From Klein's Platonic Solids to Kepler's Archimedean Solids: Elliptic Curves and *Dessins d'Enfants* Part I

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Abstract

In 1884, Felix Klein wrote his influential book, "Lectures on the Icosahedron," where he explained how to express the roots of any quintic polynomial in terms of elliptic modular functions. His idea was to relate rotations of the icosahedron with the automorphism group of 5-torsion points on a suitable elliptic curve. In fact, he created a theory which related rotations of each of the five regular solids (the tetrahedron, cube, octahedron, icosahedron, and dodecahedron) with the automorphism groups of 3-, 4-, and 5-torsion points.

Using modern language, the functions which relate the rotations with elliptic curves are Belyĭ maps. In 1984, Alexander Grothendieck introduced the concept of a *Dessin d'Enfant* in order to understand Galois groups via such maps. We will complete a circle of ideas by reviewing Klein's theory with an emphasis on the octahedron; explaining how to realize the five regular solids (the Platonic solids) as well as the thirteen semi-regular solids (the Archimedean solids) as *Dessins d'Enfant*; and discussing how the corresponding Belyĭ maps relate to moduli spaces of elliptic curves.



Research Experiences for Undergraduate Faculty (June 4 – 8, 2012)

http://aimath.org/ARCC/workshops/reuf4.html

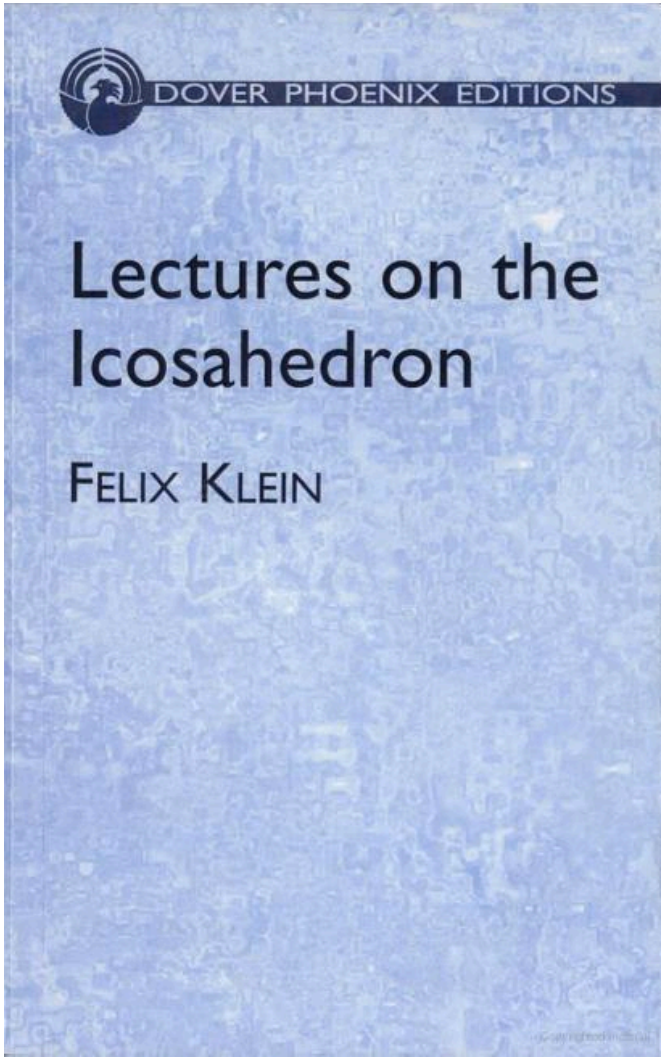
Outline of Talk

1. Lectures on the Icosahedron, Part I
   - The Icosahedron
   - Solving the Quintic
   - Examples
2. Lectures on the Icosahedron, Part II
   - Platonic Solids
   - Klein's Theory of Covariants
   - Solving Cubics, Quartics, and Quintics
3. Dessin d'Enfants
   - Belyĭ's Theorem
   - Children's Drawings
   - Examples



Christian Felix Klein (April, 25 1849 – June 22, 1925)

http://en.wikipedia.org/wiki/Felix_Klein



http://www.amazon.com/Lectures-Icosahedron-Dover-Phoenix-Editions/dp/0486495280

What is an Icosahedron?

The icosahedron is the collection of 12 vertices

$$V = \left\{ (z_1 : z_0) \in \mathbb{P}^1(\mathbb{C}) \;\middle|\; z_1\, z_0 \left({z_1}^{10} - 11\, {z_1}^5\, {z_0}^5 - {z_0}^{10} \right) = 0 \right\}$$

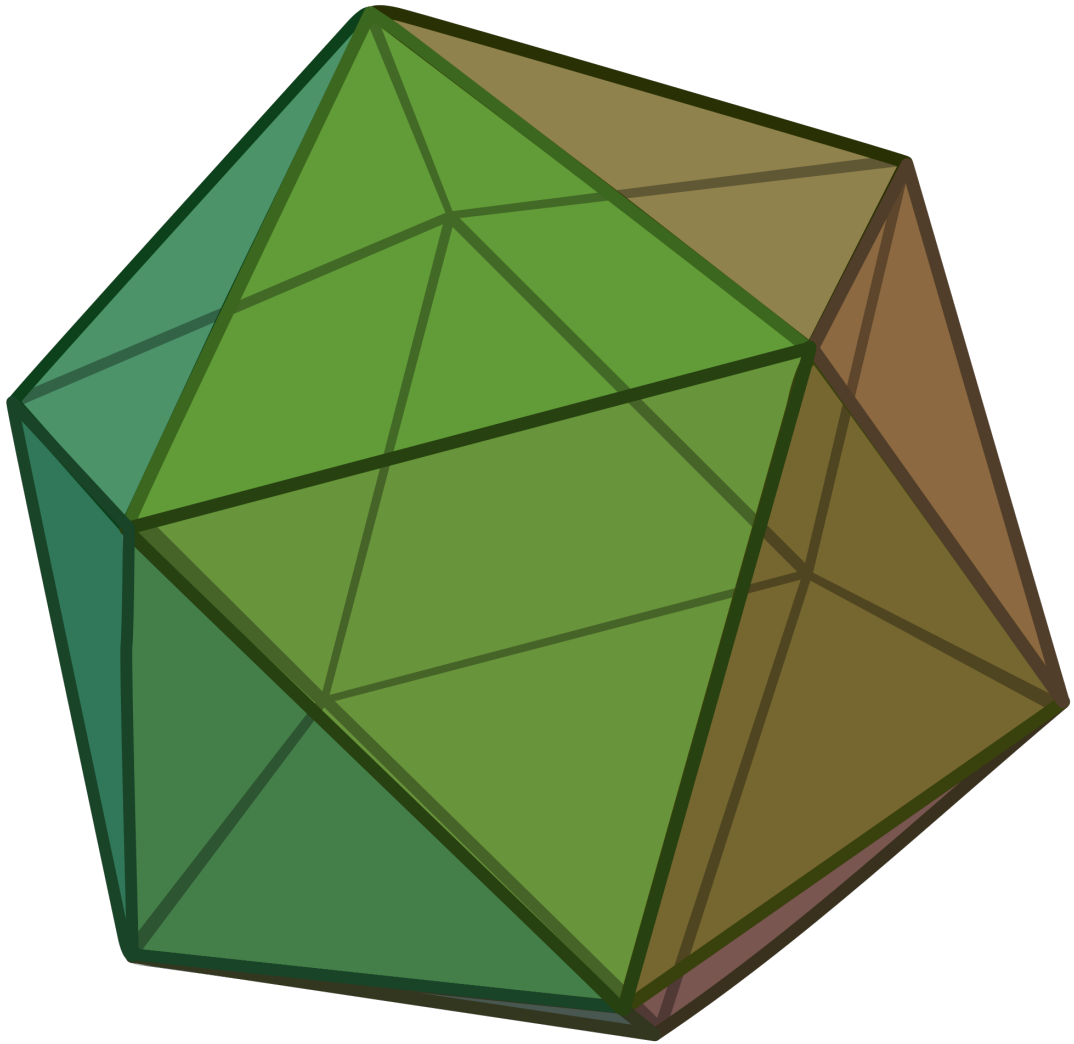
$$= \left\{ 0,\ \left(\zeta_5 + {\zeta_5}^4 \right) {\zeta_5}^{\nu},\ \left({\zeta_5}^2 + {\zeta_5}^3 \right) {\zeta_5}^{\nu},\ \infty \;\middle|\; \nu \in \mathbb{F}_5 \right\}.$$

It is a regular solid having 30 faces and 20 edges. We embed this object $V \hookrightarrow S^2(\mathbb{R})$ into the unit sphere via stereographic projection

$$\mathbb{P}^1(\mathbb{C}) \xrightarrow{\sim} S^2(\mathbb{R}) = \left\{ (u,\, v,\, w) \in \mathbb{A}^3(\mathbb{R}) \;\middle|\; u^2 + v^2 + w^2 = 1 \right\}$$

defined as that bijection which sends

$$(z_1 : z_0) \quad \mapsto \quad \left(\frac{2\,\mathrm{Re}\,(z_1\,\overline{z_0})}{|z_1|^2 + |z_0|^2},\ \frac{2\,\mathrm{Im}\,(z_1\,\overline{z_0})}{|z_1|^2 + |z_0|^2},\ \frac{|z_1|^2 - |z_0|^2}{|z_1|^2 + |z_0|^2} \right).$$



http://en.wikipedia.org/wiki/Icosahedron

Rigid Rotations of the Icosahedron

We have an action $\circ : PSL_2(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ defined by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \circ z = \frac{a\,z + b}{c\,z + d}.$$

By restriction, the alternating group $A_5 = \langle r, s \mid s^2 = r^3 = (s\,r)^5 = 1 \rangle$ acts on the icosahedron, that is, $\circ : A_5 \times V \to V$ via the fractional linear transformations

$$r(z) = \frac{\left(\zeta_5 + {\zeta_5}^4\right)\zeta_5 - z}{\left(\zeta_5 + {\zeta_5}^4\right) z + \zeta_5} \qquad \text{and} \qquad s(z) = \frac{\left(\zeta_5 + {\zeta_5}^4\right) - z}{\left(\zeta_5 + {\zeta_5}^4\right) z + 1}.$$

The following rational function is invariant under action of A_5:

$$j(z) = \frac{(z^{20} + 228\,z^{15} + 494\,z^{10} - 228\,z^5 + 1)^3}{z^5\,(z^{10} - 11\,z^5 - 1)^5}.$$

Note that the algebraic extension $\mathbb{Q}(\zeta_5, z)/\mathbb{Q}(\zeta_5, j)$ is Galois.

Principal Quintic Form

Proposition (Felix Klein, 1884)

- $j = (\lambda + 3)^3 (\lambda^2 + 11\lambda + 64) = (\mu^2 + 10\mu + 5)^3/\mu$ in terms of

$$\lambda(z) = \frac{\left[z^2 + 1\right]^2 \left[z^2 + 2\left(\zeta_5 + \zeta_5{}^4\right) z - 1\right]^2 \left[z^2 + 2\left(\zeta_5{}^2 + \zeta_5{}^3\right) z - 1\right]^2}{z\left(z^{10} - 11\, z^5 - 1\right)}$$

$$\mu(z) = \frac{125\, z^5}{z^{10} - 11\, z^5 - 1}$$

- For each m, $n \in \overline{\mathbb{Q}}$, the five resolvents

$$x_\nu = \frac{m}{\lambda(\zeta_5{}^\nu z) + 3} + \frac{n}{\left[\lambda(\zeta_5{}^\nu z) + 3\right]\left[\lambda(\zeta_5{}^\nu z)^2 + 10\,\lambda(\zeta_5{}^\nu z) + 45\right]}$$

are roots of the quintic $x^5 + A_{m,n,j}\, x^2 + B_{m,n,j}\, x + C_{m,n,j}$ where

$$A_{m,n,j} = -\frac{20}{j}\left[\left(2\, m^3 + 3\, m^2\, n\right) + 432\, \frac{6\, m\, n^2 + n^3}{1728 - j}\right]$$

$$B_{m,n,j} = -\frac{5}{j}\left[m^4 - 864\, \frac{3\, m^2\, n^2 + 2\, m\, n^3}{1728 - j} + 559872\, \frac{n^4}{(1728 - j)^2}\right]$$

$$C_{m,n,j} = -\frac{1}{j}\left[m^5 - 1440\, \frac{m^3\, n^2}{1728 - j} + 62208\, \frac{15\, m\, n^4 + 4\, n^5}{(1728 - j)^2}\right]$$

Elliptic Curves Associated to Quintic Polynomials

Theorem (Felix Klein, 1884; Annette Klute, 1997; G, 2003)

Let $q(x) = x^5 + A\,x^2 + B\,x + C$ be over $\mathbb{Q}$. Set $K = \mathbb{Q}(\sqrt{5 \cdot Disc(q)})$, and denote L as its splitting field.

- *$A = A_{m,n,j}$, $B = B_{m,n,j}$ and $C = C_{m,n,j}$ for some m, n, $j \in K$.*
- *There exists an elliptic curve E over K such that $L\,K$ is the field generated by sum $x_P + x_{2P}$ of x-coordinates of the 5-torsion of E.*

Proof: We find $\delta^5\,j^2 - 1728\,\left(\gamma_4^3 - \gamma_6^2 + \delta^5\right)\,j + 1728^2\,\gamma_4^3 = 0$ upon eliminating m and n, where

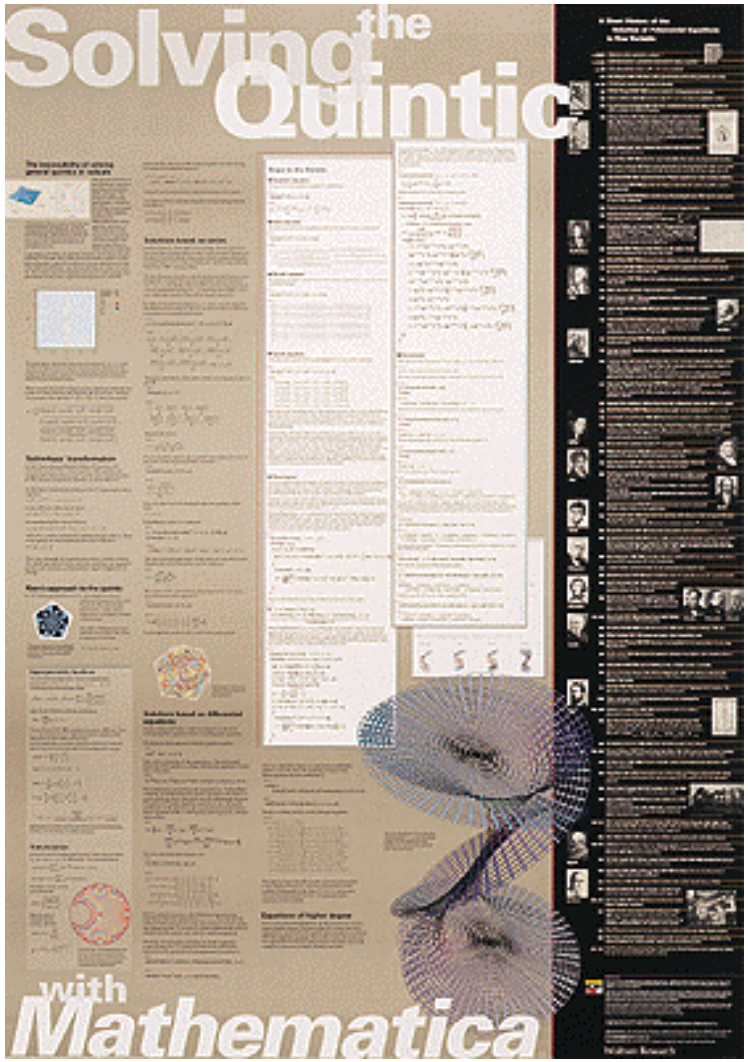
$$5^4 \cdot \delta = A^4 - 5B^3 + 25\,A\,B\,C$$

$$12^2\,5^5 \cdot \gamma_4 = 128\,A^4\,B^2 - 144\,B^5 - 192\,A^5\,C - 600\,A\,B^3\,C + 1000\,A^2\,B\,C^2 + 3125\,C^4$$

$$12^3\,5^{10} \cdot \gamma_6 = 1728\,A^{10} + 10400\,A^6\,B^3 + 405000\,A^2\,B^6 - 180000\,A^7\,B\,C - 1170000\,A^3\,B^4\,C$$
$$+ 1725000\,A^4\,B^2\,C^2 - 2025000\,B^5\,C^2 - 1800000\,A^5\,C^3 + 2812500\,A\,B^3\,C^3 - 4687500\,A^2\,B\,C^4 - 9765625\,C^6$$

Then $L\,K$ is the splitting field of $(\mu^2 + 10\,\mu + 5)^3 - j\,\mu$. For $P \in E[5]$ on $E : y^2 = x^3 + 3\,j/(1728 - j)\,x + 2\,j/(1728 - j)$, set $x = x_P + x_{2P}$. Then

$$x = -2\,\frac{\mu^2 + 10\,\mu + 5}{\mu^2 + 4\,\mu - 1} \qquad \Longleftrightarrow \qquad \mu = \frac{31104\,x^3}{(x-2)^5\,j - 1728\,x^3\,(x^2 - 10\,x + 34)}.$$ □



http://library.wolfram.com/examples/quintic/

Bring-Jerrard Quintic Form

Corollary (Kuang-yen Shih, 1978; G, 2003)

For any $t \in \mathbb{Q}$, consider the quintic polynomials

$$q_{p,t}(x) = \begin{cases} x^5 - 5\left(5\,t^2 - 1\right)x - 4\left(5\,t^2 - 1\right) & \text{for } p = 2, \\ x^5 - 20\left(5\,t^2 - 1\right)^2 x^2 - 48\left(5\,t^2 - 1\right)^3 & \text{for } p = 3; \end{cases}$$

and define the elliptic curves

$$E_{p,t}: \quad \begin{cases} y^2 = x^3 + 2\,x^2 + \frac{1}{2}(1 + \sqrt{5}\,t)\,x & \text{for } p = 2, \\ y^2 + 3\,x\,y + \frac{1}{2}(1 + \sqrt{5}\,t)\,y = x^3 & \text{for } p = 3. \end{cases}$$

- $q_{p,t}(x)$ has Galois group contained in A_5.
- $E_{p,t}$ is associated to $q_{p,t}(x)$, and is a p-isogenous $\mathbb{Q}$-curve defined over $\mathbb{Q}(\sqrt{5})$. (A $\mathbb{Q}$-curve is an elliptic curve without complex multiplication which is isogenous to each of its Galois conjugates.)
- Conversely, given $q(x) = x^5 + A\,x^2 + B\,x + C$ a quintic over $\mathbb{Q}$ with Galois group A_5 where $A\,B = 0$, set $t = \sqrt{\mathrm{Disc}(q)}/(125\,C^2)$ and $p = 2$ if $A = 0$ or $p = 3$ if $B = 0$. Then $E_{p,t}$ is associated to $q(x)$.

Can We

Generalize?

What is a Platonic Solid?

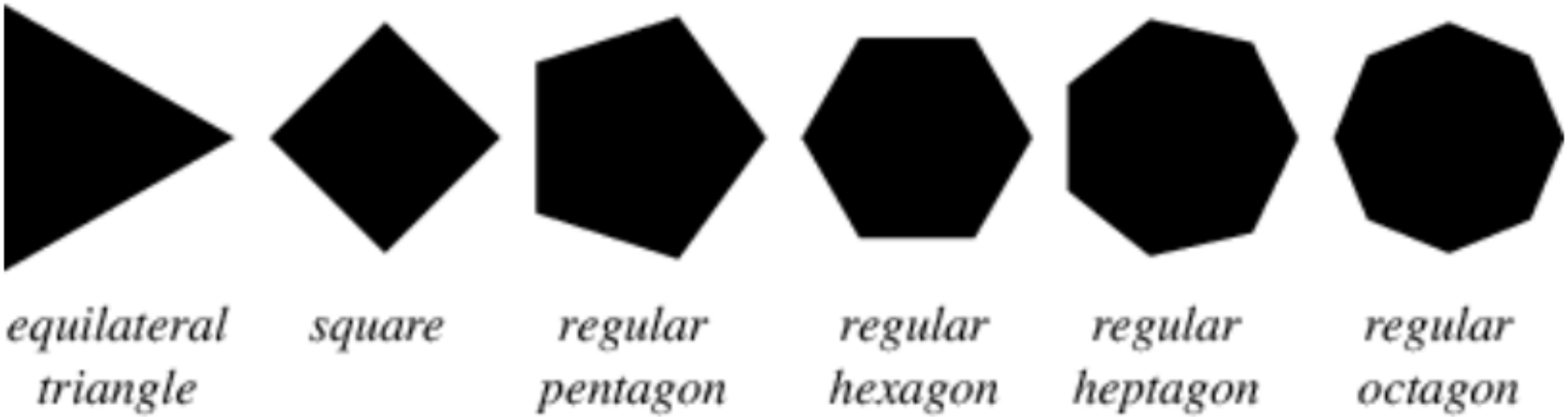
A regular, convex polyhedron is one of the collections of vertices

$$V = \left\{ (z_1 : z_0) \in \mathbb{P}^1(\mathbb{C}) \;\middle|\; \delta(z_1, z_0) = 0 \right\}$$

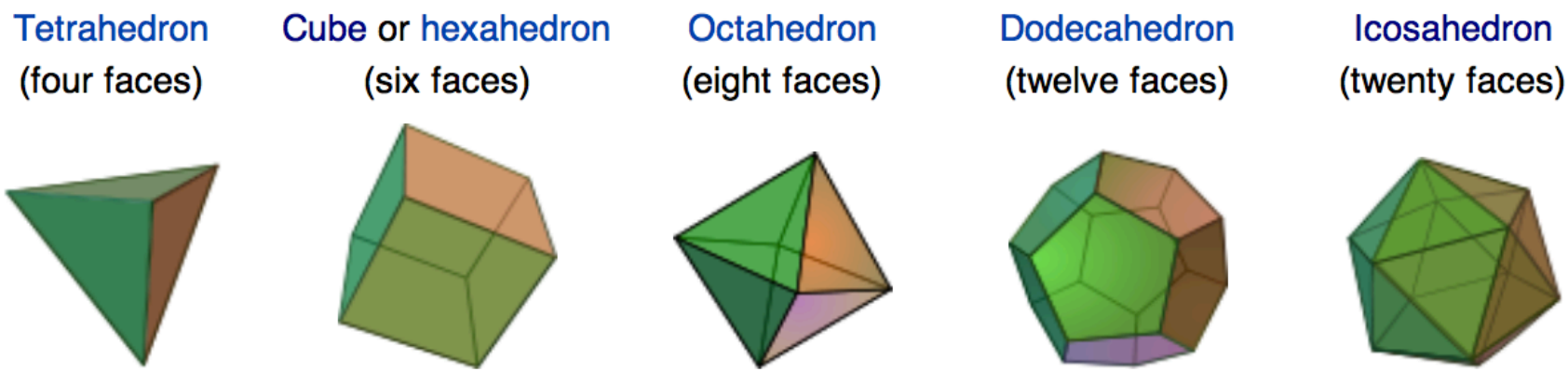
in terms of the homogeneous polynomials

$$\delta(z_1, z_0) = \begin{cases} {z_1}^n + {z_0}^n & \text{for the regular polygon,} \\ z_1 \left({z_1}^3 - {z_0}^3 \right) & \text{for the tetrahedron,} \\ z_1 z_0 \left({z_1}^4 - {z_0}^4 \right) & \text{for the octahedron,} \\ {z_1}^8 + 14 {z_1}^4 {z_0}^4 + {z_0}^8 & \text{for the cube,} \\ z_1 z_0 \left({z_1}^{10} - 11 {z_1}^5 {z_0}^5 - {z_0}^{10} \right) & \text{for the icosahedron,} \\ {z_1}^{20} + 228 {z_1}^{15} {z_0}^5 + 494 {z_1}^{10} {z_0}^{10} & \\ \quad - 228 {z_1}^5 {z_0}^{15} + {z_0}^{20} & \text{for the dodecahedron.} \end{cases}$$

We embed $V \hookrightarrow S^2(\mathbb{R})$ into the unit sphere via stereographic projection.



http://mathworld.wolfram.com/RegularPolygon.html



http://en.wikipedia.org/wiki/Platonic_solids

Rigid Rotations of the Platonic Solids

Recall the action $\circ : PSL_2(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$. We seek Galois extensions $\mathbb{Q}(\zeta_n, z)/\mathbb{Q}(\zeta_n, j)$ for rational $j(z)$.

- $Z_n = \langle r \,|\, r^n = 1 \rangle$ and $D_n = \langle r, s \,|\, s^2 = r^n = (s\,r)^2 = 1 \rangle$ are the rigid rotations of the regular convex polygons, with

$$r(z) = \zeta_n\, z, \qquad s(z) = \frac{1}{z}, \qquad \text{and} \qquad j(z) = 1728\, z^n \quad \text{or} \quad \frac{6912\, z^n}{(z^n+1)^2}.$$

- $A_4 = \langle r, s \,|\, s^2 = r^3 = (s\,r)^3 = 1 \rangle \simeq PSL_2(\mathbb{F}_3)$ are the rigid rotations of the tetrahedron, with

$$r(z) = \zeta_3\, z, \qquad s(z) = \frac{1-z}{2\,z+1}, \qquad \text{and} \qquad j(z) = -\frac{27\,(8\,z^3+1)^3}{z^3\,(z^3-1)^3}.$$

- $S_4 = \langle r, s \,|\, s^2 = r^3 = (s\,r)^4 = 1 \rangle \simeq PGL_2(\mathbb{F}_3) \simeq PSL_2(\mathbb{Z}/4\,\mathbb{Z})$ are the rigid rotations of the octahedron and the cube, with

$$r(z) = \frac{\zeta_4 + z}{\zeta_4 - z}, \qquad s(z) = \frac{1-z}{1+z}, \qquad \text{and} \qquad j(z) = \frac{16\,(z^8+14\,z^4+1)^3}{z^4\,(z^4-1)^4}.$$

- $A_5 = \langle r, s \,|\, s^2 = r^3 = (s\,r)^5 = 1 \rangle \simeq PSL_2(\mathbb{F}_4) \simeq PSL_2(\mathbb{F}_5)$ are the rigid rotations of the icosahedron and the dodecahedron, with

$$r(z) = \frac{(\zeta_5 + {\zeta_5}^4)\,\zeta_5 - z}{(\zeta_5 + {\zeta_5}^4)\,z + \zeta_5}, \quad s(z) = \frac{(\zeta_5 + {\zeta_5}^4) - z}{(\zeta_5 + {\zeta_5}^4)\,z + 1}, \quad j(z) = \frac{(z^{20} + 228\,z^{15} + 494\,z^{10} - 228\,z^5 + 1)^3}{z^5\,(z^{10} - 11\,z^5 - 1)^5}.$$

Klein's Theory of Covariants

Each $\delta = \delta(z_1, z_0)$ is invariant under a group. Choose nonzero α, β, γ, and n such that $c_4^3 - c_6^2 = 1728\,\Delta$ for the homogeneous polynomials

$$c_4(z_1, z_0) = \alpha \cdot \det \begin{bmatrix} \dfrac{\partial^2 \delta}{\partial z_1^2}(z_1, z_0) & \dfrac{\partial^2 \delta}{\partial z_1 \partial z_0}(z_1, z_0) \\ \dfrac{\partial^2 \delta}{\partial z_0 \partial z_1}(z_1, z_0) & \dfrac{\partial^2 \delta}{\partial z_0^2}(z_1, z_0) \end{bmatrix}$$

$$c_6(z_1, z_0) = \beta \cdot \det \begin{bmatrix} \dfrac{\partial \delta}{\partial z_1}(z_1, z_0) & \dfrac{\partial \delta}{\partial z_0}(z_1, z_0) \\ \dfrac{\partial c_4}{\partial z_1}(z_1, z_0) & \dfrac{\partial c_4}{\partial z_0}(z_1, z_0) \end{bmatrix}$$

$$\Delta(z_1, z_0) = \gamma \cdot \delta(z_1, z_0)^n$$

These are also invariant under the specified groups. For each z in the inverse image of the thrice punctured sphere

$$j^{-1}\Big(\mathbb{P}^1(\mathbb{C}) - \{0,\, 1728,\, \infty\}\Big) = \left\{ z \in \mathbb{P}^1(\mathbb{C}) \;\middle|\; c_4(z)\, c_6(z)\, \Delta(z) \neq 0 \right\}$$

we form the elliptic curve

$$E: \;\; y^2 = x^3 + \frac{3\,j(z)}{1728 - j(z)}\, x + \frac{2\,j(z)}{1728 - j(z)} \quad \text{where} \quad j(z) = \frac{c_4(z)^3}{\Delta(z)}.$$

$$
\begin{aligned}
D_n: \quad c_4(z_1, z_0) &= -144\, {z_1}^{n-2}\, {z_0}^{n-2} \\
c_6(z_1, z_0) &= -864\, {z_1}^{n-3}\, {z_0}^{n-3} \left({z_1}^n - {z_0}^n\right) \\
\Delta(z_1, z_0) &= -432\, {z_1}^{2n-6}\, {z_0}^{2n-6} \left({z_1}^n + {z_0}^n\right)^2 \\
j(z_1, z_0) &= 6912\, {z_1}^n\, {z_0}^n \Big/ \left({z_1}^n + {z_0}^n\right)^2
\end{aligned}
$$

$$
\begin{aligned}
A_4: \quad c_4(z_1, z_0) &= 9\, z_0 \left(8\, {z_1}^3 + {z_0}^4\right) \\
c_6(z_1, z_0) &= 27 \left(8\, {z_1}^6 + 20\, {z_1}^3\, {z_0}^3 - {z_0}^6\right) \\
\Delta(z_1, z_0) &= -27\, z_1 \left({z_1}^3 - {z_0}^3\right)^3 \\
j(z_1, z_0) &= -27\, {z_0}^3 \left(8\, {z_1}^3 + {z_0}^3\right)^3 \Big/ \left[{z_1}^3 \left({z_1}^3 - {z_0}^3\right)^3\right]
\end{aligned}
$$

$$
\begin{aligned}
S_4: \quad c_4(z_1, z_0) &= 4 \left({z_1}^8 + 14\, {z_1}^4\, {z_0}^4 + {z_0}^8\right) \\
c_6(z_1, z_0) &= -8 \left({z_1}^{12} - 33\, {z_1}^8\, {z_0}^4 - 33\, {z_1}^4\, {z_0}^8 + {z_0}^{12}\right) \\
\Delta(z_1, z_0) &= 4\, {z_1}^4\, {z_0}^4 \left({z_1}^4 - {z_0}^4\right)^4 \\
j(z_1, z_0) &= 16 \left({z_1}^8 + 14\, {z_1}^4\, {z_0}^4 + {z_0}^8\right)^3 \Big/ \left[{z_1}^4\, {z_0}^4 \left({z_1}^4 - {z_0}^4\right)^4\right]
\end{aligned}
$$

$$
\begin{aligned}
A_5: \quad c_4(z_1, z_0) &= {z_1}^{20} + 228\, {z_1}^{15}\, {z_0}^5 + 494\, {z_1}^{10}\, {z_0}^{10} - 228\, {z_1}^5\, {z_0}^{15} + {z_0}^{20} \\
c_6(z_1, z_0) &= {z_1}^{30} - 522\, {z_1}^{25}\, {z_0}^5 - 10005\, {z_1}^{20}\, {z_0}^{10} - 10005\, {z_1}^{10}\, {z_0}^{20} + 522\, {z_1}^5\, {z_0}^{25} + {z_0}^{30} \\
\Delta(z_1, z_0) &= {z_1}^5\, {z_0}^5 \left({z_1}^{10} - 11\, {z_1}^5\, {z_0}^5 - {z_0}^{10}\right)^5 \\
j(z_1, z_0) &= \left({z_1}^{20} + 228\, {z_1}^{15}\, {z_0}^5 + 494\, {z_1}^{10}\, {z_0}^{10} - 228\, {z_1}^5\, {z_0}^{15} + {z_0}^{20}\right)^3 \Big/ \left[{z_1}^5\, {z_0}^5 \left({z_1}^{10} - 11\, {z_1}^5\, {z_0}^5 - {z_0}^{10}\right)^5\right]
\end{aligned}
$$

Klein's Theory of Covariants (cont'd.)

Theorem (Felix Klein, 1884)

The rational function

$$j(z) = \begin{cases} -27\,\dfrac{\left(8\,z^3+1\right)^3}{z^3\left(z^3-1\right)^3} & \text{for } n=3, \\ 16\,\dfrac{\left(z^8+14\,z^4+1\right)^3}{z^4\left(z^4-1\right)^4} & \text{for } n=4, \\ \dfrac{\left(z^{20}+228\,z^{15}+494\,z^{10}-228\,z^5+1\right)^3}{z^5\left(z^{10}-11\,z^5-1\right)^5} & \text{for } n=5; \end{cases}$$

is invariant under the group $G = \left\langle r,\, s \,\middle|\, s^2 = r^3 = (s\,r)^n = 1 \right\rangle$ *expressed in terms of the generators*

$$r(z) = \begin{cases} \zeta_3\,z & \text{for } n=3, \\ \dfrac{\zeta_4+z}{\zeta_4-z} & \text{for } n=4, \\ \dfrac{\left(\zeta_5+{\zeta_5}^4\right)\zeta_5-z}{\left(\zeta_5+{\zeta_5}^4\right)z+\zeta_5} & \text{for } n=5; \end{cases} \qquad s(z) = \begin{cases} \dfrac{1-z}{2\,z+1} & \text{for } n=3, \\ \dfrac{1-z}{1+z} & \text{for } n=4, \\ \dfrac{\left(\zeta_5+{\zeta_5}^4\right)-z}{\left(\zeta_5+{\zeta_5}^4\right)z+1} & \text{for } n=5. \end{cases}$$

In particular, $Gal\big(\mathbb{Q}(\zeta_n, z)/\mathbb{Q}(\zeta_n, j)\big) \simeq G \simeq PSL_2(\mathbb{Z}/n\,\mathbb{Z})$.

Elliptic Curves Associated to Principal Polynomials

Theorem (Felix Klein, 1884; G, 1999)

Set $n = 3, 4, 5$. Let $q(x) = x^n + A\,x^{n-3} + \cdots + B\,x + C$ be over $K = \mathbb{Q}(\zeta_n)$ with splitting field L, and assume that $Gal(L/K) \simeq PSL_2(\mathbb{Z}/n\,\mathbb{Z})$. Then there exists $j \in K$ such that $L\,K = K(E[n]_x)$ is the field generated by the x-coordinates of the n-torsion of an elliptic curve E with invariant j.

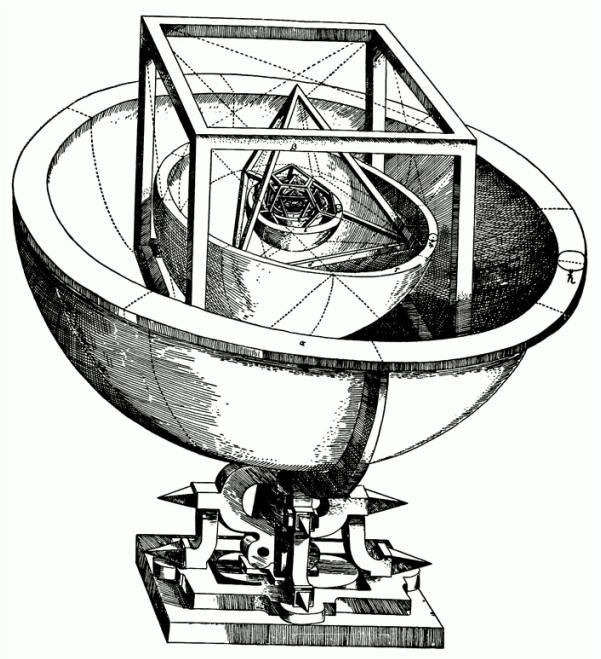
If we define the rational functions

$$\lambda(z) = \begin{cases} \dfrac{8\,z^3 + 1}{z\,(z^3 - 1)} & \text{for } n = 3, \\ \dfrac{(1+\zeta_4)\left[z^2 - (1+\zeta_4)\,z - \zeta_4\right]\left[z^2 - (1-\zeta_4)\,z + \zeta_4\right]\left[z^2 + (1+\zeta_4)\,z - \zeta_4\right]}{z\,(z^4 - 1)} & \text{for } n = 4, \\ \dfrac{\left[z^2+1\right]^2\left[z^2 + 2\left(\zeta_5 + {\zeta_5}^4\right)z - 1\right]^2\left[z^2 + 2\left({\zeta_5}^2 + {\zeta_5}^3\right)z - 1\right]^2}{z\,\left(z^{10} - 11\,z^5 - 1\right)} + 3 & \text{for } n = 5; \end{cases}$$

then we have the polynomials

$$q(x) = \prod_{\nu}\left[x - \frac{1}{\lambda\left({\zeta_n}^{\nu}\,z\right)}\right] = \begin{cases} x^3 + \dfrac{1}{j} & \text{for } n = 3, \\ x^4 + \dfrac{32}{j}\,x + \dfrac{4}{j} & \text{for } n = 4, \\ x^5 - \dfrac{40}{j}\,x^2 - \dfrac{5}{j}\,x - \dfrac{1}{j} & \text{for } n = 5. \end{cases}$$

Johannes Kepler's *Mysterium Cosmographicum* (1596)



"Kepler's principal goal was to explain the relationship between the existence of five planets (and their motions) and the five regular solids. It is customary to sneer at Kepler for this. . . . It is instructive to compare this with the current attempts to 'explain' the zoology of elementary particles in terms of irreducible representations of Lie groups."

– Shlomo Sternberg, *Celestial Mechanics* (1969), pg. 95

Motivating Question

Let X be a compact Riemann surface. Fix a function $\phi : X \to \mathbb{P}^1(\mathbb{C})$. For each z in the inverse image of the thrice punctured sphere

$$\phi^{-1}\Big(\mathbb{P}^1(\mathbb{C}) - \{0, 1, \infty\}\Big) \subseteq X$$

form the elliptic curve

$$E: \; y^2 = x^3 + \frac{3}{\phi(z) - 1}\, x + \frac{2}{\phi(z) - 1} \qquad \text{where} \qquad j(E) = \frac{1728}{\phi(z)}.$$

What are the properties of this elliptic curve?

- Which types of functions $\phi : X \to \mathbb{P}^1(\mathbb{C})$ are allowed?
- If $\phi(z)$ has "lots of symmetries," how does this translate into properties of the torsion elements $K(E[n]_x)$?

Belyĭ's Theorem

Let $\phi : X \to \mathbb{P}^1(\mathbb{C})$ be a meromorphic function on a Riemann surface X. We say $z \in X$ is a critical point if $\phi'(z) = 0$, and $w \in \mathbb{P}^1(\mathbb{C})$ is a critical value if $w = \phi(z)$ for some critical point $z \in X$.

Theorem (André Weil, 1956; Gennadiĭ Vladimirovich Belyĭ, 1979)

Let X be a compact, connected Riemann surface.

- *X is a smooth, irreducible, projective variety of dimension 1. In particular, X is an algebraic variety; that is, it can be defined by polynomial equations.*
- *If X can be defined by a polynomial equation $\sum_{i,j} a_{ij}\, z^i\, w^j = 0$ where the coefficients a_{ij} are not transcendental, then there exists a rational function $\phi : X \to \mathbb{P}^1(\mathbb{C})$ which has at most three critical values.*
- *Conversely, if there exists rational function $\phi : X \to \mathbb{P}^1(\mathbb{C})$ which has at most three critical values, then X can be defined by a polynomial equation $\sum_{i,j} a_{ij}\, z^i\, w^j = 0$ where the coefficients a_{ij} are not transcendental.*

Proof: Fix a meromorphic $f : X \to \mathbb{P}^1(\mathbb{C})$ such that $|f^{-1}(z)| = n$ for almost all $z \in \mathbb{P}^1(\mathbb{C})$. For any meromorphic $g : X \to \mathbb{P}^1(\mathbb{C})$, consider

$$\prod_{P \in f^{-1}(z)} \big(w - g(P)\big) = \sum_{j=0}^{n} c_j(z)\, w^j.$$

Each coefficient $c_j = c_j(z)$ is a well-defined rational function.

$$c_j(z) = \frac{\displaystyle\sum_{i=0}^{m} a_{ij}\, z^i}{\displaystyle\sum_{i=0}^{m} a_{in}\, z^i} \qquad \Longrightarrow \qquad \sum_{j=0}^{n} c_j(z)\, w^j = \frac{\displaystyle\sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij}\, z^i\, w^j}{\displaystyle\sum_{i=0}^{m} a_{in}\, z^i}.$$

Hence we have a bijective map

$$\begin{array}{ccc} X & \xrightarrow{\ \sim\ } & \left\{ (z_1 : z_2 : z_0) \in \mathbb{P}^2(\mathbb{C}) \;\middle|\; \displaystyle\sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij}\, {z_1}^i\, {z_2}^j\, {z_0}^{e-i-j} = 0 \right\} \\ z & \longrightarrow & (f(z) : g(z) : 1) \end{array}$$

Proof (cont'd.): Let $f : X \to \mathbb{P}^1(\mathbb{C})$ be a rational function. Following Wushi Goldring, we create a series of compositions

$$\phi : \qquad X \xrightarrow{f} \mathbb{P}^1(\mathbb{C}) \xrightarrow{g} \mathbb{P}^1(\mathbb{C}) \xrightarrow{h} \mathbb{P}^1(\mathbb{C})$$

Define $g_0(z)$ as that polynomial over $\mathbb{Q}$ which contains the affine critical points of $f(z)$ as its roots; and define Σ_0 as the set of finite critical values of g_0. Recursively define

$$\Sigma_k = \left\{ w \in \mathbb{A}^1(\mathbb{C}) \;\middle|\; w = g_k(z) \text{ for some } z \text{ with } g_k'(z) = 0 \right\}$$

$$g_{k+1}(z) = \prod_{w \in \Sigma_k} (z - w).$$

Each $g_k(z)$ has rational coefficients, distinct roots, and $\deg g_{k+1} < \deg g_k$. As $\deg g_{k_0+1} = 1$ for some positive integer k_0, define the polynomial $g = g_{k_0} \circ \cdots \circ g_2 \circ g_1$ which has rational coefficients. Hence $g \circ f$ is a rational function whose critical points are the critical points of $f(z)$, and whose critical values lie in $\mathbb{P}^1(\mathbb{Q})$.

Proof (cont'd.): Say that the critical values of $g \circ f$ lie in the set

$$\Sigma = \{(m_1 : m_0),\ (m_2 : m_0),\ \ldots,\ (m_n : m_0),\ (1 : 0)\} \subseteq \mathbb{P}^1(\mathbb{Q})$$

where each $m_k \in \mathbb{Z}$. Consider the rational function

$$h(z) = \prod_{k=1}^{n} (m_0\, z - m_k)^{e_k} \qquad \text{where} \qquad e_k = \frac{\prod_{i<j}(m_j - m_i)}{\prod_{i \neq k}(m_k - m_i)}.$$

One checks that we have the identities

$$\frac{1}{m_0}\,\frac{h'(z)}{h(z)} = \sum_{k=1}^{n} \frac{e_k}{m_0\, z - m_k} = \frac{\prod_{i<j}(m_j - m_i)}{\prod_k (m_0\, z - m_k)} \qquad \text{and} \qquad \sum_{k=1}^{n} e_k = 0$$

In particular, Σ is the set of critical points of h, so that $\{0,\ 1,\ \infty\}$ is the see of its critical values. Hence $\phi = h \circ g \circ f$ has critical values only at $\{0,\ 1,\ \infty\}$.

The converse to these statements was shown by André Weil in a 1956 paper entitled "The Field of Definition of a Variety". □

Belyĭ Maps

Theorem (André Weil, 1956; Gennadiĭ Vladimirovich Belyĭ, 1979)

A compact, connected Riemann surface X can be defined by a polynomial equation $\sum_{i,j} a_{ij}\, z^i\, w^j = 0$ where the coefficients a_{ij} are not transcendental if and only if there exists a rational function $\phi : X \to \mathbb{P}^1(\mathbb{C})$ which has at most three critical values $\{0, 1, \infty\}$.

"This discovery, which is technically so simple, made a very strong impression on me, and it represents a decisive turning point in the course of my reflections, a shift in particular of my centre of interest in mathematics, which suddenly found itself strongly focused. I do not believe that a mathematical fact has ever struck me quite so strongly as this one, nor had a comparable psychological impact.

– Alexander Grothendieck, *Esquisse d'un Programme* (1984)

Definition

A rational function $\phi : X \to \mathbb{P}^1(\mathbb{C})$ which has at most three critical values $\{0, 1, \infty\}$ is called a Belyĭ map.

Dessins d'Enfant

Fix a Belyĭ map $\phi : X \to \mathbb{P}^1(\mathbb{C})$. Denote the preimages

$$B = \phi^{-1}(\{0\}) \qquad W = \phi^{-1}(\{1\}) \qquad E = \phi^{-1}([0,1]) \qquad X \xrightarrow{\phi} \mathbb{P}^1(\mathbb{C})$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\left\{\begin{matrix}\text{black}\\ \text{vertices}\end{matrix}\right\} \qquad \left\{\begin{matrix}\text{white}\\ \text{vertices}\end{matrix}\right\} \qquad \left\{\text{edges}\right\} \qquad \mathbb{R}^3$$

The bipartite graph $\Gamma_\phi = (V, E)$ with vertices $V = B \cup W$ and edges E is called Dessin d'Enfant. We embed the graph on X in 3-dimensions.

> I do not believe that a mathematical fact has ever struck me quite so strongly as this one, nor had a comparable psychological impact. This is surely because of the very familiar, non-technical nature of the objects considered, of which any child's drawing scrawled on a bit of paper (at least if the drawing is made without lifting the pencil) gives a perfectly explicit example. To such a *dessin* we find associated subtle arithmetic invariants, which are completely turned topsy-turvy as soon as we add one more stroke.
>
> – Alexander Grothendieck, *Esquisse d'un Programme* (1984)

Properties of Dessins

Theorem

Let $\phi : X \to \mathbb{P}^1(\mathbb{C})$ be a Belyĭ map on a compact, connected Riemann Surface X. Define a bipartite graph $\Gamma_\phi = (V, E)$ by choosing

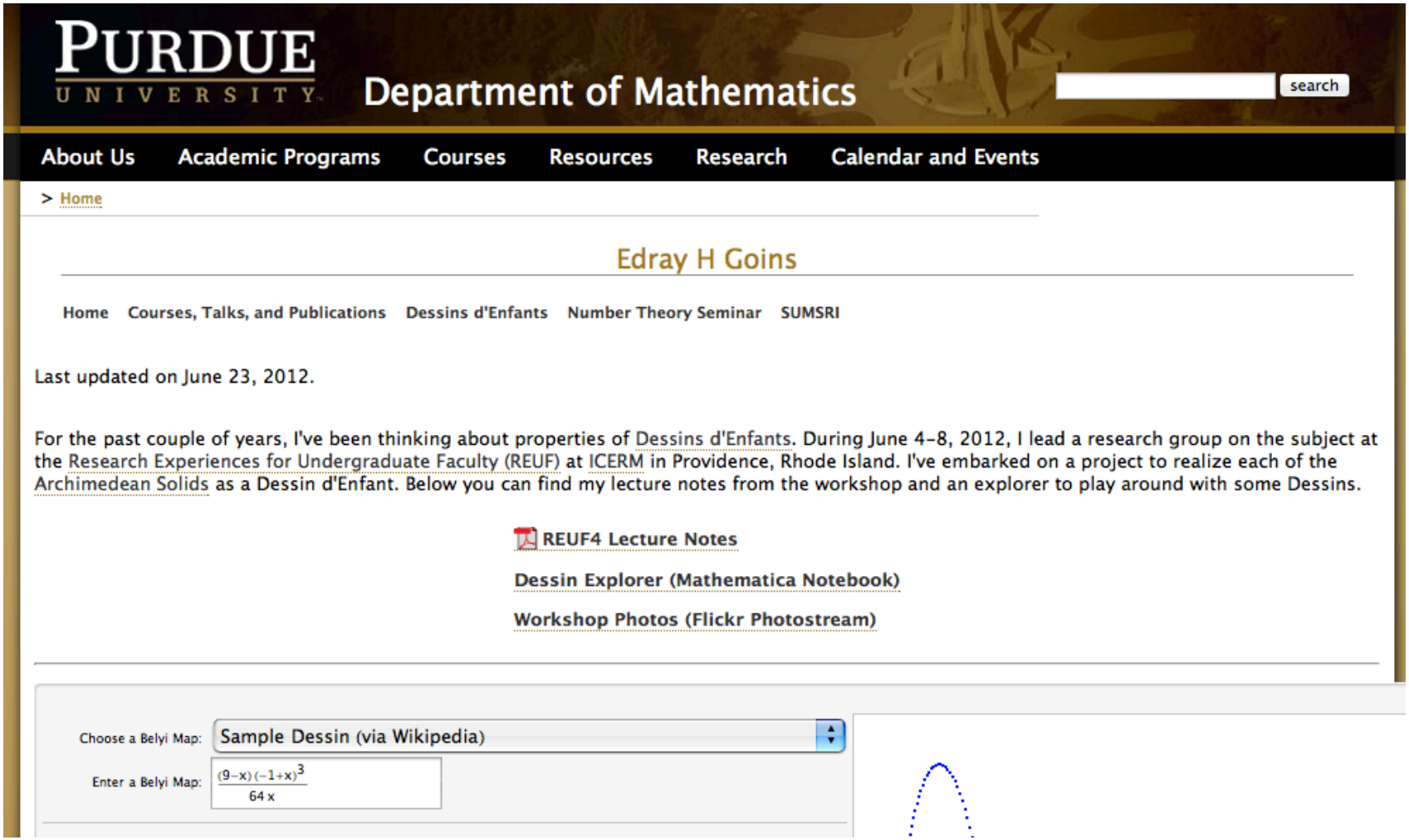
$$V = \phi^{-1}(\{0, 1\}) \qquad \text{and} \qquad E = \phi^{-1}([0, 1]).$$

- *If $z \in \phi^{-1}\left(\mathbb{P}^1(\mathbb{C}) - \{0, 1, \infty\}\right) \subseteq X$ then z is not a critical point for ϕ.*
- *The graph Γ_ϕ can be embedded in X without crossings.*
- *The number of vertices, edges, and faces is*

$$v = \left|\phi^{-1}(\{0, 1\})\right| \qquad e = \deg \phi \qquad f = \left|\phi^{-1}(\{\infty\})\right|.$$

- *We have $v - e + f = 2 - 2g$ in terms of the genus*

$$g = 1 + \frac{1}{2}\left[\deg \phi - \left|\phi^{-1}(\{0, 1, \infty\})\right|\right].$$



http://www.math.purdue.edu/~egoins/site//Dessins_d'Enfants.html

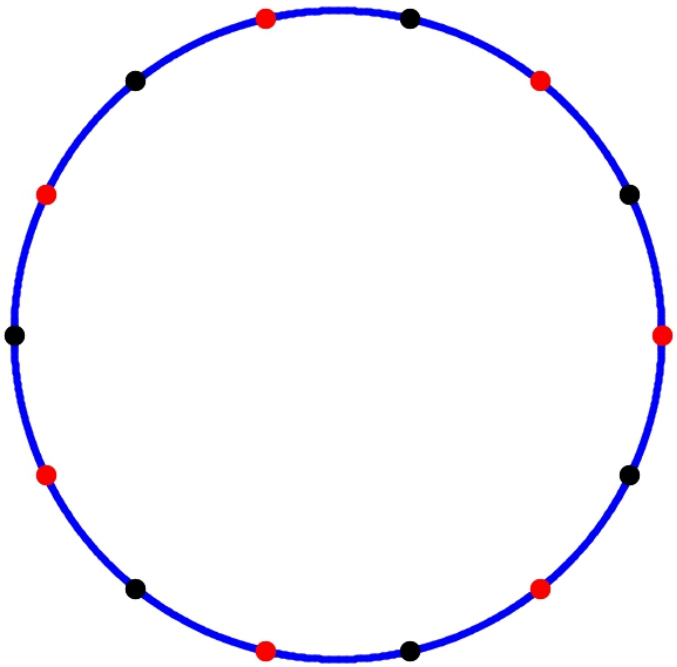
Examples

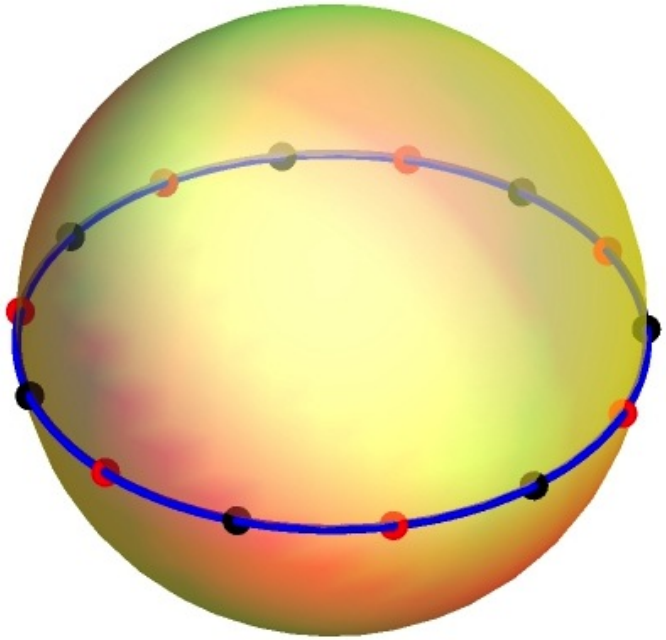
Klein constructed the Platonic Solids as the inverse images of the origin $V = \phi^{-1}(\{0\})$ in terms of the rational functions $\phi : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ given by

$$\phi(z) = \begin{cases} \dfrac{(z^n+1)^2}{4\,z^n} & \text{for the regular polygons,} \\ -\dfrac{64\,z^3\left(z^3-1\right)^3}{\left(8\,z^3+1\right)^3} & \text{for the tetrahedron,} \\ \dfrac{108\,z^4\left(z^4-1\right)^4}{\left(z^8+14\,z^4+1\right)^3} & \text{for the octahedron,} \\ \dfrac{\left(z^8+14\,z^4+1\right)^3}{108\,z^4\left(z^4-1\right)^4} & \text{for the cube,} \\ \dfrac{1728\,z^5\left(z^{10}-11\,z^5-1\right)^5}{\left(z^{20}+228\,z^{15}+494\,z^{10}-228\,z^5+1\right)^3} & \text{for the icosahedron,} \\ \dfrac{\left(z^{20}+228\,z^{15}+494\,z^{10}-228\,z^5+1\right)^3}{1728\,z^5\left(z^{10}-11\,z^5-1\right)^5} & \text{for the dodecahedron.} \end{cases}$$

These are examples of Belyĭ maps on $X = \mathbb{P}^1(\mathbb{C}) \simeq S^2(\mathbb{R})$.

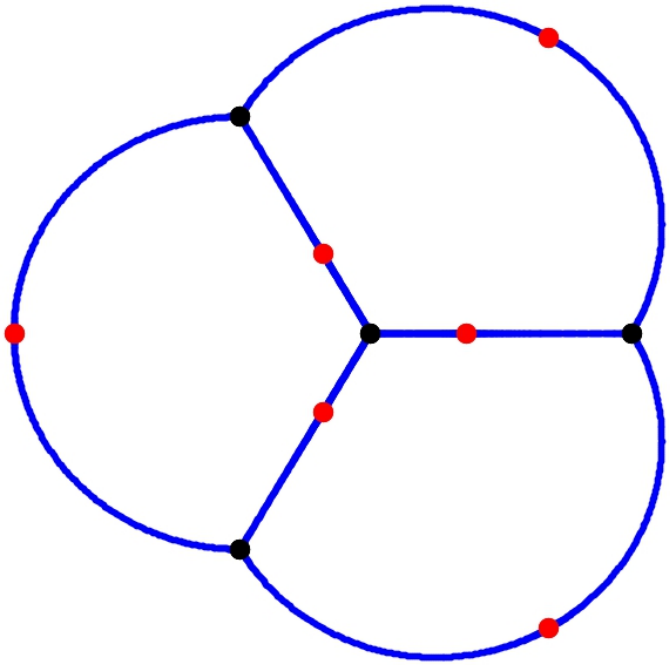
Rotation Group D_n: Regular Convex Polygon

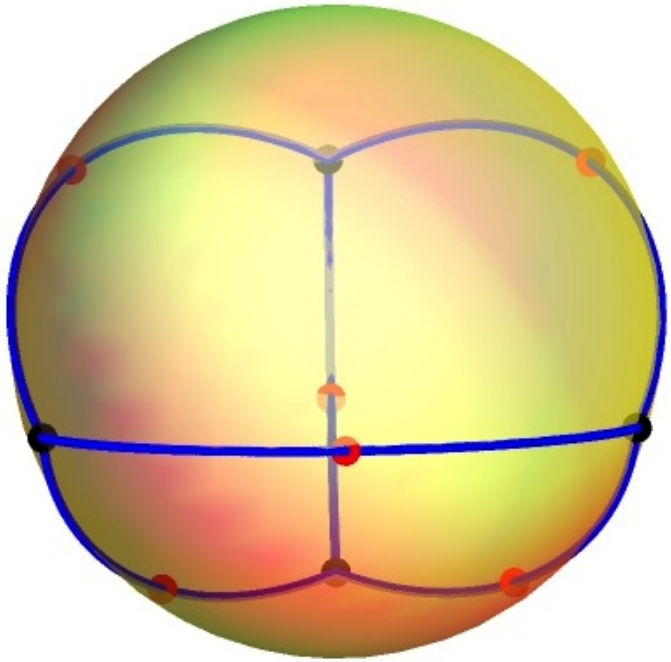




$$\phi(z) = \frac{\left(z^n + 1\right)^2}{4\, z^n} : \qquad v = n + n, \quad e = 2 \cdot n, \quad f = 2$$

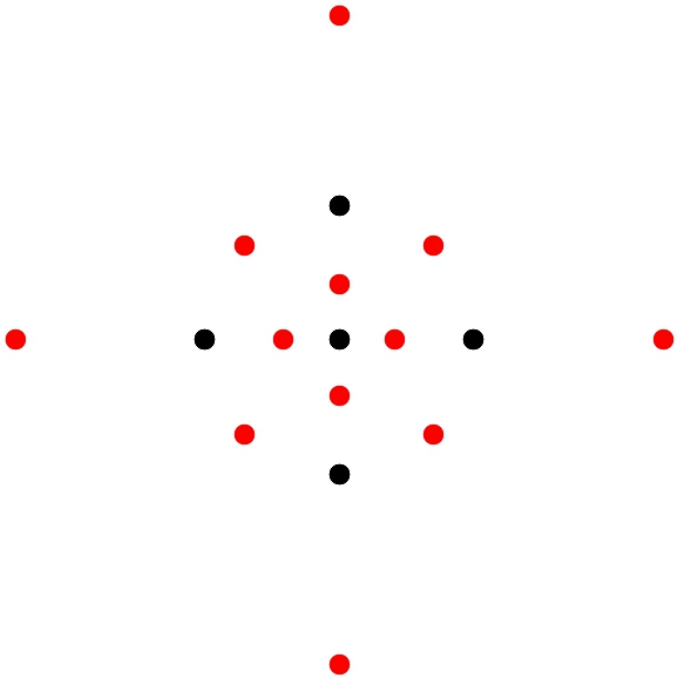
Rotation Group A_4: Tetrahedron

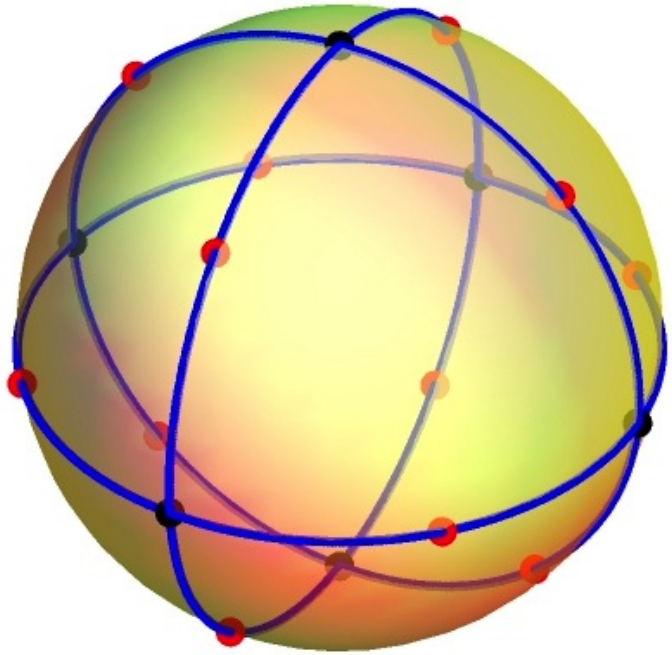




$$\phi(z) = -\frac{64\,z^3\left(z^3-1\right)^3}{\left(8\,z^3+1\right)^3}: \qquad v = 4+6, \quad e = 2\cdot 6, \quad f = 4$$

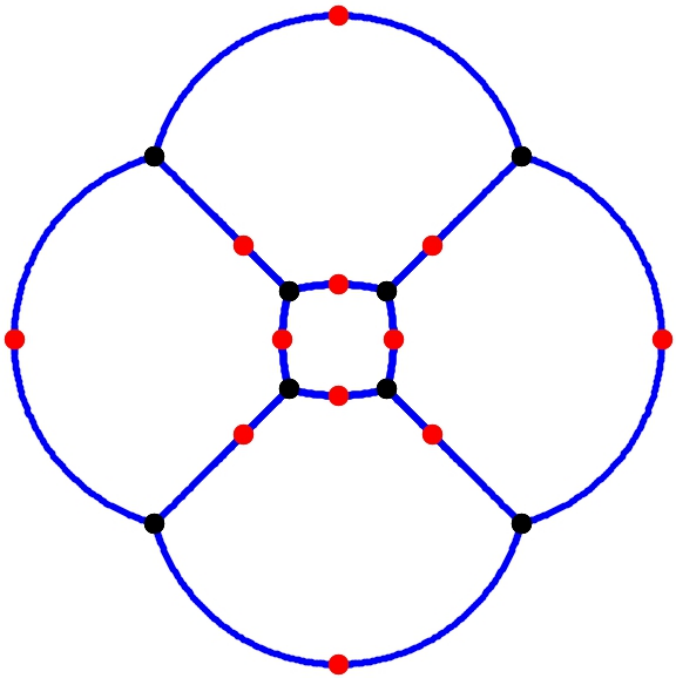
Rotation Group S_4: Octahedron

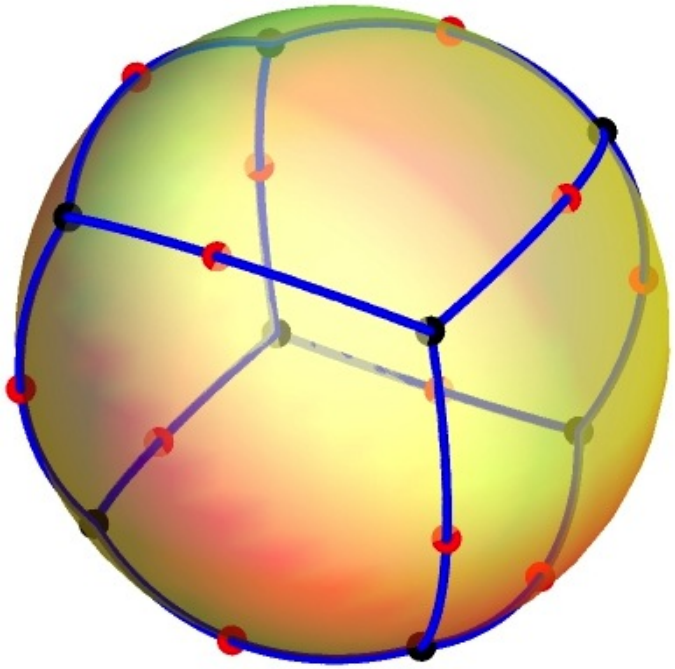




$$\phi(z) = \frac{108\, z^4 \left(z^4-1\right)^4}{\left(z^8+14\, z^4+1\right)^3} : \qquad v = 6 + 12, \quad e = 2 \cdot 12, \quad f = 8$$

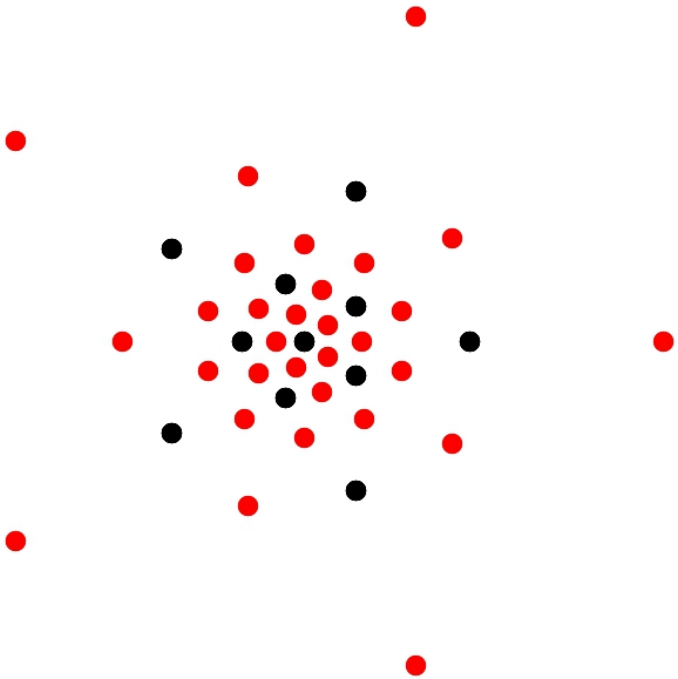
Rotation Group S_4: Cube

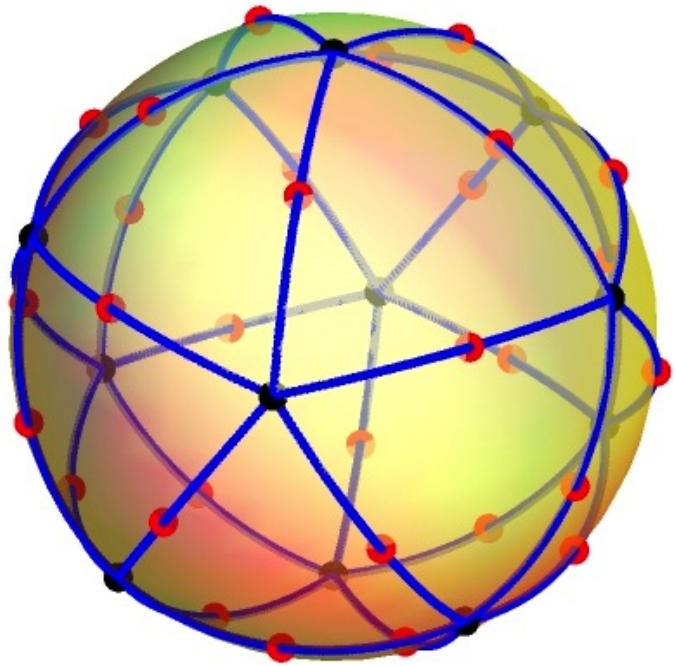




$$\phi(z) = \frac{\left(z^8 + 14\,z^4 + 1\right)^3}{108\,z^4\left(z^4 - 1\right)^4} : \qquad v = 8 + 12, \quad e = 2 \cdot 12, \quad f = 6$$

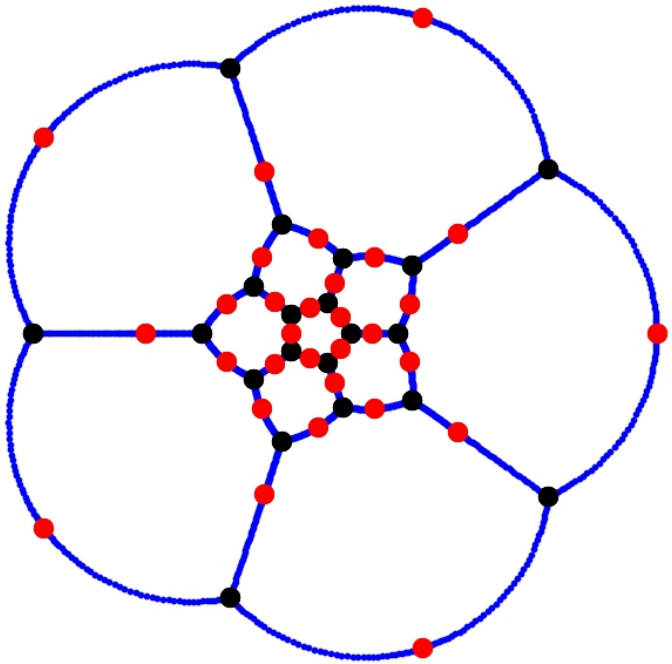
Rotation Group A_5: Icosahedron

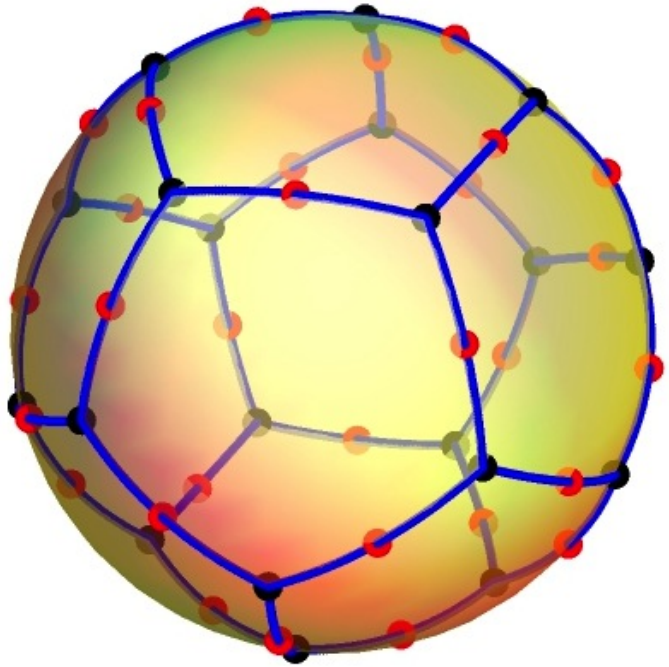




$$\phi(z) = \frac{1728\, z^5 \left(z^{10} - 11\, z^5 - 1\right)^5}{\left(z^{20} + 228\, z^{15} + 494\, z^{10} - 228\, z^5 + 1\right)^3} \; : \; v = 12+30, e = 2\cdot 30, f = 20$$

Rotation Group A_5: Dodecahedron





$$\phi(z) = \frac{(z^{20} + 228\,z^{15} + 494\,z^{10} - 228\,z^5 + 1)^3}{1728\,z^5\,(z^{10} - 11\,z^5 - 1)^5} \;:\; v = 20+30,\, e = 2\cdot 30,\, f = 12$$

Thank You!

Questions?