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THE EVOLUTION OF ...

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The Genesis of the Abstract Ring Concept

Israel Kleiner

Algebra textbooks usually give the definition of a ring first and follow it with examples. Of course, the examples came first, and the abstract definition later—much later. So we begin with examples.

Among the most elementary examples of rings are the integers, polynomials, and matrices. "Simple" extensions of these examples are at the roots of ring theory. Specifically, we have in mind the following three examples:

(a) The integers Z can be thought of as the appropriate subdomain of the field Q of rationals in which to do number theory. (The rationals themselves are unsuitable for that purpose: every rational is divisible by every other (nonzero) rational.) Take a simple extension field $Q(\alpha)$ of the rationals, where α is an algebraic number, that is, a root of a polynomial with integer coefficients. $Q(\alpha)$ is called an algebraic number field; it consists of polynomials in α with rational coefficients (e.g., $Q(\sqrt{3}) = \{a + b\sqrt{3} : a, b \in Q\}$). The appropriate subdomain of $Q(\alpha)$ in which to do number theory—the "integers" of $Q(\alpha)$—consists of those elements that are roots of *monic* polynomials with integer coefficients (the integers of $Q(\sqrt{3})$ are $\{a + b\sqrt{3} : a, b \in Z\}$). This is our first example.

(b) The polynomial rings $\mathbb{R}[x]$ and $\mathbb{R}[x, y]$ in one and in two variables, respectively, share important properties but also differ in significant ways. In particular, while the roots of a polynomial in one variable constitute a discrete set of real numbers, the roots of a polynomial in two variables constitute a curve in the plane—a so-called algebraic curve. Our second example, then, is the ring of polynomials in two (or more) variables.

(c) Square $m \times m$ matrices (for example, over the reals) can be viewed as m^2-tuples of real numbers with coordinate-wise addition and appropriate multiplication obeying the axioms of a ring. Our third example consists, more generally, of n-tuples $\mathbb{R}^n$ of real numbers with coordinate-wise addition and appropriate multiplication, so that the resulting system is a (not necessarily commutative) ring. Such systems are extensions of the complex numbers—in the 19th and early 20th centuries they were called hypercomplex number systems.

In what contexts did these examples arise? What was their importance? The answers will lead us to the genesis of the abstract ring concept.

The abstract ring concept emerged in the context of a theory—in fact, in the context of two theories: commutative ring theory and noncommutative ring theory.

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The abstract theories of these two categories came from distinct sources and developed in different directions. Commutative ring theory originated in algebraic number theory and algebraic geometry. Central to the development of these subjects were, respectively, the rings of integers in algebraic number fields and the rings of polynomials in two or more variables. Noncommutative ring theory began with attempts to extend the complex numbers to various hypercomplex number systems. We consider first the evolution of the "simpler" theory of noncommutative rings.

A. NONCOMMUTATIVE RING THEORY. In a strict sense, noncommutative ring theory originated from a single example—the quaternions—invented (discovered?) by Hamilton in 1843. These are "numbers" of the form $a + bi + cj + dk$ (a, b, c, d real numbers) that are added componentwise and in which multiplication is subject to the relations $i^2 = j^2 = k^2 = ijk = -1$. This was the first example of a noncommutative number system, obeying all the (algebraic) laws of the real and complex numbers except for commutativity of multiplication. Such a system is now called a skew field or a division algebra. Hamilton's motivation was to define an algebra of vectors in 3-space so that multiplication would represent composition of rotations (just as multiplication of complex numbers represents composition of rotations in the plane). Having failed in this task, he turned to quadruples of reals and created the algebra of quaternions. The "pure" quaternions did, in fact, yield the required computing tool for rotations in 3-space.

Examples. Hamilton's invention of the quaternions was conceptually groundbreaking, but like all revolutions, it was initially received with less than universal approbation. Most mathematicians, however, soon came around to Hamilton's point of view. The quaternions acted as a catalyst for the exploration of diverse "number systems", with properties that departed in various ways from those of the real and complex numbers. Among the examples of such hypercomplex number systems are octonions, exterior algebras, group algebras, matrices, and biquaternions. See [6].

Structure. The first example of a noncommutative algebra was given by Hamilton in 1843. During the next forty years mathematicians introduced other examples, began to bring some order into them, and singled out certain types for special attention. For example, Frobenius and C. S. Peirce showed in 1880 that the reals, the complex numbers, and the quaternions are the only finite-dimensional (associative) division algebras over the reals. The stage was set for the founding of a general theory of finite-dimensional, noncommutative, associative algebras (important examples of rings).

In the 1890s, Cartan, Frobenius, and Molien proved (independently) the following fundamental structure theorem for finite-dimensional semi-simple algebras over the real or complex numbers: Any such algebra is a finite unique direct sum of simple algebras. These, in turn, are isomorphic to matrix algebras with entries in division algebras. An algebra is "semi-simple" if it has no nontrivial nilpotent ideals and it is "simple" if it has no nontrivial ideals.

In 1907, Wedderburn extended that result to algebras over arbitrary fields. This was no mere generalization as it necessitated the introduction of such fundamental algebraic concepts as ideal, quotient algebra, nilpotent algebra, radical, semi-simple and simple algebra, direct sum, and tensor product. Wedderburn's theorem is one of the basic results in ring theory, and it serves as a model for many

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ring-theoretic structure theorems. It is also central in group-representation theory. See [6].

B. COMMUTATIVE RING THEORY. *Commutative* ring theory originated in algebraic number theory and algebraic geometry and has in turn been applied mainly to these two subjects. Invariant theory, with roots in both number theory and geometry, also had a role in these developments.

Algebraic number theory. Several of the central areas of number theory, principally Fermat's Last Theorem, binary quadratic forms, and reciprocity laws, were instrumental in the emergence of algebraic number theory. Although the key problems in these areas were expressed in terms of integers, it gradually became apparent that the solutions called for embedding the integers in domains of what came to be known as algebraic integers. The following examples give an idea of what is involved.

(i) To show that $x^3 + y^3 = z^3$ has no nonzero integer solutions, factor the left side as $(x + y)(x + yw)(x + yw^2) = z^3$, $w^3 = 1$, $w \neq 1$. This is now an equation in the domain $D_3 = \{a + bw\colon a, b \in Z\}$. Assuming the existence of a solution in D_3 one can arrive at a contradiction, showing in particular that $x^3 + y^3 = z^3$ has no solutions. A similar approach turned out to be fruitful in the general case of Fermat's Last Theorem, $x^p + y^p = z^p$. See [5].

(ii) A conceptual way to determine which integers are sums of two squares is to factor the right side of $n = x^2 + y^2$ and consider the equation $n = (x + yi)(x - yi)$ in the domain $G = \{a + bi\colon a, b \in Z\}$ of Gaussian integers. (Gauss introduced G in order to *state* the biquadratic reciprocity law.) This problem is but an instance of the problem of representing integers by binary quadratic forms $ax^2 + bxy + cy^2$ $(a, b, c \in Z)$. The general approach is to factor $ax^2 + bxy + cy^2$ and to consider the resulting equation in a domain of "complex integers". See [1].

(iii) The diophantine equation $x^2 + 2 = y^3$ is a special case of the famous Bachet equation $x^2 + k = y^3$. It was in the margins of Bachet's Latin translation of Diophantus' *Arithmetica* that Fermat made his famous remark about the equation $x^n + y^n = z^n$. While the general Bachet equation is still a topic of intensive investigation, the equation $x^2 + 2 = y^3$ can readily be solved using "complex integers". It is easy to see that $x = \pm 5$, $y = 3$ are solutions. To find *all* solutions, we write $x^2 + 2 = y^3$ as $(x + \sqrt{2}\,i)(x - \sqrt{2}\,i) = y^3$. This is now an equation in the domain $D = \{a + b\sqrt{2}\,i\colon a, b \in Z\}$. We can show that D is a unique factorization domain (UFD) and that $x + \sqrt{2}\,i$ and $x - \sqrt{2}\,i$ are relatively prime in D. Since their product is a cube, each factor must be a cube (in D). In particular, $x + \sqrt{2}\,i = (a + b\sqrt{2}\,i)^3$, $a, b \in Z$. Cubing and then equating coefficients, we can easily show that $x = \pm 5$, $y = 3$ are the *only* solutions of $x^2 + 2 = y^3$—no easy feat to accomplish without the use of complex integers. See [1].

What is common to these examples? Additive problems in Z have been transformed to multiplicative problems in the domains D_3, G, D respectively (these domains are important examples of rings). In the latter settings the problems can be dealt with effectively provided that the domains in question are unique

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factorization domains (UFDs). Now, D_3, G, and D *are* UFDs, but the domains arising from the respective general problems (Fermat's Last Theorem, binary quadratic forms, the Bachet equation) are (as a rule) not. For example, $\{a_0 + a_1 w + a_2 w^2 + \cdots + a_{22} w^{22}: a_i \in Z, w$ a primitive 23rd root of unity$\}$, arising from the equation $x^{23} + y^{23} = z^{23}$, is not a UFD; neither is the domain $\{a + b\sqrt{5}\, i: a, b \in Z\}$ resulting from factoring the left side of the Bachet equation $x^2 + 5 = y^3$. The problem then becomes one of restoring, *in some sense*, unique factorization in such domains. Kummer dealt with it by means of ideal numbers, Dedekind by means of ideals, and Kronecker by means of divisors. We consider briefly Dedekind's contribution.

Dedekind's rings and ideals. The main result of Dedekind's groundbreaking 1871 work, which appeared as Supplement X of Dirichlet's *Vorlesungen über Zahlentheorie*, was that every nonzero ideal in the domain of integers of an algebraic number field is a unique product of prime ideals. Before one could state this theorem one had, of course, to define the concepts in its statement, namely "the domain of integers of an algebraic number field", "ideal", and "prime ideal". It took Dedekind about twenty years to formulate them.

Given an algebraic number field $Q(\alpha)$, all its elements are roots of polynomials with integer coefficients. Dedekind defined the *domain of integers* of $Q(\alpha)$ to be the subset of elements that are roots of *monic* polynomials with integer coefficients. This notion is an extension of the domain of integers (of Q), whose elements are the roots of monic *linear* polynomials. He showed that these elements "behave" like integers—they are closed under addition, subtraction, and multiplication. For example, the integers of $Q(\sqrt{5}) = \{a + b\sqrt{5}: a, b \in Q\}$ are $\{\frac{a + b\sqrt{5}}{2}: a, b \in Z, a \equiv b (\mathrm{mod}\, 2)\}$ rather than $\{a + b\sqrt{5} = a, b \in Z\}$, as seems perhaps more natural. See [1].

Having defined the domain R of algebraic integers of $Q(\alpha)$ in which he would formulate and prove his result on unique decomposition of ideals, Dedekind considered, more generally, sets of integers of $Q(\alpha)$ closed under addition, subtraction, and multiplication. He called these "orders". The domain R of integers of $Q(\alpha)$ is the largest order. Here, then, was an algebraic first for Dedekind—an essentially axiomatic definition of a (commutative) ring, albeit in a concrete setting. The *term* ring, also in the setting of domains of algebraic integers, was coined by Hilbert in 1897.

The second fundamental concept of Dedekind's theory, that of ideal, derived its motivation (and name) from Kummer's ideal numbers (see [2]). Dedekind defined it essentially as we do today. Having then defined the notion of prime ideal, he proved his fundamental theorem that every nonzero ideal in the ring of integers of an algebraic number field is a unique product of prime ideals. See [2].

How did Dedekind's theory relate to the number-theoretic problems (e.g., Fermat's Last Theorem, reciprocity laws, binary quadratic forms) from which it drew inspiration? It did shed important light on these problems and resolved special cases (see [1], [5]). But as often happens, the ideas Dedekind put forth acquired great significance independent of the original problems that stimulated their introduction. (Galois theory also far superseded in importance the problem of solution of equations that gave it birth.)

Algebraic geometry. Algebraic geometry is the study of algebraic curves and their generalizations to n dimensions, algebraic varieties. An algebraic curve is the set of

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roots of an algebraic function; that is, a function $y = f(x)$ defined implicitly by the polynomial equation $P(x, y) = 0$.

Several approaches were used in the study of algebraic curves, notably the analytic, the geometric-algebraic, and the algebraic-arithmetic. In the analytic approach, to which Riemann (in the 1850s) was the major contributor, the main objects of study were algebraic functions $f(w, z) = 0$ (of a complex variable) and their integrals, the so-called abelian integrals, which are closely related to the important notion of the genus of an algebraic curve. It was in this connection that Riemann introduced the fundamental notion of a Riemann surface, on which algebraic functions become single valued. Riemann's methods were brilliant but nonrigorous, and relied heavily on the physically obvious Dirichlet Principle, which was mathematically incorrect in its unrestricted form.

(i) Algebraic function fields. Dedekind and Weber, in their groundbreaking 1882 paper "Theory of algebraic functions of a single variable", proposed to "provide a basis for the theory of algebraic functions, the major achievement of Riemann's researches, in the simplest and at the same time rigorous and most general manner". The fundamental idea of their algebraic-arithmetic approach was to carry over to algebraic function fields the ideas that Dedekind had earlier introduced for algebraic number fields.

Just as an algebraic number field is a finite extension $Q(\alpha)$ of the field Q of rationals, so an algebraic function field is a finite extension $K = \mathbb{C}(z)(w)$ of the field $\mathbb{C}(z)$ of rational functions (in the indeterminate z). That is, w is a root of a polynomial $a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_n\alpha^n$, where $a_i \in \mathbb{C}(z)$ (we can take $a_i \in \mathbb{C}[z]$). Thus $w = f(z)$ is an algebraic function defined implicitly by the polynomial equation $P(z, w) = a_0 + a_1 w + a_2 w^2 + \cdots + a_n w^n = 0$. In fact, all elements of $K = \mathbb{C}(z, w)$ are algebraic functions.

Now let A be the "ring of integers" of K over $\mathbb{C}(z)$; that is, A consists of the elements of K that are roots of *monic* polynomials over $\mathbb{C}[z]$. As for algebraic numbers, here too every nonzero ideal of A is a unique product of prime ideals. Incidentally, in the case of the field of meromorphic functions on a Riemann surface, the role of the integers is played by the entire functions.

Dedekind and Weber were now ready to give a rigorous, algebraic definition of the Riemann surface S of the algebraic function field K: it is (in our terminology) the set of nontrivial discrete valuations on K. The finite points of S correspond to ideals of A; to deal with points at infinity of S Dedekind and Weber introduced the notions of "place" and "divisor". Many of Riemann's ideas about algebraic functions were here developed algebraically and rigorously. In particular, a rigorous proof was given of the important Riemann-Roch theorem. See [3].

Beyond Dedekind and Weber's technical achievements in putting major parts of Riemann's algebraic function theory on solid ground, their conceptual breakthrough lay in pointing to the strong analogy between algebraic number fields and algebraic function fields, hence between algebraic number theory and algebraic geometry. This analogy proved extremely fruitful for both theories. For example, the use of power series in algebraic geometry inspired Hensel in 1897 to introduce p-adic numbers ("power series" in the prime p). The resulting idea of p-adic completion proved important in both algebraic number theory and algebraic geometry. Another noteworthy aspect of Dedekind and Weber's work was its generality and applicability to arbitrary fields, in particular Q and Z_p, which were important in number-theoretic contexts. Thus, ideas from algebraic geometry could be applied to number theory.

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(ii) Polynomial rings and their ideals. Polynomial ideals in algebraic geometry had their implicit beginnings in M. Noether's work in the 1870s. Important advances were made by Kronecker in the 1880s and especially by Lasker and Macauley in 1905 and 1913, respectively.

The need for polynomial ideals in the study of algebraic varieties is manifest. An algebraic variety V is defined as the set of points in $\mathbb{R}^n$ (or $\mathbb{C}^n$) satisfying a system of polynomial equations $f_i(x_1, \ldots, x_n) = 0$, $i = 1, 2, 3, \ldots$. The Hilbert Basis Theorem implies that finitely many equations will do. But different systems of polynomial equations may give rise to the same set of roots. For example, the circle V in $\mathbb{R}^3$ of radius 2 lying in the plane parallel to the (x, y) plane and two units above it may be described as $V = \{(x, y, z)\colon\ x^2 + y^2 - 4 = 0,\ z - 2 = 0\}$, as $V = \{(x, y, z)\colon\ x^2 + y^2 + z^2 - 8 = 0,\ z - 2 = 0\}$, or as $V = \{(x, y, z)\colon\ x^2 + y^2 - 4 = 0,\ x^2 + y^2 - 2z = 0\}$. Is there a canonical set of polynomials that describes the variety (circle) V?

It is easy to see that if $f_1, \ldots, f_m$ are polynomials that vanish on the points of V, then so do all polynomials of the set $I = \{g_1 f_1 + \cdots + g_m f_m\colon\ g_i \in \mathbb{R}[x, y, z]\}$. But I is an ideal of the polynomial ring $\mathbb{R}[x, y, z]$. In fact, the set of *all* polynomials of $\mathbb{R}[x, y, z]$ that vanish on the points of V is also an ideal—and it is evidently the "canonical" set of polynomials to describe V.

Note that the preceding remarks point to a correspondence between ideals of $\mathbb{R}[x_1, \ldots, x_n]$ (or of $\mathbb{C}[x_1, \ldots, x_n]$) and varieties in $\mathbb{R}^n$ (or $\mathbb{C}^n$): If V is a variety, let $I(V) = \{f(x_1, \ldots, x_n) \in \mathbb{R}[x_1, \ldots, x_n]\colon\ f(a_1, \ldots, a_n) = 0 \text{ for all } (a_1, \ldots, a_n) \in V\}$, and if J is an ideal of $\mathbb{R}[x_1, \ldots, x_n]$, let $V(J) = \{(b_1, \ldots, b_n) \in \mathbb{R}^n\colon\ g(b_1, \ldots, b_n) = 0 \text{ for all } g \in J\}$. The Hilbert Nullstellensatz, in one of its incarnations, says that $V(J) \neq 0$ if the variety is in $\mathbb{C}^n$, or K^n for any algebraically closed field K. This correspondence is central in algebraic geometry. It is, in fact, a one-to-one correspondence between varieties (over an algebraically closed field K) and their largest defining ideals (the so-called radical ideals). Under this correspondence, prime ideals correspond to irreducible varieties (those that cannot be non-trivially decomposed into finite unions of other varieties). See [3].

Lasker and Macauley exploited this correspondence in the early 20th century by undertaking a thorough study of ideals in polynomial rings in order to shed light on algebraic varieties. Lasker's major result was the "primary decomposition" of ideals: Every ideal in a polynomial ring $F[x_1, \ldots, x_n]$ is a finite intersection of primary ideals. Primary ideals, first defined by Lasker, are generalizations of prime ideals; the former are to the latter what prime powers are to primes in the ring of integers. Translated into the language of algebraic geometry, the result says that every variety is a finite union of irreducible varieties. Macauley proved the uniqueness of the primary decomposition, which implied that every variety can be expressed uniquely as a union of irreducible varieties—a type of fundamental theorem of arithmetic for varieties. By the way, it is no easy matter to determine *geometrically* when a curve is irreducible; it is the algebra that comes to the geometer's aid here.

C. THE ABSTRACT DEFINITION OF A RING. In the first decade of the 20th century there were well-established, flourishing, concrete theories of both commutative and noncommutative rings and their ideals. Their roots were in algebraic number theory, algebraic geometry, and the theory of hypercomplex number systems. Moreover, abstract (axiomatic) definitions of groups, fields, and vector spaces had then been in existence for about two decades. The time was ripe for the abstract ring concept to emerge.

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The first abstract definition of a ring was given by Fraenkel (of set-theory fame) in a 1914 paper entitled "On zero divisors and the decomposition of rings" [4]. He defines a ring as "a system" with two (abstract) operations, to which he gives the names addition and multiplication. Under one of the operations (addition) the system forms a group (he gives its axioms). The second operation (multiplication) is associative and distributes over the first. Two axioms give the closure of the system under the operations, and there is the requirement of an identity in the definition of the ring. Commutativity under addition does *not* appear as an axiom but is proved! So are other elementary properties of a ring such as $a \times 0 = 0$, $a(-b) = (-a)b = -(ab)$, and $(-a)(-b) = ab$.

Fraenkel's work exerted little influence since it was not grounded in the major concrete theories that had earlier been established. Its main significance was that rings now began to be studied as independent, abstract objects, not just as rings of polynomials, as rings of algebraic integers, or as rings (algebras) of hypercomplex numbers.

D. EMMY NOETHER AND EMIL ARTIN. Yet rings of polynomials, rings of algebraic integers, and rings of hypercomplex numbers remained central in ring theory. In the hands of the master algebraists Noether and Artin their study was transformed in the 1920s into powerful, abstract theories. Noether's two seminal papers of 1921 and 1927 extended and abstracted the decomposition theories of polynomial rings on the one hand and of the rings of integers of algebraic number fields and algebraic function fields on the other, to abstract commutative rings with the ascending chain condition—now called *noetherian rings*.

More specifically, Noether showed in her 1921 paper, "Ideal theory in rings", that the results of Hilbert, Lasker, and Macauley on primary decomposition in polynomial rings hold for any (abstract) ring with the ascending chain condition. Thus results which seemed inextricably connected with the properties of polynomial rings were shown to follow from a single axiom! In her 1927 paper, "Abstract development of ideal theory in algebraic number fields and function fields", she characterized abstract commutative rings in which every nonzero ideal is a unique product of prime ideals. These are now called *dedekind domains*.

Artin, inspired by Noether's work on commutative rings with the ascending chain condition, generalized Wedderburn's structure theorems in his 1927 paper, "On the theory of hypercomplex numbers", to noncommutative semi-simple rings with the descending chain condition. In particular, he showed that such rings (now called *artinian rings*) can be decomposed into direct sums of simple rings that, in turn, are matrix rings over division rings.

While Fraenkel gave the first abstract definitions of a ring, Noether and Artin made the abstract ring concept central in algebra by framing in an abstract setting the theorems that were its major inspirations. In this context they introduced, and gave prominence to, such fundamental algebraic notions as ideal (including one-sided ideal), module, and chain conditions (both ascending and descending). Ring theory now took its rightful place along the by then well-established theories of groups and fields as one of the pillars of abstract algebra.

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A method to trisect a series of angles having relation to each other; also another to trisect any given angle. By James Sabben. 1848 (two quarto pages). "The consequence of years of intense thought."

De Morgan's comment on this effort: Very likely, and very sad.

A. De Morgan, *A Budget of Paradoxes*, Vol. 2, p. 10, Open Court Edition, 1915.

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