major reference is [2]; our presentation of the definitions is slightly different to the usual in order to pave the way for later definitions of tricategories and enriched bicategories. Then the tricategories, introduced in [20], are defined; once more the approach is slightly different, moreover in the case of the morphisms the definition is substantively different, this change being made in order to ensure that these morphisms compose. Several co-

herence results are quoted from [20]; fundamental is that every tricategory is triequivalent to a **Gray**-category [21, 20]. Just as a monoidal category is a one-object bicategory, so a monoidal bicategory is a one-object tricategory; once again the coherence result is crucial: it says that every monoidal bicategory is monoidally biequivalent (*i.e.* triequivalent as a tricategory) to a **Gray**-monoid, or one-object **Gray**-category.

Chapter 2 gives some coherence results for bicategories. The slickest proof that every bicategory is biequivalent to a 2-category is the observation of Gordon and Power [19] that Street's theorem that the Yoneda embedding for a bicategory is fully faithful [38] implies that it is biequivalent to the 2-category which is its image. What seems to be new is that using left Kan extensions allows us to extend this argument to morphisms, transformations, and modifications: given a modification between pseudo-natural transformations between homomorphisms of bicategories, one can deduce the existence of a modification between 2-natural transformations between 2-functors, equivalent at each level in the appropriate "pseudo" sense. Unfortunately the functoriality of this construction is also only of the pseudo variety, making the final theorem of the chapter rather weaker than one might be tempted to hope for.

In Chapter 3 we turn to enriched bicategories. A definition was given in [20], however it seems to involve *too much* structure, and we follow the definition of [9]. We then make the non-trivial step of defining morphisms, transformations, and modifications, and various compositions of such; also the underlying bicategory of an enriched bicategory, and the canonical $\mathcal{W}$-bicategory $\overline{\mathcal{W}}$, for $\mathcal{W}$ a monoidal closed bicategory. All this is done first in the simplified context where $\mathcal{W}$ is a **Gray**-monoid, and then, using the coherence result of Chapter 1 and [20], for general $\mathcal{W}$.

The second part, comprising the remaining four chapters, is where we turn to the distributive and extensive categories themselves, applying the material

of the first part of the dissertation. In Chapter 4 we give basic definitions and results for distributive and extensive categories, including the cancellation lemma referred to above. Especially important is the explicit construction of the extensive reflection of a distributive category.

In Chapter 5 coinverters are treated. These are the categorical version of quotients and so of course crucial to algebraic analyses such as ours. Unfortunately, for an arbitrary category the construction is rather complicated; it becomes simpler if the category has pullbacks [3], and one can even make do with only certain pullbacks. This chapter depends heavily both on [3] and its predecessor [18]; some of it appeared as [27] and [28] and some of it is completely new.

Chapter 6 defines the tensor product of categories with finite coproducts, and proves that this makes the 2-category of such categories into a monoidal closed bicategory. Once more this is done using coherence methods; first proving the theorem for a restricted class of categories, and then using a coherence result to deduce it for all such. The definition of the tensor product, but not the further structure, was presented in the author's undergraduate thesis [29], written under the supervision of R. F. C. Walters. The final section of the chapter adapts these results to the case of extensive categories.

Finally in Chapter 7 we move to what might be termed the algebra of distributive categories. We define the tensor product of distributive categories, show that the bicategory of distributive categories enriches in the monoidal bicategory of categories with finite coproducts. This analysis is repeated with distributive categories being replaced, variously, by distributive and extensive categories, locally distributive categories, and lextensive categories. We then adapt some of the basic definitions, constructions, and results of commutative ring theory to this setting. Theorem 7.3.6 also appeared in [29].

There are also two appendices. The first may prove helpful in under-

standing some aspects of Chapters 1 and 3; the second shows how certain other ideas [15] can be re-expressed in terms of the language and machinery developed above.

We should perhaps make a few comments about assumed knowledge and preliminary reading. Basic category theory is assumed throughout; as usual [33] is the standard reference. Basic enriched category theory is not assumed as such, but doubtless some familiarity with this subject would greatly facilitate understanding the ideas contained here; for this one might consult [31, 26]. Definitions and therems needed for bicategories are always at least stated, albeit sometimes a little tersely. Still the only real *introduction* to bicategories is the aptly named [2], for further reading [38] is highly recommended; as well is giving a good general framework for thinking about bicategories, the important definition of a module over a distributive category, which appears in Chapter 7, is directly inspired by it. Two more general sources of insight into questions of higher-dimensional coherence can be found in [39] and [40]. Finally the entire project of looking at distributive categories in this way can be seen as being inspired by Schanuel's short and beautiful paper [37], the reading of which would surely clarify much that remains muddy or opaque below.

# Chapter 1

# Bicategories and tricategories

## 1.1 Bicategories and their maps

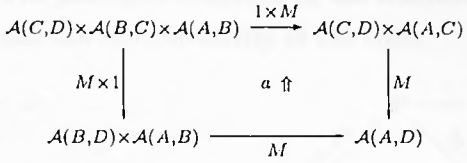
Bicategories were defined in [2], where two equivalent definitions were given, called there the local and global definitions. Here we present several basic aspects of the theory of bicategories in a third and different framework, which one might call the "geometric approach". It may or may not be preferable when dealing only with bicategories, but it certainly will be helpful later on, both in the context of tricategories, and in the context of enriched bicategories, and so we shall introduce it here by way of preparation.

A bicategory $\mathcal{A}$, then, comprises data

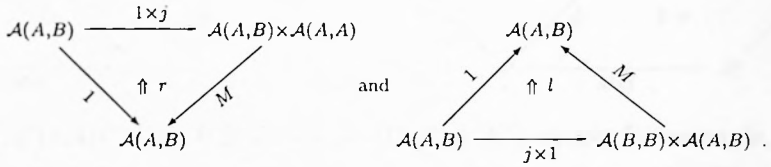
- a set $Ob\mathcal{A}$ of objects
- a hom-category $\mathcal{A}(A, B)$ for each pair of objects $A, B \in \mathcal{A}$
- a composition functor $M : \mathcal{A}(B, C) \times \mathcal{A}(A, B) \to \mathcal{A}(A, C)$ for each triple $A, B, C \in \mathcal{A}$
- a functor "identity" $j : 1 \to \mathcal{A}(A, A)$ for each $A \in \mathcal{A}$
- an "associativity" natural isomorphism



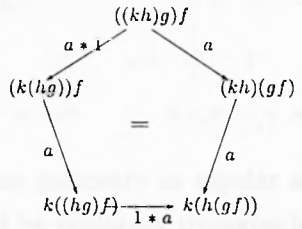
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- natural isomorphisms "the identity isomorphisms"

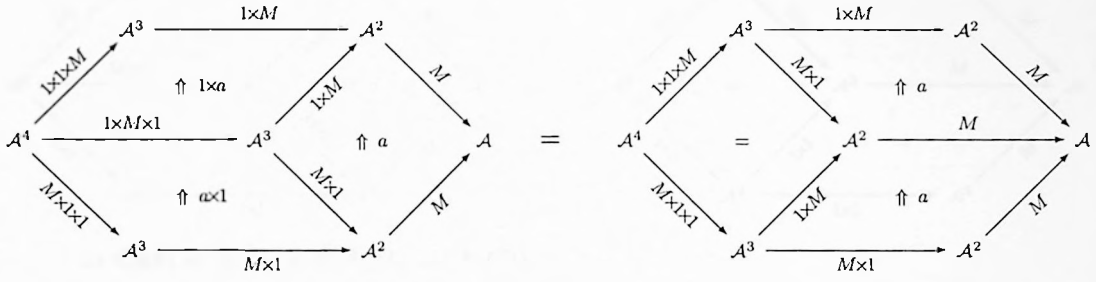



These data are required to satisfy certain coherence conditions; normally these conditions are expressed as the equality of certain composites of the components of the natural isomorphisms, such as the famous pentagon condition:



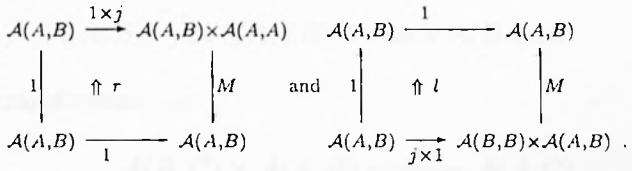

We take a slightly different approach, and continue to deal with the natural isomorphisms as entities in themselves, rather than considering only their components — calling them invertible 2-cells rather than natural isomorphisms is perhaps more suggestive in this context. The identity isomorphisms we rewrite so as to be squares commuting up to isomorphism, and then paste these squares together into polyhedra, which we ask to commute, in the sense that, breaking one of these polyhedra into two halves, the pasting composites of the halves are equal. Commutativity of the pentagon then becomes commutativity of a cube; the five sides of the pentagon become five faces of the cube, the sixth is an identity 2-cell and thus lost when working at the

level of components. The pentagon condition in the traditional definition of bicategories thus become the commutativity of the cube:




in $\mathbf{Cat}(\mathcal{A}(D,E)\times\mathcal{A}(C,D)\times\mathcal{A}(B,C)\times\mathcal{A}(A,B),\mathcal{A}(A,E))$ where, for example, $\mathcal{A}^4$ stands for $\mathcal{A}(D,E)\times\mathcal{A}(C,D)\times\mathcal{A}(B,C)\times\mathcal{A}(A,B)$ and $\mathcal{A}$ for $\mathcal{A}(A,E)$.

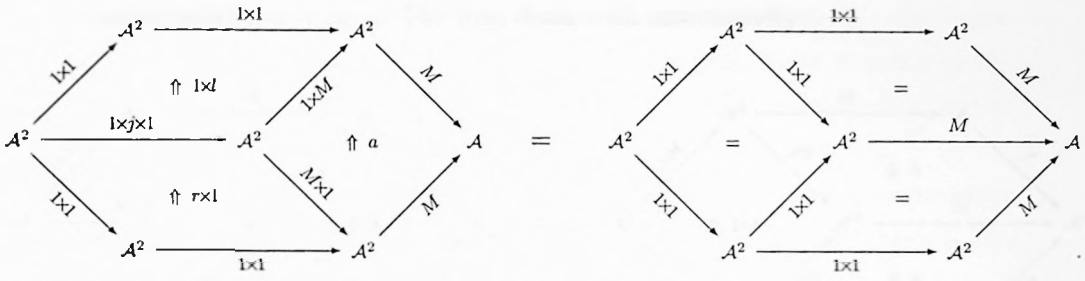
The other change is to take the triangles of $r$ and $\ell$ and artificially replace them by squares:




The point here is to keep the geometry as regular as possible. While there is in itself little to be gained by replacing triangles by squares, in a moment we shall give the coherence condition for the constraints $r$ and $l$, and by our artificial construct this condition will become commutativity of a cube, rather than of a square pyramid, which is already some saving. As we progress to higher dimensions, the greater simplicity of dealing with higher dimensional cubes rather than higher dimensional pyramids soon becomes significant.

The other coherence condition, then, is commutativity of the following

cube:




in $\mathbf{Cat}(\mathcal{A}(B,C)\times\mathcal{A}(A,B),\mathcal{A}(A,C))$.

We once more use the "geometric" approach in the definition of morphisms of bicategories. A morphism $\mathcal{A}\to\mathcal{A}'$ of bicategories comprises the data

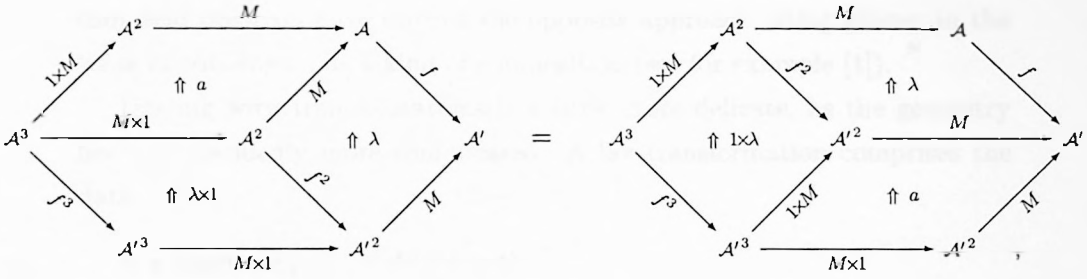
- an object function $f : Ob\mathcal{A}\to Ob\mathcal{A}'$
- functors $f : \mathcal{A}(A,B)\to\mathcal{A}'(fA,fB)$ for each $A,B\in\mathcal{A}$
- natural transformations

$$\begin{array}{ccc} \mathcal{A}(B,C)\times\mathcal{A}(A,B) & \xrightarrow{\quad M\quad} & \mathcal{A}(A,C) \\ {\scriptstyle f\times f}\downarrow & \lambda\ \Uparrow & \downarrow{\scriptstyle f} \\ \mathcal{A}'(fB,fC)\times\mathcal{A}'(fA,fB) & \xrightarrow[\quad M\quad]{} & \mathcal{A}'(fA,fC) \end{array}$$

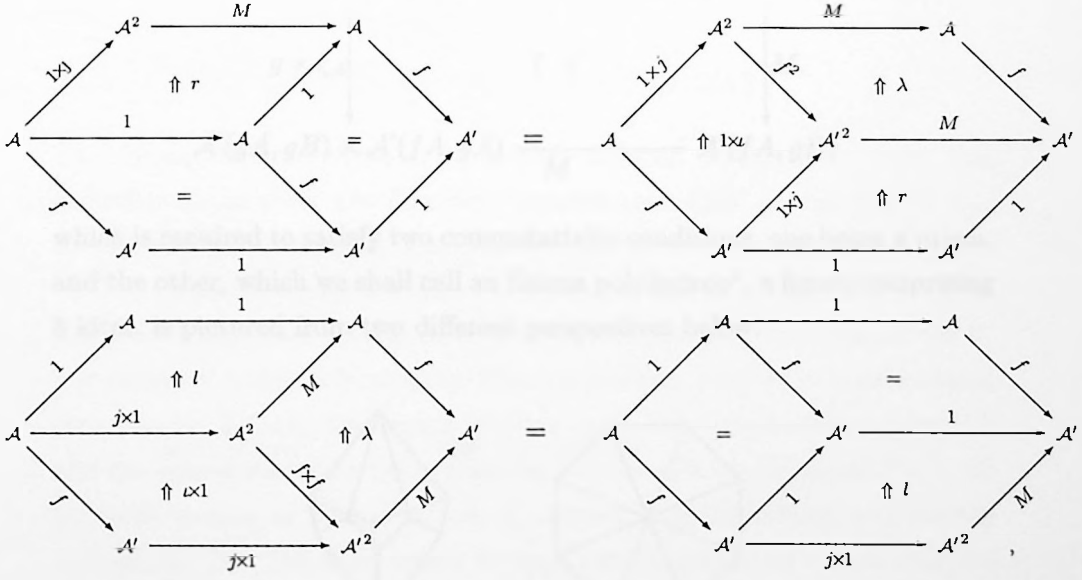
- natural transformations

$$\begin{array}{ccc} 1 & \xrightarrow{\quad j\quad} & \mathcal{A}(A,A) \\ {\scriptstyle 1}\downarrow & \iota\ \Uparrow & \downarrow{\scriptstyle f} \\ 1 & \xrightarrow[\quad j\quad]{} & \mathcal{A}'(fA,fA) \end{array}$$

which are now required to satisfy three conditions — all three being the commutativity of a cube. The first deals with associativity:




in $\mathbf{Cat}(\mathcal{A}(C,D)\times\mathcal{A}(B,C)\times\mathcal{A}(A,B),\mathcal{A}'(fA,fD))$; and the other two with the identity constraints




both in $\mathbf{Cat}(\mathcal{A}(A,B),\mathcal{A}'(fA,fB))$.

Composition also can be viewed geometrically: the 2-cells $\lambda$ and $\iota$ for the composite of two morphisms are obtained as pasting composites of the corresponding 2-cells for the individual morphisms, and moreover the coherence

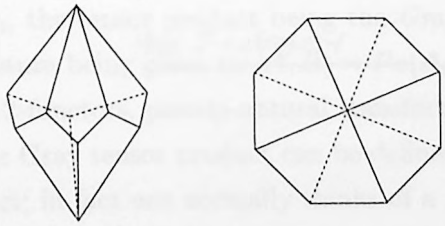
conditions can immediately be seen to hold by pasting the appropriate cubes. This idea of "composition as gluing" was already present in [2], where the composite of transformations was defined in this way; more recently quantum field theorists have utilized the opposite approach, using gluing, in the sense of cobordism, as a kind of composition (see for example [1]). *

Dealing with transformations is a little more delicate, as the geometry becomes decidedly more complicated. A lax transformation comprises the data

- a functor $\xi_A : 1 \to \mathcal{A}'(fA, gA)$
- a natural transformation

$$\begin{array}{ccc} \mathcal{A}(A,B) & \xrightarrow{\xi_B \times f} & \mathcal{A}'(fB,gB) \times \mathcal{A}(fA,fB) \\ {\scriptstyle g \times \xi_A}\downarrow & \xi \Uparrow & \downarrow{\scriptstyle M} \\ \mathcal{A}'(gA,gB) \times \mathcal{A}'(fA,gA) & \xrightarrow[M]{} & \mathcal{A}'(fA,gB) \end{array}$$

which is required to satisfy two commutativity conditions, one being a prism, and the other, which we shall call an Emma polyhedron[1], a figure comprising 8 kites, is pictured from two different perspectives below.



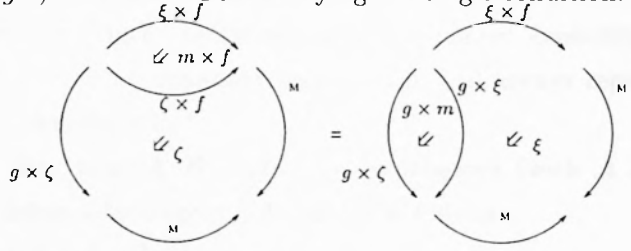
(1.1.1)

[1]named after the person who spent several Sunday afternoons helping me to build models of them

* We shall write Bicat for the category of bicategories and homomorphisms.

Appendix A contains instructions for constructing Emma polyhedra. In fact just as earlier we saw that hexagons arise as two-dimensional "projections" of cubes, so Emma polyhedra can be seen as arising as three-dimensional "projections" of more regular higher dimensional objects [14]. We shall not go into this point of view here.

Finally a modification comprises a natural transformation $m_A : \xi_A \to \zeta_A : 1 \to \mathcal{A}'(fA, gA)$ for each $A \in \mathcal{A}$ satisfying the single condition:




## 1.2 The "closed" structure of Bicat

The (mere) category $\mathbf{Cat}_0$ is of course cartesian closed; it is using this cartesian-closed structure that one "enriches over **Cat**" — for that is just to do $\mathcal{V}$-category theory for the cartesian closed category **Cat**.

The (mere) category $\mathbf{2\text{-}Cat}_0$ of 2-categories and 2-functors is also cartesian closed, and when we take it, with the cartesian-closed structure, as our $\mathcal{V}$, a $\mathcal{V}$-category is then a 3-category. There is however another monoidal-closed structure on $\mathbf{2\text{-}Cat}_0$, the tensor product being the *Gray* tensor product $\square$, and the ~~closed structure~~ being ~~given by $[A, B] = Ps(A, B)$ where~~ $Ps(A, B)$ ~~is the 2-category~~ of 2-functors, pseudo-natural transformations, and modifications [20, 21]. The Gray tensor product can be defined as a coidentifier of the cartesian product; in fact one normally thinks of a map out of $A \square B$ as being a map out of $A \times B$ satisfying a certain property (where this property is precisely that it coidentify certain 2-cells).

In this section we extend the results of [20, 21] in a straightforward manner to deal with bicategories and homomorphisms, rather than the 2-categories and 2-functors of [20] or the 2-categories and lax functors of [21].*

In an effort to keep the notation within control, we try as far as possible not to give names to isomorphisms which arise in the context of something "happening-up-to-isomorphism". Thus the basic isomorphisms in the definitions of bicategories, homomorphisms, strong transformations, and also the structure cells introduced below will never be named explicitly unless ambiguity would arise; an unnamed isomorphism will always represent one of these "basic isomorphisms".

Suppose then that $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are bicategories (with $\mathcal{A}$ and $\mathcal{B}$ being small). We define a bicategory $\{\mathcal{A} \wedge \mathcal{B}, \mathcal{C}\}$ as follows.

An object of $\{\mathcal{A} \wedge \mathcal{B}, \mathcal{C}\}$ comprises

- homomorphisms $\mathcal{F}(-, B) : \mathcal{A} \to \mathcal{C}$ for each $B \in \mathcal{B}$
- homomorphisms $\mathcal{F}(A, -) : \mathcal{B} \to \mathcal{C}$ for each $A \in \mathcal{A}$

such that $\mathcal{F}(A, -)(B) = \mathcal{F}(-, B)(A)$ which we then write as $\mathcal{F}(A, B)$;

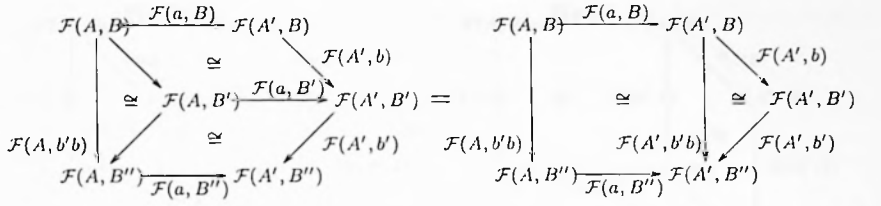
- invertible 2-cells in $\mathcal{C}$ (called structure cells)

$$\begin{array}{ccc} \mathcal{F}(A,B) & \xrightarrow{\mathcal{F}(a,B)} & \mathcal{F}(A',B) \\ {\scriptstyle \mathcal{F}(A,b)}\downarrow & \cong & \downarrow{\scriptstyle \mathcal{F}(A',b)} \\ \mathcal{F}(A,B') & \xrightarrow[\mathcal{F}(a,B')]{} & \mathcal{F}(A',B') \end{array}$$

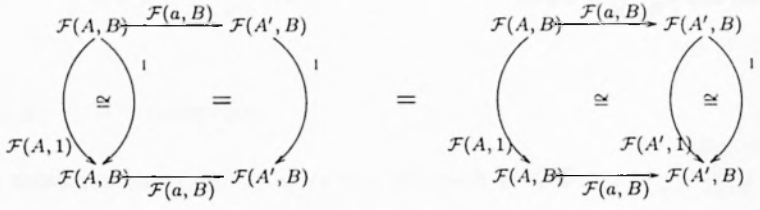
for each $a : A \to A'$ in $\mathcal{A}$, $b : B \to B'$ in $\mathcal{B}$

satisfying the three conditions

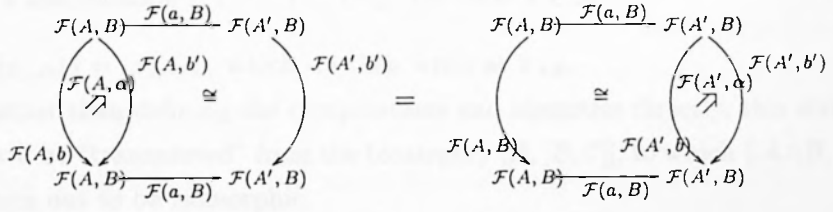
* We shall write $[\mathcal{A}, \mathcal{B}]$, or sometimes $\mathrm{Hom}(\mathcal{A}, \mathcal{B})$, for the bicategory of homomorphisms, strong transformations, and modifications between given bicategories $\mathcal{A}$ and $\mathcal{B}$.




(1.2.1)




(1.2.2)



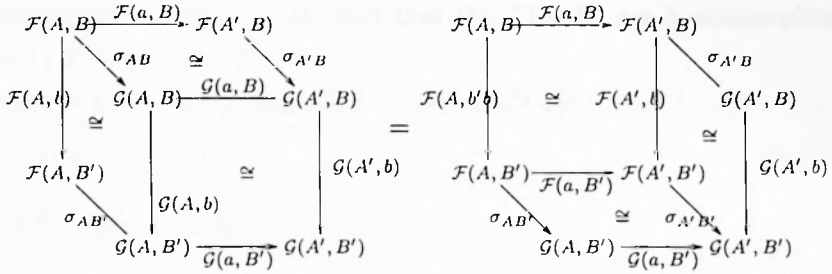

(1.2.3)

and the three corresponding conditions in which the roles of the first and second variables are reversed; we shall call these (1.2.1′), (1.2.2′), and (1.2.3′).

An arrow $\sigma : \mathcal{F} \to \mathcal{G}$ comprises

- a strong transformation $\sigma_{-B} : \mathcal{F}(-, B) \to \mathcal{G}(-, B)$ for each $B \in \mathcal{B}$
- a strong transformation $\sigma_{A-} : \mathcal{F}(-, A) \to \mathcal{G}(-, A)$ for each $A \in \mathcal{A}$

such that $(\sigma_{-B})_A = (\sigma_{A-})_B$, which we then write as $\sigma_{AB}$; and satisfying




(1.2.4)

A 2-cell $x : \sigma \to \tau$ comprises

- a modification $x_{-B} : \sigma_{-B} \to \tau_{-B}$ for each $B \in \mathcal{B}$
- a modification $X_{A-} : \sigma_{A-} \to \tau_{A-}$ for each $A \in \mathcal{A}$

such $(x_{-B})_A = (x_{A-})_B$, which we then write as $x_{AB}$.

Rather than defining the compositions and identities directly, this structure will be "transported" from the bicategory $[\mathcal{A}, [\mathcal{B}, \mathcal{C}]]$, to which $\{\mathcal{A} \wedge \mathcal{B}, \mathcal{C}\}$ will turn out to be isomorphic.

**Lemma 1.2.1** *The objects of* $[\mathcal{A}, [\mathcal{B}, \mathcal{C}]]$ *are in bijection with those of* $\{\mathcal{A} \wedge \mathcal{B}, \mathcal{C}\}$.

PROOF: Given $\mathcal{F} \in \{\mathcal{A} \wedge \mathcal{B}, \mathcal{C}\}$, we define $\mathcal{F}^* : \mathcal{A} \to [\mathcal{B}, \mathcal{C}]$ as

- $\mathcal{F}^* A = \mathcal{F}(A, -)$
- $(\mathcal{F}^* a)_B = \mathcal{F}(a, B)$; the structure cells and (1.2.1), (1.2.2), and (1.2.3) making this a strong transformation
- $(\mathcal{F}^* \alpha)_B = \mathcal{F}(\alpha, B)$; here (1.2.3′) makes this a modification

and then functoriality in $\alpha$ follows as the $\mathcal{F}(-,B)$ are homomorphisms, while functoriality in $a$ follows by the fact that the $\mathcal{F}(-,B)$ are homomorphisms and by (1.2.1′) and (1.2.2′).

Given $\mathcal{H} \in [\mathcal{A},[\mathcal{B},\mathcal{C}]]$ define $\mathcal{H}^{\#} \in \{\mathcal{A} \wedge \mathcal{B},\mathcal{C}\}$ by

- $\mathcal{H}^{\#}(A,-) = \mathcal{H}A$
- $\mathcal{H}^{\#}(-,B) = ev_B\mathcal{H}$
- structure cell for $a : A \to A'$ and $b : B \to B'$ from the naturality isomorphism of $\mathcal{H}a$ for $b$

and then (1.2.1), (1.2.2), and (1.2.3) hold because $\mathcal{H}a$ is a strong transformation; (1.2.1′) and (1.2.2′) hold because $\mathcal{H}$ is a homomorphism, and (1.2.3′) holds because $\mathcal{H}\alpha$ is a modification. The assignments $^{\star}$ and $^{\#}$ are now mutually inverse: this is completely obvious for the "assignment" parts, giving the value of the homomorphisms at an object, arrow, or 2-cell; just a little care must be taken that the functoriality constraints also agree. □

**Lemma 1.2.2** *Arrows $\mathcal{F} \to \mathcal{G}$ are in bijection with arrows $\mathcal{F}^{\star} \to \mathcal{G}^{\star}$.*

PROOF: Given $\sigma : \mathcal{F} \to \mathcal{G}$, define the component of $\sigma^{\star} : \mathcal{F}^{\star} \to \mathcal{G}^{\star} : \mathcal{A} \to [\mathcal{B},\mathcal{C}]$ at $A$ to be $\sigma^{\star}A = \sigma A$, and the naturality isomorphism for $a : A \to A'$ to be

$$\begin{array}{ccc} \mathcal{F}^{\star}A & \xrightarrow{\sigma^{\star}_A} & \mathcal{G}^{\star}A \\ {\scriptstyle \mathcal{F}^{\star}a}\downarrow & \cong & \downarrow{\scriptstyle \mathcal{G}^{\star}a} \\ \mathcal{F}^{\star}A' & \xrightarrow[\sigma^{\star}_{A'}]{} & \mathcal{G}^{\star}A' \end{array}$$

the invertible modification having component at $B$ the naturality isomorphism of $\sigma_B$ for $a$; this is indeed a modification by commutativity of the cube (1.2.4); it follows that $\sigma^{\star}$ is a strong transformation as the $\sigma_{-B}$ are.

On the other hand, given $\lambda : \mathcal{F}^{\star} \to \mathcal{G}^{\star}$, define $\lambda^{\#} : \mathcal{F} \to \mathcal{G}$ by



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- $\lambda^{\#}_B = ev_B\lambda$
- $\lambda^{\#}_A = \lambda_A$

and then $(\lambda^{\#}_B)_A = (ev_B\lambda)_A = (\lambda A)_B = (\lambda^{\#}_A)_B$ and (1.2.4) holds by the 2-dimensional part of the naturality condition for $\lambda$, making $\lambda^{\#}$ into an arrow $\mathcal{F} \to \mathcal{G}$ as desired.

The assignments $^*$ and $^{\#}$ are again mutually inverse. □

**Lemma 1.2.3** *The 2-cells from $\sigma : \mathcal{F} \to \mathcal{G}$ to $\tau : \mathcal{F} \to \mathcal{G}$ are in bijection with the modifications $\sigma^* \to \tau^*$.*

PROOF: Given $x : \sigma \to \tau$, define $x^* : \sigma^* \to \tau^*$ by $x^*A = x_{A-}$; then $x^*$ is a modification because the $x_B$ are. Given $y : \sigma^* \to \tau^*$ define $y^{\#} : \sigma \to \tau$ by

- $y^{\#}_A = y_A$
- $y^{\#}_B = ev_B y$

and then $(y^{\#}_A)_B = y_{AB} = (ev_B y)_A = (y^{\#}_B)_A$ and so $y^{\#}$ is indeed a 2-cell.

Once more $^*$ and $^{\#}$ are clearly mutually inverse. □

The following theorem is now immediate.

**Theorem 1.2.4** *The bicategorical structure of $[\mathcal{A}, [\mathcal{B}, \mathcal{C}]]$ now "transports" to make $\{\mathcal{A} \wedge \mathcal{B}, \mathcal{C}\}$ into a bicategory* isomorphic *to $[\mathcal{A}, [\mathcal{B}, \mathcal{C}]]$.*

It is now possible to "read off" the compositions and identities for $\{\mathcal{A} \wedge \mathcal{B}, \mathcal{C}\}$. The 2-cells $y : \rho \to \sigma$ and $x : \sigma \to \tau$ have vertical composite

- $(yx)_A = (y_A)(x_A)$
- $(yx)_B = (y_B)(x_B)$ ,

and the identity 2-cell on $\sigma$ has components

- $(1_\sigma)_A = 1_{\sigma_A}$
- $(1_\sigma)_B = 1_{\sigma_B}$ .

The composite of arrows $\tau : T \to S$ and $\sigma : S \to R$ is given by

- $(\sigma\tau)_A = (\sigma^*\tau^*)^{\#}_{A-} = (\sigma^*\tau^*)_A = (\sigma^*_A)(\tau^*_A) = (\sigma_A)(\tau_A)$
- $(\sigma\tau)_B = (\sigma^*\tau^*)^{\#}_A = ev_B(\sigma^*\tau^*) = (ev_B\sigma^*)(ev_B\tau^*) = (\sigma_B)(\tau_B)$

and structure cells are pasting composites of the structure cells for $\sigma$ and $\tau$. Identities on objects and horizontal composites of 2-cells are also "defined in the obvious way".

**Corollary 1.2.5** *There is a natural isomorphism* $[\mathcal{A}, [\mathcal{B}, \mathcal{C}]] \cong [\mathcal{B}, [\mathcal{A}, \mathcal{C}]]$.

PROOF: This result, which was stated in [38], is now immediate as the definition of $\{\mathcal{A} \wedge \mathcal{B}, \mathcal{C}\}$ is completely symmetric with respect to $\mathcal{A}$ and $\mathcal{B}$. □

We can obtain further isomorphisms as in the previous theorem by restricting to various sub-bicategories of $[\mathcal{A}, [\mathcal{B}, \mathcal{C}]]$ and using the original isomorphism to see which parts of $\{\mathcal{A} \wedge \mathcal{B}, \mathcal{C}\}$ arise. For example by considering the case where $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are 2-categories, and then restricting $[\mathcal{A}, [\mathcal{B}, \mathcal{C}]]$ to the sub-bicategory $Ps(\mathcal{A}, Ps(\mathcal{B}, \mathcal{C}))$ we recover the cubical functors of [20].

There are two particular cases that will be of use in subsequent chapters. As the assumptions made on $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ will simplify things significantly, we shall write out explicitly the results for the first case; the second will then be completely analogous to the first.

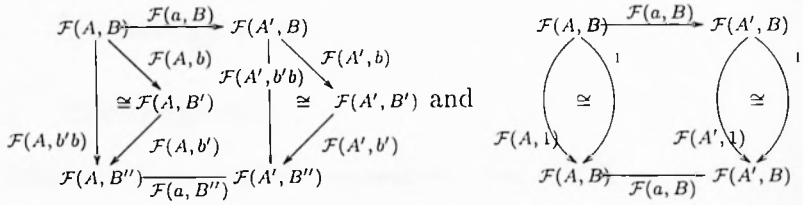
First we consider the case where $\mathcal{A}$ and $\mathcal{C}$ are 2-categories and we restrict to $\mathbf{2\text{-}Cat}(\mathcal{A}, [\mathcal{B}, \mathcal{C}])$. The objects of the corresponding sub-bicategory (in fact 2-category) of $\{\mathcal{A} \wedge \mathcal{B}, \mathcal{C}\}$ comprise 2-functors $\mathcal{F}(-, B) : \mathcal{A} \to \mathcal{C}$ and homomorphisms $\mathcal{F}(A, -) : \mathcal{B} \to \mathcal{C}$ satisfying the three conditions

$$\mathcal{F}(-, B)(A) = \mathcal{F}(A, -)(B)$$

$$\mathcal{F}(A', b)\mathcal{F}(a, B) = \mathcal{F}(a, B')\mathcal{F}(A, b)$$

$$\mathcal{F}(\alpha, B') * \mathcal{F}(A, \beta) = \mathcal{F}(A', \beta) * \mathcal{F}(\alpha, B) \ ,$$

(writing $*$ in the last of these for horizontal composition of 2-cells); and commutativity of (2-cells in) the diagrams

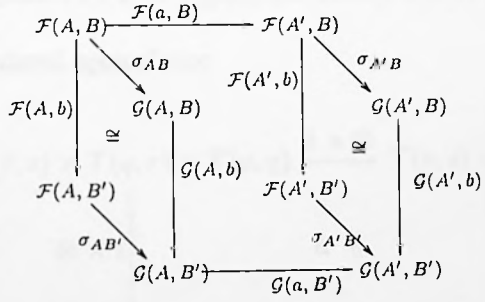



(1.2.5)

Then an arrow $\mathcal{F} \to \mathcal{G}$ in this sub-bicategory of $\{\mathcal{A} \wedge \mathcal{B}, \mathcal{C}\}$ comprises

- 2-natural transformations $\sigma_B : \mathcal{F}(-, B) \to \mathcal{G}(-, B)$
- strong transformations $\sigma_A : \mathcal{F}(A, -) \to \mathcal{G}(A, -)$

satisfying $(\sigma_B)_A = (\sigma_A)_B$ and commutativity (of 2-cells in) the diagram




(1.2.6)

Finally a 2-cell $x : \sigma \to \tau$ in $\{\mathcal{A} \wedge \mathcal{B}, \mathcal{C}\}$ comprises

- modifications $x_B : \sigma_B \to \tau_B$

- modifications $x_A : \sigma_A \to \tau_A$

such that $(x_B)_A = (x_A)_B$.

We shall also need the case, for $\mathcal{B}$ and $\mathcal{C}$ 2-categories, of $[\mathcal{A}, \mathbf{2\text{-}Cat}(\mathcal{B}, \mathcal{C})]$. This is just as for $\mathbf{2\text{-}Cat}(\mathcal{A}, [\mathcal{B}, \mathcal{C}])$ but with the roles of $\mathcal{A}$ and $\mathcal{B}$ reversed.

## 1.3 Tricategories

### 1.3.1 Tricategories

The definition of tricategories is essentially that of [20].

**Definition 1.3.1** *A* tricategory *comprises the data*

*(TD1) a set* $Ob\mathcal{T}$

*(TD2) a bicategory* $\mathcal{T}(p,q)$ *for each* $p, q \in Ob\mathcal{T}$

*(TD3) a homomorphism* $\otimes : \mathcal{T}(q,r) \times \mathcal{T}(p,q) \to \mathcal{T}(p,r)$ *for each* $p, q, r \in Ob\mathcal{T}$

*(TD4) a homomorphism* $I : 1 \to \mathcal{T}(p,p)$ *for each* $p \in Ob\mathcal{T}$

*(TD5) a pseudo-natural equivalence*

$$\begin{array}{ccc} \mathcal{T}(r,s) \times \mathcal{T}(q,r) \times \mathcal{T}(p,q) & \xrightarrow{1 \times \otimes} & \mathcal{T}(r,s) \times \mathcal{T}(p,r) \\ \Big\downarrow \otimes \times 1 & a \Uparrow & \Big\downarrow \otimes \\ \mathcal{T}(q,s) \times \mathcal{T}(p,q) & \xrightarrow[\otimes]{} & \mathcal{T}(p,s) \end{array}$$

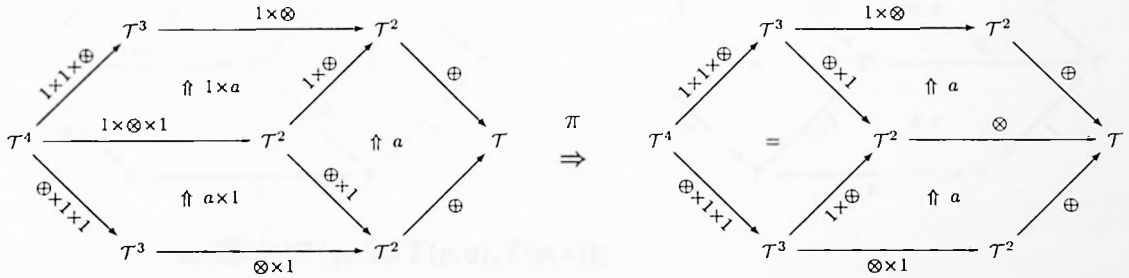
*for each* $p, q, r, s \in Ob\mathcal{T}$

*(TD6) pseudo-natural equivalences*

$$\begin{array}{ccccc}
\mathcal{T}(q,q)\times\mathcal{T}(p,q) & \xrightarrow{I\times 1} & \mathcal{T}(p,q) & \xrightarrow{1\times I} & \mathcal{T}(p,q)\times\mathcal{T}(p,p) \\
\Big\downarrow{\scriptstyle\otimes} & \overset{l}{\Rightarrow} & \Big\downarrow{\scriptstyle 1} & \overset{r}{\Rightarrow} & \Big\downarrow{\scriptstyle\otimes} \\
\mathcal{T}(p,q) & \xrightarrow[1]{} & \mathcal{T}(p,q) & \xrightarrow[1]{} & \mathcal{T}(p,q)
\end{array}$$

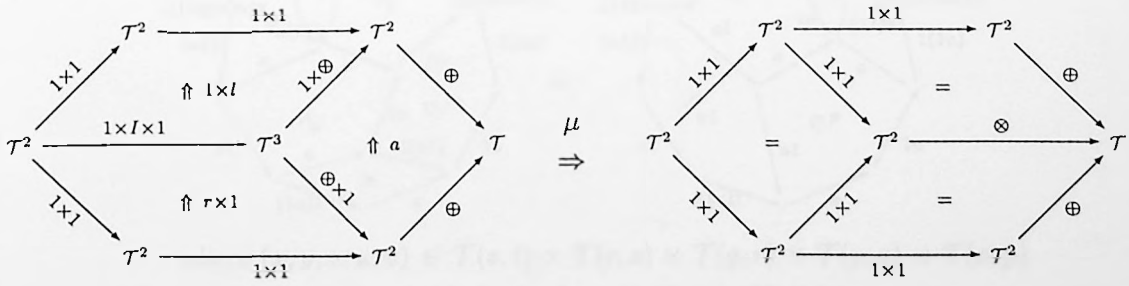
*for each* $p, q \in Ob\mathcal{T}$

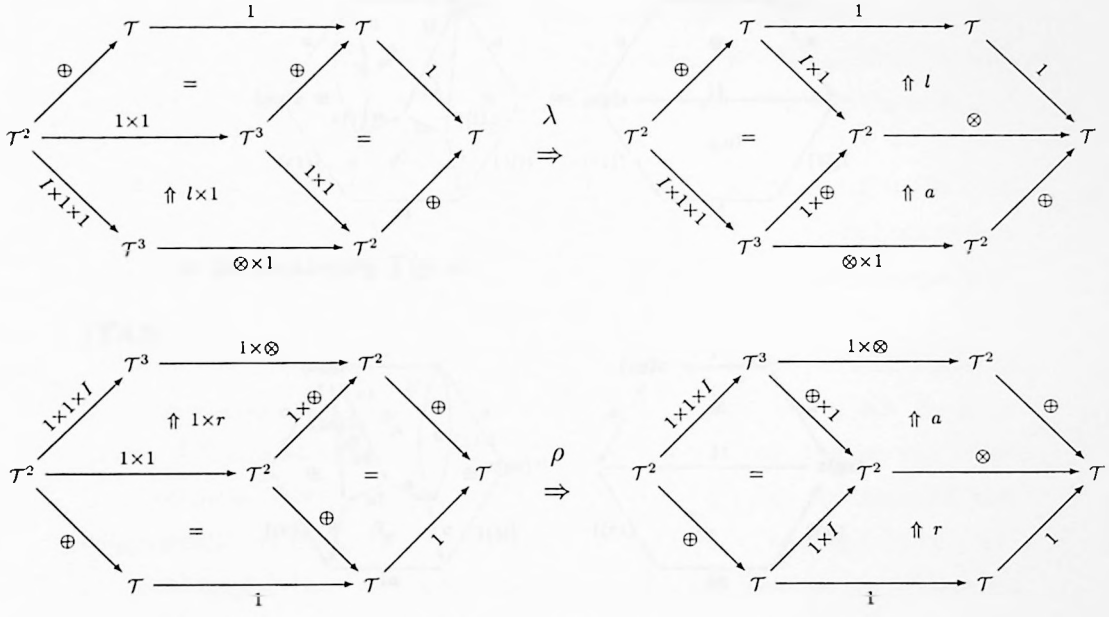
*(TD7) invertible modifications*




*in* $\mathcal{H}om(\mathcal{T}(s,t)\times\mathcal{T}(r,s)\times\mathcal{T}(q,r)\times\mathcal{T}(p,q), \mathcal{T}(p,t))$, *(where* $\mathcal{T}^4$ *is an abbreviation for* $\mathcal{T}(s,t)\times\mathcal{T}(r,s)\times\mathcal{T}(q,r)\times\mathcal{T}(p,q)$ *for example);*

*(TD8) invertible modifications*

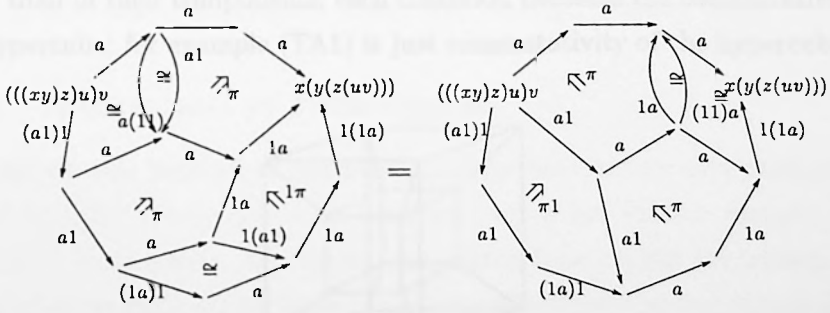





*in* $Hom(\mathcal{T}(q,r)\times\mathcal{T}(p,q),\mathcal{T}(p,r))$;

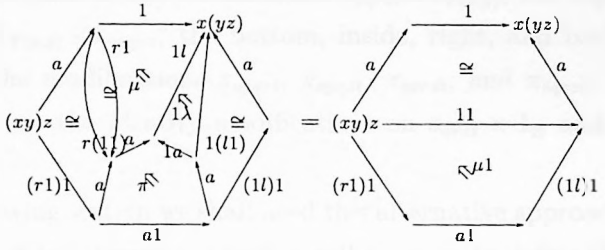
*satisfying the following coherence conditions on the components of* $\pi$, $\mu$, $\lambda$, *and* $\rho$:

*(TA1)*



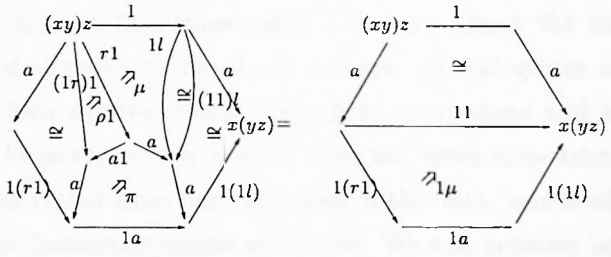
*where* $(x,y,z,u,v)\in\mathcal{T}(s,t)\times\mathcal{T}(r,s)\times\mathcal{T}(q,r)\times\mathcal{T}(p,q)\times\mathcal{T}(o,p)$ *and this equation holds in the hom-bicategory* $\mathcal{T}(o,t)$

*(TA2)*



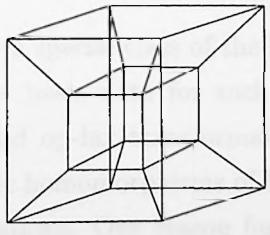

*in the bicategory* $\mathcal{T}(p, s)$

*(TA3)*




*in the bicategory* $\mathcal{T}(p, s)$

The coherence conditions are given above in terms of the equality of pasting composites in various hom-bicategories of $\mathcal{T}$. We note here that there is another approach, which involves pasting composites of the modifications, rather than of their components; each condition becomes the commutativity of a hypercube; for example (TA1) is just commutativity of the hypercube



(1.3.1)

where the front cube is the modification $\pi_{pqrst} \times 1_{\mathcal{T}(o,p)}$, the top cube is the modification $1_{\mathcal{T}(s,t)} \times \pi_{opqrs}$; the bottom, inside, right, and back cubes are, respectively, the modifications $\pi_{opqrt}$, $\pi_{opqst}$, $\pi_{oprst}$, and $\pi_{oqrst}$; the left and outside cubes are the identity modifications on $a_{qrst} \times 1_{\otimes}$ and $1_{\otimes} \times a_{opqrs}$, respectively.

In the following section we shall need this alternative approach in the case of morphisms of tricategories, where it will allow us to define the composite of these morphisms. We shall describe there in more detail the passage from one approach to the other.

We should also at this stage make a remark about the isomorphisms between curved arrows in the above axioms. Tricategories are given in terms of hom-bicategories, composition homomorphisms and so on; these isomorphsims between curved arrows arise not from non-strictness of this composition but rather from non-strictness of the data "one level down", *i.e.* the bicategories, homomorphisms and so on. We can however use coherence results for bicategories and their morphisms (see for example [30, 25, 20]) to eliminate the need for these isomorphisms; this considerably simplifies the diagrams involved, and we shall do so without comment from now on. One important consequence of this convention is that, given $x$ an object in a bicategory, we can write $x$ indiscriminately for the object and its identity 1-cell, as is standard in category theory and 2-category theory alike.

### 1.3.2 Morphisms of tricategories

We shall consider here only a special class of the lax functors of tricategories defined in [20]. There the basic data for such a lax functor include lax functors of bicategories, and op-lax transformations; in our *morphisms of tricategories* we shall ask for homomorphisms of bicategories and strong (i.e. pseudo-natural) transformations. One reason for this restriction is that we

want to be able to compose our morphisms; the more general lax functors of [20] do not compose. They fail to do so because, although one can define the composite of lax morphisms of bicategories, one cannot define a horizontal composite of lax transformations between them, even in the case where the second is an identity transformation (i.e. lax morphism).

**Definition 1.3.2** *A morphism of tricategories $\mathcal{T} \to \mathcal{T}'$ comprises*

*(HTD1) a function $H : Ob\mathcal{T} \to Ob\mathcal{T}'$*

*(HTD2) homomorphisms $H : \mathcal{T}(p,q) \to \mathcal{T}'(Hp, Hq)$ of the hom-bicategories*

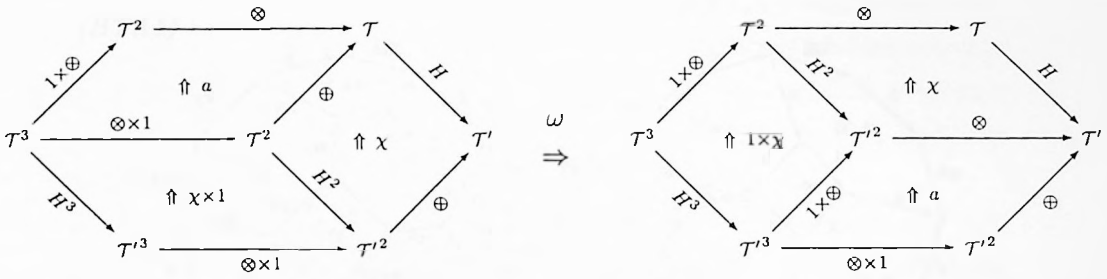
*(HTD3) strong transformations*

$$\begin{array}{ccc} \mathcal{T}(q,r) \times \mathcal{T}(p,q) & \xrightarrow{\otimes} & \mathcal{T}(p,r) \\ {\scriptstyle H \times H}\downarrow & \Uparrow \chi & \downarrow{\scriptstyle H} \\ \mathcal{T}'(Hq,Hr) \times \mathcal{T}'(Hp,Hq) & \xrightarrow[\otimes]{} & \mathcal{T}'(Hp,Hr) \end{array}$$

*(HTD4) strong transformations*

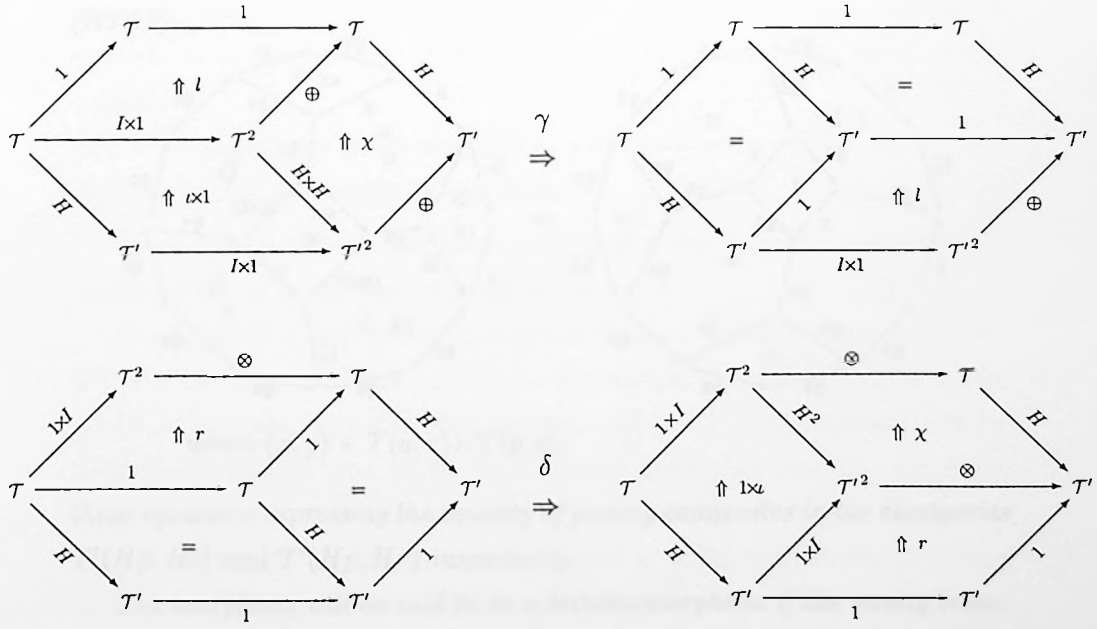
$$\begin{array}{ccc} 1 & \xrightarrow{I} & \mathcal{T}(p,p) \\ {\scriptstyle 1}\downarrow & \iota \Uparrow & \downarrow{\scriptstyle H} \\ 1 & \xrightarrow[I]{} & \mathcal{T}'(Hp,Hp) \end{array}$$

*(HTD5) invertible modifications*




*in* $\mathcal{HOM}\,(\mathcal{T}(r,s)\times\mathcal{T}(q,r)\times\mathcal{T}(p,q),\mathcal{T}'(Hp,Hs))$*; here* $\mathcal{T}^3$*, for example, is an abbreviation for* $\mathcal{T}(r,s)\times\mathcal{T}(q,r)\times\mathcal{T}(p,q)$*;*

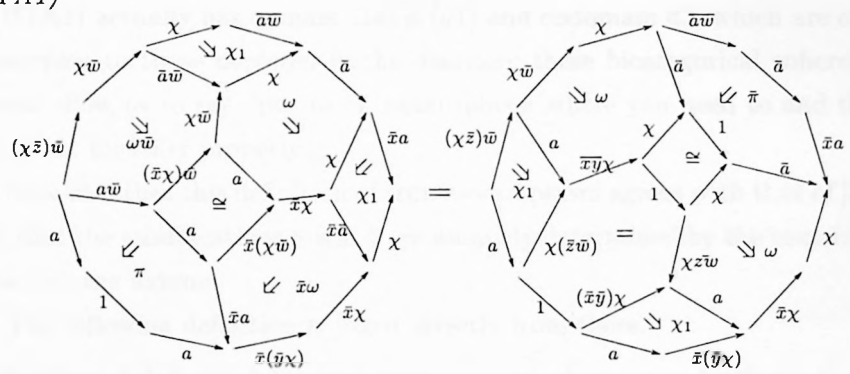
*(HTD6) invertible modifications*




*in* $\mathcal{HOM}\,(\mathcal{T}(p,q),\mathcal{T}'(Hp,Hq))$

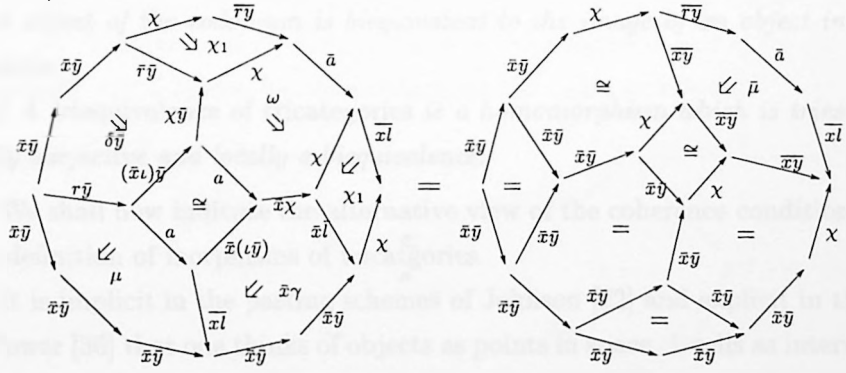
*subject to two equations*

*(HTA1)*



*where* $(x, y, z, w) \in \mathcal{T}(s,t)\times\mathcal{T}(r,s)\times\mathcal{T}(q,r)\times\mathcal{T}(p,q)$, $(^{-}) = H$, *and* $\chi_1$ *is a naturality isomorphism for* $\chi$

*(HTA2)*



*where* $(x, y) \in \mathcal{T}(q,r)\times\mathcal{T}(p,q)$;

*these equations expressing the equality of pasting composites in the bicatgories* $\mathcal{T}'(Hp, Ht)$ *and* $\mathcal{T}'(Hp, Hr)$ *respectively.*

*The morphism will be said to be a trihomomorphism if the strong transformations* $\chi$ *and* $\iota$ *are equivalences, and the modifications* $\omega$, $\gamma$, *and* $\delta$ *are invertible.*

Note here that we are using the bicategorical coherence results as described at the end of Section 1.3.1; for example the 2-cell $\overline{\pi}$ on the right side of (HTA1) actually has domain $\overline{(1a).a.(a1)}$ and codomain $\overline{a}.\overline{a}$ which are only isomorphic to those depicted in the diagram; these bicategorical coherence results allow us to say "put in an isomorphism where you need to and they will all fit together properly".

Note also that this definition of trihomomorphism agrees with that of [20], and that the modifications $\gamma$ and $\delta$ are uniquely determined by the remaining data and the axioms.

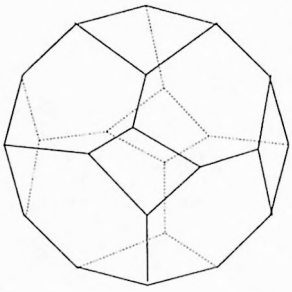
The following definition is taken directly from there.

**Definition 1.3.3** *(i) A* biequivalence *in a tricategory is a 1-cell $g : a \to b$ for which there is a 1-cell $f : b \to a$ and 2-cells $I_b \to g \otimes f$ and $f \otimes g \to I_a$ which are equivalences in the hom-bicategories in which they lie.*
*(ii) A homomorphism of tricategories is said to be* triessentially surjective *if each object of the codomain is biequivalent to the image of an object in the domain.*
*(iii) A* triequivalence of tricategories *is a homomorphism which is triessentially surjective and locally a biequivalence.*

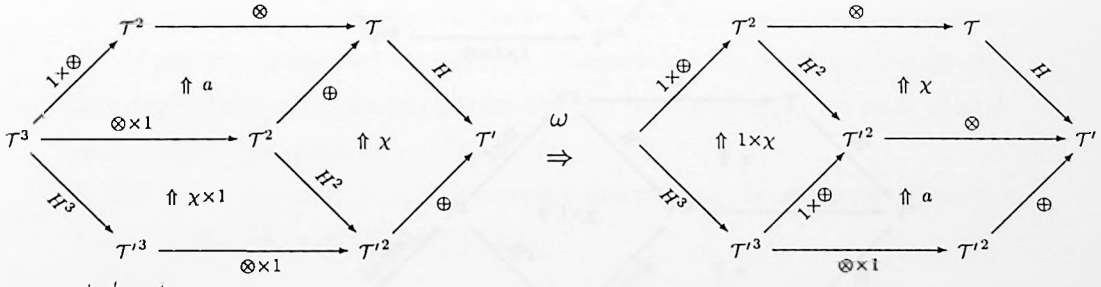
We shall now indicate the alternative view of the coherence conditions in the definition of morphisms of tricategories.

It is implicit in the pasting schemes of Johnson [22] and explicit in those of Power [36] that one thinks of objects as points in space, 1-cells as intervals, 2-cells as surfaces, 3-cells as solid regions, and so on. This approach can be particularly fruitful here, where the diagrams are rather complex, and some concrete intuition is useful. In particular, the modification $\omega$ in the definition of tricategory morphisms can be viewed as being a cube, rather than a collection of hexagons (its components) satisfying a naturality condition.

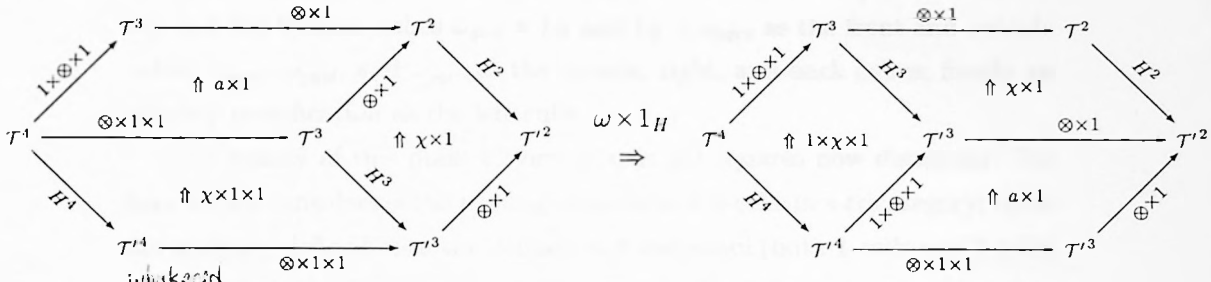
Consider now the first axiom (HTA1) for a tricategory morphism, which asks for commutativity of a "truncated octahedron":



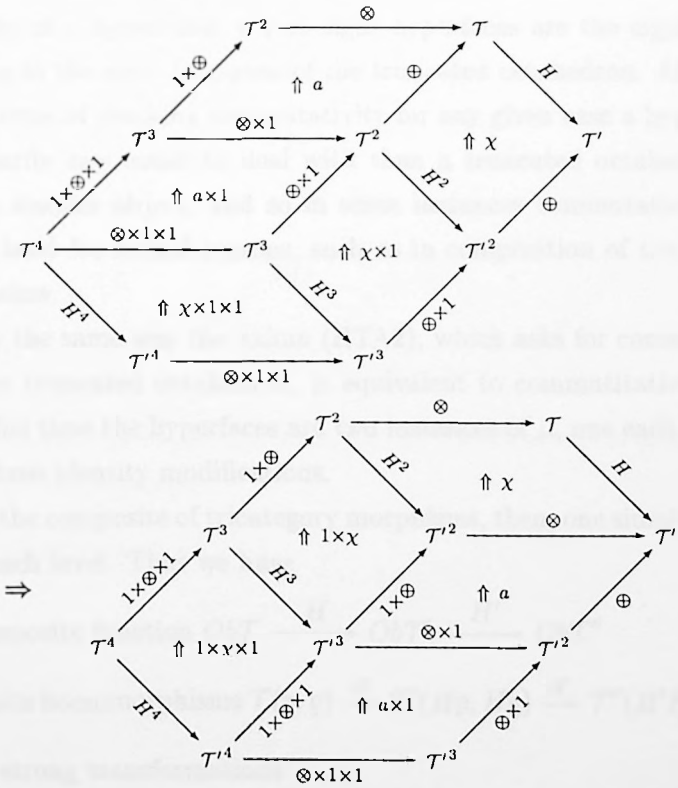
the surface of which comprises eight hexagons and six squares. Each hexagon, however, represents a component of a modification; if we consider the modification in its own right, rather than reducing it to its components, it becomes not a hexagon but a cube; the six faces of the cube corresponding to the six sides of the hexagon. Moreover, the pasting composite of hexagons corresponds to the pasting composite of the corresponding cubes. Thus for example the case of the truncated octahedron of axiom (HTA1), consider the two upper hexagons on the lefthand side of the equation (one is $\omega\bar{w}$ and the other $\omega$). They are components of the modifications represented by the two "~~whisked~~ whiskered" cubes below; for the hexagon on the right, $\omega$, the cube




~~whisked~~ whiskered on the left by $\otimes \times 1 \times 1$, and for the hexagon on the left, $\omega\bar{w}$, the cube




whisked on the right by $\otimes$. Their pasting composite (as formed in the truncated octahedron) is then a component of the pasting composite of the cubes:




It is possible to consider the entire truncated octahedron as the pasting composite of eight cubes. Using again the hypercube drawn as (1.3.1), we

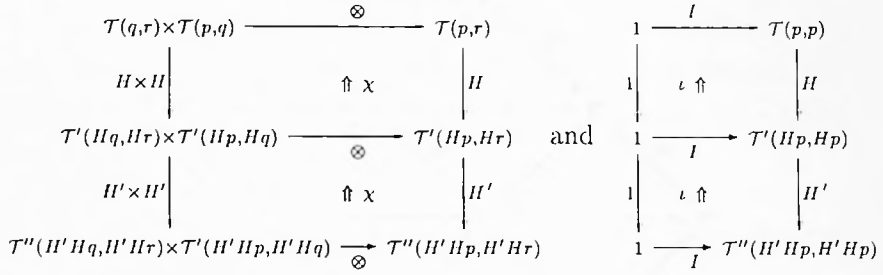
find that (HTA1) corresponds to taking $\pi$ for $\mathcal{T}$ and $\mathcal{T}'$ respectively as the top and the bottom cubes $\omega_{qrsl} \times 1_H$ and $1_H \times \omega_{pqrs}$ as the front and outside cubes; $\omega_{prsl}$, $\omega_{pqsl}$, and $\omega_{pqrl}$ as the inside, right, and back cubes; finally an identity modification as the left cube.

The beauty of this point of view is that the squares now disappear. For here we are considering the pasting composite of 3-cells in a tricategory; these are uniquely defined once the domain and codomain (both 1-cells and 2-cells) are specified; the function of the squares (all of which are isomorphisms) in the truncated octahedron is simply to makes these domains and codomains agree. Then commutativity of the truncated octahedron becomes "just" commutativity of a hypercube, whose eight hyperfaces are the eight cubes corresponding to the eight hexagons of the truncated octahedron. Although certainly in terms of checking commutativity for any given case a hypercube is not necessarily any easier to deal with than a truncated octahedron, it is *formally* a simpler object, and so in some instances commutativity will quite clearly hold for formal reasons, such as in composition of tricategory morphisms below.

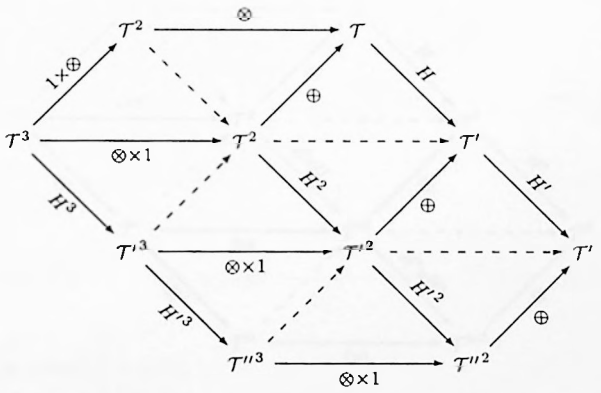
In exactly the same way the axiom (HTA2), which asks for commutativity of another truncated octahedron, is equivalent to commutitativity of a hypercube; this time the hyperfaces are two instances of $\mu$, one each of $\omega$, $\delta$, and $\gamma$, and three identity modifications.

To define the composite of tricategory morphisms, then, one simply pastes the data at each level. Thus we have

- the composite function $Ob\mathcal{T} \xrightarrow{H} Ob\mathcal{T}' \xrightarrow{H'} Ob\mathcal{T}''$
- composite homomorphisms $\mathcal{T}(p,q) \xrightarrow{H} \mathcal{T}'(Hp, Hq) \xrightarrow{H'} \mathcal{T}''(H'Hp, H'Hq)$
- pasted strong transformations

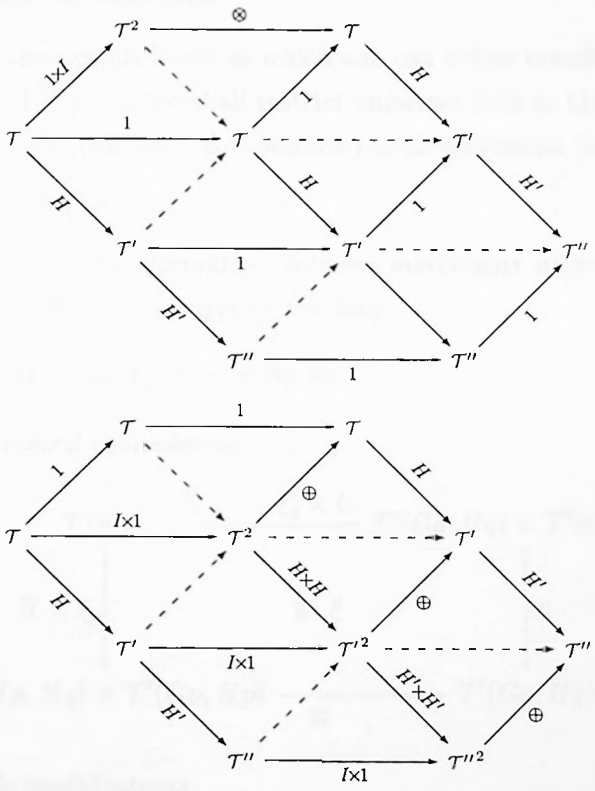



- pasted modifications $\omega$




where the two cubes are the cubes for the modifications $\omega$ attached to $H$ and $H'$ respectively;

- pasted modifications $\gamma$ and $\delta$




comprising the cubes for the modifications $\gamma$ and $\delta$ attached to $H$ and $H'$;

and when we view axioms (HTA1) and (HTA2) as asking for the commutativity of two hypercubes, they follow immediately by pasting the individual hypercubes.

**Proposition 1.3.4** *The above data define a morphism of tricategories; this is defined to be the composite of the morphisms $H$ and $H'$.*

### 1.3.3 Tritransformations

Once again there are various levels at which one can define transformations in the tricategorical context; we shall restrict ourselves here to the tritransformations of [20]. The definition is formulated in an equivalent, but slightly different manner.

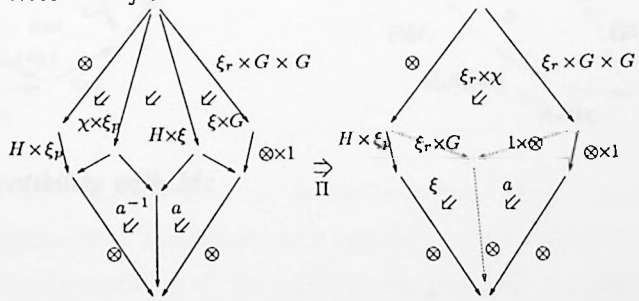
**Definition 1.3.5** *A tritransformation between morphisms of tricategories $G : \mathcal{T} \to \mathcal{T}'$ and $H : \mathcal{T} \to \mathcal{T}'$ comprises the data*

*(TTD1) a family of 1-cells $\xi_p : Gp \to Hp$ in $\mathcal{T}'$*

*(TTD2) pseudo-natural equivalences*

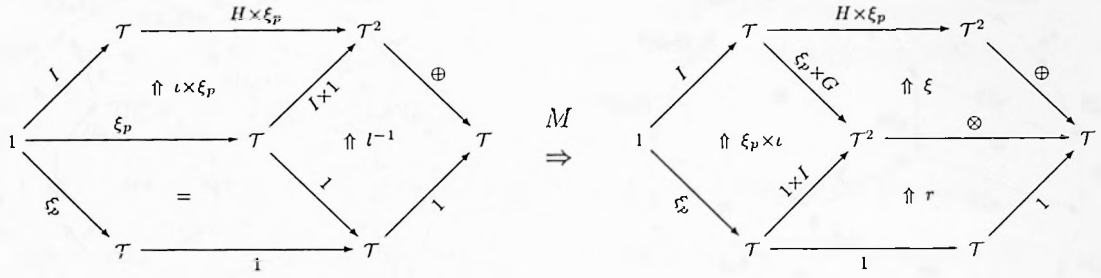
$$\begin{array}{ccc} \mathcal{T}(p,q) & \xrightarrow{\xi_q \times G} & \mathcal{T}'(Gq,Hq) \times \mathcal{T}'(Gp,Gq) \\ {\scriptstyle H \times \xi_p}\downarrow & \Downarrow \xi & \downarrow{\scriptstyle \otimes} \\ \mathcal{T}'(Hp,Hq) \times \mathcal{T}'(Gp,Hp) & \xrightarrow[\otimes]{} & \mathcal{T}'(Gp,Hq) \end{array}$$

*(TTD3) invertible modifications*



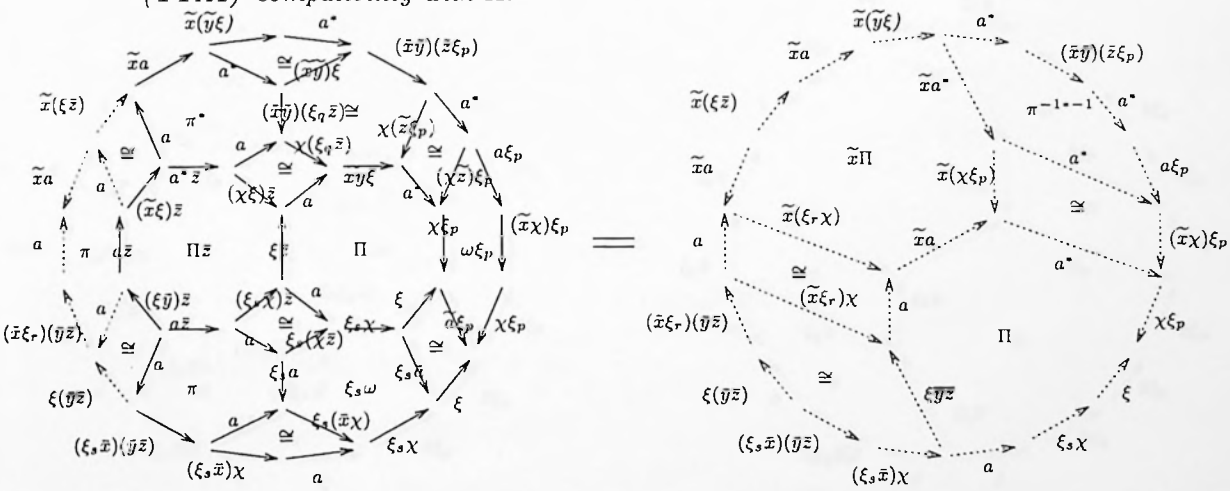

*in* $Hom(\mathcal{T}(q,r) \times \mathcal{T}(p,q), \mathcal{T}'(Gp,Hr))$

*(TTD4) invertible modifications*

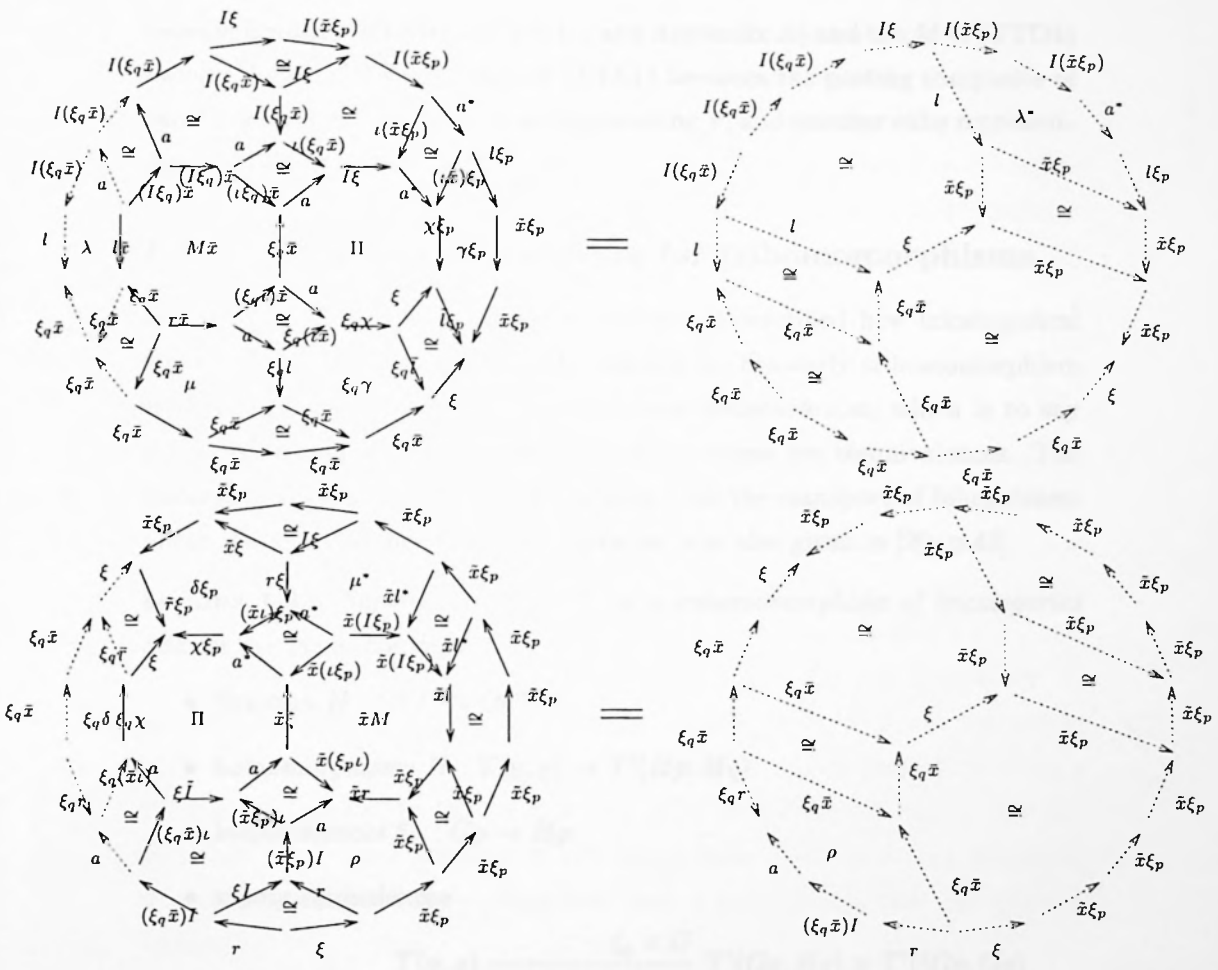



*satisfying the conditions*

*(TTA1) compatibility with* $\Pi$*:*




*(TTA2) compatibility with* $M$*:*



The way to view these conditions is the following. They represent a solid, having been broken into two halves only to enable their two-dimensional representation. Commutativity of the solid means that two pasting composites are equal; these are distinguished by the use of solid and dashed lines.

Once more we can reinterpret the axioms on the components of the various modifications given in this definition in terms of the pasting of various 3-dimensional figures. Note that the invertible modifications $\Pi$ of (TTD3)

become Emma polyhedra (as in 1.1.1 and Appendix A) and the $M$ of (TTD4) become cubes. Then each side of (TTA1) becomes the pasting composite of two Emma polyhedra, two cubes representing $\pi$, and another cube representing $\omega$.

### 1.3.4 Transport of structure for trihomomorphisms

In [20, p.28] there was a lemma proved which described how tricategorical structure is transported along triequivalences. Similarly trihomomorphism structure can be transported: along strong biequivalences, which is to say along strong tritransformations whose components are biequivalences. The *bicategorical* version of our result, dealing with the transport of bihomomorphism structure along strong equivalences, was also given in [20, p.43].

**Lemma 1.3.6** *Suppose $G : \mathcal{T} \to \mathcal{T}'$ is a trihomomorphism of tricategories and we are given the data*

- *function $H : Ob\mathcal{T} \to Ob\mathcal{T}'$*
- *homomorphisms $H : \mathcal{T}(p,q) \to \mathcal{T}'(Hp, Hq)$*
- *biequivalences $\xi_P : Gp \to Hp$*
- *strong equivalences*

$$\begin{array}{ccc} \mathcal{T}(p,q) & \xrightarrow{\xi_q \times G} & \mathcal{T}'(Gq,Hq) \times \mathcal{T}'((Gp,Gq) \\ {\scriptstyle H \times \xi_p}\downarrow & \Downarrow \xi & \downarrow{\scriptstyle \otimes} \\ \mathcal{T}'(Hp,Hq) \times \mathcal{T}'(Gp,Hp) & \xrightarrow[\otimes]{} & \mathcal{T}'(Gp,Hq) \end{array}$$

*then the function and homomorphisms $H$ form part of the data for a trihomomorphism $H : \mathcal{T} \to \mathcal{T}'$ such that the biequivalences and equivalences $\xi$ form part of the data for a strong biequivalence $\xi : G \to H$.*

PROOF: As the $\xi_p$ are biequivalences, we can simultaneously provide the data for (TTD3) and (HTD3); likewise for (TTD4) and (HTD4); the choice is determined only up to biequivalence, but any of these possibilities will do. (TTA1) tells us how to define (HTD5) and (TTA2) tells us how to define (HTD6). Finally (HTA1) and (HTA2) follow from the corresponding axioms for the trihomomorphism $H$ and the fact that the $\xi_p$ are biequivalences. □

### 1.3.5 Coherence for tricategories

In this section are outlined some of the results needed from [20].

Given a bicategory $\mathcal{B}$, one can associate its *strictification* — the 2-category called $\mathbf{st}\mathcal{B}$ with the same objects as $\mathcal{B}$, 1-cells the paths of 1-cells in $\mathcal{B}$, and 2-cells between given paths the 2-cells in $\mathcal{B}$ between the composites of the paths. There is then a homomorphism $\xi : \mathcal{B} \to \mathbf{st}\mathcal{B}$, which takes 1-cell to the trivial path containing only it, and is trivial on objects and 2-cells; this $\xi$ is a biequivalence, and has a universal property: for any 2-category $\mathcal{X}$, composition with $\xi$ induces a bijection, natural in $X$, between 2-functors $\mathbf{st}\mathcal{B} \to \mathcal{X}$ and homomorphisms $\mathcal{B} \to \mathcal{X}$, which is to say that $\mathbf{st}$ becomes a functor left adjoint to the inclusion of *mere categories* $\mathbf{2\text{-}Cat}_0 \to \mathbf{Bicat}_0$, and that the $\xi$ form the unit of this adjunction. Note here that this is an honest adjunction of mere categories, with a genuine bijection and genuine naturality.

Moreover, given 2-functors $F, G : \mathbf{st}\mathcal{B} \to \mathcal{X}$, composition with $\xi$ induces a bijection between strong transformations $F \to G$ and strong transformations $F\xi \to G\xi$ and similarly a bijection between modifications, which gives the result

**Proposition 1.3.7** *Composition with $\xi$ now induces an* isomorphism *of 2-categories*

$$\mathbf{Gray}(\mathbf{st}\mathcal{B}, \mathcal{X}) \longrightarrow [\mathcal{B}, \mathcal{X}] .$$

In [20], a triequivalence from **Bicat** to **Gray**, essentially just **st**, is then deduced.[2] It is using this triequivalence that they then go on to prove the main result, that every tricategory is triequivalent to a **Gray**-category; moreover this triequivalence can be chosen to be a bijection on objects.

## 1.4 Monoidal bicategories

A monoidal bicategory is simply a tricategory with one object; in fact the monoidal bicategories were given in [20] as motivating examples for tricategories, just as, in [2], the monoidal categories were given as motivating examples for bicategories. Monoidal bicategories were also defined in [7] for the restricted case where the underlying bicategory is locally posetal.

Given two monoidal bicategories thought of as tricategories, a morphism of tricategories between them will be called a monoidal homomorphism, a trihomomorphism will be called a strong monoidal homomorphism, and a triequivalence will be called a monoidal biequivalence. By the previous section, every monoidal bicategory is monoidally biequivalent to a one-object **Gray**-category; a one-object **Gray**-category is precisely a monoid in the monoidal category **Gray**, of 2-categories and 2-functors, equipped with the **Gray** tensor product $\Box$; such a monoid we shall call a **Gray**-monoid. Equivalently we have

**Proposition 1.4.1** *Every monoidal bicategory is monoidally biequivalent to a monoidal bicategory comprising a 2-category $\mathcal{W}$, a cubical functor $\otimes : \mathcal{W} \times \mathcal{W} \to \mathcal{W}$ which is strictly associative, and a strict unit $I \in \mathcal{W}$.*

Often we shall first prove results for the restricted case of **Gray**-monoids, and then use the restricted theorem to obtain a more general one, such as is done in the next section.

[2] This differs slightly from the notation of [20] but for our purposes does no harm.

### 1.4.1 A little lemma

We prove here just one result that will be of use later.

**Lemma 1.4.2** *If $\mathcal{W}$ is a monoidal bicategory then the homomorphism of bicategories $\mathcal{W}(I,-) : \mathcal{W} \to \mathbf{Cat}$ can be made into a monoidal homomorphism, where the monoidal structure on* **Cat** *is the cartesian structure. If $\mathcal{W}$ is moreover a* **Gray***-monoid then $\mathcal{W}(I,-)$ is a* **Gray***-functor.*

Proof: Suppose first that $\mathcal{W}$ is a **Gray**-monoid.

The strong transformation

$$\begin{array}{ccc} \mathcal{W} \times \mathcal{W} & \xrightarrow{\otimes} & \mathcal{W} \\ {\scriptstyle \mathcal{W}(I,-)\times\mathcal{W}(I,-)}\downarrow & \chi \Uparrow & \downarrow{\scriptstyle \mathcal{W}(I,-)} \\ \mathbf{Cat} \times \mathbf{Cat} & \xrightarrow[\times]{} & \mathbf{Cat} \end{array}$$

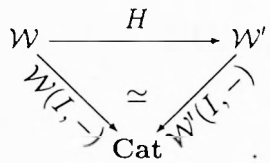
has components $\otimes : \mathcal{W}(I,U) \times \mathcal{W}(I,V) \to \mathcal{W}(I, U \otimes V)$; the pseudo-naturality isomorphisms then come from functoriality constraints for $\otimes$. The strong transformation

$$\begin{array}{ccc} 1 & \xrightarrow{I} & \mathcal{W} \\ {\scriptstyle 1}\downarrow & \Uparrow \iota & \downarrow{\scriptstyle \mathcal{W}(I,-)} \\ 1 & \xrightarrow[I]{} & \mathbf{Cat} \end{array}$$

has component $j : 1 \to \mathcal{W}(I,I)$.

The fact that $\mathcal{W}$ is a **Gray**-monoid gives all the necessary equations on arrows that allow us to choose the modifications $\omega$, $\gamma$, and $\delta$ to be identities; the axioms for a morphism then clearly hold. In this case, the morphism is in fact a **Gray**-functor.

Now consider the case of a general $\mathcal{W}$. Choose a **Gray**-monoid $\mathcal{W}'$ and a monoidal bicategorical equivalence $H : \mathcal{W} \to \mathcal{W}'$. We then have a composite monoidal homomorphism, which we may write as $\mathcal{W}'(I, H-) : \mathcal{W} \to \mathbf{Cat}$, and by Lemma 1.3.6 on the transport of trihomomorphism structure applied to the one-object case of monoidal bicategories, to make the homomorphism $\mathcal{W}(I, -) : \mathcal{W} \to \mathbf{Cat}$ into a monoidal homomorphism it will suffice to find a strong equivalence between the homomorphisms of bicategories




The desired strong equivalence has as components the equivalences

$$\mathcal{W}(I, U) \xrightarrow{H} \mathcal{W}'(HI, HU) \xrightarrow{\mathcal{W}'(\iota, HU)} \mathcal{W}'(I, HU) ;$$

and, as naturality isomorphisms, pasting composites of functoriality isomorphisms for $H$ and associativity isomorphisms for $\mathcal{W}'$. □

### 1.4.2 Monoidal closed bicategories

**Definition 1.4.3** *A monoidal bicategory $\mathcal{W}$ is said to be* right closed *if each of the homomorphisms $- \otimes W : \mathcal{W} \to \mathcal{W}$ has a right biadjoint. A* **Gray**-*monoid is right closed if it is right closed as a monoidal bicategory.*

# Chapter 2

# Some coherence results

## 2.1 Colimits in 2-categories

Throughout this section $\mathcal{K}$ will be a fixed 2-category. We shall describe some of the notions of colimits that exist in $\mathcal{K}$.

Let $\mathcal{A}$ be a small bicategory and let $G : \mathcal{A}^{op} \to \mathcal{K}$ and $F : \mathcal{A} \to \mathbf{Cat}$ be homomorphisms. The bicolimit of $G$ weighted by $F$ can be defined as an object $F *_b G$ of $\mathcal{K}$ together with equivalences

$$\mathcal{K}(F *_b G, X) \simeq [\mathcal{A}, \mathbf{Cat}](F, \mathcal{K}(G, X)) \tag{2.1.1}$$

which are strong in $X$; or equivalently by giving $F *_b G$ and an equivalence

$$\mathcal{K}(F *_b G, -) \simeq [\mathcal{A}, \mathbf{Cat}](F, \mathcal{K}(G, -))$$

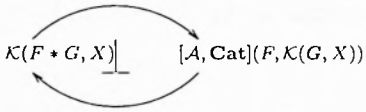
in the 2-category $[\mathcal{K}, \mathbf{Cat}]$. As $\mathcal{K}$ is a 2-category, both sides of this last equivalence are in fact 2-functors, and so we could ask not for an equivalence in $[\mathcal{K}, \mathbf{Cat}]$, but an isomorphism in $\mathbf{2\text{-}Cat}(\mathcal{K}, \mathbf{Cat})$ . In this case we write $F * G$, dropping the subscript $b$, and speak of the pseudocolimit as opposed to the bicolimit. As $\mathbf{2\text{-}Cat}(\mathcal{K}, \mathbf{Cat})$ sits inside $[\mathcal{K}, \mathbf{Cat}]$ as a sub-2-category, the pseudocolimit, if it exists, is *a fortiori* the bicolimit. In this case, the

equivalences (2.1.1) become isomorphisms

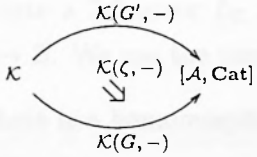
$$\mathcal{K}(F * G, X) \cong [\mathcal{A}, \mathbf{Cat}](F, \mathcal{K}(G, X))$$

which are not just pseudo-natural, but 2-natural. If all such $F * G$ exist, for small $\mathcal{A}$, we say that $\mathcal{K}$ is pseudococomplete, and we shall assume henceforth that $\mathcal{K}$ is so.

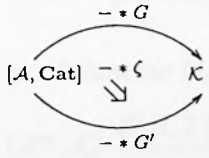
In the usual way, 2-naturality of the isomorphisms allows us to define a 2-functor $- * G$ from $[\mathcal{A}, \mathbf{Cat}]$ to $\mathcal{K}$ with value $F * G$ at $F$, and this forms part of a genuine 2-adjunction




comprising 2-functors and 2-natural transformations satisfying triangle equations "on-the-nose." Now, given another homomorphism $G' : \mathcal{A} \to \mathcal{K}$ and a strong transformation $\zeta : G \to G'$ we can form the 2-natural transformation




and then use the mate calculus to obtain a 2-natural transformation




Similarly a modification $x : \zeta \to \zeta'$ induces a modification $\mathcal{K}(x, -)$ and hence a modification $- * x : (- * \zeta) \to (- * \zeta')$.

Passing from $G$, $\zeta$, and $x$ to $\mathcal{K}(G, -)$, $\mathcal{K}(\zeta, -)$, and $\mathcal{K}(x, -)$ is strictly functorial — essentially because $\mathcal{K}(-, X)$ is a 2-functor — and similarly the mate calculus is strictly functorial, and so we have:

**Proposition 2.1.1** *Forming colimits in a pseudococomplete 2-category $\mathcal{K}$ is a 2-functor*

$$[\mathcal{A}, \mathbf{Cat}] \times [\mathcal{A}^{op}, \mathcal{K}] \xrightarrow{- * -} \mathcal{K}.$$

We should note that the standard definition of colimits has $\mathcal{A}$ interchanged with $\mathcal{A}^{op}$; we have chosen to present it this way purely for convenience in what follows. The notation $F * G$ is sometimes used for the mere bicolimit, and sometimes (in the case where $F$ and $G$ are 2-functors) for the 2-colimit, but we choose the pseudocolimit as basic, being the notion used in subsequent sections. As always with different levels of strictness, it is the context that will determine which notion should be taken to be basic.

## 2.2 The assignments $L$

Throughout this section $\mathcal{K}$ will be a pseudococomplete 2-category. Given bicategories $\mathcal{A}$ and $\mathcal{B}$ we can associate 2-categories $[\mathcal{A}^{op}, \mathcal{K}]$ and $[\mathcal{B}^{op}, \mathcal{K}]$; we would like similarly to associate a 2-functor $L_T : [\mathcal{A}^{op}, \mathcal{K}] \to [\mathcal{B}^{op}, \mathcal{K}]$ to a given homomorphism $T : \mathcal{A} \to \mathcal{B}$. We use the results of Section 1.2:

- for each $G \in [\mathcal{A}^{op}, \mathcal{K}]$ there is a homomorphism $L_T G$ given by

$$\mathcal{B}^{op} \xrightarrow{\mathcal{B}(-,T)} [\mathcal{A}, \mathbf{Cat}] \xrightarrow{- * G} \mathcal{K}$$

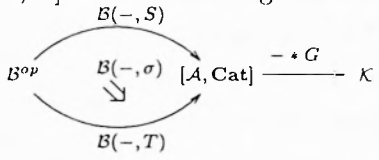
- for each $B \in \mathcal{B}^{op}$ there is a 2-functor $(L_T)_B$ given by

$$[\mathcal{A}^{op}, \mathcal{K}] \xrightarrow{\mathcal{B}(B,T) * -} \mathcal{K}$$

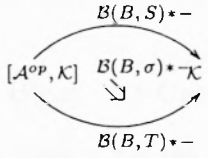
- clearly $(L_T G)(B) = (L_T)_B(G) = \mathcal{B}(B,T) * G$
- by 2-naturality of $- * \zeta$ for a strong transformation $\zeta : G \to G'$ we can take the structure cells to be the identities and the various coherence conditions hold;

and thus there is a 2-functor $L_T : [\mathcal{A}^{op}, \mathcal{K}] \to [\mathcal{B}^{op}, \mathcal{K}]$. Given now another homomorphism $S : \mathcal{A} \to \mathcal{B}$ and a strong transformation $\sigma : S \to T$ then we can define a 2-natural transformation $L_\sigma : L_S \to L_T$:

- for each $G \in [\mathcal{A}^{op}, \mathcal{K}]$ there is a strong transformation $L_\sigma G$



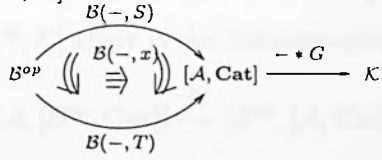

- for each $B \in \mathcal{B}^{op}$ there is a 2-natural transformation $(L_\sigma)_B$




- clearly $(L_\sigma G)_B = (L_\sigma)_B(G) = \mathcal{B}(B,\sigma) * G$ and the coherence conditions again hold by 2-naturality of the $- * \zeta$;

then by Section 1.2 these give the desired 2-natural transformations $L_\sigma$. Finally, given a modification $x : \sigma \to \sigma'$ we define a modification $L_x : L_\sigma \to L_{\sigma'}$:

- for each $G \in [\mathcal{A}^{op}, \mathcal{K}]$ there is a modification $L_x G$




- for each $B \in \mathcal{B}^{op}$ there is a modification $(L_x)_B = \mathcal{B}(B,x) * -$
- again clearly $(L_x G)_B = (L_x)_B(G) = \mathcal{B}(B,x) * G$

giving the required modification $L_x$.

## 2.3 Some functoriality properties of the assignments $L$

We have now defined a series of assignments

$$\begin{array}{rcl} \text{bicategory } \mathcal{A} & \longmapsto & \text{2-category } [\mathcal{A}^{op}, \mathcal{K}] \\ \text{homomorphism } T & \longmapsto & \text{2-functor } L_T \\ \text{strong transformation } \sigma & \longmapsto & \text{2-natural transformation } L_\sigma \\ \text{modification } x & \longmapsto & \text{modification } L_x \end{array}$$

and so we might hope to show that these form part of the data for a trihomomorphism

$$\mathbf{Bicat} \to \mathbf{2\text{-}Cat} \ ;$$

unfortunately, as we shall see, this is not the case.

First we would need to give a homomorphism of bicategories

$$[\mathcal{A}, \mathcal{B}] \xrightarrow{L} \mathbf{2\text{-}Cat}([\mathcal{A}^{op}, \mathcal{K}], [\mathcal{B}^{op}, \mathcal{K}]) \ ;$$

this in fact can be done, again using Section 1.2:

- for each $T \in [\mathcal{A}, \mathcal{B}]$ there is the 2-functor $L_T$
- for each $G \in [\mathcal{A}^{op}, \mathcal{K}]$ there is the homomorphism $L_G$:

$$[\mathcal{A}, \mathcal{B}] \xrightarrow{[\mathcal{A}, Y]} [\mathcal{A}, [\mathcal{B}^{op}, \mathbf{Cat}]] \xrightarrow{\cong} [\mathcal{B}^{op}, [\mathcal{A}, \mathbf{Cat}]] \xrightarrow{[\mathcal{B}^{op}, - * G]} [\mathcal{B}^{op}, \mathcal{K}]$$

- then $L_T(G) = L_G(T)$ and the remaining conditions follow as the $- * \zeta$ are 2-natural and the $- * x$ are modifications,

whence the required homomorphism $L$.

The next step would be to give strong equivalences

$$\begin{array}{ccc} [\mathcal{B},\mathcal{C}] \times [\mathcal{A},\mathcal{B}] & \longrightarrow & [\mathcal{A},\mathcal{C}] \\ {\scriptstyle L\times L}\downarrow & \Uparrow & \downarrow{\scriptstyle L} \\ \mathbf{2\text{-}Cat}([\mathcal{B}^{op},\mathcal{K}],[\mathcal{C}^{op},\mathcal{K}]) \times \mathbf{2\text{-}Cat}([\mathcal{A}^{op},\mathcal{K}],[\mathcal{B}^{op},\mathcal{K}]) & \longrightarrow & \mathbf{2\text{-}Cat}([\mathcal{A}^{op},\mathcal{K}],[\mathcal{C}^{op},\mathcal{K}]) \end{array}$$

and

$$\begin{array}{ccc} & 1 & \\ {\scriptstyle 1}\swarrow & \Downarrow & \searrow{\scriptstyle 1} \\ [\mathcal{A},\mathcal{A}] & \xrightarrow[L]{} & \mathbf{2\text{-}Cat}([\mathcal{A}^{op},\mathcal{K}],[\mathcal{A}^{op},\mathcal{K}]) \end{array}$$

but the components of such transformations would have to be 2-natural transformations, and the comparison maps $L_T L_S \Rightarrow L_{TS}$ and $1 \Rightarrow L_1$, while they are indeed equivalences, are not 2-natural, but only pseudo-natural (i.e. strong). We could attempt to salvage the situation by showing that the above data form part of a trihomomorphism from **Bicat** not to **2-Cat** but rather to **Gray**, and note that its "image" lies in **2-Cat**, in the sense precisely that its value at a 0-cell, 1-cell, or 2-cell in **Bicat** lies in **2-Cat**. However as the coherence conditions involved in checking such a claim are horrendous, we shall not attempt further progress at this level of generality.

## 2.4 The case $\mathcal{K} = \mathbf{Cat}$

We now specialize to the case where $\mathcal{K} = \mathbf{Cat}$. **Cat** is of course a pseudo-cocomplete 2-category, and so comes within the scope of the previous two sections. What is special about the case $\mathcal{K} = \mathbf{Cat}$ is the existence of the Yoneda homomorphisms $Y : \mathcal{A} \to [\mathcal{A}^{op}, \mathbf{Cat}]$. Recall that the assignments $L$ "look like" part of the data for a trihomomorphism **Bicat** $\cdots\cdots\!\!\rightarrow$ **Gray**; in

the same way, these $Y$ look like the components for a tritransformation from the identity trihomomorphism on **Bicat** to the composite of this would-be-trihomomorphism **Bicat** $\cdots\cdots\rightarrow$ **Gray** with the inclusion **Gray** $\rightarrow$ **Bicat**.

The first part to proving such a claim would be to give a strong equivalence

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{Y} & [\mathcal{A}^{op}, \mathbf{Cat}] \\ {\scriptstyle T}\downarrow & \simeq & \downarrow{\scriptstyle L_T} \\ \mathcal{B} & \xrightarrow[Y]{} & [\mathcal{B}^{op}, \mathbf{Cat}] \end{array}$$

which we shall now do.

Given $A, C \in \mathcal{A}$ and $B \in \mathcal{B}$ there is a functor

$$\mathcal{B}(B, TC) \times \mathcal{A}(C, A) \xrightarrow{1 \times T} \mathcal{B}(B, TC) \times \mathcal{B}(TC, TA) \xrightarrow{M} \mathcal{B}(B, TA)$$

and then, given a map $\gamma : C \rightarrow C'$ in $\mathcal{A}$, there is an isomorphism

$$\begin{array}{ccccc} \mathcal{B}(B,TC)\times\mathcal{A}(C,A) & \xrightarrow{1\times T} & \mathcal{B}(B,TC)\times\mathcal{B}(TC,TA) & \xrightarrow{M} & \mathcal{B}(B,TA) \\ {\scriptstyle 1\times\mathcal{A}(\gamma,A)}\downarrow & \cong & \uparrow{\scriptstyle 1\times\mathcal{B}(T\gamma,TA)} & \cong & \downarrow{\scriptstyle M} \\ \mathcal{B}(B,TC)\times\mathcal{A}(C',A) & \xrightarrow[1\times T]{} & \mathcal{B}(B,TC)\times\mathcal{B}(TC',TA) & \xrightarrow[\mathcal{B}(B,T\gamma)\times 1]{} & \mathcal{B}(B,TC')\times\mathcal{B}(TC',TA) \\ {\scriptstyle 1}\uparrow & & \uparrow{\scriptstyle 1\times T} & & \uparrow{\scriptstyle 1\times T} \\ \mathcal{B}(B,TC)\times\mathcal{A}(C',A) & \xrightarrow[1]{} & \mathcal{B}(B,TC)\times\mathcal{A}(C',A) & \xrightarrow[\mathcal{B}(B,T\gamma)\times 1]{} & \mathcal{B}(B,TC')\times\mathcal{A}(C',A) \end{array}$$

where the top left isomorphism comes from the functoriality constraint for $T$, the top right isomorphism comes from the associativity constraint for $\mathcal{B}$, and the bottom part of the diagram commutes; these functors and natural isomorphisms then correspond, respectively, to functors

$$\mathcal{B}(B, TC) \longrightarrow \mathbf{Cat}(\mathcal{A}(C, A), \mathcal{B}(B, TA)) \ ,$$

and natural isomorphisms

$$\begin{array}{ccc} \mathcal{B}(B,TC) & \longrightarrow & \mathbf{Cat}\big(\mathcal{A}(C,A),\mathcal{B}(B,TA)\big) \\ \Big\downarrow{\scriptstyle \mathcal{B}(B,T\gamma)} & \cong & \Big\downarrow{\scriptstyle \mathbf{Cat}\big(\mathcal{A}(\gamma,A),\mathcal{B}(B,TA)\big)} \\ \mathcal{B}(B,TC') & \longrightarrow & \mathbf{Cat}\big(\mathcal{A}(C,A),\mathcal{B}(B,TA)\big) \end{array}$$

which satisfy the three conditions for these data to give a strong transformation

$$\mathcal{B}(B,T-) \longrightarrow \mathbf{Cat}(\mathcal{A}(-,A),\mathcal{B}(B,TA))$$

and hence an arrow

$$\mathcal{B}(B,T) * \mathcal{A}(-,A) \longrightarrow \mathcal{B}(B,TA)\ . \tag{2.4.1}$$

Now given a map $b : B \to B'$ in $\mathcal{B}$ there are isos

$$\begin{array}{ccccc} \mathcal{B}(B',TC)\times\mathcal{A}(C,A) & \xrightarrow{1\times T} & \mathcal{B}(B',TC)\times\mathcal{B}(TC,TA) & \xrightarrow{M} & \mathcal{B}(B',TA) \\ \Big\downarrow{\scriptstyle \mathcal{B}(b,TC)\times 1} & & \Big\downarrow{\scriptstyle \mathcal{B}(b,TC)\times 1} & \cong & \Big\downarrow{\scriptstyle \mathcal{B}(b,TA)} \\ \mathcal{B}(B,TC)\times\mathcal{A}(C,A) & \xrightarrow{1\times T} & \mathcal{B}(B,TC)\times\mathcal{B}(TC,TA) & \xrightarrow{M} & \mathcal{B}(B,TA) \end{array}$$

where the left square commutes and the isomorphism in the right square comes from the associativity constraint for $\mathcal{B}$; these then correspond to isos

$$\begin{array}{ccc} \mathcal{B}(B',TC) & \longrightarrow & \mathbf{Cat}\big(\mathcal{A}(C,A),\mathcal{B}(B',TA)\big) \\ \Big\downarrow{\scriptstyle \mathcal{B}(b,TC)} & \cong & \Big\downarrow{\scriptstyle \mathbf{Cat}\big(\mathcal{A}(C,A),\mathcal{B}(b,TA)\big)} \\ \mathcal{B}(B,TC) & \longrightarrow & \mathbf{Cat}\big(\mathcal{A}(C,A),\mathcal{B}(B,TA)\big) \end{array}$$

which form an invertible modification

$$\begin{array}{ccc} \mathcal{B}(B',T-) & \longrightarrow & \mathbf{Cat}\big(\mathcal{A}(-,A),\mathcal{B}(B',TA)\big) \\ \Big\downarrow{\scriptstyle \mathcal{B}(b,T-)} & \cong & \Big\downarrow{\scriptstyle \mathbf{Cat}\big(\mathcal{A}(-,A),\mathcal{B}(b,TA)\big)} \\ \mathcal{B}(B,T-) & \longrightarrow & \mathbf{Cat}\big(\mathcal{A}(-,A),\mathcal{B}(B,TA)\big) \end{array}$$

and so an invertible 2-cell

$$\begin{array}{ccc} \mathcal{B}(B',T) * \mathcal{A}(-,A) & \longrightarrow & \mathcal{B}(B',TA) \\ \Big\downarrow{\scriptstyle \mathcal{B}(b,T) * \mathcal{A}(-,A)} & \cong & \Big\downarrow{\scriptstyle \mathcal{B}(b,TA)} \\ \mathcal{B}(B,T) * \mathcal{A}(-,A) & \longrightarrow & \mathcal{B}(B,TA) \end{array}$$

in $\mathcal{K}$, and these 2-cells form a strong transformation

$$\begin{array}{ccc} \mathcal{B}^{op} & \xrightarrow{\mathcal{B}(-,T)} & [\mathcal{A},\mathbf{Cat}] \\ & {\scriptstyle \mathcal{B}(-,TA)}\searrow \quad \Downarrow \quad \swarrow{\scriptstyle -*\mathcal{A}(-,A)} & \\ & \mathbf{Cat} & \end{array} \tag{2.4.2}$$

Similarly, we can show pseudonaturality in $A$ of the arrows

$$\mathcal{B}(B,T) * \mathcal{A}(-,A) \longrightarrow \mathcal{B}(B,TA).$$

and thus see that they form the components of a strong transformation

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{Y} & [\mathcal{A}^{op},\mathbf{Cat}] \\ & {\scriptstyle \mathcal{B}(B,T)}\searrow \quad \Downarrow \quad \swarrow{\scriptstyle \mathcal{B}(B,T)*-} & \\ & \mathbf{Cat}. & \end{array} \tag{2.4.3}$$

and the cube (1.2.4) commutes and so the two collections (2.4.2) and (2.4.3) of strong transformations fit together to give a strong transformation

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{Y} & [\mathcal{A}^{op}, \mathbf{Cat}] \\ {\scriptstyle T}\downarrow & \Downarrow & \downarrow{\scriptstyle L_T} \\ \mathcal{B} & \xrightarrow[Y]{} & [\mathcal{B}^{op}, \mathbf{Cat}] \, . \end{array} \tag{2.4.4}$$

We wish to show that this strong transformation (2.4.4) is in fact an equivalence in $[\mathcal{A}, [\mathcal{B}^{op}, \mathbf{Cat}]]$; this reduces simply to the component strong transformations (2.4.2) being equivalences in $[\mathcal{B}^{op}, \mathbf{Cat}]$; this in turn reduces to their component strong transformations

$$\mathcal{B}(B, T-) * \mathcal{A}(-, A) \longrightarrow \mathcal{B}(B, TA))$$

being equivalences in **Cat**, i.e. equivalences of categories.

Writing $\varphi$ for this last functor, we shall construct for it an inverse equivalence $\theta$.

For any category $X$ and any homomorphism $F : \mathcal{A} \to \mathbf{Cat}$ there is a functor

$$\mathbf{Cat}(F*\mathcal{A}(-,A),X) \xrightarrow{\cong} [\mathcal{A},\mathbf{Cat}](F,\mathbf{Cat}(\mathcal{A}(-,A),X)) \xrightarrow{ev_A} \mathbf{Cat}(FA,\mathbf{Cat}(\mathcal{A}(A,A),X)) \xrightarrow{\mathbf{Cat}(FA,ev_{1_A})} \mathbf{Cat}(FA,X)$$

which is 2-natural in $X$ and so of the form $\mathbf{Cat}(\theta_F, X)$ for some functor $\theta_F : FA \to F * \mathcal{A}(-, A)$. Define $\theta$ to be $\theta_{\mathcal{B}(B,T)}$.

Taking $X = FA$ we can express $\varphi\theta$ as the top path of the diagram

$$\begin{array}{ccccc} & & FA \times \mathcal{A}(A,A) & \xrightarrow{1 \times F} & FA \times \mathbf{Cat}(FA, FA) \\ & \overset{\left(\begin{smallmatrix}1\\id\end{smallmatrix}\right)}{\nearrow} & \cong & & \searrow^{ev} \\ FA & & \xrightarrow[1]{\left(\begin{smallmatrix}1\\id\end{smallmatrix}\right)} & & FA \, , \end{array}$$

where the isomorphism in the left triangle comes from the identity constraint for $F$ and the right triangle commutes, and so pasting then gives an isomorphism $\varphi\theta \cong 1$. The isomorphism $\theta\varphi \cong 1$ comes from the isomorphisms

$$\begin{array}{ccccc}
\mathcal{B}(B,TC)\times\mathcal{A}(C,A) & & \longrightarrow & & \mathcal{B}(B,T-)*\mathcal{A}(-,A) \\
{\scriptstyle 1\times M}\downarrow & & \cong & & \uparrow \\
\mathcal{B}(B,TC)\times\mathcal{A}(C,A)\times\mathcal{A}(A,A) & \xrightarrow[1\times T\times 1]{} & \mathcal{B}(B,TC)\times\mathcal{B}(TC,TA)\times\mathcal{A}(A,A) & \xrightarrow[M\times 1]{} & \mathcal{B}(B,TA)\times\mathcal{A}(A,A)
\end{array}$$

which in turn come from the "strong dinaturality" in $C$ of $\mathcal{B}(B,TC) \times \mathcal{A}(C,A) \to \mathcal{B}(B,T-) * \mathcal{A}(-,A)$.

Thus $\theta$ is inverse equivalent to $\varphi$ and so (2.4.1) are strong equivalences, and so (2.4.2) are strong equivalences, and so (2.4.4) are strong equivalences, which is to say that

**Proposition 2.4.1** *The functors $\varphi : \mathcal{B}(B,T-) * \mathcal{A}(-,A) \to \mathcal{B}(B,TA)$ constitute the basic data out of which can be made a strong equivalence*

$$\begin{array}{ccc}
\mathcal{A} & \xrightarrow{Y} & [\mathcal{A}^{op}, \mathbf{Cat}] \\
{\scriptstyle T}\downarrow & \simeq & \downarrow{\scriptstyle L_T} \\
\mathcal{B} & \xrightarrow[Y]{} & [\mathcal{B}^{op}, \mathbf{Cat}] \ .
\end{array}$$

In fact we can say more: the equivalence of this proposition has a kind of "naturality in $T$". More precisely, we shall construct a strong equivalence

$$\begin{array}{ccc}
\big[[\mathcal{A}^{op}, \mathbf{Cat}], [\mathcal{B}^{op}, \mathbf{Cat}]\big] & & \\
{\scriptstyle L}\nearrow & \simeq & \searrow{\scriptstyle \big[Y, [\mathcal{B}^{op}, \mathbf{Cat}]\big]} \\
[\mathcal{A}, \mathcal{B}] & \xrightarrow[[\mathcal{A},Y]]{} & \big[\mathcal{A}, [\mathcal{B}^{op}, \mathbf{Cat}]\big]
\end{array} \qquad (2.4.5)$$

whose component at $T$ is the equivalence of Proposition 2.4.1, directly above; as usual we use the techniques of Section 1.2. Take then a homomorphism $S : \mathcal{A} \to \mathcal{B}$ and a strong transformation $\sigma : S \to T$. Pseudo-naturality of $\sigma$ gives isomorphisms

$$\begin{array}{ccc} \mathcal{B}(B,SC) \times \mathcal{A}(C,A) & \xrightarrow{1 \times S} & \mathcal{B}(B,SC) \times \mathcal{B}(SC,SA) \xrightarrow{M} \mathcal{B}(B,SA) \\ {\scriptstyle \mathcal{B}(B,\sigma_C) \times 1}\downarrow & \cong & \downarrow{\scriptstyle \mathcal{B}(B,\sigma_A)} \\ \mathcal{B}(B,TC) \times \mathcal{A}(C,A) & \xrightarrow{1 \times T} & \mathcal{B}(B,TC) \times \mathcal{B}(TC,TA) \xrightarrow{M} \mathcal{B}(B,TA) \end{array}$$

and so isomorphisms

$$\begin{array}{ccc} \mathcal{B}(B,S) & \longrightarrow & \mathbf{Cat}\big(\mathcal{A}(-,A), \mathcal{B}(B,TA)\big) \\ {\scriptstyle \mathcal{B}(B,\sigma)}\downarrow & \cong & \downarrow{\scriptstyle \mathbf{Cat}\left(\mathcal{A}(-,A), \mathcal{B}(B,\sigma_A)\right)} \\ \mathcal{B}(B,T) & \longrightarrow & \mathbf{Cat}\big(\mathcal{A}(-,A), \mathcal{B}(B,SA)\big) \end{array}$$

which, combined with the maps (2.4.1) give, by Section 1.2, isomorphisms

$$\begin{array}{ccccc} [\mathcal{A},\mathcal{B}] & \xrightarrow{L} & \big[[\mathcal{A}^{op}, \mathbf{Cat}], [\mathcal{B}^{op}, \mathbf{Cat}]\big] & \xrightarrow{\big[Y, [\mathcal{B}^{op}, \mathbf{Cat}]\big]} & \big[\mathcal{A}, [\mathcal{B}^{op}, \mathbf{Cat}]\big] \\ {\scriptstyle [\mathcal{A},Y]}\downarrow & & \cong & & \downarrow{\scriptstyle ev_A} \\ \big[\mathcal{A}, [\mathcal{B}^{op}, \mathbf{Cat}]\big] & & \xrightarrow[ev_A]{} & & [\mathcal{B}^{op}, \mathbf{Cat}] \end{array}$$

which, combined with the maps (2.4.3) give, by Section 1.2, the desired equivalences (2.4.5):

**Proposition 2.4.2** *The strong equivalences of Proposition 2.4.1 form the*

*components for a strong equivalence*

$$\begin{array}{ccc} \mathbf{Gray}\big([\mathcal{A}^{op}, \mathbf{Cat}], [\mathcal{B}^{op}, \mathbf{Cat}]\big) & \longrightarrow & \big[[\mathcal{A}^{op}, \mathbf{Cat}], [\mathcal{B}^{op}, \mathbf{Cat}]\big] \\ \Big\downarrow{\scriptstyle L} & \simeq & \Big\downarrow{\scriptstyle [Y, [\mathcal{B}^{op}, \mathbf{Cat}]]} \\ [\mathcal{A}, \mathcal{B}] & \xrightarrow[[\mathcal{A}, Y]]{} & \big[\mathcal{A}, [\mathcal{B}^{op}, \mathbf{Cat}]\big] \ . \end{array}$$

## 2.5 The 2-categories $\bar{\mathcal{A}}$

It was first observed in [38] that the Yoneda embeddings

$$Y : \mathcal{A} \to [\mathcal{A}^{op}, \mathbf{Cat}]$$

were local equivalences; then Gordon and Power noticed in [19] the consequence that the full sub-2-category of $[\mathcal{A}^{op}, \mathbf{Cat}]$ determined by the objects $YA$ (i.e. the "strict" representables) is a 2-category biequivalent to the bicategory $\mathcal{A}$. Here we shall consider the full sub-2-category $\bar{\mathcal{A}}$ determined not just by the $YA$, but by all objects equivalent to them.

Then Proposition 2.4.2 tells us in particular that the 2-functor

$$L_T : [\mathcal{A}^{op}, \mathbf{Cat}] \to [\mathcal{B}^{op}, \mathbf{Cat}]$$

restricts to a 2-functor $\bar{T} : \bar{\mathcal{A}} \to \bar{\mathcal{B}}$. Given another homomorphism, $S : \mathcal{A} \to \mathcal{B}$, then

$$\mathbf{Gray}(\bar{\mathcal{A}}\ , \bar{\mathcal{B}})(\bar{S}, \bar{T}) = \mathbf{Gray}\big([\mathcal{A}^{op}, \mathbf{Cat}], [\mathcal{B}^{op}, \mathbf{Cat}]\big)(L_S, L_T)$$

where I write equality because the data involved are really *exactly* the same, thus in particular strong (resp. 2-natural) transformations $L_\sigma : L_S \to L_T$ "are" strong (resp. 2-natural) transformations $\bar{\sigma} : \bar{S} \to \bar{T}$ and modifications $L_\varkappa$ "are" modifications $\bar{\varkappa}$. Then the next proposition is immediate.

**Proposition 2.5.1** *There is a homomorphism of bicategories*

$$(\bar{\ }) : [\mathcal{A}, \mathcal{B}] \to \mathbf{Gray}(\bar{\mathcal{A}}\ , \bar{\mathcal{B}})$$

*which takes the values* $\bar{T}$, $\bar{\sigma}$, *and* $\bar{x}$ *on* $T$, $\sigma$, *and* $x$ *respectively, and with constraints inherited from those of* $L$.

This result "reconstructs" those of Section 2.3; we shall now reconstruct the results of Section 2.4.

By definition of $\bar{\mathcal{A}}$ , the Yoneda isomorphism $Y : \mathcal{A} \to [\mathcal{A}^{op}, \mathbf{Cat}]$ lands in it — $\bar{\mathcal{A}}$ is "the (bicategorical) repletion of $Y(\mathcal{A})$". Thus we write $Y$ also for the homomorphism (and indeed biequivalence) $\mathcal{A} \to \bar{\mathcal{A}}$ determined by the original $Y$.

Then Proposition 1.3.7 gives immediately

**Proposition 2.5.2** *There is a strong equivalence*

$$\begin{array}{ccc} \mathbf{Gray}(\bar{\mathcal{A}}\ , \bar{\mathcal{B}}) & \longrightarrow & [\bar{\mathcal{A}}\ , \bar{\mathcal{B}}] \\ \uparrow{\scriptstyle (\bar{\ })} & \simeq & \downarrow{\scriptstyle [Y, \bar{\mathcal{B}}]} \\ [\mathcal{A}, \mathcal{B}] & \xrightarrow[[\bar{\mathcal{A}}\ , Y]]{} & [\mathcal{A}, \bar{\mathcal{B}}] \end{array}$$

*whose data are exactly those of the strong equivalence in Proposition 2.4.2.*

## 2.6 A trihomomorphism from Bicat to Gray

We now apply Lemma 1.3.6, dealing with the transport of trihomomorphism structure, to the trihomomorphism given by the composite

$$\mathbf{Bicat} \xrightarrow{\mathbf{st}} \mathbf{Gray} \hookrightarrow \mathbf{Bicat} \qquad (2.6.1)$$

(where **st** is the trihomomorphism of [20] and Section 1.3.5 and the second map is the inclusion.) Then the required data are

- the function mapping $\mathcal{B}$ to $\bar{\mathcal{B}}$
- the homomorphisms $[\mathcal{A}, \mathcal{B}] \xrightarrow{\bar{(\ )}} \mathbf{Gray}(\bar{\mathcal{A}}\ , \bar{\mathcal{B}}) \longrightarrow [\bar{\mathcal{A}}\ , \bar{\mathcal{B}}]$
- the homomorphism (which is in fact a 2-functor) $\tilde{Y} : \mathbf{st}\mathcal{B} \to \bar{\mathcal{B}}$, defined by $\tilde{Y}\xi = Y$ using the universal property of $\xi$
- the strong equivalences

$$\begin{array}{ccc} [\mathcal{A}, \mathcal{B}] & \xrightarrow{\bar{(\ )}} & [\bar{\mathcal{A}}\ , \bar{\mathcal{B}}] \\ \Big| \mathbf{st} & \simeq\ \tilde{Y} & \Big| [\tilde{Y}, \bar{\mathcal{B}}] \\ [\mathbf{st}\mathcal{A}, \mathbf{st}\mathcal{B}] & \xrightarrow[[\mathbf{st}\mathcal{A}, \tilde{Y}]]{} & [\mathbf{st}\mathcal{A}, \bar{\mathcal{B}}] \end{array}$$

  which are built out of the equivalences of Proposition 2.5.2, the isomorphism of Proposition 1.3.7 and the definiting property of $\tilde{Y}$.

giving a trihomomorphism from **Bicat** to itself, strongly biequivalent to (2.6.1). Note that the trihomomorphism so-defined maps cells in **Bicat** of arbitrary dimension into **Gray**, all constraints are in **Gray** (as **Gray** is locally fully faithful), and the coherence conditions still hold as **Gray** is a "strict subtricategory". Thus this trihomomorphism can be factorized through **Gray** giving a trihomorphism

$$\mathbf{Bicat} \xrightarrow{\bar{(\ )}} \mathbf{Gray}\ .$$

There is still a tritransformation $\mathbf{st} \to \bar{(\ )}$ with components $\tilde{Y}$ (this fact is not entirely trivial, but follows by reapplying the lemma along with judicious use of Proposition 1.3.7). These $\tilde{Y}$, however, are not biequivalences in **Gray** as the maps which try to be their inverses are not in general 2-functors, thus the trihomomorphisms **st** and $\bar{(\ )}$ need no longer be triequivalent.

Summing up we have

**Theorem 2.6.1** *The above definitions give a trihomomorphism of tricategories* $(\bar{\ }) : \mathbf{Bicat} \to \mathbf{Gray}$ *such that the value at an arbitrary cell of* **Bicat** *lies in* **2-Cat**; *there is a tritransformation from the triequivalence* **st** *to* $(\bar{\ })$ *which under post-composition by the inclusion* $\mathbf{Gray} \to \mathbf{Bicat}$ *becomes a biequivalence, whence it follows that* $(\bar{\ }) : \mathbf{Bicat} \to \mathbf{Gray}$ *post-composed by the inclusion* $\mathbf{Gray} \to \mathbf{Bicat}$ *is biequivalent to the identity.*

# Chapter 3

# Enriched bicategories

For most of this chapter we shall be occupied with the definitions and some basic results of $\mathcal{W}$-bicategory theory for $\mathcal{W}$ a **Gray**-monoid; in the final section we shall extend these to the case of a general monoidal bicategory. Until that final section, then, we take $\mathcal{W}$ to be a fixed **Gray**-monoid, that is, a one-object **Gray**-category.

## 3.1 Enriched categories

If $\mathcal{V}$ is a monoidal category, then a $\mathcal{V}$-category comprises

- a set $Ob\mathcal{A}$
- a "hom-object" $\mathcal{A}(A, B) \in \mathcal{V}$ for each $A, B \in Ob\mathcal{A}$
- a "composition" $M : \mathcal{A}(B, C) \otimes \mathcal{A}(A, B) \to \mathcal{A}(A, C)$ in $\mathcal{V}$ for each $A, B, C \in Ob\mathcal{A}$
- an "identity map" $j : I \to \mathcal{A}(A, A)$ for each $A \in Ob\mathcal{A}$

these data satisfying the associativity and identity laws [26].

In [2], Bénabou gave an alternative description in terms of lax functors of bicategories. Given a set $X$, there is a bicategory $X_{ch}$ with object-set $X$ and with each hom-category the trivial category 1. On the other hand, a monoidal category $\mathcal{V}$ is exactly a one-object bicategory (which we shall sometimes write $\mathcal{V}_b$ when we wish to emphasize that it is being thought of as a bicategory). Then to give a $\mathcal{V}$-category is just to give the set $Ob\mathcal{A}$ of objects and a lax functor $Ob\mathcal{A}_{ch} \to \mathcal{V}_b$.

## 3.2 Enriched bicategories

Following his lead, Gordon, Power, and Street defined in [20] the notion of a $\mathcal{T}$-category, for a tricategory $\mathcal{T}$, using chaotic tricategories and lax functors of tricategories. Given a set $X$, there is a tricategory $X_{ch}$ with object set $X$ and each hom-bicategory the trivial bicategory 1. Then they define a $\mathcal{T}$-category to be a lax functor $X_{ch} \to \mathcal{T}$. We here restrict our attention to the case of one-object tricategories, *i.e.* monoidal bicategories; in fact at this stage, as mentioned above, we restrict to one-object **Gray**-categories, leaving the case for the general monoidal bicategories to the final section of the chapter. Given a monoidal bicategory $\mathcal{W}$, we shall sometimes write $\mathcal{W}_t$ if we wish to emphasize that it is being thought of as a tricategory.

**Definition 3.2.1** *A* lax $\mathcal{W}$-bicategory *with object set $Ob\mathcal{A}$ is a lax functor (of tricategories) $Ob\mathcal{A}_{ch} \to \mathcal{W}_t$.*

This is in some sense the weakest (most general) definition, and doubtless genuine examples in this generality will arise — at the risk of sounding trivial, whenever such lax functors do — however this definition has moved somewhat beyond the classical idea of enrichment given in the previous section. For to give such a lax functor one gives not *hom-objects* $\mathcal{A}(A, B) \in \mathcal{W}$, but rather *lax functors* (of bicategories) $1 \to \mathcal{W}$ which in fact includes *more* structure than

just an object of $\mathcal{W}$. Similarly there is more structure involved in giving the composition and identities. Thus we propose as the basic notion that given in [9], which amounts to an explicit version of the following definition.

**Definition 3.2.2** *A* $\mathcal{W}$-bicategory *with object set* $Ob\mathcal{A}$ *is a morphism (of tricategories)* $Ob\mathcal{A}_{ch} \to \mathcal{W}_t$ *such that the homomorphisms (of bicategories)* $1 \to \mathcal{W}$ *giving the hom-objects are in fact normal.*

**Remark 3.2.3** Enrichment in a monoidal bicategory is quite different from the notion of enrichment in a *bicategory*, given in [4]. In our case the homs are given by *objects* of the bicategory in question, in theirs by *arrows*. Ordinary enrichment in a monoidal category is a special case of enrichment in a bicategory exactly as our enrichment in a monoidal bicategory is a special case of enrichment in a tricategory.

This then amounts, as expected, to giving

(WBD1) a set $Ob\mathcal{A}$

(WBD2) an object $\mathcal{A}(A, B) \in \mathcal{W}$ for each $A, B \in Ob\mathcal{A}$

(WBD3) a 1-cell $M : \mathcal{A}(B, C) \otimes \mathcal{A}(A, B) \to \mathcal{A}(A, C)$ in $\mathcal{W}$ for each $A, B, C \in Ob\mathcal{A}$

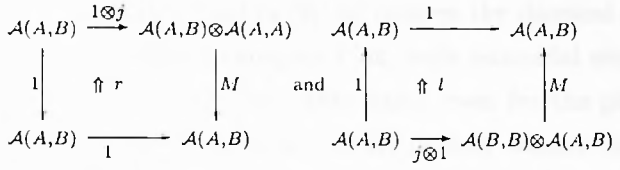
(WBD4) a 1-cell $j : I \to \mathcal{A}(A, A)$ in $\mathcal{W}$ for each $A \in Ob\mathcal{A}$

(WBD5) an invertible 2-cell

$$\begin{array}{ccc} \mathcal{A}(C,D)\otimes\mathcal{A}(B,C)\otimes\mathcal{A}(A,B) & \xrightarrow{1\otimes M} & \mathcal{A}(C,D)\otimes\mathcal{A}(A,C) \\ {\scriptstyle M\otimes 1}\downarrow & \Uparrow a & \downarrow{\scriptstyle M} \\ \mathcal{A}(B,D)\otimes\mathcal{A}(A,B) & \xrightarrow[M]{} & \mathcal{A}(A,D) \end{array}$$

in $\mathcal{W}$ for each $A, B, C, D \in Ob\mathcal{A}$

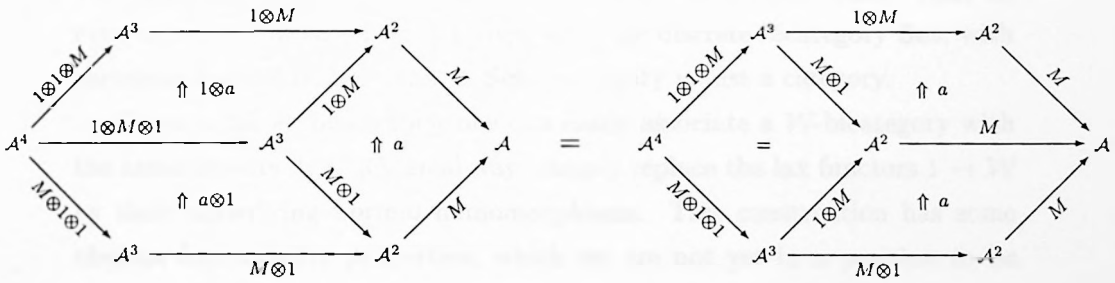
(WBD6) invertible 2-cells




in $\mathcal{W}$ for each $A, B \in Ob\mathcal{A}$

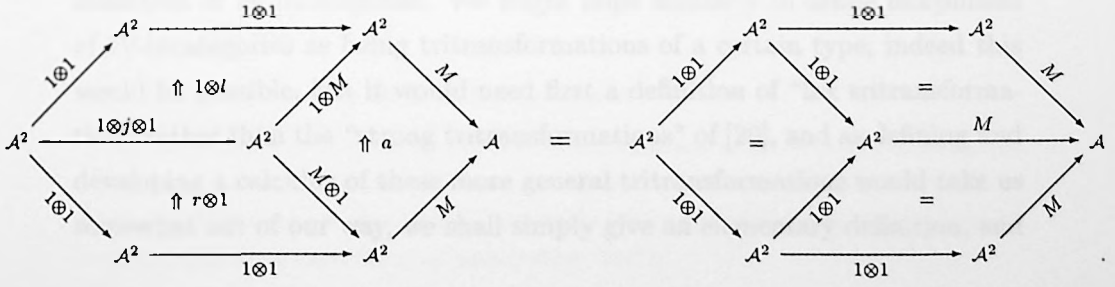
such that the following two cubes commute

(WBA1)




in $\mathcal{W}(\mathcal{A}(D,E)\otimes\mathcal{A}(C,D)\otimes\mathcal{A}(B,C)\otimes\mathcal{A}(A,B), \mathcal{A}(A,E))$ where, for example, $\mathcal{A}^4$ stands for $\mathcal{A}(D,E)\otimes\mathcal{A}(C,D)\otimes\mathcal{A}(B,C)\otimes\mathcal{A}(A,B)$ and $\mathcal{A}$ for $\mathcal{A}(A,E)$.

(WBA2)



in $\mathcal{W}(\mathcal{A}(B,C)\otimes\mathcal{A}(A,B),\mathcal{A}(A,C))$.

Using this definition, as pointed out in [9], we recover the classical notion of bicategory on taking for $\mathcal{W}$ the 2-category **Cat**, with monoidal structure given by the categorical product. On the other hand, even for the paradigmatic case of $\mathcal{W} = \mathbf{Cat}$, the lax $\mathcal{W}$-bicategories are strictly weaker than the $\mathcal{W}$-bicategories, providing further evidence for the correctness of this definition.

Another important special case is that where we take $\mathcal{W}$ to be discrete as a bicategory, *i.e.* to be a monoidal *category* $\mathcal{V}$, with only identity 2-cells. Then a $\mathcal{V}$-bicategory is just a $\mathcal{V}$-category in the traditional sense. Thus an even more special case is given when $\mathcal{W}$ is the discrete bicategory **Set**, with cartesian monoidal structure. A **Set**-bicategory is just a category.

Given a lax-$\mathcal{W}$-bicategory, one can easily associate a $\mathcal{W}$-bicategory with the same objects in a canonical way: simply replace the lax functors $1 \to \mathcal{W}$ by their underlying normal homomorphisms. This construction has some obvious functoriality properties, which we are not yet in a position to be able to formulate in a sensible manner.

## 3.3 Morphisms of $\mathcal{W}$-bicategories

The notion of lax functor of tricategories provides to some extent a convenient conceptual framework for the disparate data and conditions involved in the definition of $\mathcal{W}$-bicategories. We might hope similarly to define morphisms of $\mathcal{W}$-bicategories as being tritransformations of a certain type; indeed this would be possible, but it would need first a definition of "lax tritransformation" rather than the "strong tritransformations" of [20], and as defining and developing a calculus of these more general tritransformations would take us somewhat out of our way, we shall simply give an elementary definition, and

note here that a systematic treatment of the more general "lax tritransformations", analogous to that given in [20] would facilitate even the most basic considerations of the $\mathcal{W}$-morphisms to be defined below[1].

**Definition 3.3.1** *A (lax) $\mathcal{W}$-morphism of $\mathcal{W}$-bicategories $f : \mathcal{A} \to \mathcal{A}'$ comprises the data:*

*(WMD1) a function $f : Ob\mathcal{A} \to Ob\mathcal{A}'$*

*(WMD2) a 1-cell $f : \mathcal{A}(A, B) \to \mathcal{A}'(fA, fB)$ for each $A, B \in Ob\mathcal{A}$*

*(WMD3) a 2-cell*

$$\begin{array}{ccc} \mathcal{A}(B,C) \otimes \mathcal{A}(A,B) & \xrightarrow{\quad M \quad} & \mathcal{A}(A,C) \\ \Big\downarrow{\scriptstyle f \otimes f} & \Uparrow \lambda & \Big\downarrow{\scriptstyle f} \\ \mathcal{A}'(fB,fC) \otimes \mathcal{A}(fA,fB) & \xrightarrow[\quad M \quad]{} & \mathcal{A}'(fA,fC) \end{array}$$

*for each $A, B, C \in Ob\mathcal{A}$*

*(WMD4) a 2-cell*

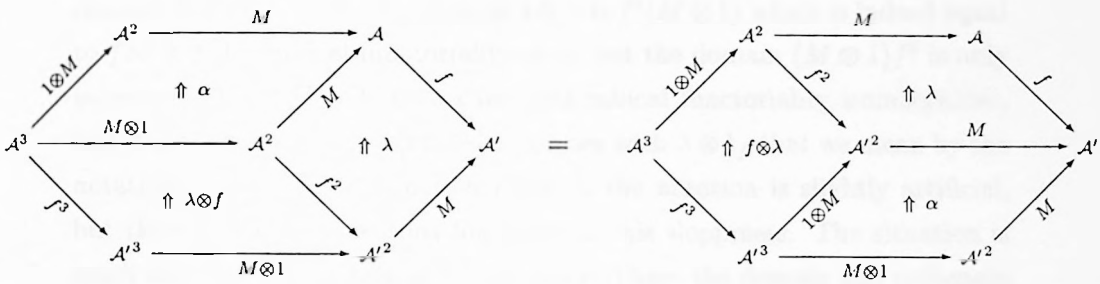
$$\begin{array}{ccc} I & \xrightarrow{\quad j \quad} & \mathcal{A}(A,A) \\ \Big\downarrow{\scriptstyle 1} & \Uparrow \iota & \Big\downarrow{\scriptstyle f} \\ I & \xrightarrow[\quad j \quad]{} & \mathcal{A}'(fA,fA) \end{array}$$

*for each $A \in \mathcal{A}$*

*such that the following three cubes commute:*

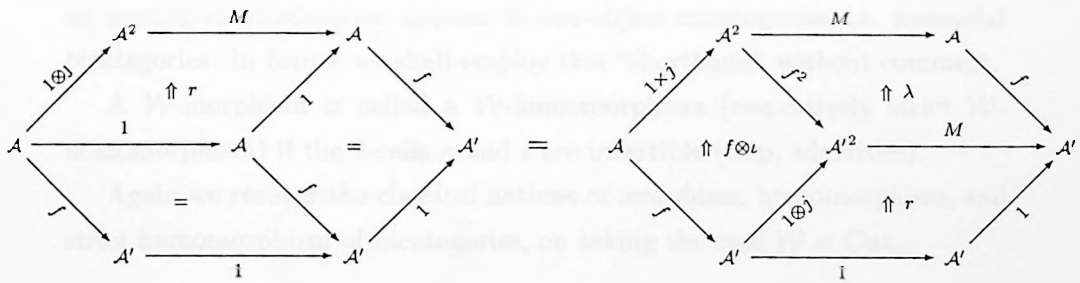
[1] In the notation of [20] (pp.18-19), a $\mathcal{W}$-transformation would be a tritransformation where the 1-cells $\theta_p$ of $\mathcal{W}_t$ are all the unit $I$ of $\mathcal{W}$, and the transformations $\theta$ would be 2-natural rather than pseudo-natural, but would not be required to be equivalences (which is what would make them "strong" rather than "lax").

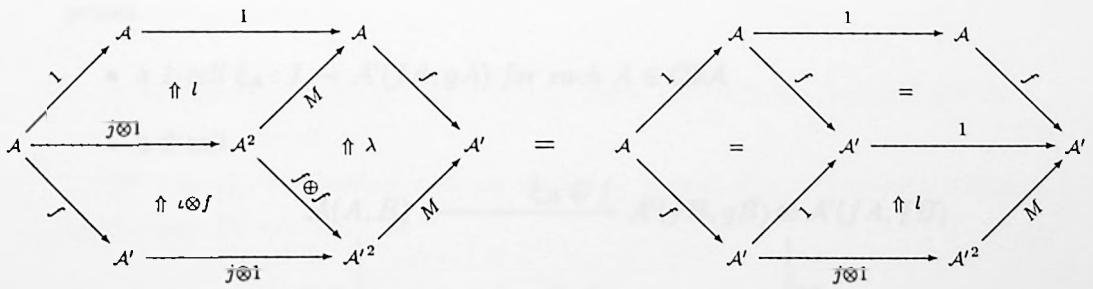
*(WMA1)*




*in* $\mathcal{W}(\mathcal{A}(C,D)\otimes\mathcal{A}(B,C)\otimes\mathcal{A}(A,B),\mathcal{A}'(fA,fD))$

*(WMA2)*




*both in* $\mathcal{W}(\mathcal{A}(A,B),\mathcal{A}'(fA,fB))$.

In the axioms of this last definition we have slyly introduced a shorthand notation. Consider for example the 2-cell $\lambda \otimes f$ of (WMA1). The tensor

product of the 2-cell $\lambda$ and the identity 2-cell on $f$ has codomain $fM \otimes f$ and domain $Mf^2 \otimes f$. The codomain of $\lambda \otimes f$ is $f^2(M \otimes 1)$ which is indeed equal to $fM \otimes f$, by cubical functoriality of $\otimes$, but the domain $(M \otimes 1)f^3$ is only isomorphic to $Mf^2 \otimes f$, by a canonical cubical functoriality isomorphism, and it is the composite of this isomorphism with $\lambda \otimes 1_f$ that we mean by the notation $\lambda \otimes f$. The distinction made in the notation is slightly artificial, but there are general reasons for allowing this sloppiness. The situation is much like that for pasting in a bicategory. There the domain and codomain of a pasting composite are defined only up to isomorphism, but once chosen the 2-cell is unique if it makes sense. In fact this is more than just reasoning by analogy, it is a special case of the tricategorical version of this result on pasting in bicategories, applied to one-object tricategories, i.e. monoidal bicategories. In future we shall employ this "shorthand" without comment.

A $\mathcal{W}$-morphism is called a $\mathcal{W}$-homomorphism (respectively strict $\mathcal{W}$-homomorphism) if the 2-cells $\kappa$ and $\iota$ are invertible (resp. identities).

Again we recover the classical notions of morphism, homomorphism, and strict homomorphism of bicategories, on taking the case $\mathcal{W} = \mathbf{Cat}$.

**Definition 3.3.2** *A $\mathcal{W}$-transformation of $\mathcal{W}$-morphisms $\xi : f \to g$ comprises*

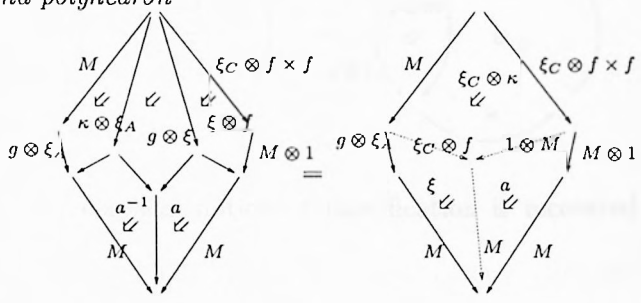
- *a 1-cell $\xi_A : I \to \mathcal{A}'(fA, gA)$ for each $A \in Ob\mathcal{A}$*
- *a 2-cell*

$$\begin{array}{ccc} \mathcal{A}(A,B) & \xrightarrow{\xi_B \otimes f} & \mathcal{A}'(fB,gB) \otimes \mathcal{A}'(fA,fB) \\ {\scriptstyle g \otimes \xi_A}\downarrow & \Downarrow \xi & \downarrow{\scriptstyle M} \\ \mathcal{A}'(gA,gB) \otimes \mathcal{A}'(fA,gA) & \xrightarrow[M]{} & \mathcal{A}'(fA,gB) \end{array}$$

*for each $A, B \in Ob\mathcal{A}$*

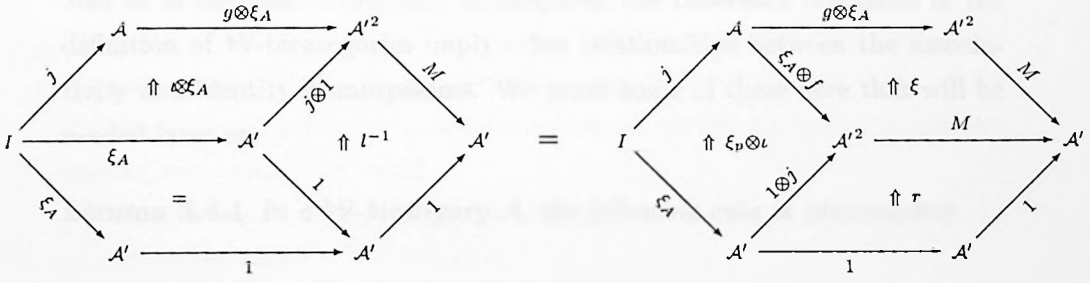
*satsifying commutativity of*

- *the Emma polyhedron*




*in* $\mathcal{W}(\mathcal{A}(B,C)\otimes\mathcal{A}(A,B),\mathcal{A}'(fA,gB))$

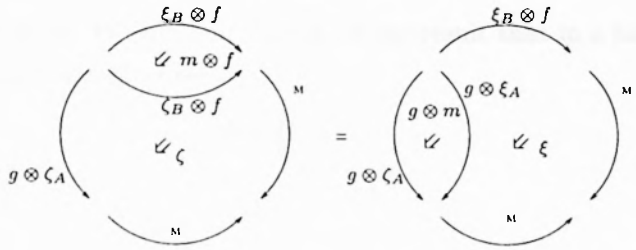
- *and the cube*




A $\mathcal{W}$-transformation is called strong (resp. strict) if the 2-cells $\xi$ are invertible (resp. identities).

We recover the classical notions of (mere, strong, strict) transformations for the case $\mathcal{W} = \mathbf{Cat}$.

**Definition 3.3.3** *A $\mathcal{W}$-modification of $\mathcal{W}$-transformations $\chi : \xi \to \zeta$ comprises, for each $A \in Ob\mathcal{A}$, a 2-cell $m_A : \xi_A \to \zeta_A : I \to \mathcal{A}'(fA, gA)$ satisfying the condition*

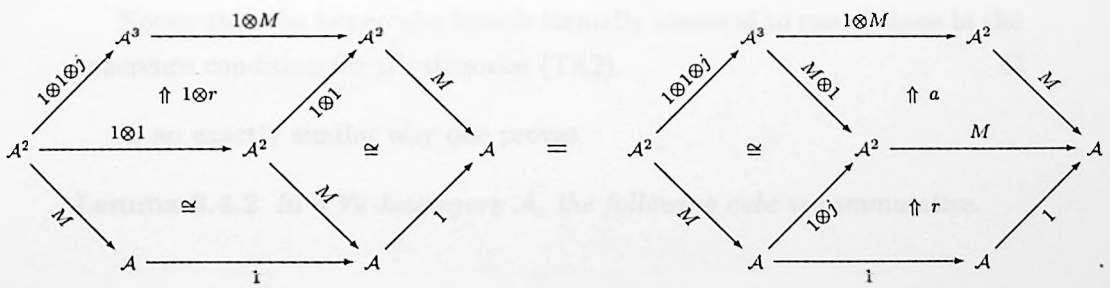



Once more the classical notion of modification is recovered on taking $\mathcal{W} = \mathbf{Cat}$.

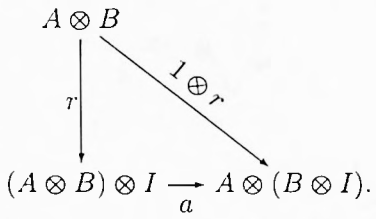
## 3.4 Some coherence results for enriched bicategories

Just as in the case of ordinary bicategories, the coherence conditions in the definition of $\mathcal{W}$-bicategories imply other relationships between the associativity and identity isomorphisms. We prove some of these here that will be needed later on.

**Lemma 3.4.1** *In a $\mathcal{W}$-bicategory $\mathcal{A}$, the following cube is commutative.*



PROOF: This is the $\mathcal{W}$-enriched version of the result that in a bicategory the following diagram commutes




Here we use for the first time an important technique in this subject — one might call it "the hypercube method". One constructs a hypercube of which the vertices are objects in $\mathcal{W}$, the edges are 1-cells, the faces are 2-cells, and 7 of the 8 hyperfaces are commutative cubes, the aim being to prove that the cube that is the final hyperface itself commutes.

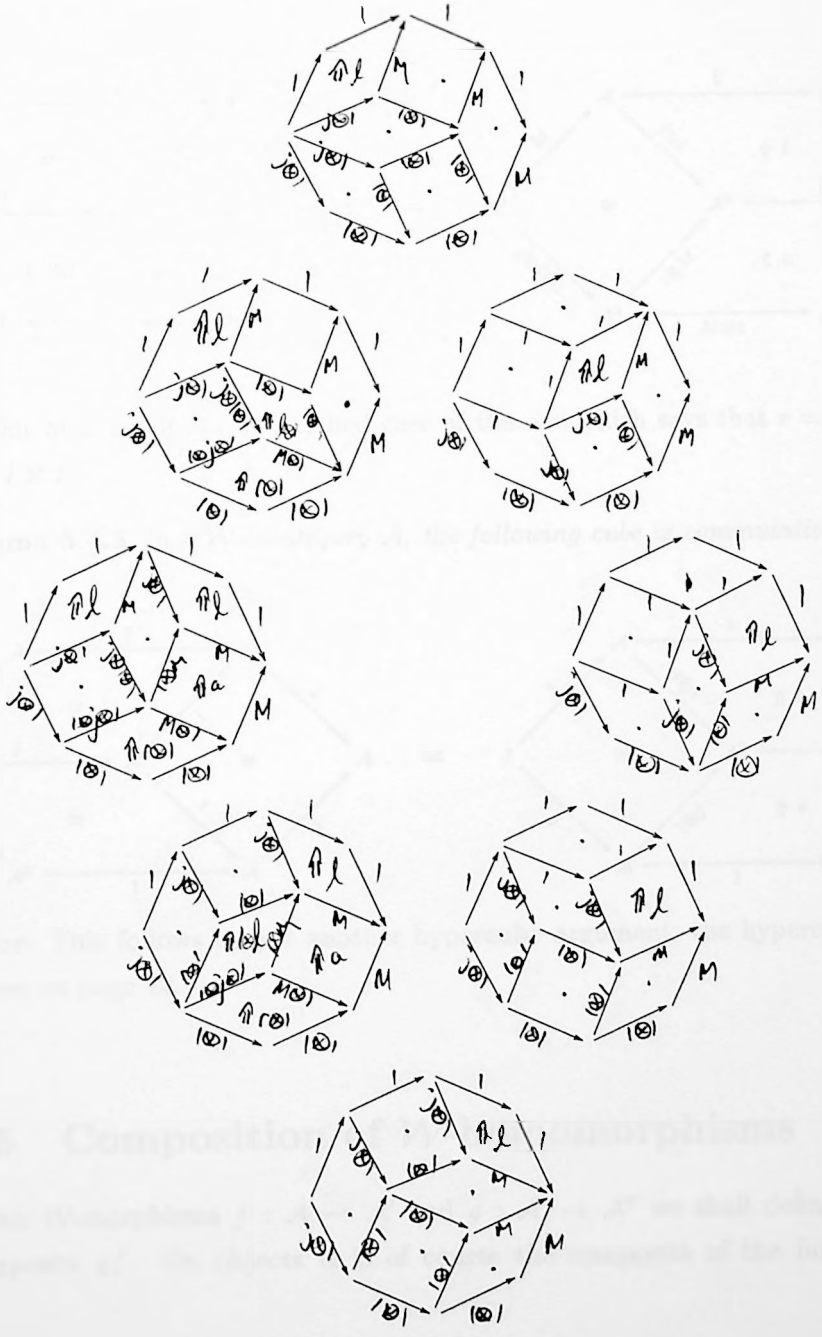
The hypercube in question here is shown on page 70, presented in the manner of [40]. Starting at the top and working clockwise each step corresponds to using the fact that one of the hyperfaces is commutative, which implies that the pasting composites are equal. It follows that the last two pasting composites are equal.

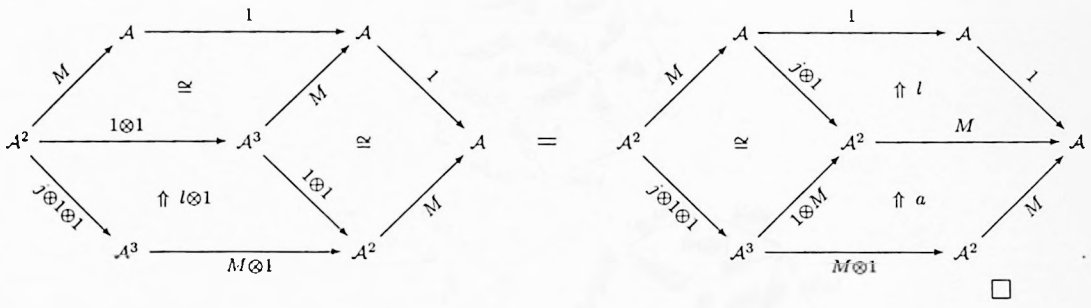
We now use the invertiblility of $1 \otimes 1 \otimes l$, $1 \otimes a$, and $a$, and naturality of $l$ to deduce the result.

Notice that the hypercube here is formally identical to one of those in the coherence condition for tricategories (TA2). □

In an exactly similar way one proves

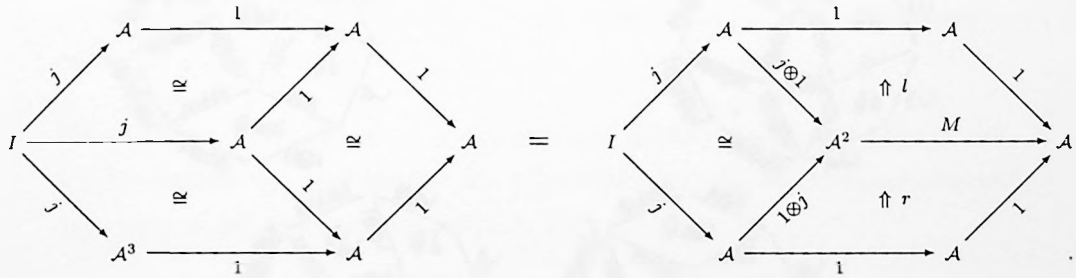
**Lemma 3.4.2** *In a $\mathcal{W}$-bicategory $\mathcal{A}$, the following cube is commutative.*






Our final result is the enriched case of the one which says that $r = l^{-1} : I \to I \otimes I$.

**Lemma 3.4.3** *In a $\mathcal{W}$-bicategory $\mathcal{A}$, the following cube is commutative.*

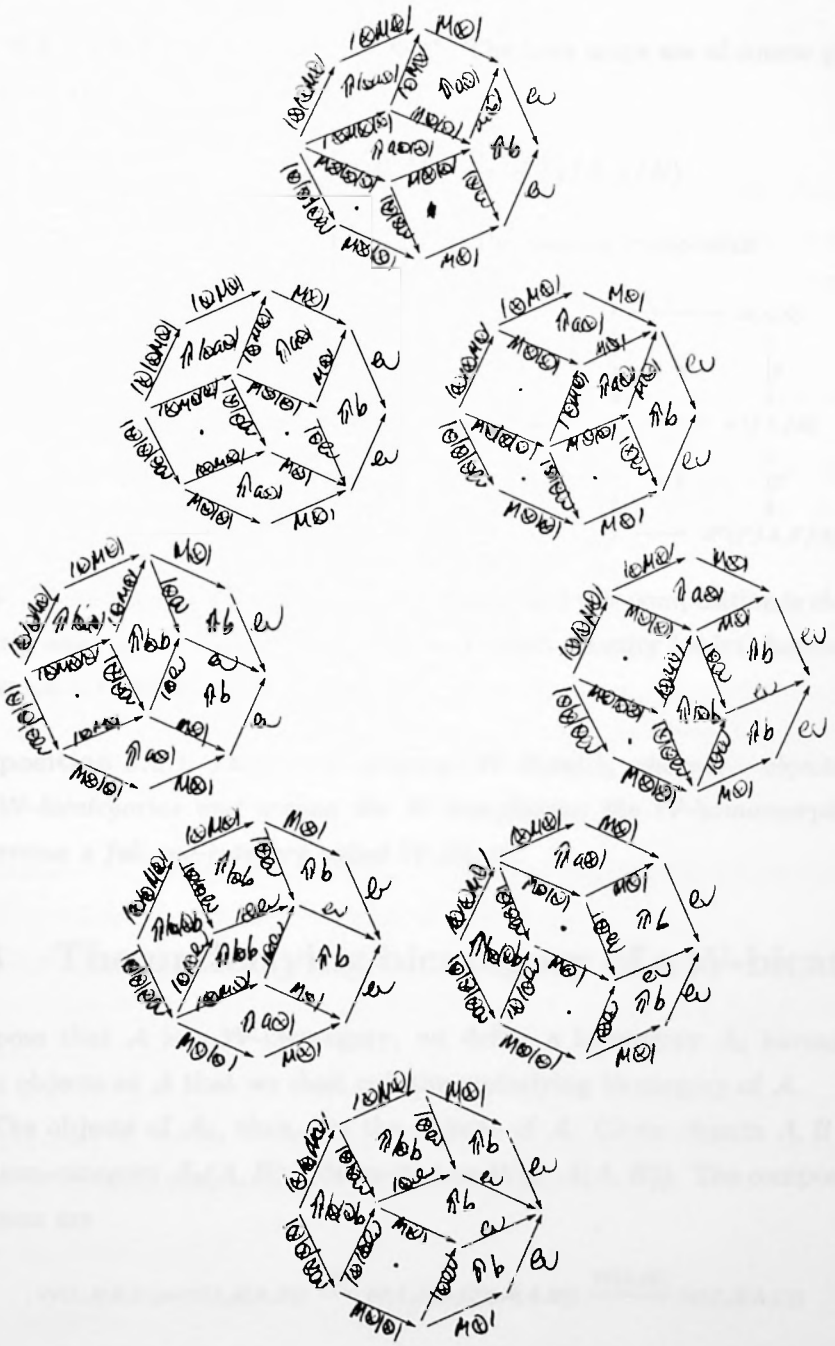



PROOF: This follows by yet another hypercube argument; the hypercube is shown on page 72.

□

## 3.5 Composition of $\mathcal{W}$-homomorphisms

Given $\mathcal{W}$-morphisms $f : \mathcal{A} \to \mathcal{A}'$ and $g : \mathcal{A}' \to \mathcal{A}''$ we shall define their composite $gf$. On objects it is of course the composite of the functions



$f : Ob\mathcal{A} \to Ob\mathcal{A}'$ and $g : Ob\mathcal{A}' \to Ob\mathcal{A}''$. The hom maps are of course given by composition of 1-cells in $\mathcal{W}$:

$$\mathcal{A}(A,B) \xrightarrow{f} \mathcal{A}'(fA,fB) \xrightarrow{g} \mathcal{A}''(gfA,gfB)\ .$$

The functoriality constraints are given by the pasting composites

$$\begin{array}{ccc}
\mathcal{A}(B,C)\otimes\mathcal{A}(A,B) & \xrightarrow{M} & \mathcal{A}(A,C) \\
{\scriptstyle f\times f}\downarrow & \Uparrow\kappa & \downarrow{\scriptstyle f} \\
\mathcal{A}'(fB,fC)\otimes\mathcal{A}'(fA,fB) & \xrightarrow[M]{} & \mathcal{A}'(fA,fC) \\
{\scriptstyle f'\times f'}\downarrow & \Uparrow\kappa & \downarrow{\scriptstyle f'} \\
\mathcal{A}''(f'fB,f'fC)\otimes\mathcal{A}'(f'fA,f'fB) & \xrightarrow[M]{} & \mathcal{A}''(f'fA,f'fC)
\end{array}
\quad\text{and}\quad
\begin{array}{ccc}
I & \xrightarrow{j} & \mathcal{A}(A,A) \\
{\scriptstyle 1}\downarrow & \iota\Uparrow & \downarrow{\scriptstyle f} \\
I & \xrightarrow[j]{} & \mathcal{A}'(fA,fA) \\
{\scriptstyle 1}\downarrow & \iota\Uparrow & \downarrow{\scriptstyle f'} \\
I & \xrightarrow[j]{} & \mathcal{A}''(f'fA,f'fA)
\end{array}$$

in $\mathcal{W}$. These then give a composite morphism, and this composition is clearly strictly associative, moreover there is an evident identity (strict) homomorphism on a $\mathcal{W}$-bicategory.

**Proposition 3.5.1** *There is a category $(\mathcal{W}\text{-}Bicat_l)_0$ whose objects are the $\mathcal{W}$-bicategories and arrows the $\mathcal{W}$-morphisms; the $\mathcal{W}$-homomorphisms determine a full sub-category called $\mathcal{W}\text{-}Bicat_o$.*

## 3.6 The underlying bicategory of a $\mathcal{W}$-bicategory

Suppose that $\mathcal{A}$ is a $\mathcal{W}$-bicategory; we define a bicategory $\mathcal{A}_0$ having the same objects as $\mathcal{A}$ that we shall call the underlying bicategory of $\mathcal{A}$.

The objects of $\mathcal{A}_0$, then, are the objects of $\mathcal{A}$. Given objects $A, B \in \mathcal{A}$ the hom-category $\mathcal{A}_0(A,B)$ is defined to be $\mathcal{W}(I, \mathcal{A}(A,B))$. The composition functors are

$$\mathcal{W}(I,\mathcal{A}(B,C))\times\mathcal{W}(I,\mathcal{A}(A,B)) \xrightarrow{\otimes} \mathcal{W}(I,\mathcal{A}(B,C)\otimes\mathcal{A}(A,B)) \xrightarrow{\mathcal{W}(I,M)} \mathcal{W}(I,\mathcal{A}(A,C))$$

and the identities are given by

$$1 \xrightarrow{\quad j \quad} \mathcal{W}(I,I) \xrightarrow{\mathcal{W}(I,j)} \mathcal{W}(I,\mathcal{A}(A,A)) \ .$$

Similarly the associativity and identity isomorphisms come from $\mathcal{W}(I,-)$ applied to the corresponding isomorphisms for the $\mathcal{W}$-bicategory $\mathcal{A}$, and the coherence conditions then hold because $\mathcal{W}(I,-)$ is a monoidal bihomomorphism from $\mathcal{W}$ to the bicategory **Cat**, as proved in Lemma 1.4.2.

**Definition 3.6.1** *The bicategory $\mathcal{A}_0$ is called the* underlying bicategory *of the $\mathcal{W}$-bicategory $\mathcal{A}$.*

There is another way of looking at the underlying bicategory of $\mathcal{A}$. We have seen that a $\mathcal{W}$-bicategory $\mathcal{A}$ is just a morphism

$$\mathcal{A}_{obch} \to \mathcal{W}_t$$

for which the data comprise normal homomorphisms, strong transformations and invertible modifications. We compose this with the **Gray**-functor

$$\mathcal{W}_t \to \mathbf{Cat}_t$$

of Lemma 1.4.2 giving another such morphism, or equivalently a **Cat**-bicategory, which is to say an ordinary unenriched bicategory.

There is a $\mathcal{W}$-bicategory $\mathcal{I}$, also called the trivial $\mathcal{W}$-bicategory, with a single object 0 and with hom-object $\mathcal{I}(0,0) = I$. As $\mathcal{W}$ is a **Gray**-monoid all other data can taken to be identities.

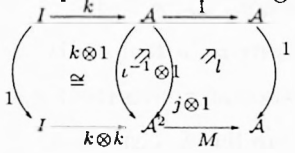
It is useful in this context to consider the various $\mathcal{W}$-enriched entities that live between the trivial $\mathcal{W}$-bicategory $\mathcal{I}$ and an arbitrary $\mathcal{W}$-bicategory $\mathcal{A}$.

A $\mathcal{W}$-homomorphism from $\mathcal{I}$ to $\mathcal{A}$ comprises

- an object $A$ of $\mathcal{A}$
- an arrow $k : I \to \mathcal{A}(A,A)$ in $\mathcal{W}$

- an invertible 2-cell $\kappa : M(k \otimes k) \to k$ in $\mathcal{W}$
- an invertible 2-cell $\iota : j \to k$ in $\mathcal{W}$

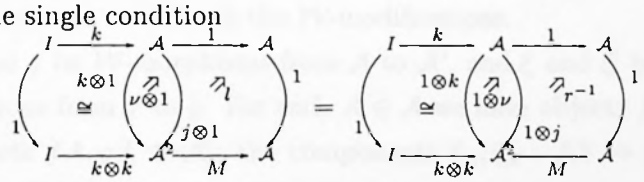
satisfying three conditions (which are given in Section 3.3). The third of these tells us that $\kappa$ must be equal to the pasting composite




where $\mathcal{A}$ stands for $\mathcal{A}(A, A)$ and $\mathcal{A}^2$ for $\mathcal{A}(A, A) \otimes \mathcal{A}(A, A)$, and the first then follows from the second. Thus to give a $\mathcal{W}$-homomorphism $\mathcal{I} \to \mathcal{A}$ actually amounts to giving

- an object $A$ of $\mathcal{A}$
- an arrow $k : I \to \mathcal{A}(A, A)$ in $\mathcal{W}$
- an invertible 2-cell $\nu : k \to j$ in $\mathcal{W}$ (where $\nu$ is $\iota^{-1}$ of the old notation)

satisfying the single condition




Given an object $A$ of $\mathcal{A}$, there is a canonical associated homomorphism $\ulcorner A \urcorner : \mathcal{I} \to \mathcal{A}$ made up of $A$, $j : I \to \mathcal{A}(A, A)$, and the identity 2-cell on $j$. The single coherence condition then follows by Lemma 3.4.3.

A strong $\mathcal{W}$-transformation between such $\mathcal{W}$-homomorphisms $\ulcorner A \urcorner, \ulcorner B \urcorner : \mathcal{I} \to \mathcal{A}$ comprises

- an arrow $x : I \to \mathcal{A}(A, B)$ in $\mathcal{W}$
- an invertible 2-cell $\xi : M(x \otimes j) \to M(j \otimes x)$ in $\mathcal{W}$

satisfying 2 conditions (again to be found in Section 3.3), the second of which states that $\xi$ must be given by

$$M(x \otimes j) \xrightarrow{\sim} M(1 \otimes j)x \xrightarrow{\rho x} x \xrightarrow{\lambda^{-1}x} M(j \otimes 1)x \xrightarrow{\sim} M(j \otimes x)$$

and the first of which will then follow automatically. Thus a strong $\mathcal{W}$-transformation from $\ulcorner A \urcorner$ to $\ulcorner B \urcorner$ is just an arrow $x : I \to \mathcal{A}(A, B)$ in $\mathcal{W}$.

Now if $x$ and $y$ are strong $\mathcal{W}$-transformations from $\ulcorner A \urcorner$ to $\ulcorner B \urcorner$, then a $\mathcal{W}$-modification from $x$ to $y$ comprises a 2-cell $m : x \Rightarrow y : I \to \mathcal{A}(A, B)$; the coherence condition of Section 3.3 is in this case vacuous. Thus at the level of graphs we have an isomorphism $\mathcal{W}\text{-}Bicat(\mathcal{I}, \mathcal{A})(\ulcorner A \urcorner, \ulcorner B \urcorner) \cong \mathcal{W}(I, \mathcal{A}(A, B))$, which we shall see in the next section lifts to an isomorphism of categories.

## 3.7 The structure of $\mathcal{W}$-Bicat

Suppose that $\mathcal{A}$, $\mathcal{A}'$ are $\mathcal{W}$-bicategories; we wish to construct a bicategory $\mathcal{W}\text{-}Bicat(\mathcal{A}, \mathcal{A}')$ whose objects are the $\mathcal{W}$-homomorphisms, arrows the strong $\mathcal{W}$-transformations, and 2-cells the $\mathcal{W}$-modifications.

Let $f$ and $g$ be $\mathcal{W}$-morphisms from $\mathcal{A}$ to $\mathcal{A}$', and $\xi$ and $\xi'$ be strong $\mathcal{W}$-transformations from $f$ to $g$. For each $A \in \mathcal{A}$ we have objects $fA, gA \in \mathcal{A}'$, and so objects $fA, gA \in \mathcal{A}'_0$; the components $\xi_A, \zeta_A : fA \to gA$ are then arrows $I \to \mathcal{A}'(fA, gA)$ in $\mathcal{W}$, which is to say arrows $fA \to gA$ in $\mathcal{A}'_0$, and a modification $m : \xi \to \xi'$ is just a collection of 2-cells between these $\xi_A, \xi'_A$ in $\mathcal{A}'_0$ satisfying a certain property. Then writing $\mathcal{C}(f, g)(\xi, \xi')$ for the set of $\mathcal{W}$-modifications from $\xi$ to $\xi'$, we have functions

$$E_A : \mathcal{C}(f, g)(\xi, \xi') \to \mathcal{A}'_0(fA, gA)(\xi_A, \xi'_A)$$

which are jointly injective; thus we need only observe that composing $\mathcal{W}$-modifications componentwise gives another $\mathcal{W}$-modification and the local

structure of $\mathcal{A}'_0$ gives us categories $\mathcal{C}(f,g)$ and jointly faithful functors

$$E_A : \mathcal{C}(f,g) \to \mathcal{A}'_0(fA, gA)$$

where $\mathcal{C}(f,g)$ comprises the strong $\mathcal{W}$-transformations and $\mathcal{W}$-modifications between $f$ and $g$.

Specializing to the case where $\mathcal{A}$ is $\mathcal{I}$, we have a faithful functor

$$E_* : \mathcal{W}\text{-}Bicat(\mathcal{I}, \mathcal{A}')(\ulcorner A \urcorner, \ulcorner B \urcorner) \to \mathcal{A}'_0(A, B)$$

which moreover, by Section 3.6, is an isomorphism of categories: we saw there that a strong $\mathcal{W}$-transformation from $\ulcorner A \urcorner$ to $\ulcorner B \urcorner$ was just an arrow $I \to \mathcal{A}(A, B)$ and a $\mathcal{W}$-modification just a 2-cell.

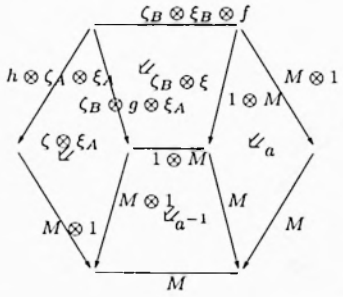
We now wish to define composition functors $M : \mathcal{C}(g,h) \otimes \mathcal{C}(f,g) \to \mathcal{C}(f,h)$ such that the following diagram commutes "on-the-nose".

$$\begin{array}{ccc} & \overset{M}{\cdots\cdots\cdots\cdots\to} & \\ \mathcal{C}(g,h)\times\mathcal{C}(f,g) & & \mathcal{C}(f,h) \\ \Big\downarrow{\scriptstyle E_A \times E_A} & & \Big\downarrow{\scriptstyle E_A} \\ \mathcal{A}'_0(\ulcorner gA \urcorner, \ulcorner hA \urcorner)\times\mathcal{A}'_0(\ulcorner fA \urcorner, \ulcorner gA \urcorner) & \xrightarrow{M} & \mathcal{A}'_0(\ulcorner fA \urcorner, \ulcorner hA \urcorner) \end{array}$$

First we define the object part of the functor: given strong $\mathcal{W}$-transformations $\zeta : g \to h$ and $\xi : f \to g$ we must define a composite transformation $\zeta\xi$. In order that this diagram commute, the composite must have component $(\zeta\xi)_A$ given by

$$I \xrightarrow{\zeta_A \otimes \xi_A} \mathcal{A}'(gA, hA) \otimes \mathcal{A}'(fA, gA) \xrightarrow{M} \mathcal{A}'(fA, hA) \ ;$$

we then define the pseudo-naturality isomorphisms as the pasting composites




in $\mathcal{W}(\mathcal{A}(A,B), \mathcal{A}'(fA, gB))$.

Using joint faithfulness of the $E_A$, there is a unique way to define $M$ on arrows to ensure that the $E_A$ preserve composition; we must only check that this unique choice is indeed well-defined, which amounts to the simple fact that forming the componentwise vertical composite of $\mathcal{W}$-modifications gives a $\mathcal{W}$-modification.

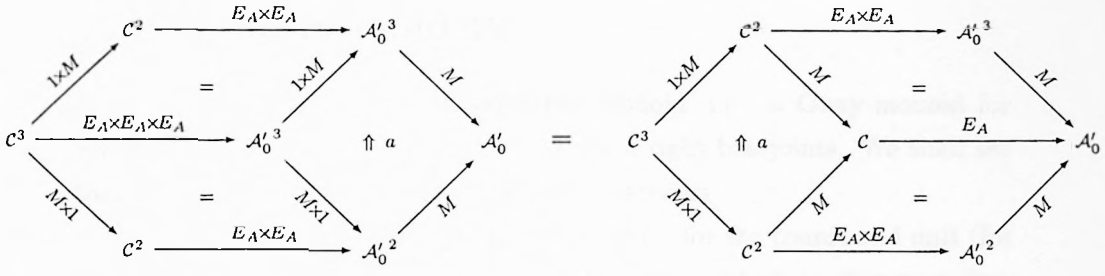
The identity strong $\mathcal{W}$-transformation on $f$ has component at $A$ given by $j : I \to \mathcal{A}'(fA, fA)$ and pseudo-naturality isomorphisms coming from $r$ and $l$; this identity is again strictly preserved by the $E_A$.

We are trying to define a bicategory $\mathcal{W}\text{-}Bicat(\mathcal{A}, \mathcal{A}')$; so far we have given the hom-categories, the composition functors, and the identities, and shown that the jointly faithful functors $E_A$ strictly preserve this structure. We shall now give the associativity isomorphisms. Thus we need an invertible 2-cell

$$\begin{array}{ccc} \mathcal{C}(h,k)\times\mathcal{C}(g,h)\times\mathcal{C}(f,g) & \xrightarrow{1\times M} & \mathcal{C}(h,k)\times\mathcal{C}(f,h) \\ {\scriptstyle M\times 1}\downarrow & \Uparrow\ \alpha & \downarrow{\scriptstyle M} \\ \mathcal{C}(g,k)\times\mathcal{C}(f,g) & \xrightarrow[M]{} & \mathcal{C}(f,k). \end{array}$$

This will have components in $\mathcal{C}(f,k)$, i.e. $\mathcal{W}$-modifications, which as we have seen are uniquely determined by *their* components. We choose these so

as to render equal




where $\mathcal{C}^3$, for example, stands for $\mathcal{C}(h,k) \times \mathcal{C}(g,h) \times \mathcal{C}(f,g)$, and $\mathcal{A}_0'^3$ for $\mathcal{A}_0'(\ulcorner hA \urcorner, \ulcorner kA \urcorner) \times \mathcal{A}_0'(\ulcorner gA \urcorner, \ulcorner hA \urcorner) \times \mathcal{A}_0'(\ulcorner fA \urcorner, \ulcorner gA \urcorner)$.

It is straightforward to confirm that these components do indeed form $\mathcal{W}$-modifications. The associativity coherence condition for bicategories holds because it does in $\mathcal{A}_0'$ and by joint faithfulness of the $E_A$. The identity constraints are constructed similarly, giving the following proposition.

**Proposition 3.7.1** *The above definitions give bicategories $\mathcal{W}$-Bicat$(\mathcal{A}, \mathcal{A}')$ of $\mathcal{W}$-homomorphisms, strong $\mathcal{W}$-transformations, and $\mathcal{W}$-modificatoins; and strict homomorphisms $E_A : \mathcal{W}\text{-}Bicat(\mathcal{A}, \mathcal{A}') \to \mathcal{A}_0'$ which are jointly locally faithful. Moreover $\mathcal{A}_0'$ sits inside $\mathcal{W}$-Bicat$(\mathcal{I}, \mathcal{A}')$ as a full sub-bicategory (determined by the objects of the form $\ulcorner A \urcorner$), and we have the following diagram of strict homomorphisms of bicategories:*

$$\begin{array}{ccc} \mathcal{W}\text{-}Bicat(\mathcal{A}, \mathcal{A}') & \xrightarrow{\overline{E}_A} & \mathcal{W}\text{-}Bicat(\mathcal{I}, \mathcal{A}') \\ & \searrow^{E_A} & \uparrow I \;\; \downarrow E_* \\ & & \mathcal{A}_0' \end{array}$$

*where $E_* I = 1$, $I E_A = \overline{E}_A$, and $E_* \overline{E}_A = E_A$.*

## 3.8 The $\mathcal{W}$-bicategory $\overline{\mathcal{W}}$ for a right-closed Gray-monoid $\mathcal{W}$

Suppose that $\mathcal{W}$ is a right-closed **Gray**-monoid, i.e. a **Gray**-monoid for which the 2-functors $- \otimes W : \mathcal{W} \to \mathcal{W}$ have right biadjoints. We shall see that $\mathcal{W}$ can naturally be made into a $\mathcal{W}$-bicategory.

Write $[W, -]$ for the right biadjoint; $ev$ and $\eta$ for the counit and unit (for arbitrary $W$, we shall rely on context to distinguish which biadjunction they form a part of).

The objects for this $\mathcal{W}$-bicategory will be those of $\mathcal{W}$. Given $U, V \in \mathcal{W}$ we define $\overline{\mathcal{W}}(U, V) = [U, V]$. The composition $M$ is defined to be

$$[V,W] \otimes [U,V] \xrightarrow{\eta} [U,[V,W] \otimes [U,V] \otimes U] \xrightarrow{[1,1\otimes ev]} [U,[V,W] \otimes V] \xrightarrow{[1,ev]} [U,W]$$

and the identity to be

$$I \xrightarrow{\eta} [U, I \otimes U] = [U,U] \ .$$

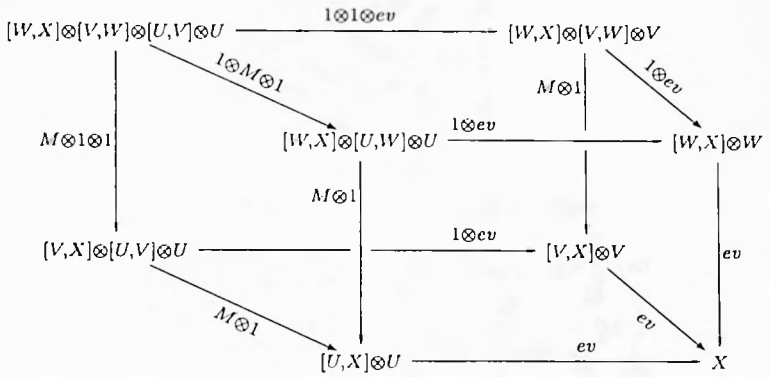
We now form the pasting composite

$$\begin{array}{ccccccc}
[V,W]\otimes[U,V]\otimes U & \xrightarrow{\eta\otimes 1} & [U,[V,W]\otimes[U,V]\otimes U]\otimes U & \xrightarrow{[1,1\otimes ev]\otimes 1} & [U,[V,W]\otimes U]\otimes U & \xrightarrow{[1,ev]\otimes 1} & [U,W]\otimes U \\
\Big\downarrow{\scriptstyle 1} & \cong & \Big\downarrow{\scriptstyle ev} & \cong & \Big\downarrow{\scriptstyle ev} & \cong & \Big\downarrow{\scriptstyle ev} \\
[V,W]\otimes[U,V]\otimes U & \xrightarrow[1]{} & [V,W]\otimes[U,V]\otimes U & \xrightarrow[1\otimes ev]{} & [V,W]\otimes V & \xrightarrow[ev]{} & W
\end{array}$$

coming (left to right) from a triangle isomorphism and two pseudo-naturality isomorphisms, and thus obtain an isomorphism

$$\begin{array}{ccc}
[V,W] \otimes [U,V] \otimes U & \xrightarrow{M \otimes 1} & [U,W] \otimes U \\
\Big\downarrow{\scriptstyle 1 \otimes ev} & \cong\ b & \Big\downarrow{\scriptstyle ev} \\
[V,W] \otimes V & \xrightarrow[ev]{} & W
\end{array} \ .$$

In the cube




we can place the isomorphism $b$ in the front, bottom and right faces; its tensor product with $1_{[W,X]}$ in the top face, and the isomorphism expressing cubical functoriality of $\otimes$ in the back face; then use the biadjunction to obtain an associativity isomorphism

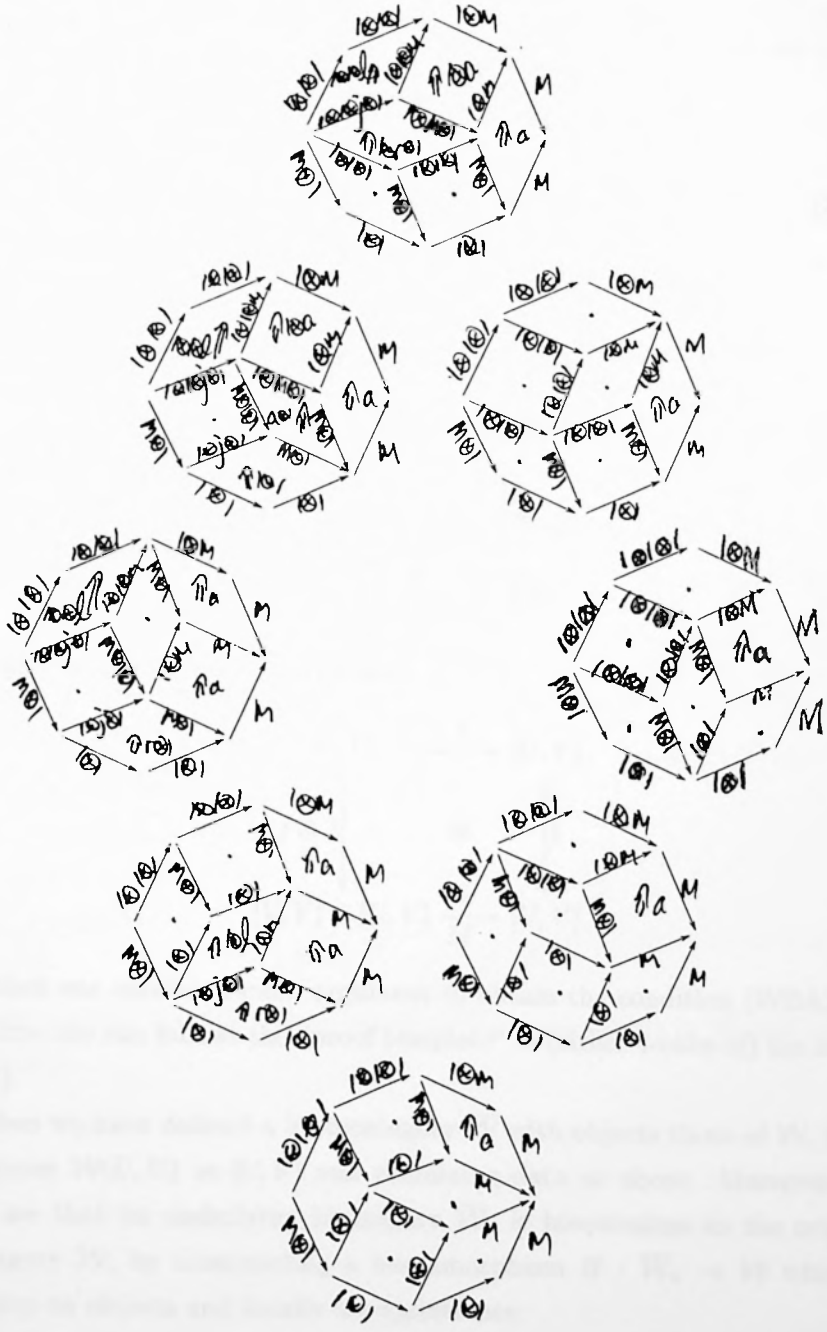
$$\begin{array}{ccc} [W,X]\otimes[V,W]\otimes[U,V] & \xrightarrow{M\otimes 1} & [V,X]\otimes[U,V] \\ \Big\downarrow{\scriptstyle 1\otimes M} & \overset{\alpha}{\cong} & \Big\downarrow{\scriptstyle M} \\ [W,X]\otimes[U,W] & \xrightarrow[M]{} & [U,X] \end{array}$$

rendering the cube commutative.

To prove the associativity coherence axiom (WBA1) we have to resort once more to a "hypercube argument", as in Section 3.4.

The hypercube is shown on 82; this time one starts at the top, works anticlockwise, and then uses the biadjunction.

Note this time the formal similarity of this 4-cube and that for the associativity coherence axiom for tricategories (TA1).



The triangle isomorphism

$$\begin{array}{ccc} U & \xrightarrow{j \otimes 1} & [U,U] \otimes U \\ {\scriptstyle 1}\downarrow & \cong & \downarrow{\scriptstyle ev} \\ U & \xrightarrow[1]{} & U \end{array} \qquad (3.8.1)$$

gives via the biadjunction an isomorphism

$$\begin{array}{ccc} [U,V] & \xrightarrow{1 \otimes j} & [U,V] \otimes [U,U] \\ {\scriptstyle 1}\downarrow & \cong & \downarrow{\scriptstyle M} \\ [U,V] & \xrightarrow[1]{} & [U,V]; \end{array}$$

similarly we obtain an isomorphism

$$\begin{array}{ccc} [U,V] & \xrightarrow{1} & [U,V] \\ {\scriptstyle j \otimes 1}\downarrow & \cong & \downarrow{\scriptstyle 1} \\ [V,V] \otimes [U,V] & \xrightarrow[M]{} & [U,V] \end{array}$$

and then use another 4-cube argument to obtain the condition (WBA2) — this time one can follows the "proof template" of (either 4-cube of) the axiom (TA2).

Thus we have defined a $\mathcal{W}$-bicategory $\overline{\mathcal{W}}$ with objects those of $\mathcal{W}$, hom-categories $\mathcal{W}(U,V) = [U,V]$ and remaining data as above. Moreover, we shall see that its underlying bicategory $\overline{\mathcal{W}}_0$ is biequivalent to the original bicategory $\mathcal{W}$, by constructing a homomorphism $H : \overline{\mathcal{W}}_0 \to \mathcal{W}$ which is bijective on objects and locally an equivalence.

The objects of $\overline{\mathcal{W}}_0$ and $\mathcal{W}$ are the same and so we can take the object part of $H$ to be an identity. The hom-categories of $\overline{\mathcal{W}}_0$ are

$$\overline{\mathcal{W}}_0(U,V) = \mathcal{W}(I, \overline{\mathcal{W}}(U,V)) = \mathcal{W}(I,[U,V]);$$

but the biadjunction making $\mathcal{W}$ right-closed as a **Gray**-monoid tells us precisely that $\mathcal{W}(I,[U,V])$ is equivalent to $\mathcal{W}(U,V)$, and we take this equivalence for the "hom" part of $H$. Explicitly it can be written as

$$\mathcal{W}(I,[U,V]) \xrightarrow{1\times j} \mathcal{W}(I,[U,V])\times\mathcal{W}(U,U) \xrightarrow{\otimes} \mathcal{W}(U,[U,V]\otimes U) \xrightarrow{\mathcal{W}(1,ev)} \mathcal{W}(U,V).$$

In proving functoriality of this $H$, the crucial step is to construct the pasting composite

$$\begin{array}{ccccccc}
[U,V]\otimes U & \xrightarrow{j\otimes 1\otimes 1} & [V,V]\otimes[U,V]\otimes U & \xrightarrow{[V,y]\otimes 1} & [V,W]\otimes[U,V]\otimes U & \xrightarrow{M\otimes 1} & [U,W]\otimes U \\
\downarrow{\scriptstyle ev} & \cong & \downarrow{\scriptstyle 1\otimes ev} & \cong & \downarrow{\scriptstyle 1\otimes ev} & \cong\ b & \downarrow{\scriptstyle ev} \\
V & \xrightarrow[j\otimes 1]{} & [V,V]\otimes V & \xrightarrow[[V,y]\otimes 1]{} & [V,W]\otimes V & \xrightarrow[ev]{} & W
\end{array}$$

note that the bottom row is isomorphic to $y$, and deduce by the biadjunction an isomorphism

$$\begin{array}{ccc}
[U,V] & \xrightarrow{\quad[U,y]\quad} & [U,W] \\
\downarrow{\scriptstyle j\otimes 1} & \cong & \downarrow{\scriptstyle M} \\
[V,V]\otimes[U,V] & \xrightarrow[[V,y]]{} & [V,W]\otimes[U,V]\ ;
\end{array}$$

the rest reduces to using associativity of the composition in $\overline{\mathcal{W}}$ and the biadjunction.

**Proposition 3.8.1** *Given a right-closed* **Gray**-*monoid $\mathcal{W}$ there is a $\mathcal{W}$-bicategory with the same objects as $\mathcal{W}$, hom-objects given by $\mathcal{W}(U,V) = [U,V]$, and remaining data as above; its underlying bicategory is biequivalent to $\mathcal{W}$.*

□

## 3.9 $\mathcal{W}$-bicategory theory for a general $\mathcal{W}$

We now wish to define $\mathcal{W}$-bicategories for $\mathcal{W}$ a general monoidal bicategory, extending the restricted case of **Gray**-monoids considered in the rest of this chapter. The most concise definition is again in terms of morphisms of tricategories.

**Definition 3.9.1** *A* $\mathcal{W}$-bicategory *with object set* $Ob\mathcal{A}$ *is a morphism (of tricategories)* $Ob\mathcal{A}_{ch} \to \mathcal{W}_t$ *such that the homomorphisms (of bicategories)* $1 \to \mathcal{W}$ *giving the hom-objects are in fact normal.*

Again one can express this in a more elementary manner in terms of hom-objects $\mathcal{A}(A, B) \in \mathcal{W}$, composition arrows in $\mathcal{W}$, and so on. This is done in [9]. The geometry is of course now more complicated than the cubes of the previous section; they are replaced by 3-dimensional objects called Stasheff polytopes, which are precisely the objects which appear, for example, in the associativity coherence condition (TA1) for tricategories. Just as there, they can in fact be viewed as 3-dimensional projections of 4-cubes, as indeed we would expect on remembering that $\mathcal{W}$-bicategories can be viewed as particular instances of tricategory morphisms.

Just as in Section 3.6, for $\mathcal{W}$ a **Gray**-monoid, we gave the underlying bicategory of a $\mathcal{W}$-bicategory $\mathcal{A}$ as the bicategory associated to the composite

$$\mathcal{A}_{obch} \xrightarrow{Hom_{\mathcal{A}}} \mathcal{W}_t \xrightarrow{\mathcal{W}(I,-)} \mathbf{Cat}$$

so can we do so here. Of course we now have not a **Gray**-functor $\mathcal{W}(I, -)$ but only a monoidal homomorphism $\mathcal{W}(I, -)$, and so we get in the first instance only a "lax bicategory" — the homomorphisms $1 \to \mathbf{Cat}$ giving the hom-objects may not be normal — but we can then take the associated genuine bicategory as described in Section 3.2, simply by replacing these homomorphisms by the associated normal homomorphisms.

### 3.9.1 Change of base for enriched bicategories

In enriched category theory one wants to be able to consider "change of base"; i.e. to allow the monoidal category $\mathcal{V}$ over which one enriches to vary. Similarly for enriched bicategory theory one wants to vary the monoidal bicategory $\mathcal{W}$. We have already seen one example of change-of-base — in Section 3.6 where we considered the underlying bicategory of a $\mathcal{W}$-bicategory, which amounts to changing base from $\mathcal{W}$ to **Cat**. Although it is beyond the scope of this thesis to deal with this topic in any systematic way, we shall develop just enough to deal with the problem of extending results proved for $\mathcal{W}$ a **Gray**-category to the case of a general monoidal bicategory.

Given a monoidal homomorphism $H : \mathcal{W} \to \mathcal{W}'$ we can assign in a natural way a $\mathcal{W}'$-bicategory $\mathcal{A}^H$ to each $\mathcal{W}$-bicategory $\mathcal{A}$. For $\mathcal{A}$ is just a morphism $Ob\mathcal{A}_{ch} \to \mathcal{W}_t$ satisfying a certain property; we compose it with $H$ to get a lax $\mathcal{W}'$-bicategory, and then use the construction given at the end of Section 3.2 to obtain a $\mathcal{W}'$-bicategory.

Explicitly, $\mathcal{A}^H$ has the same objects as $\mathcal{A}$ and

- $\mathcal{A}^H(A,B) = H\mathcal{A}(A,B)$
- $M^H = H\mathcal{A}(B,C) \otimes H\mathcal{A}(A,B) \xrightarrow{\kappa} H(\mathcal{A}(B,C) \otimes \mathcal{A}(A,B)) \xrightarrow{HM} H\mathcal{A}(A,C)$
- $j^H = I \xrightarrow{\iota^{-1}} HI \xrightarrow{Hj} H\mathcal{A}(A,A)$

and the other data is constructed similarly. The advantage of treating enriched bicategories as lax functors is that we can immediately bring to bear the general theory of tricategories for which most of the coherence conditions that need to be checked already have been. Indeed were the theory of tricategories developed already at the level of generality described in Section 3.3, we could now immediately deduce the functoriality properties of this construction.

Of particular interest to us is the case where $H$ is a monoidal biequivalence (*i.e.* triequivalence of one-object tricategories) between a monoidal bicategory and a **Gray**-category, as is constructed for each monoidal bicategory in [20]. For then, given a monoidal bicategory $\mathcal{W}$, we can choose a monoidal biequivalence $H : \mathcal{W} \to \mathcal{W}'$, where $\mathcal{W}'$ is a **Gray**-monoid, and then move freely between $\mathcal{W}$-bicategories and $\mathcal{W}'$-bicategories using the above change-of-base construction. In this way we can prove results in the simpler case of enrichment over a **Gray**-monoid, and then deduce corresponding results in the general context.

**Proposition 3.9.2** *Change of base preserves the underlying bicategory.*

PROOF: More precisely, given a $\mathcal{W}$-bicategory $\mathcal{A}$, and a monoidal biequivalence $H : \mathcal{W} \to \mathcal{W}'$ we give a biequivalence $H_0$ between $\mathcal{A}_0$ and $\mathcal{A}_0^H$.

The objects of $\mathcal{A}_0$ and $\mathcal{A}_0^H$ are the same, and the object part of $H_0$ is taken to be the identity.

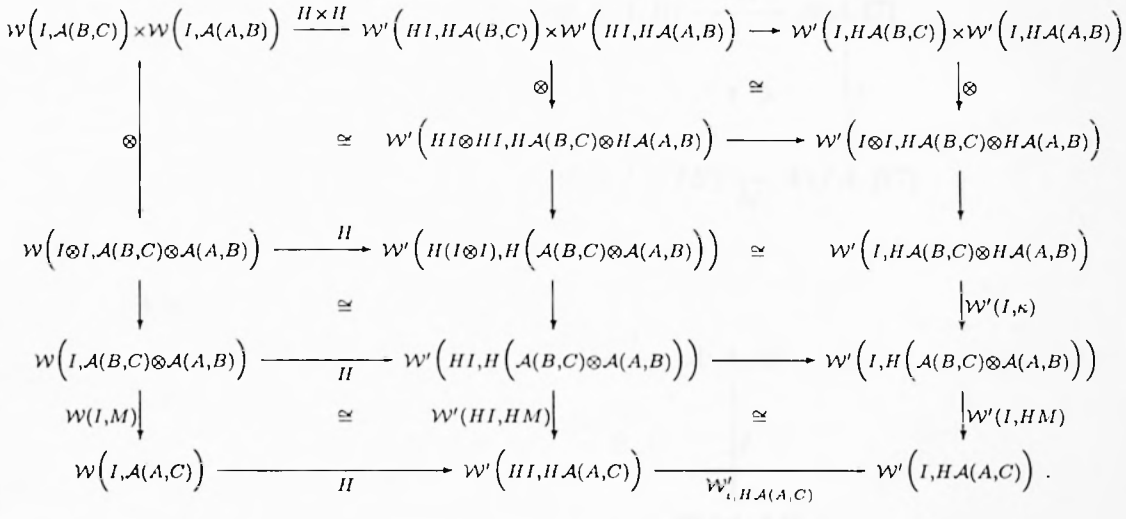
The hom categories of the two bicategories are

$$\mathcal{A}_0(A, B) = \mathcal{W}(I, \mathcal{A}(A, B))$$
$$\mathcal{A}_0^H(A, B) = \mathcal{W}'(I, H\mathcal{A}(A, B))$$

and so we let $H_0$ act on the homs by

$$\mathcal{W}(I, \mathcal{A}(A, B)) \xrightarrow{H} \mathcal{W}'(HI, H\mathcal{A}(A, B)) \xrightarrow{\mathcal{W}'(\iota, H\mathcal{A}(A, B))} \mathcal{W}'(I, H\mathcal{A}(A, B));$$

the functoriality constraints are formed, for example, as



This is bijective on objects and locally an equivalence, hence a biequivalence of bicategories. □

### 3.9.2 $\mathcal{W}$-morphisms

We now define the $\mathcal{W}$-morphisms of $\mathcal{W}$-bicategories for a general monoidal bicategory $\mathcal{W}$; it is only slightly more complicated than in the case of $\mathcal{W}$ a Gray-monoid.

**Definition 3.9.3** *A (lax) $\mathcal{W}$-morphism of $\mathcal{W}$-bicategories $f : \mathcal{A} \to \mathcal{A}'$ comprises the data:*

*(WMD1) a function $f : Ob\mathcal{A} \to Ob\mathcal{A}'$*

*(WMD2) a 1-cell $f : \mathcal{A}(A, B) \to \mathcal{A}'(fA, fB)$ for each $A, B \in Ob\mathcal{A}$*

*(WMD3) a 2-cell*

$$\begin{array}{ccc} \mathcal{A}(B,C)\otimes\mathcal{A}(A,B) & \xrightarrow{M} & \mathcal{A}(A,C) \\ {\scriptstyle f\otimes f}\downarrow & \Uparrow\ \lambda & \downarrow{\scriptstyle f} \\ \mathcal{A}'(fB,fC)\otimes\mathcal{A}(fA,fB) & \xrightarrow[M]{} & \mathcal{A}'(fA,fC) \end{array}$$

*for each* $A, B, C \in Ob\mathcal{A}$

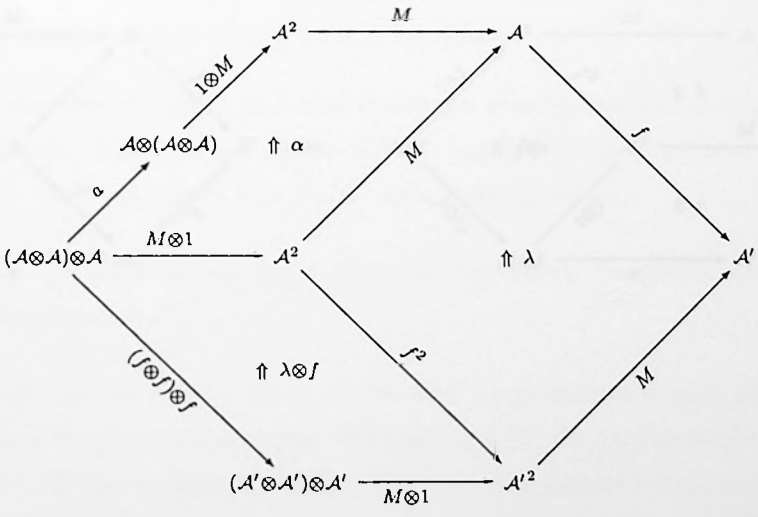
*(WMD4) a 2-cell*

$$\begin{array}{ccc} I & \xrightarrow{j} & \mathcal{A}(A,A) \\ {\scriptstyle 1}\downarrow & \Uparrow\ \iota & \downarrow{\scriptstyle f} \\ I & \xrightarrow[j]{} & \mathcal{A}'(fA,fA) \end{array}$$

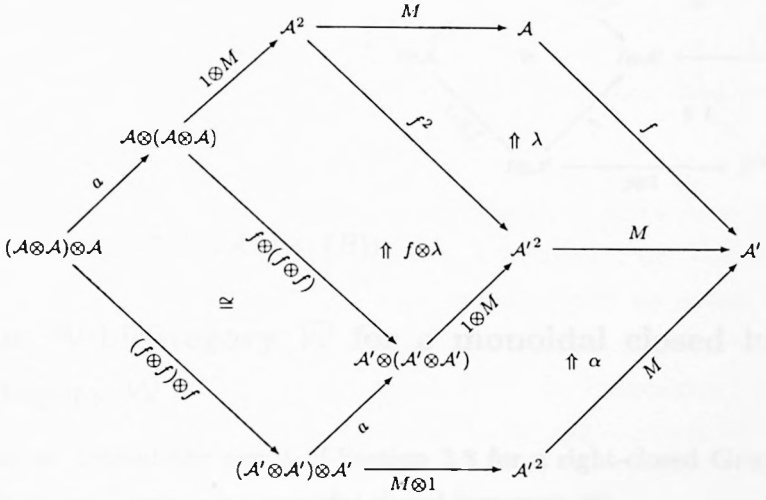
*for each* $A \in \mathcal{A}$

*such that the following three cubes commute:*

*(WMA1)*

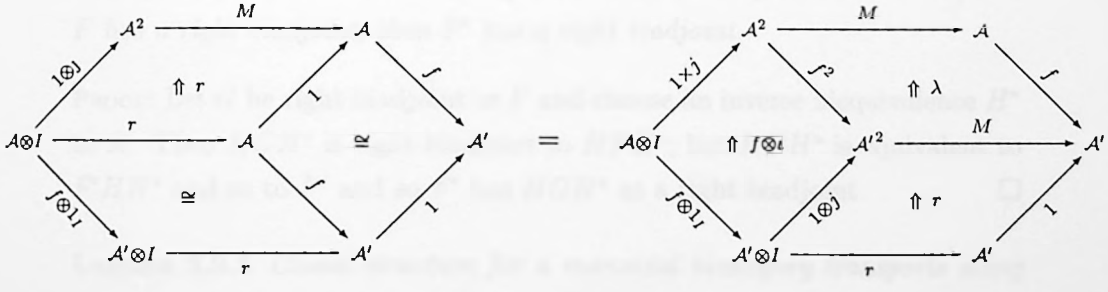


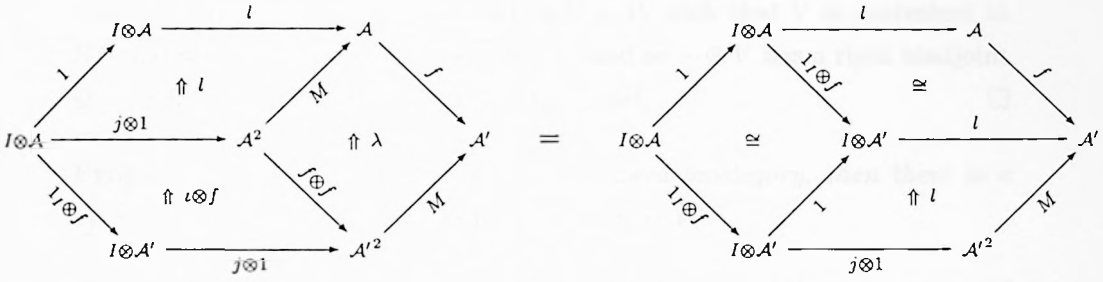
=




$$in\ \mathcal{W}\Big(\big((\mathcal{A}(C,D)\otimes\mathcal{A}(B,C)\big)\otimes\mathcal{A}(A,B),\mathcal{A}'(fA,fD)\Big)$$

*(WMA2)*






*both in* $\mathcal{W}(\mathcal{A}(A,B),\mathcal{A}'(fA,fB))$.

### 3.9.3 The $\mathcal{W}$-bicategory $\overline{\mathcal{W}}$ for a monoidal closed bicategory $\mathcal{W}$

In this section we extend the result of Section 3.8 for a right-closed **Gray**-monoid to the general case of a monoidal closed bicatgory $\mathcal{W}$.

First we need a lemma about biadjunctions.

**Lemma 3.9.4** *If* $F : \mathcal{W} \to \mathcal{W}$ *and* $F' : \mathcal{W}' \to \mathcal{W}'$ *are homomorphisms of bicategories,* $H : \mathcal{W} \to \mathcal{W}'$ *is a biequivalence,* $HF$ *is equivalent to* $F'H$*, and* $F$ *has a right biadjoint, then* $F'$ *has a right biadjoint.*

Proof: Let $G$ be right biadjoint to $F$ and choose an inverse biequivalence $H^*$ to $H$. Then $HGH^*$ is right biadjoint to $HFH^*$; but $HFH^*$ is equivalent to $F'HH^*$ and so to $F'$ and so $F'$ has $HGH^*$ as a right biadjoint. □

**Lemma 3.9.5** *Closed structure for a monoidal bicategory transports along a monoidal biequivalence.*

Proof: Suppose that $H : \mathcal{W} \to \mathcal{W}'$ is a monoidal biequivalence of monoidal bicategories, and $\mathcal{W}$ is monoidal closed. For each $U \in \mathcal{W}$, the homomorphism $- \otimes U : \mathcal{W} \to \mathcal{W}$ has a right biadjoint. Thus by the previous lemma the

homomorphism $- \otimes HU : \mathcal{W}' \to \mathcal{W}'$ has a right biadjoint. But as $H$ is a biequivalence, given $V \in \mathcal{W}'$ there is a $U \in \mathcal{W}$ such that $V$ is equivalent to $HU$ and so $- \otimes V$ equivalent to $- \otimes HU$, and so $- \otimes V$ has a right biadjoint as $- \otimes HU$ does. Thus $\mathcal{W}'$ is monoidal closed. □

**Proposition 3.9.6** *If $\mathcal{W}$ is a monoidal closed bicategory, then there is a $\mathcal{W}$-bicategory $\overline{\mathcal{W}}$ whose underlying bicategory is $\mathcal{W}$.*

Proof: Choose a monoidal biequivalent **Gray**-monoid $\mathcal{W}'$ and a monoidal biequivalence $H : \mathcal{W} \to \mathcal{W}'$. Then $\mathcal{W}'$ is closed as a **Gray**-monoid. Thus we can construct a $\mathcal{W}'$-bicategory $\overline{\mathcal{W}'}$ with underlying bicategory equivalent to $\mathcal{W}'$ as in Section 3.8. Now let $H^*$ be an inverse monoidal biequivalence to $H$ and change base along $H^*$ to give a $\mathcal{W}$-bicategory $\mathcal{W}'^{H^*}$. By Proposition 3.9.2 the underlying bicategory of $\overline{\mathcal{W}'}^{H^*}$ is biequivalent to $\mathcal{W}'$ and so also to $\mathcal{W}$. □

# Chapter 4

# Distributive and extensive categories

In the previous chapters the necessary background theory of enriched bicategories has been developed to allow some elementary considerations of the algebra of distributive and extensive categories, as promised in the title of the thesis. Before we can turn to this, however, we must first develop some basic facts about distributive and extensive categories themselves, and it is just that which we set out to do in this chapter. An introduction to distributive and extensive categories can be found in [6] or [11], or for a more general philosophical viewpoint one might consult [32].

## 4.1 Distributive categories

Most of this section can be found for example in [6].

**Definition 4.1.1** *A category is said to be* distributive *if it possesses finite products and finite coproducts, and if the canonical map*

$$\delta : A \times B + A \times C \to A \times (B + C)$$

*is invertible for all objects $A$, $B$, and $C$.*

Here $\delta$ is the map out of the coproduct $A \times B + A \times C$ induced by $1_A \times i_B : A \times B \to A \times (B + C)$ and $1_A \times i_C : A \times C \to A \times (B + C)$. There is an obvious analogy here with commutative rings: the categorical products are like the multiplication of a ring, the coproducts like the addition, and the isomorphisms $\delta$ are a form of distributivity law. On the other hand there is nothing corresponding to additive inverses, and the following result shows that we can never hope to find objects behaving as such.

**Lemma 4.1.2** *In any category, if $A + B \cong 0$ then $A \cong 0$ and $B \cong 0$.*

PROOF: If $A + B \cong 0$ then there is a coproduct diagram

$$A \xrightarrow{\ i\ } 0 \xleftarrow{\ j\ } B$$

and so, writing $!_A$ for the unique map 0 to $A$, the maps $1_A : A \to A$ and $!_A.j : B \to A$ induce a unique map from $\theta : 0 \to A$, such that $\theta.i = 1_A$ and $\theta.j = !_A.j$. As 0 is initial, $i.\theta = 1_0$ and so $i$ is inverse to $\theta$ and $A \cong 0$. Similarly $B \cong 0$. □

Thus a distributive category is more precisely a categorical analogue to a *rig* or semiring, that is to say a set with two binary unitary associative operations, one distributing over the other. A rig *need not* have additive inverses; distributive categories correspond to rigs which *do not* have (non-trivial) additive inverses, nor, by the dual of the previous lemma, non-trivial multiplicative inverses.

The category **Set** of sets and functions is distributive, as is $\mathbf{Sets}_f$, the category of finite sets and functions. Similarly so is any topos, or indeed any cartesian closed category with finite coproducts. The category **Top**, of topological spaces and continuous functions is distributive, as is the opposite of the category of commutative rings and ring homomorphisms. Another

important class of distributive categories is the distributive lattices (thought of as partially ordered, or more generally preordered, sets).

The category $\mathbf{Sets}_f$ is of course closely associated to the rig $\mathbb{N}$ of natural numbers, via the process of taking the cardinality of a finite set, and in passing by this construction from $\mathbf{Sets}_f$ to $\mathbb{N}$, a categorical product becomes a numerical product, and likewise a coproduct becomes a numerical sum. This construction in fact can be generalized to any distributive category by means of the Burnside rig [37]. The Burnside rig of a distributive category is the set of isomorphism classes of objects of the category, made into a rig with multiplication and addition being given by categorical product and coproduct and multiplicative and additive identities by the (isomoprhism class of the) terminal and initial objects.

A first exercise in commutative rings is the proof that multiplication by 0 always gives 0. A first (only slightly harder) exercise in distributive categories is the following.

**Lemma 4.1.3** *In a distributive category the projection* $\pi : A \times 0 \to 0$ *is inverse to the unique map* $! : 0 \to A \times 0$.

Proof: Certainly $0 + 0 \cong 0$, then by distributivity

$$A \times 0 \xrightarrow{1} A \times 0 \xleftarrow{1} A \times 0$$

is a coproduct diagram. As the two injections are the same there can be at most one map from $A \times 0$ to a given other object and so $!.\pi = 1_{A\times 0}$. On the other hand there is at most one map from 0 to another given object and so $\pi.! = 1_0$. □

In a topos one can prove many things that say essentially that certain properties are shared with the category **Set**. A distributive category is a much weaker structure than a topos, but some such properties are still true, as the next two results show.

**Proposition 4.1.4** *In a distributive category coproduct injections are monic.*

PROOF: Suppose that $i : B \to B+C$ is a coproduct injection, and $f, g : A \to B$ are such that $if = ig$. We must show that $f = g$.

Consider the map $1_B \times i : B \times B \to B \times (B + C)$. This is a split monic, with retraction given by the composite

$$B \times (B + C) \xrightarrow{\delta^{-1}} B \times B + B \times C \xrightarrow{(1_{B\times B}\ \Delta\pi_1)} B \times B.$$

Then as $if = ig$, also $(B \times i)\binom{f}{f} = (B \times i)\binom{f}{g}$, and so $\binom{f}{f} = \binom{f}{g}$, which in turn gives $f = g$. □

**Proposition 4.1.5** *In a distributive category initial objects are* strict, *which is to say that any arrow into an initial object is invertible.*

PROOF: Suppose that $\alpha : A \to 0$. Then the composite $\pi_1\binom{1_A}{\alpha}$ is equal to the identity $1_A$, but as $A \times 0$ is initial also $\binom{1_A}{\alpha}\pi_1 = 1$, and so $A \cong 0$. As $A$ is now initial, $\alpha$ must be the isomorphism. □

Another approach to distributive categories is that they can be seen as a natural setting in which to interpret "if, then ... else" commands. Suppose for example that one has arrows $f : A \to B$ and $g : A \to C$ in a category and a "control" arrow $x : A \to 1+1$. One can then form an arrow which can be interpreted as "if $x$ lands in the first copy of 1 then do $f$; if it lands in the second copy then so does $g$".[1] Such an arrow can be constructed as

$$A \xrightarrow{\binom{1_A}{x}} A \times (1 + 1) \xrightarrow{\delta^{-1}} A + A \xrightarrow{f+g} B + C.$$

Similarly one can deal with nested control structures using distributivity.

Before leaving this section of distributive categories, we should of course not forget the categorical maxim that in introducing a new structure one should always also introduce the morphisms of that structure.

[1] On the other hand in an arbitrary distributive category one must exercise caution in talking about being "in" the first copy or the second copy of 1 in $1 + 1$.

**Definition 4.1.6** *A functor between distributive categories is said to be* distributive *if it preserves finite products and finite coproducts.*

The extensive categories are closely related to distributive categories. We shall give the definition here, but then return to them only after the following section, which contains some general considerations on adjuntions.

**Definition 4.1.7** *A category $\mathcal{E}$ with finite coproducts is said to be* extensive *if the functors*

$$\mathcal{E}/X \times \mathcal{E}/Y \xrightarrow{+} \mathcal{E}/(X+Y)$$

*are equivalences, for all objects $X$ and $Y$ in $\mathcal{E}$.*
*A functor between extensive categories is said to be extensive just when it preserves finite coproducts.*

## 4.2 Local equivalences and Frobenius adjunctions

The material in this section on local equivalences was presented by Thiébaud [41] subject to the additional hypothesis that the categories involved have all pullbacks, which simplifies some of the proofs; in that context, he uses the name Cartesian adjunction for what we have called a local equivalence. The connection with Frobenius adjunctions is new.

Suppose that

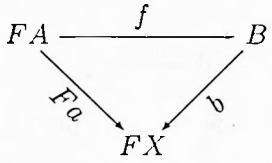
$$\mathcal{A} \underset{U}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{B} \tag{4.2.1}$$

is an adjunction. Given an object $X \in \mathcal{A}$, we can form slice categories $\mathcal{A}/X$ and $\mathcal{B}/FX$, and localize the functor $F$ to give $F_X : \mathcal{A}/X \to \mathcal{B}/FX$. One can then ask under what conditions this $F_X$ itself has a right adjoint.

If $\mathcal{A}$ has pullbacks, then $F_X$ has a right adjoint $U_X$, which can be constructed as follows. Given $b : B \to FX$, form the pullback

$$\begin{array}{ccc} P & \xrightarrow{q} & UB \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow{\scriptstyle Ub} \\ X & \xrightarrow[\eta_X]{} & UFX \end{array} \tag{4.2.2}$$

and then $U_X b$ is taken to be $p$. This is extended to arrows in the obvious way, using maps induced by the pullbacks, giving a functor $U_X$. Now a map $f$ in $\mathcal{B}/FX$ is a commutative triangle




and so the maps $a$ and $Uf.\eta_A$ induce a map $f'$ in $\mathcal{A}/X$ with domain $a : A \to X$ and codomain $p : P \to X$. This assignment gives a natural isomorphism $\mathcal{A}/X(a, U_X b) \cong \mathcal{B}/FX(F_X a, b)$ and so we have the desired adjunction. In fact to form the adjunction in this way there is no need to have all pullbacks in $\mathcal{A}$, but only pullbacks of the form of (4.2.2), *i.e.* pullbacks of an arrow in the image of $U$ by a component of $\eta$. Then $F_X$ has a right adjoint precisely when these pullbacks exist, in which case the adjunction is as described above.

**Definition 4.2.1** *The adjunction* (4.2.1) *is called a* local equivalence *if the functors $F_X$ are equivalences.*

**Example 4.2.2** *A category with finite coproducts is extensive if and only if the adjunction*

$$C \times C \underset{\Delta}{\overset{+}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} C$$

*is a local equivalence.*

**Theorem 4.2.3** *The adjunction* (4.2.1) *is a local equivalence if and only if* $\mathcal{A}$ *has pullbacks of the form* (4.2.2), $F$ *preserves them, and the unit* $\eta$ *and the counit* $\epsilon$ *of* (4.2.1) *are cartesian.*

PROOF: A natural transformation is called cartesian if its naturality squares are cartesian, *i.e.* are pullback squares.

We have seen that $F_X$ has a right adjoint if and only if $\mathcal{A}$ has pullbacks of the form (4.2.2), and so it remains to show that in this case the unit and counit $\eta'$ and $\epsilon'$ of the induced adjunction are invertible if and only if $F$ preserves pullbacks (4.2.2) and $\eta$ and $\epsilon$ are cartesian.

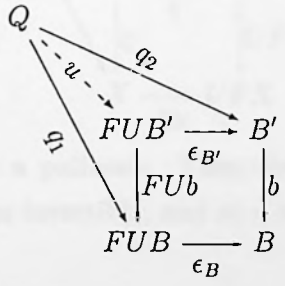
Suppose that $\epsilon'$ is invertible and let

$$\begin{array}{ccc} Q & \xrightarrow{q_2} & B' \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle b} \\ FUB & \xrightarrow[\epsilon_B]{} & B \end{array}$$

be a commutative diagram. Then we can form the pullback

$$\begin{array}{ccc} P & \xrightarrow{r} & UQ \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow{\scriptstyle Uq_1} \\ UB & \xrightarrow[\eta_{UB}]{} & UFUB \end{array}$$

where $p$ is $U_{UB}(q_1)$, and $r$ is adjoint transpose under (4.2.1) to $\epsilon'_{q_1}$ as a map in $\mathcal{B}$. Now there is a unique map $u$ rendering commutative



and it can be formed as the composite

$$Q \xrightarrow{(\epsilon'_b)^{-1}} FP \xrightarrow{Fr} FUQ \xrightarrow{FUq_2} FUB'$$

which proves that $\epsilon$ is cartesian.

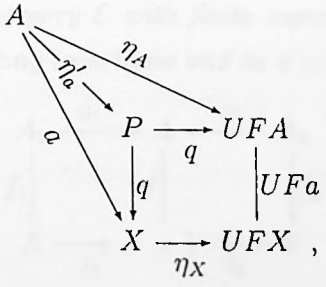
In the commutative diagram

$$\begin{array}{ccccc} FP & \xrightarrow{Fq} & FUB & \xrightarrow{\epsilon_B} & B \\ {\scriptstyle Fp}\downarrow & & {\scriptstyle FUb}\downarrow & & \downarrow{\scriptstyle b} \\ FX & \xrightarrow[F\eta_X]{} & FUFX & \xrightarrow[\epsilon_{FX}]{} & FX \end{array}$$

the right hand square is a pullback as $\epsilon$ is cartesian, the composite arrow along the bottom is the identity on $FX$, and the one along the top is $\epsilon'_b$, and so invertible, and so the composite rectangle is a pullback; then by cancellation of pullbacks the left square is a pullback, which says exactly that $F$ preserves pullbacks of the form (4.2.2).

Conversely suppose that $F$ preserves pullbacks of the form (4.2.2) and $\epsilon$ is cartesian. Then in the last diagram, the two small squares are pullbacks, and so the composite rectangle is a pullback, and so $\epsilon'$ is invertible.

Given an object $a : A \to X$ in $\mathcal{A}/X$, the map $\eta'_a$ is the unique map rendering commutative the diagram




where the inner square is a pullback. Then clearly the outer square is a pullback if and only if $\eta'_a$ is invertible, and so $\eta$ is cartesian if and only if $\eta'$ is invertible. □

**Remark 4.2.4** Distributivity too can be presented in terms of a simple condition on the adjunction

$$C \times C \underset{\Delta}{\overset{+}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} C \tag{4.2.3}$$

expressing the existence of binary coproducts. Recall that an adjunction

$$A \underset{U}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} B$$

is said to be *Frobenius* when the composite

$$F(A \times UB) \xrightarrow{\binom{F\pi_1}{F\pi_2}} FA \times FUB \xrightarrow{1 \times \epsilon} FA \times B$$

is invertible. Then a category $\mathcal{C}$ with finite products and coproducts is distributive just when the adjunction (4.2.3) is Frobenius.

## 4.3 Extensive categories

From the results of the previous section we can now form various equivalent conditions for a category to be extensive, the most useful of which is contained in the following proposition.

**Proposition 4.3.1** *A category $\mathcal{E}$ with finite coproducts is extensive if and only if it has pullbacks along injections and in a commutative diagram*

$$\begin{array}{ccccc} A_1 & \xrightarrow{a_1} & A & \xleftarrow{a_2} & A_2 \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_2} \\ X & \xrightarrow[i_1]{} & X+Y & \xleftarrow[i_2]{} & Y \end{array}$$

*the top row is a coproduct diagram if and only if the two squares are pullbacks. In the case where $\mathcal{E}$ has a terminal object 1, it is enough that these conditions hold for $X = Y = 1$.*

PROOF: To have pullbacks along injections is just to have pullbacks of the form (4.2.2). That in the above diagram the top row being a coproduct diagram implies that the squares are pullbacks is just the condition that $\eta$ be cartesian. Similarly that in the above diagram the squares being pullbacks implies the top row a coproduct is just the condition that $\epsilon'$ be invertible.

Finally if $\mathcal{E}$ has a terminal object 1 and satisfies the hypotheses of the "if" part of the proposition for the case $X = Y = 1$, then in the diagram

$$\begin{array}{ccccc}
A_1 & \xrightarrow{a_1} & A & \xleftarrow{a_2} & A_2 \\
{\scriptstyle f_1}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_2} \\
X & \xrightarrow[i_1]{} & X+Y & \xleftarrow[i_2]{} & Y \\
{\scriptstyle !}\downarrow & & {\scriptstyle !+!}\downarrow & & \downarrow{\scriptstyle !} \\
1 & \xrightarrow[i_1]{} & 1+1 & \xleftarrow[i_2]{} & 1
\end{array}$$

the lower squares are coproducts and so pullbacks, and thus the upper squares are pullbacks if and only if the tall rectangles are. □

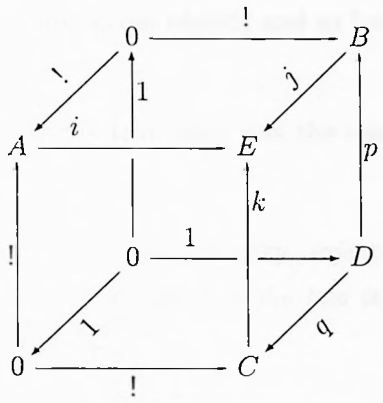
We saw that one can never hope for (non-trivial) additive inverses in a category; often failing the existence of inverses, one seeks cancellation results; the following is a "categorical version" of a cancellation result.

**Lemma 4.3.2** *If $\mathcal{E}$ is an extensive category and $A + B \cong A + C$, by an isomorphism commuting with the injections of $A$, then $B \cong C$.*

PROOF: Under the above hypotheses we can form coproduct diagrams

$$\begin{array}{ccccc}
A & \xrightarrow{i} & E & \xleftarrow{j} & B \\
A & \xrightarrow{i} & E & \xleftarrow{k} & C
\end{array}$$

whence a commutative cube



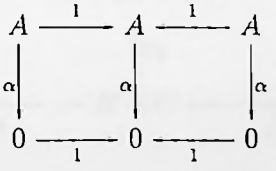

whose right face is a pullback. The front and top faces are also pullbacks by disjointness of sums, and so, twice using extensivity, $C \cong 0 + D$ and $B \cong 0 + D$, and so $C \cong B$. □

There is another, more logical, approach to extensive categories. Often one talks about the "logic of a category" (usually with some extra structure) in terms of the lattice of subobjects of an object of the category. In the case of extensive categories, one can also consider the lattice of *direct summands* of an object, and this lattice is now a Boolean algebra. In this context the last result says exactly that in the lattice of direct summands, complements are unique, which is of course one of the requirements for a Boolean algebra.

The extensive categories share many properties with the distributive categories.

**Proposition 4.3.3** *In an extensive category, initial objects are strict.*

Proof: In the diagram



the squares are pullbacks and so the top row a coproduct, and so there is at most one map from $A$ to any given object, and so $!\alpha = 1_A$. Clearly $\alpha ! = 1_0$ and thus $\alpha$ is invertible. □

In the next result, more is true than was the case for distributive categories.

**Proposition 4.3.4** *In an extensive category, injections are monic and coproducts are disjoint, i.e. the pullback of the two injection of a binary coproduct exists and is initial.*

PROOF: In the diagram

$$\begin{array}{ccccc} A & \xrightarrow{1} & A & \xrightarrow{!} & 0 \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle !} \\ A & \xrightarrow[i]{} & A+B & \xleftarrow[j]{} & B \end{array}$$

the top row is a coproduct and so the two squares are pullbacks. The right square being a pullback is precisely disjointness of coproducts; the left square being a pullback is precisely monicity of injections. □

The most important connection between extensive and distributive categories is the following one.

**Theorem 4.3.5** *An extensive category with products is distributive.*

PROOF: In the diagram

$$\begin{array}{ccccc} A\times B & \xrightarrow{1_A\times i} & A\times(B+C) & \xrightarrow{1_A\times j} & A\times C \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_2} \\ B & \xrightarrow[i]{} & B+C & \xleftarrow[j]{} & C \end{array}$$

both squares are pullbacks, and so the top square is a coproduct diagram, which is exactly the distributivity condition. □

In fact this last theorem admits a generalization to the level of Frobenius adjunctions and local equivalences: an adjunction between categories with finite products that is a local equivalence is Frobenius. We omit the proof.

If an extensive category has not just finite products, but all finite limits, then it is said to be a *lextensive* category. A distributive category is said to be locally distributive if every slice is distributive.

**Theorem 4.3.6** *A category is lextensive if and only if it is locally distributive and has disjoint coproducts.*

Proof: Extensivity is clearly a local property, in the sense that if a category is extensive, so are all of its slices. Likewise possession of finite limits is local, and so a lextensive category is locally lextensive and so locally distributive; also we have seen that an extensive category, and hence a lextensive category, has disjoint coproducts.

Conversely suppose that a category $\mathcal{E}$ is locally distributive with disjoint coproducts. Local distributivity implies the existence of all finite limits, and, in particular, pullbacks along injections, and also that coproducts are stable. Thus it will suffice to show that the square

$$\begin{array}{ccc} A & \xrightarrow{i_A} & A+B \\ {\scriptstyle !}\downarrow & & \downarrow{\scriptstyle !+!} \\ 1 & \xrightarrow[i_1]{} & 1+1 \end{array}$$

is a pullback. Given then an arrow $g : C \to A + B$ whose composite with $!+!$ factorizes through the first injection $i_1 : 1 \to 1 + 1$, we can pull $g$ back along the injections of $A + B$. Disjointness of coproducts guarantees that the pullback along $i_B$ must be 0 and so stability of coproducts guarantees

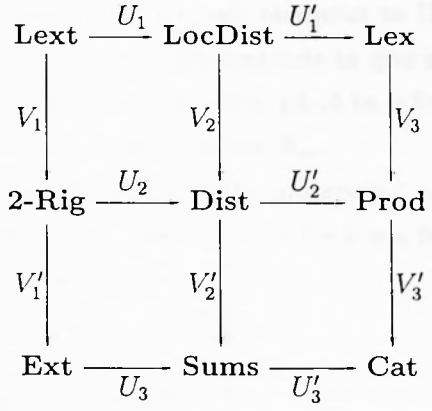
that the pullback along $i_A$ must be isomorphic to $C$, giving the required factorization. □

Once again this theorem can be generalized to one concerning Frobenius adjunctions and local equivalences.

## 4.4 The extensive reflection of a distributive category

We shall now introduce certain 2-categories of categories with various structures. We write **Sums** for the 2-category whose objects, arrows, and 2-cells, are the categories with finite coproducts, the finite-coproduct-preserving functors, and the natural transformations between such, the compositions being the usual ones of functors or natural transformations; likewise **Prod** for the case where coproducts are replaced by products. Then **Sums** has a full sub-2-category **Ext** whose objects are the extensive categories; recall that an extensive functor was taken to mean any finite-coproduct-preserving functor between extensive categories — by extensivity of the categories such a functor then preserves pullbacks along injections. Next we write **Dist** for the 2-category of distributive categories, distributive functors, and natural transformations between such, and **2-Rig** for its full sub-2-category determined by the extensive categories with finite products. Finally we write **Lex** for the 2-category of categories with finite limits, finite-limit-preserving functors, and natural transformations; **LocDist** for the 2-category of locally distributive categories, functors preserving finite limits and finite coproducts, and natural transformations; and **Lext** for its full sub-2-category determined by the lextensive categories.

There is a diagram of forgetful 2-functors




each of which, for general reasons, has a left biadjoint; some of these biadjoints we are able to describe explicitly.

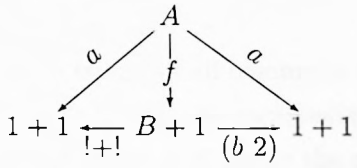
The left biadjoint to $U_3'$ is well-known: it is generally called $Fam$. $Fam(\mathcal{A})$ has as objects finite families $(A_i)_{i\in I}$ of objects of $\mathcal{A}$; a morphism from $(A_i)_{i\in I}$ to $(B_j)_{j\in J}$ is a function $f : I \to J$ and a family of morphisms $\alpha_i : A_i \to B_{f(i)}$ in $\mathcal{A}$. The left biadjoints to $U_2'$ and to $U_1'$ are restrictions of $Fam$, and the left biadjoint to $V_3'$ is a dual of $Fam$: the objects are the same, but a morphism $(A_i)_{i\in I}$ to $(B_j)_{j\in J}$ is a function $f : J \to I$ and a family of morphisms $\alpha_j : A_{f(j)} \to B_j$. Restricting this to **Sums** (respectively **Ext**), however, does not give the left biadjoint to $V_2'$ or $V_1'$; the resulting categories would not in general be distributive: their opposites will be. A construction of Andy Pitts [34] provides a left biadjoint to $V_3$ and $V_2$; it does not restrict to a left biadjoint to $V_1$, for the Pitts construction never results in an extensive category unless one starts with **1**, but one can then apply the extensive reflection, described below, to obtain a lextensive category; this then gives the left biadjoint to $V_1$. I do not know of explicit descriptions of the left biadjoints to $U_3$, $V_2'$, or $V_1'$.

It is an important fact that $Fam(\mathcal{A})$ is extensive, and so $Fam$ factors through $U_3$ giving the left biadjoint to the composite $U_3'U_3$; likewise the

*Fam* construction gives the left biadjoint to the composites $U_2'U_2$ and $U_1'U_1$. There is a simple description of the left biadjoint to $U_2$, unlike $U_3$, and we shall now describe it; moreover it also restricts to give a left biadjoint to $U_1$.

For the remainder of the section, then, let $\mathcal{A}$ be a fixed distributive category. The 2-rig constructed will be called $\mathcal{A}_{ex}$.

The objects of $\mathcal{A}_{ex}$ are maps in $\mathcal{A}$ with codomain $1+1$. An arrow between two such objects $a : A \to 1+1$ and $b : B \to 1+1$ is a map $f : A \to B+1$ in $\mathcal{A}$, such that the triangles




commute, where we write simply $i$ for the $i$th injection of a coproduct if no confusion will arise. The composite of such arrows $f$ and $g$ is the composite map $A \xrightarrow{f} B+1 \xrightarrow{(g\ 2)} C+1$ in $\mathcal{A}$; this composition is clearly associative. The identity arrow on an object $a$ is the map

$$\iota_a = A \xrightarrow{\binom{1_A}{a}} A \times (1+1) \xrightarrow{\delta^{-1}} A + A \xrightarrow{1_A+!} A + 1.$$

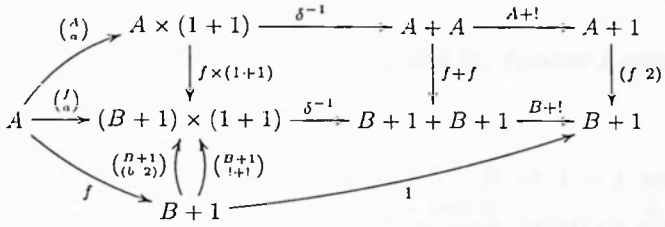
**Proposition 4.4.1** *The above data make $\mathcal{A}_{ex}$ into a category; there is a functor $I : \mathcal{A} \to \mathcal{A}_{ex}$ which is faithful and injective on objects.*

Proof: First we must show that $\iota_a$ is an arrow from $a$ to $a$; one condition is obvious, on the other hand we have commutativity of

$$\begin{array}{ccccccc}
A & \xrightarrow{\binom{1_A}{a}} & A \times (1+1) & \xrightarrow{\delta^{-1}} & A + A & \xrightarrow{A+!} & A + 1 \\
{\scriptstyle a}\downarrow & & \downarrow{\scriptstyle a \times (1+1)} & & \downarrow{\scriptstyle a+a} & & \downarrow{\scriptstyle (a\ 2)} \\
1+1 & \xrightarrow[\Delta]{} & (1+1) \times (1+1) & \xrightarrow[\delta^{-1}]{} & 1+1+1+1 & \xrightarrow[(1\ 2\ 2\ 2)]{} & 1+1
\end{array}$$

and the composite along the bottom is the identity, which gives the second condition.

To show that $\mathcal{A}_{ex}$ is a category, it remains to prove the identity law. In the diagram




the two squares and the three triangles all commute, and $\binom{B+1}{(b\ 2)} f = \binom{B+1}{!+!} f$. The composite along the top is $f \circ \iota_a$, the composite through the middle is $\iota_b \circ f$, and the one along the bottom is $f$, hence the result.

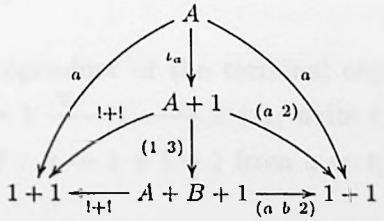
We define $I$ on objects to take $A$ to $A \xrightarrow{!} 1 \xrightarrow{i_1} 1+1$ and on arrows $f : A \to B$ to $i_B f : A \to B+1$. The only thing that is not immediate is faithfulness of $I$; this holds as $\mathcal{A}$ is distributive and so coproduct injections are monic. □

**Proposition 4.4.2** *The category $\mathcal{A}_{ex}$ has, and the functor $I$ preserves, finite coproducts.*

Proof: Given objects $a : A \to 1+1$ and $b : B \to 1+1$ we form their coproduct as $(a\ b) : A+B \to 1+1$ with injections

$$A \xrightarrow{\iota_a} A+1 \xrightarrow{(1\ 3)} A+B+1 \xleftarrow{(2\ 3)} B+1 \xleftarrow{\iota_b} B\ ,$$

where commutativity of



and the companion diagram for $b$ ensures that these injections are indeed arrows in $\mathcal{A}_{ex}$. The universal property is immediate from the universal property of coproducts in $\mathcal{A}$. Clearly $! : 0 \to 1+1$ is initial in $\mathcal{A}_{ex}$, and so $\mathcal{A}_{ex}$ has finite coproducts, and by their construction $I$ preserves finite coproducts. □

**Proposition 4.4.3** *The category $\mathcal{A}_{ex}$ has, and the functor $I$ preserves, finite products.*

PROOF: Given objects $a : A \to 1+1$ and $b : B \to 1+1$ we form their product as $a \wedge b = A{\times}B{+}1 \xrightarrow{i} (A{+}1){\times}(B{+}1) \xrightarrow{(a\ 2){\times}(b\ 2)} (1{+}1){\times}(1{+}1) \xrightarrow{\wedge} 1{+}1$ where $\wedge$ stands for the map $(1{+}1){\times}(1{+}1) \xrightarrow{\delta^{-1}} 1{+}1{+}1{+}1 \xrightarrow{(1\ 2\ 2\ 2)} 1{+}1$ and $i$ for the map $A{\times}B{+}1 \xrightarrow{(1\ 4)} A{\times}B{+}A{\times}1{+}1{\times}B{+}1{\times}1 \xrightarrow{\delta} (A{+}1){\times}(B{+}1)$,, and the projections as

$$A + 1 \xrightarrow{\pi_A + 1} A \times B + 1 \xleftarrow{\iota_{a\wedge b}} A \times B \xrightarrow{\iota_{a\wedge b}} A \times B + 1 \xrightarrow{\pi_2 + 1} B + 1 \ .$$

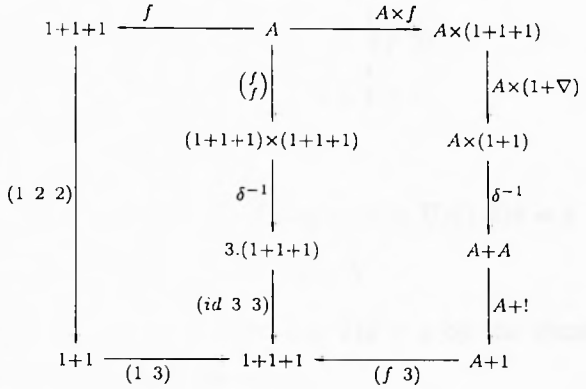
Given $c : C \to 1+1$ and arrows $f$ from $c$ to $a$ and $g$ from $c$ to $b$, the map $\binom{g}{f} : C \to (A+1) \times (B+1)$ factorizes uniquely as $ih$, and this $h$ is then the required map from $c$ to $a \wedge b$ proving that $a \wedge b$ is the product of $a$ and $b$ with the above projections.

The injection $i_1 : 1 \to 1+1$ in $\mathcal{A}$ is initial as an object of $\mathcal{A}_{ex}$, for given $a : A \to 1+1$ another object in $\mathcal{A}_{ex}$, $a$ is the unique arrow from $a$ to $i_1$ in $\mathcal{A}_{ex}$. Thus $\mathcal{A}_{ex}$ has finite products, and by construction, $I$ preserves finite products. □

**Proposition 4.4.4** *The category $\mathcal{A}_{ex}$ is extensive, and so, as it has finite products, is distributive.*

PROOF: The (binary) coproduct of the terminal object with itself is given by the map $(1\ 1) = 1 + 1 \xrightarrow{\nabla} 1 \xrightarrow{i_1} 1 + 1$; write $t$ for this map. Suppose that we have a arrow $f : A \to 1 + 1 + 1$ from $a$ to $t$; we are looking for the

pullback of $f$ along the first injection of $t$. Write $f_1$ for the composite of $f$ with $1 + \nabla : 1 + 1 + 1 \to 1 + 1$; then commutativity of




(where 3.1 stands for the 3-fold coproduct of $1 + 1 + 1$ with itself and $id$ for the identity map on $1 + 1 + 1$) implies commutativity of

$$\begin{array}{ccc} A & \xrightarrow{\iota_{f_1}} & A + 1 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle (f\ 3)} \\ 1 + 1 & \xrightarrow[(1\ 3)]{} & 1 + 1 + 1. \end{array}$$

We claim that this "is" the desired pullback, in the sense that these maps are the arrows in $\mathcal{A}_{ex}$, and the objects are $f_1$, $a$, $i_1$, and $t$. Of course $f_1$ is an arrow from $f_1$ to $i_1$, and

$$f_1 = (\nabla + 1)(1\ 3)f_1 = (\nabla + 1)(f\ 3)\iota_{f_1} = (a\ 2)\iota_{f_1}$$

and so $\iota_{f_1}$ is an arrow from $f_1$ to $t$. The square commutes in $\mathcal{A}$ and so also in $\mathcal{A}_{ex}$, thus it remains only to show universality.

Suppose then that $b : B \to 1 + 1$ is an object of $\mathcal{A}_{ex}$ and $g$ an arrow from

$b$ to $a$, and that the following square commutes

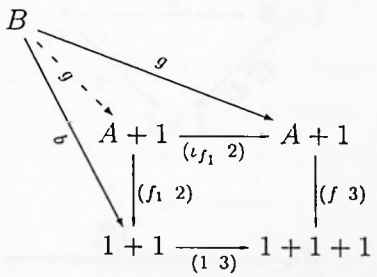
$$\begin{array}{ccc} B & \xrightarrow{g} & A+1 \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle (f\ 3)} \\ 1+1 & \xrightarrow[(1\ 3)]{} & 1+1+1. \end{array}$$

Then

$$(f_1\ 2)g = (1+\nabla)(f\ 3)g = (1+\nabla)(1\ 3)b = b$$

$$(!+!)g = b$$

and so $g$ is a map from $b$ to $f_1$; also $(\iota_{f_1}\ 2)g = g$ by the identity law and so we have a commutative diagram




in which moreover $g$ is the unique such arrow from $b$ to $f_1$, by monicity of $\iota_{f_1}$ *as an arrow from* $f_1$ *to* $f_1$. Thus the square is indeed a pullback.

Write $f_2$ for the composite of $f$ with $(2\ 1\ 2) : 1+1+1 \to 1+1$. To say that coproducts are stable under pullback is just to say that the arrow $(\iota_{f_1}\ \iota_{f_2} : A+A \to A+1$ from $(f_1\ f_2)$ to $a$ is invertible. It is indeed so, with inverse the composite

$$A \xrightarrow{\binom{A}{f}} A \times (1+1+1) \xrightarrow{\delta^{-1}} A+A+A \xrightarrow{A+A+!} A+A+1.$$

The other half of the extensivity condition amounts to invertibility of the arrow $(i_A+1)\iota_a : A \to A+B+1$ from $a$ to $(a\ 2) : A+B \to 1+1$; it is so with inverse $(\iota_a\ 2) : A+B \to A+1$.

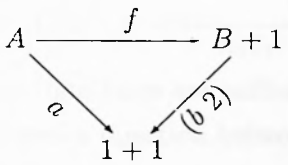
This completes the proof of extensivity of $\mathcal{A}_{ex}$. As it also has finite products, it is distributive. □

**Theorem 4.4.5** *$\mathcal{A}_{ex}$ is the extensive reflection of $\mathcal{A}$, in the sense that composition with $I$ induces an equivalence of categories*

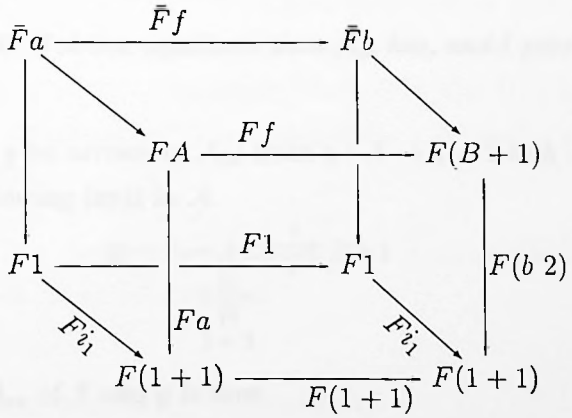
$$\mathbf{Dist}(\mathcal{A}, \mathcal{B}) \simeq \mathbf{2\text{-}Rig}(\mathcal{A}_{ex}, \mathcal{B})$$

*for each extensive category $\mathcal{B}$ with finite products.*

PROOF: Given a distributive functor $F : \mathcal{A} \to \mathcal{B}$ we define $\bar{F} : \mathcal{A}_{ex} \to \mathcal{B}$, on an arrow

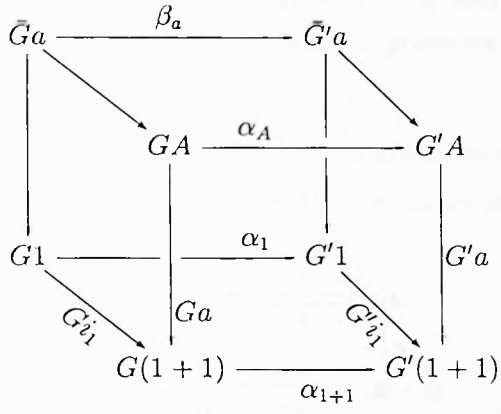



in $\mathcal{A}_{ex}$, by the cube




where the left and right faces are pullbacks and all faces commute. This $\bar{F}$ is now a distributive functor and $\bar{F}I \cong F$, and $\bar{F}$ is clearly unique up to isomorphism with this property.

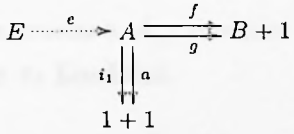
Given distributive functors $G$ and $G'$ from $\mathcal{A}_{ex}$ to $\mathcal{B}$, and a natural transformation $\alpha : GI \to G'I$, there is a natural transformation $\beta : G \to G'$ whose component at $a$ is the unique map rendering commutative the cube




where once more the left and right faces are pullbacks and all faces commute. Then $\beta I = \alpha$, and this provides a bijection between natural transformations $G \to G'$ and natural transformations $GI \to G'I$, *i.e.* proves that composition with $I$ is fully faithful, and so an equivalence. □

**Proposition 4.4.6** *If $\mathcal{A}$ has equalizers then $\mathcal{A}_{ex}$ has, and $I$ preserves, equalizers.*

PROOF: Let $f$ and $g$ be arrows in $\mathcal{A}_{ex}$ from $a : A \to 1+1$ to $b : B \to 1+1$. Then form the following limit in $\mathcal{A}$.




The equalizer in $\mathcal{A}_{ex}$ of $f$ and $g$ is now

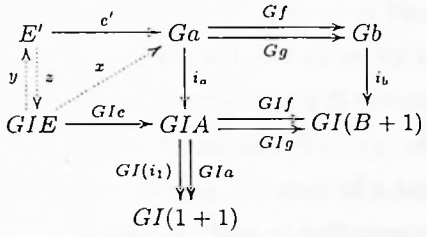
$$E \xrightarrow{\ e\ } A \xrightarrow{\ i_A\ } A+1$$

with domain $I(E)$. Thus $\mathcal{A}_{ex}$ has equalizers; that $I$ preserves equalizers is immediate: for a pair of arrows with domain $a : A \to 1+1$ in the image of

$I$ then the parallel pair $a, i_1 : A \to 1+1$ in the above limit are equal, and so $E$ is just the equalizer of $f$ and $g$ in $\mathcal{A}$. □

**Proposition 4.4.7** *If $\mathcal{A}$ has equalizers, $\mathcal{B}$ is a 2-rig with equalizers, and $G : \mathcal{A}_{ex} \to \mathcal{B}$ a distributive functor such that $GI$ preserves equalizers, then $G$ preserves equalizers.*

Proof: Let $f : A \to B+1$ and $g : A \to B+1$ be arrows from $a : A \to 1+1$ to $b : B \to 1+1$ in $\mathcal{A}_{ex}$; let $e : E \to A$ be their equalizer in $\mathcal{A}_{ex}$. Consider the diagram




where $e'$ is the equalizer of $Gf : Ga \to Gb$ and $Gg : Ga \to Gb$ in $\mathcal{B}$. By construction, $i_a$ is the equalizer of $GIa$ and $GIi_1$ and so $GIe$ factorizes as $i_a x$. But then using the fact that $i_b$ is monic, $x$ equalizes $Gf$ and $Gg$, and so factorizes as $e'y$. Finally $i_a e'$ equalizes both $GIf$ and $GIg$ and also $GIa$ and $GIi_1$, and so factorizes as $GIe.z$, and then $z$ is inverse to $y$, and $E'$ isomorphic to $GIE$. □

**Corollary 4.4.8** *The construction $\mathcal{A}_{ex}$ provides a left biadjoint to the forgetful 2-functor from* **Lext** *to* **LocDist**.

## 4.5 An embedding theorem

Cockett [10] proves that every locally distributive category can be subdirectly decomposed into a distributive preorder and a lextensive category (in fact he does not assume the existence of an initial object; note that he uses

distributive to mean what we mean by lextensive). Explicitly, given a locally distributive category $\mathcal{A}$, he forms the pullback $\bar{0}$ of the injections $i_1 : 1 \to 1 + 1$ and $i_2 : 1 \to 1 + 1$ and shows that the categories $\bar{0}/\mathcal{A}$ and $\mathcal{A}/\bar{0}$ are, respectively, lextensive, and a distributive preorder; that $\bar{0} + (-) : \mathcal{A} \to \bar{0}/\mathcal{A}$ and $\bar{0} \times (-) : \mathcal{A} \to \mathcal{A}/\bar{0}$ are distributive functors; and that the induced functor $\mathcal{A} \to \mathcal{A}/\bar{0} \times \bar{0}/\mathcal{A}$ is fully faithful.

These constructions fail if $\mathcal{A}$ is only distributive: in particular we cannot even guarantee the existence of $\bar{0}$. However we can nonetheless subdirectly decompose a distributive category in this way. The functor $\bar{0} + (-) : \mathcal{A} \to \bar{0}/\mathcal{A}$ has a characterization not given in [10]: it is the unit of a bireflection for the full sub-2-category **LocDist** of **Lext** given by the locally distributive categories. This is easily seen by noting that if $\mathcal{B}$ is lextensive and $F : \mathcal{A} \to \mathcal{B}$ preserves finite limits then $F\bar{0} = 0$, and that $\bar{0}+(-) : \mathcal{A} \to \bar{0}/\mathcal{A}$ is "natural in $\mathcal{A}$". We found in the previous section the unit of a bireflection $I : \mathcal{A} \to \mathcal{A}_{ex}$ for the full sub-2-category **2-Rig** of **Dist**; it will turn out that it can be used instead. In place of $\bar{0} \times (-) : \mathcal{A} \to \mathcal{A}/\bar{0}$ we use the preorder reflection of $\mathcal{A}$— this has the same objects as $\mathcal{A}$ and $A \leq B$ if and only if there is some arrow from $A$ to $B$ in $\mathcal{A}$. We write $\mathcal{A}_{pr}$ for the preorder reflection, it is clearly distributive and the reflection a distributive functor.

We now have a distributive functor $J : \mathcal{A} \to \mathcal{A}_{pr} \times \mathcal{A}_{ex}$ which we must show is fully faithful. Faithfulness of $J$ is immediate from faithfulness of $I$. Suppose that $A, B \in \mathcal{A}$ and that there exist maps

$$A \xrightarrow{\quad x \quad} B$$

$$A \xrightarrow{\quad f \quad} B + 1$$

such that $f$ is an arrow in $\mathcal{A}_{ex}$ from $IA$ to $IB$. If we write $g$ for the composite

$$A \xrightarrow{\binom{f}{x}} (B + 1) \times B \xrightarrow{\delta^{-1}} B \times B + B \xrightarrow{(\pi_1\ 1_B)} B$$

and $i_B$ for the injection $B \to B + 1$, then $i_B g = f$, giving fullness.

This completes the decomposition; before leaving this section we make a few comments.

First we note that if $\mathcal{A}$ is of the form $\mathcal{P} \times \mathcal{E}$ to start with, with $\mathcal{P}$ a distributive preorder and $\mathcal{E}$ a 2-rig, then $\mathcal{A}_{ex} = \mathcal{E}$ and $I$ is the projection. On the other hand, the preorder reflection of $\mathcal{P} \times \mathcal{E}$ is *not* in general $\mathcal{P}$ — for example consider the case $\mathbf{1} \times \mathbf{Set}$. There is a distributive preorder $\mathcal{A}_p$ and distributive functor $L : \mathcal{A} \to \mathcal{A}_p$ such that $(\mathcal{P} \times \mathcal{E})_p = \mathcal{P}$ and $L$ is the projection. The objects of $\mathcal{A}_p$ are the objects of $\mathcal{A}$; there is a map $A \to B$ if and only if for every preinitial object $Q$ of $\mathcal{A}$ with a map $Q \to A$ there is a map $Q \to B$ (an object is preinitial if it has at most one map to any given object). Unfortunately there seems no reason why $\binom{L}{I} : \mathcal{A} \to \mathcal{A}_p \times \mathcal{A}_{ex}$ should in general be full. It is full in the case where $\mathcal{A}$ is locally distributive; in that case $\binom{L}{I}$ will give (up to equivalence) the embedding of [10].

We have covered here a more general case than [10], but there is also a more specific case — with a correspondingly stronger result [24]. If $\mathcal{A}$ is not just locally distributive but a quasitopos, then we can capture $\mathcal{A}$ more precisely in terms of $\bar{0}/\mathcal{A}$ and $\mathcal{A}/\bar{0}$. In that case $\mathcal{A}$ is (up to equivalence) the comma category $(\bar{0}/\mathcal{A})/F$ where $F$ is the cartesian closed functor from the Heyting algebra $\mathcal{A}/\bar{0}$ to the topos $\bar{0}/\mathcal{A}$, taking $A \to \bar{0}$ to $A^{\bar{0}} + \bar{0}$; conversely if $\mathcal{P}$ is a Heyting algebra, $\mathcal{E}$ a topos, and $F : \mathcal{P} \to \mathcal{E}$ a cartesian closed functor, then $\mathcal{E}/F$ is a quasitopos.

# Chapter 5

# Coinverters

## 5.1 Categories of fractions and coinverters

An important construction in all areas of mathematics is that of forming quotients. In the case of sets, or groups, or other one-dimensional structures, this is done by "universally" making certain objects equal. With a category, however, one is interested not in objects being equal, but rather in their being isomorphic — isomorphic, of course, by a particular isomorphism. Thus the natural notion of forming quotients of categories is to "universally invert" a collection of arrows.

**Definition 5.1.1** *If $\mathcal{A}$ is a category and $\Sigma$ a collection of arrows of $\mathcal{A}$, a functor $P_\Sigma : \mathcal{A} \to \mathcal{A}[\Sigma^{-1}]$ is said to exhibit $\mathcal{A}[\Sigma^{-1}]$ as* the category of fractions of $\mathcal{A}$ for $\Sigma$ *if $P_\Sigma$ sends each arrow of $\Sigma$ to an isomorphism, and if, for all $\mathcal{B}$, composition with $P_\Sigma$ induces an isomorphism between the functor category $\mathbf{Cat}(\mathcal{A}[\Sigma^{-1}], \mathcal{B})$ and the full subcategory of $\mathbf{Cat}(\mathcal{A}, \mathcal{B})$ given by those functors which map the arrows of $\Sigma$ to isomorphisms.*

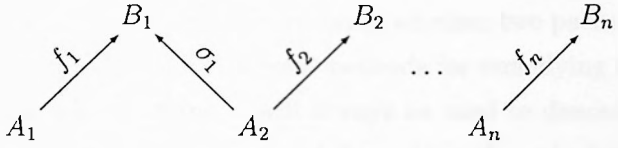
Categories of fractions exist, and can be constructed as follows. Take the underlying graph of $\mathcal{A}$. Add "formal inverses" of the arrows in $\Sigma$, *i.e.*

for each $\sigma \in \Sigma$ add an arrow $\sigma'$ with domain and codomain reversed. Now $\mathcal{A}[\Sigma^{-1}]$ is the free category on this graph subject to the relations

$$g.f \sim gf$$
$$f.1_A \sim f \sim 1_A.f$$
$$\sigma.\sigma' \sim 1$$
$$\sigma'.\sigma \sim 1$$

where we use "." for composition of paths in the graph and juxtaposition for composition in $\mathcal{A}$. The projection $P_\Sigma$ is the evident functor induced by the inclusion of graphs.

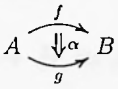
The arrows of $\mathcal{A}[\Sigma^{-1}]$ can be represented by paths




where the $f_i$ are in $\mathcal{A}$ and the $\sigma_i$ are in $\Sigma$; at this stage however it is not straightforward to determine whether two such paths represent the same arrow.

Another way of viewing categories of fractions is provided by the more general *coinverters*; they can be defined in any 2-category (or indeed bicategory).

**Definition 5.1.2** *In a bicategory $\mathcal{B}$, the* coinverter *of a 2-cell*




*is a 1-cell $q : B \to C$ such that the composite $q\alpha$ is invertible, and for any object $D$, composition with $q$ induces an isomorphism between $\mathcal{B}(C, D)$ and the full subcategory of $\mathcal{B}(B, D)$ determined by those 1-cells whose composite with $\alpha$ is invertible.*

Coinverters exist in **Cat**: given a natural transformation $\alpha$ between functors $F, G : \mathcal{S} \to \mathcal{A}$, the coinverter is given by the category of fractions $\mathcal{A}[\Sigma_\alpha^{-1}]$,

where $\Sigma_\alpha$ is the collection of those arrows of $\mathcal{A}$ which appear as components of $\alpha$. Conversely, given a collection of arrows $\Sigma$, one can realize $\mathcal{A}[\Sigma^{-1}]$ as a coinverter by taking $\mathcal{S}$ to be the discrete category on the set $\Sigma$, taking $F$ and $G$ to give the domain and codomain of an arrow in $\Sigma$, and then $\alpha$ to be the natural transformation whose component at an arrow $\sigma \in \Sigma$ is $\sigma$, thought of as being an arrow in $\mathcal{A}$.

## 5.2 The calculus of fractions

The construction of the previous section is rather unwieldy; in particular there is no practical method for determining whether two paths represent the same arrow. We shall consider various methods for remedying this situation. Throughout this section $\sigma$ and $\tau$ will always be used to denote arrows of $\Sigma$.

Gabriel and Zisman [18] introduced the notion of a calculus of right fractions.

**Definition 5.2.1** *A collection $\Sigma$ of arrows of a category $\mathcal{A}$ is said to admit a* calculus of right fractions *if it satisfies the following conditions*

(*a*) *$\Sigma$ contains the identities of $\mathcal{A}$.*

(*b*) *$\Sigma$ is closed under composition.*

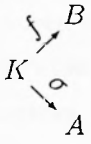
(*c*) *if $\sigma : D \to B$ and $f : A \to B$ in $\mathcal{A}$, then there exists a commutative square*

$$\begin{array}{ccc} C & \xrightarrow{g} & D \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\sigma} \\ A & \xrightarrow[f]{} & B \end{array}$$

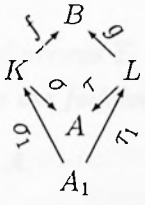
*(where of course $\tau \in \Sigma$).*

(*d*) *if* $f, g : A \to B$ *and* $\sigma : B \to C$ *with* $\sigma f = \sigma g$, *then there exists a* $\tau \in \Sigma$ *such that* $f\tau = g\tau$.

In this case there is a simple description of the arrows of $\mathcal{A}[\Sigma^{-1}]$. If $A$ and $B$ are objects of $\mathcal{A}[\Sigma^{-1}]$, then the arrows of $\mathcal{A}[\Sigma^{-1}](A, B)$ can be represented by pairs

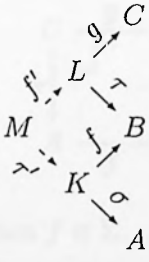



where $(\sigma, f)$ and $(\tau, g)$ represent the same arrow if there exist maps $\sigma_1$ and $\tau_1$ such that in the diagram




we have $f\sigma_1 = g\tau_1$ and $\sigma\sigma_1 = \tau\tau_1$.

Composites are formed as in




;

given arrows represented by $(f, \sigma)$ and $(g, \tau)$, one extends $f$ and $\tau$ to a commutative square by $f'$ and $\tau'$, and the composite is then the arrow represented by $(gf', \sigma\tau')$. Note that this arrow does not depend on the choice of $f'$ and $\tau'$. The identity on $A$ is represented by $(1_A, 1_A)$, and $P_\Sigma$ takes $f$ to $(f, 1)$.

**Proposition 5.2.2** *If $\Sigma$ admits a calculus of right fractions, then $P_\Sigma$ preserves whatever limits exist in $\mathcal{A}$.*

PROOF: The dual case (called a calculus of left fractions) is proved by Gabriel and Zisman [18, p. 16]. □

As Bénabou points out in [3], even classes of arrows admitting a calculus of right fractions do not behave as well as one might hope, and he introduces the *pullback congruences*, defined below, which turn out to be precisely the kernels of pullback-preserving functors.

We shall deal with an intermediate concept, partly because we shall be considering categories that need not have all pullbacks, and partly because it will be of use in the actual computation of coinverters.

**Definition 5.2.3** *A collection of arrows $\Sigma$ is said to admit a* cartesian precalculus of fractions *if it satisfies the following conditions*

($a'$) *$\Sigma$ contains the identities of $\mathcal{A}$.*

($c'$) *if $\sigma : D \to B$ and $f : A \to B$ in $\mathcal{A}$, then there exists a cartesian (i.e. pullback) square*

$$\begin{array}{ccc} C & \xrightarrow{g} & D \\ {\scriptstyle\tau}\downarrow & \lrcorner & \downarrow{\scriptstyle\sigma} \\ A & \xrightarrow[f]{} & B \end{array} \quad .$$

($e'$) *if $\sigma \in \Sigma$ and $\sigma f \in \Sigma$ then $f \in \Sigma$.*

We do not assume the existence of pullbacks beyond those demanded in condition ($c'$), nor do we demand that $\Sigma$ be stable under pullbacks — there may be other squares which are pullbacks of $\sigma$ and $f$ such that the arrow parallel to $\sigma$ is not in $\Sigma$. Although making these further assumptions would simplify

much of the following (see the pullback congruences of [3] and below) without changing the category of fractions induced, we start with weaker assumptions as we are interested specifically in calculating categories of fractions for collections of arrows as general as possible.

**Lemma 5.2.4** *If $\Sigma$ admits a cartesian precalculus of fractions then it satisfies condition $(d)$ in the definition of a calculus of fractions.*

PROOF: The proof is exactly that given by Benabou [3, p. 133] for the special case of pullback congruences. Suppose then that we are given $f, g : A \to B$ and $\sigma : B \to C$ with $\sigma f = \sigma g$. Form the pullback square

$$\begin{array}{ccc} D & \xrightarrow{m} & B \\ {\scriptstyle \sigma'}\downarrow & \lrcorner & \downarrow{\scriptstyle \sigma} \\ B & \xrightarrow[\sigma]{} & C \end{array}$$

as given in the definition of a cartesian precalculus of functions. Then there is a unique map $\delta : B \to D$ such that $\sigma'\delta = m\delta = 1_B$, and so by condition $(e')$ above $\delta \in \Sigma$, whence a pullback

$$\begin{array}{ccc} E & \xrightarrow{h} & B \\ {\scriptstyle \tau}\downarrow & \lrcorner & \downarrow{\scriptstyle \delta} \\ A & \xrightarrow[(f,g)]{} & D \end{array}$$

with $\tau \in \Sigma$ and $f\tau = g\tau$. □

**Corollary 5.2.5** *If $\Sigma$ is closed under composition and admits a cartesian precalculus of fractions then it admits a calculus of right fractions; we shall also say in this case that it admits a* cartesian calculus of fractions.

□

We now reduce the hypotheses on $\Sigma$ even further. Write $Iso(\mathcal{A})$ for the collection of isomorphisms of $\mathcal{A}$, and $\overline{\Sigma}$ for the arrows of $\mathcal{A}$ which can be formed as paths in $\Sigma \cup Iso(\mathcal{A})$.

Consider the following conditions on $\Sigma$:

($c''$) if $\sigma : D \to B$ and $f : A \to B$ in $\mathcal{A}$, then there exists a cartesian (i.e. pullback) square

$$\begin{array}{ccc} C & \xrightarrow{g} & D \\ {\scriptstyle s}\downarrow & \lrcorner & \downarrow{\scriptstyle \sigma} \\ A & \xrightarrow[f]{} & B \end{array}$$

with $s \in \overline{\Sigma}$.

($e''$) if $\sigma \in \Sigma$ and $\sigma f = 1$ then $f \in \overline{\Sigma}$.

**Proposition 5.2.6** *If $\Sigma$ is a class of arrows satisfying conditions ($c''$) and ($e''$) above, then $\overline{\Sigma}$ admits a cartesian calculus of fractions, and the categories of fractions for $\Sigma$ and $\overline{\Sigma}$ are isomorphic.*

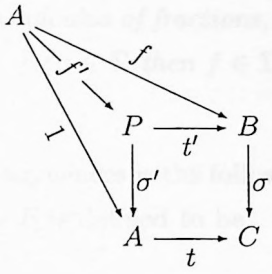
PROOF: The last part is trivial: a functor $F : \mathcal{A} \to \mathcal{B}$ inverts the arrows of $\Sigma$ if and only if it inverts the arrows of $\overline{\Sigma}$.

Suppose then that $\Sigma$ satisfies ($c''$) and ($e''$); we shall show that $\overline{\Sigma}$ admits a cartesian precalculus of fractions, and as $\overline{\Sigma}$ is by definition closed under composition, it will then admit a cartesian calculus of fractions.

Condition ($a'$) is of course true for $\overline{\Sigma}$ and condition ($c'$) holds because pullbacks compose and pullbacks of isomorphisms exist and are isomorphisms.

Now consider condition ($e'$). Suppose first that $\sigma \in \Sigma$ and $\sigma f \in \overline{\Sigma}$. Write

$t = \sigma f$; then there is a commutative diagram




where the square is a pullback, $\sigma' \in \Sigma$ and $t' \in \overline{\Sigma}$. Then $\sigma' f' = 1$, and so $f' \in \overline{\Sigma}$, which in turn gives $f \in \overline{\Sigma}$. If now $s \in \overline{\Sigma}$ and $sf \in \overline{\Sigma}$, then an easy induction gives $f \in \overline{\Sigma}$. □

**Corollary 5.2.7** *If $\Sigma$ satisfies the above hypotheses then the coinverter $P_\Sigma : \mathcal{A} \to \mathcal{A}[\Sigma^{-1}]$ preserves whatever limits exist in $\mathcal{A}$.*

□

## 5.3 Pullback congruences

The pullback congruences of [3] provide a context in which to understand cartesian calculi of fractions, and will also be of some use to us later on.

**Definition 5.3.1** *If $\mathcal{A}$ is a category with pullbacks, then a pullback congruence on $\mathcal{A}$ is a class $\Sigma$ of maps satisfying*

($P_1$) *Every isomorphism of $\mathcal{A}$ belongs to $\Sigma$.*

($P_2$) *If $f_0 = f_1 f_2$ and two of the maps $f_i$ are in $\Sigma$, then so is the third.*

($P_3$) *$\Sigma$ is stable under pullbacks, i.e. for any pullback*




*if $\sigma \in \Sigma$ then $\tau \in \Sigma$.*

**Proposition 5.3.2** *If $\mathcal{A}$ has pullbacks then $\Sigma$ is a pullback congruence if and only if it admits a cartesian calculus of fractions, contains the isomorphisms, and satisfies the condition: if $f\sigma \in \Sigma$ then $f \in \Sigma$ for all $\sigma \in \Sigma$.*

□

The point of pullback congruences is the following beautiful theorem. The kernel of a functor $F : \mathcal{A} \to \mathcal{B}$ is defined to be

$$Ker(F) = \{a : A \to A' : Fa \text{ is invertible}\}.$$

**Theorem 5.3.3 (Bénabou)** *Let $\mathcal{A}$ be a category with pullbacks, and $\Sigma$ a class of arrows of $\mathcal{A}$. Then the following are equivalent.*

*(i)* *$P_\Sigma$ preserves finite limits, $\mathcal{A}[\Sigma^{-1}]$ has pullbacks, and $\Sigma = Ker(P_\Sigma)$.*

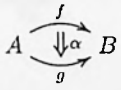
*(ii)* *There exists a pullback-preserving functor $F : \mathcal{A} \to \mathcal{B}$ such that $\Sigma = Ker(F)$ (where $\mathcal{B}$ need not have all pullbacks).*

*(iii)* *$\Sigma$ is a pullback congruence on $\mathcal{A}$.*

## 5.4 Reflexive coinverters

Much of the material in this section appeared as part of [28].

We have seen in the previous sections that in forming categories of fractions $\mathcal{A}[\Sigma^{-1}]$ the condition that $\Sigma$ contain the identities is very important. This condition can also be expressed conveniently in terms of coinverters. Recall that the coinverter of a 2-cell

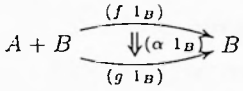



in **Cat** can be formed as the category of fractions for those arrows which appear as components of $\alpha$. We can ask that these contain the identities by

requiring that there be a 1-cell $i : B \to A$ such that $fi = gi = 1_B$ and $\alpha i$ is the identity 2-cell on $1_B$.

**Definition 5.4.1** *A 2-cell $\alpha$ with such an $i$ is said to be reflexive.*

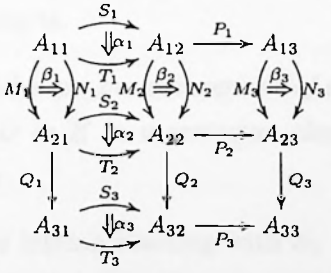
In a 2-category with coproducts, there is a canonical reflexive 2-cell $\alpha_r$ associated to a 2-cell $\alpha$; it is formed as the following coproduct




and then $i$ is taken to be the injection of the coproduct. It is clear that the coinverter of $\alpha_r$ is isomorphic to the coinverter $\alpha$ as a 1-cell coinverts $\alpha_r$ if and only if it coinverts $\alpha$, which corresponds in turn to the obvious fact that the image of an identity under any functor is invertible.

By an abuse of notation, we shall sometimes say that the coinverter of a reflexive 2-cell is a *reflexive coinverter*. There is an obvious formal analogy between coinverters of reflexive 2-cells and coequalizers of reflexive pairs, and the "3-by-3 lemma" for such coequalizers [23, p. 16] can be adapted to the case of coinverters.

Suppose that $\mathcal{A}$ is a category and we have a diagram




satisfying

$$M_2\alpha_1 = \alpha_2 M_1,\ N_2\alpha_1 = \alpha_2 N_1,\ S_2\beta_1 = \beta_2 S_1,\ T_2\beta_1 = \beta_2 T_1$$

where implicit in these equations are equalities of the domains and codomains; it then follows that $\beta_2\alpha_1 = \alpha_2\beta_1$. Suppose further that $P_1$, $P_2$, $Q_1$, $Q_2$ are the coinverters of $\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2$ respectively; whereupon $M_3$, $N_3$, and $\beta_3$ are the maps and the 2-cell induced in the evident way, as are $S_3$, $T_3$, and $\alpha_3$. Finally let $P_3$ be the coinverter of $\alpha_3$, and $Q_3$ the induced map; it is easy to see that $Q_3$ is then the coinverter of $\beta_3$.

**Lemma 5.4.2** *Suppose that in the above circumstances the 2-cells $\alpha_1$ and $\beta_1$ are reflexive. Then $Q_3P_2$ $(= P_3Q_2)$ is the coinverter of $\beta_2\alpha_1$ $(= \alpha_2\beta_1)$.*

PROOF: Let $I : A_{12} \to A_{11}$ and $J : A_{21} \to A_{11}$ be the reflections. Suppose that $F : A_{22} \to T$ coinverts $\beta_2 \circ \alpha_1$. Then also $F$ coinverts $\beta_2\alpha_1 I = \beta_2$, and so $F$ factorizes uniquely as $F = F_1Q_2$. Now $F_1$ coinverts $Q_2\beta_2\alpha_1 = Q_2\alpha_2\beta_1$, and so also coinverts $Q_2\alpha_2\beta_1 J = Q_2\alpha_2 = \alpha_3 Q_1$. But $Q_1$ is a coinverter, and so by the two-dimensional part of the universal property of coinverters, $F_1$ coinverts $\alpha_3$. Thus $F_1$ factorizes uniquely as $F'P_3$. We now have a unique factorization $F = F'P_3Q_2$. Thus $P_3Q_2$ is the coinverter. □

The major application of the $3 \times 3$ lemma for reflexive coinveters is, just as in the case of reflexive coequalizers, the proof that reflexive coinverters commute with finite products.

**Corollary 5.4.3** *If $p$ and $q$ are the coinverters of reflexive 2-cells $\rho : s \to t : A \to B$ and $\sigma : u \to v : D \to E$ in a cartesian closed 2-category, then $p \times q$ is the coinverter of $\rho \times \sigma$.*

PROOF: Apply the above lemma starting with $\alpha_1 = \rho \times id_D$, $\alpha_2 = \rho \times id_E$, $\beta_1 = id_A \times \sigma$, and $\beta_2 = id_B \times \sigma$. □

**Remark 5.4.4** The same thing clearly applies in a monoidal bicategory, with $p \otimes q$ replacing $p \times q$, provided that each $A \otimes -$ and each $- \otimes B$ preserves colimits.

**Remark 5.4.5** Given coinverters as in Corollary 5.4.3, which are not necessarily reflexive, and converting them as before into reflexive ones, we see that $p \times q$ is the coinverter of $(id_A, \rho) \times (id_D, \sigma)$.

If we return to the case of the cartesian closed 2-category **Cat**, we can reinterpret this last result as saying that if $\Sigma$ is a class of arrows in $\mathcal{A}$ which contains the identities, and $\Omega$ is a class of arrows in $\mathcal{B}$ which contains the identities, then $\mathcal{A}[\Sigma^{-1}] \times \mathcal{B}[\Omega^{-1}] \cong (\mathcal{A} \times \mathcal{B})[(\Sigma \times \Omega)^{-1}]$.

## 5.5 Strongly-finitary 2-monads

Much of the material in this section appeared as part of [27] or [28].

A number of structures that categories might admit can be expressed in terms of $n$-ary operations and equations. Such a structure, on a given category $\mathcal{A}$, is understood to include various functors $\mathcal{A}^n \to \mathcal{A}$, called the *basic operations*, from which further functors may be derived; and also natural transformations between these "derived operations"; finally these natural transformations may be required to satisfy certain equations.

The possession by a category of (assigned) binary sums is an example of such a structure: it amounts to giving a functor $+ : \mathcal{A}^2 \to \mathcal{A}$ as the only basic operation; natural transformations between derived operations $\epsilon : +\Delta \to 1$ and $\eta : 1 \to \Delta+$, where $\Delta$ stands for the diagonal $\mathcal{A} \to \mathcal{A}^2$; and the two triangular equations which make $+$ left adjoint to $\Delta$. Similarly the possession of finite sums is an example of such a structure. For a category to be extensive is *not* an example of such a structure, as it requires operations whose arity is not discrete, such as forming pullbacks in the category or taking slices of the category.

The structures expressible in this way can be delimited with the notion of strong finitariness. In the following we write $\mathcal{S}_f$ for the sub-2-category of **Cat** determined by the finite sets $n \in \mathbf{N}$, seen as discrete categories.

**Definition 5.5.1** *An endo-2-functor $T$ on* **Cat** *is said to be* strongly finitary *if it is the left Kan extension of its restriction to $\mathcal{S}_f$; in terms of coends we can write this coindition as*

$$T = \int^{n \in \mathcal{S}_f} (-)^n \times Tn.$$

*A 2-monad on* **Cat** *is said to be strongly finitary if its underlying endo-2-functor is so.*

**Theorem 5.5.2** *A strongly-finitary endo-2-functor on* **Cat** *preserves reflexive coinverters.*

Proof: The endo-2-functors $(-)^n$ preserve reflexive coinverters by a straightforward extension of Corollary 5.4.3, the endo-2-functors $- \times Tn$ preserve all colimits as **Cat** is cartesian closed, and coends commute with reflexive coinverters, both being colimits. □

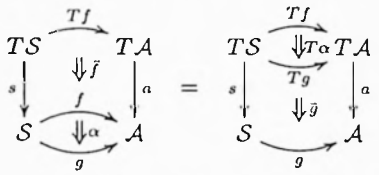
The strongly-finitary 2-monads encode precisely the structures of the type described above; to say that a category possesses such a structure is precisely to say that it is the underlying category of a $T$-algebra for the corresponding stronly-finitary 2-monad $T$. This correspondence is described in detail in [27].

We shall wish to consider two types of maps between $T$-algebras $(\mathcal{S}, s)$ and $(\mathcal{A}, a)$: the ordinary ones, and the strict. An (ordinary) map comprises a functor $f : \mathcal{S} \to \mathcal{A}$ and a natural isomorphism

$$\begin{array}{ccc} T\mathcal{S} & \xrightarrow{Tf} & T\mathcal{A} \\ {\scriptstyle s}\downarrow & \Downarrow \bar{f} & \downarrow{\scriptstyle a} \\ \mathcal{S} & \xrightarrow[g]{} & \mathcal{A} \end{array}$$

satisfying two coherence conditions found in [5]; a strict map is a map for which $\bar{f}$ is an identity. The maps and the strict maps are the 1-cells, respectively, for 2-categories **T-Alg** and **T-Alg**$_s$ whose objects are the $T$-algebras.

If $f$ and $g$ are two such 1-cells then a 2-cell from $f$ to $g$ is a natural transformation $\alpha : f \to g$ satisfying




(5.5.1)

In the case of categories with sums, the ordinary maps are precisely the ordinary sum-preserving functors, and the strict maps are those which preserve the (assigned) sums "on-the-nose". The condition on 2-cells is vacuous.

**Theorem 5.5.3** *The forgetful 2-functor* $U$ : **T-Alg** $\to$ **Cat** *creates the coinverter of reflexive 2-cells; these coinverters are always strict maps of* $T$*-algebras, from which it follows that the forgetful 2-functor* $U_s$ : $\mathbf{T\text{-}Alg}_s \to$ **Cat** *also creates such coinverters.*

PROOF: Let $\alpha$ be a 2-cell as above in **T-Alg**, and suppose that $\alpha$ has a reflection $i : \mathcal{A} \to \mathcal{S}$ (in **T-Alg** and so also in **Cat**). Let $p : \mathcal{A} \to \mathcal{C}$ be the coinverter of $\alpha$ in **Cat**. Now $p\alpha$ and $\bar{f}$ are invertible, and so the composite of $p$ with the left, and so likewise the right, side of (5.5.1) is. But as $\bar{g}$ is invertible so too is $p.a.T\alpha$. Now $T$ preserves reflexive coinverters by Theorem 5.5.2, and so $Tp$ is the coinverter of $T\alpha$ and so there is a unique $c : T\mathcal{C} \to \mathcal{C}$ such that $c.Tp = p.a$; this $c$ can easily be seen to provide $\mathcal{C}$ with the structure of a $T$-algebra, using the fact that $p$ and $T^2p$ are both coinverters and so epimorphic. Moreover by definition of $c$, the functor $p$ is a strict map of $T$-algebras. That it is the coinverter of $\alpha$ in **T-Alg** follows as $p$ and $Tp$ are both coinverters. □

# Chapter 6

# The monoidal closed bicategories Sums and Ext

## 6.1 The closed structure of Sums

We think of a category with finite coproducts as being the categorical version of an abelian group, or more precisely, a commutative monoid. The category **CMon** of commutative monoids and monoid homomorphism is monoidal closed, with monoidal structure given by the tensor product; and we shall construct an analogous structure on the 2-category **Sums**.

The closed structure on **CMon** is given by

$$[M, N] = \mathbf{CMon}(M, N) \text{ with pointwise addition}$$

and we mimic this for the case of **Sums** by setting $[\mathcal{B}, \mathcal{C}]$ equal to the hom-category $\mathbf{Sums}(\mathcal{B}, \mathcal{C})$; this has finite coproducts which as we shall see are formed pointwise.

To give the structure of a category with finite coproducts on a category $\mathcal{C}$ is to give a left adjoint to the diagonal functor $\Delta : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$; that is, to give a functor $+ : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and natural transformations $\eta : 1 \to \Delta+$

and $\epsilon : +\Delta \to 1$ satisfying the triangle equations. If we now pass from the category $\mathcal{C}$ to $\mathbf{Cat}(\mathcal{B},\mathcal{C})$ for arbitrary $\mathcal{B}$, we have

$$\mathbf{Cat}(\mathcal{B},\mathcal{C}) \times \mathbf{Cat}(\mathcal{B},\mathcal{C}) = \mathbf{Cat}(\mathcal{B},\mathcal{C} \times \mathcal{C})$$

with the diagonal being $\mathbf{Cat}(\mathcal{B},\Delta)$; this diagonal has left adjoint $\mathbf{Cat}(\mathcal{B},+)$ by 2-functoriality of $\mathbf{Cat}(\mathcal{B},-)$. If now $\mathcal{B}$ has finite coproducts, all of this restricts from $\mathbf{Cat}(\mathcal{B},\mathcal{C})$ to its sub-2-category $\mathbf{Sums}(\mathcal{B},\mathcal{C})$. We shall write $[\mathcal{B},\mathcal{C}]$ to denote $\mathbf{Sums}(\mathcal{B},\mathcal{C})$ when we wish to think of it as an object of **Sums**.

One could now go on to define a 2-functor

$$\mathbf{Sums}^{op} \times \mathbf{Sums} \xrightarrow{[-,-]} \mathbf{Sums}$$

and show that **Sums** is *closed* in the sense of [17], but we shall follow a different line, showing that **Sums** is monoidal in the bicategorical sense, and that the above closed structure then makes it into a monoidal closed bicategory.

## 6.2 Coherence for categories with sums

Writing $\mathbf{Sums}_s$ for the subcategory of **Sums** containing only the *strict* maps, we saw in the previous chapter that the forgetful 2-functor $U_s : \mathbf{Sums}_s \to \mathbf{Cat}$ is 2-monadic, and the associated 2-monad is strongly finitary. Thus $U_s$ and also $U : \mathbf{Sums} \to \mathbf{Cat}$ create coinverters of reflexive 2-cells.

We write $T$ for the evident 2-monad on **Cat** with underlying endo-2-functor $Fam$. This is also strongly finitary and the $T$-algebras, forming the objects of the Eilenberg-Moore 2-category $\mathbf{T\text{-}Alg}_s$, are the categories with strictly associative finite coproducts (and initial objects as units in the strict sense).

This $T$ is strong, and we write

$$\lambda : T\mathcal{A} \times \mathcal{B} \to T(\mathcal{A} \times \mathcal{B})$$

$$\lambda' : \mathcal{A} \times T\mathcal{B} \to T(\mathcal{A} \times \mathcal{B})$$

for the maps expressing this strength. For a category $\mathcal{A}$ with finite coproducts we shall write $m_{\mathcal{A}} : T\mathcal{A} \to \mathcal{A}$ for the functor which forms the coproduct of a family.

We now have three different levels of strictness being considered, but in fact the 2-categories involved are all biequivalent, as the following result shows. It can be deduced from various other known theorems; in particular the construction of a strict category with coproducts equivalent to a given category with coproducts is an instance of that given in the proof of Power's "general coherence result" in [35].

**Proposition 6.2.1** *The inclusions of 2-categories*

$$\mathbf{T\text{-}Alg}_s \longrightarrow \mathbf{Sums}_s \longrightarrow \mathbf{Sums}$$

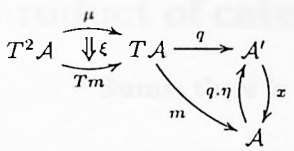
*are biequivalences.*

Proof: In fact we prove that the composite is a biequivalence, and then the result follows, as inclusions are monic.

First we consider biessential surjectivity; this is just the fact that every category with finite coproducts is equivalent to a strict one. Given $\mathcal{A} \in \mathbf{Sums}$, we can form the reflexive 2-cell

$$T^2\mathcal{A} \underset{Tm_{\mathcal{A}}}{\overset{\mu}{\Downarrow \epsilon}} T\mathcal{A}$$

where the components of $\xi$ comprise families of injections; this is the underlying natural transfomation of a 2-cell in $\mathbf{T\text{-}Alg}_s$. Its coinverter $q : T\mathcal{A} \to \mathcal{A}'$ therefore lifts to $\mathbf{T\text{-}Alg}_s$, and so we have the following diagram in $\mathbf{Cat}$.




Here

$$x.q.\eta_{\mathcal{A}} = m.\eta_{\mathcal{A}} = 1$$

and

$$q.\eta_{\mathcal{A}}.x.q = q.\eta_{\mathcal{A}}.m = q.Tm.\eta_{T\mathcal{A}} \cong q.\mu_{\mathcal{A}}.\eta_{T\mathcal{A}} = q$$

and so

$$q.\eta_{\mathcal{A}}.x \cong 1$$

making $q.\eta_{\mathcal{A}}$ inverse equivalent to $x$. An equivalence of course preserves finite coproducts, and so we have biessential surjectivity.

As we are dealing with strict inclusions, it remains only to show local essential surjectivity. Suppose then that $\mathcal{A}, \mathcal{B} \in \mathbf{T\text{-}Alg}_s$, and $F : \mathcal{A} \to \mathcal{B}$ preserves finite coproducts in the usual sense. Then form the composite $m_{\mathcal{B}}.TF$ which preserves them strictly, and which moreover inverts $\xi$, and so factors as $m_{\mathcal{B}}.TF = F'.m_{\mathcal{A}}$, where $F'$ is equivalent to $F$ and strictly preserves finite coproducts. □

Before moving on to define the tensor product of categories with finite coproducts, we note here that the 2-functor $[-,-] : \mathbf{Sums}^{op} \times \mathbf{Sums} \to \mathbf{Sums}$ restricts to give 2-functors

$$[-,-] : \mathbf{Sums}_s^{op} \times \mathbf{Sums}_s \to \mathbf{Sums}_s$$

$$[-,-] : \mathbf{T\text{-}Alg}_s^{op} \times \mathbf{T\text{-}Alg}_s \to \mathbf{T\text{-}Alg}_s.$$

## 6.3 The tensor product of categories with sums

We saw in Section 6.1 that for $\mathcal{B} \in \mathbf{Sums}$, there is a 2-functor

$$[\mathcal{B}, -] : \mathbf{Sums} \to \mathbf{Sums};$$

one might then attempt to construct some sort of left adjoint $- \otimes \mathcal{B}$. It is not the case that $[\mathcal{B}, -]$ has a left 2-adjoint, nor even does its underlying functor have an ordinary left adjoint. We shall see however, that the 2-functor $[\mathcal{B}, -]$ does have a left *biadjoint*, which we construct using coinverters.

Given $\mathcal{A}, \mathcal{B} \in \mathbf{Sums}$, we construct $\mathcal{A} \otimes \mathcal{B}$ very much like the tensor product of commutative monoids (or abelian groups). First form the cartesian product $\mathcal{A} \times \mathcal{B}$ and then then freely add finite coproducts, using the $Fam$ construction, giving $Fam(\mathcal{A} \times \mathcal{B})$. The tensor product $\mathcal{A} \otimes \mathcal{B}$ will then be formed as the coinverter of finite coproducts of the arrows

$$(A_i, B)_{i \in I} \to (\sum_{i \in I} A_i, B) \tag{6.3.1}$$

$$(A, B_j)_{j \in J} \to (A, \sum_{j \in J} B_j) \tag{6.3.2}$$

induced by the injections.

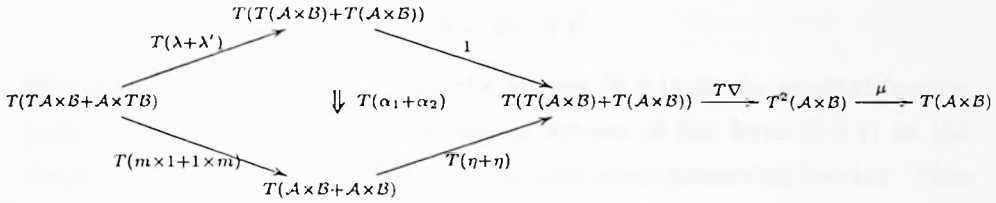
Now the arrows (6.3.1) form the components of a natural transformation

$$T(T\mathcal{A} \times \mathcal{B}) \;\underset{\eta(m_{\mathcal{A}} \times 1)}{\overset{\lambda}{\Downarrow \alpha_1}}\; T(\mathcal{A} \times \mathcal{B})$$

and the arrows (6.3.2) form the components of a natural transformation

$$T(\mathcal{A} \times T\mathcal{B}) \;\underset{\eta(1 \times m_{\mathcal{B}})}{\overset{\lambda'}{\Downarrow \alpha_2}}\; T(\mathcal{A} \times \mathcal{B})$$

and we can from them form another natural transformation




which we shall call $\alpha$. This is clearly the underlying 2-cell of a reflexive 2-cell in **Sums** (which is in fact also in $\mathbf{Sums}_s$ and $\mathbf{T\text{-}Alg}_s$) and so the coinverter

$$T(TA\times B+A\times TB) \underset{\Downarrow\alpha}{\rightrightarrows} T(A\times B) \xrightarrow{q} A\otimes B$$

lifts to **Sums**; this is the definition of $\mathcal{A}\otimes\mathcal{B}$, and clearly gives a 2-functor

$$\otimes : \mathbf{Sums}\times\mathbf{Sums}\to\mathbf{Sums}$$

which restricts to 2-functors

$$\otimes : \mathbf{Sums}_s\times\mathbf{Sums}_s\to\mathbf{Sums}_s$$

$$\otimes : \mathbf{T\text{-}Alg}_s\times\mathbf{T\text{-}Alg}_s\to\mathbf{T\text{-}Alg}_s.$$

A finite-coproduct-preserving functor

$$\mathcal{A}\to[\mathcal{B},\mathcal{C}]$$

gives, upon composition with the inclusion functor into $\mathbf{Cat}[\mathcal{B},\mathcal{C}]$, a finite-coproduct-preserving functor

$$\mathcal{A}\to\mathbf{Cat}(\mathcal{B},\mathcal{C})$$

and so by cartesian-closedness of **Cat** a functor

$$\mathcal{A}\times\mathcal{B}\to\mathcal{C}$$

and so a finite-coproduct-preserving functor

$$T(\mathcal{A} \times \mathcal{B}) \to \mathcal{C}.$$

Moreover this last functor inverts the arrows (6.3.1) as the original functor preserved finite coproducts; and inverts arrows of the form (6.3.2) as the image of this original functor was a finite-coproduct-preserving functor. Thus it induces a finite-coproduct-preserving functor

$$\mathcal{A} \otimes \mathcal{B} \to \mathcal{C}.$$

This process is clearly functorial, and so we have a functor

$$\mathbf{Sums}(\mathcal{A}, [\mathcal{B}, \mathcal{C}]) \to \mathbf{Sums}(\mathcal{A} \otimes \mathcal{B}, \mathcal{C})$$

which is clearly fully faithful as each step of the above construction is so. Moreover it is essentially surjective. Given a finite-coproduct-preserving functor $\mathcal{A} \otimes \mathcal{B} \to \mathcal{C}$, one can retrace the above process to obtain a functor $\mathcal{A} \to \mathbf{Cat}(\mathcal{B}, \mathcal{C})$; that this factors to give a functor $\mathcal{A} \to \mathbf{Sums}(\mathcal{B}, \mathcal{C})$ follows from (6.3.2), and that this resulting functor preserves finite coproducts follows from (6.3.1).

This equivalence

$$\mathbf{Sums}(\mathcal{A}, [\mathcal{B}, \mathcal{C}]) \to \mathbf{Sums}(\mathcal{A} \otimes \mathcal{B}, \mathcal{C})$$

is itself finite-coproduct-preserving, and so is in fact an arrow

$$[\mathcal{A}, [\mathcal{B}, \mathcal{C}]] \to [\mathcal{A} \otimes \mathcal{B}, \mathcal{C}]$$

in **Sums**. Any equivalence preserves finite coproducts, and so any equivalence between categories with finite coproducts is an equivalence in **Sums**.

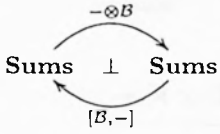
**Proposition 6.3.1** *The above data give an equivalence*

$$[\mathcal{A} \otimes \mathcal{B}, \mathcal{C}] \to [\mathcal{A}, [\mathcal{B}, \mathcal{C}]]$$

*in the 2-category* **Sums**.

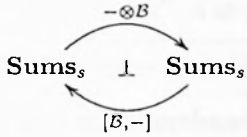
Moreover this equivalence is natural and so we have:

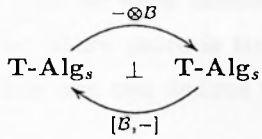
**Proposition 6.3.2** *For any category $\mathcal{B}$ with finite coproducts there is a biadjunction*




□

**Corollary 6.3.3** *This biadjunction restricts to genuine 2-adjunctions*




PROOF: The equivalence $\mathbf{Sums}(\mathcal{A}\otimes\mathcal{B},\mathcal{C}) \simeq \mathbf{Sums}(\mathcal{A},[\mathcal{B},\mathcal{C}])$ restricts to isomorphisms

$$\mathbf{Sums}_s(\mathcal{A}\otimes\mathcal{B},\mathcal{C}) \simeq \mathbf{Sums}_s(\mathcal{A},[\mathcal{B},\mathcal{C}])$$
$$\mathbf{T\text{-}Alg}_s(\mathcal{A}\otimes\mathcal{B},\mathcal{C}) \simeq \mathbf{T\text{-}Alg}_s(\mathcal{A},[\mathcal{B},\mathcal{C}]).$$

□

## 6.4 The monoidal closed bicategory Sums

We have so far given a 2-category **Sums**, and a 2-functor $\otimes$ : **Sums** $\times$ **Sums** $\to$ **Sums**. The next step is to consider the associativity of the tensor product. Given $\mathcal{A},\mathcal{B},\mathcal{C} \in$ **Sums** we shall give a functor

$$Ass : (\mathcal{A}\otimes\mathcal{B})\otimes\mathcal{C} \to \mathcal{A}\otimes(\mathcal{B}\otimes\mathcal{C})$$

which will turn out both to preserve finite coproducts and to be an equivalence.

It is possible to deduce the existence of such a map using the biadjunction of Proposition 6.3.2 in the previous section:

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{A}\otimes(\mathcal{B}\otimes\mathcal{C}) \longrightarrow \mathcal{A}\otimes(\mathcal{B}\otimes\mathcal{C})}{\mathcal{A} \longrightarrow [\mathcal{B}\otimes\mathcal{C}, (\mathcal{A}\otimes(\mathcal{B}\otimes\mathcal{C})]}}{\mathcal{A} \longrightarrow [\mathcal{B}, [\mathcal{C}, (\mathcal{A}\otimes(\mathcal{B}\otimes\mathcal{C}))]]}}{\mathcal{A}\otimes\mathcal{B} \longrightarrow [\mathcal{C}, \mathcal{A}\otimes(\mathcal{B}\otimes\mathcal{C})]}}{(\mathcal{A}\otimes\mathcal{B})\otimes\mathcal{C} \longrightarrow \mathcal{A}\otimes(\mathcal{B}\otimes\mathcal{C})}$$

where here the horizontal lines are shorthand not for a bijection between such maps but rather a bijection between isomorphism classes of such maps — although in fact of course rather more is true than this. Then starting with an identity in the top line one can deduce the existence of a functor

$$Ass_{\mathcal{A},\mathcal{B},\mathcal{C}} : (\mathcal{A}\otimes\mathcal{B})\otimes\mathcal{C} \to \mathcal{A}\otimes(\mathcal{B}\otimes\mathcal{C})$$

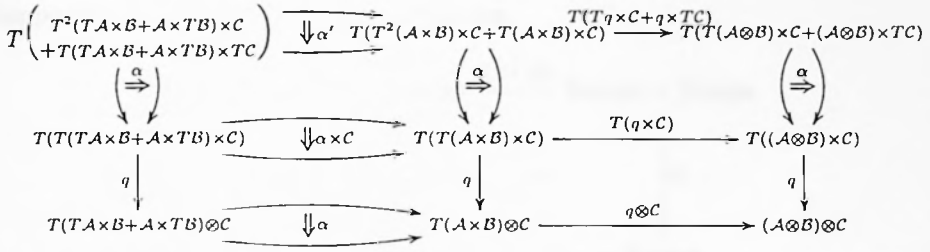
and all of these functors preserve finite coproducts; moreover by symmetry there is clearly a functor going in the opposite direction, and then by naturality and the two-dimensional part of the biadjunction these are mutually inverse equivalences.

However to consider coherence questions of this equivalence we need to have an explicit description, and we need to choose a particular functor, not just allow it to be determined up to isomorphism.

The tensor product $\mathcal{A}\otimes\mathcal{B}$ was constructed as the coinverter

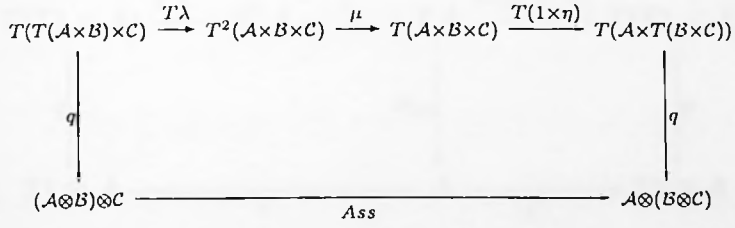
$$T(TA\times B + A\times TB) \underset{\Downarrow\alpha}{\rightrightarrows} T(A\times B) \xrightarrow{q} A\otimes B$$

Now consider $(\mathcal{A}\otimes\mathcal{B})\otimes\mathcal{C}$. We can construct it using the following "3-by-3 diagram".



where we have written $\alpha'$ as shorthand for $T(T\alpha \times \mathcal{C} + \alpha \times T\mathcal{C})$. That the columns are coinverters follows by the definition of $\otimes$. Noting that all coinverters involved are of reflexive 2-cells , the top two rows are coinverters by cartesian-closedness of **Cat**, strong finitariness of $T$, and the results of Chapter 5. It then follows that the bottom row is a coinverter; alternatively we could deduce this by the fact that all arrows in this row strictly preserve coproducts and the fact that $- \otimes \mathcal{B} : \mathbf{Sums}_s \to \mathbf{Sums}_s$ has a right 2-adjoint. It now follows by the 3-by-3 lemma (Lemma 5.4.2) that the diagonal of this diagram itself forms a coinverter. It lifts to all three of $\mathbf{T\text{-}Alg}_s$, $\mathbf{Sums}_s$, and **Sums**.

Similarly we can construct $\mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C})$ as a coinverter of $T(\mathcal{A} \times T(\mathcal{B} \times \mathcal{C}))$, and now we can describe the map $Ass$ as the unique map rendering commutative



(6.4.1)

and $Ass$ has the evident map $T(\mathcal{A} \times T(\mathcal{B} \times \mathcal{C})) \to T(T(\mathcal{A} \times \mathcal{B}) \times \mathcal{C})$ as inverse equivalence.

Everything here is 2-natural in $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, and so these maps form the

components of a 2-natural transformation

$$\begin{array}{ccc} \mathbf{Sums} \times \mathbf{Sums} \times \mathbf{Sums} & \xrightarrow{1 \times \otimes} & \mathbf{Sums} \times \mathbf{Sums} \\ \downarrow{\scriptstyle \otimes \times 1} & \Uparrow Ass & \downarrow{\scriptstyle \otimes} \\ \mathbf{Sums} \times \mathbf{Sums} & \xrightarrow[\otimes]{} & \mathbf{Sums} \end{array}$$

which clearly restricts to give 2-natural transformations

$$\begin{array}{ccc} \mathbf{Sums}_s^3 & \xrightarrow{1 \times \otimes} & \mathbf{Sums}_s^2 \\ \downarrow{\scriptstyle \otimes \times 1} & \Uparrow Ass & \downarrow{\scriptstyle \otimes} \\ \mathbf{Sums}_s^2 & \xrightarrow[\otimes]{} & \mathbf{Sums}_s \end{array} \qquad \begin{array}{ccc} \mathbf{T\text{-}Alg}_s^3 & \xrightarrow{1 \times \otimes} & \mathbf{T\text{-}Alg}_s^2 \\ \downarrow{\scriptstyle \otimes \times 1} & \Uparrow Ass & \downarrow{\scriptstyle \otimes} \\ \mathbf{T\text{-}Alg}_s^2 & \xrightarrow[\otimes]{} & \mathbf{T\text{-}Alg}_s \end{array} .$$

The strict maps

$$T(T1 \times \mathcal{A}) \xrightarrow{T\lambda} T^2\mathcal{A} \xrightarrow{\mu} T\mathcal{A}$$

$$T\mathcal{A} \xrightarrow{T(\eta \times \mathcal{A})} T(T1 \times \mathcal{A})$$

induce strict maps $l$ and $l^*$ rendering commutative

$$\begin{array}{ccccccc} T(T1 \times \mathcal{A}) & \xrightarrow{T\lambda} & T^2\mathcal{A} & \xrightarrow{\mu} & T\mathcal{A} & \xrightarrow{T(\eta \times \mathcal{A})} & T(T1 \times \mathcal{A}) \\ \downarrow{\scriptstyle q} & & & & \downarrow{\scriptstyle m_{\mathcal{A}}} & & \downarrow{\scriptstyle q} \\ T1 \otimes \mathcal{A} & & & \xrightarrow[l]{} & \mathcal{A} & \xrightarrow[l^*]{} & T1 \otimes \mathcal{A} \end{array}$$

Now $\mu.T\lambda.T(\eta \times \mathcal{A}) = \mu.T\eta = 1$, and so

$$ll^* = 1.$$

On the other hand

$$\begin{aligned} l^*.l.q &= q.T(\eta \times \mathcal{A}).\mu.T\lambda \\ &= q.\mu.T^2(\eta \times \mathcal{A}).T\lambda \\ &= q.\mu.T\lambda.T(T\eta \times \mathcal{A}) \\ &\cong q.T(\mu \times \mathcal{A}).T(T\eta \times \mathcal{A}) \\ &= q \end{aligned}$$

and so $l^*l \cong 1$, and $l$ and $l^*$ form an equivalence pair.

Similarly there is an equivalence

$$\mathcal{A} \underset{r^*}{\overset{r}{\rightleftarrows}} \mathcal{A} \otimes T1$$

where, for example, $r$ is induced by

$$T(\mathcal{A} \times \eta) : T\mathcal{A} \to T(\mathcal{A} \times T1).$$

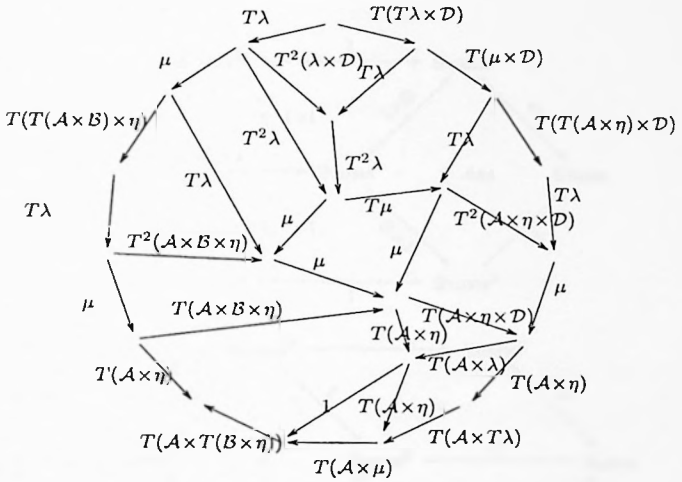
We now turn to the question of coherence. Thus far we have dealt with the 2-categories $\mathbf{T\text{-}Alg}_s$, $\mathbf{Sums}_s$, and **Sums** in parallel; we shall now consider only $\mathbf{T\text{-}Alg}_s$, and later use the biequivalences of Proposition 6.2.1 to extend the results of the special case.

To consider the coherence of $Ass$ we need to consider quadruple tensor products, such as $((\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C}) \otimes \mathcal{D}$. This is (the codomain of) a coinverter

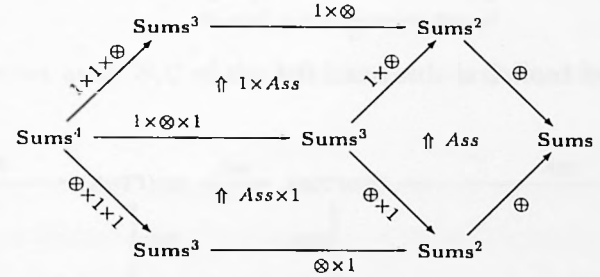
$$q : T(T(T(\mathcal{A} \times \mathcal{B}) \times \mathcal{C}) \times \mathcal{D}) \to ((\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C}) \otimes \mathcal{D}.$$

We think of it as a coinverter in **Cat**, but of course it lifts to $\mathbf{T\text{-}Alg}_s$, $\mathbf{Sums}_s$, and **Sums**.

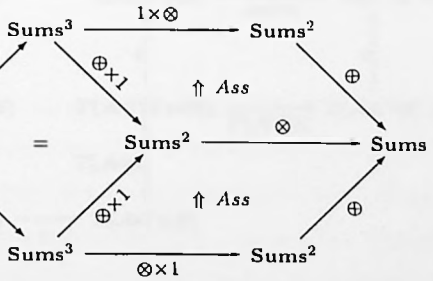
Commutativity of




gives commutativity of the cube

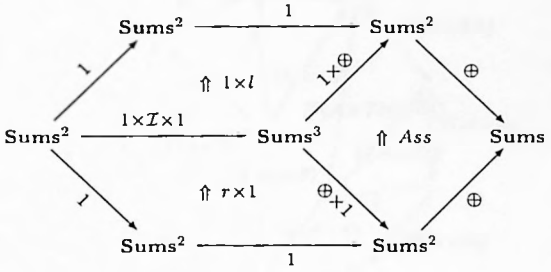



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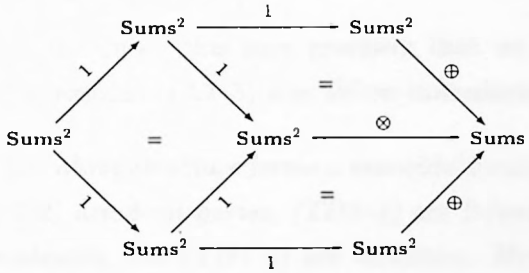



which says exactly that we may choose the modification (TD7) to be an identity; condition (TA1) now follows immediately.

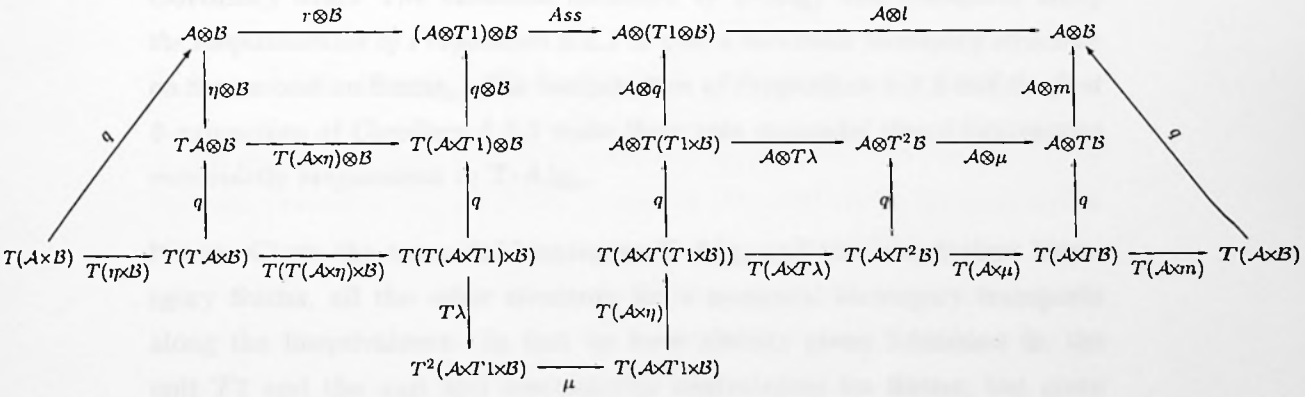
Next we look at coherence for $r$ and $l$; we shall show commutativity of the following cube.



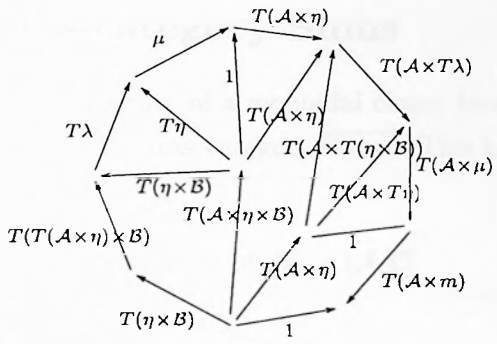

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The component at $\mathcal{A}, \mathcal{B}, \mathcal{C}$ of the left hand side is defined by the diagram




and then commutativity of




gives commutativity of the cube; this says precisely that we may choose (TD8) to be an identity; axioms (TA2–3) now follow immediately.

**Proposition 6.4.1** *The above structure forms a monoidal bicategory* $\mathbf{T\text{-}Alg}_s$*, for which the data (TD2) are 2-categories, (TD3–4) are 2-functors, (TD5–6) are 2-natural equivalences, and (TD7–8) are identities. Moreover the 2-adjunction of Corollary 6.3.3 make it into a monoidal closed bicatgory.*

**Corollary 6.4.2** *The monoidal structure of* $\mathbf{T\text{-}Alg}_s$ *now transports along the biequivalences of Proposition 6.2.1 to give a monoidal bicategory structure on* $\mathbf{Sums}$ *and on* $\mathbf{Sums}_s$*. The biadjunction of Proposition 6.3.2 and the first 2-adjunction of Corollary 6.3.3 make them into monoidal closed bicategories monoidally biequivalent to* $\mathbf{T\text{-}Alg}_s$*.*

Proof: Given the monoidal bicategory $\mathbf{T\text{-}Alg}_s$ and the biequivalent bicategory $\mathbf{Sums}$, all the other structure for a monoidal bicategory transports along the biequivalence. In fact we have already given 2-functors $\otimes$, the unit $T1$ and the unit and associativity equivalences for $\mathbf{Sums}$, but given that the biequivalence $\mathbf{T\text{-}Alg}_s \to \mathbf{Sums}$ is a 2-functor strictly preserving all this structure we can choose to retain this structure already given, and thus monoidal biequivalence $\mathbf{T\text{-}Alg}_s \to \mathbf{Sums}$ becomes a strict map of monoidal bicategories. The argument for $\mathbf{Sums}_s$ is identical.

## 6.5 The Sums-category Sums

We have now given the structure of a monoidal closed bicategory to **Sums**, and so it can be made into a **Sums**-category $\overline{\mathbf{Sums}}$. This has hom-categories $[\mathcal{A}, \mathcal{B}]$, composition maps

$$M : [\mathcal{B}, \mathcal{C}] \otimes [\mathcal{A}, \mathcal{B}] \to [\mathcal{A}, \mathcal{C}]$$

induced by

$$\begin{array}{ccc} T([\mathcal{B}, \mathcal{C}] \times [\mathcal{A}, \mathcal{B}]) & \xrightarrow{TM} & T[\mathcal{A}, \mathcal{C}] \\ \Big\downarrow{\scriptstyle q} & & \Big\downarrow{\scriptstyle m} \\ [\mathcal{B}, \mathcal{C}] \otimes [\mathcal{A}, \mathcal{B}] & \xrightarrow[M]{} & [\mathcal{A}, \mathcal{C}] \end{array}$$

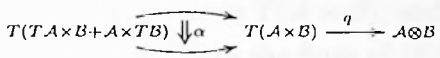
identities given by the maps $T1 \xrightarrow{Tj} T[\mathcal{A}, \mathcal{A}] \xrightarrow{m} [\mathcal{A}, \mathcal{A}]$.

## 6.6 The monoidal closed bicategory Ext

In this section we move from the 2-category **Sums** to the full sub-2-category **Ext** determined by the extensive categories. We shall see that the monoidal bicategory structure on **Sums** restricts to give a monoidal bicategory structure on **Ext**. As **Ext** is a full sub-2-category, one need only show that the tensor product of extensive categories is extensive; the unit $Fam1$ is of course extensive.

To prove this result we shall use some of the technical results on coinverters from Chapter 5.

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are extensive categories. Recall that the category $\mathcal{A} \otimes \mathcal{B}$ is formed as the coinverter




in **Cat**. Write $\Sigma_\alpha$ for the collection of arrows in $T(\mathcal{A} \times \mathcal{B})$ which appear as components of $\alpha$. As in Chapter 5, write $\overline{\Sigma_\alpha}$ for the closure of $\Sigma_\alpha$ under isomoprhisms and composition.

**Proposition 6.6.1** *$\Sigma_\alpha$ satisfies the hypotheses of Proposition 5.2.6.*

PROOF: Suppose that $s \in \Sigma_\alpha$ and $h$ in $T(\mathcal{A} \times \mathcal{B})$ have the same codomain; we must find a pullback square such that the pullback of $s$ by $h$ is in $\Sigma_\alpha$. By extensivity of $T(\mathcal{A} \times \mathcal{B})$ it suffices to consider the case where $h$ is in the image of $\eta : \mathcal{A} \times \mathcal{B} \to T(\mathcal{A} \times \mathcal{B})$ and $s$ is of the form (6.3.1) or (6.3.2); we treat the former case, the latter clearly follows from it by symmetry. Thus we are faced with a diagram

$$\begin{array}{ccc} & & (C,D) \\ & & \Big\downarrow{\scriptstyle (f,g)} \\ (A_i,B)_{i\in I} & \xrightarrow[\sigma]{} & (\sum_{i\in I} A_i, B) \end{array}$$

in $T(\mathcal{A} \times \mathcal{B})$. By extensivity both of $T(\mathcal{A} \times \mathcal{B})$ and of $\mathcal{A}$, this has pullback

$$\begin{array}{ccccc} (C_i,D)_{i\in I} & \xrightarrow{\sigma'} & (\sum_{i\in I} C_i, D) & \xrightarrow{\phi} & (C,D) \\ \Big\downarrow & & & & \Big\downarrow{\scriptstyle (f,g)} \\ (A_i,B)_{i\in I} & & \xrightarrow[\sigma]{} & & (\sum_{i\in I} A_i, B) \end{array}$$

where $\sigma$ is of the form (6.3.1) and $\phi$ is an isomorphism. This proves condition $(c')$.

Finally suppose that $t \in \Sigma_\alpha$, $k$ is in $T(\mathcal{A} \times \mathcal{B})$, and $th = 1$. Once more we can restrict to the case where $t$ is of the form of (6.3.1):

$$\begin{array}{ccc} (\sum_{i\in I} A_i, B) & \xrightarrow{k} & (A_i, B)_{i\in I} \\ & \searrow^{1} & \downarrow \sigma \\ & & (\sum_{i\in I} A_i, B) \end{array}$$

and so we have a diagram in $\mathcal{A}$

$$\begin{array}{ccc} \sum_{i\in I} A_i & \xrightarrow{k_1} & A_j \\ & \searrow^{1} & \downarrow \sigma_j \\ & & \sum_{i\in I} A_i \end{array}$$

for some $j \in I$, where $\sigma_j$ is the injection. But then as $\mathcal{A}$ is extensive, this $k_1$ is inverse to the injection, and $A_i \cong 0$ if $i \neq j$.

We can now write $k$ as the composite

$$(\sum_{i\in I} A_i, B) \xrightarrow{(k_1, 1)} (A_j, B) \xrightarrow{\alpha_x} (A'_i, B)_{i\in I} \xrightarrow{\varphi} (A_i, B)_{i\in I}$$

where $A'_j = A_j$ and $A'_i = 0$ if $i \neq j$, the map $\alpha_x$ is a component of $\alpha$, and $\varphi$ is invertible. Thus $k$ is in $\overline{\Sigma}$ as claimed and proves condition $(e')$. □

**Corollary 6.6.2** *The coinverter $q : T(\mathcal{A} \times \mathcal{B}) \to \mathcal{A} \otimes \mathcal{B}$ preserves any limits that exist in $T(\mathcal{A} \times \mathcal{B})$.*

**Corollary 6.6.3** *The tensor product of extensive categories is extensive.*

PROOF: The category $T(\mathcal{A} \times \mathcal{B})$ is extensive, and the functor $q : T(\mathcal{A} \times \mathcal{B}) \to \mathcal{A} \otimes \mathcal{B}$ preserves coproducts and whatever limits exist, and is bijective on objects; moreover the injections of coproducts in $\mathcal{A} \otimes \mathcal{B}$ are in the image of $q$. Thus $\mathcal{A} \otimes \mathcal{B}$ has pullbacks of injections along maps in the image of $q$ and

coproducts are stable under pullbacks along such maps; likewise coproducts in $\mathcal{A} \otimes \mathcal{B}$ are disjoint. Moreover every arrow in $\mathcal{A} \otimes \mathcal{B}$ is the composite of an arrow in the image of $q$ and an isomorphism, and so $\mathcal{A} \otimes \mathcal{B}$ has pullbacks along injections, and coproducts are stable under pullback. Thus $\mathcal{A} \otimes \mathcal{B}$ has disjoint stable coproducts and so is extensive. □

**Theorem 6.6.4** *The 2-category* **Ext** *admits the structure of a monoidal closed bicategory; the inclusion of* **Ext** *into* **Sums** *strictly preserves all of this structure.*

PROOF: Given extensive categories $\mathcal{B}$ and $\mathcal{C}$, we have seen that $\mathbf{Sums}(\mathcal{B}, \mathcal{C})$ has pointwise sums; it is in fact extensive with the remaining structure required of an extensive category also being given pointwise. The biadjunction $- \otimes \mathcal{B} \dashv [\mathcal{B}, -]$ on **Sums** then restricts to **Ext**. □

# Chapter 7

# 2-Dimensional rig theory

## 7.1 Tensor products of distributive categories and 2-rigs

In the previous chapter a monoidal structure on **Sums** and **Ext** was presented; we now turn to the case of the 2-categories **Dist** and **2-Rig**, and then to **LocDist** and **Lext**. Because limits and colimits in a category are unique up to isomorphism if they exist, they are citizens of a nebulous world somewhere between structures and properties; there are times when it is more useful to think of finite coproducts, say, as being equationally defined structure, as we did in the preceding chapter; at other times, one thinks instead in terms of the property of possessing certain limits or colimits, and in this chapter we lean more towards this latter point of view. Thus in describing the monoidal structure on **Dist** we begin with that on **Sums** and then show that if the categories in **Sums** are in fact distributive then their tensor product is too, and likewise for maps. The result for **2-Rig** will then follow easily.

Whereas in the previous chapter we treated the coherence problem in a

very concrete way, describing a strict category with finite coproducts equivalent to a given category with finite coproducts in terms of coinverters, presenting a slightly different approach to Power's "general coherence result" [35], here we simply use the result to ignore wholesale the problem of non-strictness.

The general result of [35] is a still more general version of the following theorem, whose proof we omit.

**Theorem 7.1.1 (Power)** *Let $T$ be a monad on* **Cat** *which preserves the property of being bijective on objects. Then every pseudo-$T$-algebra is equivalent in* **Ps-T-Alg** *to a strict algebra.*

The definition of pseudo-$T$-algebras and their morphisms can be found in [35]; we simply remark that in the case where $T$ is the monad "dual" to *Fam* described in Section 4.4 then the (strict) $T$-algebras are the strict categories with finite products, the pseudo-$T$-algebras are categories with finite products, and their morphisms are the ordinary finite-product-preserving functors. Moreover this $T$ is strongly-finitary in the sense of [27] and Section 5.5, and so the forgetful 2-functor from **T-Alg** to **Cat** creates coinverters of reflexive 2-cells. We now move from the "coinverter" point of view back toward the "category of fractions" point of view, in concerning ourselves more with the arrows inverted than a specific 2-cell. Suppose then that in the coinverter

$$T(T\mathcal{A}\times\mathcal{B}+\mathcal{A}\times T\mathcal{B}) \underset{}{\overset{}{\Downarrow\alpha}} T(\mathcal{A}\times\mathcal{B}) \xrightarrow{q} \mathcal{A}\otimes\mathcal{B}$$

defining the tensor product the categories $\mathcal{A}$ and $\mathcal{B}$ are in fact distributive. We shall show that finite products of arrows which are components of $\alpha$ are inverted by $q$ and so $\mathcal{A}\otimes\mathcal{B}$ has finite products, $q$ preserves finite products, and $r : \mathcal{A}\otimes\mathcal{B} \to \mathcal{C}$ is finite-product-preserving if $rq$ is.

A component of $\alpha$ is a family of components of $\alpha_1$ and $\alpha_2$ and so by

distributivity of $T(\mathcal{A} \times \mathcal{B})$ it suffices to consider finite products of components of $\alpha_1$ and $\alpha_2$, for which it suffices in turn to consider binary products of components of $\alpha_1$ and $\alpha_2$. There are three cases to consider: the product of two components of $\alpha_1$, the product of two components of $\alpha_2$, and the product of a component of $\alpha_1$ and a component of $\alpha_2$.

Let

$$(A_i, B)_{i \in I} \xrightarrow{\sigma} (\sum_{i \in I} A_i, B)$$
$$(C_j, D)_{j \in J} \xrightarrow{\sigma'} (\sum_{j \in J} C_j, D)$$

be components of $\alpha_1$. Then their product is

$$(A_i \times C_j, B \times D)_{i \in I, j \in J} \xrightarrow{\sigma \times \sigma'} (\sum_{i \in I} A_i \times \sum_{j \in J} C_j, B \times D)$$

which, by distributivity of $\mathcal{A}$ is the composite of a component of $\alpha_1$ with an isomorphism, and so inverted by $q$. The second case is exactly the same. For the third case suppose that $\sigma$ is as above and

$$(C, D_k)_{k \in K} \xrightarrow{\tau} (C, \sum_{k \in K} D_k)$$

is a component of $\alpha_2$. Then the product of $\sigma$ and $\tau$

$$(A_i \times C, B \times D_k)_{i \in I, k \in K} \xrightarrow{\sigma \times \tau} \Big( \big( \sum_{i \in I} A_i \big) \times C, B \times \sum_{k \in K} D_k \Big),$$

can be formed as the composite of a family of components of $\alpha_1$, a component of $\alpha_2$, and isomorphisms, and so is inverted by $q$.

**Proposition 7.1.2** *If $\mathcal{A}$ and $\mathcal{B}$ are distributive categories then $\mathcal{A} \otimes \mathcal{B}$ has finite products, $q$ preserves finite products, and a functor $r : \mathcal{A} \otimes \mathcal{B} \to \mathcal{C}$ is finite-product-preserving if $rq$ is.*

□

**Corollary 7.1.3** *If $\mathcal{A}$ and $\mathcal{B}$ are distributive categories, respectively 2-rigs, then so is $\mathcal{A} \otimes \mathcal{B}$.*

PROOF: Suppose first that $\mathcal{A}$ and $\mathcal{B}$ are distributive. By the previous proposition $\mathcal{A} \otimes \mathcal{B}$ has, and $q$ preserves, finite products. But then $q$ is a functor from a distributive category to a category with finite products and coproducts, preserves those products and coproducts, and is bijective on objects. It follows immediately that $\mathcal{A} \otimes \mathcal{B}$ must also be distributive. On the other hand we have already seen that $\mathcal{A} \otimes \mathcal{B}$ is extensive if $\mathcal{A}$ and $\mathcal{B}$ are. □

**Proposition 7.1.4** *If $\mathcal{A}$ and $\mathcal{B}$ are locally distributive, respectively lextensive, categories, then $\mathcal{A} \otimes \mathcal{B}$ is locally distributive, respectively lextensive.*

PROOF: The arrows inverted by $q : T(\mathcal{A} \times \mathcal{B}) \to \mathcal{A} \otimes \mathcal{B}$ admit a cartesian precalculus of fractions; $T(\mathcal{A} \times \mathcal{B})$ has equalizers as $\mathcal{A}$ and $\mathcal{B}$ do, and so, by Proposition 5.2.7, the category $\mathcal{A} \otimes \mathcal{B}$ has, and the functor $q$ preserves, equalizers. Once more local distributivity of $\mathcal{A} \otimes \mathcal{B}$ follows from local distributivity of $T(\mathcal{A} \times \mathcal{B})$, the fact that $q$ preserves finite coproducts and finite limits and is bijective on objects, and the fact that an arrow in $\mathcal{A} \otimes \mathcal{B}$ can be expressed as the composite of a map in the image of $q$ with an isomorphism. We have already seen that $\mathcal{A} \otimes \mathcal{B}$ is extensive if $\mathcal{A}$ and $\mathcal{B}$ are. □

In the case of commutative rigs $R$ and $S$, the multiplication on $R \otimes S$ can be given as the composite

$$R \otimes S \otimes R \otimes S \xrightarrow{1_R \otimes tw \otimes 1_S} R \otimes R \otimes S \otimes S \xrightarrow{m_R \otimes m_S} R \otimes S$$

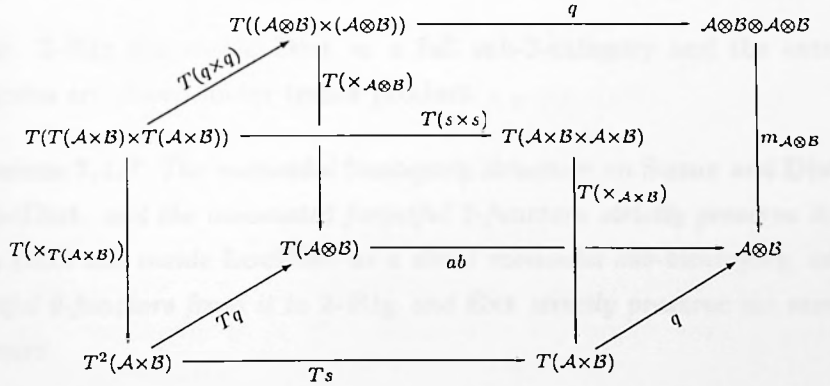
where $tw$ is the symmetry isomorphism for $\otimes$, and $m_R$ and $m_S$ are the multiplications on $R$ and $S$. We have not discussed the symmetric structure on the tensor product of distributive categories, so we are forced to approach products in $\mathcal{A} \otimes \mathcal{B}$ in a different way.

Exactly as in the case of commutative rigs (or equally rings) if $\mathcal{C}$ is a distributive category then "product" is a bilinear functor $\times : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and so induces a finite-coproduct-preserving functor $m : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$; it is defined by commutativity of

$$\begin{array}{ccc} T(\mathcal{C} \times \mathcal{C}) & \xrightarrow{q} & \mathcal{C} \otimes \mathcal{C} \\ {\scriptstyle T(\times)}\downarrow & & \downarrow{\scriptstyle m} \\ T\mathcal{C} & \xrightarrow[c]{} & \mathcal{C} \end{array}$$

where $c$ is the structure map for $\mathcal{C}$, *i.e.* where $c$ forms the coproduct of a family. By Proposition 7.1.2 this $m$ preserves finite products and so is a distributive functor.

Consider now the case of $\mathcal{A} \otimes \mathcal{B}$. Write $ab$ for the structure map for $\mathcal{A} \otimes \mathcal{B}$ and $s$ for the structure map for $\mathcal{A} \times \mathcal{B}$. Then the following diagram commutes




and $\times_{\mathcal{A}\times\mathcal{B}}$ we *can* calculate explicitly using the symmetry isomorphism for $\times$ and the products in $\mathcal{A}$ and $\mathcal{B}$.

We remark in passing that it is possible to fill in the "missing edge" of the cube; it will be a "biuniversal quadrilinear map", but not yet having defined such a concept we shall not take the trouble to do so here.

**Theorem 7.1.5** *The monoidal bicategory structure on* **Sums** *lifts through the forgetful functor* $U$ : **Dist** $\to$ **Sums**; *this* $U$ *then becomes a strict morphism of monoidal bicategories.*

PROOF: The last two results allow the lifting of the homomorphism $\otimes$ : **Sums** $\times$ **Sums** $\to$ **Sums** to a homomorphism $\otimes$ : **Dist** $\times$ **Dist** $\to$ **Dist**. The components of the associativity and unit strong equivalences, being equivalences, of course preserve finite products, and so they lift too. The lifting of the remaining structure and the axioms are immediate as $U$ is locally fully faithful. □

**Theorem 7.1.6** *The 2-category* **2-Rig** *admits the structure of a monoidal bicategory and the inclusion of* **2-Rig** *into* **Dist** *strictly preserves this structure. Similarly the forgetful 2-functor from* **2-Rig** *to* **Ext** *strictly preserves the monoidal structure.*

PROOF: **2-Rig** sits inside **Dist** as a full sub-2-category and the extensive categories are closed under tensor product. □

**Theorem 7.1.7** *The monoidal bicategory structure on* **Sums** *and* **Dist** *lifts to* **LocDist**, *and the associated forgetful 2-functors strictly preserve it; similarly* **Lext** *sits inside* **LocDist** *as a strict monoidal sub-bicategory, and the forgetful 2-functors from it to* **2-Rig** *and* **Ext** *strictly preserve the monoidal structure.*

PROOF: Once more this is immediate from previous results; this time we use Propositions 7.1.4 and 5.2.7. □

~~**Theorem 7.1.8** **Dist** *and* **LocDist** *admit enrichment in the monoidal bicatgory* **Sums**; **2-Rig** *and* **Lext** *admit enrichments in the monoidal bicategories* **Sums** *and* **Ext**.~~

PROOF: We have seen that **Sums** admits an enrichment over itself; also the distributive functors are closed under composition. From these two facts and the locally fully faithful strict monoidal homomorphism from **Dist** to **Sums** we can deduce the **Sums**-enriched structure for **Dist**; the other cases are similar.

## 7.2 Bidistributive bicategories

Like most notions involving limits and colimits, distributivity admits a bicategorical version, which we call bidistributivity. The notion of product and coproduct used is of course that of *bi*product and *bi*coproduct, given in [38].

Note that the bidistributive bicategories are quite distinct from the *distributive bicategories* of [7].

**Definition 7.2.1** *A bicategory with finite biproducts and finite bicoproducts is said to be* bidistributive *if the canonical arrows*

$$\delta : A \times B + A \times C \to A \times (B + C)$$

*are equivalences for all objects $A$, $B$, and $C$.*

Here $\delta$ is induced from the bicoproduct $A \times B + A \times C$ just as in Chapter 4 except that in this case it is defined only up to isomorphism; of course if an arrow in a bicategory is an equivalence then any arrow isomorphic to it is likewise an equivalence.

Given distributive categories $\mathcal{A}$ and $\mathcal{B}$, there is a distributive functor $i_1$ equal to the composite

$$\mathcal{A} \xrightarrow{r} \mathcal{A} \otimes T1 \xrightarrow{1_{\mathcal{A}} \otimes z} \mathcal{A} \otimes \mathcal{B}$$

where $r$ is the equivalence coming from the monoidal bicategory structure of **Sums**, and $z$ is induced as a coproduct-preserving functor by the constant

functor $1 \to \mathcal{B}$ at the terminal object of $\mathcal{B}$; this $i_1$ is also equal to

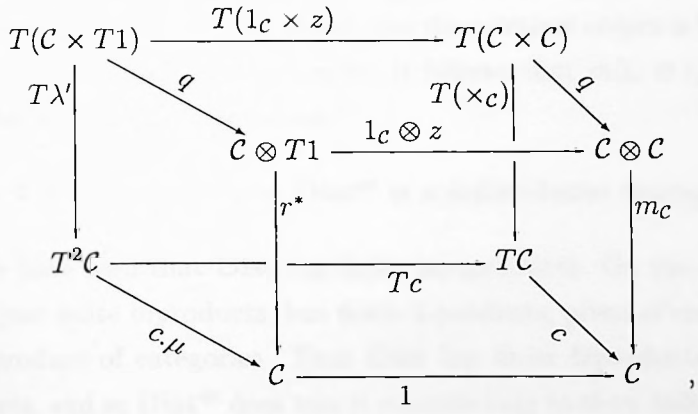
$$\mathcal{A} \xrightarrow{\binom{1_{\mathcal{A}}}{\ulcorner 1 \urcorner}} \mathcal{A} \times \mathcal{B} \xrightarrow{\eta} T(\mathcal{A} \times \mathcal{B}) \xrightarrow{q} \mathcal{A} \otimes \mathcal{B}$$

Similarly there is a distributive functor $i_2 : \mathcal{B} \to \mathcal{A} \otimes \mathcal{B}$.

**Proposition 7.2.2** *The functors $i_1$ and $i_2$ exhibit $\mathcal{A} \otimes \mathcal{B}$ as the bicoproduct of $\mathcal{A}$ and $\mathcal{B}$ in* **Dist**.

Proof: We shall show that $\otimes$ is left biadjoint to $\Delta : \mathbf{Dist} \to \mathbf{Dist} \times \mathbf{Dist}$. The component of the unit at $(\mathcal{A}, \mathcal{B})$ is the pair of injections $i_1 : \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}$ and $i_2 : \mathcal{B} \to \mathcal{A} \otimes \mathcal{B}$. The counit at $\mathcal{C}$ is $m_{\mathcal{C}} : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$.

In the diagram




as $q : T(\mathcal{C} \times T1) \to \mathcal{C} \otimes T1$ is epimorphic, $m(\mathcal{C} \otimes z) \cong r^*$ and so $m\, i_1 \cong 1_{\mathcal{C}}$, and similarly $m\, i_2 \cong 1_{\mathcal{C}}$. This is one of the triangle equations; for the other

consider the commutative diagram

$$\begin{array}{ccccc}
T(\mathcal{A}\times\mathcal{B}\times\mathcal{A}\times\mathcal{B}) & \xrightarrow{T\left(\binom{1_{\mathcal{A}}}{\ulcorner 1\urcorner}\times\binom{\ulcorner 1\urcorner}{1_{\mathcal{B}}}\right)} & T(\mathcal{A}\times\mathcal{B}) & \xrightarrow{q} & \mathcal{A}\otimes\mathcal{B} \\
\downarrow{\scriptstyle T(\eta\times\eta)} & & \downarrow{\scriptstyle T(i_1\times i_2)} & & \downarrow{\scriptstyle i_1\otimes i_2} \\
T(T(\mathcal{A}\times\mathcal{B})\times T(\mathcal{A}\times\mathcal{B})) & \xrightarrow{T(q\times q)} & T((\mathcal{A}\otimes\mathcal{B})\times(\mathcal{A}\otimes\mathcal{B})) & \xrightarrow{q} & \mathcal{A}\otimes\mathcal{B}\otimes\mathcal{A}\otimes\mathcal{B} \\
\downarrow{\scriptstyle T(s\times s)} & & \downarrow{\scriptstyle T(\times_{\mathcal{A}\otimes\mathcal{B}})} & & \downarrow{\scriptstyle m_{\mathcal{A}\otimes\mathcal{B}}} \\
T(\mathcal{A}\times\mathcal{B}\times\mathcal{A}\times\mathcal{B}) & \xrightarrow[T_{\mathcal{A}\times\mathcal{B}}]{} & T(\mathcal{A}\times\mathcal{B}) & \xrightarrow[q]{} & \mathcal{A}\otimes\mathcal{B}
\end{array}$$

where we write $\ulcorner 1\urcorner$ for the constant map at the terminal object of a category.

The left hand edge is an identity — the sum of a singleton family is just the element of the family, and we may as well take this to be true "on-the-nose". Then the composite of $\binom{1_{\mathcal{A}}}{\ulcorner 1\urcorner} \times \binom{\ulcorner 1\urcorner}{1_{\mathcal{B}}}$ with $T(\times_{\mathcal{A}\times\mathcal{B}})$ is isomorphic to the identity as the product of an object and the terminal object is isomorphic to the object. Then as $q$ is epimorphic it follows that $m(i_1 \otimes i_2) \cong 1_{\mathcal{A}\otimes\mathcal{B}}$, which is the other triangle equation. □

**Theorem 7.2.3** *The 2-category* $\mathbf{Dist}^{op}$ *is a bidistributive bicategory.*

Proof: We have seen that **Dist** has finite bicoproducts. On the other hand it has not just finite biproducts, but finite 2-products, given of course by the ordinary product of categories. Thus **Dist** has finite biproducts and finite bicoproducts, and so $\mathbf{Dist}^{op}$ does too; it remains only to show bidistributivity of $\mathbf{Dist}^{op}$.

Consider again the 2-category **Sums**; the product of two categories with finite coproducts is of course their 2-product in **Sums**; but it is moreover their bicoproduct. This is entirely analogous to the case of abelian groups or commutative monoids; the injection $\mathcal{A} \to \mathcal{A}\times\mathcal{B}$ is $\binom{1_{\mathcal{A}}}{\ulcorner 1\urcorner}$ and similarly for the other injection; the codiagonal $\mathcal{C}\times\mathcal{C} \to \mathcal{C}$ is the coproduct functor. But **Sums** is (bi)closed with respect to the the monoidal structure $\otimes$ and so $\mathcal{A}\otimes -$

preserves bicolimits, and in particular bicoproducts. Thus $\otimes$ distributes over $\times$ in **Sums**, and equally in **Dist**, and so $\mathbf{Dist}^{op}$ is bidistributive. □

**Corollary 7.2.4** *The 2-categories* $\mathbf{2\text{-}Rig}^{op}$, $\mathbf{LocDist}^{op}$ *and* $\mathbf{Lext}^{op}$ *are bidistributive bicategories, and the various forgetful 2-functors between them are strict morphisms of bidistributive bicategories, i.e. they are 2-functors strictly preserving biproducts and bicoproducts.*

PROOF: **2-Rig** is a full sub-2-category of **Dist** closed under bicoproducts and products.

**LocDist** is a locally full sub-2-category of **Dist** closed under products and tensor products. The maps $i_1 : \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}$ and $i_2 : \mathcal{B} \to \mathcal{A} \otimes \mathcal{B}$ preserve equalizers, as do the multiplication maps $c : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ — each of these facts follows from Proposition 5.2.7 — and so the tensor product still gives the bicoproduct in **LocDist**, and **LocDist** is also closed under bicoproducts.

**Lext** is a full sub-2-category of **LocDist** closed under bicoproducts and products. □

## 7.3 Polynomial 2-rigs

Most of the material in this section appeared in [29], and was developed jointly with R.F.C.Walters during an undergraduate project at the University of Sydney.

An important construction in the theory of commutative rings is the formation of the *polynomial ring* $R[X]$ of a ring $R$. We can mimic this construction very closely in the case of 2-rigs, and the resulting 2-rig will satisfy bicategorical versions of the various universal properties satisfied by $R[X]$. Extensivity is important here, for in the case of distributive categories one could define universal properties satisfied by a polynomial distributive category, and while a category satisfying these properties would exist for

general reasons, it would not admit a simple concrete description as in the case of commutative rings, commutative rigs, or 2-rigs.

Suppose then that $\mathcal{A}$ is a 2-rig; we shall describe the *polynomial 2-rig* $\mathcal{A}[X]$ of $\mathcal{A}$. We first present the construction, and then describe the various properties it possesses. Given a natural number $n$ and an object $A$ of $\mathcal{A}$ write $n.A$ for the coproduct of $n$ copies of $A$.

An object of $\mathcal{A}[X]$ will be a finite family $(A_i, m_i)_{i\in I}$, where $A_i$ is an object of $\mathcal{A}$, and $m_i$ a natural number. We think of $(A_i, m_i)_{i\in I}$ as representing $\sum_{i\in I} A_i X^{m_i}$, and so in particular we shall sometimes write $X$ for the singleton family $(1, 1)$. An arrow from $(A_i, m_i)_{i\in I}$ to $(B_j, n_j)_{j\in J}$ comprises an arrow $\alpha : \sum_{i\in I} A_i \to \sum_{j\in J} B_j$ in $\mathcal{A}$, called the *coefficient map*, and an $(I\times J)$-indexed family $f_{ij} : n_j.A_{ij} \to m_i.1$ of arrows in $\mathcal{A}$, called the *indexing maps*, where $A_{ij}$ stands for the pullback

$$\begin{array}{ccccc} A_{ij} & & \xrightarrow{\alpha_{ij}} & & B_j \\ \downarrow & \lrcorner & & & \downarrow inj \\ A_i & \xrightarrow[inj]{} \sum_{i\in I} A_i & \xrightarrow[\alpha]{} & \sum_{j\in J} B_j & . \end{array}$$

The identity on an object $(A_i, m_i)$ has as coefficient map the identity arrow of $\sum_{i\in I} A_i$ in $\mathcal{A}$. If $i \neq j$ then $A_{ij} \cong 0$, giving a unique choice for the indexing map; on the other hand $A_{ii} \cong A_i$ and for $i = j$ we take $m_i.! : m_i.A_i \to m_i.1$ as the indexing map.

Suppose now that we have objects $(A_i, m_i)_{i\in I}$, $(B_j, n_j)_{j\in J}$, and $(C_k, p_k)_{k\in K}$, which we shall call $A$, $B$, and $C$, respectively; also two arrows $f : A \to B$ and $g : B \to C$, where $f$ has coefficient map $\alpha : \sum_{i\in I} A_i \to \sum_{j\in J} B_j$ and indexing maps $f_{ij} : n_j.A_{ij} \to m_i.1$ as above, and $g$ has coefficient map $\beta : \sum_{j\in J} B_j \to \sum_{k\in K} C_k$ and indexing maps $g_{jk} : p_k.B_{jk} \to n_j.1$. The coefficient map of the composite is simply $\beta\alpha : \sum_{i\in I} A_i \to \sum_{k\in K} C_k$. Next form

the diagram of pullbacks $A_{ijk}$:

$$\begin{array}{ccccc}
C_k & \longrightarrow & \sum_{k\in K} C_k & & \\
{\scriptstyle\beta_{jk}}\uparrow & \lrcorner & \uparrow{\scriptstyle\beta_j} & & \\
B_{jk} & \longrightarrow & B_j & \longrightarrow & \sum_{j\in J} B_j \\
{\scriptstyle\alpha_{ijk}}\uparrow & \lrcorner & \uparrow{\scriptstyle\alpha_{ij}} & \lrcorner & \uparrow{\scriptstyle\alpha_i} \\
A_{ijk} & \longrightarrow & A_{ij} & \longrightarrow & A_i
\end{array} ,$$

where we write $\alpha_i$ for the composite of $\alpha$ with the injection $A_i \to \sum_{i\in I} A_i$. Now write $(g * f)_{ijk}$ for the map

$$p_k.A_{ijk} \xrightarrow{p_k.\binom{\alpha_{ijk}}{1}} p_k.B_{jk} \times A_{ijk} \xrightarrow{g_{ijk} \times inj} n_j.A_{ij} \xrightarrow{f_{ij}} m_i.1$$

and then the indexing maps of the composite arrow $gf$ are the $(g * f)_{ik}$ induced by the $(g * f)_{ijk}$ using the fact that $A_{ijk} \cong \sum_{j\in J} A_{ik}$.

**Proposition 7.3.1** *The above data make $\mathcal{A}[X]$ into a category.*

PROOF: Suppose $f$ and $g$ are as above, and $h = (\gamma, (h_{kl}))$ is an arrow in $\mathcal{A}[X]$ from $C$ to $D = (D_l, q_l)_{l\in L}$. We must show that $(hg)f = h(gf)$. Of course $(\gamma\beta)\alpha = \gamma(\beta\alpha)$, and so it remains to show that $((h * g) * f)_{il} = (h * (g * f))_{il}$.

Form the pullbacks

$$\begin{array}{ccccccc}
D_l & \longrightarrow & \sum_{l\in L} D_l & & & & \\
{\scriptstyle\gamma_{kl}}\uparrow & \lrcorner & \uparrow{\scriptstyle\gamma_k} & & & & \\
C_{kl} & \longrightarrow & C_k & \longrightarrow & \sum_{k\in K} C_k & & \\
{\scriptstyle\beta_{jkl}}\uparrow & \lrcorner & {\scriptstyle\beta_{jk}}\uparrow & \lrcorner & \uparrow{\scriptstyle\beta_j} & & \\
B_{jkl} & \longrightarrow & B_{jk} & \longrightarrow & B_j & \longrightarrow & \sum_{j\in J} B_j \\
{\scriptstyle\alpha_{ijkl}}\uparrow & \lrcorner & {\scriptstyle\alpha_{ijk}}\uparrow & \lrcorner & {\scriptstyle\alpha_{ij}}\uparrow & \lrcorner & \uparrow{\scriptstyle\alpha_i} \\
A_{ijkl} & \longrightarrow & A_{ijk} & \longrightarrow & A_{ij} & \longrightarrow & A_i
\end{array} ;$$

then the composite

$$q_l.A_{ijkl} \xrightarrow{q_l.\binom{\beta_{jkl}\alpha_{ijkl}}{\alpha_{ijkl}\ \ 1}} q_l.C_{kl}\times B_{jkl}\times A_{ijkl} \xrightarrow{h_{kl}\times inj\times inj} p_k.B_{jk}\times A_{ijk} \xrightarrow{g_{jk}\times inj} n_j.A_{ij} \xrightarrow{f_{ij}} m_i.1$$

induces a map from $q_l.A_{il}$ to $m_i.1$ which is equal to each of $(h*(g*f))_{il}$ and $((h*g)*f)_{il}$.

Thus the composition is associative. The identity law is immediate. □

**Remark 7.3.2** There is in fact a subtlety here that has not been dealt with specifically. To give the maps $f_{ij}$ it is not sufficient to know that pullbacks along injections exist in $\mathcal{A}$, one must *choose* specific objects to be those pullbacks. But now the object chosen becomes part of the data for the map $f_{ij}$, and unless we make some strong assumptions on compatibility of the choices made, the composition may now not be strictly associative. The solution is *not* to choose a specific object to be the desired pullback, but rather to give an $f_{ij}$ for each such choice, and require these $f_{ij}$ to satisfy the obvious compatibility conditions with respect to the pullback chosen. As the $f_{ij}$ for one choice determines that for all others, we need never do more than specify $f_{ij}$ for an arbitrary choice, and ignore the existence of the others.

In the case of commutative rings there is a homomorphism $R \to R[X]$ which takes an element of $R$ to the constant polynomial at that element. Similarly there is a functor $I : \mathcal{A} \to \mathcal{A}[X]$ which takes $A$ to the singleton family $(A, 0)$ and an arrow $f : A \to B$ to the pair $(f, 1_0)$. Functoriality is obvious.

**Proposition 7.3.3** *The category $\mathcal{A}[X]$ has, and the functor $I$ preserves, finite coproducts.*

Proof: The initial object in $\mathcal{A}[X]$ is the empty family, which is isomorphic to the singleton family $(0, 0)$ as the coproduct in $\mathcal{A}$ of the empty family of objects is the initial object of $\mathcal{A}$.

Given objects $A$ and $B$ as above, their coproduct $A + B$ is the disjoint union of the families $(A_i, m_i)_{i \in I}$ and $(B_j, n_j)_{j \in J}$, with the injection $i_A : A \to A + B$ as follows. The coefficient map is

$$\sum_{i \in I} A_i \to \sum_{i \in I} A_i + \sum_{j \in J} B_j$$

and then, writing $k$ for an arbitrary element of $I + J$ we have

$$A_{ik} \cong \begin{cases} A_i & \text{if } i = k \\ 0 & \text{otherwise} \end{cases}$$

and so we can take the indexing maps of the injection to be $m_i.! : m_i.A_i \to m_i.1$ and $! : 0 \to m_k.1$. The second injection is similar.

Preservation of coproducts by $I$ amounts to the fact that the arrows $(A_i, 0)_{i \in I} \to (\sum_{i \in I} A_i, 0)$, given by the injections in $\mathcal{A}$, are invertible. Their inverse has as coefficient map the identity on $\sum_{i \in I} A_i$ and trivial indexing maps. Alternatively one can deduce that $I$ preserve coproducts by the fact that it has a right adjoint; this takes a family $(A_i, m_i)_{i \in I}$ to $\sum_{i \in I} A_i$. □

By the proof of this last proposition the coproduct of $(A_i, m_i)_{i \in I}$ and $(A'_i, m_i)_{i \in I}$ can be formed as the disjoint union of the two families, but it can also be formed as $(A_i + A'_i, m_i)_{i \in I}$ in keeping with our intuition from the case of commutative rings.

**Proposition 7.3.4** *The category $\mathcal{A}[X]$ has, and the functor $I$ preserves, finite products.*

PROOF: The terminal object in $\mathcal{A}[X]$ is the singleton family $(1, 0)$; the functor $I$ then preserves the terminal object "on-the-nose".

Given again $A$ and $B$ their product is formed just as for the product of polynomials: it is $(A_i \times B_j, m_i + n_j)_{i \in I, j \in J}$. The first projection has coefficient map

$$\sum_{i \in I, j \in J} A_i \times B_j \to (\sum_{i \in I} A_i) \times (\sum_{j \in J} B_j) \to \sum_{i \in I} A_i$$

of a distributivity isomorphism and a projection in $\mathcal{A}$. We now must give an $(I \times J \times I)$-indexed family of indexing maps; these have domain 0 and so are uniquely determined except for those $(i, j, i')$ with $i = i'$, in which case the indexing maps are the composites

$$m_i.A_i \times B_j \xrightarrow{m_i.!} m_i.1 \xrightarrow{inj} (m_i + n_j).1$$

where $inj$ is the obvious injection.

This time perhaps a little justification is appropriate that this is indeed the product. Of course $\sum_{i \in I, j \in J} A_i \times B_j \cong (\sum_{i \in I} A_i) \times (\sum_{j \in J} B_j)$ and so insofar as the coefficient maps go the universal property is immediate. We concentrate therefore on the indexing maps. Suppose that we are given arrows in $\mathcal{A}[X]$ from $C$ to $A$ and $C$ to $B$ with coefficient maps $\alpha$ and $\beta$ and indexing maps $f_{ki} : m_i.C_{ki} \to p_k.1$ and $g_{kj} : n_j.C_{ji} \to p_k.1$. The induced arrow $C \to A \times B$ of course has coefficient map $\binom{\alpha}{\beta}$, and then the $C_{kij}$ can be formed as the pullbacks

$$\begin{array}{ccccc} C_k & \xrightarrow{\Delta} & C_k \times C_k & \xrightarrow{\binom{\alpha}{\beta}_k \times \binom{\alpha}{\beta}_k} & \sum_{i \in I} A_i \times \sum_{j \in J} B_j \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \uparrow \\ C_{kij} & \longrightarrow & C_{ki} \times C_{kj} & \xrightarrow[\alpha_{ki} \times \beta_{kj}]{} & A_i \times B_j \end{array}$$

using distributivity of $\mathcal{A}$, and commutativity of pullbacks with products, and then the indexing maps for $\binom{f}{g}$ can be constructed as

$$(m_i + n_j).C_{kij} \to m_i.C_{ki} + n_j.C_{kj} \to p_k.1$$

in the obvious way.

That $I$ preserves products follows immediately from distributivity. □

**Proposition 7.3.5** *$\mathcal{A}[X]$ is extensive, i.e. it is in object of* **2-Rig**; *thus $I : \mathcal{A} \to \mathcal{A}[X]$ is certainly an arrow of* **2-Rig**.

PROOF: We shall show that the functor

$$+ : \mathcal{A}[X] \times \mathcal{A}[X] \to \mathcal{A}[X]/(1+1)$$

is an equivalence. For this proof we shall write $E$ for this functor to avoid confusion.

The terminal object in $\mathcal{A}[X]$ is the singleton family $(1,0)$, and the coproduct of it with itself is the doubleton $(1,0)_{k\in 2}$. A map in $\mathcal{A}[X]$ from $(A_i, m_i)_{i\in I}$ to $(1,0)_{k\in 2}$ is just a coefficient map $\sum_{i\in I} A_i \to 1+1$; the indexing maps are all uniquely determined.

The functor $E$ takes a pair of objects $A_1 = (A_i, m_i)_{i\in I_1}$ and $A_2 = (A_i, m_i)_{i\in I_2}$ to the arrow from $(A_i, m_i)_{i\in I_1+I_2}$ to $(1,0)_{k\in 2}$ with coefficient map $!+! : \sum_{i\in I_1} A_i + \sum_{i\in I_2} A_i \to 1+1$. An arrow $a$ from $(A_i, m_i)_{i\in I}$ to $(1,0)_{k\in 2}$ is, once more, a map in $\mathcal{A}$ from $\sum_{i\in I} A_i$ to $1+1$; each of the component maps $A_i \to 1+1$ can be pulled back, using extensivity of $\mathcal{A}$, to give objects $U_i$ and $V_i$, with $A_i \cong U_i + V_i$. Thus we have objects $U = (U_i, m_i)_{i\in I}$ and $V = (V_i, m_i)_{i\in I}$ of $\mathcal{A}[X]$ such that $E(U,V) \cong a$, giving essential surjectivity of $E$. Full fidelity follows by the construction of $U$ and $V$ and extensivity of $\mathcal{A}$. □

The objects of $\mathcal{A}[X]$, and the products and coproducts there have been constructed so as to make clear that the Burnside rig of $\mathcal{A}[X]$ is the polynomial rig $\mathfrak{B}(\mathcal{A})[X]$, where $\mathfrak{B}(\mathcal{A})$ is the Burnside rig of $\mathcal{A}$. Just as in the case of polynomial rings, the polynomial 2-rig "freely adjoins" an object to a given 2-rig, in the sense of the following theorem.

**Theorem 7.3.6** *Given a map $F : \mathcal{A} \to \mathcal{B}$ of 2-rigs and an object $Y \in \mathcal{B}$, there is a 2-rig map $\overline{F} : \mathcal{A}[X] \to \mathcal{B}$, unique up to isomorphism, such that $FX = Y$ and $\overline{F}I \cong F$.*

PROOF: We have seen that an object $A = (A_i, m_i)_{i\in I}$ is the coproduct in $\mathcal{A}[X]$ of the singleton families $(A_i, m_i)$ as $i$ ranges through $I$. Moreover this can

be decomposed as a product, to give

$$(A_i, m_i)_{i\in I} \cong \sum_{i\in I}\Big((A_i, 0) \times (1,1)^{m_i}\Big) \cong \sum_{i\in I}\Big((A_i, 0) \times X^{m_i}\Big)$$

and so as $\overline{F}$ is to preserve finite products and coproducts it must take $A$ to $\sum_{i\in I}(FA_i \times Y^{m_i})$.

If $B = (B_j, n_j)_{j\in J}$ is another object and $f : A \to B$ an arrow of $\mathcal{A}[X]$, with coefficient map $\alpha : \sum_{i\in I} A_i \to \sum_{j\in J} B_j$ and indexing maps $f_{ij}$, then we must define an arrow

$$\overline{F}f : \sum_{i\in I}(A_i \times X^{m_i}) \to \sum_{j\in J}(B_j \times X^{n_j})$$

in $\mathcal{B}$. Decomposing $A_i$ as $\sum_{j\in J} A_{ij}$, it will suffice to give maps

$$A_{ij} \times X^{m_i} \to B_j \times X^{n_j}$$

for each $i$ and $j$; this in turn reduces to giving a map

$$A_{ij} \times X^{m_i} \to B_j,$$

for which we take the projection onto $A_{ij}$ followed by $\alpha_{ij}$, and $n_j$ maps

$$A_{ij} \times X^{m_i} \to X,$$

for which we take projections: the factor of $X^{m_i}$ on which to project is determined by the indexing map $f_{ij} : n_j.A_{ij} \to m_i.1$ which gives $n_j$ coproduct-decompositions of $A_{ij}$ into $m_i$ parts.

This $\overline{F}$ is functorial, and by construction is unique up to isomorphism. □

Now just as $\mathbb{N}$ is initial in the category of commutative rigs, so $\mathbf{Sets}_f$ is biinitial in **2-Rig**, and $\mathbf{Sets}_f[X]$ has the property that given a 2-rig $\mathcal{B}$ and an object $Y \in \mathcal{B}$, there is a unique (up to isomorphism) distributive functor from $\mathbf{Sets}_f[X]$ to $\mathcal{B}$ mapping $X$ to $Y$. Thus $\mathbf{Sets}_f[X]$ is the free 2-rig on

the trivial category **1**; but free 2-rigs can be constructed as in Section 4.4 by the "double-Fam" construction, and so in this case we get equivalences of categories

$$\mathbf{Sets}_f[X] \simeq Fam(\mathbf{Sets}_f) \simeq \mathbf{Sets}_f^{\mathbf{2}};$$

the second equivalence being well-known. On the other hand, returning to the universal property of $\mathcal{A}[X]$ described in the preceding theorem, we now see that $\mathcal{A}[X]$ is the bicoproduct of $\mathcal{A}$ and $\mathbf{Sets}_f[X]$, and, bicoproducts being tensor products, we have an equivalence

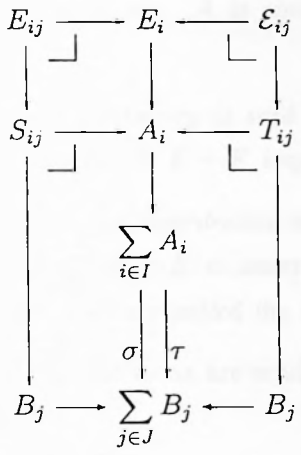
$$\mathcal{A}[X] \simeq \mathcal{A} \otimes \mathbf{Sets}_f[X].$$

**Proposition 7.3.7** *If $\mathcal{A}$ is lextensive then $\mathcal{A}[X]$ is lextensive and $I$ a lextensive functor; if also $\mathcal{B}$ is lextensive and $G : \mathcal{A}[X] \to \mathcal{B}$ a distributive functor, then $G$ preserves equalizers if $GI$ does.*

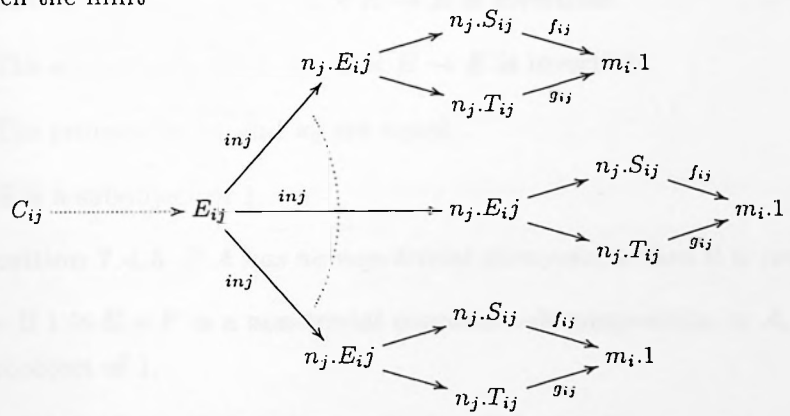
PROOF: Suppose that $A = (A_i, m_i)_{i \in I}$ and $B = (B_j, n_j)_{j \in J}$ are objects of $\mathcal{A}[X]$, and $f$ and $g$ are arrows from $A$ to $B$ with coefficient maps $\sigma$ and $\tau$ and indexing maps $f_{ij}$ and $g_{ij}$, having domains $S_{ij}$ and $T_{ij}$; these $S_{ij}$ and $T_{ij}$ are summands of $A_i$. Form the equalizer

$$E_i \cdots\cdots\cdots\!\!\rightarrow A_i \longrightarrow \sum_{i \in I} A_i \underset{\tau}{\overset{\sigma}{\rightrightarrows}} \sum_{j \in J} B_j$$

and then we can form pullbacks




and then the limit




The equalizer is now $(C_{ij}, m_i)_{i\in I, j\in J}$ with the evident injection. □

## 7.4 Connected 2-rigs and simple 2-rigs

The following definitions all make sense in an arbitrary distributive category, but we shall see that they are generally more useful in the context of 2-rigs.

**Definition 7.4.1** *A distributive category is said to be* simple *if it has no non-trivial product decompositions, i.e. $\mathcal{A}$ is simple if $\mathcal{A} \simeq \mathcal{B} \times \mathcal{C}$ implies $\mathcal{B} \simeq 1$ or $\mathcal{C} \simeq 1$.*

**Definition 7.4.2** *A distributive category is said to be* connected *if its terminal object 1 is connected, i.e. if $1 \cong E + F$ implies $E \cong 0$ or $F \cong 0$.*

**Definition 7.4.3** *An object $E$ of a distributive category is said to be* idempotent *if the diagonal map $E \to E \times E$ is invertible. Initial and terminal objects are always idempotent, and are called the trivial idempotents.*

**Remark 7.4.4** The following conditions are equivalent.

- $E$ is idempotent.
- The first projection $\pi_1 : E \times E \to E$ is invertible.
- The second projection $\pi_2 : E \times E \to E$ is invertible.
- The projections $\pi_1$ and $\pi_2$ are equal.
- $E$ is a subobject of 1.

**Proposition 7.4.5** *If $\mathcal{A}$ has no non-trivial idempotents then it is connected.*

PROOF: If $1 \cong E + F$ is a non-trivial coproduct-decomposition in $\mathcal{A}$, then $E$ is a subobject of 1. □

**Proposition 7.4.6** *If $\mathcal{A}$ is connected then it is simple.*

PROOF: Suppose that $\mathcal{A} \simeq \mathcal{B} \times \mathcal{C}$ is a non-trivial product decomposition of $\mathcal{A}$. Then, writing $E$ and $F$ for the image under this isomorphism of $(1_\mathcal{B}, 0_\mathcal{C})$ and $(0_\mathcal{B}, 1_\mathcal{C})$ respectively, we have a non-trivial coproduct decomposition $1_\mathcal{A} \cong E + F$. □

If we restrict to the case of 2-rigs then the converse of this last result is true.

**Proposition 7.4.7** *For a 2-rig $\mathcal{A}$ the following conditions are equivalent:*

*(i) The distributive functor* $\mathbf{Sets}_f \to \mathcal{A}$, *always faithful for an extensive category, is full.*

*(ii) $\mathcal{A}$ is connected.*

*(iii) $\mathcal{A}$ is simple.*

PROOF: A non-trivial decomposition $1 \cong E + F$ gives maps $1 \to 1 + 1$ other than the injections, *i.e.* not in the image of $\mathbf{Sets}_f$. Conversely a map $n \to m$ in $\mathcal{A}$ not in the image of $\mathbf{Sets}_f$ implies the existence of a map $1 \to m$ in $\mathcal{A}$ not in the image of $\mathbf{Sets}_f$, and so a non-trivial coproduct decomposition of 1. Thus $(i)$ is equivalent to $(ii)$.

We have already seen that $(ii)$ implies $(iii)$. Conversely if $1 \cong E + F$ then consider the composite

$$\mathcal{A} \longrightarrow \mathcal{A}/1 \longrightarrow \mathcal{A}/(E+F) \longrightarrow \mathcal{A}/E \times \mathcal{A}/F \ ,$$

where the first two maps are the evident isomorphisms and the third is the is the extensivity equivalence. As $E$ and $F$ are subobjects of 1, pullbacks under $E$ or $F$ are just products, and so $\mathcal{A}/E$ and $\mathcal{A}/F$ are still 2-rigs, and we have exhibited $\mathcal{A}$ as a non-trivial product of 2-rigs. □

## 7.5 Ideals and kernels

We shall define ideals for arbitrary distributive categories, but the concept seems to be most useful in the presence of all finite limits.

**Definition 7.5.1** *A class $\Sigma$ of arrows in a distributive category $\mathcal{D}$, will be called an* ideal *if it satisfies the following conditions*

- *The isomorphisms are in $\Sigma$*

- *If any two sides of a commutative triangle are in $\Sigma$ then so is the third*
- *$\Sigma$ is stable under pullback by arbitrary maps in $\mathcal{D}$*
- *$\Sigma$ is closed under finite coproducts, in the sense that if $\sigma : A \to B$ and $\sigma' : A' \to B'$ are in $\Sigma$ then so too is $\sigma + \sigma' : A + A' \to B + B'$.*

If $\mathcal{D}$ has equalizers, and hence all finite limits, then an ideal is precisely a pullback congruence closed under finite coproducts.

**Proposition 7.5.2** *(i) If $\mathcal{D}$ is a distributive category and $\Sigma$ is an ideal of $\mathcal{D}$ then $\mathcal{D}[\Sigma^{-1}]$ is distributive and $P_\Sigma$ is a distributive functor with kernel equal to $\Sigma$.*

*(ii) If $\mathcal{D}$ is a locally distributive category and $\Sigma$ an ideal of $\mathcal{D}$ then $\mathcal{D}[\Sigma^{-1}]$ is locally distributive and $P_\Sigma$ a morphism of locally distributive categories, i.e. a functor which preserves finite limits and finite coproducts; if $\mathcal{D}$ is not just locally distributive but also lextensive then $\mathcal{D}[\Sigma^{-1}]$ too is lextensive.*

*(iii) If $\mathcal{D}$ is a distributive category with equalizers then a collection $\Sigma$ of arrows in $\mathcal{D}$ is an ideal if and only if there is another distributive category $\mathcal{D}'$, and a distributive functor from $\mathcal{D}$ to $\mathcal{D}'$ which preserves equalizers and has $\Sigma$ as its kernel.*

PROOF:

(i) The distributive categories are the algebras for a strongly-finitary 2-monad on **Cat**, and $\Sigma$ contains the identities, and is closed under coproducts and products — closed under products because it is pullback stable and closed under composition — and so $\mathcal{D}[\Sigma^{-1}]$ is distributive and $P_\Sigma$ a distributive functor. The argument of [3, Theorem 1.4] depends only on the existence of pullbacks along arrows of $\Sigma$ and so it holds here, giving $Ker(P_\Sigma) = \Sigma$.

(ii) If $\mathcal{D}$ is locally distributive then $\mathcal{D}[\Sigma^{-1}]$ has, and $P_\Sigma$ preserves, finite products and finite coproducts, by (i), and equalizers by Proposition 5.2.7. That $\mathcal{D}[\Sigma^{-1}]$ is locally distributive then follows as $\mathcal{D}$ is locally distributive, $P_\Sigma$ preserves finite coproducts and finite limits, and is bijective on objects. Similarly disjointness of coproducts, and so extensivity, is inherited by $\mathcal{D}[\Sigma^{-1}]$ from $\mathcal{D}$.

(iii) If $\Sigma$ is an ideal then by (ii) it is the kernel of such a functor. Conversely if $\Sigma$ is the kernel of any functor then it contains the isomorphisms and satisfies the "commutative triangle condition" for an ideal. If it is moreover the kernel of a pullback-preserving functor then it must be stable under pullback and if it is the kernel of a distributive functor then it must be closed under finite coproducts. □

Thus the 2-categories **LocDist** and **Lext** have coinverter-conservative factorizations; in fact these 2-categories are *conservational bicategories* in the sense of [8].

## 7.6 Modules

Once more we make the definition for an arbitrary distributive category, although some of the "nice results" that one would like only hold in more restricted cases.

**Definition 7.6.1** *If $\mathcal{D}$ is a distributive category then a* left $\mathcal{D}$-module *is a finite-coproduct-preserving functor $M : \mathcal{A} \to \mathcal{D}$ such that each of the induced functors $M_X : \mathcal{A}/X \to \mathcal{D}/MX$ has a fully faithful finite-coproduct-preserving right adjoint.*

This is precisely a fibration from **1** to $\mathcal{D}$ in the bicategory **Sums**, in the sense of [38]; that is to say that it is the bicategorical version of a fibration for

which the fibration preserves finite coproducts, and given a choice of cleavage functors, they do so too.

Given such an $M$, one gets a bilinear functor $\mathcal{D} \times \mathcal{A} \to \mathcal{A}$, and so a finite-coproduct-preserving functor $\mathcal{D} \otimes \mathcal{A} \to \mathcal{A}$, satisfying the usual associativity and unit conditions. This can be constructed as follows: starting with a pair of objects $D \in \mathcal{D}$ and $A \in \mathcal{A}$, one forms the product of $MA$ and $D$, and then domain of the cartesian lifting of the projection $D \times MA \to D$ is taken to be $D.A$.

**Proposition 7.6.2** *If $\mathcal{D}$ is locally distributive then*

$$\mathcal{D}^2 \xrightarrow{\partial_1} \mathcal{D}$$

*is a left $\mathcal{D}$-module.*

PROOF: This is well-known to be a fibration; of course $\partial_1$ always preserves finite coproducts; that the cleavages do is just stability of coproducts by pullback. □

**Proposition 7.6.3** *If $\mathcal{D}$ is locally distributive and $\Sigma$ is an ideal of $\mathcal{D}$ then*

$$\Sigma \hookrightarrow \mathcal{D}^2 \xrightarrow{\partial_1} \mathcal{D}$$

*is a sub-$\mathcal{D}$-module of $\partial_1 : \mathcal{D}^2 \to \mathcal{D}$.*

PROOF: It preserves finite coproducts as $\Sigma$ is closed under finite coproducts; it is stable under pullback and so is a fibration; the cleavages still preserve finite coproducts. □

**Proposition 7.6.4** *If $\mathcal{D}$ is locally distributive, then*

$$\mathcal{D}^2 \xrightarrow{+} \mathcal{D}$$

*is a left $\mathcal{D}$-module.*

PROOF: The cleavages are given by pulling back along the injections. □

In the case of modules over a commutative ring, to give a $\mathbb{N}$-module is just to give an abelian group; here, however the analogous result is not quite true. The $\mathbf{Sets}_f$-modules, up to equivalence, are the functors

$$Fam(\mathcal{X}) \xrightarrow{\pi_0} \mathbf{Sets}_f$$

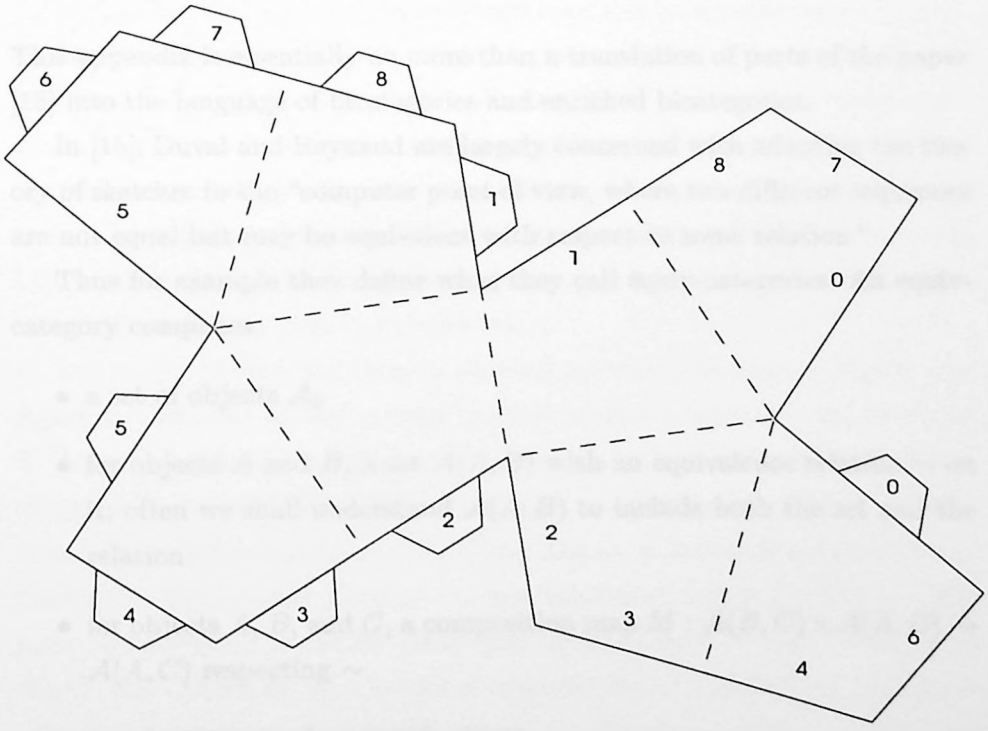
which forget the data associated with $\mathcal{X}$, and only remember the indexing families and reindexing functions.

One can now, following [38], define left $\mathcal{D}$- right $\mathcal{C}$-modules to be fibrations from $\mathcal{C}$ to $\mathcal{D}$ in the bicategory **Sums**; *i.e.* finite-coproduct-preserving functors $\binom{M}{N} : \mathcal{A} \to \mathcal{D} \times \mathcal{C}$ such that each of the induced functors $M_X \times N_X : \mathcal{A}/X \times X/\mathcal{A} \to \mathcal{D}/MX \times NX/\mathcal{C}$ has a fully faithful finite-coproduct-preserving right adjoint. Likewise one could define algebras to be fibrations in the bicategory **Dist**. Also important, of course will be the fibrations in **Ext**, **2-Rig**, and the various other related 2-categories.

# Appendix A

# Emma polyhedra

Cut along the straight lines, fold along the dotted lines, and join up the tabs.



# Appendix B

# Locally bidiscrete bicategories and equiv-categories

This appendix is essentially no more than a translation of parts of the paper [15] into the language of bicategories and enriched bicategories.

In [15], Duval and Reynaud are largely concerned with adapting the theory of sketches to the "computer point of view, where two different sequences are not equal but may be equivalent with respect to some relation."

Thus for example they define what they call equiv-categories. An equiv-category comprises

- a set of objects $\mathcal{A}_0$
- for objects $A$ and $B$, a set $\mathcal{A}(A,B)$ with an equivalence relation $\sim$ on it; often we shall understand $\mathcal{A}(A,B)$ to include both the set and the relation
- for objects $A$, $B$, and $C$, a composition map $M : \mathcal{A}(B,C) \times \mathcal{A}(A,B) \to \mathcal{A}(A,C)$ respecting $\sim$
- an identity map $1_A$ on each object

such that the associativity and identity laws hold up to $\sim$.

Such a structure is precisely what one might call a *locally bidiscrete bicategory*, following [38] in calling a category bidiscrete if it is posetal and a groupoid, and [2] in saying that a bicategory has a property locally if each of its hom-categories has the property. For an equiv-category is simply a bicategory with at most one 2-cell between given 1-cells and with every 2-cell invertible. The theory of locally posetal bicategories, and in particular the theory of locally bidiscrete bicategories is much simpler than the usual theory of bicategories, as we need not worry about coherence conditions.

Duval and Reynaud then go on to consider functors, products and coproducts, distributivity, sketches and so on, in each case replacing the classical version by a corresponding bicategorical one; all in the context of locally bidiscrete bicategories, so that coherence need not be considered.

In fact apart from describing this program as "developing the bicategorical version of sketches in the restricted case of locally bidiscrete bicategories", there is another (less elementary) description possible, already suggested by their terminology. For one can view the restriction from bicategories to locally bidiscrete bicategories rather as transferring attention from the **Cat**-bicategories to the **Equiv**-bicategories, *i.e.* bicategories enriched in the monoidal bicategory **Equiv** defined below.

Let us define then the (locally posetal) monoidal bicategory **Equiv**. An object is a set with an equivalence relation (almost always denoted by $\sim$) on it. A 1-cell is a function between the two sets which respects the equivalence relation. There is an (invertible) 2-cell between 1-cells $f, g : X \to Y$ exactly when $fx \sim gx$ for all $x \in X$. Composition is ordinary composition of functions. This clearly gives not just a bicategory but a 2-category. The monoidal structure is the cartesian one: given objects $X$ and $Y$ one forms $X \times Y$ and defines $(x, y) \sim (x', y')$ if and only if $x \sim y$ and $x' \sim y'$. We have associativity *isomorphisms* satisfying strictly the usual coherence conditions,

and so **Equiv** is in fact a 2-monoidal 2-category — a monoid object in the monoidal category **2-Cat**.

Even more is true: **Equiv** is monoidal closed, and not just in the bicategorical sense of the first three chapters, but in the strict 2-categorical sense. The closed structure is given by: $[X, Y]$ is the set of maps from $X$ to $Y$ in **Equiv**; two such are equivalent in $[X, Y]$ just when they are equivalent in **Equiv** itself. Thus **Equiv** can be made into an **Equiv**-bicategory, where the hom-objects are given by $\mathbf{Equiv}(X, Y) = [X, Y]$. Of course **Equiv** is just the intersection in **Cat** of its sub-2-categories comprising the preorders and the groupoids; each of these is welll-known to be closed under the cartesian closed structure of **Cat**.

Now an **Equiv**-bicategory in the sense of Chapter 3 is precisely an equiv-category in the sense of [15], which is to say a locally bidiscrete bicatgory.

Similarly they define a functor between **Equiv**-categories to be what according to the terminology of Chapter 3 would be called a normal **Equiv**-homomorphism, but amounts simply to a normal homomorphism of locally bidiscrete bicategories. Their equiv-products and equiv-coproducts are likewise simply bi-products and bi-coproducts (particular instances of bilimits and bicolimits) although the conditions for such are again simplified by local bidiscreteness of the bicategories involved.

In the case of ordinary categories, products (and dually coproducts) are determined up to (a unique) isomorphism; sometimes, however, one wishes to have *chosen* products, where we of course mean that both the object and the projections are chosen. For a category $C$ to possess (binary) products is exactly for the diagonal functor $\Delta : C \to C \times C$ to possess a right adjoint; to choose the products is then to choose an adjunction. Of course even when the products have been chosen, one does not in general expect product preserving functors to respect this choice; the product preserving functors in real life do so only up to isomorphism.

The case of biproducts (and dually bicoproducts) in bicategories is similar but slightly more complicated. Biproducts are determined up to equivalence; again one sometimes wants to have them chosen once and for all. However in the case of biproducts, the factorization of a cone through a product is itself only determined up to isomorphism, and once more one sometimes wants to have these factorizations "chosen". As with the classical case however, existence of biproducts corresponds to the existence of a right *biadjoint*, and a given choice of biproducts, projections, and factorizations corresponds to a chosen biadjunction. The natural maps to consider in this case are those that preserve the products up to equivalence. The interest in *chosen* products and factorizations is in keeping with the "computer point of view" of [15]. Similarly when considering bidistributive bicategories, as defined in Section 7.2, they ask that the inverse equivalence to the canonical distributivity maps $\delta$ be chosen.

Wolff [43] defined $\mathcal{V}$-graphs for a symmetric monoidal closed category $\mathcal{V}$. The special case where $\mathcal{V}$ is the cartesian closed category **Cat** was treated in [20] under the name of CE-graphs (CE for category enriched). Graphs, enriched or otherwise, can be thought of as giving generators for a categorical structure, and in [43] the free $\mathcal{V}$-category on a $\mathcal{V}$-graph is constructed. CE-graphs are a way of giving generators, equally for 2-categories or for bicategories — as there is as yet no composition, there is no distinction to be made between the two. One also wishes to consider *relations*, which desire gives rise to the *compositive graphs*. There are a number of (very similar) possible notions here; one can choose to look at such a structure as a category with composition only partially defined, or a graph with partial composition, or a graph with equations, or a system of generators and relations for a category. In the formulation of [15], an *equiv-compositive graph* is a graph with identities, with a family $Compose(S,T)$ of pairs of paths in S with domain $S$ and codomain $T$, for each pair of objects $S$ and $T$. For such a pair of paths

they write

$$s_m \circ \ldots \circ s_1 \circ s_0 \sim t_n \circ \ldots \circ t_1 \circ t_0$$

and call it a composition relation. For every arrow $s : S \to T$ they ask that the *identity composition relations* $s \sim s \circ 1_S$ and $s \sim 1_T \circ s$ be among the composition relations $Compose(S, T)$. Functors are defined not just for equiv-categories but for equiv-graphs as the obvious extension of normal homomorphism to the case of graphs. They identify an equiv-category with the underlying equiv-compositive graph for which $Compose(S, T)$ comprises all pairs of paths from $S$ to $T$ whose composites are isomorphic in the bicategory. On the other hand they identify a *category* with the equiv-compositive graph whose composition relations are just those of the form $t.s \sim ts$.

A *finite discrete cone* in an equiv-compositive graph $\mathbf{S}$ is a finite family of arrows $\sigma_i : S \to S_i$ in $\mathbf{S}$; the object $S$ is said to be the vertex and the set of the $S_i$ to be the base of the cone. All cones considered are finite and discrete and henceforth will be called simply cones. Dually there are (finite discrete) cocones.

Now an equiv-sketch (understood to be finite and discrete) is a triple $/\!/\mathbf{S}/\!/ = (\mathbf{S}, \mathcal{P}, \mathcal{I})$ where $\mathbf{S}$ is an equiv-compositive graph, $\mathcal{P}$ is a set of cones in $\mathbf{S}$, and $\mathcal{I}$ a set of cocones. A model of the sketch is a functor from $\mathbf{S}$ to (the equiv-compositive graph identified with) the category **Set** such that the cones of $\mathcal{P}$ become products in **Set**, and the cocones of $\mathcal{I}$ become coproducts in **Set**.

A projective equiv-sketch is an equiv-sketch for which $\mathcal{I}$ is empty. This is a bicategorical version of the projective sketches of [16]. In the categorical version, there is then a general result [16, 12, 13] that every projective sketch induces a lim-theory with the same models; in the special case of finite discrete projective sketches this conatians as a further special case the free category on a multigraph of [42], where Walters proposes a precise 2-dimensional universal property of this construction; projective sketches generalize multi-

graphs to allow putting relations on the data.

When we move to (locally bidiscrete) bicategories, a finite projective equiv-sketch still "generates" a bicategory with finite biproducts, in the sense that the evident forgetful functor has a left biadjoint, but as it is a *bi*adjoint rather than a genuine one, the bicategory is determined only up to equivalence rather than isomorphism. Duval and Reynaud define a *substitution category* to be "a projective sketch such that the sublying [equiv-]graph is an equiv-category with chosen factorizations and each distinguished cone is an equiv-product" [15, p. 205], which appears to mean a locally bidiscrete bicategory with finite biproducts, with some but not all of the biproduct structure being chosen, possibly non-uniquely. The point of these complications seems to ensure that a projective sketch "generate" substitution categories up to isomorphism rather than just equivalence, but as it is never made precise what is meant by "generate", this is not entirely clear.

The general (finite discrete) equiv-sketches, for which $\mathcal{I}$ need not be empty are of course a bicategorical version of finite discrete sketches; they include distributive graphs as the case where there are "no relations"; we might call them (locally bidiscrete) bidistributive graphs. An *evaluation category* is defined to be a bidistributive locally bidiscrete bicategory, with all of this structure "chosen", and an equiv-sketch generates an evaluation category; once more in the case of categories rather than bicategories this contains as a special case the free distributive category on a distributive graph.

The existence of the free distributive category on a distributive graph with equations (i.e. the distributive category generated by a finite discrete sketch) means that one doesn't lose descriptive power by restricting attention to finite discrete sketches which are in fact distributive categories. The existence of the extensive reflection of a distributive category means that one doesn't lose descriptive power by further restricting to 2-rigs. One can similarly restrict attention to those equiv-sketches which are evaluation categories, and even

further to those which are biextensive — the natural bicategorical version of the notion of extensivity.

The key definition of [15] is that of a quasi-projective equiv-sketch; these turn out to be more or less those sketches for which there is an initial model. An equiv-sketch $/\!/\mathbf{S}/\!/$ is quasi-projective if the morphism

$$/\mathbf{S}/ \xrightarrow{sub} Subst(/\mathbf{S}/) \xrightarrow{Hom(1,-)} \mathbf{Set}$$

is a model of $/\mathbf{S}/$, the underlying projective equiv-sketch of $/\!/\mathbf{S}/\!/$, obtained by simply removing $\mathcal{I}$, and where $sub : /\mathbf{S}/ \to /\!/\mathbf{S}/\!/$ is the inclusion of $/\mathbf{S}/$ in the substitution category it generates. In terms of bicategories, one might say that a bidistributive graph $\mathcal{G}$ is quasi-projective if the composite

$$\mathcal{G} \xrightarrow{\eta} F\mathcal{G} \xrightarrow{F\mathcal{G}(1,-)} \mathbf{Set}$$

maps the designated cones and cocones to biproducts and bicoproducts; here $F\mathcal{G}$ is the free bidistributive bicategory on $\mathcal{G}$, and $\eta$ is the unit of the bi-adjunction. Now this $\eta$ certainly turns the designated cones and cocones to biproducts and bicoproducts, and so the condition amounts to $F\mathcal{G}(1,-)$ preserving biproducts and bicoproducts; of course it preserves biproducts, being birepresentable, and so the condition reduces to $F\mathcal{G}(1,-)$ preserving bicoproducts.

We have in the case of an extensive category $\mathcal{E}$ that $\mathcal{E}(1,-)$ preserves coproducts if and only if 1 is connected and similarly we can view quasi-projectivity as being the condition that the biextensive evaluation category generated by an equiv-sketch be biconnected.

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