Alexander Grothendieck

Grothendieck's greatest contribution to mathematics started from a vast and harmonious framework he developed for understanding the duality between spaces and number systems, or geometry and algebra.

Over the millenia following Euclid, notions of space that required sophisticated intuition, such as projective or hyperbolic geometries, were gradually investigated and understood by many generations of scholars, culminating in a grand synthesis around the time of Bernhard Riemann (1826-1866) with the notion of an *abstract manifold*. Most famously, the tools developed by Riemann enabled Einstein to construct his geometric theory of gravitation and the large scale structure of the universe. Over roughly the same period, mathematicians laid the foundations of algebraic structures like the real number system, discovering along the way the complex numbers, which became connected in the twentieth century to the microscopic description of matter. Motivation from number theory even led to curiousities like finite number systems, such as might be formed by the set $\mathbb{F}_5 = \{0, 1, 2, 3, 4\}$ of remainders after dividing by 5, or strings of digits of 0's and 1's. It is one of the miracles of twentieth century science that intricate ways of multiplying binary code, first developed in the process of solving polynomial equations, eventually became critical to the processing of large quantities of information.

As the objects of geometry and algebra became progressively more general and powerful in parallel, a tenacious observer would have seen a question emerging in the history of ideas as to which language is more fundamental in the description of reality. There is still no detailed consensus on this matter even among physicists, since the most difficult problems concerning the ultimate constituents of space touch upon versions of this question. As far as the broad conceptual framework is concerned, largely through the efforts of Grothendieck, it has been clear for some time that these two perspectives are essentially equivalent, or *dual*, in a sense that can be made quite precise, so that a structure of one type determines the other. Within this context, the technical name for a number system is a *commutative ring*, that is, any structure in which elements can be added, subtracted, and multiplied in the usual manner. Grothendieck realized that to any such ring A, one can associate a space in a completely natural fashion, referred to as the *spectrum* of A and denoted $\mathrm{Spec}(A)$. The original ring A is then interpreted as the set of functions on $\mathrm{Spec}(A)$. Among the remarkable consequences of this new way of thinking, one notes that the usual integers $\mathbb{Z}$ are casually and fruitfully regarded as functions on its spectrum $\mathrm{Spec}(\mathbb{Z})$ in present day number theory. To get a further sense of the strange geometries comprised by this concept, one might consider the system $D = \{(a, b)\}$ formed by pairs of real numbers. We are here taking the addition to be commonplace componentwise addition $(a, b) + (c, d) = (a + c, b + d)$, but the multiplication causes the components to interact: $(a, b) \times (c, d) = (ab, ad + bc)$. The geometry $Spec(D)$ turns out to be an *infinitesimal space* consisting of a single point with arrows of arbitrary magnitude pointing out of it in two directions. Exotic as this may appear, number systems of this sort are necessary in order to assign values to fermionic fields in particle physics. Recent efforts to ground spacetime in supersymmetry, notably by string-theorists, require the mixture of geometries like $\mathrm{Spec}(D)$ with the usual four dimensions in such a way that all geometries are dealt with on an equal footing. Without the conceptual machinery wrought by Grothendieck, it would be hard not just to carry out such a procedure, but even to know what it would mean to do so.

The spectrum of a ring belongs to a general class of geometries referred to as *schemes*, and Grothendieck emphasised greatly the importance of studying any single scheme via the interaction

between *all* of them. One might start with a simple equation like $x^2 + y^2 = 1$, which defines a scheme via the quotient ring $\mathbb{Z}[x, y]/(x^2 + y^2 - 1)$. Grothendieck's remarkable idea was that the geometry of such an equation, indeed the very identity of the corresponding scheme, is determined by its solutions in various number systems. So in the real number system, the solutions willl define a circle, while in the number system $\mathbb{F}_5$, the solution set is $\{(0, 1), (1, 0), (0, 4), (4, 0)\}$. The solutions in the complex numbers, on the other hand, acquires the geometry of a sphere with two punctures in it. A basic insight of Grothendieck's was that the variation of this solution set as we change the number system that surrounds it, like the values of a function responding to changes in its argument, is a computationally dynamic tool for probing a geometry more fundamental than the kind that is easily visible, somewhat as a bat flying in the dark might use the intricate trajectory and frequency of sound waves to access the configuration of space around it.

The interactive view of a mathematical object, whereby the internal structure becomes secondary to the way it is situated inside a network, or a *category*, of objects of similar type, became a hallmark of the Grothendieck style, and ended up changing profoundly almost all areas of mathematics in the later part of the 20th century. Close to Grothendieck's original program, the great expansion of the notion of space that became possible through his powers of essential abstraction continues to call forth deep reflection and fierce dialectic within the global community of mathematicians, and will undoubtedly enrich the evolution of mathematics at the deepest levels of language, philosophy, methodology, and applications for the foreseeable future.