KLEIN-WEYL'S PROGRAM AND THE ONTOLOGY OF GAUGE AND QUANTUM SYSTEMS

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Abstract. We distinguish two orientations in Weyl's analysis of the fundamental role played by the notion of symmetry in physics, namely an orientation inspired by Klein's Erlangen program and a phenomenological-transcendental orientation. By privileging the former to the detriment of the latter, we sketch a group(oid)-theoretical program—that we call the *Klein-Weyl program*—for the interpretation of both gauge theories and quantum mechanics in a single conceptual framework. This program is based on Weyl's notion of a "*structure-endowed entity*" equipped with a "*group of automorphisms*". First, we analyze what Weyl calls the "*problem of relativity*" in the frameworks provided by special relativity, general relativity, and Yang-Mills theories. We argue that both general relativity and Yang-Mills theories can be understood in terms of a *localization* of Klein's Erlangen program: while the latter describes the *group-theoretical automorphisms of a single structure* (such as homogenous geometries), local gauge symmetries and the corresponding gauge fields (Ehresmann connections) can be naturally understood in terms of the *groupoid-theoretical isomorphisms in a family of identical structures*. Second, we argue that quantum mechanics can be understood in terms of a *linearization* of Klein's Erlangen program. This stance leads us to an interpretation of the fact that quantum numbers are "*indices characterizing representations of groups*" ([81], p.xxi) in terms of a correspondence between the ontological categories of *identity* and *determinateness*.

Key words: Hermann Weyl, Klein's Erlangen Program, General Relativity, Gauge Theories, Quantum Mechanics, Symmetries.

1. Introduction

H. Weyl was a principal actor in the two most important revolutions in the 20^{th} century physics, namely the geometrization of the fundamental interactions in the framework of the *gauge theories* on the one hand and *quantum mechanics* on the other. Regarding the former, Weyl gave the first steps to extend Einstein's geometrization of gravitation to the non-gravitational interactions. In particular, he provided—in the framework of his gauge theories of 1918 and 1929 [76, 79]—the first formulation of the *gauge argument*, i.e. of the argument according to which the requirement of local gauge invariance dictates both the introduction of the gauge fields and the form of the interaction between the latter and the matter fields.[1] Weyl's first attempts finally led to the formulation of the *Yang-Mills theories* of non-gravitational fundamental interactions in

[1]For a conceptual discussion of the gauge argument see Refs.[8, 12, 43, 69]. For a discussion of the history of the gauge argument and a collection of the main articles in this history see Ref.[48]. It is worth noting that the term *gauge theory* has two interrelated meanings. First, it denotes Dirac's theory of constrained Hamiltonian systems [14, 22]. Second, it denotes the geometric description of the fundamental interactions in terms of connections on principal fiber bundles over spacetime. While the gauge theories of fundamental interactions have a constrained

1954 [91]. From a mathematical viewpoint, C. Ehresmann gave an important step in this history by unveiling the fundamental geometric structure underpinning Yang-Mills theories, namely the *Ehresmann connections* on principal fiber bundles [17].

Regarding quantum mechanics, Weyl was one of the first—together with Wigner—to apply the theory of group representations to this theory [78, 81] (see also Refs.[37, 44, 61, 62, 68] and Ref.[36] for a comparison of Weyl's and Wigner's applications of the theory of group representations to quantum mechanics). What we could call—following Mackey [38]—*Weyl's program* amounts to understanding one of the most fundamental features of quantum mechanics—namely the commutation relations and the resulting Heisenberg indeterminacy principle—from a group-theoretical perspective. Weyl's attempt to ground quantum mechanics on a group-theoretical basis had—in spite of the negative reactions initially elicited against the *Gruppenpest*—a glorious posterity. Among the principal hallmarks in the history of the relations between quantum physics and group theory we can mention the group-theoretical classification of elementary particles [67, 89], Mackey's systems of imprimitivity [39, 74], Kirillov's orbit method [27], and the Kostant-Souriau geometric quantization formalism [32, 67]. Nonetheless, it is worth stressing that in spite of these ground-breaking formal achievements the group-theoretical foundational program launched by Weyl does not play (at least to the knowledge of the author) any *central* role in the leading interpretations of quantum mechanics (such as the Copenhagen interpretation, the hidden variables interpretations, the many-worlds interpretation, or the information-based interpretations).

Weyl's contributions to these "revolutions" in 20^{th} century physics were oriented by two independent regulative leitmotifs.[2] On the one hand, Weyl's "*purely infinitesimal program*" (*Reine Infinitesimalgeometrie*), which underpinned his contributions to the development of gauge theory, was philosophically guided by what we shall generally call a *transcendental leitmotif*.[3] The influence of transcendental idealism in Weyl's thought, far from being homogenous, combines influences from both Husserl's transcendental phenomenology (see Refs.[50, 52, 53]) and the post-Kantian "constructivist" philosophy of Fichte (see Ref.[63, 64, 65]).[4] Regarding the notion

Hamiltonian formulation, there are Hamiltonian theories with constraints not arising from the description of connections on principal fiber bundles.

[2]In this article, we shall not analyze the possible chronological periodization of these two orientations. For an analysis of the evolution of Weyl's philosophical ideas see Refs.[15, 59, 63].

[3]Even if transcendental arguments do not play a central role in the framework of Weyl's works on quantum mechanics, he endorses the view that the "*central problem*" of this theory is an epistemic problem. In *The Theory of Groups and Quantum Mechanics*, Weyl explicitly uses a Kantian terminology to endorse a transcendental interpretation of quantum mechanics: "But scientists have long held the opinion that such *constructive concepts* [such as the Galilean concept of mass] *were nevertheless intrinsic attributes of the 'Ding an sich,'* even when the manipulations necessary for their determination were not carried out. *In quantum theory we are confronted with a fundamental limitation to this metaphysical standpoint.*" ([81], p.76). In *Philosophy of Mathematics and Natural Science*, Weyl endorses an instrumentalist interpretation of the Heisenberg indeterminacy principle ([85], p.256, p.263). He concludes his summary of the features of quantum physics which seem to him of "paramount philosophical significance" by saying that "the meaning of quantum physics [...] is not yet clarified [...] The relation of reality and observation is the central problem. We seem to need a deeper epistemological analysis of what constitutes an experiment, a measurement, and what sort of of language is used to communicate its result." ([85], p.264).

[4]It is worth noting the existence of a tension in Weyl's thought between the Husserlian "intuitionism" and the Fichtean "constructivism" (see Ref.[66] for an analysis of this point). However, we think that this tension can be sublated *within a transcendental framework* once we take into account that Kantian transcendental philosophy is essentially based on the articulation between what Kant calls *receptivity* and *spontaneity*.

of symmetry, this transcendental leitmotif leads Weyl to the claim that the fundamental role played by symmetries in physics can be explained by means of *a priori* considerations regarding the transition—mediated by "*symbolic construction*"—from absolute subjective experience (of a qualitative *suchness* placed in an extended *Here-Now*) to relative objective knowledge.

On the other hand, Weyl's work and reflections on both the "*problem of relativity*" and quantum mechanics were also influenced by a *Kleinian leitmotif*. This regulative orientation results from the influence of both Klein's Erlangen program [28] (see also Ref.[57], Chap.4, and Ref.[18] for a historical discussion) and the development of group theory in the framework of crystallography.[5] Rather than addressing the *a priori* conditions of possibility of objective representation, this orientation is based on the (*a posteriori*) consideration of "*structure-endowed entities*" endowed with non-trivial *automorphisms* ([84], p.144). In other terms, the analysis of the transcendental constitution of physical objectivity is substituted with the analysis of the intrinsic structures of particular (formal, physical, and cultural) objects, namely regular objects endowed with intrinsic symmetries (such as for instance Klein geometries, crystals, and ornaments). Weyl's book *Symmetry*—being a sort of naturalist compendium of regular structures found in mathematics, nature, and culture—can be understood as a paradigmatic expression of this "empirical" attitude regarding the notion of symmetry.

In what follows we propose a particular articulation of a certain number of ideas and statements extracted from Weyl's reflections on the notion of symmetry, namely

(1) that the *epistemic symmetries* related to the free election of a frame of reference are dependent upon the *intrinsic symmetries* resulting from the fact that the corresponding "*structure-endowed entity*" has non-trivial automorphisms (see for instance the Section III.13 "*The Problem of Relativity*" in Ref.[86]).

(2) that the knowledge of the group of automorphisms of a "*structure-endowed entity*" Σ provides a "*deep insight into the constitution of* Σ" ([84], p.144),

(3) that the quantum numbers are "*indices characterizing representations of groups*" ([81], p.xxi),

(4) that "*objectivity means invariance with respect to the group of automorphisms*" ([84], p.132).

We argue that the articulation of these statements points towards a *groupoid-theoretical comprehension* of both gauge and quantum systems based on the notion of a *structure-endowed entity* equipped with a *group of automorphisms*, i.e. with a group of transformations which leave the structure of the entity unchanged (see for instance [84], p.42). To do so, we address the inner tension between the two aforementioned regulative orientations in Weyl's reflections by privileging a Klein-oriented interpretation of these statements to the detriment of the transcendental orientation. In order to stress this Kleinian inflection of what Mackey called *Weyl's program*, we call the resulting "program" for the interpretation of gauge and quantum systems *Klein-Weyl's program*. Transcendental arguments provide an extremely elegant and compelling explanation

[5]With respect to the influence of Klein's Erlangen program in Weyl's work on quantum mechanics see Ref.[15]. Regarding the historical importance of crystallography, Weyl writes in the introduction to *The Theory of Groups and Quantum Mechanics* that "*until the present, the most important application* [of the group concept] *to natural science lay in the description of the symmetries of crystals*." ([81], p.xxi).

of the *fundamental* role played by the notion of *symmetry* in physics.[6] However, we argue that Klein's Erlangen program provides an alternative conceptual framework for understanding the fundamental role played by symmetries in physics.[7] In this alternative conceptual framework, gauge theories and quantum mechanics can be placed in a very different philosophical and scientific history whose Ariadne's thread is provided by a progressive enrichment of the notion of *identity*. Briefly, this history results in a detrivialization of the notion of identity, one which substitutes the *principle of identity* $a = a$ (which does not depend on the particular features of the entity a) with a notion of identity that—depending on the entity at stake—can be used to identify the specific entity. A fundamental notion in the contemporary phase of this history is the notion of $(\infty-)$*groupoid*. This notion provides the formal tool encoding the transition from a trivial notion of identity to the notion of *identity-qua-identification*.

The structure of the paper is as follows. First, we analyze what Weyl calls the "problem of relativity" for the case of homogeneous structures such as the Minkowski spacetime of special relativity (Section N°2). In Section N°3, we extend the analysis of the "problem of relativity" to the formal and conceptual framework provided by the theory of general relativity. In particular, we use Cartan's "localization" of Klein's Erlangen program—in the framework of the theory of *Cartan geometries*—to criticize Weyl's phenomenological-transcendental interpretation of general relativity. In Sections N°4, N°5, and N°6 we address the "problem of relativity" in the framework of Yang-Mills theories. In particular, we criticize the thesis according to which gauge symmetries can be understood as a mere mechanism for getting rid of the "surplus structure" arising from descriptive freedom in the utilization of arbitrary coordinate systems. We argue that local gauge symmetries naturally arise when we pass from the Kleinian consideration of a single structure-endowed entity (endowed with its *group* of automorphisms) to the consideration of a family of identical structures (endowed with the *groupoid* of self- and hetero-identifications). This description allows us to provide a groupoid-theoretic analysis of the gauge argument, i.e. of the argument according to which local gauge symmetries require that we introduce gauge fields (or, in geometric terms, Ehresmann connections). Since this groupoid-theoretical description of Yang-Mills theories is not standard in the physics literature, we propose a more or less informal description of the underlying mathematical structures. In Section N°7, we propose the heuristic conjecture according to which quantum mechanics can be understood as a "linearization" of Klein's Erlangen program. We then interpret Weyl's statements (2) and (3) in terms of a correspondence between the ontological categories of *identity* and *determinateness*. Finally, we interpret statement (4) in terms of a compatibility condition between the non-trivial identity of

[6]See notably the articles in the collected volume [5]. See also Refs.[26, 52] for an analysis of the role played by transcendental philosophy with respect to quantum mechanics and gauge theories respectively

[7]The historical fact that transcendental arguments played a key role in the gestation period of gauge theories—at least in Weyl's work (see for instance Ref.[53])—is a strong argument in favor of a transcendental interpretation of gauge symmetries. However, the recognition of the historical fact that transcendental arguments philosophically grounded Weyl's work on gauge theories is not incompatible with the thesis according to which the "revolutions" associated with gauge theories and quantum mechanics are not of a transcendental nature. As Hegel clearly states, "*the course of a science's origin and the preliminaries of its construction are one thing, while the science itself is another. In the latter, the former can no longer appear as the foundation of the science; here, the foundation must be the necessity of the Notion.*" ([21], §246, p.6). Of course, the fact that Weyl's transcendental orientation prevented him (as we shall see) from accepting Cartan's generalization of Klein's Erlangen Program is not, for exactly the same reasons, an argument *against* a transcendental comprehension of the theories at stake.

a non-rigid structure and the observables that can be consistently valuated on such structure. In the final Section, we summarize the proposed discussion.

2. The "Problem of Relativity"

Let us start by considering what Weyl calls the "*problem of relativity*". According to Weyl, this "*problem*" is associated with the "*symbolic construction*" of an "*objective world*" "*from what is immediately given to my intuition*" ([85], p.113). We could say that the problem of relativity is nothing but the *problem of objectification*, i.e. of the conditions underpinning the transition from *subjective experience* to *objective knowledge.*[8] Whereas "*immediate experience is* subjective and absolute [...] *this* objective *world is of necessity* relative*; it can be represented by definite things (numbers or other symbols) only after a system of coordinates has been arbitrarily carried into the world.*" ([85], p.116). In this way, the constitution of objectivity proceeds by replacing the possible subjects of experience with a class of coordinate systems.[9] This objectification process, rather than simply eliminating the subject of experience, incorporates the multiplicity of possible subjects in the form of a class of coordinate systems. It is then necessary to impose the condition according to which only the properties that do not depend on the *arbitrary* election of a coordinate system define *objective* (or intrinsic) properties of the corresponding physical system. In this way, the very necessity of explicitly taking into account the requirement of "gauge invariance" attests to the impossibility of completely concealing the subjective underpinning of any objective knowledge. Briefly, gauge invariance is the "*necessary residue of the ego-extinction*".

In *Philosophy of Mathematics and Natural Science*, Weyl distinguishes two phases of the "*problem of relativity*" ([85], Section III.13). In the first phase of the analysis, Weyl introduces the notion of *automorphism* of a "*point-field*" (*Punktfeldes*), where an automorphism is defined as a "*one-to-one mapping of the point field into itself which leaves the basic relations undisturbed*" ([85], p.72). In other terms, an automorphism is a symmetry of a "*structure-endowed entity*", i.e. a transformation of the structure into itself. We shall not address here the problem of providing a rigorous definition of the term *structure*. It is enough to point out that Weyl characterizes the notion of *structure* both *internally* (i.e. as a "*point-field*" endowed with relations) and *externally* (in the sense that "*isomorphic domains may be said to posses the same* structure", [85], p.25).[10]

Let us introduce some definitions and ideas that we shall use in the rest of the article. A structure endowed with a non-trivial group of automorphisms will be called a *non-rigid structure*. A non-rigid structure a is a structure that can be transformed into itself—or *identified* with itself—in different non-trivial manners. In what follows, we shall understand this multiplicity of possible self-identifications of an entity a as an enrichment of the fact that every entity is trivially identical to itself. In this line of reasoning, the so-called *principle of identity* $a = a$ can be understood as an *abstract* statement whose *concrete* proofs are given by all the automorphisms of the structure a. In the Peircean jargon of homotopy type theory, we could say that the automorphisms of a are the tokens of the identity type $a = a$ [23].

According to Weyl, the analysis of the "*problem of relativity*" in terms of the automorphisms of the corresponding structure is only the first phase of such a problem: "*Having explained*

[8]In Ref.[87] (p.1), Weyl writes that "relativity *is another name for the same matter*", that is "*the central epistemological question of* objectivity, *of what constitutes an objective statement*".

[9]In Weyl's own terms: "*The objectification, by elimination of the ego and its immediate life of intuition, does not fully succeed, and the coordinate system remains as the necessary residue of the ego-extinction.*" ([85], p.75).

[10]Regarding the history of the relations between the notions of *structure* and *isomorphism* see Ref.[71].

automorphism we now come to a second phase of the relativity problem [...] *to assign to the points of a point-field labels which could serve for their identification*", i.e. to choose a frame of reference ([85], p.75). Now, Weyl states that "*a class* Σ *of frames of reference* [...] *should be objectively distinguished, i.e. if* $\mathfrak{f}$ *belongs to it, so does any similar frame* $\mathfrak{f}\sigma = \mathfrak{f}'$ *arising from* $\mathfrak{f}$ *by an automorphism* σ." ([85], p.76). This means that the class of possible frames of reference and the group of coordinate transformations between them depend on the automorphisms of the entity, and thereby on the *intrinsic* (or "*objective*") structure of the latter. Therefore, the requirement of invariance under changes of the coordinate system can be understood as a derived "epistemic" consequence of the intrinsic "ontological" structure of the entity at stake. In other terms, the fact that a structure can afford a certain class of coordinate presentations, far from being an *a priori* necessity, depends on an intrinsic feature of the structure, namely on its particular form of non-rigidity. In the article *Relativity theory as a stimulus in mathematical research*, Weyl summarizes the "*two phases*" of the "*relativity problem*" in the following terms:

> "Suppose a realm of objects, which may be called points, is given to you. Those transformations, those one-to-one mappings of the point field into itself which leave all relations of objective significance between points undisturbed, form a group, the group of automorphisms. If in some way coordinates, self-created reproducible symbols like numbers, are assigned to the points then any automorphism carries this assignment over into a new one from which it is objectively indiscernible (in Leibniz's word). Hence, any coordinate assignment requires an act of choice by which one picks out one from a class of equally admissible coordinate systems. The class is objectively characterized, but not the individual coordinate assignment." ([86], p.394)

This text makes clear that the group of automorphisms—and thereby the class of "*equally admissible*" and "*objectively indiscernible*" coordinate systems—must be understood as an "*objectively characterized*" feature of the corresponding structure (in this case a "*point field*" endowed with relations). The interpretation of these statements depends on the (pre-critical or critical) scope of the term *objective(ly)*: whereas in a pre-critical framework *objective* can be interpreted as *intrinsic*, in a critical framework *objective* means correlated to a transcendental type of subjectivity (or *transcendental subject*). We could rephrase Kant by saying that an objective determination is not a determination of the "*thing in itself*", but rather a determination that is *transcendentally intersubjective*, i.e. valid for all the empirical subjects of the same transcendental type. Now, we think that Weyl's use of the term *objective* in this context remains pre-critical. For Weyl, the problem of deciding which is the group of automorphisms of a given physical entity (for instance spacetime) is not an *a priori* problem that could be decided by means of a transcendental analysis of the conditions of possibility of human experience, but rather an empirical problem. According to Weyl, "*objectivity is an issue decidable on the ground of experience only*" ([87], p.2).[11] Therefore, the particular way according to which the "*problem of relativity*" is

[11]Regarding the limits of *a priori* arguments for determining the structure of physical space, see also the following texts: "*when we said* [...] *that left and right are indiscernible, we want to express that the inner structure of space does not permit us, except by arbitrary choice, to distinguish a left from a right screw.*" ([84], p.17); "*In science the belief in the equivalence of left and right has been upheld even in the face of certain biological facts* [...] *which seem to suggest their inequivalence* [...] *The same problem of equivalence arises with respect to past and future* [...] *and with respect to positive and negative electricity. In these cases, especially*

effectively posited depends upon empirical features of physical nature, such as for instance the group of automorphisms of spacetime. For instance, the problem of relativity as it is effectively posited in the framework of quantum field theory on a Minkowski spacetime depends upon the particular intrinsic structure of this background. This means that any "objective" statement in this theory must be invariant under the action of the Poincaré group of symmetries of this particular spatiotemporal background.

3. Revisiting Weyl's Phenomenological-Transcendental Interpretation of General Relativity

In the previous section we have argued that what Weyl calls the "*problem of relativity*" can be understood as an "epistemic" consequence of the fact that we are dealing with non-rigid structures, i.e. with structures intrinsically (or objectively) endowed with non-trivial automorphisms. The relation between the group of passive transformations between different coordinate systems (epistemic reading of symmetries) and the group of active automorphisms of the structure at stake (ontological reading of symmetries) has been widely discussed in the philosophy of physics literature. Now, there is a widespread tendency in physics and the philosophy of physics to interpret the fundamental symmetries in physics (notably the gauge symmetries in the theory of constrained Hamiltonian systems and the phase symmetries in quantum mechanics) by means of what we could generally call "transcendental arguments" regarding the general conditions of scientific objectivity. One version of such arguments can be summarized as follows. Scientific knowledge has to be *intersubjective* knowledge, that is (in Eddington's terms) knowledge constructed "*from the point of view of no one in particular*" ([16], p.27). This requirement of intersubjectivity is concomitant with the condition according to which experiments performed by observers related by a transformation in the group of displacements and boosts in space and time (i.e. by observers in different spatiotemporal positions and different states of inertial motion) should give the same results.[12] In Wigner's work, this amounts to the statement that the natural laws have to be invariant under the action of the Poincaré group. In this way, the invariance under the action of this group could seem to play a necessary *a priori* role in physics. Now, this argument is valid under the hypothesis according to which spacetime is a particular homogeneous manifold, in this case the Minkowski spacetime of special relativity. In Anderson's terms, the Poincaré group—far from being endowed with some sort of transcendental aprioricity—is just the symmetry group of the "absolute object" presupposed by the theory of special relativity [1].

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in the second, it is perhaps clearer than for the pair left-right that a priori *evidence is not sufficient to settle the question; the empirical facts have to be consulted.*" ([84], p.24); "[it is] *the inherent symmetry of the four-dimensional continuum of space and time that relativity deals with.*" ([84], p.132); "*We have seen that the notions of similarity and invariance are relative to a group. The physicist will question Nature to reveal to him her true group of automorphisms.*" ([87], p.4); "*they* [the real occurrences and their laws] *disclose a certain structure of the space-time continuum* [...] *particular 'allowable' coordinate systems may be introduced which are adapted to that structure in a certain manner.*" ([83], p.129).

[12]In Wigner's terms: "It is not necessary to look deeper into the situation to realize that laws of nature could not exist without principles of invariance. [...] If the correlations between events changed from day to day, and would be different for different points of space, it would be impossible to discover them. Thus the invariance of the laws of nature with respect to displacements in space and time are almost necessary prerequisites that it be possible to discover, or even catalogue, the correlations between events which are the laws of nature." ([90], p.29). For a discussion of invariance principles in Wigner (and its relation to Kantian transcendental philosophy) see Ref.[51].

More generally, any field theory defined on a homogeneous spacetime manifold has to be invariant under the action of the group of symmetries of this "background". But in the general case of a non-homogenous background there is no reason to expect (for instance) that an experiment performed within a black hole will give the same results as an experiment realized in a region of spacetime with a flat local curvature. Therefore, Wigner's argument has to be modified so as to make it compatible with general relativity.[13]

It was in fact Weyl who tried to extend the transcendental understanding of invariance principles to general relativity. From a philosophical standpoint, Weyl's *purely infinitesimal geometry* program is based on two interrelated ideas:

(1) the Kantian thesis according to which Euclidean space and time (or Minkowski spacetime) are *a priori* forms of intuition ([25], First Part. Transcendental Aesthetic),

(2) a *phenomenological localization* of this Kantian stance.

We could say that Husserl's transcendental phenomenology introduces an important refinement of Kant's transcendental aesthetic by arguing that the *a priori* forms of intuition only concern what is *immediately* given to consciousness. An act of consciousness—far from being an experience of an object placed in a *global* spatio-temporal manifold—is an experience of the immediate vicinity of the corresponding experiencing subject.[14] In Weyl's own terms, "*only the spatio-temporally coinciding and the immediate spatio-temporal neighborhood have a directly clear meaning exhibited in intuition*" [82] (translated by Ryckman in Ref.[52], p.148; see also Ref.[54] for a discussion of these ideas). As Ryckman writes, Weyl argues that the "*world geometry*" must be "*entirely built up from these basic* [Minkowskian] *geometrical relations immediately evident within a purely infinitesimal space of intuition.*" ([52], p.149). In this way, the Kantian thesis according to which Euclidean space and time (or Minkowski spacetime) are *a priori* forms of intuition has to be restricted to the infinitesimal neighborhood of the experiencing subject. In Weyl's terms, "*the philosophers may have been correct that our space of intuition bears a Euclidean structure, regardless of what physical experience says. I only insist* [...] *that the ego-center* (Ich Zentrum) *belongs to this space of intuition and that the coincidences, the relations of the space of intuition to that of physics, becomes vaguer the further one distances oneself from the ego-center.*" ([82], p.52; translated by Ryckman in Ref.[52], p.148). According to this phenomenologically corrected transcendental aesthetic, the *infinitesimal* spatiotemporal neighborhood of any possible observer necessarily carries a Minkowski structure. Since this (phenomenologically inflected) transcendental stance does not say anything about the

[13]The inapplicability of Wigner's notion of "geometrical invariance" to general relativity has been explicitly recognized by Wigner *et al.* in Ref.[24], p.603 (see also Ref.[90], p.22).

[14]According to Weyl, the general form of an act of consciousness is an interpenetration of a *thisness* (continuous extension) and a *suchness* (continuous quality): "The world only enters into our consciousness in the general form of consciousness, a mutual penetration of being (*Seins*) and of essence (*Wesens*), of the 'this' (*'Dies'*) and 'thus' (*'So'*)" (quoted in Ref.[52], p.147). In *Philosophy of Mathematics and Natural Science*, Weyl writes: "The penetration of the Here-Now (*Hier-jetzt*) and the Thus (*So*) is the general form of consciousness; something is only in the insoluble unity of intuition and sensation, in which continuous extension and continuous quality overlap. Phenomenologically one cannot get beyond this." ([85], p.130). The important point in the present context is that the notion of a "thisness" placed in a "Here-Now" only requires, as its *a priori* condition of possibility, an *infinitesimal* spatiotemporal form of intuition.

local and the *global* structures of spacetime, the door is open to different *empirical* determinations of these structures. The "world geometry" has a structure that is locally and globally *a posteriori*, but whose infinitesimal *a priori* structure is given by Minkowski spacetime.[15] In this way, the phenomenological refinement of Kant's transcendental aesthetic succeeds in saving the transcendental stance from the menace posed by Einstein's theory of general relativity. We could say that the *infinitesimal localization* of Euclidean (Minkoswki) geometry put forward by Riemannian (Lorentzian) geometry can be understood as a formalization of the *phenomenological localization* of Kant's transcendental aesthetic.[16]

The conceptual elegance of this attempt to save the *a priori* nature of Euclidean or Minkowski geometry from the threat introduced by general relativity is compelling. However, we can observe that Weyl's phenomenologically inflected defence of Euclidean geometry is, to some degree, in tension with the other main source of inspiration for Weyl's own ideas, namely Klein's Erlangen program. Klein's generalization of Euclidean geometry is based on the observation that—from the standpoint provided by the notion of symmetry—there is no reason to privilege Euclidean geometry with respect to other possible *homogeneous* (or *maximally symmetric*) spaces associated with other symmetry groups G. According to Klein, Euclidean geometry is just one homogeneous geometry among other possible homogeneous geometries, which are defined by other "*principal groups*" of symmetries (or "*groups of motions*" in Cartan's terms). This means that the supposed *a priori* privilege of Euclidean geometry is, in the framework of Klein's Erlangen program, democratically abolished. By endorsing the Kantian thesis according to which Euclidean geometry is endowed with an (infinitesimally valid) *a priori* privilege, Weyl lags behind the relativisation of the homogeneous geometries introduced by Klein. However, we could still argue that the relevant infinitesimal concept in Riemannian geometry—and thereby in general relativity—is not the concept of (maximal) *symmetry*, but rather the concept of *flatness*. We could then claim that Euclidean geometry, being a *flat* Klein geometry, keeps its *a priori* privilege. Let us consider this point in more detail.

There is a tension between Riemannian geometry and Klein's Erlangen program (see notably the preface to Ref.[57] for a discussion of this point; see also Refs.[10, 45]). Both Riemannian geometry and Klein's Erlangen program generalize Euclidean geometry. However, they do it in different directions: while Riemannian geometry breaks the homogeneity of Euclidean geometry by introducing the notion of local curvature (which "mades it lumpy" in Sharpe's terms; [57], p.x), Klein's Erlangen program considers homogeneous geometries whose principal groups of symmetries are not the Euclidean group. While Riemannian geometry only uses infinitesimal Euclidean models (thereby lagging behind Klein's Erlangen program), Klein only considers

[15] See Ref.[41] (p.329) for a discussion of Weyl's distinction between the aprioristic "*metrical nature of space*" and the *a posteriori* form of the metric (which depends on the contingent distribution of mater and energy).

[16] A similar rescue operation of Kant's transcendental aesthetic (which does not rely on a phenomenological justification of the localization of the spatiotemporal forms of intuition) is attempted by Cassirer in *Einstein's Theory of Relativity* [9]. In this text, Cassirer argues in favor of the "logical simplicity" of Euclidean geometry. Moreover, Cassirer maintains that "it is precisely the fundamental doctrine of the general theory of relativity, that [...] can establish and render intelligible this special position. [...] This logical simplicity belonging to Euclidean space in the system of our intellectual meanings wholly independently of its relations to experience, is shown, e.g., in the fact that we can make any "given" space, that possesses any definitive curvature, into Euclidean by regarding sufficiently small fields of it from which the difference conditioned by the curvature disappears. Euclidean geometry shows itself herein as the real geometry of infinitely small areas [...] The development of the general theory of relativity leaves this methodic advantage of Euclidean geometry unaffected." ([9], pp.435-436).

homogeneous geometries (thereby lagging behind Riemannian geometry). We can here cling to Riemannian geometry—as the mathematical underpinning of general relativity—and maintain the infinitesimal privilege of Euclidean geometry. This would amount to maintaining the physical primacy of the notion of *flatness* over the notion of *symmetry*. A more general way to resolve the tension between these two paradigms is to overcome them in a new synthesis by privileging the gauge-theoretic notion of symmetry over the notion of flatness. This strategy is more general in the sense that a flat Euclidean space is a particular case of the notion of a maximally symmetric space. This is exactly what E. Cartan achieved in the theory of the *Cartan geometries* or *Cartan connections* [10, 57]. Contrary to Weyl, Cartan remained faithful to the legacy of Klein's Erlangen program to the point of accepting the necessity of generalizing Riemannian geometry (see Ref.[52], Sec.6.5, for a discussion of Weyl's initial opposition to Cartan's ideas and Ref.[58] for a comparison of Weyl's and Cartan's ways of dealing with the "problem of space"). Cartan realized that it is possible to sublate the divergence between Klein's Erlangen program and Riemannian geometry by using Klein geometries as *infinitesimal models* (alternatively, Cartan's "*generalized spaces*" can be understood, as Sharpe writes, as Klein geometries "*made lumpy by the introduction of curvature*" [57], p.x). The important insight of Cartan's approach is that—rather than limiting the scope of Klein's Erlangen program—we can make it compatible with the existence of non-homogenenous geometries by simply *localizing* it. According to Weyl's description of Cartan's ideas, "*Klein's notions were applied to the tangent plane and not to the n-dimensional manifold M itself*" ([80], p.207). An interesting symptom of the tension in Weyl's work between the influence of Klein's Erlangen program and the phenomenological-transcendental orientation is given by the fact that Weyl initially criticized Cartan's generalization of Klein's Erlangen program *for phenomenological reasons* [77, 80].[17] Overcoming his initial criticism, Weyl finally recognized his mistake and praised the importance of Cartan's approach for the comprehension of the gravitational interaction [86].

In the framework of a transcendental orientation, the fundamental role played by gauge symmetries in physics acquires a straightforward and elegant interpretation: gauge symmetries are fundamental because they play an *a priori* role related to the very conditions of possibility of objective knowledge. Now, if we no longer endorse such a transcendental interpretation, how can we explain the fundamental role played by gauge symmetries? As a first approximation, one could think that the *fundamental* scope of Klein's Erlangen program is limited, since homogenous structures are very particular kinds of structures. For instance, homogeneous solutions of the Einstein equations—such as Minkoswki spacetime—are just particular solutions among others. Why should they play any privileged role? Now, Cartan's theory of connections suggests that the fundamental role played by homogenous structures stems from the fact that they provide the *elementary building blocks* out of which more general structures (not necessarily symmetric) can be constructed. We could say that in Cartan's theory homogeneous geometries are the

[17]Weyl criticized Cartan's theory in the following terms: *"The [Klein] space Σ_p is not a pure product of the manifold M (as is the tangent space); it requires grounds of determination lying still outside M. As long as that is the case, the Cartan schema is quite out of the question as a theory of a single manifold and accordingly as a world-geometric foundation for physics."*(quoted by Ryckman in Ref.[52], p.171). In Ref.[80] (p.210), Weyl writes: "[...] *the 'tangent plane' T_p is not as yet uniquely determined by the nature of M, and so long as this is not accomplished we can not say that Cartan's theory deals only with the manifold M. Conversely, the tangent plane in P in the ordinary sense, that is, the linear manifold of line elements in P, is a centered affine space; its group $\mathfrak{G}$ is not a matter of convention. This has always appeared to me to be a deficiency of the theory* [...]".

elements—in the Platonic sense of the term—of geometry. These homogenous "elements" can be "connected" in a "curved" manner in order to produce more general structures, i.e. *Cartan geometries* in which the maximal symmetry of the building blocks might be broken. The *Cartan connection* is the additional structure that gives "transversal" coherence to this multiplicity of homogenous "elements". While the elementary building blocks are maximally symmetric, this is not necessarily the case for the resulting Cartan geometry, wherein the notion of *Cartan curvature* is the corresponding obstruction.[18] We could say that Riemannian geometry and Cartan theory convey two different conceptions of metric manifolds: whereas in Riemannian geometry a manifold is *infinitesimally modeled* by Euclidean tangent spaces, in Cartan's theory a manifold is *constructed* by "connecting" (possibly in a "curved" manner) homogeneous (but not necessarily Euclidean) building blocks.

Cartan's theory constitutes an important step towards the understanding of the gravitational interaction in terms of a gauge theory, i.e. of a theory endowed with a *local symmetry* describing the dynamics of a *connection*. Even if a Cartan geometry is not necessarily homogenous (i.e. maximally symmetric), the fact that the elementary local Klein geometries are homogenous implies that the resulting theory will be endowed with a *local* (infinitesimal) *symmetry*. In other terms, the *global* symmetry of the Klein geometry used as basic "element" becomes a *local* symmetry of the Cartan geometry constructed by "connecting" copies of these "elements". By passing from Riemannian geometry (with its *flat* infinitesimal models) to Cartan geometries (with its *maximally symmetric* "elements") the notion of *local flatness* loses its preeminence in favor of the gauge-theoretic notion of *local symmetry*.[19] Moreover, the fundamental geometric structure in Cartan's theory is the notion of Cartan connections, where the metric is now a derived geometric structure.[20]

4. Towards a Klein-Oriented Interpretation of Yang-Mills Theories

We shall now apply Weyl's analysis of the *problem of relativity* to the case of Yang-Mills theories. To do so, we shall stress the groupoid-theoretical aspects of these theories. Since this formulation of Yang-Mills theories is not standard in the physics literature, we shall propose—in this Section and the next one—a summarized description of the basic definitions and ideas (see Ref.[35] for a detailed treatment of the theory of Lie groupoids and Lie algebroids).

The notion of a *G-principal bundle* endowed with an *Ehresmann connnection* constitutes the main geometric structure of gauge theories. This geometric structure underpins both the Yang-Mills theories of non-gravitational interactions and the linear part of the Cartan description of the gravitational interaction (i.e. the gauge field associated with the local Lorentz gauge symmetry) [10]. Now, gauge theories are characterized by the fact that the corresponding phase space is endowed with an action of the *gauge group*. The presence of these gauge symmetries implies that the phase space is too big. This means that the phase space is parameterized by more variables than the variables which are strictly needed to separate (or individualize) the *physical* states. The

[18]Since the Cartan "curvature" does not measure a deviation from *flatness*, but rather a deviation from the maximal *symmetry* of the Klein geometries used as building blocks, the term "curvature " might be misleading.

[19]It is worth noting that in the framework of gauge theories the adjective *local* does not make reference to an open neighborhood of the manifold, but rather to the fibers above each single point. In particular, the expression *local symmetry* makes reference to the symmetries of the different fibers in the corresponding fiber bundle.

[20]More precisely, the metric can be constructed from the *soldering form*, which is the translational component of the Cartan connection (see Ref.[10] for details).

crucial conceptual problem in the "philosophy" of gauge theories is to understand the rationale behind the presence of these non-physical variables. The standard answer—going back at least to Dirac [14] (see also Ref.[22])—is that these non-physical variables are (in Redhead's terms [49]) a sort of "*surplus structure*" resulting from the arbitrary election of a coordinate system.[21] A transformation of the values of these variables is nothing but a *gauge transformation*, i.e. a transformation that merely changes the coordinate system by means of which the corresponding *unique* physical situation is framed. If gauge symmetries result from descriptive freedom in the utilization of arbitrary coordinate systems, then the relevant physical information is encoded by the gauge invariant quantities. Gauge symmetries would then be a mere mechanism for getting rid of the surplus "subjective" structure associated with the different re-coordinatizations of a unique physical (i.e. coordinate-independent or "objective") situation. According to this interpretative scheme, gauge symmetries have no empirical content: in a coordinate-free description of the intrinsic physical situation there should be neither coordinate-dependent surplus structures nor gauge symmetries.[22]

This "epistemic" interpretation of gauge symmetries is strongly supported by the fact that the *gauge group* is (in geometric terms) the *group of vertical automorphisms* of the G-principal bundle $P \to M$ over spacetime M on which the corresponding gauge field (the Ehresmann connection) is defined. Such a principal bundle can be understood as a *frame bundle*, i.e. as a bundle such that each fiber is the set of *frames* (interrelated by the transitive and free action of G) in the fibers of the *associated bundles* defined by the representations of G (such as for instance a vector bundle or a spinor bundle). In turn, the gauge group acts on the gauge fields by changing these local frames of reference. Hence, it might seem completely natural to understand the gauge symmetries as a way to get rid of the arbitrariness in the election of local frames of reference. However, we can here apply Weyl's prescription and try to understand the "relativity problem" at stake in an active manner, that is in terms of the intrinsic automorphisms of certain structures.

In Section N°3, we have used the conceptual framework provided by Cartan's theory to sketch an alternative Klein-oriented interpretation of the local symmetries associated with the gravitational interaction. According to the proposed interpretative scheme, local gauge symmetries are a natural consequence of the fact that we are dealing with families of (inter-)connected homogeneous structures. In order to apply this idea to Yang-Mills theories, let us consider the notion of a *G-bundle* ([30], p.86). A G-bundle $(E \xrightarrow{\pi} M, S, G)$ is a bundle $E \xrightarrow{\pi} M$ such that the *standard fiber* S is endowed with a left G-action and the transition functions on the intersections of an open covering $\{U_i\}$ of M are valued in G. In other terms, the fibers of E are copies of a standard G-space S that can be pasted on the intersections of an open covering $\{U_i\}$ by using the action of G. In order to fix ideas, let us consider a vector space V endowed with the left action of its group of automorphisms $G = Gl(V)$ as standard fiber. A vector G-bundle $E \xrightarrow{\pi} M$ is completely determined by fixing a *cocycle of transition functions* $\{\psi_{ij}\}$ with respect to an atlas

[21] In Dirac's own terms, gauge symmetries result from the fact that "*we are using a mathematical framework containing arbitrary features, for example, a coordinate system which we can choose in some arbitrary way* [...]" ([14], p.17).

[22] For instance, Healey endorses this position in the following terms: "*But if we do continue to employ gauge-dependent terminology, we must constantly bear in mind that we do so purely for convenience, and not be fooled into thinking that the consequent gauge symmetry of our theories has any empirical content.*" ([20], p.221). Regarding the hypothetical empirical content of gauge symmetries see also Refs.[6, 19, 31] and the references therein.

$(U_i, \psi_i : \pi^{-1}(U_i) \xrightarrow{\simeq} U_i \times V)$.[23] Roughly speaking, a cocyle $\{\psi_{ij}\}$ specifies how the fibers are identified on the intersections U_{ij} of the open covering $\{U_i\}$ in a manner that is consistent on the triple intersections U_{ijk}.

In the context of the present discussion, it is important to stress the following point: since the standard fiber V—and therefore its copies $E_x = \pi^{-1}(x)$—have non-trivial automorphisms, we can identify the fibers in non-trivial manners on the intersections U_{ij}. By doing so, we can introduce topological twists in the fibration. All the topological phenomena associated with Yang-Mills theories—such as Dirac monopoles, solitons, and Yang-Mills instantons—depend on the topological non-triviality of the corresponding bundles. The topological richness of Yang-Mills theories can therefore be understood as a direct consequence of the fact that the corresponding fibers are non-rigid structures that can be mutually identified in non-trivial manners.[24]

Let us consider now what we shall call the *V-identifications* of each fiber E_x of E, where a *V-identification* is defined as an isomorphism $p : V \to E_x$. In particular, we can take $V = \mathbb{R}^n$ with its canonical basis. In this case, a V-identification identifies each vector in E_x with a n-tuple of elements in $\mathbb{R}^n$, i.e. with a set of n coordinates. For this reason, the isomorphisms $\mathbb{R}^n \to E_x$ are usually called *frames* (as we have done before). We prefer to use here the term *V-identification* in order to stress that in the general case an isomorphism $p : V \to E_x$ is just a map by means of which each vector in E_x is *represented* by means of a vector in the standard fiber V in a bijective manner. It is worth stressing that each fiber E_x, being a copy of V, is isomorphic to V *in a non-canonical way* (see Ref.[12] for a discussion of this point). In what follows, we shall denote $Id(V, E_x)$ the set of V-identifications of the fiber E_x. The important point in the present context is that the multiplicity of possible identifications between the standard vector space V and each copy E_x is a direct consequence of the fact that V is a non-rigid structure. The left action of G on the standard fiber V induces a right (free and transitive) action on the set $Id(V, E_x)$ of V-identifications of E_x given by

$$(p : V \to E_x) \mapsto (pg : V \xrightarrow{g} V \xrightarrow{p} E_x).$$

In other terms, each set $Id(V, E_x)$ is a right *G-torsor*. In Weyl's terms, we could say that the fact that the standard fiber V has a group of automorphisms G implies that we can introduce a G-torsor of "*objectively distinguished frames*" in each copy E_x of V. By using a cocycle $\{\psi_{ij}\}$ associated with the vector bundle $E \to M$ we can glue together the G-torsors $Id(V, E_x)$ for all $x \in M$ into a G-principal bundle $P \to M$ such that the fiber over $x \in M$ is $Id(V, E_x)$. We can now go back and recover the vector G-bundle $E \to M$ as a bundle *associated* to P by the action of G on V.[25]

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[23] More precisely, a vector bundle $E \xrightarrow{\pi} M$ is by definition a bundle such that there are isomorphisms $\psi_i : \pi^{-1}(U_i) \to U_i \times V$ for each open set U_i of an open covering $\{U_i\}$ of M. The transition functions $\psi_{ij} : U_{ij} \to G$ on each intersection $U_{ij} \doteq U_i \cap U_j$ are defined by the expressions $\psi_i \circ \psi_j^{-1}(x, v) = (x, \psi_{ij}(x)v)$ and satisfy the *cocycle conditions* $\psi_{ii}(x) = id_G$ (for each $x \in U_i$) and $\psi_{ij}(x)\psi_{jk}(x) = \psi_{ik}(x)$ (for each $x \in U_{ijk}$).

[24] Schreiber has clearly stressed this point in the following terms: "[...] often one sees texts claiming that gauge invariance is just a 'redundancy' in the description of a physics, insinuating that one might just as well pass to the set of gauge equivalence classes. And this is not true: passing to gauge equivalence classes leads to violation of [...] the principle of locality. For reconstructing non-trivial global gauge field configurations [...] from local data, it is crucial to retain all the information about the gauge equivalences [...]." ([55], p.18).

[25] More precisely, $E \simeq P \times_G V$, where the right action of G on $P \times V$ is given by $(p, v) \cdot g = (pg, g^{-1}v)$. According to this description, each vector in E is described as a G-equivalence class of the form $[(p, v)]$, where the element $v \in V$ *represents* the vector $[(p, v)]$ in the "frame" p.

As it is well-known, the *gauge group* $\mathcal{G}$ of *local gauge transformations* is—from a mathematical standpoint—the group of vertical automorphisms of the G-principal bundle $P \to M$. This group acts on the sections of E. An element of the gauge group $\mathcal{G} \doteq Aut_v P$ acts on a section $s : M \to E$ by acting either on the "frames" $p : V \to E_x$ of the classes $s(x) = [(p, v)]$ (passive viewpoint) or on the representatives $v \in V$ of these classes (active viewpoint). In both cases, the action of the gauge group on the sections of $E \simeq P \times_G V$ just transfers the action of $Gl(V)$ on V to the different copies E_x of V. The important point that we want to stress here is that the "epistemic" invariance of a gauge theory under local gauge transformations is just a consequence of the intrinsic fact that we are dealing with families of identical non-rigid structures E_x that cannot be *canonically* identified.

5. On Groups and Groupoids in Yang-Mills Theories

In the previous section we have argued that the geometric underpinning of Yang-Mills theories is given by *families*—parameterized by spacetime—of *identical non-rigid structures*. Now, the transition from the automorphisms of a single structure (in the present case the standard fiber V) to the isomorphisms—i.e. both the self- and the hetero-identifications—in a family of structures (in the present case the bundle of copies E_x of V) is formally encoded by the generalization from the notion of *group* to the category-theoretic notion of *groupoid*. We shall now consider the geometric structure of Yang-Mills theories from the standpoint provided by the notion of groupoid.

Before continuing, it is worth noting that this groupoid-theoretical description is not standard in the literature since Yang-Mills theories can be entirely formulated in terms of groups (namely, in terms of the structural group and the gauge group). However, we shall adopt here a groupoid-theoretical perspective for the following reasons. The main point of the presentation of gauge theories proposed here is to plead—in the wake of Weyl's analysis of the *two phases of the relativity problem*—in favor of an *intrinsic* or *ontological* interpretation of gauge symmetries. To do so, we argue that these symmetries are a direct consequence of the fact that we are dealing with copies of a non-rigid structure (the standard fiber). Now, *copies of a non-rigid structure cannot be canonically identified, even if they are identical* (this point has been discussed at length in [12]). This statement presupposes a detrivialization of the notion of identity, in the sense that an entity can be identical (i.e. identified) to itself in many non-trivial ways. More generally, two entities can be identical in many different ways. The notion of groupoid is the mathematical notion that formalizes the situations in which one deals with multiple possible identifications between identical structures. In turn, the notion of group is a particular case of the notion of groupoid, in the sense that a group (understood as a category) is a groupoid with a unique object. Whereas groups encode the multiple identities (or, in more usual terms, the symmetries) of non-rigid structures, groupoids encode the (self- and hetero-)identifications in a family of non-rigid structures. The formalism of groupoids permits us to understand the transition between Klein's Erlangen program and Ehresmann's theory of fiber bundles—made possible by Cartan's generalization of the former—as a transition from the *group-theoretical* self-identifications of a single structure (e.g. Klein geometries) to the *groupoid-theoretical* (self- and

hetero-)identifications in a family of identical structures.[26] In other terms, Ehresmann's extension of Klein's Erlangen program was—in the wake of Cartan (as Pradines stresses)—an extension from "*group geometries to groupoid geometries*" [47]. While the group-theoretical perspective (based on the notion of *gauge group*) puts the accent on the (vertical) automorphisms of a single structure (the principal bundle taken as a whole), the groupoid-theoretical perspective (based on the notion of *structural groupoid*) puts the accent on the fact that this non-rigid structure (the bundle) is in turn a parameterization of identical structures (the fibers).[27]

Given a G-principal bundle $P \xrightarrow{\pi} M$, we can define the *structural Lie groupoid* associated with P:

$$P \times_G P \rightrightarrows M, \tag{1}$$

where the two arrows denote the source and the target maps between the manifold of arrows $P \times_G P$ and the manifold of objects M, and where $P \times_G P$ denotes the set of equivalence classes $[(p_2, p_1)]$ under the right G-action given by $(p_2, p_1) \cdot g = (p_2 g, p_1 g)$ ([35], p.9).[28] This means that an arrow in this groupoid can be represented by the pairs $(p_2, p_1) \cdot g$ for any $g \in G$. By using the characterization of the points in P as V-identifications of the different copies E_x of V—in particular, $p_1 : V \xrightarrow{\simeq} E_1$ and $p_2 : V \xrightarrow{\simeq} E_2$—we can interpret an arrow $[(p_2, p_1)]$ as a map

$$p_2 \circ p_1^{-1} : E_1 \xrightarrow{p_1^{-1}} V \xrightarrow{p_2} E_2, \tag{2}$$

i.e. as an identification between the copies E_1 and E_2 of V.[29] This interpretation of the arrows in the groupoid (1) explains why Ehresmann denoted this groupoid PP^{-1}. The partial multiplication of the groupoid is defined on the pairs of composable arrows, i.e. on the pairs $([p_3, p_2'], [p_2, p_1])$ such that p_2' and p_2 are on the same fiber (i.e. such that $p_2' = p_2 g$ for some

[26] In Pradines' terms: "The *geometric understanding* of *Cartan connections* led Charles Ehresmann from the Erlangen program of (abstract) *transformation groups* to the enlarged program of *Lie groupoid actions*, via the basic concept of *structural groupoid* acting through the fibres of a (smooth) *principal fibre bundle* [...]" [47].

[27] It is also worth noting that is was the same person—Ehresmann—who introduced the notions of *fiber bundle*, (Ehresmann) *connection*, and *groupoid* in the same theoretical context. The detrivialization of the notion of identity made possible by the notion of $(\infty-)$groupoid has been the condition of possibility of an ongoing "paradigm shift" (that we could characterize as a *homotopization of mathematics*) that is currently taking place in contemporary mathematics, mainly in the fields of *higher category theory* (see Chap.I in [34]), *homotopy type theory* [23], and *derived geometry* (see [2] and references therein). Given that Ehresmann's notion of a *principal fiber bundle characterized by a structural groupoid of symmetries and endowed with a connection* is the geometric underpinning of Yang-Mills theories, we could say that the ongoing homotopization of mathematics has started *avant la lettre* in the framework of fundamental physics (as Schreiber has repeatedly stressed [55]). This also explains why we think that it is important to revisit gauge theories in the light of this new groupoid-theoretical paradigm.

[28] The notion of *structural groupoid* was introduced by C. Ehresmann (see Ref.[47] for a discussion of this notion in the framework of Ehresmann's work). A *Lie groupoid* is a groupoid $G_1 \rightrightarrows G_0$ (i.e. a category in which every arrow is invertible, being G_1 and G_0 the sets of arrows and objects respectively) in which G_1 and G_0 are manifolds, the source $s : G_1 \to G_0$ and target $t : G_1 \to G_0$ maps are surjective submersions, the identity map $i : G_0 \to G_1$ is smooth and the partial multiplication $G_1 * G_1 \to G_1$ is smooth (where $G_1 * G_1 = \{(h, g) \in G \times G / s(h) = t(g)\}$ is the set of pairs of composable arrows).

[29] If we consider another representative $(p_2 g, p_1 g)$ of the arrow $[(p_2, p_1)]$ we obtain the same map. Indeed,

$$(p_2 g) \circ (p_1 g)^{-1} : E_1 \xrightarrow{p_1^{-1}} V \xrightarrow{g^{-1}} V \xrightarrow{g} V \xrightarrow{p_2} E_2 = E_1 \xrightarrow{p_1^{-1}} V \xrightarrow{p_2} E_2.$$

$g \in G$). The multiplication of these arrows gives an arrow from E_1 to E_3, which can be represented by the pair $(p_3, p_1 g)$. The structural groupoid (1) can be understood as the *symmetry groupoid* of the family $\{E_x\}_{x \in M}$ of isomorphic fibers. This means that the structural groupoid encodes both the groups of automorphisms (or self-identifications) of each fiber E_x and the hetero-identifications between different fibers—i.e. all the ways according to which two fibers can be said to be identical—(see Ref.[4], Section 1.1.1, for a nice discussion of the notion of *symmetry groupoid*).

Let us now introduce the *pair Lie groupoid*:

$$M \times M \rightrightarrows M, \tag{3}$$

where there is one arrow (x_2, x_1) for any pair of points $x_2, x_1 \in M$. The pair groupoid encodes the *maximal equivalence relation* between the points of M, where *maximal* means that any $x \in M$—i.e. each label of a fiber E_x of the G-bundle $E \to M$—is equivalent to any other. We could say that the pair groupoid encodes the *abstract* statements according to which any two fibers of the G-bundle $E \to M$ are isomorphic. On the other hand, the structural groupoid (1) encodes all the *concrete* isomorphisms that witness to the truth of these abstract statements. Now, there is a natural projection from the structural groupoid (1) to the pair-groupoid (3)—a projection that we could call *abstraction*—given by the identity on the objects and the following map (called *anchor*) between the arrows

$$\begin{array}{rcl} P \times_G P & \to & M \times M \\ [(p_2, p_1)] & \mapsto & (\pi(p_2), \pi(p_1)) \end{array} \tag{4}$$

where $\pi : P \to M$. Roughly speaking, all the different *concrete* identifications between the fibers E_1 and E_2 collapse to the unique *abstract* statement according to which E_1 and E_2 are isomorphic. We can now consider what we lose by passing to the abstract standpoint provided by the pair groupoid (3), i.e. the kernel of the projection (4). This kernel is given by the *gauge groupoid*

$$P \times_G G \rightrightarrows M, \tag{5}$$

where the right action of G on $P \times G$ is given by $(p, g) \cdot h = (ph, h^{-1} g h)$. We then have the following exact sequence of Lie groupoids ([35], Prop.1.4.2, p.25)

$$P \times_G G \hookrightarrow P \times_G P \twoheadrightarrow M \times M, \tag{6}$$

where the inclusion on the left is given by $[(p, g)] \mapsto [(p, pg)]$.[30] By means of this inclusion, an arrow $[(p, g)] \in P \times_G G$ can be interpreted as a map

$$p \circ (pg)^{-1} : E_x \xrightarrow{p^{-1}} V \xrightarrow{g^{-1}} V \xrightarrow{p} E_x, \qquad x = \pi(p),$$

i.e. as an automorphism of the G-torsor E_x. In this way, the gauge groupoid $P \times_G G$ encodes the concrete self-identifications of each fiber E_x. It is worth noting that all these concrete automorphisms of E_x are projected—by means of the anchor map (4)—to the trivial abstract statement $E_x \simeq E_x$.

The notion of groupoid can be understood as a synthesis of two "orthogonal" (so to speak) mathematical notions, namely the notion of *group* and the notion of *equivalence relation*. In other

[30] This map is independent of the chosen representative of the class in $P \times_G G$. Indeed $[(ph, h^{-1} g h)] \mapsto [(ph, phh^{-1} gh)] = [(ph, pgh)] = [(p, pg)]$.

terms, groups and equivalence relations can be understood as particular "degenerated" cases of the notion of groupoid (this point has been emphasized in Ref.[47]). Whereas a group can encode the *multiple* self-identifications of a *unique* object, an equivalence relation encodes the *unique* identifications between *multiple* objects (in the sense that there is a unique self-identification for each object and no more than one identification between different objects). The notion of groupoid, by unifying both notions, permits us to deal with both *multiple objects* and *multiple identifications* between these objects. Given a groupoid $G_1 \rightrightarrows G_0$, the arrows a in G_1 such that $s(a) = t(a) = x \in G_0$ define the groups of automorphisms of the object x. In our example, the gauge groupoid (5) is the family of groups of automorphisms of the different fibers of the G-bundle $E \to M$. According to the terminology proposed in Ref.[47], the gauge group is a *pluri-group*, i.e. a disjoint union of groups. In turn (for a general groupoid $G_1 \rightrightarrows G_0$) the anchor

$$\begin{array}{rcl}(s,t) : G_1 & \to & G_0 \times G_0 \\ g & \mapsto & (s(g), t(g))\end{array}$$

defined by the source and the target maps gives the underlying "abstract" equivalence relation. We can then understand a groupoid as a class of multiple self- and hetero-identifications in a family of objects. These identifications can always be projected to the underlying abstract equivalences between the corresponding objects of the family.

From this groupoid-theoretical description of fiber bundles, we can reobtain the usual description in terms of the *gauge group*. To do so, let us consider the bisections of the groupoids in the sequence (6).[31] It can be shown that the set $\mathcal{B}(G_\bullet)$ of bisections of a Lie groupoid has the structure of a group (see Ref.[35], Prop.1.4.2, p.22, for a detailed description of the structure of this group). By taking bisections, the sequence (6) yields the following sequence of groups:

$$\mathcal{B}(P \times_G G) \hookrightarrow \mathcal{B}(P \times_G P) \twoheadrightarrow Diff(M). \tag{7}$$

By using the fact that the structural groupoid $P \times_G P$ and the gauge groupoid $P \times_G G$ act on P[32], the groups $\mathcal{B}(P \times_G P)$ and $\mathcal{B}(P \times_G G)$ can be understood as the group of automorphisms of P and the group of *vertical* automorphisms of P respectively. The group $\mathcal{B}(P \times_G G)$ of vertical automorphisms is also called the *gauge group* and denoted $\mathcal{G}$. In this way, given a family $E \to M$ of non-rigid structures with symmetry groupoid $P \times_G P$ (encoding the identifications between the vector spaces E_x), the family as a whole can be understood as a single structure with a group of automorphisms $\mathcal{B}(P \times_G P)$. According to the sequence (7), a transformation in $\mathcal{B}(P \times_G P)$ can be projected to a diffeomorphism of M, i.e. to an "external" transformation. In turn, a transformation in the gauge group $\mathcal{B}(P \times_G G)$—being a vertical automorphism of P—is projected to the trivial diffeomorphism in $Diff(M)$ (i.e. it is a purely "internal" transformation). We could say that Ehresmann's notion of *structural groupoid* (associated with a principal fiber bundle P)

[31] Roughly speaking, a bisection of a Lie groupoid $G_\bullet \doteq G_1 \rightrightarrows G_0$ is a smooth election of a unique arrow for each object x in G_0 (starting at x) such that the resulting field of arrows on the manifold G_0 induces a diffeomorphism of G_0. More precisely, given a Lie groupoid $G_1 \rightrightarrows G_0$, a *bisection* is a smooth map $\sigma : G_0 \to G_1$ such that 1) σ is a section of the source map $s : G_1 \to G_0$ and 2) $t \circ \sigma : G_0 \to G_0$ is a diffeomorphism ([35], p.22).

[32] The structural groupoid $PP^{-1} \doteq P \times_G P$ acts on the left on P, where the action of an arrow $[(p_2, p_1)]$—understood in terms of the map (2)—on a V-identification $p \in P$ above x_1 is given by

$$p \mapsto (p_2 \circ p_1^{-1}) \circ p : V \xrightarrow{p} E_1 \xrightarrow{p_1^{-1}} V \xrightarrow{p_2} E_2. \tag{8}$$

This action is just the composition of the map (2) (defined by the arrow $[(p_2, p_1)]$) and the V-identification $p : V \to E_1$.

is a sort of *evanescent mediator* between the structural group and the gauge group of P. While these groups describe automorphisms of the corresponding structures (i.e. the automorphisms of the standard fiber and the vertical automorphisms of P respectively), the notion of structural groupoid encodes the multiple identifications in a family of identical structures (i.e. the fibers of P).

From this intrinsic description we can easily recover the epistemic characterization of the elements in the gauge group as *local gauge transformations* that change the "frame" on each location in M. If we take the vector space $V = \mathbb{R}^n$, endowed with its canonical basis as a standard fiber, then we can understand the V-identifications as *frames* $p : \mathbb{R}^n \to E_x$. We can then interpret the action of the gauge group on the V-identifications p as local "gauge" transformations of the local frames of reference. This epistemic description relies on an interpretation of the V-identifications $p : V \to E_x$ as an assignment of coordinates to the fibers E_x by means of the canonical basis in $V = \mathbb{R}^n$. However, the definition of a vector bundle just requires us to specify a standard vector space V, being the fibers of the vector bundle copies of V. Since V has a non-trivial group of automorphisms $Gl(V)$, these fibers cannot be canonically identified to V. According to this intrinsic description of vector bundles, the structural groupoid $P \times_G P$ encodes the identifications between the copies of V parameterized by M (including the "vertical" or "internal" self-identifications of each fiber E_x encoded by the gauge groupoid $P \times_G G$).

6. On Gauge Fields

Let us consider now the notion of gauge field—or Ehresmann connection—in the light of these ideas. An arrow in the pair groupoid (3) from x_1 to x_2 merely encodes the abstract proposition $E_1 \simeq E_2$ according to which E_1 and E_2 are isomorphic. Whereas we can project a concrete identification $[(p_2, p_1)] \in P \times_G P$ between two fibers E_1 and E_2—i.e. an isomorphism $p_2 \circ p_1^{-1} : E_1 \xrightarrow{\simeq} E_2$—to the underlying abstract identification in $M \times M$, there is no canonical "concretization" (or lifting) of an abstract identification. We could say that all the concrete identifications $E_1 \xrightarrow{\simeq} E_2$ democratically witness to the same abstract identification $E_1 \simeq E_2$. In order to select a concrete realization of each abstract identification it is necessary to introduce an additional geometric structure, namely an *Ehresmann connection*.

Given a Lie groupoid one can canonically construct an associated Lie algebroid.[33] Applying this construction to the exact sequence of Lie groupoids (6) one obtains the following sequence of Lie algebroids on M (see Ref.[35], def.3.3.1, p.100):

$$P \times_G \mathfrak{g} \to TP/G \xrightarrow{\pi} TM, \tag{9}$$

where the actions of G on $\mathfrak{g}$ and TP are given by the adjoint action $(X, g) \mapsto Ad_{g^{-1}} X$ and the differential of the action of G on P respectively. The algebroid defined by the vector bundle $TP/G \to M$ and an anchor map given by the projection π is called the *Atiyah algebroid* of the G-principal bundle $P \to M$. Roughly speaking, the arrows in the groupoids $G_1 \rightrightarrows M$ of the sequence (6) are now substituted with their infinitesimal generators (namely the tangent vectors to the identity arrows id_x for all x in M which are in the kernel of the differential map $Ts : TG_1 \to TM$ defined by the source map $s : G_1 \to M$). The anchor projection (4) to the pair

[33] A Lie algebroid on M is a vector bundle $A \to M$ endowed with a vector bundle map $\alpha : A \to TM$ (called *anchor*) and equipped with a bracket $[\![\cdot, \cdot]\!] : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ on the sections of $A \to M$. The Lie bracket and the anchor map are required to satisfy $\alpha([\![\sigma, \varsigma]\!]) = [\alpha(\sigma), \alpha(\varsigma)]$ (where $[\cdot, \cdot]$ is the Lie bracket in TM) and the Leibniz rule $[\![\sigma, f\varsigma]\!] = f[\![\sigma, \varsigma]\!] + \alpha(\sigma)(f)\varsigma$, for $\sigma, \varsigma \in \Gamma(\mathcal{A})$ and $f \in \mathcal{C}^\infty(M)$ ([35], def.3.5.8, p.123).

groupoid of abstract equivalences is now substituted with a projection to the tangent bundle TM. This anchor map can be understood as a projection of the infinitesimal generators of concrete identifications to the infinitesimal generators of the corresponding abstract identifications.

An *Ehresmann connection* is a section of the anchor π of the Atiyah algebroid, i.e. a morphism of vector bundles

$$\gamma : TM \to TP/G \tag{10}$$

such that $\pi \circ \gamma = id_{TM}$ ([35], p.186). This definition of an Ehresmann connection can be understood as the translation of the standard definition of a connection—as a *G-equivariant horizontal distribution* in P—into the framework provided by the theory of Lie groupoids and Lie algebroids (see for instance Ref.[29], Chap.II; see also Ref.[35], Proposition 5.3.2, p.193, for a detailed description of the relation between both definitions). The conceptual elegance of this groupoid-theoretical definition of an Ehresmann connection relies on the fact that it stresses that connections are the additional geometric structures by means of which we can overcome the lack of *canonical* identifications between the members of a family $E \to M$ of isomorphic non-rigid structures. We could say that the connection γ "connects" the different fibers by selecting, for each infinitesimal abstract identification v in T_xM, an infinitesimal concrete identification $\gamma(v)$ between the fibers E_x and E_{x+v}. In other terms, the connection γ is the geometric structure thanks to which the abstract identifications between the fibers of $E \to M$ can be lifted—generally in a path-dependent manner—to *concrete* identifications. While the projection that defines the *abstraction*—given by the map (4)—is natural, the *concretization* of abstract equivalences is a structure (given by the Ehresmann connection).

It is easy to see that the gauge group $\mathcal{G}$ acts on the space of connections. Briefly, if one describes an Ehresmann connection by means of a $\mathfrak{g}$-valued 1-form on P, the vertical automorphisms of P act on these forms by pullback. Conceptually, a geometric structure (in this case an Ehresmann connection) defined on a non-rigid structure (in this case a principal bundle $P \to M$) inherits the automorphisms of the latter. In the framework of the (constrained) Hamiltonian description of gauge theories, one considers the phase space $\mathcal{P}$ defined by the connections and their conjugated momenta. The phase space $\mathcal{P}$ is endowed with an action of the gauge group $\mathcal{G}$. According to Dirac's theory of constrained Hamiltonian systems, the phase space of *physical* gauge fields and conjugate momenta is not the phase space $\mathcal{P}$, but rather the *reduced phase space*

$$\mathcal{P}_{red} = \Sigma/\mathcal{G},$$

where $\Sigma \subset \mathcal{P}$ is the *constraint surface* defined by the common zeros of the first-class constraints of the theory (i.e. the generators of the gauge group action) [14, 22, 42].[34] In turn, the *Dirac observables* are given by the observables on the original phase space $\mathcal{P}$ whose restriction to the

[34]It is worth noting that these set-theoretic quotient spaces can be ill-defined. In the finite-dimensional case, the reduced phase space $\mathcal{P}_{red}$ is a manifold if and only if the group acts *freely* and *properly* on Σ [42]. The problem with the set-theoretic notion of quotient is that it simply "forgets" the identifications between the points of Σ that define the projection to the quotient space Σ/G (including the isotropy groups). Therefore, the quotient space Σ/G might have singularities. In order to solve this problem, it is necessary to refine the notion of quotient by keeping track—by means of groupoid-theoretic tools—of all the invertible arrows (or isomorphisms) by means of which the points of the original space are identified. This can be done for instance by means of the theory of *stacks* or by using tools coming from *noncommutative geometry*. We could then maintain (in Timmmermans' words) that it is not in general possible "*to conceal the 'pluralism of transformations' for the sole benefit of the 'realism of invariants*" ([70], p.292).

constraint surface Σ is gauge invariant. The Dirac observables are the observables on $\mathcal{P}_{red}$, i.e. the observables by means of which one can separate the different *physical* configurations of the corresponding fields. Rather than working on the reduced phase space, one might alternatively choose a section of the projection $\Sigma \to \mathcal{P}_{red} = \Sigma/\mathcal{G}$ (i.e. one can "fix the gauge") and impose the condition according to which the physical predictions should not depend on this election.

According to the standard epistemic interpretation of gauge theories described before, the condition of gauge invariance that defines the Dirac observables just encodes the fact that a physical field should not depend on the arbitrary election of a particular coordinate system. Now, this description of the gauge invariance is—in Weyl's terms—just the "second phase" of the "relativity problem" as it is posited in the particular framework of Yang-Mills theories. According to the proposed description of the geometric structure that underpins Yang-Mills theories, we can trace back the action of the gauge group on the phase space $\mathcal{P}$ to its geometric source, namely the fact that a G-bundle $E \to M$ is a parametrization of identical copies of a non-rigid structure V. The important point is that copies of a non-rigid structure cannot be canonically identified, even if they are identical. These intrinsic symmetries of the G-bundle $E \to M$ are transferred to the G-principal bundle $P \to M$ and thereby to the Ehresmann connection (gauge field) defined on P.

The fact that a geometric structure such as a gauge field is endowed with non-trivial automorphisms imposes a compatibility condition on the observables that can be consistently valuated on the structure, namely the condition of being invariant under these automorphisms. In more philosophical terms, we could say that the requirement of gauge invariance is a condition on the *determinateness* of a structure (i.e. on the properties that can be attributed to the structure) resulting from its *non-trivial identity* (i.e. from the fact that it can be identified with itself in non-trivial manners). In other terms, the non-trivial identity of an entity entails a compatibility condition on the predicates that can be attributed to the entity. As Schulman clearly states with respect to Minkowski spacetime, this spacetime "*can be identified with itself in many ways, such as translations, 3D rotations, and Lorentz boosts. These extra added identifications force everything we say about 'Minkowski spacetimes' to be invariant under their action.*" ([56], p.12). In particular, a gauge field can be interpreted as a non-rigid structure whose non-trivial identity is encoded by the gauge group $\mathcal{G}$. It follows that all the properties that we can attribute to a gauge field must satisfy the compatibility condition of being invariant under the gauge group $\mathcal{G}$ that encodes this non-trivial identity.[35]

7. On the Quantum Relation Between Identity and Determinateness

We have thus far discussed the gauge theories of fundamental interactions (i.e. general relativity and Yang-Mills theories) as a generalization—in the form of a *localization*—of Klein's Erlangen program. We shall now argue that the other pillar of fundamental physics—namely quantum mechanics—can also be understood in the light of Klein's Erlangen program, this time as a *linearization* of this program (in sense that the group of automorphisms are now given by linear representations of groups). We shall argue that this linearization of Klein's Erlangen

[35]This reconceptualization of gauge symmetries in terms of an extended notion of identity is the basis of a far-reaching formal and conceptual reformulation of quantum field theory developed by Urs Schreiber. In Schreiber's approach, the notion of *moduli space* of gauge fields (i.e. the quotient space defined by the gauge equivalent classes of gauge fields) is substituted with the refined notion of quotient given by the *moduli stack* of gauge fields [55].

program provides a formal realization of the Kleinian principle—generalized by Weyl—according to which the automorphisms of a structure carry information about its constitution. This last Section intertwines the four key points endorsed by Weyl regarding the notion of symmetry that we have mentioned in the paper's introduction, namely 1) Weyl's analysis of the "two phases" of the relativity problem (Section III.13, "*The Problem of Relativity*", in Ref.[86]), 2) the relation between the group of automorphisms of a "*structure-endowed entity* Σ" and the intrinsic constitution of Σ ([84], p.144), 3) the fact that quantum numbers are "*indices characterizing representations of groups*" ([81], p.xxi), and 4) the definition of *objectivity* as "*invariance with respect to the group of automorphisms*" ([84], p.132).

One of the central heuristic ideas of Klein's Erlangen program is—in Weyl's terms—"*that a* [homogeneous] *geometry is determined by its group of transformations*" ([85], p.74). Since the group of automorphisms of a Klein geometry encodes (what we have characterized as) its non-trivial identity, we could rephrase Klein's idea by saying that the non-trivial *identity* of a geometry carries information concerning its *determinateness*. In other terms, the *being-itself* of a homogenous geometry—far from being a mere instantiation of a trivial "principle of identity"—should be somehow related to its *being-thus*. Weyl concludes the discussion of symmetric structures in arts, nature, and mathematics proposed in his book *Symmetry* by extracting the following Kleinian "*guiding principle in modern mathematics*":

> "Whenever you have to do with a structure-endowed entity Σ try to determine its group of automorphisms, the group of those element-wise transformations which leave all structural relations undisturbed. You can expect to gain a deep insight into the constitution of Σ in this way." ([84], p.144)

According to Weyl, we could expect the group of automorphisms of a structure-endowed entity Σ to encode information concerning the constitution or the determinateness of Σ. In what follows, we shall argue that certain aspects of quantum mechanics can be understood in the light of this Klein-Weyl correspondence between the ontological categories of *identity* and *determinateness*. As we shall see, the quantum correspondence between these categories depends on the relation (discovered by Weyl and Wigner) between quantum mechanics and the theory of linear representations of groups.

Weyl's program (as Mackey calls it) amounts to explaining the commutation relations (and thereby the Heisenberg indeterminacy relations) in terms of group-theoretical considerations.[36] Weyl had the idea to prove the uniqueness of the representation of the exponential version of the commutation relations by considering two abelian groups G and $\hat{G} \doteq Hom_{Group}(G, U(1))$ in duality (see Ref.[81], Chapter IV, §15; see also Refs.[72], p.4 and [73], p.36). By using the unitary operators

$$U(a) = e^{ia\hat{p}} \qquad V(b) = e^{ib\hat{q}} \tag{11}$$

Weyl recasted the commutation relations $[\hat{p}, \hat{q}] = -i\hbar$ in the form

$$U(a)V(b) = e^{-iab}V(b)U(a).$$

[36]As Mackey writes: "It is perhaps not too far from the truth to assert that the essential idea of Weyl is that one does not have to assume the truth of the Heisenberg commutation rules—rather that they may be deduced from plausible a priori symmetry considerations." ([37], p.613).

While the operator $U(a)$ generates a translation in the position by an amount a, the operator $V(b)$ generates a translation in the momentum by an amount b.[37] Therefore, the groups of translations in the position and the momentum (that we shall denote G and $\hat{G}$ respectively) are *unitary duals*. This means that $\hat{G}$ is the abelian group of characters classifying the 1-dimensional unirreps of G (and viceversa, in the sense that $G \simeq \hat{\hat{G}}$).[38]

The important point in the present context is that the position and the momentum—far from being independent variables as we are used to thinking of them in the framework of classical mechanics—are intertwined by the fact that they belong to abelian groups that are unitary dual. This means that the values of each variable—q or p—define unirreps of the abelian group of translations in the conjugated variable. The fact that the numbers that allow us to distinguish the different quantum states fix unitary representations of groups has been explicitly stated by Weyl in the introduction to *The Theory of Groups and Quantum Mechanics* in the following terms:

> "*All quantum numbers [...] are indices characterizing representations of groups.*" ([81], p.xxi)

In particular, the quantum number p that defines the quantum state $[|p\rangle]$ (i.e. a normalized vector $|p\rangle$ modulo phase factors) is an index that fixes a 1-dimensional unirrep of the group G of translations in the position q. This representation is defined by the character

$$\begin{aligned} \chi_p : G &\rightarrow U(1), \\ a &\mapsto e^{iap}. \end{aligned}$$

A quantum state $[|p\rangle]$ defined by a particular value p of the momentum can be represented by a normalized vector $|p\rangle$ in the 1-dimensional Hilbert space associated with the unirrep of G defined by the character χ_p. A translation in the position—far from physically changing the state $[|p\rangle]$—just multiplies each representative of the state by an overall phase factor e^{iap}. In this way, the different values of q—far from defining different physical states—correspond to different representatives of the same physical state $[|p\rangle]$. In more usual terms, the value of q is completely undetermined. The phase invariance of physical states can therefore be understood as the group-theoretical version of the Heisenberg indeterminacy principle for the extreme cases of

[37] Strictly speaking, Weyl analyzed the discrete case $G = \hat{G} = \mathbb{Z}/n\mathbb{Z}$ ([81], IV.D.15, [72], p.4). In Weyl's description ([81], p.279), the operator $\hat{p}$ (called A by Weyl) generates discrete translations in the position in the sense that $\hat{p} : |q_k\rangle \mapsto |q_{k+1}\rangle$. On the other side, the operator $\hat{q}$ (called B by Weyl) generates discrete translations in the momentum in the sense that $\hat{q} : |q_k\rangle \mapsto e^{2\pi ik/n}|q_k\rangle = \varepsilon^k|q_k\rangle$, where $\varepsilon \doteq e^{2\pi i/n}$. This means that each value q_k of the position defines a map sending the translation of the momentum in one discrete step (understood as an element of what Weyl call an "abstract group") to the representation $e^{2\pi ik/n} = \varepsilon^k$. In general, each value q_k defines a character

$$\begin{aligned} \chi_k : G &\rightarrow U(1), \\ t &\mapsto e^{2\pi ikt/n} = \varepsilon^{kt}. \end{aligned}$$

sending an "abstract" translation of the momentum in t steps to a representation defined by a multiplication by the phase $e^{2\pi ikt/n}$. Briefly, each value of the position q_k defines a 1-dimensional unitary irreducible representation (*unirrep* in short) of the "abstract group" of translations in the momentum (and viceversa).

[38] This relation between the abelian groups of translations in the position and the momentum was systematized by Pontryagin in 1934 in the framework of the *Pontryagin duality* [46]. An interesting philosophical-oriented discussion of the Pontryagin duality can be found in Ref.[40].

a well-determined variable and the completely undetermined conjugate variable: if the variable p that defines the unirreps of G is well-determined, then the variable q acted upon by G is completely undetermined.

The interpretation of quantum phases is an important problem in the foundations of quantum mechanics. To a certain extent, the debate is analogous to the debate regarding the physical meaning of gauge symmetries. On the one hand, one could argue that phase symmetries are just an artifact of the Hilbert space formalism by means of which we can eliminate the "surplus structure" resulting from the fact that a quantum state (i.e. a ray in the Hilbert space) can be represented by any element in the corresponding $U(1)$-class of normalized vectors. We could get rid of both the surplus structure and the phase symmetries by passing to a coordinate-free description, i.e. by working in the corresponding projective Hilbert space (see for instance Ref.[3]). According to this line of reasoning, phase symmetries might be understood as a mere symptom of the pathologies of the Hilbert space formulation of quantum mechanics (this was for instance the viewpoint of von Neumann; see Ref.[75], p.59). Now, in the Hilbert space formulation of quantum mechanics phase symmetries are related to important physical aspects of the quantum formalism (which can appear in different guises in other formulations of the theory), namely the presence of interference effects (related to the relative phases among the terms of a linear superposition) and (as we have just explained) the Heisenberg indeterminacy relations. In Refs.[11, 13], we have argued that phase symmetries in quantum mechanics can be understood in gauge-theoretic terms.[39] If this idea is correct, then the problem of interpreting gauge symmetries and phase symmetries are one and the same problem. The importance of this conjecture in the present context is that it opens the possibility of extending the proposed Klein-oriented interpretation of gauge symmetries in terms of non-rigid structures to the quantum phase symmetries.

Let us briefly consider the relation between gauge symmetries and phase symmetries. In a gauge theory, the action of the gauge group $\mathcal{G}$ on the phase space $\mathcal{P}$ is generated by a family of observables $G_i(q,p) \in \mathcal{C}^\infty(\mathcal{P})$ (called first-class constraints) satisfying

$$\{G_i, G_j\} = c_{ij}^k G_k,$$

where the c_{ij}^k are functions [14, 22]. Among the Hamilton equations, there are equations of the form

$$G_i(q,p) = 0, \tag{12}$$

for all i. The states in $\mathcal{P}$ that satisfy these *constraint equations* define the *constraint surface* $\Sigma \subset \mathcal{P}$. As we have said before, the reduced phase space $\mathcal{P}_{red}$ is the quotient space defined by the action of the gauge group $\mathcal{G}$ (generated by the constraints) on Σ. According to the *Dirac method*, the physical states $[|\psi\rangle]$ of the corresponding quantum theory must satisfy the quantum

[39] It is worth noting that in the geometric formulation of quantum mechanics proposed in Ref.[3], phase symmetries are also understood gauge-theoretically, albeit in a different direction. Briefly, the normalization condition $\langle\psi|\psi\rangle$ (with $|\psi\rangle \in \mathcal{H}$) is interpreted as a constraint. This constraint defines the "constraint surface" $S\mathcal{H} \subset \mathcal{H}$ containing the normalized vectors. The gauge transformations generated by this constraint are nothing but the $U(1)$-phase transformations between the different representatives of a unique normalized state. It can be shown that the projective Hilbert space $P\mathcal{H} \simeq S\mathcal{H}/U(1)$ can be obtained from the original Hilbert space $\mathcal{H}$ (which is endowed with a symplectic stucture [associated with] the imaginary part of the Hermitian inner-product of $\mathcal{H}$) by means of the symplectic reduction formalism (see also Ref.[33], I.2.5 and IV.1.5).

constraint equations

$$\hat{G}_i|\psi\rangle = 0, \tag{13}$$

where $\hat{G}_i$ are the quantum operators associated with the constraints G_i. These equations imply the following *strong gauge invariance conditions*

$$e^{i\hat{G}_i\theta}|\psi\rangle = |\psi\rangle. \tag{14}$$

In group-theoretical terms, the representatives $|\psi\rangle$ of the physical state $[|\psi\rangle]$ belong to the trivial unitary representation of the gauge group. The point to stress is that the gauge transformations generated by each $\hat{G}_i$ do not modify the physical state $[|\psi\rangle]$ *in the strong sense that they leave invariant its representatives* $|\psi\rangle$.

In Ref.[11] we have argued that the phase symmetries of a quantum theory obtained by quantizing a classical system endowed with a Hamiltonian action of an abelian Lie group can be formally and conceptually interpreted in analogous terms. Briefly, the role played by the quantum constraint equations (13) is now played by the eigenvalue equations

$$\hat{f}|f_a\rangle = f_a|f_a\rangle \tag{15}$$

associated with the *non-constrained* observables $f \in \mathcal{C}^\infty(\mathcal{P})$. Whereas all the solutions to (13) are defined by the same eigenvalue of $\hat{G}_i$ (the eigenvalue 0), the non-constrained eigenvalue equations of the form (15) are solved by eigenvectors corresponding to different eigenvalues. Now, expression (14)—which encodes the *strong gauge invariance* of the physical states of a quantum gauge theory—is replaced by the following expression

$$e^{i\hat{f}\xi}|f_a\rangle = e^{if_a\xi}|f_a\rangle. \tag{16}$$

Since quantum states are given by normalized vectors of the Hilbert space *modulo overall phase factors*, the vector $|f_a\rangle$ and the transformed vector $e^{if_a\xi}|f_a\rangle$ are different representatives of the same physical state $[|f_a\rangle]$. The operator $e^{i\hat{f}\xi}$ merely changes the representative of the state $[|f_a\rangle]$, thereby *leaving the physical state as such invariant*. We can then interpret expression (16) as a weakening of the strong gauge invariance condition (14). If physical states are elements of the projective Hilbert space $P\mathcal{H}$, then the strong gauge invariance on the normalized vectors in $\mathcal{H}$ can be relaxed to an *invariance modulo phase factors* (or *phase invariance*). As we have just seen, when an observable f is not constrained—i.e. when the theory includes states corresponding to different eigenvalues of $\hat{f}$—the corresponding eigenvectors $|f_a\rangle$ are phase invariant under the action of $\hat{f}$. We could then say that the *strong gauge invariance* associated with the presence of *constrained* observables is just a particular case (when the eigenvalue is constrained to zero) of the more general notion of *phase invariance*.

If quantum gauge symmetries can be understood as a particular case of phase symmetries, then we can try to extend the Klein-oriented interpretation of the former to the latter. This amounts to interpreting the phase transformations generated by an operator $\hat{f}$ on its eigenvectors $|f_a\rangle$ as automorphisms of the corresponding quantum states $[|f_a\rangle]$. For instance, the group $U(1)$ acting in the linear representation defined by the character χ_p should be interpreted as the group of automorphisms of the state $[|p\rangle]$. This means that the undetermined variable q (i.e. the variable whose variations are generated by $\hat{p}$) must be understood as a variable spanning what we could call the *internal structure* of the state. The state $[|p\rangle]$ should then be understood as a non-rigid structure which is completely delocalized along the variable q, i.e. a structure that *envelops* (so to speak) the extension spanned by q.

The group of automorphisms of a quantum state is not just a group, but rather a *linear representation* of a group. In this sense, we can understand quantum mechanics as a *linearization* of Klein's Erlangen program. Now, the corresponding linear representation is *defined by the very variable that determines the state*. We can thus say that a quantum state $[|p\rangle]$ is *completely determined* by an index—the value p of the momentum—that fixes a linear representation of the group of translations in q. Therefore, Weyl's rephrasing of the fundamental idea of Klein's Erlangen program (reproduced at the beginning of this Section) acquires in this context a precise formal realization: the knowledge of the (linear) group of automorphisms of a quantum system provides a complete characterization of the latter. We can thus paraphrase Weyl's description of Klein's Erlangen program ([85], p.74) by stating that *a homogeneous quantum state is determined by its linear group of transformations*. If—according to the proposed conceptual framework—we understand the (linear) group of automorphisms of a state as the algebraic structure that encodes its non-trivial identity, then we can state that the index that fixes the non-trivial identity of the state $[|p\rangle]$ (its *being-itself*) is nothing but the quantum number p that defines its *determinateness* (its *being-thus*). In other terms, we can use the *identity* of a state to *identify* it.

This conceptual explanation of the fact that "*quantum numbers* [...] *are indices characterizing representations of groups*" ([81], p.xxi) provides an elegant interpretation of the two-fold role played by observables in physics. As it was stressed in Ref.[13], an observable—for instance the position or the momentum—plays two fundamental roles in mechanics: they can be evaluated on the states—thereby assigning numbers to the states—and they generate transformations of the states. In the geometric formulation of quantum mechanics proposed in Ref.[3], an Hermitian operator $\hat{v}_f$

(1) can be evaluated on a quantum state $[|\psi\rangle]$ in $P\mathcal{H}$ by means of the expectation value function
$$[|\psi\rangle] \mapsto \langle\psi|\hat{v}_f\psi\rangle$$
(endowed with the corresponding dispersion $\triangle_{[|\psi\rangle]}f$),
(2) and it acts on the states $[|\psi\rangle] \in P\mathcal{H}$ by means of the Hamiltonian vector field on $P\mathcal{H}$ defined by the expectation value function (where we are using the fact that $P\mathcal{H}$ inherits a symplectic structure from $\mathcal{H}$).

If the dispersion $\triangle_{[|\psi\rangle]}f$ is zero—i.e. if the state is an eigenstate $[|f_i\rangle]$ of the operator $\hat{v}_f$—then the transformations generated by $\hat{v}_f$ are automorphisms of the state. Moreover (as we have already explained), the quantum number f_i that determines the state has a clear interpretation: it is the index that fixes the linear group of automorphisms of the state, i.e. its non-trivial identity. If the dispersion $\triangle_{[|\psi\rangle]}f$ is not zero (i.e. if the state is not an eigenstate of the operator $\hat{v}_f$), then the transformations generated by this operator—far from being automorphisms—are "external" transformations that transform the state into a physically different state. As it was nicely stated by Zalamea, "*the indeterminacy of a physical property-as-number is precisely a measure of how much a state is changed by the property-as-transformation*" [92]. In the conceptual framework that we have proposed, we could say that the dispersion $\triangle_{[|\psi\rangle]}f$ measures the extent to which the transformations generated by $\hat{v}_f$ cannot be "absorbed", "neutralized" or "endured" by the non-rigid structure of the state.

It is worth noting that the linearization of Klein's Erlangen program opens the possibility of introducing states—such as for instance $[|p_1\rangle]$ and $[|p_2\rangle]$—determined by different linear representations *of the same abstract group* (i.e. the group of translations in q). Moreover, the non-rigid

structures represented by the states $[|p_1\rangle]$ and $[|p_2\rangle]$ can be linearly superposed. Since the groups of automorphisms of $[|p_1\rangle]$ and $[|p_2\rangle]$ are given by different linear representations of the group of translations in q, the superposed states of the form $c_1[|p_1\rangle] + c_2[|p_2\rangle]$ will no longer be invariant under translations in q. In this way, the introduction of a dispersion in the value of the momentum p by means of a linear superposition has as a consequence that the transformations generated by $\hat{p}$ are no longer automorphisms of the resulting state.

As is the case in gauge theories, the fact that a quantum state is characterized by a non-trivial identity imposes a *compatibility condition* on the observables that define properties of the state: only the observables that are invariant under the automorphisms of the state define properties of the latter. In the simplest case, the observable q does not define a property of the state $[|p\rangle]$ since it is not invariant under the group of automorphisms of this state, namely the group of translations in q. This is just a restatement of the claim according to which the position q—far from defining possible numerical properties of the state $[|p\rangle]$—should be interpreted as a variable spanning (what we have called) the internal structure of the state. In this way, both the notion of the Dirac observable in gauge theories and the Heisenberg indeterminacy principle (in the extreme cases in which one of the variables is sharply determined) appear as instantiations of the same compatibility condition between a non-rigid structure and the observables that define properties of the latter.

In this line of reasoning, Weyl's famous statement "*objectivity means invariance with respect to the group of automorphisms*" ([84], p.132) acquires what we could call a non-transcendental scope. According to the "second [epistemic] phase" of the "relativity problem", this statement makes reference to the fact that "objective" statements should not depend on the arbitrary "subjective" election of a coordinate system. Now, by stressing—in the wake of Weyl—the "first ontological phase" of the "relativity problem", it is important to note that the group at stake is not described by Weyl as a *group of coordinate transformations*, but rather as a *group of automorphisms*. In turn (as we have argued in Section N°2), the term *objectivity* seems to have in Weyl a pre-critical meaning, in the sense that it does not refer to the correlate of a transcendental constitution, but rather to what properly belongs to the intrinsic structure of the entity at stake. We can then understand the correlation between *objectivity* and *invariance* as a restatement of the compatibility condition between the predicates that define the determinateness of a state and its non-trivial identity.

8. Conclusion

In this article we have sketched a group(oid)-theoretical program–that we have called *Klein-Weyl's program*—for the conceptual interpretation of both gauge systems and quantum systems. In short, this program amounts to understanding gauge theories and quantum mechanics in terms of a *localization* and a *linearization* of Klein's Erlangen program respectively. In order to unpack these ideas, we have started by stressing the influence of Klein's Erlangen program on Weyl's discussion of the "problem of relativity". The analysis that we have proposed downgrades the importance of the "transcendental" consideration of frames of reference in order to stress that the "problem of relativity" arises in the first place from the consideration of non-rigid structures, i.e. of structures—like for instance Klein geometries and crystals—*intrinsically* endowed with non-trivial automorphisms. In order to stress this *ontological* comprehension of the notion of symmetry, we have interpreted the non-rigidity of a structure as an extension of its otherwise

trivial identity. In this conceptual framework, each automorphism of a non-rigid structure a can be understood as a *concrete* witness of the truth of the *abstract* identity principle $a = a$.

On the one hand, we have stressed the fact that the geometric underpinning of gauge theories such as general relativity (in the gauge-theoretic reformulation proposed by Cartan) and Yang-Mills theories is provided by families of copies of a non-rigid structure V. The fact that V has a non-trivial identity implies that its copies cannot be canonically identified, i.e. that there are many equivalent identifications (i.e. isomorphisms) between them. We could say that the transition from Klein's Erlangen program to gauge theories is analogous to the transition from *groups* (encoding the automorphisms of a single structure) to *groupoids* (encoding the multiple self- and hetero-identifications in a family of structures). The notion of Ehresmann connection (or gauge field) can be conceptually understood as the supplementary geometric structure that defines concrete witnesses for the abstract identities between infinitesimally closed copies of V. In this conceptual framework, gauge theories can be understood—in the wake of Cartan's theory of connections—as a sort of *localization* of Klein's Erlangen program. We have argued that this remark brings forth a new possibility for understanding the *fundamental* role played by symmetries in physics in a manner that does not rely on the "transcendental" consideration of the role played by frames of reference. In this alternative Klein-oriented conceptual framework, homogeneous structures can be understood as the *elementary building blocks* (or the *elements* in the Platonic sense of the term) out of which more general structures—such as non-homogenous geometries or bundles which are topologically non-trivial—can be constructed. By construction, these more general structures are endowed with a local symmetry associated with the symmetry of the local building blocks.

On the other hand, we have argued that quantum mechanics can be understood as a sort of *linearization* of Klein's Erlangen program. This characterization results from the fact that the non-trivial identity of a homogenous quantum state is not simply encoded by a group, but rather by a *linear representation* of a group. We have argued that this linearization of Klein's Erlangen program encodes a formal realization of the idea according to which a homogenous structure—such as a Klein geometry—"*is* determined [we underline] *by its group of transformations*" ([85], p.74). Indeed, the quantum numbers that encode the *determinateness* of a quantum state (its *being-thus*) are nothing but the "indices" that define the linear group of automorphisms of the state, i.e. its non-trivial *identity* (its *being-itself*). In short, *the determinateness of a quantum entity is encoded in its non-trivial identity.* We have also argued that the rationale behind the Heisenberg indeterminacy principle (in the extreme cases in which one of the variables is sharply determined) can be understood in terms of the compatibility condition between the non-trivial identity of a state and the properties that can be consistently attributed to it. Indeed, an observable (such as q) that is not invariant under the automorphisms of a state (such as $[|p\rangle]$) cannot define an "objective" property of the latter. Hence, the expectation value function will have a non-zero dispersion. We have argued that this dispersion measures the extent to which the transformations generated by q transform the state into a physically different state, i.e. the extent to which the non-rigidity of $[|p\rangle]$ cannot "endure" the transformations generated by q.

In order to conclude, it is worth stressing that there are at least two important directions along which the Klein-Weyl program that we have sketched here could be tested and (in the best of cases) developed. First, the analysis of the relations between the ontological categories of *identity* and *determinateness* in quantum mechanics should be extended to the cases of *non-abelian* groups of symmetries. Second, the Klein-oriented conceptual interpretations of both gauge theories and

quantum mechanics should be combined under the form of a Kleinian analysis of *quantum gauge theories* (such as for instance quantum Yang-Mills theories).

Acknowledgments

The research leading to these results has received funding from the European Research Council under the European Community's Seventh Framework Programme (FP7/2007-2013 Grant Agreement N°263523, ERC project *Philosophy of Canonical Quantum Gravity*). I would like to thank Mathieu Anel for introducing me to the "homotopization of mathematics" that results from the substitution of the notion of *identity* with the notion of *identification qua isomorphisms*.

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