ON CLUBS AND DOCTRINES

by

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Doctrines (= strict 2-monads) were introduced, under the name of equational doctrines, by Lawvere [16], and are discussed to some extent in [12] above, which is our general reference for notation, terminology, and basic concepts. Clubs were introduced, in connection with coherence problems, by the present author in [5], [6], and [7], where covariant clubs were distinguished from mixed-variance clubs; only the first concern us in this paper. Such a club K gives rise to a doctrine $K\circ -$ on the 2-category CAT of all categories. I suggested in [5] §4.1 and in [6] §4.1 that the simple concept of (covariant) club used there doubtless admitted of wide generalization, and indicated the way I expected this to go. Such a widening of the meaning of "club" would increase the number of doctrines writable in the form $K\circ -$ for a "club" K, and I was in effect toying there with the idea that perhaps every doctrine was so writable.

The purpose of the present paper (some aspects of which were announced in [11]) is to carry out this generalization of the "club" idea to what seem to be its natural limits. I must say at once that I withdraw from the position that every doctrine then comes from some club; in particular I withdraw my assertion in [6] §4.1 that we can get in this way the doctrine whose algebras are categories-with-equalizers. I, at any rate, can see no generalization wide enough to achieve this.

What we do construct is a monoidal 2-category $CAT\int *CAT$, an object of which is a category K together with an "augmentation functor" $\Gamma\colon K \to CAT$, and the "tensor product" for which is denoted by $\circ$. We embed CAT fully in $CAT\int *CAT$ by assigning to $A \in CAT$ the "trivial augmentation" $A \to CAT$ sending each object of A to the empty category; then $K\circ A \in CAT$ whenever $A \in CAT$. Thus $K \mapsto K\circ -$ gives a 2-functor Φ

from $CAT\int^{*}CAT$ to the 2-category $[CAT,CAT]$ of endo-2-functors of CAT. This 2-functor Φ is 2-fully-faithful, and carries the tensor product $\circ$ into composition of endo-2-functors. A club is now by definition a $\circ$-monoid K in $CAT\int^{*}CAT$, and is carried by Φ into the doctrine $K\circ -$.

The doctrines arising thus from a club have the advantage of a very concrete presentation: in the sense that a category K with its augmentation $\Gamma\colon K \to CAT$ is a more concrete object for study than a general 2-functor $D\colon CAT \to CAT$. Still more special are the doctrines $K\circ -$ where K is a $\circ$-monoid in certain sub-monoidal-2-categories of $CAT\int^{*}CAT$, such as $CAT/\mathcal{S}et$. It is easy to recognize certain structures on a category as being actions of a doctrine of the latter kind, and to describe in theory the process of constructing K from the basic operations and axioms; finding K explicitly in practice is what we may call the coherence problem for the given structure. Again, it is to doctrines D of this latter kind that the work of Day [1] seems to extend, giving a D-structure on $[A^{op},\mathcal{S}et]$ from one on A, and leading to the notions of a pro-D-structure and of a closed D-structure - one in which each structural functor has a right adjoint in every argument.

There are lots of things I still do not understand. I have no criterion in terms of the doctrine D for it to be $K\circ -$ where K is a club in $CAT\int^{*}CAT$, or a club in $CAT/\mathcal{S}et$; although as I said I can recognize the latter case from certain presentations of the structure. I know that clubs in $CAT/\mathcal{S}et$ play an important role, but have no special feeling for those in $CAT\int^{*}CAT$, except that one is ineluctibly led to this generalization. In the $CAT/\mathcal{S}et$ case, I know what it means to ask the coherence question "which diagrams commute?"; for more general structures I do not. In the still more restricted $CAT/\underline{\underline{P}}$ case, where $\underline{\underline{P}}$ is the category of natural numbers and permutations, I know how to ask the coherence question even for closed D-categories (e.g. closed monoidal categories), in terms of mixed-variance functors and the generalized natural transformations of Eilenberg-Kelly [2]; but in the $CAT/\mathcal{S}et$ case

I do not. For instance, what kind of coherence result does one hope for in the case of cartesian closed categories?

Because of these uncertainties, which may attract the interest of others, I choose to devote §1 to a brief account of the original club idea as suggested by coherence problems, before embarking on the above generalizations; besides providing motivation, it may suggest avenues that I have not seen. The substance of the present paper begins in §2 where, changing gear completely, we begin with the most general before passing to the particular. Our path differs in two ways from the brief outline above: first, some things loosely stated there have to be made more precise; and secondly, to minimize the long verifications so common to 2-categorical studies, we take a round-about route.

1. THE ORIGINAL CLUB IDEA

1.1 It arose naturally, originally in the mixed-variance case, from an attempt to refine the formulation of certain coherence problems - such as that for a closed category [9], that for two closed categories connected by a closed functor [17], and that for a closed category V together with a V-natural transformation between two V-functors between two V-categories [10].

In each of these cases the diagrams whose commutativity was in question could be conceived of, to a first approximation, as having functors (of many variables and of mixed variances) for vertices, and natural transformations (of the generalized kind introduced by Eilenberg-Kelly [2]) for edges. Precision however required, as was already recognized in [9], that the vertices be not actual functors in the model but their abstract descriptions in the theory; for otherwise one would have unwanted composites of $f: T \to S$ and $g: S' \to R$, where S and S' although formally different had identical realizations in a

particular model. Yet in [9] the edges were still actual natural transformations.

This served well enough while only positive results - certain diagrams do commute - were sought. In Mac Lane's original coherence theorem [19] for a monoidal or for a symmetric monoidal category, the result had in fact been that all diagrams commute. For a closed category [9] it was no longer so; some diagrams failed to commute in general, although they might do so in a sufficiently degenerate model. My student Lewis [17] found such non-commutativity even in the purely covariant case of two monoidal categories connected by a monoidal functor. What was crucial was that Lewis found a perfectly correct necessary and sufficient condition for commutativity in this case - which made no sense at the level of models. We clearly had to go further and replace the edges too by their abstract descriptions; it is noteworthy that Isbell independently suggested doing this in his review [4] of [9]. (The preliminary account of Lewis' work in [17] is superseded by a far better one in his forthcoming thesis [18].)

1.2 We now had a category K whose vertices were abstract descriptions of those functors obtainable by iterated composition of the basic structural ones, and whose morphisms were abstract descriptions of those natural transformations obtainable from the basic structural ones by composing them in all possible ways with the basic functors and with one another. This category K had a good deal of structure, and it was the abstraction of this structure that I called a club. Each club was an invariant description of an extra structure that a category, or a family of categories, might carry; and the coherence problem became that of determining the club K from its generators (the basic structural functors and natural transformations) and its relations (the axioms to which these were subjected). The analogy with Lawvere's theories is clear; although in fact a closer analogy is with props or operads.

The structure on K was as follows. First, each formal functor T had a type ΓT, specifying its arity, the category (when there was more than one) from which the i-th argument was to be drawn, and the variance of this argument. Next, each formal natural transformation $f: T \to S$ also had a type Γf, specifying which arguments it was to pair off; this was called by Eilenberg-Kelly [2], and has since been called by me, not too happily I think, the graph of f; I shall try to avoid this over-used word. These types ΓT and Γf may be considered as objects and morphisms of a suitable category $\mathcal{T}$, so that Γ appears as a functor $\Gamma: K \to \mathcal{T}$, and K as an object of the category $CAT/\mathcal{T}$.

The next piece of structure is the recognition that objects of K can be composed with one another - or rather substituted into one another, a better expression for functors of many variables. Thus from T of arity n and $S_1, \ldots, S_n$ of arities $m_1, \ldots, m_n$ we can form $T(S_1, \ldots, S_n)$ of arity $m_1 + \ldots + m_n$, also written $n(m_1, \ldots, m_n)$. Similarly for the morphisms of K; one can form $f(g_1, \ldots, g_n)$, with its special cases $T(g_1, \ldots, g_n)$ and $f(S_1, \ldots, S_n)$. It turns out that this can be expressed as a multiplication $\mu: K \circ K \to K$, where $\circ$ is the "tensor product" for a certain monoidal structure on $CAT/\mathcal{T}$.

Finally one must distinguish among the objects of K the formal identity functor (or functors, when many categories are involved); it turns out that this comes to giving a unit $\eta: J \to K$, where J is the identity for $\circ$. The associativity of composition (or of substitution), and the identity property of η, are then given by subjecting μ and η to the monoid axioms. Thus we end with the definition of club as "$\circ$-monoid in the monoidal category $CAT/\mathcal{T}$ ".

We can embed CAT in $CAT/\mathcal{T}$ by giving to $A \in CAT$ the trivial augmentation $\Gamma: A \to \mathcal{T}$ assigning arity zero to each object of A. Then to give to a category A the extra structure characterized by the club K is to give an action $K \circ A \to A$ of the monoid K on A. Thus the categories with this structure are the algebras for the monad $K \circ -$ on CAT.

Similarly when the structure is borne by a family of categories, indexed say by Λ; then CAT/Λ replaces CAT.

1.3 We must now point out that what we have just said is, in the mixed-variance cases from which we drew our examples, over-simplified for expository purposes: the phenomenon of incompatibility for the generalized natural transformations of [2] causes the "tensor product" $\circ$ to be not everywhere defined on CAT/T. This is discussed at length in [5], [6], and [7]; we pursue it no further here because we henceforth restrict to the purely covariant case. Then all we have said is exact, and in fact more is true. Namely, $\circ$ is a 2-functor and CAT/T is a monoidal 2-category; so that $K\circ-$ for a club K is not only a monad but a 2-monad or doctrine. Moreover the monoidal 2-category CAT/T is actually closed: its internal-hom $\{B,C\}$ is a very rich functor category, whose objects are functors $B^n \to C$ of all arities and whose morphisms are natural transformations of all types.

In the covariant case the type Γf of a natural transformation of the Eilenberg-Kelly kind is nothing but a permutation of the arguments in passing from domain to codomain, as in the symmetry $A\otimes B \to B\otimes A$ in a monoidal category, or in the associativity $(A\otimes B)\otimes C \to A\otimes(B\otimes C)$ where this permutation is the identity. If we restrict for simplicity to the one-category case the type ΓT of a functor is just its arity $n \in \underline{N}$, so that in this case the category T is just the category $\underline{P}$ with natural numbers as objects and permutations as morphisms. It is to clubs at this $CAT/\underline{P}$ level that [5] and [6] are devoted, except that the many-category case is allowed for and the corresponding mixed-variance case is also considered.

1.4 The suggestion naturally occurs that many more coherence problems can be formulated in this way if we further widen the notion of "natural transformation f of type Γf ". For instance, a category with finite coproducts has a binary functor + and a nullary functor 0, with "natural transformations" $A \to A + B$, $B \to A + B$, $A + A \to A$, $0 \to A$,

subjected to equational axioms. The types of these natural transformations are not permutations but functions: the two functions $1 \to 2$, and the unique functions $2 \to 1$ and $0 \to 1$. Similarly for a category with two monoidal structures $(\otimes, I)$ and $(\oplus, N)$, connected by distributivity maps (not isomorphisms)

(1.1) $\qquad d: (A \otimes B) \oplus (A \otimes C) \to A \otimes (B \oplus C), \quad e: N \to A \otimes N,$

for which the coherence problem has been studied in another formulation by Laplaza [14] and [15] (whose results we generalize in [8] below). In other words we expect a generalization with $CAT/\underline{\underline{P}}$ replaced by $CAT/\underline{\underline{S}}$, where $\underline{\underline{S}}$ is the skeletal category of finite sets, with natural numbers as objects and functions as morphisms.

Having gone this far it is natural to expect still more: why should the arities n be finite sets? We can still speak of a functor $\mathcal{B}^n \to C$ when n is a <u>small category</u>: and if a category A with finite coproducts has functors $A^2 \to A$ and $A^0 \to A$, so a category A with coequalizers has a functor $A^{\rightrightarrows} \to A$. So we may expect a further generalization with $CAT/\underline{\underline{S}}$ replaced by CAT/Cat.

That such a generalization does exist was announced in [11]; in the sense that we still have a monoidal structure $\circ$ on CAT/Cat, the monoids K for which give doctrines on CAT. (I am ignoring for the moment the fact that we must, like Kock in [13], use here a rather special version of Cat.) Yet this is not altogether a generalization of the "coherence problem"; for K is not now so simply given by its generators and relations. It was my failure to anticipate this last point that led me falsely to predict that the doctrine for categories-with-coequalizers would be seen as coming from a club.

The attempt to carry out the details of this generalization automatically led to a further, and final, one: the 2-functor $\Phi: CAT/Cat \to [CAT, CAT]$ is not <u>full</u>; we can make it so by adding more 1-cells and 2-cells to CAT/Cat to get a new 2-category $CAT\int^{*}Cat$; and $\circ$ can still be defined, indeed cries out to be so, on $CAT\int^{*}Cat$ or even

$CAT\int *CAT$. This is the level of generalization we promised in the introduction, to the details of which we now pass, after first adding a final remark to this section.

1.5 If $D = K\circ -$ is the doctrine for categories-with-finite--coproducts, that for categories-with-finite-products is the opposite doctrine D^* (cf. [16]) given by $D^*A = (DA^{op})^{op}$. We could if we liked define a club of the second kind, with augmentation landing not in CAT but in CAT^{op} - or in $\underline{\underline{S}}^{op}$ in this particular case - and then D^* would be $K^{op}\circ -$, where this is a new "$\circ$". For the basic natural transformations are now $A\times B \to A$, $A\times B \to B$, $A \to A\times A$, $A \to 1$, which can clearly be said to have "types" in $\underline{\underline{S}}^{op}$. So we get a second distinguished subclass of doctrines, those coming from clubs of the second kind; but they are just the opposite doctrines of those coming from clubs, and we shall not complicate the present paper by making special reference to them.

This does, however, raise the question of how we approach the coherence problem for more complicated structures, such as a category with both finite coproducts and finite products, where we have natural transformations like

$$A+A \to A \to A\times A;$$

or for the distributivity problem of §1.4 when the maps (1.1) are isomorphisms, so that $d^{-1}d$ is a natural transformation

$$(A\otimes B)\oplus(A\otimes C) \to (A\otimes B)\oplus(A\otimes C).$$

It is tempting to say that here the type of the natural transformation is neither in $\underline{\underline{S}}$ nor in $\underline{\underline{S}}^{op}$, but is something of the form $n \to p \leftarrow m$, that is, an op-span. However I have so far been unable to extend the club idea to this case; I am quite puzzled about what a good "coherence result" would be, other than the explicit determination of the doctrine; and I make no further reference in this paper to the possibility of such further extensions. (P.s. Some later thoughts on what a "coherence result" should be are contained in my paper [8] below in this volume.)

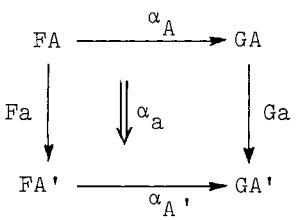
2. LAX NATURAL TRANSFORMATIONS

2.1 We use a single arrow $\to$ for 1-cells and 3-cells, in particular for functors or 2-functors and for modifications; we use a double arrow $\Rightarrow$ for 2-cells, in particular for natural or 2-natural transformations; and we use a wiggly arrow $\rightsquigarrow$ for the lax natural transformations we are about to introduce.

We identify a set with the corresponding discrete category, and a category with the corresponding 2-category with only identity 2-cells. If A is a 2-category, A_0 is the category obtained by discarding the 2-cells, and $|A| = \mathrm{ob}\, A$ is the set obtained by discarding also the 1-cells. For 2-categories A and B, we denote by $[A,B]$ the 2-category of 2-functors, 2-natural transformations, and modifications; recall that we use these terms in the sense of [12] above. It is necessary in this context to distinguish clearly A from A_0, for $[A,B]$ and $[A_0,B]$ are quite different when B is a 2-category. Therefore we shall always write CAT for the 2-category and CAT_0 for the category, and so on.

2.2 For 2-categories A and T and 2-functors $F,G: A \to T$, a lax natural transformation $\alpha: F \rightsquigarrow G$ is what Gray [3] called a "2-natural" one: it assigns to each object A of A a morphism $\alpha_A: FA \to GA$ in T, and to each morphism $a: A \to A'$ in A a 2-cell α_a in T of the form

(2.1)

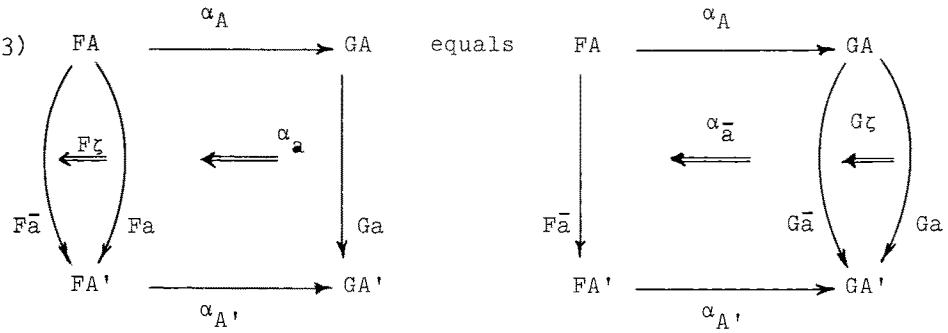


These data are to satisfy the axioms

(2.2) $\qquad \alpha_{1_A} = 1_{\alpha_A}, \qquad \alpha_{a'a} = \alpha_{a'}\alpha_a$ (pasting composite),

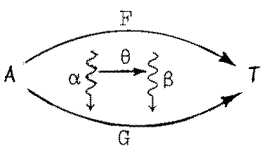
and the axiom

(2.3)



for each 2-cell $\zeta\colon a \Rightarrow \bar{a}$ in A. This last axiom (2.3) is vacuous when A, as it shall be in all our applications, is merely a category. The 2-natural transformations $\alpha\colon F \Rightarrow G$ (which are just the natural ones when A is merely a category) are those lax ones $\alpha\colon F \rightsquigarrow G$ for which each α_a is the identity 2-cell.

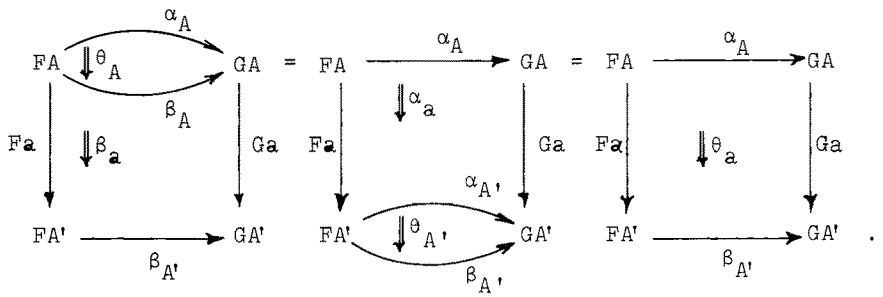
We compose lax n.t.'s by horizontal pasting of diagrams like (2.1), getting a category $[\![A,T]\!]_0$, of which $[A,T]_0$ is a subcategory. It becomes a 2-category $[\![A,T]\!]$ when we take as 2-cells the <u>modifications</u>



.

Such a modification $\theta\colon \alpha \to \beta$ assigns to each $A \in A$ a 2-cell $\theta_A\colon \alpha_A \Rightarrow \beta_A$ rendering equal the left two diagrams below, for whose common value we use the notation of the diagram on the right:

(2.4)



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Horizontal and vertical composition of modifications is defined in the obvious way. When α and β are 2-natural, the above is the usual

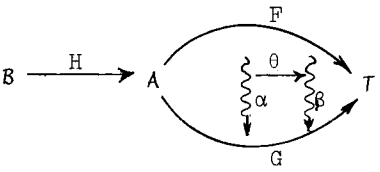
definition of modification as in [12], so that the inclusion $[A,T] \to [\![A,T]\!]$ is "full on 2-cells".

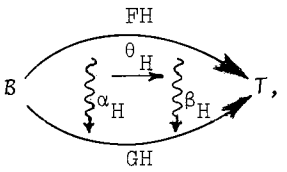
We write I for the unit category with one object $\star$ and one morphism. There is an evident isomorphism

$$\ulcorner\ \urcorner : T \xrightarrow{\cong} [I,T] = [\![I,T]\!] , \tag{2.5}$$

where for example $\ulcorner B \urcorner$, called the name of B, for $B \in T$, is the 2-functor sending $\star$ to B.

2.3 Let $A,\mathcal{B}$ be merely categories and T a 2-category. From the sit uation

(2.6) 

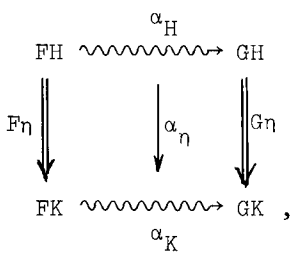
we get 

where

$$(\alpha_H)_B = \alpha_{HB},\ (\alpha_H)_b = \alpha_{Hb},\quad (\theta_H)_B = \theta_{HB}. \tag{2.7}$$

Thus the functor H induces a 2-functor $[\![H,T]\!]: [\![A,T]\!] \to [\![\mathcal{B},T]\!]$.

Given a second functor $K: \mathcal{B} \to A$ and a natural transformation $\eta: H \Rightarrow K$, we get a modification

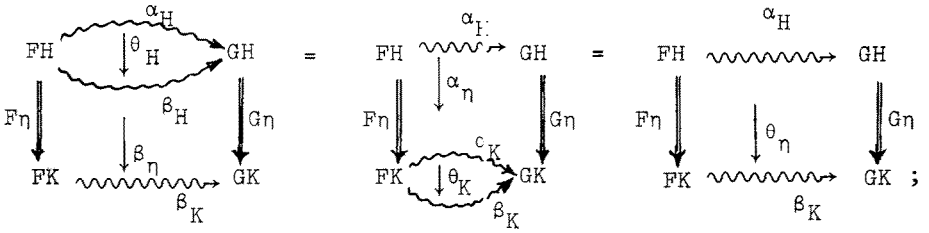
(2.8) 

where

$$(\alpha_\eta)_B = \alpha_{\eta_B} . \tag{2.9}$$

Moreover the two modifications on the left below coincide, and we denote their common value as in the diagram on the right:

(2.10)



here we have, in the notation of (2.4),

$$(\theta_\eta)_B = \theta_{\eta_B}. \tag{2.11}$$

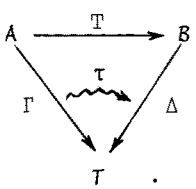
We may say, with an obvious meaning, that $\eta\colon H \Rightarrow K$ induces an op-lax--natural $[\![\eta,T]\!]\colon [\![H,T]\!] \rightsquigarrow [\![K,T]\!]$ with components

$$[\![\eta,T]\!]_F = F\eta,\ [\![\eta,T]\!]_\alpha = \alpha_\eta. \tag{2.12}$$

3. THE 2-CATEGORIES $CAT\int T$ AND $CAT\int^* T$

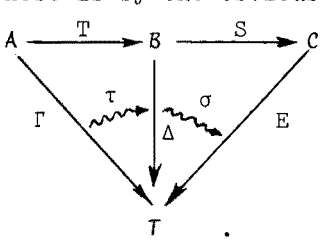
3.1 Let T be a 2-category. An object of $CAT\int T$ consists of a category A and a functor $\Gamma\colon A \to T$. A morphism from (A,Γ) to (B,Δ) consists of a functor $T\colon A \to B$ together with a lax natural transformation τ as in

(3.1)



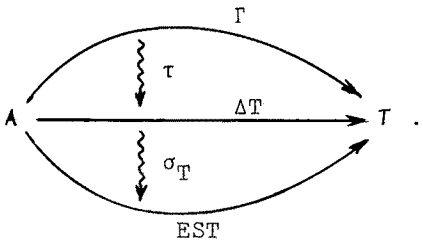
Composition of these is by the obvious pasting of diagrams:

(3.2)



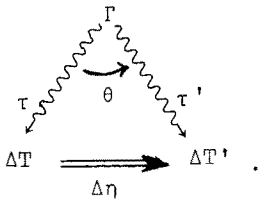
That is, the lax n.t. in the composite is the composite

(3.3)



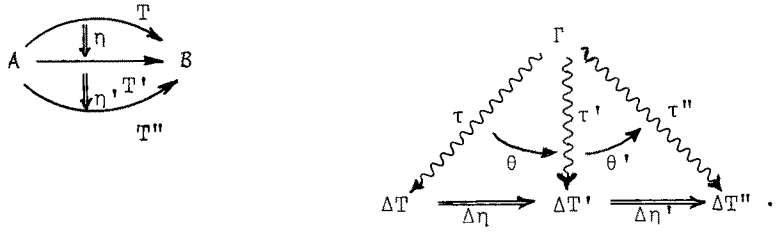
(The reader is asked for his cooperation in seeing "E" here as upper-case Greek epsilon.) A 2-cell $(T,\tau) \Rightarrow (T',\tau')\colon (A,\Gamma) \to (B,\Delta)$ consists of a natural transformation $\eta\colon T \Rightarrow T'$ together with a modification θ as in

(3.4)



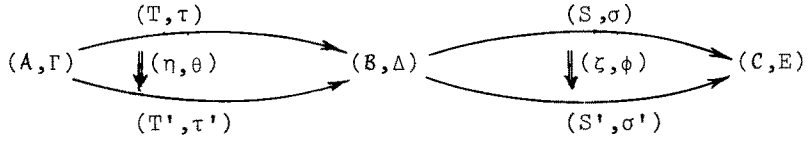
Vertical composition of these is given by

(3.5)



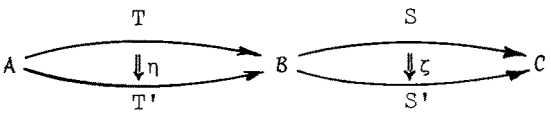
The more complicated horizontal composition of

(3.6)



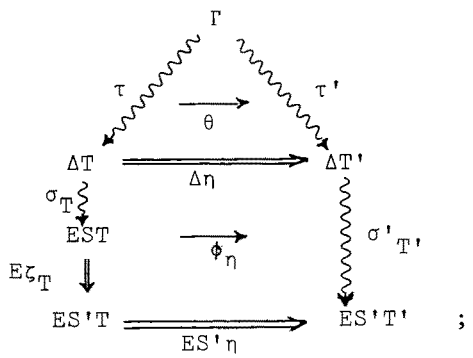
is given by

(3.7)



together with

(3.8)



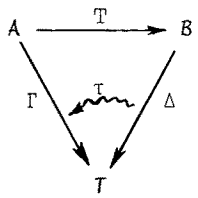
;

here ϕ_η is formed as in (2.10) from ϕ: $E\zeta.\sigma \to \sigma'$ and η: $T \Rightarrow T'$.

A direct verification that $CAT\int T$ is a 2-category is frightfully tedious, and to be manageable at all needs some lemmas about the behaviour under pasting of derived modifications like ϕ_η. Happily it is not necessary. An obvious representability argument shows that it suffices to do it for $CAT\int CAT$, and in §6.3 below we give a 2-full embedding of this into a known 2-category. Once we check that this embedding respects the various compositions and identities, the 2-category axioms are automatic. Alternatively, a different full embedding given in §5.2 below does the same for $CAT\int *CAT$ which we are about to introduce; this extends by representability arguments to $CAT\int *T$; and $(CAT\int T)^{co}$ is isomorphic to $CAT\int *T^{coop}$.

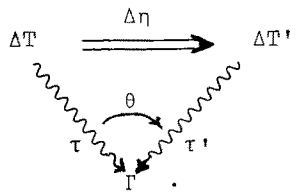
3.2 $CAT\int *T$ has the same objects $(A,\Gamma\colon A \to T)$ as $CAT\int T$. However a morphism from (A,Γ) to (B,Δ) now consists of a functor $T\colon A \to B$ together with a lax n.t. τ as in

(3.9)



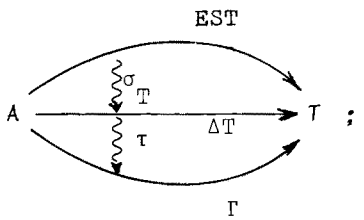
(so that T has the same sense as in (3.1), but τ the opposite sense). A 2-cell from (T,τ) to (T',τ') now consists of (η,θ) where η: T ⇒ T' as before, but θ is a modification of the form

(3.10)



With τ and σ reversed in (3.2), the new analogue of (3.3) is

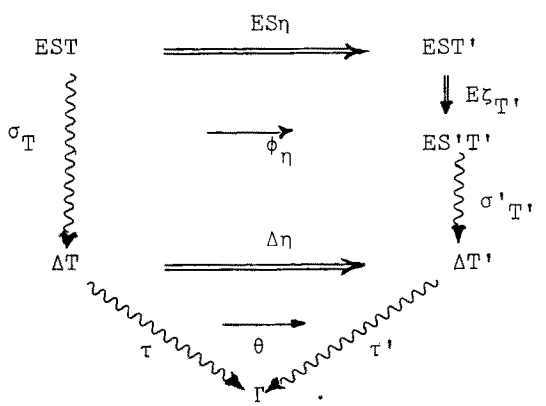
(3.11)



;

there is no change in (3.5) except the reversal of τ,τ',τ"; and (3.8) gets replaced by

(3.12)



It is easy to see that there is an isomorphism $(CAT\int T)^{co} \to CAT\int^{*}T^{coop}$ sending (in what I hope is, in the light of [12], an obvious notation) (A,Γ) to (A^{op},Γ^{op}), (T,τ) to (T^{op},τ^{coop}), and (η,θ) to (η^{op},θ^{co}). (In fact we apply coop to everything, but co is trivial on some elements and op on others.)

3.3 It is clear what is meant by $L\int T$, $L\int^{*}T$ where L is a sub-2-category of CAT.

There are various sub-2-categories of $CAT\int T$ that stand out. By fixing A and requiring that $T = 1$ and $\eta = 1$, we get $[\![A,T]\!]$ as a sub-2-category; similarly $[\![A,T]\!]^{op}$ as a sub-2-category of $CAT\int^{*}T$. In particular taking A as the unit category I we get the composite

(3.13) $\ulcorner\ \urcorner : T \cong [\![I,T]\!] \to CAT\int T$,

where the first isomorphism is what we called $\ulcorner\ \urcorner$ in (2.5).

If instead we allow all T and η, but require $\tau = 1$ and $\theta = 1$, we get CAT/T_0 (cf. [12] §1.3) as a sub-2-category both of $CAT\int T$ and of $CAT\int^{*}T$. (We should note I suppose that the strictly accurate name for CAT/T_0 is $CAT/\ulcorner T_0 \urcorner$: it is the comma-object of $1: CAT \to CAT$ and $\ulcorner T_0 \urcorner : I \to CAT$ in the 2-category of 2-categories. However we may omit the corners when no confusion is likely.)

Finally we observe that we have evident forgetful 2-functors

(3.14) $\qquad CAT\int T \to CAT \leftarrow CAT\int^{*}T$,

sending (A,Γ) to A, (T,τ) to T, and (η,θ) to η.

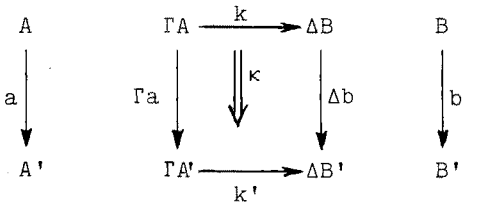
4. LAX COMMA CATEGORIES

4.1 Let T be a 2-category, A and B categories, and $\Gamma: A \to T$, $\Delta: B \to T$ functors. We define the lax comma category $\Gamma/\!/\Delta$, or more precisely $(A,\Gamma)/\!/(B,\Delta)$, to be what Gray [3] calls the "2-comma-category". An object of $\Gamma/\!/\Delta$ is a triple (A,k,B),

(4.1) $\quad A, \qquad \Gamma A \xrightarrow{k} \Delta B, \qquad B,$

where $A \in \mathcal{A}$, $B \in \mathcal{B}$, and k is a morphism in $\mathcal{T}$. A morphism from (A,k,B) to (A',k',B') is a triple (a,κ,b),

(4.2)



where $a \in \mathcal{A}(A,A')$, $b \in \mathcal{B}(B,B')$, and κ is a 2-cell in $\mathcal{T}$. Composition is the evident pasting operation.

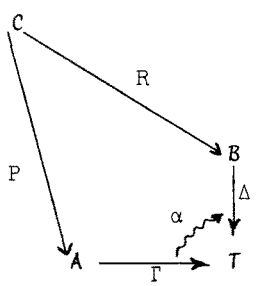
There are the obvious projection functors $d_0 : \Gamma /\!/ \Delta \to \mathcal{A}$, $d_1 : \Gamma /\!/ \Delta \to \mathcal{B}$, sending (a,κ,b) to a and to b respectively. There is also a lax natural transformation $\lambda : \Gamma d_0 \rightsquigarrow \Delta d_1$ with components $\lambda_{(A,k,B)} = k$, $\lambda_{(a,\kappa,b)} = \kappa$. The easy proofs of the following two propositions are left to the reader.

Proposition 4.1 $\Gamma /\!/ \Delta$ is determined by the following universal property. For a category $\mathcal{C}$, the equations

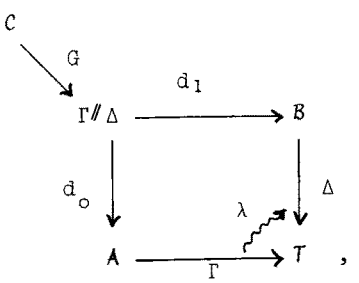
(4.3) $\quad P = d_0 G, \quad R = d_1 G, \quad \alpha = \lambda_G,$

or in brief

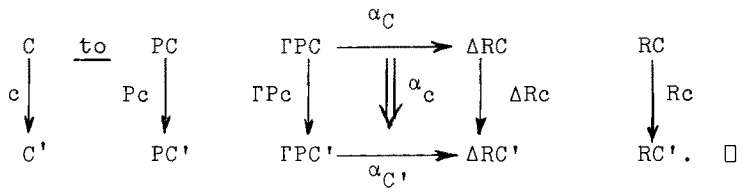
(4.4)



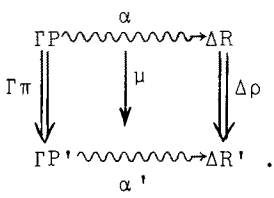
equals



set up a bijection between functors $G : \mathcal{C} \to \Gamma /\!/ \Delta$ and triples $(P : \mathcal{C} \to \mathcal{A}, R : \mathcal{C} \to \mathcal{B}, \alpha : \Gamma P \rightsquigarrow \Delta R)$. To wit, G must carry

(4.5)


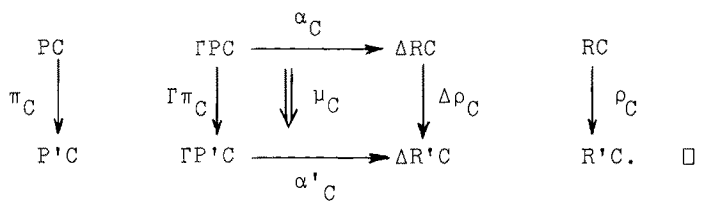
Proposition 4.2 If in the above situation we also have G' corresponding to (P',R',α'), there is a bijection between natural transformations $\gamma: G \Rightarrow G'$ and triples (π,ρ,μ) where $\pi: P \Rightarrow P'$, $\rho: R \Rightarrow R'$, and μ is a modification

(4.6)


In terms of Υ, we have in the notation of (2.8)

(4.7) $\pi = d_0\gamma, \quad \rho = d_1\gamma, \quad \mu = \lambda_\gamma;$

while the C-component of Υ is

(4.8)


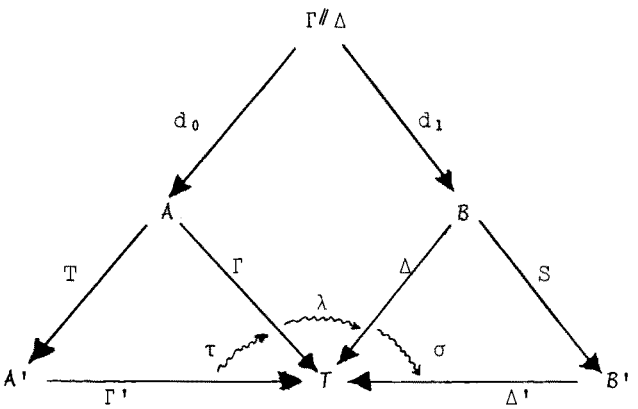
4.2 We now exhibit $/\!/$ as a 2-functor

(4.9) $/\!/: CAT\int\!*T \times CAT\int T \to CAT;$

in fact the definitions of $CAT\int T$ and $CAT\int\!*T$ were chosen to make this so.

From $(T,\tau): (A,\Gamma) \to (A',\Gamma')$ in $CAT\int\!*T$ and $(S,\sigma): (B,\Delta) \to (B',\Delta')$ in $CAT\int T$ we get

(4.10)

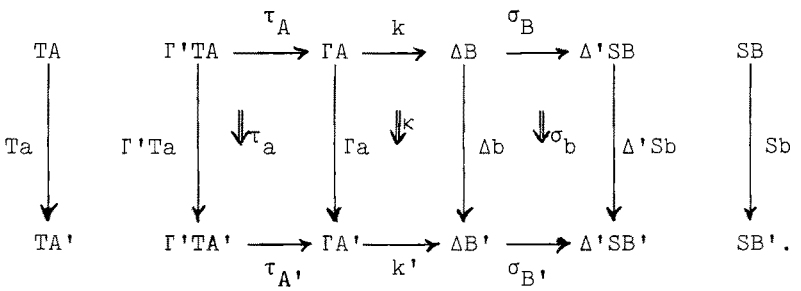


and hence, by Proposition 4.1, a functor

(4.11) $(T,\tau)/\!/(S,\sigma)\colon (A,\Gamma)/\!/(B,\Delta) \to (A',\Gamma')/\!/(B',\Delta')$

which sends the typical morphism (4.2) to

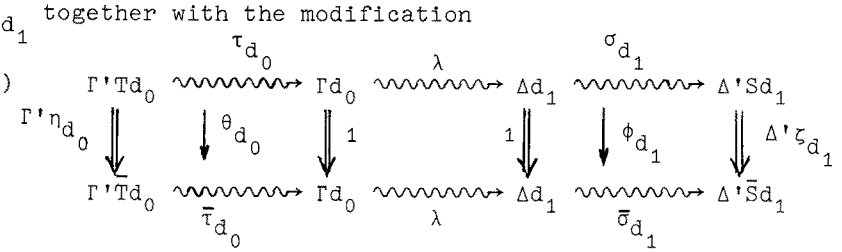
(4.12)



That the functor (4.11) depends functorially on (T,τ) and (S,σ) is perhaps best seen by pasting onto (4.10) a $(T',\tau')\colon (A',\Gamma') \to (A'',\Gamma'')$ and an $(S',\sigma')\colon (B',\Delta') \to (B'',\Delta'')$; alternatively by direct use of (4.12) in the light of (3.3) and (3.11).

Next consider $(\eta,\theta)\colon (T,\tau) \Rightarrow (\bar{T},\bar{\tau})$ in $CAT\int *T$ and $(\zeta,\phi)\colon (S,\sigma) \Rightarrow (\bar{S},\bar{\sigma})$ in $CAT\int T$. From the natural transformations η_{d_0} and ζ_{d_1} together with the modification

(4.13)



we get by Proposition 4.2 a natural transformation

(4.14) $(\eta,\theta)/\!/(\zeta,\phi)\colon (T,\tau)/\!/(S,\sigma) \Rightarrow (\bar{T},\bar{\tau})/\!/(\bar{S},\bar{\sigma})$

whose (A,k,B)-component is

(4.15)
$$\begin{array}{ccccccc}
TA & \Gamma'TA & \xrightarrow{\tau_A} & \Gamma A & \xrightarrow{k} & \Delta B & \xrightarrow{\sigma_B} \Delta'SB \quad SB \\
{\scriptstyle \eta_A}\downarrow & {\scriptstyle \Gamma'\eta_A}\downarrow & \Downarrow\theta_A & \downarrow{\scriptstyle 1} & & {\scriptstyle 1}\downarrow & \Downarrow\phi_B \quad \downarrow{\scriptstyle \Delta'\zeta_B} \quad \downarrow{\scriptstyle \zeta_B} \\
\bar{T}A & \Gamma'\bar{T}A & \xrightarrow[\bar{\tau}_A]{} & \Gamma A & \xrightarrow[k]{} & \Delta B & \xrightarrow[\bar{\sigma}_B]{} \Delta'\bar{S}B \quad \bar{S}B
\end{array}$$

It is very easy from (4.15) to check that // respects vertical and horizontal composition of 2-cells in $CAT\int{}^*T$ and $CAT\int T$; thus // is indeed a 2-functor of the form (4.9).

4.3 The projections of $CAT\int{}^*T \times CAT\int T$ onto its factors, composed with the forgetful 2-functors (3.14), give new forgetful 2-functors

(4.6) $\mathrm{forg}_0, \mathrm{forg}_1\colon CAT\int{}^*T \times CAT\int T \to CAT$,

sending $((A,\Gamma), (B,\Delta))$ to A and B respectively. The following is clear from (4.12) and (4.15):

Proposition 4.3 The functors $d_0\colon (A,\Gamma)/\!/(B,\Delta) \to A$ and $d_1\colon (A,\Gamma)/\!/(B,\Delta) \to B$ are the components of 2-natural transformations $d_0\colon /\!/ \Rightarrow \mathrm{forg}_0$ and $d_1\colon /\!/ \Rightarrow \mathrm{forg}_1$. □

5. THE 2-FUNCTOR ∘ AND THE EMBEDDING $CAT\int{}^*CAT \to [CAT,CAT]$

5.1 Form the composite 2-functor

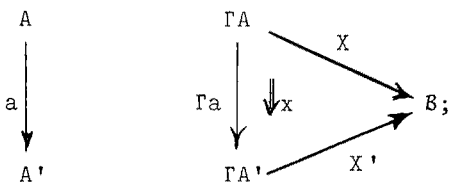
$$CAT\int{}^*T \times T \xrightarrow[1 \times \ulcorner\urcorner]{} CAT\int{}^*T \times CAT\int T \xrightarrow{/\!/} CAT,$$

where $\ulcorner\urcorner : T \to CAT\int T$ is the 2-functor (3.13) sending $B \in T$ to $(1,\ulcorner B\urcorner)$, etc. Our interest is in the special case $T = CAT$; we then write the above 2-functor as

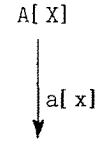
(5.1) $\circ\colon CAT\int{}^*CAT \times CAT \to CAT.$

We usually abbreviate its value $(A,\Gamma)\circ\mathcal{B}$ on objects to $A\circ\mathcal{B}$; for its value on morphisms or 2-cells we write $(T,\tau)\circ S$ and $(\eta,\theta)\circ\zeta$ in full; except that we are sometimes interested in the restriction of $\circ$ to $\mathcal{CAT}/\mathcal{CAT}_0 \times \mathcal{CAT}$, where τ and θ are identities, and then we write simply $T\circ S$ and $\eta\circ\zeta$.

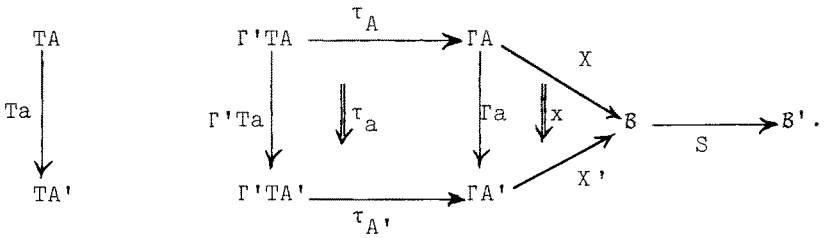
An object of $A\circ\mathcal{B}$ is a pair (A,X) where $A \in \mathcal{A}$ and X is a functor $\Gamma A \to \mathcal{B}$; we write this object as $A[X]$. A morphism $A[X] \to A'[X']$ is a pair (a,x) of the form

(5.2)


here a is a morphism in A and x is a natural transformation. We write the morphism (5.2) as

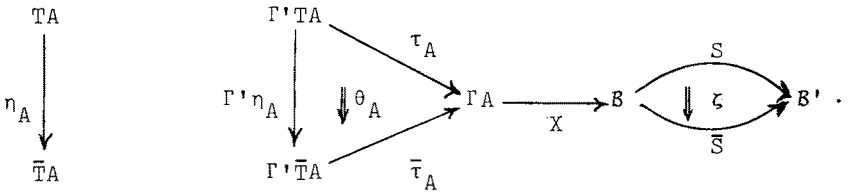
(5.3)


If $(T,\tau)\colon (A,\Gamma) \to (A',\Gamma')$ and $S\colon \mathcal{B} \to \mathcal{B}'$, the functor $(T,\tau)\circ S\colon A\circ\mathcal{B} \to A'\circ\mathcal{B}'$ sends, as a special case of (4.12), the morphism (5.2) to

(5.4)


If $(\eta,\theta)\colon (T,\tau) \Rightarrow (\bar{T},\bar{\tau})$ and $\zeta\colon S \Rightarrow \bar{S}$, the natural transformation $(\eta,\theta)\circ\zeta\colon (T,\tau)\circ S \Rightarrow (\bar{T},\bar{\tau})\circ\bar{S}$ has, as a special case of (4.15), the $A[X]$--component

(5.5)



We have as in §4.1 the projection functor

(5.6) $\quad d_0 : A \circ \mathcal{B} \to A$

sending (5.2) to $a: A \to A'$. As in Proposition 4.3 this is the component of a 2-natural transformation

(5.7) $\quad d_0 : \circ \Rightarrow \mathrm{forg}_0,$

where $\mathrm{forg}_0 : CAT\int *CAT \times CAT \to CAT$ sends $((A,\Gamma),\mathcal{B})$ to A, etc.

5.2 The 3-category of 2-categories is of course cartesian closed; in particular there is a bijection between 2-functors $K \times L \to M$ and 2-functors $K \to [L,M]$. Under this bijection the 2-functor $\circ$ of (5.2) gives a 2-functor

(5.8) $\quad \Phi : CAT\int *CAT \to [CAT, CAT],$

to wit

(5.9) $\quad \Phi(A,\Gamma) = A\circ -,\ \Phi(T,\tau) = (T,\tau)\circ -,\ \Phi(\eta,\theta) = (\eta,\theta)\circ -.$

Our first main result is:

Theorem 5.1 The 2-functor Φ is 2-fully-faithful.

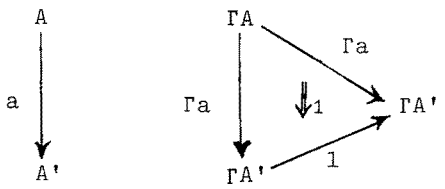
Proof. Take a 2-natural transformation $G: (A,\Gamma)\circ - \Rightarrow (A',\Gamma')\circ -$, with components $G_{\mathcal{B}}: A\circ\mathcal{B} \to A'\circ\mathcal{B}$. We are to show that $G = (T,\tau)\circ -$ for a unique (T,τ).

The uniqueness of (T,τ) is immediate; for if G is $(T,\tau)\circ -$ it follows from (5.4) that

(5.10) $\quad G_{\Gamma A}(A[1_{\Gamma A}]) = TA[\tau_A],$

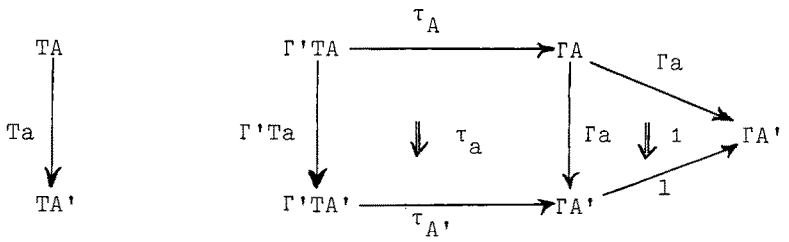
and that $G_{\Gamma A'}$ sends the morphism

(5.11)



of $A\circ\Gamma A'$ to the morphism

(5.12)



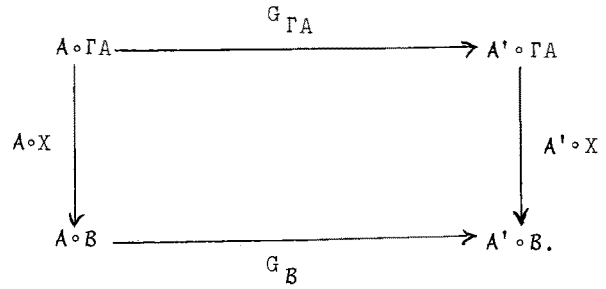
of $A'\circ\Gamma A'$; that is to say,

(5.13) $\quad G_{\Gamma A'}(a[1_{\Gamma a}]) = Ta[\tau_a]$.

So by (5.10) and (5.13) G uniquely determines T and τ.

Now let any 2-natural G be given, let $A[X] \in A\circ\mathcal{B}$, and observe that by the naturality of G we have commutativity in

(5.14)



If we <u>define</u> TA,τ_A by (5.10) and evaluate both legs of (5.14) at the object $A[1_{\Gamma A}]$ we get

(5.15) $\quad G_{\mathcal{B}}(A[X]) = TA[X.\tau_A]$.

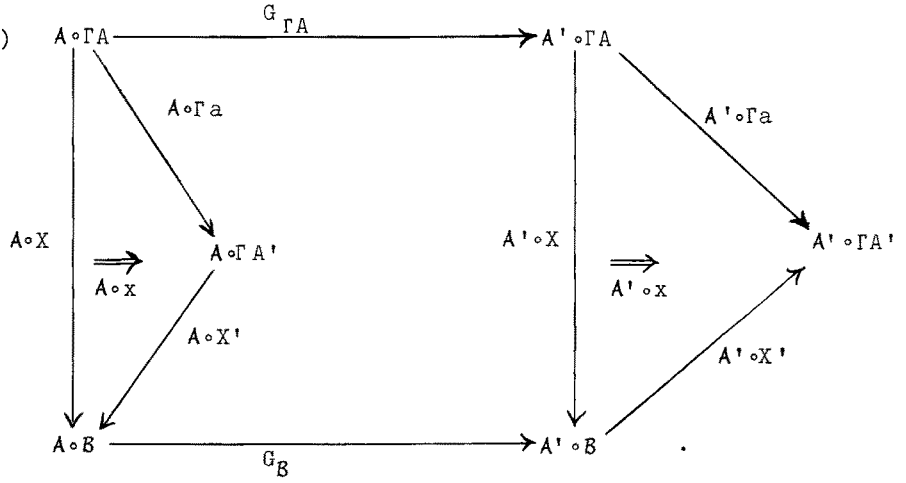
In particular, $G_{\Gamma A'}(A[\Gamma a]) = TA[\Gamma a.\tau_A]$, the domain of (5.12). With this said, it now <u>makes sense</u> to <u>define</u> Ta,τ_a by (5.13).

Next, replace A,X by A',X' in (5.14) and evaluate both legs at the morphism $a[1_{\Gamma a}]$ of (5.11). By (5.13) this gives

(5.16) $\quad G_{\mathcal{B}}(a[1_{X'.\Gamma a}]) = Ta[X'.\tau_a]$.

Now let $a[x]$ be the morphism (5.2) of $A \circ \mathcal{B}$. By the 2-naturality of G we have commutativity in

(5.17)



Taking the $A[1_{\Gamma A}]$-component of each leg we get, using (5.10) and (5.5),

$$G_{\mathcal{B}}(1_A[x]) = 1_{TA}[x.\tau_A]. \tag{5.18}$$

But the $a[x]$ of (5.2) is the composite

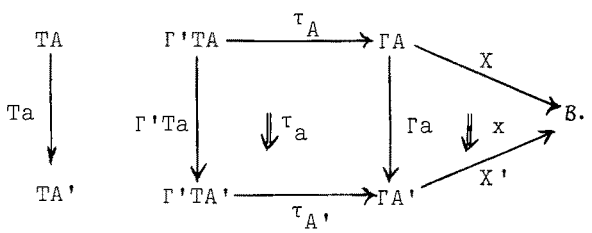
$$A[X] \xrightarrow[1_A[x]]{} A[X'.\Gamma a] \xrightarrow[a[1_{X'}.\Gamma a]]{} A'[X'],$$

and $G_{\mathcal{B}}$ is a functor; so $G_{\mathcal{B}}(a[x])$ is, by (5.16) and (5.18), the composite

$$TA[X.\tau_A] \xrightarrow[1_{TA}[x.\tau_A]]{} TA[X'.\Gamma a.\tau_A] \xrightarrow[Ta[X'.\tau_a]]{} TA'[X'.\tau_{A'}];$$

that is,

(5.19) $G_{\mathcal{B}}(a[x])$ is



Using the fact that $G_{\mathcal{B}}$ is a functor, we now easily get that T is a functor and τ a lax n.t. by applying $G_{\Gamma A}$ to $1_A[1_{1_{\Gamma A}}]$ and by applying

$G_{\Gamma A''}$ to the composite

$$A[\Gamma a'.\Gamma a] \xrightarrow[a[1]]{} A'[\Gamma a'] \xrightarrow[a'[1]]{} A''[1_{\Gamma A''}].$$

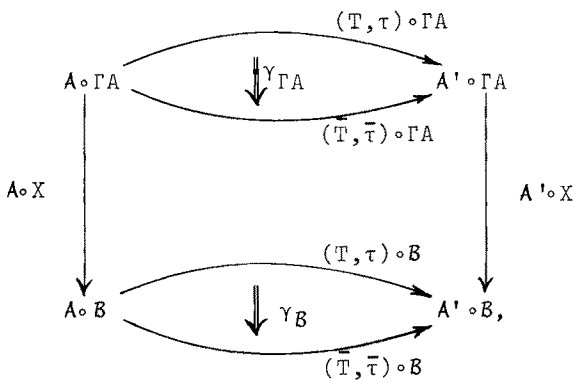
Then by comparing (5.19) with (5.4) we see that $G_{\mathcal{B}}$ is indeed $(T,\tau)\circ\mathcal{B}$.

It remains to prove that Φ is fully faithful on 2-cells. So let (T,τ), $(\bar{T},\bar{\tau})$: $(A,\Gamma) \to (A',\Gamma')$ and let γ: $(T,\tau)\circ- \to (\bar{T},\bar{\tau})\circ-$ be a modification with components $\gamma_{\mathcal{B}}$: $(T,\tau)\circ\mathcal{B} \Rightarrow (\bar{T},\bar{\tau})\circ\mathcal{B}$. We are to show that $\gamma = (\eta,\theta)\circ-$ for a unique (η,θ).

If γ is indeed of this form, it follows from (5.5) that the $A[1_{\Gamma A}]$ component of $\gamma_{\Gamma A}$ is

$$(\gamma_{\Gamma A})_{A[1_{\Gamma A}]} = \eta_A[\theta_A], \tag{5.20}$$

which proves the uniqueness of η and θ. If now for any γ we <u>define</u> η_A and θ_A by (5.20), we have because γ is a modification commutativity in



and calculating the $A[1_{\Gamma A}]$-component of both legs gives

$$(\gamma_{\mathcal{B}})_{A[X]} = \eta_A[X\theta_A], \tag{5.21}$$

in agreement with (5.5). That η is indeed natural, and that θ is indeed a modification, follow at once when we express the <u>naturality</u> of $\gamma_{\Gamma A'}$ for the particular morphism (5.11) of $A\cdot\Gamma A'$. □

<u>Corollary 5.2</u> $CAT\int T$ <u>and</u> $CAT\int{}^{*}T$ <u>are indeed</u> 2-<u>categories</u>.

<u>Proof</u> By the remarks in §3.1 and §4.2, with the above theorem. □

5.3 We devote this section to some comments, largely informal, on this full embedding Φ: $CAT\int^{*}CAT \to [CAT,CAT]$ of 2-categories.

First, its image is not the whole of $[CAT\ CAT]$. To see this, note that we have clearly

(5.22) $\qquad (A,\Gamma)\circ I \cong A,$

where I is the unit category. So for a 2-functor D: $CAT \to CAT$ we have:

(5.23) If $D \cong (A,\Gamma)\circ -$ then $A \cong DI$.

It follows that if $DI \cong I$ and if $D \cong (A,\Gamma)\circ -$ then $A \cong I$, so that $\Gamma = \ulcorner n \urcorner$ for some category n. But it is also clear that

(5.24) $\qquad (I,\ulcorner n \urcorner)\circ \mathcal{B} \cong [n,\mathcal{B}];$

so any D in the image of Φ with $DI \cong I$ must be $[n,-]$ for some n. Now the 2-functor D: $CAT \to CAT$ sending $\mathcal{B}$ to $[[\mathcal{B},2],\mathcal{B}]$, where 2 is the discrete category with 2 objects, is certainly not of the form $[n,-]$. (For $[n,\mathcal{B}] = [m,\mathcal{B}]$ when $\mathcal{B}$ is discrete, where m is the number of path--components of n; and no m is consistent with both $D2 = 2^4$ and $D3 = 3^8$.)

Even the 2-functor part of a doctrine D need not be in the image of Φ. Consider the doctrine, which can be given explicitly, for a category-with-coequalizers. There it is clear that DI is at any rate equivalent to I; this is enough to make $D\mathcal{B}$ equivalent to $[n,\mathcal{B}]$ if D were in the image of Φ; but D is not of this form. I think however that this D is a quotient of something in the image of Φ, to wit of $[n,-]$ where n is $\aleph$.

I have no real idea how big the image of Φ in fact is; chiefly because, while I can pick out the putative A by (5.23), I don't see how to fix Γ in terms of D, and hence have no test for D to be in the image of Φ.

We are going to show in §7 that the image of Φ is closed under composition of 2-functors. What else is it closed under? Certainly products and coproducts.

It suffices to illustrate by binary ones. The product in $CAT\int^{*}CAT$ of (A,Γ) and (A',Γ') is $(A \times A', \Delta)$ where $\Delta(A,A') = \Gamma A + \Gamma' A'$; and $(A \times A',\Delta)\circ\mathcal{B} \cong (A,\Gamma)\circ\mathcal{B} \times (A',\Gamma')\circ\mathcal{B}$. The coproduct in $CAT\int^{*}CAT$ of (A,Γ) and (A',Γ') is $(A + A',\Delta)$ where $\Delta|A = \Gamma$ and $\Delta|A' = \Gamma'$; and $(A + A',\Delta)\circ\mathcal{B} \cong (A,\Gamma)\circ\mathcal{B} + (A',\Gamma')\circ\mathcal{B}$.

I haven't checked whether $CAT\int^{*}CAT$ has equalizers: chiefly because it would be a long business not relevant to our main purpose. But if it does have equalizers, my rough calculations indicate that Φ would not preserve them. (If Φ preserved them it would mean that equalizers of lax n.t.'s were constructed pointwise, which seems false.) If it has coequalizers, on the other hand, they are probably preserved.

For then Φ would seem to have a left adjoint, modulo size considerations. Consider a 2-natural $G\colon D \Rightarrow \Phi(A,\Gamma) = (A,\Gamma)\circ -$, with components $G_{\mathcal{B}}\colon D\mathcal{B} \to A\circ\mathcal{B}$. By Proposition 4.1 and the definition of $A\circ\mathcal{B}$ as $(A,\Gamma)/\!/(I, \ulcorner\mathcal{B}\urcorner)$, such a $G_{\mathcal{B}}$ would correspond to $P\colon D\mathcal{B} \to A$ together with $\alpha\colon \Gamma P \rightsquigarrow \mathcal{B}!$, where $\mathcal{B}!$ is the composite

$$D\mathcal{B} \to I \xrightarrow[\ulcorner\mathcal{B}\urcorner]{} CAT,$$

that is, the constant functor at $\mathcal{B}$. In other words the $G_{\mathcal{B}}$ are in bijection with morphisms $(D\mathcal{B},\mathcal{B}!) \to (A,\Gamma)$ in $CAT\int^{*}CAT$. The G's themselves would then be in bijection with the morphisms $\int^{\mathcal{B}}(D\mathcal{B},\mathcal{B}!) \to (A,\Gamma)$ in $CAT\int^{*}CAT$ if the indicated coend exists.

On the other hand it seems that Φ is as far as it can be from having a <u>right</u> adjoint; in the sense that any $D \in [CAT,CAT]$ with a 2-reflexion in $CAT\int^{*}CAT$ already lies in the latter. For if

$$(5.25)\quad [CAT,CAT]((A,\Gamma)\circ -,D) \cong CAT\int^{*}CAT((A,\Gamma),(\mathcal{B},\Delta))$$

2-naturally in (A,Γ), take $(A,\Gamma) = (I, \ulcorner n\urcorner)$. Then the left side of (5.25) is, by (5.24) and Yoneda, Dn; while the right side is easily seen to be $(\mathcal{B},\Delta)\circ n$. So $D \cong (\mathcal{B},\Delta)\circ -$.

<u>5.4</u> We shall always henceforth use $\mathcal{T}$ to denote a sub-2-category of CAT, not necessarily full on 1-cells, but always taken to be full

on 2-cells.

We have an obvious inclusion

(5.26) $\quad CAT\int^{*}T \to CAT\int^{*}CAT\ ,$

which composed with Φ gives

(5.27) $\quad \Phi_T\colon CAT\int^{*}T \to [CAT,CAT]\ .$

It is clear that (5.26) is full on 2-cells, and is full on 1-cells precisely when $T \to CAT$ is; so that the same is true of (5.27).

6. THE GROTHENDIECK CONSTRUCTION

6.1 Our next main goal is the proof in §7 below that $CAT\int^{*}CAT$ is closed in $[CAT,CAT]$ under composition of objects (that is, of 2-functors $CAT \to CAT$). We shall do this by showing that $A\circ(B\circ -) \cong (A\circ B)\circ -$; here $A\circ B$ is the category so denoted in §5, but now provided with an augmentation $A\circ B \to CAT$ derived from the augmentations of A and B.

The considerations of §1.2 above suggest that the augmentation $n = \Gamma A$ of $A \in \mathbf{A}$ is a kind of "arity" of A. An object $A[X]$ of $A\circ B$ consists of A and $X\colon \Gamma A = n \to B$, so that X is a kind of n-ad of objects of B. If these objects of B also have arities, provided by an augmentation $\Delta\colon B \to CAT$, then $A[X]$ should have an arity $n(m)$, where m is the composite $\Delta X\colon n \to B \to CAT$, and where $n(m)$ generalizes the $n(m_1,\ldots, m_n) = m_1 +\ldots+ m_n$ of §1.2. In fact the appropriate generalization is precisely the Grothendieck construction assigning to a category n and a functor $m\colon n \to CAT$ the corresponding fibred category over n, which we may call $n(m)$.

6.2 We therefore define the Grothendieck 2-functor

(6.1) $\quad \Theta\colon CAT\int CAT \to CAT$

by setting

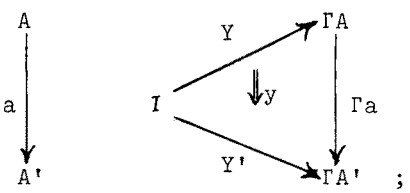
(6.2) $\Theta = \ulcorner I \urcorner /\!\!/ -$,

where of course $/\!\!/$ is the 2-functor of (4.9) with T put equal to CAT. We also use, where confusion is unlikely, the "parenthesis" notation suggested above: namely

(6.3) $\Theta(A,\Gamma) = A(\Gamma)$, $\Theta(T,\tau) = T(\tau)$, $\Theta(\eta,\theta) = \eta(\theta)$.

An object of $A(\Gamma)$ is a pair A,Y where $A \in A$ and $Y\colon I \to \Gamma A$ (that is, $Y \in \Gamma A$; but it is useful to write Y as a functor with domain I). We write this object as $A\langle Y\rangle$. A morphism $A\langle Y\rangle \to A'\langle Y'\rangle$ is a pair a,y of the form

(6.4)

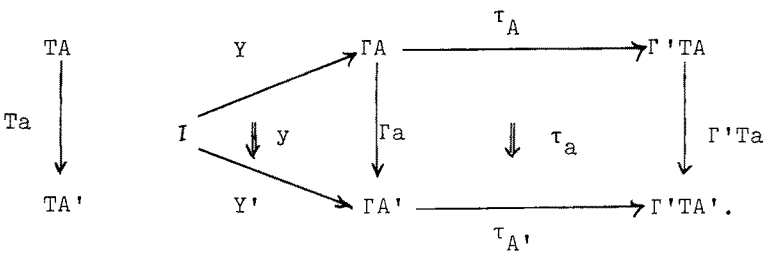


;

which is a fancy way of saying that y is a morphism $\Gamma a.Y \to Y'$ in $\Gamma A'$. We write this morphism as $a\langle y\rangle$.

If $(T,\tau)\colon (A,\Gamma) \to (A',\Gamma')$ in $CAT\int CAT$, then by (4.12) the functor $T(\tau)$ sends (6.4) to

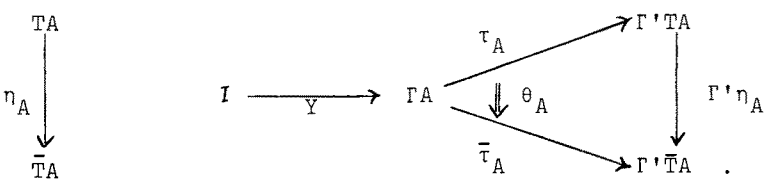
(6.5)



.

If $(\eta,\theta)\colon (T,\tau) \Rightarrow (\bar{T},\bar{\tau})$ in $CAT\int CAT$, the natural transformation $\eta(\theta)$ has by (4.15) its $A\langle Y\rangle$ component given by

(6.6)



.

We have as in §4.1 the projection functor

(6.7) $\quad d_1 : A(\Gamma) \to A$

sending (6.4) to $a: A \to A'$. As in Proposition 4.3 this is the component of a 2-natural transformation

(6.8) $\quad d_1 : \theta \Rightarrow \text{forg},$

where forg: $CAT\int CAT \to CAT$ is the forgetful 2-functor sending (A,Γ) to A, etc. Writing $\underline{2}$ for the arrow category $0 \to 1$, we can regard θ, forg, and d_1 in (6.8) as constituting a 2-functor

(6.9) $\quad \bar{\theta}: CAT\int CAT \to [\underline{2}, CAT]$

sending (A,Γ) to $d_1 : A(\Gamma) \to A$, etc. We can call $\bar{\theta}$ the augmented Grothendieck 2-functor.

6.3 The results we need about the Grothendieck construction are most easily derived from the following theorem, which is a slight extension of Gray's "Yoneda-Like Lemma" on page 290 of [3].

Theorem 6.1 The 2-functor $\bar{\theta}: CAT\int CAT \to [\underline{2}, CAT]$ is 2-fully-faithful.

Proof. Let

(6.10)
$$\begin{array}{ccc} A(\Gamma) & \xrightarrow{\quad M \quad} & A'(\Gamma') \\ {\scriptstyle d_1}\downarrow & & \downarrow{\scriptstyle d_1} \\ A & \xrightarrow[\quad T \quad]{} & A' \end{array}$$

be a morphism in $[\underline{2}, CAT]$. We have to show that $M = T(\tau)$ for a unique τ.

If M is indeed $T(\tau)$ we conclude from (6.5) with $a = 1_A$ that

(6.11) $\quad M(A\langle Y\rangle) = TA\langle \tau_A Y\rangle,$

(6.12) $\quad M(1_A\langle y\rangle) = 1_{TA}\langle \tau_A y\rangle,$

which fixes τ_A on objects and morphisms. Moreover if we define τ_A by

(6.11) and (6.12), the fact that M is a functor and the fact that the composite of $1_A\langle y'\rangle$ and $1_A\langle y\rangle$ is $1_A\langle y'y\rangle$ show that τ_A is a functor.

Again, if M is $T(\tau)$ we conclude from (6.5) with $Y' = \Gamma a.Y$ and $y = 1$ that

$$M(a\langle 1_{\Gamma a.Y}\rangle) = Ta\langle(\tau_a)_Y\rangle, \tag{6.13}$$

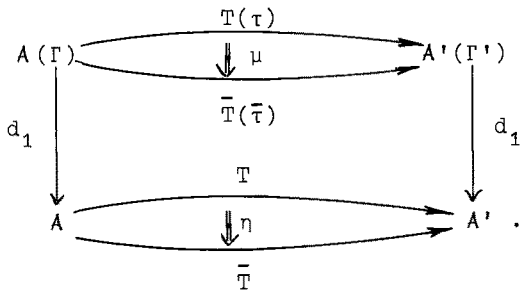
which fixes the components of τ_a. If we define these components by (6.13), the naturality of τ_a follows from the functoriality of M together with the following equality, where $y: Y \to Y'$ in ΓA:

$$1_{A'}\langle \Gamma a.y\rangle \cdot a\langle 1_{\Gamma a.Y}\rangle = a\langle 1_{\Gamma a.Y'}\rangle \cdot 1_A\langle y\rangle .$$

The fact that the typical morphism $a\langle y\rangle$ of (6.4) is the composite $1_{A'}\langle y\rangle \cdot a\langle 1_{\Gamma a.Y}\rangle$, along with the fact that M is a functor and the definitions (6.12) and (6.13), now easily gives that $M(a\langle y\rangle)$ is (6.5). The lax naturality of τ then follows by applying M to the composite $a'\langle 1_{\Gamma(a'a).Y}\rangle \cdot a\langle 1_{\Gamma a.Y}\rangle$.

Now let $\mu: T(\tau) \Rightarrow \bar{T}(\bar{\tau})$ be a natural transformation rendering commutative

(6.14)



We have to show that $\mu = \eta(\theta)$ for a unique θ.

If μ is indeed $\eta(\theta)$ we have by (6.6) that

$$\mu_{A\langle Y\rangle} = \eta_A\langle(\theta_A)_Y\rangle, \tag{6.15}$$

which fixes the Y-component of θ. If we now define $(\theta_A)_Y$ by (6.15), the naturality of θ_A follows from the naturality of μ with respect to a morphism $1_A\langle y\rangle$; and the fact that θ is a modification follows from the naturality of μ with respect to a morphism $a\langle 1_{\Gamma a.Y}\rangle$. □

Remark 6.2 The above theorem could serve as well as Theorem 5.1 to complete the verification, in Corollary 5.2, that $CAT\int T$ and $CAT\int^* T$ are 2-categories.

6.4 As a result of the above theorem we have an isomorphism of categories

$$(6.16)\quad CAT\int CAT((A,\Gamma),(A',\Gamma')) \cong [\underline{2},CAT]\begin{pmatrix} A(\Gamma) & , & A'(\Gamma') \\ {\scriptstyle d_1}\downarrow & & \downarrow{\scriptstyle d_1} \\ A & & A' \end{pmatrix},$$

which is moreover 2-natural because it is given by the 2-functor $\bar{\Theta}$.

There is an evident 2-functor (not full)

$$(6.17)\quad CAT \times CAT \to CAT\int CAT$$

sending A,B to $(A,B!)$ etc., where $B!\colon A \to CAT$ is the constant functor at B, composite of $A \to I$ and $\ulcorner B \urcorner\colon I \to CAT$. It is also clear that

$$(6.18)\quad \bar{\Theta}(A,B!) = \begin{array}{c} A \times B \\ \downarrow{\scriptstyle pr_1} \\ A \end{array},$$

and similarly on 1-cells and 2-cells. Putting $(A',\Gamma') = (I,\ulcorner B \urcorner)$ in (6.16) therefore gives a 2-natural isomorphism

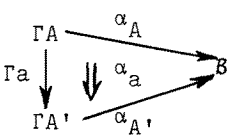
$$(6.19)\quad CAT\int CAT((A,\Gamma),(I,\ulcorner B \urcorner)) \cong CAT(A(\Gamma),B),$$

which exhibits Θ as the left-2-adjoint of the 2-functor $\ulcorner\ \urcorner$ of (3.13). (Note that, for categories C and B, we mean exactly the same by $CAT(C,B)$ and by $[C,B]$.) The left side of (6.19) can equally be written as the left side of

$$(6.20)\quad [\![A,CAT]\!](\Gamma,B!) \cong CAT(A(\Gamma),B).$$

An object $\alpha\colon \Gamma \rightsquigarrow B!$ of the left side of (6.20) is called a lax cocone over Γ with vertex B; its components look like

(6.21)

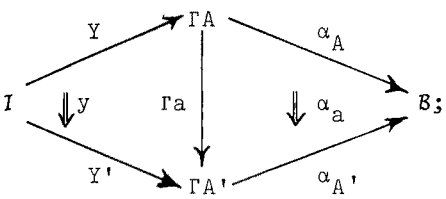


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So in view of (6.20) it is reasonable to call $A(\Gamma)$ the lax colimit of $\Gamma\colon A \to CAT$; this is in effect Gray's theorem on page 289 of [3].

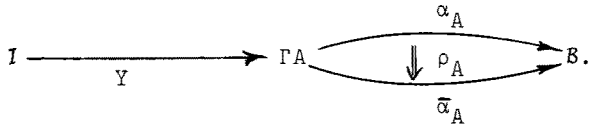
We give the explicit form of (6.20). By (6.5), the image of $\alpha\colon \Gamma \rightsquigarrow B!$ is the functor $P\colon A(\Gamma) \to B$ sending a $\langle y\rangle$ to

(6.22)



and by (6.6) the image of a modification $\rho\colon \alpha \to \bar{\alpha}$ is the natural transformation $\pi\colon P \Rightarrow \bar{P}$ whose $A\langle Y\rangle$ component is

(6.23)



By (6.11) - (6.13), or directly from the above, α is refound in terms of P by

(6.24) $\alpha_A Y = P(A\langle Y\rangle)$, $\alpha_A y = P(1_A\langle y\rangle)$, $(\alpha_a)_Y = P(a\langle 1_{\Gamma a.Y}\rangle)$;

and ρ is refound in terms of π by

(6.25) $(\rho_A)_Y = \pi_{A\langle Y\rangle}$.

6.5 In (6.20), we describe functors with domain $A(\Gamma)$. To get information about functors with codomain $A(\Gamma)$, we can either go back to the definition of $A(\Gamma)$ as $\ulcorner I \urcorner /\Gamma$ and use Propositions 4.1 and 4.2 directly; or use (6.16) again, this time replacing (A,Γ) by $(C,I!)$ and (A',Γ') by (A,Γ). In either case we get

(6.26) $CAT\int CAT((C,I!),(A,\Gamma)) \cong CAT(C,A(\Gamma))$.

An object of the left side of (6.26) consists of a functor $T\colon C \to A$ together with a lax n.t. $\tau\colon I! \rightsquigarrow \Gamma T$ (that is, a lax cone over ΓT with vertex I). A nicer-looking result is given by looking at the subcategories of both sides of (6.16) obtained by fixing the T of

(6.10).

In the general case this gives, for $T: A \to A'$,

$$(6.27)\quad [\![A,CAT]\!](\Gamma,\Gamma'T) \cong CAT/A' \left(\begin{array}{ccc} A(\Gamma) & , & A'(\Gamma') \\ {\scriptstyle Td_1}\downarrow & & \downarrow{\scriptstyle d_1} \\ A' & & A' \end{array} \right).$$

Making again the above substitutions of $(C,I!)$ for (A,Γ) and (A,Γ) for (A',Γ') this gives

$$(6.28)\quad [\![C,CAT]\!](I!,\Gamma T) \cong CAT/A \left(\begin{array}{ccc} C & , & A(\Gamma) \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle d_1} \\ A & & A \end{array} \right).$$

We remark, without making use of the fact, that another expression for the right side of (6.28) is classical; by e.g. the proposition on page 255 of Gray [3] we have

$$(6.29)\quad [A,CAT](\Omega T,\Gamma) \cong CAT/A \left(\begin{array}{ccc} C & , & A(\Gamma) \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle d_1} \\ A & & A \end{array} \right),$$

where the functor Ω sends T to the functor $\Omega T: A \to CAT$ given by $(\Omega T)A = T/\ulcorner A \urcorner$, the ordinary comma category of $T: C \to A$ and $\ulcorner A \urcorner: I \to A$.

6.6 Taking $A = A'$ and $T = 1$ in (6.27) gives

$$(6.30)\quad [\![A,CAT]\!](\Gamma,\Gamma') \cong CAT/A \left(\begin{array}{ccc} A(\Gamma) & , & A(\Gamma') \\ {\scriptstyle d_1}\downarrow & & \downarrow{\scriptstyle d_1} \\ A & & A \end{array} \right).$$

If we define the 2-functor Θ_A as the restriction of Θ given by

$$(6.31)\quad [\![A,CAT]\!] \to CAT\int CAT \xrightarrow[\Theta]{} CAT,$$

and define $\bar{\Theta}_A: [\![A,CAT]\!] \to CAT/A$ as the 2-functor defined by $\bar{\Theta}$ and sending Γ to $d_1: A(\Gamma) \to A$, then (6.30) is the special case of Theorem 6.1 asserting

Proposition 6.3 *$\bar{\Theta}_A$ is 2-fully-faithful.* □

We can look upon (6.20) as asserting that Θ_A has the right adjoint $B \mapsto B!$; and on (6.29) as asserting that the composite

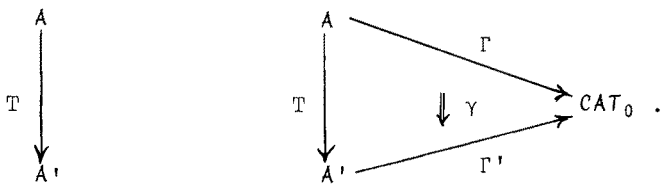
(6.32) $\quad [A,CAT] \to [\![A,CAT]\!] \xrightarrow[\bar{\Theta}_A]{} CAT/A$

has left adjoint Ω.

7. THE MONOIDAL STRUCTURE ON $CAT\int *CAT$

7.1 Consider the category $CAT_0 \circ CAT_0$, where the first CAT_0 has as augmentation the inclusion $CAT_0 \to CAT$, and the second CAT_0 is merely a category. Consider a typical morphism $T[\gamma]: A[\Gamma] \to A'[\Gamma']$ in $CAT_0 \circ CAT_0$; it has the form

(7.1)



Here A is a category, and Γ a functor from A to CAT_0, or equally to CAT; so we may identify the objects $A[\Gamma]$ with the objects (A,Γ) of $CAT\int CAT$. We can then identify the morphism $T[\gamma]$ in (7.1) with the morphism $(T,\gamma): (A,\Gamma) \to (A',\Gamma')$ of $CAT\int CAT$. Thus $CAT_0 \circ CAT_0$ may be identified with a subcategory of $(CAT\int CAT)_0$: to wit, it has all the objects, but only those morphisms (T,τ) for which the lax n.t. τ is in fact an honest natural transformation γ. This being so, there should be no confusion if we write

(7.2) $\quad \Theta: CAT_0 \circ CAT_0 \to CAT$

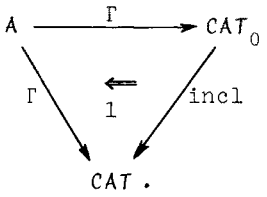
for the functor obtained by restricting the 2-functor Θ of (6.1).

Given functors $\Gamma: A \to CAT$ and $\Delta: B \to CAT$, we write as usual $A \circ B$ for the more precise $(A,\Gamma) \circ B$, and we define an augmentation functor $E: A \circ B \to CAT$ as the composite

(7.3) $\quad A \circ B \xrightarrow[\Gamma \circ \Delta]{} CAT_0 \circ CAT_0 \xrightarrow[\Theta]{} CAT.$

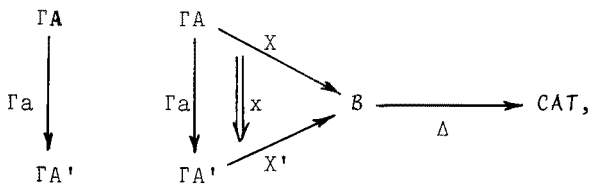
Here Γ stands for the morphism $(\Gamma, 1)$ of $CAT\int *CAT$ shown in

(7.4)



Thus $\Gamma\circ\Delta$ in (7.3) sends the typical morphism $a[x]$ of $A\circ B$ exhibited in (5.2) to the morphism

(7.5)



so that E sends $a[x]$ to

(7.6) $\qquad \Gamma a(\Delta x)\colon \Gamma A(\Delta X) \to \Gamma A'(\Delta X')$.

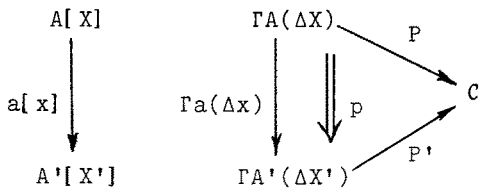
The basic result asserting that the image of the embedding Φ of (5.8) is closed under composition of endo-2-functors of CAT is now the following:

Proposition 7.1 Given categories A and B with augmentations Γ and Δ, assign to $A\circ B$ the augmentation E above. Then for $C \in CAT$ there is, 2-naturally in C, an isomorphism

(7.7) $\qquad a\colon (A\circ B)\circ C \to A\circ(B\circ C)$.

Proof A typical morphism of $(A\circ B)\circ C$ is an $a[x][p]\colon A[X][P]\to A'[X'][P']$ of the form

(7.8)



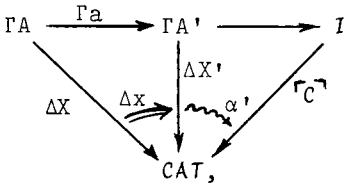
where a[x] itself is the typical morphism (5.2) of $\mathbb{A}\circ\mathcal{B}$.

By (6.19) or (6.20), to give P is equivalent to giving a lax n.t.

(7.9) $\alpha\colon \Delta X \rightsquigarrow C!$

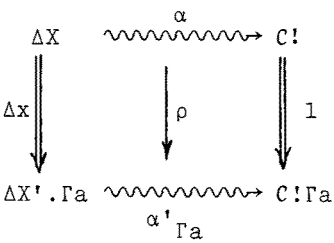
where $\Delta X, C!\colon \Gamma\mathbb{A} \to \mathcal{CAT}$. If similarly P' corresponds to α', the 2-naturality of (6.19) shows that $P'.\Gamma a(\Delta x)$ corresponds to the composite

(7.10)



that is, to $\alpha'_{\Gamma a}.\Delta x$. By (6.19) or (6.20) again, to give p is then equivalent to giving a modification

(7.11)



Since however $\mathcal{B}\circ\mathcal{C}$ is $\Delta/\!\!/\ulcorner C\urcorner$, to give the pair $X\colon \Gamma\mathbb{A} \to \mathcal{B}$ and α is equivalent by Proposition 4.1 to giving a functor

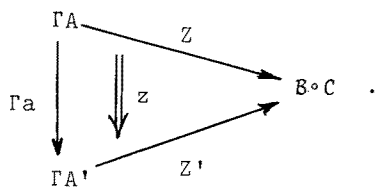
(7.12) $Z\colon \Gamma\mathbb{A} \to \mathcal{B}\circ\mathcal{C}$.

If similarly the pair X', α' corresponds to $Z'\colon \Gamma\mathbb{A}' \to \mathcal{B}\circ\mathcal{C}$, then by (4.4) the pair $X'.\Gamma a$, $\alpha'_{\Gamma a}$ corresponds to the composite

(7.13) $$\Gamma\mathbb{A} \xrightarrow[\Gamma a]{} \Gamma\mathbb{A}' \xrightarrow[Z']{} \mathcal{B}\circ\mathcal{C}.$$

Now by Proposition 4.2 to give the pair x,ρ as in (7.11) is equivalent to giving a natural transformation z of the form

(7.14)



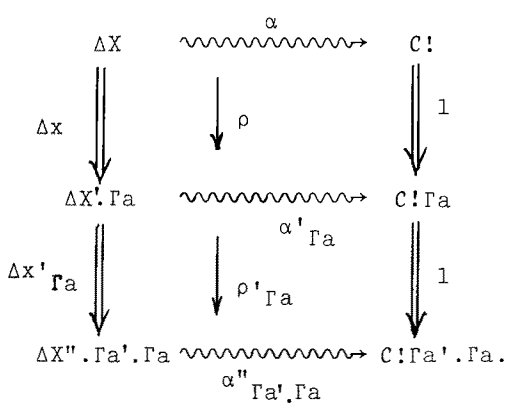
We henceforth write Z,z as

(7.15) $\qquad Z = X[P], \quad z = x[p];$

and we define the isomorphism α by

(7.16) $\qquad \alpha(a[x][p]) = a[x[p]].$

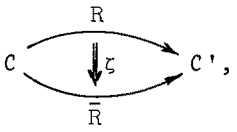
It has of course to be verified that α respects identies - this is obvious - and composites. If we compose (7.8) with $a'[x'][p']: A'[X'][P'] \to A''[X''][P'']$ to get $a''[x''][p'']$, we actually have $p'' = p'_{\Gamma A(\Delta X)}.p$. By the 2-naturality of (6.19), $p'_{\Gamma A(\Delta X)}$ corresponds to $\rho'_{\Gamma a}.\Delta x$, when we take account of what horizontal composition of 2-cells means in $CAT\int CAT$. Since (6.19) is functorial, it follows that the ρ'' corresponding to p'' is the composite



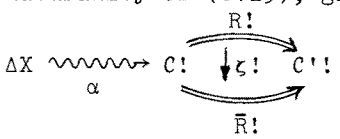
It now follows easily from (4.8) that $z'' = z'_{\Gamma a}.z$, as required.

Finally we have to verify the 2-naturality of α in C. First, if we compose (7.8) with $R: C \to C'$ then α,ρ are replaced by their composites with $R!: C! \Rightarrow C'!$ by the 2-naturality of (6.19), and it

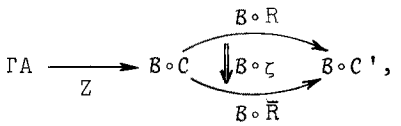
follows by §4.2 that (7.14) is replaced by its composite with $\mathcal{B}\circ R$. This is the naturality of a. For its 2-naturality we just take the domain $A[X][P]$ of (7.8) and compose it with



to get the morphism $1_A[1_X][\zeta_P]$, and apply a to this. The corresponding "ρ" is, by the 2-naturality of (6.19), given by



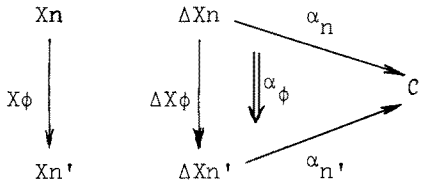
so that by §4.2 it easily follows that a sends $1_A[1_X][\zeta_P]$ to



as required. □

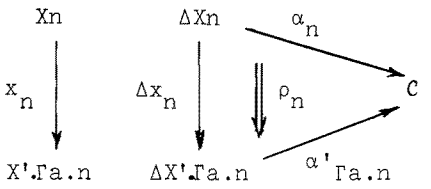
We record the detailed form of a more explicitly than by (7.16) by using Propositions 4.1 and 4.2, along with (6.24) and (6.25), to describe explicitly $Z = X[P]$ and $z = x[p]$. Namely Z sends the morphism $\phi\colon n \to n'$ of ΓA to

(7.17)



and z has n-component

(7.18)



where, for $y\colon Y \to Y'$ in ΔXn,

(7.19) $\alpha_n Y = P(n\langle Y\rangle)$, $\alpha_n y = P(1_n\langle y\rangle)$, $(\alpha_\phi)_Y = P(\phi\langle 1_{\Delta X\phi.Y}\rangle)$,

(7.20) $(\rho_n)_Y = p_{n\langle Y\rangle}$.

7.2 We now get automatically the desired monoidal structure on $CAT\int{*}CAT$. First, by Theorem 5.1, there is a unique way of making

(7.21) $\circ: CAT\int{*}CAT \times CAT\int{*}CAT \to CAT\int{*}CAT$,

which is so far only defined on objects, into a 2-functor such that the a of (7.7) is 2-natural in (A,Γ) and (B,Δ) as well as in C. Using the explicit forms (7.17) - (7.20) above, and (5.10), (5.13), and (5.20), it is a purely mechanical process to calculate

(7.22) $(T,\tau)\circ(S,\sigma): A\circ B \to A'\circ B'$

and

(7.23) $(\eta,\theta)\circ(\zeta,\phi): (T,\tau)\circ(S,\sigma) \Rightarrow (\bar{T},\bar{\tau})\circ(\bar{S},\bar{\sigma})$.

One part of the calculation is easy: if

(7.24) $(T,\tau)\circ(S,\sigma) = (N,\nu)$ and $(\eta,\theta)\circ(\zeta,\phi) = (\xi,\psi)$

then

(7.25) $N = (T,\tau)\circ S$ and $\xi = (\eta,\theta)\circ\zeta$;

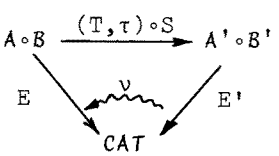
this follows on observing that the I-component of the a of (7.7) is clearly

(7.26) $$(A\circ B)\circ I \xrightarrow[r]{} A\circ B \xrightarrow[A\circ r^{-1}]{} A\circ(B\circ I)$$

where $r: A\circ I \cong A$ is the 2-natural isomorphism of (5.22). For the calculation of ν and ψ, however, we have to follow through the above process, and I merely give the results.

ν is to be a lax n.t.

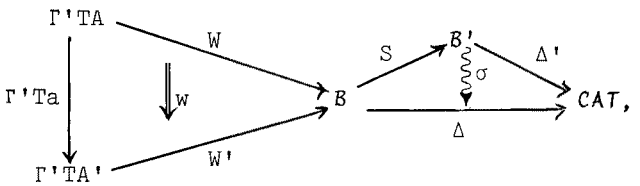
(7.27)



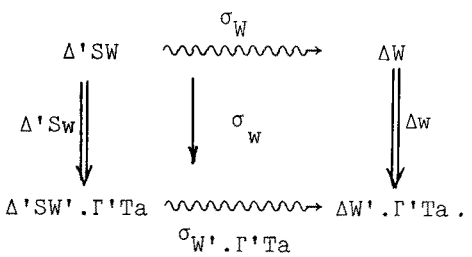
where E is given by (7.6) and E' analogously. Take $a[x]: A[X] \to A'[X']$ in $\mathcal{A}\circ\mathcal{B}$ as in (5.2), and write $Ta[w]: TA[W] \to TA'[W']$ for its image under $(T,\tau)\circ\mathcal{B}$, as displayed on the right side of (5.19). Then its image under $(T,\tau)\circ S$ is $Ta[Sw]: TA[SW] \to TA'[SW']$ as in (5.4). In view of (7.6), the ν we want is to have components

(7.28)
$$\begin{array}{ccc} \Gamma'TA(\Delta'SW) & \xrightarrow{\nu_{A[X]}} & \Gamma A(\Delta X) \\ {\scriptstyle \Gamma'Ta(\Delta'Sw)}\downarrow & \Downarrow \nu_{a[x]} & \downarrow{\scriptstyle \Gamma a(\Delta x)} \\ \Gamma'TA'(\Delta'SW') & \xrightarrow[\nu_{A'[X']}]{} & \Gamma A'(\Delta X') \end{array} .$$

Now we have

(7.29)


whence as in (2.8) we get

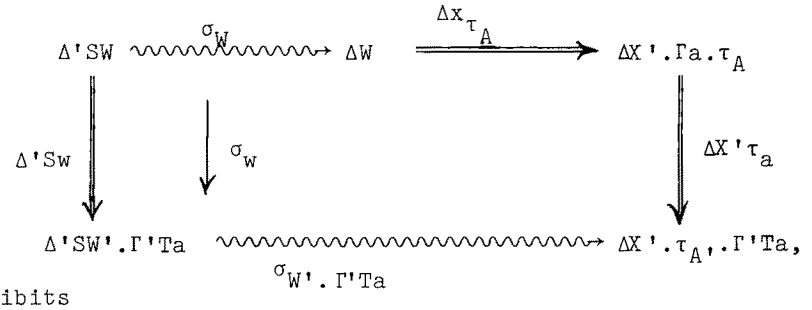
(7.30)


Since $W = X\tau_A$ by (5.19), the pair (τ_A,σ_W) is a morphism $(\Gamma'TA,\Delta'SW) \to (\Gamma A,\Delta X)$ in $\mathcal{CAT}\int\mathcal{CAT}$, and it turns out that we have

(7.31) $\quad \nu_{A[X]} = \tau_A(\sigma_W).$

The top leg of (7.28) is therefore (recalling the meaning of composition in $\mathcal{CAT}\int\mathcal{CAT}$) $\Gamma a.\tau_A(\Delta x_{\tau_A}.\sigma_W)$, while the bottom leg is $\tau_{A'}.\Gamma'Ta(\sigma_{W'}.\Gamma'Ta.\Delta'Sw)$. Since by (5.19) $w = X'\tau_a.x_{\tau_A}$, we can write (7.30) as

(7.32)



which exhibits

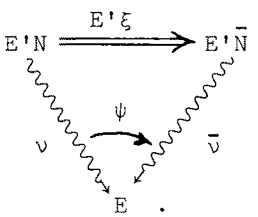
$$(\tau_a,\sigma_w)\colon (\Gamma a.\tau_A,\Delta x_{\tau_A}.\sigma_W) \to (\tau_{A'}.\Gamma'Ta,\sigma_{W'}.\Gamma'Ta.\Delta'Sw)$$

as a 2-cell in $CAT\int CAT$. It turns out that we have

(7.33) $$\nu_{a[X]} = \tau_a(\sigma_w).$$

It remains to give the modification ψ in

(7.34)



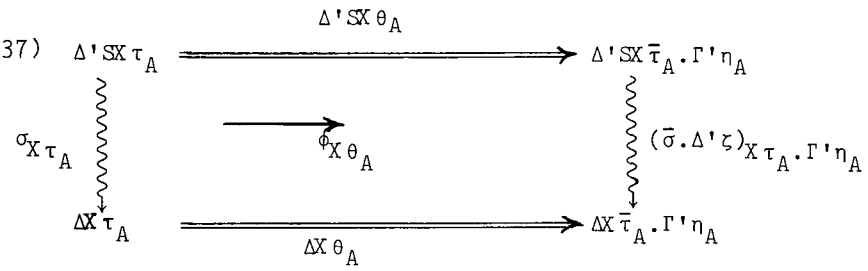
The A[X]-component of ν is $\tau_A(\sigma_W)$ by (7.31); that of $\bar{\nu}$ is $\bar{\tau}_A(\bar{\sigma}_{\bar{W}})$; here $W = X\tau_A$, $\bar{W} = X\bar{\tau}_A$. The A[X]-component of ξ, that is of $(\eta,\theta)\circ\zeta$ by (7.25), is $\eta_A[\zeta_{X\bar{\tau}_A}.\Gamma'\eta_A.SX\theta_A]$ by (5.5); hence that of E'ξ is $\Gamma'\eta_A(\Delta'\zeta_{X\bar{\tau}_A}.\Gamma'\eta_A.\Delta'SX\theta_A)$ by (7.6). Thus the A[X]-component of $\bar{\nu}.E'\xi$ is

(7.35) $$\bar{\tau}_A.\Gamma'\eta_A(\bar{\sigma}_{X\bar{\tau}_A}.\Gamma'\eta_A.\Delta'\zeta_{X\bar{\tau}_A}.\Gamma'\eta_A.\Delta'SX\theta_A).$$

The A[X]-component of ψ is to be a 2-cell from $\tau_A(\sigma_W)$ to (7.35); it is in fact given by

(7.36) $$\psi_{A[X]} = \theta_A(\phi_{X\theta_A}).$$

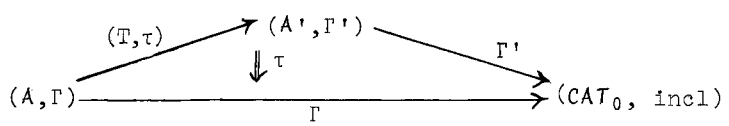
Here $\phi_{X\theta_A}$ is the modification

(7.37)


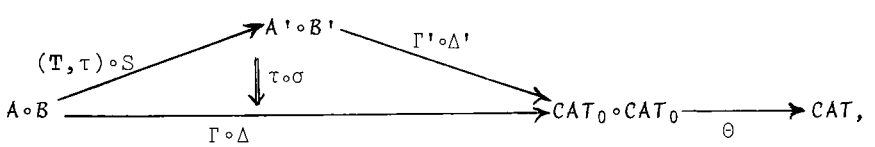
obtained as in (2.10) from $\phi\colon \sigma \to \bar{\sigma}.\Delta\zeta$ and $X\theta_A\colon X\tau_A \Rightarrow X\bar{\tau}_A.\Gamma'\eta_A$; it is clear that the pair $(\theta_A,\phi_{X\theta_A})$ is indeed a 2-cell in $CAT\int CAT$ with the right domain and codomain.

7.3 Some comments now about the 2-functor (7.21) whose explicit form we have just given. Let us embed CAT 2-fully in $CAT\int{}^*CAT$ by identifying A with $(A,0!)$ where $0!$ is the constant functor at the empty category 0. Then if $B \in CAT$, so that $\Delta = 0!$, we have for all $A[X]$ that $\Gamma A(\Delta X) = \Gamma A(0!) = 0$, so that $E\colon A\circ B \to CAT$ is also $0!$, and $A\circ B \in CAT$. Hence the 2-functor $\circ$ of (5.1) is just the restriction of the 2-functor $\circ$ of (7.21).

The expressions we have given for $(T,\tau)\circ(S,\sigma)$ and $(\eta,\theta)\circ(\zeta,\phi)$ above simplify when, as in many important examples, τ and σ are honest natural transformations, not lax ones, and θ and ϕ are identities; this is necessarily so for instance when the augmentations Γ and Δ of A and B factorize through $SET \subset CAT$. For then, using Γ in the two senses of (7.4), it is the case that $\tau = (\tau,1)$ is a 2-cell

(7.38)


in $CAT\int{}^*CAT$, and the ν of (7.27) is the composite

(7.39)


while the ψ of (7.34) is, by (7.36), the identity again.

There is a further simplification when we suppose that τ and

σ also are identities, so that we are working in the sub-2-category CAT/CAT of $CAT\int^{*}CAT$; then ν too is an identity by (7.31) and (7.33). In other words CAT/CAT is closed under $\circ$; and $T\circ S$, $\eta\circ\zeta$ unambiguously denote the composites in (7.24) or (7.25).

7.4 That $\circ$ is coherently associative with an identity, making $CAT\int^{*}CAT$ a monoidal 2-category, follows from Proposition 7.1 since $[CAT,CAT]$ is a strict monoidal 2-category with composition as its "tensor product". Turning first to the identity for $\circ$, we see at once that it may be taken to be the object

$$J = (I, \ulcorner I \urcorner) \tag{7.40}$$

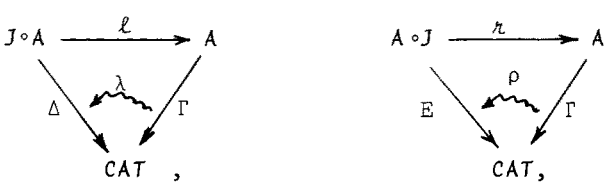
of $CAT\int^{*}CAT$; for (5.24) gives a 2-natural isomorphism $\ell\colon J\circ C \cong C$ for $C \in CAT$, so that $J\circ -$ is isomorphic to $1 \in [CAT,CAT]$. (Note that, while in general we write n for the discrete category with n objects, and in particular 0 for the empty category, we write I rather than 1 to avoid confusion with identity maps. Note also that, since we identify $A \in CAT$ with $(A,0!) \in CAT\int^{*}CAT$, and in particular I with $(I,\ulcorner 0\urcorner)$, we need to distinguish I from $J = (I,\ulcorner I\urcorner)$.)

From Proposition 7.1 we now get well-determined 2-natural isomorphisms

$$\ell\colon J\circ A \cong A, \quad r : A\circ J \cong A \tag{7.41}$$

for $A \in CAT\int^{*}CAT$. It is clear that the functor-part of ℓ is the ℓ above, that of (5.24), sending $*[\ulcorner a\urcorner]$ to a; and that the functor-part of r is what we called r in (7.26), that of (5.22), sending $a[1_{*!}]$ to a. However, as morphisms in $CAT\int^{*}CAT$, ℓ and r involve also lax n.t.'s λ and ρ as in

(7.42)



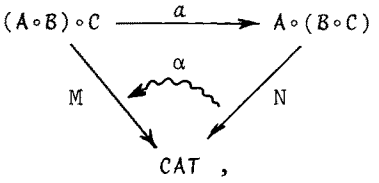
where Δ,E are the appropriate augmentations. As we have defined things λ and ρ are not identities, only isomorphisms; for e.g. by (7.6) $\Delta(*[\ulcorner A\urcorner]) = I(\ulcorner \Gamma A\urcorner)$ which, while isomorphic to ΓA, is not the <u>same</u> as it.

Again Proposition 7.1 gives a well-determined 2-natural isomorphism

(7.43) $\qquad a\colon (A\circ B)\circ C \to A\circ(B\circ C)$

for $A,B,C \in CAT\int^{*}CAT$; the coherence of a,ℓ,r is automatic; and clearly the functor-part of a is just the a of (7.7). Once again, however, its other part α ,

(7.44)

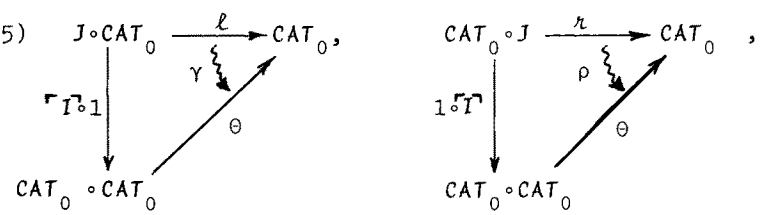


is only an isomorphism, not an identity. This is true even if the augmentations Γ,Δ,E of A,B,C factorize through $SET \subset CAT$; $M(A[X][P])$ and $N(A[X[P]])$ are different sets, in the same sense that, for sets U,V,W, the sets $(U \times V) \times W$ and $U \times (V \times W)$ are different sets.

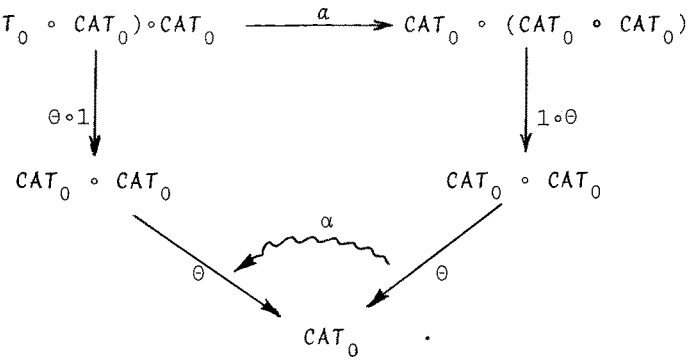
<u>7.5</u> This failure of λ,ρ,α to be identities is of importance, and we look at it more closely.

The whole failure arises, as it were, from the special case $A = B = C = CAT_0$, where (7.42) and (7.44) take the forms

(7.45)

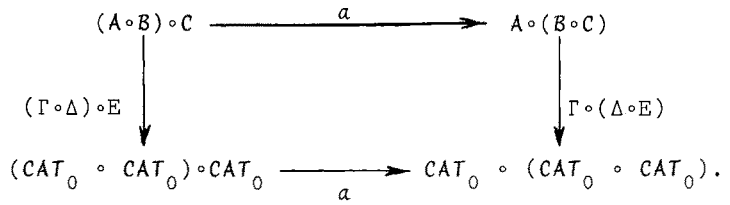


(7.46)



For example, the diagram (7.44) is just the result of pasting on to the top of (7.46) the commutative diagram

(7.47)



To see this, recall that M is $\Theta.\Theta\circ 1.(\Gamma\circ\Delta)\circ E$ by (7.3), with a corresponding value for N; recall that the a of (7.43) is natural; express this naturality with respect to Γ,Δ,E regarded as morphisms as in (7.4); and use the observation at the end of §7.3 that $(\Gamma,1)\circ(\Delta,1) = (\Gamma\circ\Delta,1)$, etc. Similar observations can be made for λ and ρ.

It is easy to calculate precisely the λ,ρ,α of (7.45) and (7.46). First, they are not merely lax natural, but honest natural isomorphisms; so we need only look at their values on objects. First, for $A \in CAT$, the top leg of the left diagram in (7.45) sends $*[A]$ to A, and the bottom leg sends it to $I(\ulcorner A \urcorner)$; $\lambda_{*[A]}$ is the canonical isomorphism $A \to I(\ulcorner A \urcorner)$. Next, the top leg of the right diagram in (7.45) sends $A[!]$ to A, while the bottom leg sends it to $A(I!)$; $\rho_{A[!]}$ is the canonical isomorphism $A \to A(I!)$. Finally, consider the effect of the two legs of (7.46) on $A[\Gamma][\Delta]$ where $\Gamma\colon A \to CAT$ and

$\Delta\colon A(\Gamma) \to CAT$. The top leg sends it to $A(E)$, where $E = \Theta(\Gamma[\Delta])$; the bottom leg to $A(\Gamma)(\Delta)$. If we write Z for $\Gamma[\Delta]\colon A \to CAT_0 \circ CAT_0$, it follows from (7.17) that $E = \Theta Z$ sends $a\colon A \to A'$ to $\Gamma a(\beta_a)\colon \Gamma A(\beta_A) \to \Gamma A'(\beta_{A'})$ where, for $y\colon Y \to Y'$ in ΓA,

$$(7.48)\quad \beta_A(Y) = \Delta(A\langle Y\rangle),\ \beta_A(y) = \Delta(1_A\langle y\rangle),\ (\beta_a)_Y = \Delta(a\langle 1_{\Gamma a.Y}\rangle).$$

The isomorphism $\alpha = \alpha_{A[\Gamma][\Delta]}\colon A(E) \to A(\Gamma)(\Delta)$ is now easily written down; an object of $A(E)$ is $A\langle K\rangle$ where $A \in \mathcal{A}$, $K\colon \mathcal{I} \to EA = \Gamma A(\beta_A)$; K corresponds to $Y\langle L\rangle$ where $Y \in \Gamma A$, $L\colon \mathcal{I} \to \beta_A Y$; an object of $A(\Gamma)(\Delta)$ is $A\langle Y\rangle\langle L\rangle$ where $A\langle Y\rangle \in A(\Gamma)$ and $L\colon \mathcal{I} \to \Delta(A\langle Y\rangle) = \beta_A Y$; similarly for morphisms.

So λ, ρ, α fail to be identities in the most trivial way; if we take $\mathcal{A}$ to be discrete, Γ to be constant at the discrete category $\mathcal{B}$, and Δ to be constant at the discrete category $\mathcal{C}$, the above component of α is in effect the isomorphism $(\mathcal{A} \times \mathcal{B}) \times \mathcal{C} \to \mathcal{A} \times (\mathcal{B} \times \mathcal{C})$. However they do fail to be identities, and as a result, as we see by (7.45) and (7.46), CAT_0 fails to be a monoid in $CAT\int *CAT$; so that $CAT_0\circ -$ fails to be a doctrine; it is only a pseudo-doctrine (cf. [12], §1.5 and §3.5). The same is true of the more realistic "doctrine" $\mathcal{C}at_0\circ -$; and hence of the "doctrine" whose algebras are categories with small colimits, which Kock [13] shows to be a quotient doctrine of $\mathcal{C}at_0\circ -$.

The pseudoness here is a fact of life; Zöberlein in his thesis [20] accepts it and gets involved in the whole complexity of pseudo-doctrines. Perhaps this is the right thing to do; but it daunts me totally, and I shall avoid it as long as I am able. To do this I follow Kock's thesis [13], where he considers categories with small colimits indexed only by specially chosen isomorphs of small categories. I am the more inclined to do so since, when we specialize the small categories to be finite sets, we get what is obviously the "right" setting for the corresponding coherence problems.

We therefore set about to modify the above in such a way that

α,λ,ρ become identities; we do so by replacing $CAT\int{}^*CAT$ by a sub-2-category $CAT\int{}^*cat$. We could define a club to be a $\circ$-monoid in $CAT\int{}^*CAT$, but it will be simpler if we use the name only for a $\circ$-monoid in $CAT\int{}^*cat$.

8. THE MONOIDAL STRUCTURES ON $CAT\int{}^*cat$ AND $CAT\int{}^*T$

8.1 Up until now size considerations have clearly been of no importance, and we have not hesitated for instance to write CAT/CAT (meaning of course $CAT/\ulcorner CAT\urcorner$). As we now wish to move back step-by-step towards more practical situations, we should in any case want at this stage to consider $CAT\int{}^*Cat$ rather than $CAT\int{}^*CAT$; that is, to deal with those (A,Γ) for which each ΓA is a small category. This however would not be enough to make identities of the isomorphisms λ,ρ,α of §7.5, so we go further.

We write set for the full sub-category of Set determined by those small sets that are in fact ordinal numbers. We write cat for the full sub-2-category of Cat determined by those small categories A such that $|A| = obA$ is an ordinal number and moreover each hom-set $A(A,B)$ is an ordinal number. We are led to use this 2-category cat because Kock's thesis [13] showed its success in overcoming our difficulty about λ,ρ,α; but it is in fact an entirely reasonable thing to do in analogy with §1.4 above, where the "types" ΓA were not arbitrary finite sets but elements of $\underline{N}$, i.e. finite ordinals.

More generally we are going to let T denote a sub-2-category of cat, not necessarily full on morphisms, but at any rate full on 2-cells. As in §5.4, $CAT\int{}^*T$ is then a sub-2-category of $CAT\int{}^*CAT$, full on 2-cells, and full on 1-cells when $T \to cat$ is so. Similarly $CAT\int T$ is a sub-2-category of $CAT\int CAT$. Recall too that, if L is a full sub-2-category of CAT, $L\int T$ denotes the obvious full sub-2-category of $CAT\int T$, and similarly for $L\int{}^*T$.

Of course we get the restriction

(8.1) $\circ$: $CAT\int *T \times CAT \to CAT$

of the 2-functor $\circ$ of (5.1); and as pointed out in §5.4 nothing in §5 needs changing when T is full in cat. The non-trivial modifications we are going to make concern the Grothendieck construction of §6.

8.2 In place of the Θ of (6.1) we define a new Grothendieck 2-functor

(8.2) Θ': $cat\int cat \to cat$

which is an isomorph of the restriction of Θ to $cat\int cat$. To do this it suffices, when $A \in cat$ and Γ: $A \to cat \subset CAT$, to give an object $\Theta'(A,\Gamma)$ in cat and an isomorphism ω: $\Theta(A,\Gamma) \to \Theta'(A,\Gamma)$. We then use this isomorphism to transfer to Θ' the 2-functor structure of Θ.

We begin by giving the restriction of Θ' to $set\int set$. For $A \in set$ and Γ: $A \to set$, $\Theta(A,\Gamma)$ is the discrete category with objects $A\langle Y\rangle$ where $A \in A$, Y: $I \to \Gamma A$. These objects can be ordered lexicographically: $A\langle Y\rangle < A'\langle Y'\rangle$ if $A < A'$ or if $A = A'$ and $Y < Y'$ (regarding Y,Y' as elements of ΓA). Then $\Theta(A,\Gamma)$ is a well-ordered set. We define $\Theta'(A,\Gamma)$ to be the corresponding ordinal, and ω: $\Theta(A,\Gamma) \to \Theta'(A,\Gamma)$ to be the unique order-isomorphism.

Now let $A \in cat$, Γ: $A \to cat$. Define $|\Gamma|$: $|A| \to set$ by $|\Gamma|A = |\Gamma A|$. We define $\Theta'(A,\Gamma)$ to be B where

(8.3) $$|B| = \Theta'(|A|,|\Gamma|),$$

(8.4) $$B(\omega A\langle Y\rangle, \omega A'\langle Y'\rangle) = \Theta'(A(A,A'),Z),$$

where $Za = \Gamma A'(\Gamma a.Y,Y')$.

There is an obvious isomorphism of categories ω: $\Theta(A,\Gamma) \to \Theta'(A,\Gamma)$, as desired. Clearly the Θ' we have defined on $set\int set$ is the restriction of this Θ' on $cat\int cat$.

We now simplify by changing our notation. When there is no

danger of confusion, we henceforth write θ for θ', $A\langle Y\rangle$ for $\omega A\langle Y\rangle$, and $a\langle y\rangle$ for $\omega a\langle y\rangle$. We also use the parenthesis notation again, with $A(\Gamma)$ for $\theta'(A,\Gamma)$, etc.

8.3 Returning now to §6, we see that there is little to change when the θ of (6.1) is replaced by the θ', now itself called θ, of (8.2). Of course $a\langle y\rangle$ is now not (6.4) but its image under ω, but (6.4) continues to represent it well. The $\bar{\theta}$ of (6.9) becomes $\bar{\theta}$: $cat\int cat \to [\underline{\underline{2}},cat] \subset [\underline{\underline{2}},CAT]$; and Theorem 6.1 continues to hold for this new $\bar{\theta}$. Similarly we still have (6.16) and its consequences (6.19), (6.20), and (6.26) - (6.31), as well as Proposition 6.3.

We can go further. Let $\mathcal{T}$ be a full sub-2-category of cat that is closed under the Grothendieck construction. By this we mean that $A(\Gamma) \in \mathcal{T}$ if $A \in \mathcal{T}$ and if Γ: $A \to cat$ factorizes through $\mathcal{T} \subset cat$ (we then write Γ: $A \to \mathcal{T}$). Then for the same reasons all the above results of §6 remain true with CAT replaced not only by cat but by $\mathcal{T}$. Examples of such $\mathcal{T}$ are set; the full sub-2-category ord of cat whose objects are categories $A \in cat$ which are ordered sets (i.e. $A(A,A') = 0$ or 1); and the full subcategory $\underline{\underline{S}}$ of set determined by the finite ordinals (the case of §1.4).

Those results of §6 depend of course on the fullness of $\mathcal{T}$ in cat. But we can at least define θ: $\mathcal{T}\int\mathcal{T} \to \mathcal{T}$ and $\bar{\theta}$: $\mathcal{T}\int\mathcal{T} \to [\underline{\underline{2}},\mathcal{T}]$ for certain non-full sub-2-categories $\mathcal{T}$ of cat: although we always require fullness on 2-cells. For this we need not only that $A(\Gamma) \in \mathcal{T}$ when $A \in \mathcal{T}$ and Γ: $A \to \mathcal{T}$, but also that $T(\tau)$ is a morphism in $\mathcal{T}$ when T is a morphism in $\mathcal{T}$ and each τ_A is a morphism in $\mathcal{T}$. The only cases that will concern us are that where $\mathcal{T} = \underline{\underline{P}}$, the subcategory of $\underline{\underline{S}}$ with the same object-set $\underline{\underline{N}}$ but with permutations as its only morphisms - this is the case of §1.3 - and the case where $\mathcal{T}$ is the discrete category $\underline{\underline{N}}$ itself.

8.4 Coming now to §7, and taking a $\mathcal{T} \subset cat$ as above, not necessarily full, we can again regard $\mathcal{T}_0 \circ \mathcal{T}_0$ as a subcategory of $(\mathcal{T}\int{}^*\mathcal{T})_0$, and as in (7.2) define the restriction

(8.5) $\quad \Theta: T_0 \circ T_0 \to T_0$

of Θ; of course when T is just a category, as in the cases $\mathcal{set}$, $\underline{S}$, $\underline{P}$, this is no restriction of Θ but is Θ itself. Note in particular that (8.5) is the restriction of Θ: $\mathcal{cat}_0 \circ \mathcal{cat}_0 \to \mathcal{cat}_0$.

Then for (A,Γ), $(B,\Delta) \in CAT\int^*T$, the analogue of (7.3) becomes $E: A \circ B \to T$ given by

(8.6) $$A \circ B \xrightarrow[\Gamma \circ \Delta]{} T_0 \circ T_0 \xrightarrow[\Theta]{} T,$$

exhibiting $A \circ B$ as an object of $CAT\int^*T$, where $\circ$ is the restriction (8.1) of (5.1). Since we have done no more than replace $A \circ B$ by an isomorph in $CAT\int^*CAT$, we still have Proposition 7.1; even the explicit description in (7.17) - (7.20) of the a of (7.16) does not change, modulo of course the new meaning of $n\langle Y\rangle$ etc. in (7.19) and (7.20).

As a result we get in analogy to (7.21) an extension

(8.7) $\quad \circ: CAT\int^*T \times CAT\int^*T \to CAT\int^*T$

of (8.1) when T is full in $\mathcal{cat}$; and moreover the $\circ$ of (8.7) is just the restriction of its value

(8.8) $\quad \circ: CAT\int^*\mathcal{cat} \times CAT\int^*\mathcal{cat} \to CAT\int^*\mathcal{cat}$

for $T = \mathcal{cat}$. Of course the detailed descriptions of this 2-functor in §7.2 also hold unchanged, again modulo the new meanings of $\tau_A(\sigma_W)$, etc.

When T is not full in $\mathcal{cat}$ we cannot conclude the same directly since we then lack the analogue of Theorem 5.1. Nevertheless we still do get (8.7); we have only to check that, for the ν of (7.24), each $\nu_{A[X]}$ lies in T; but by (7.31) this is $\tau_A(\sigma_W)$; and τ_A is a morphism in T by hypothesis; each $(\sigma_W)_n = \sigma_{X\tau_A n}$ for $n \in \Gamma'TA$ is also a morphism in T by hypothesis; whence the same is true of $\tau_A(\sigma_W)$ since T is closed under the Grothendieck construction.

The expressions for a,ℓ,r are also unchanged, as far as their functor parts go. However the α,λ,ρ of §§7.4 and 7.5 are now

identities, since before replacing Θ by Θ' they are order-preserving isomorphisms of categories. Hence we have what we want. In particular a,ℓ,r actually live in CAT/cat_0, or in CAT/T_0; and we recall from §7.3 that these latter 2-categories are also closed under the 2-functor $\circ$. We sum all this up in the following theorem:

Theorem 8.1 $CAT\int^* cat$ is a monoidal 2-category with tensor product $\circ$ given by (8.8) and described explicitly in §7.2; with identity $J = (I, \ulcorner I \urcorner)$; and with a,ℓ,r as described in §7.4. It contains CAT as a full sub-2-category when we identify A with $(A,0!)$; and $\circ$ restricts to $\circ: CAT\int^* cat \times CAT \to CAT$. The corresponding

(8.9) $\Phi: CAT\int^* cat \to [CAT,CAT]$

is 2-fully-faithful, and is a strong monoidal functor (sending $\circ$ to composition, to within isomorphism). The monoidal structure on $CAT\int^* cat$ restricts to the sub-2-category $CAT\int^* T$, and also to the sub-2-categories CAT/cat_0 and CAT/T_0, when T is closed under the Grothendieck construction. Moreover the restriction of (8.9) given by

(8.10) $\Phi: CAT\int^* T \to [CAT,CAT]$

is still 2-fully-faithful when T is 2-full in cat. □

Note that the $\circ$ of (8.8) is not the restriction of that of (7.21); it is only isomorphic to this.

9. THE CLOSED STRUCTURE ON CAT/T_0

9.1 We have said that we shall use the word club in general to denote a $\circ$-monoid in $CAT\int^* cat$, which may happen to lie in some smaller $CAT\int^* T$, and which by Theorem 8.1 is essentially the same thing as a doctrine whose functor-part lies in the image of the Φ of (8.9) (or of (8.10)). The original clubs however, described in §1 above, and arising from coherence problems, were $\circ$-monoids in the

non-full sub-2-category CAT/cat_0, often in a smaller CAT/T_0, and in particular in $CAT/\underline{\underline{S}}$ or $CAT/\underline{\underline{P}}$. Because these clubs are of especial practical importance, we now look more closely at the monoidal structure on CAT/T_0, and show that it is in fact closed (but not biclosed).

Consider first the relation between several such 2-categories. Let $T \subset T' \subset cat$ where T, T' are closed under the Grothendieck construction. Then, whether T and T' are full in cat or not, the inclusion 2-functors

$$(9.1) \qquad CAT/T_0 \to CAT/T'_0 \to CAT/cat_0$$

are full (which is not the case when / is replaced by $\int^*$). Moreover each has a right 2-adjoint. If we write $i: T \to T'$ for the inclusion, and write the first 2-functor in (9.1) as i_*, then it has the right 2-adjoint i^* sending (C,E) to its pullback (B,Δ) along i as in

$$(9.2) \qquad \begin{array}{ccc} B & \xrightarrow{\bar{i}} & C \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle E} \\ T & \xrightarrow[i]{} & T'. \end{array}$$

Next, the restriction

$$(9.3) \qquad \Phi': CAT/cat_0 \to [CAT,CAT]$$

of the Φ of (8.9) is certainly not full; nor of course is its further restriction

$$(9.4) \qquad \Phi': CAT/T_0 \to [CAT,CAT].$$

What is the case, however, is that (9.3), and hence also (9.4), have right 2-adjoints — recall from the end of §5.3 that this was not the case for Φ itself; rather there it was "as far from true as it could be".

We define a 2-functor

(9.5) $\Psi\colon [CAT,CAT] \to CAT/T_0$

which shall be the right adjoint of the Φ' of (9.4). Let $k\colon T_0 \to CAT$ be the inclusion, and define Ψ as the composite of the 2-functors

(9.6) $$[CAT,CAT] \xrightarrow[[k,1]]{} [T_0,CAT] \xrightarrow[q]{} [\![T_0,CAT]\!] \xrightarrow[\bar{\Theta}]{} CAT/T_0,$$

where the middle 2-functor q is just the inclusion, and $\bar{\Theta}$ is $\bar{\Theta}_{T_0}$ in the sense of (6.31) (the original, unmodified, $\bar{\Theta}$). We observe that, in view of (6.32), we are asserting that Φ' is the composite of $\Omega\colon CAT/T_0 \to [T_0,CAT]$ and the Kan right 2-adjoint of $[k,1]$; but it seems to be just as easy to establish the present result without reference to Ω. We state it formally:

Theorem 9.1 Ψ is right 2-adjoint to Φ'.

Proof We have to give a 2-natural isomorphism of categories

(9.7) $$\pi\colon CAT/T_0 \left(\begin{array}{ccc} A & & T_0(Dk) \\ \downarrow & \Gamma & \downarrow d_1 \\ T_0 & , & T_0 \end{array} \right) \to [CAT,CAT](A_0-,D).$$

In view of (6.28), we need therefore an isomorphism

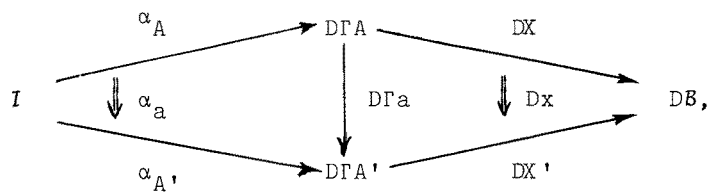
(9.8) $$\pi_1\colon [\![A,CAT]\!](I!,Dk\Gamma) \cong [CAT,CAT](A\circ-,D).$$

We might as well, as we have constantly been doing, write Γ for $k\Gamma$. So now $\Gamma\colon A \to CAT$, and we seek an isomorphism

(9.9) $$\pi_1\colon [\![A,CAT]\!](I!,D\Gamma) \cong [CAT,CAT](A\circ-,D).$$

For $\alpha\colon I! \rightsquigarrow D\Gamma$ we define $\pi_1\alpha$ to be $G\colon A\circ- \Rightarrow D$ whose B-component $G_B\colon A\circ B \to DB$ is the functor sending the typical morphism a[x] of (5.2) to

(9.10)



conceived of as a morphism in DB. For a modification

$\rho: \alpha \longrightarrow \alpha': I! \rightsquigarrow D\Gamma$ we define $\pi_1\rho$ to be the modification $\sigma: G \to G'$ whose $\mathcal{B}$-component $\sigma_{\mathcal{B}}: G_{\mathcal{B}} \Rightarrow G'_{\mathcal{B}}$ is the natural transformation with $A[X]$-component given by

(9.11) $$I \underset{\alpha'_A}{\overset{\alpha_A}{\Downarrow \rho_A}} D\Gamma A \xrightarrow[DX]{} D\mathcal{B}.$$

It is clear enough that $G_{\mathcal{B}}$ is indeed a functor, and scarcely harder to see that $\sigma_{\mathcal{B}}$ is indeed a natural transformation. We leave to the reader the easy proof that G is 2-natural, that σ is a modification, and that π_1 is a functor.

One next shows that π_1 is an isomorphism; given $G: A\circ- \Rightarrow D$, one argues pretty much as in the proof of Theorem 5.1 to show that G arises as above from a unique α; for example (9.10) shows we must have

(9.12) $$\alpha_A = G_{\Gamma A}(A[1_{\Gamma A}]),\ \alpha_a = G_{\Gamma A'}(a[1_{\Gamma a}]),$$

and we use the 2-naturality of G to show that α is lax natural and that G is $\pi_1(\alpha)$. Similarly (9.11) shows that we must have

(9.13) $$\rho_A = (\sigma_{\Gamma A})_{A[1_{\Gamma A}]};$$

and we verify that ρ is a modification with $\pi_1(\rho) = \sigma$.

I have no really short way of verifying the 2-naturality of the isomorphism π thus established; conceptual proofs would take us into extensions for which we have no other use; I myself have found it easiest to verify the 2-naturality of the inverse π^{-1}, by direct calculation using (9.12), (9.13), and our detailed knowledge of (6.28).[

I have given the above proof in outline only, because I make no really central use of the theorem. I note here that the above theorem provides a new proof of Theorem 5.1, by specialization to the case where $D = \mathcal{B}\circ-$; it is a kind of generalization of Theorem 5.1. I note further that, just as Gray on page 290 of [3] calls our Theorem 6.1 "Yoneda-like", so too is this theorem "Yoneda-like"; it purports

to give 2-natural transformations $A\circ- \Rightarrow D$ in terms of special "universal" components as in (9.12), just as the V-Yoneda lemma allows us to determine a V-natural transformation $A\otimes- \Rightarrow D$ in terms of a universal element $A \to DI$.

9.2 We have said that CAT/T_0 is a closed category; $-\circ B\colon CAT/T_0 \to CAT/T_0$ has a right adjoint $\{B,-\}\colon (CAT/T_0)^{op} \to CAT/T_0$. We have also said that it is not biclosed; $A\circ-$ has no right adjoint. Let us dispose of this latter point first.

If $A\circ-$ had a right adjoint, it would preserve the initial object $0 \to T_0$ of CAT/T_0. Yet $A\circ 0$ has as objects those pairs $A[1_0]$ where $\Gamma A = 0$; it is therefore isomorphic to the fibre of $\Gamma\colon A \to T_0$ above 0; which need by no means be 0.

What we are claiming is that there is a 2-functor

$$\{\ ,\ \}\colon (CAT/T_0)^{op} \times CAT/T_0 \to CAT/T_0 \tag{9.14}$$

such that we have 2-naturally

$$CAT/T_0(A\circ B,C) \cong CAT/T_0(A,\{B,C\}). \tag{9.15}$$

The case of greatest importance for us is that where $B,C \in CAT \subset CAT/T_0$; and here it is particularly simple to give $\{B,C\}$. So for the moment we are looking at

$$CAT(A\circ B,C) \cong CAT/T_0(A,\{B,C\}) \tag{9.16}$$

for $A \in CAT/T_0$, $B,C \in CAT$.

The left side of (9.16) is the same as $[CAT,CAT](A\circ-,D)$, where D is the right 2-Kan-extension of $\ulcorner C\urcorner\colon I \to CAT$ along $\ulcorner B\urcorner\colon I \to CAT$, that is to say,

$$D = [[-,B],C]\colon CAT \to CAT. \tag{9.17}$$

This is obvious without explicit reference to Kan extensions; we have using 2-Yoneda, and writing E for $A\circ-$,

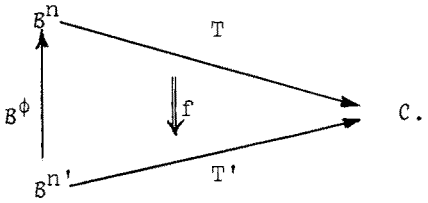
(9.18) $$\begin{aligned} [CAT,CAT](E,[[-,\mathcal{B}],\mathcal{C}]) &\cong 2\text{-nat}(En,[[n,\mathcal{B}],\mathcal{C}]) \\ &\cong 2\text{-nat}([n,\mathcal{B}],[En,\mathcal{C}]) \\ &\cong [E\mathcal{B},\mathcal{C}] \\ &= CAT(A\circ\mathcal{B},\mathcal{C}). \end{aligned}$$

It follows from Theorem 9.1 that we have (9.16) where

(9.19) $$\{\mathcal{B},\mathcal{C}\} = \Psi D = \Psi[[-,\mathcal{B}],\mathcal{C}].$$

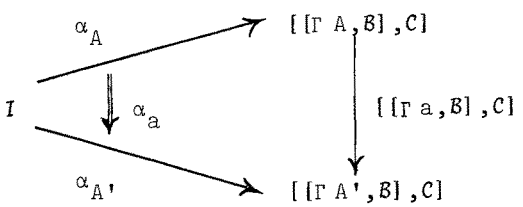
In view of the definition (9.6) of Ψ, we have the following description of $\{\mathcal{B},\mathcal{C}\}$. An object consists of $n \in \mathcal{T}_0$ together with a functor $T\colon [n,\mathcal{B}] \to \mathcal{C}$; which latter we may equally write as $T\colon \mathcal{B}^n \to \mathcal{C}$. A morphism consists of $\phi\colon n \to n'$ in $\mathcal{T}_0$ together with a natural transformation

(9.20)

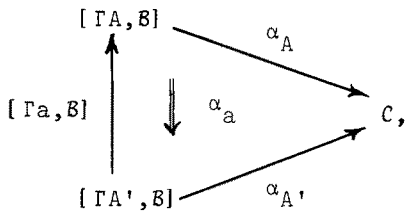


This last is precisely what we called, in §1.3 in the case $\mathcal{T} = \underline{\underline{P}}$, or in §1.4 in the case $\mathcal{T} = \underline{\underline{S}}$, a <u>natural transformation</u> $T \to T'$ <u>of type</u> $\phi : n \to n'$. So $\{\mathcal{B},\mathcal{C}\}$ is a "rich functor category", with objects $(n,T\colon \mathcal{B}^n \to \mathcal{C})$ and morphisms (ϕ,f) as in (9.20). The augmentation $\{\mathcal{B},\mathcal{C}\} \to \mathcal{T}_0$ of course sends (n,T) to n and (ϕ,f) to ϕ.

In fact one scarcely needs to appeal to Theorem 9.1, whose proof after all we have only sketched, in this case. With $\{\mathcal{B},\mathcal{C}\}$ defined by (9.19), we know from (6.28) that a functor $H\colon \mathcal{A} \to \{\mathcal{B},\mathcal{C}\}$ in $CAT/\mathcal{T}_0$ corresponds to an $\alpha\colon I! \rightsquigarrow [[\Gamma -,\mathcal{B}],\mathcal{C}]$; the components



clearly may be written as



and evidently correspond to a functor F: $A \circ B \to C$, sending $A[X]$ to $\alpha_A X$, etc. We leave the reader to verify the following explicit description of the isomorphism (9.16), which can be used to provide a third, totally direct and elementary, proof of the isomorphism.

9.3 The F: $A \circ B \to C$ and the corresponding H: $A \to \{B,C\}$ are connected as follows. H sends a: $A \to A'$, where $\Gamma A = n$, $\Gamma A' = n'$, $\Gamma a = \phi$, to (9.20) where

$$(9.21) \qquad TX = F(A[X]), \quad Tx = F(1_A[x]), \quad f_{X'} = F(a[1_{X'\phi}]).$$

To give F in terms of H, let H send a: $A \to A'$ to (9.20). Then (9.21) gives F on objects; it also gives F on morphisms since the generic morphism $a[x]$ of (5.2) is the composite

$$(9.22) \qquad A[X] \xrightarrow[1_A[x]]{} A[X'\phi] \xrightarrow[a[1_{X'\phi}]]{} A'[X'].$$

This suggests making an abbreviation which goes against the usual convention that the name of an object can be used as the name of its identity morphism. We do this because, especially in §10 below, the notational advantages outweigh the disadvantages.

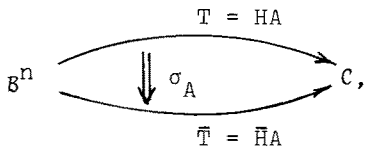
We elect then to abbreviate the morphism $1_A[x]$ of $A \circ B$ to $A[x]$, but the morphism $a[1_{X'\phi}]$, where a: $A \to A'$ with $\Gamma a = \phi$, to $a[X']$. Then (9.22) gives the generic $a[x]$ as

$$(9.23) \qquad A[X] \xrightarrow[A[x]]{} A[X'\phi] \xrightarrow[a[X']]{} A'[X'],$$

and (9.21) gets re-written as

(9.24) $\quad TX = F(A[X]),\ Tx = F(A[x]),\ f_{X'} = F(a[X'])$.

Next, we consider a natural transformation $\rho\colon F \Rightarrow \bar{F}$, and the corresponding $\sigma\colon H \Rightarrow \bar{H}$ in CAT/T_0. Writing again T for HA and $\bar{T}$ for $\bar{H}A$, we have

(9.25)


and it is given by

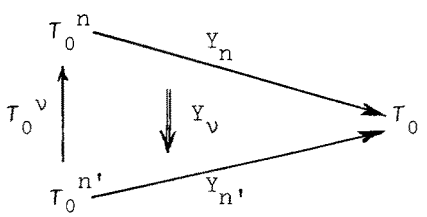
(9.26) $\quad (\sigma_A)_X = \rho_{A[X]}$.

9.4 We now give $\{\mathcal{B},\mathcal{C}\}$, or more exactly $\{(\mathcal{B},\Delta),(\mathcal{C},E)\}$, in the general case where $(\mathcal{B},\Delta)$, $(\mathcal{C},E) \in CAT/T_0$. Just for the moment write $\overline{\{\mathcal{B},\mathcal{C}\}}$ for the original $\{\mathcal{B},\mathcal{C}\}$ of §9.2 when the augmentations of $\mathcal{B}$ and $\mathcal{C}$ are discarded (or replaced by the trivial ones), so that $\mathcal{B},\mathcal{C} \in CAT$. Then $\{\mathcal{B},\mathcal{C}\} = \{(\mathcal{B},\Delta),(\mathcal{C},E)\}$ is to be a subcategory of $\overline{\{\mathcal{B},\mathcal{C}\}}$.

Recall that we have $\Theta\colon T\int T \to T$. For $n \in T_0$ define a functor $Y_n\colon T_0{}^n \to T_0$ by

(9.27) $\quad Y_n(m) = n(m),\ \ Y_n(\mu) = n(\mu) = 1_n(\mu)$,

where $\mu\colon m \Rightarrow m'$ in $T_0{}^n$. For $\nu\colon n \to n'$ in T_0 define a natural transformation

(9.28)


whose m'-component $(Y_\nu)_{m'}\colon n(m'\nu) \to n'(m')$, where $m'\colon n' \to T_0$, is given by

(9.29) $\quad (Y_\nu)_{m'} = \nu(m') = \nu(1_{m'\nu})$.

Note that in (9.27) and (9.29) we have used the same conventions

with round brackets as we introduced in §9.3 with square ones: $1_n(\mu)$ is written $n(\mu)$ but $\nu(1_{m',\nu})$ as $\nu(m')$.

Now an object of $\{\mathcal{B},\mathcal{C}\}$ is a pair n,T where $n \in \mathcal{T}_0$ and where $T\colon \mathcal{B}^n \to \mathcal{C}$ renders commutative

(9.30)
$$\begin{array}{ccc} \mathcal{B}^n & \xrightarrow{\ T\ } & \mathcal{C} \\ {\scriptstyle \Delta^n}\downarrow & & \downarrow{\scriptstyle E} \\ \mathcal{T}_0{}^n & \xrightarrow[Y_n]{} & \mathcal{T}_0 \end{array} ;$$

a morphism from n,T to n',T' consists of $\phi\colon n \to n'$ in $\mathcal{T}_0$ and a natural transformation f as in (9.20), satisfying

(9.31)
$$\begin{array}{ccccc} \mathcal{B}^n & & \xrightarrow{\ T\ } & & \\ {\scriptstyle \mathcal{B}^\phi}\uparrow & & \Downarrow f & & \mathcal{C} \xrightarrow[E]{} \mathcal{T}_0 \\ \mathcal{B}^{n'} & & \xrightarrow[T']{} & & \end{array} \quad \text{equals} \quad \begin{array}{ccccc} \mathcal{T}_0{}^n & & \xrightarrow{\ Y_n\ } & & \\ {\scriptstyle \mathcal{T}_0{}^\nu}\uparrow & & \Downarrow Y_\nu & & \mathcal{T}_0 \\ \mathcal{B}^{n'} & \xrightarrow[\Delta^{n'}]{} & \mathcal{T}_0{}^{n'} & \xrightarrow[Y_{n'}]{} & \end{array} .$$

We now state formally:

Theorem 9.2 There is a 2-adjunction

(9.24) $$\mathcal{CAT}/\mathcal{T}_0(\mathcal{A}\circ\mathcal{B},\mathcal{C}) \cong \mathcal{CAT}/\mathcal{T}_0(\mathcal{A},\{\mathcal{B},\mathcal{C}\}). \quad \square$$

We have not given the proof in the general case; it is a very easy matter to get it once we have the special case where $\mathcal{B},\mathcal{C} \in \mathcal{CAT}$; one has only to check that a functor $\mathcal{A}\circ\mathcal{B} \to \mathcal{C}$ (or a natural transformation) is consistent with the augmentations if and only if the corresponding $\mathcal{A} \to \overline{\{\mathcal{B},\mathcal{C}\}}$ lands in the subcategory $\{\mathcal{B},\mathcal{C}\}$ given by (9.30) and (9.31). A detailed treatment in the case $\mathcal{T}_0 = \underline{P}$ is given in [5], to which we refer any reader wanting more details.

Note of course that, while the $\circ$ of $\mathcal{CAT}/\mathcal{T}_0$ does not depend on $\mathcal{T}_0$, being just the restriction of that of $\mathcal{CAT}\int *_{cat}$, the internal-hom $\{\mathcal{B},\mathcal{C}\}$ depends heavily on $\mathcal{T}_0$. Of course if i is the inclusion $\mathcal{T}_0 \subset \mathcal{T}_0'$, then $\{\mathcal{B},\mathcal{C}\}$ is just $i^*\{\mathcal{B},\mathcal{C}\}'$ in the sense of the second

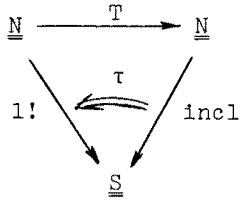
paragraph of §9.1.

9.5 We end this section with some examples, or rather counter--examples, before passing in the next section to the still-more-special properties of $CAT/\underline{\underline{S}}$.

The terminal object in any monoidal category is a monoid; hence T_0, with its identity augmentation, is a club in CAT/T_0, with multiplication the Θ of (8.5), and with identity the inclusion $J \to T$. Thus we have as examples of clubs cat, set, ord, $\underline{\underline{S}}, \underline{\underline{P}}, \underline{\underline{N}}$. Other examples of clubs in $CAT/\underline{\underline{P}}$ were given in [6]. If K is any club in CAT/T_0, then $\Gamma\colon K \to T_0$ is a map of monoids in CAT/T_0, inducing a doctrine-map $\Gamma\circ -\colon K\circ - \Rightarrow T_0\circ -$.

For an example of a club <u>not</u> in CAT/cat_0 we can take any $n \in cat$ and set $K = (I, \ulcorner n \urcorner)$. Then $K\circ A \cong A^n$, and $K\circ -$ is the 2-functor $A \mapsto A^n$. We know that this becomes a doctrine if we define its multiplication $(A^n)^n \to A^n$ and its unit $A \to A^n$ via the diagonal $n \to n \times n$ and the unique map $n \to 1$. The corresponding multiplication $\mu\colon K\circ K \to K$ and identity $\eta\colon J \to K$ for K certainly do not lie in CAT/cat_0.

Now we give an example of two clubs (K,Γ), (L,Δ) in $CAT/\underline{\underline{S}}$, indeed in $CAT/\underline{\underline{N}}$, and a doctrine-map $K\circ - \Rightarrow L\circ -$, where the $(T,\tau)\colon (K,\Gamma) \to (L,\Delta)$ inducing this does not lie in $CAT/\underline{\underline{S}}$, but only in $CAT\int^*\underline{\underline{S}}$ (and not in $CAT\int^*\underline{\underline{N}}$ or $CAT\int^*\underline{\underline{P}}$ either). We take $K = L = \underline{\underline{N}}$, with $\Gamma\colon \underline{\underline{N}} \to \underline{\underline{S}}$ as the constant functor at 1, and with $\Delta\colon \underline{\underline{N}} \to \underline{\underline{S}}$ as the inclusion. The 2-functor $K\circ -$ is $\underline{\underline{N}}\times -$; it becomes a doctrine via the sum $+\colon \underline{\underline{N}} \times \underline{\underline{N}} \to \underline{\underline{N}}$ and $\ulcorner 0 \urcorner\colon I \to \underline{\underline{N}}$; a K-structure on A consists in giving a single endofunctor $P\colon A \to A$. On the other hand L is the club (see [6]) whose algebras are strict monoidal categories. From an L-structure on A we get a K-structure on A by setting $PA = A \otimes A$. This algebraic 2-functor must arise from a doctrine-map $K\circ - \Rightarrow L\circ -$, and hence from a <u>club-map</u> $(T,\tau)\colon K \to L$. The latter is easily seen to be



where $Tn = 2^n$ and τ_n is the unique map $2^n \to 1$; it lies in $CAT\int *\underline{\underline{S}}$ but neither in $CAT/\underline{\underline{S}}$ nor in $CAT\int *\underline{\underline{P}}$.

In general, by a club-map we mean a map of $\circ$-monoids in $CAT\int *cat$. The above shows that, even if the clubs in $CAT/\underline{\underline{S}}$, to which we now turn, are of especial importance, we cannot handle all club maps without going to $CAT\int *\underline{\underline{S}}$. However in some cases a club map too can be seen from the basic data to lie in $CAT/\underline{\underline{S}}$.

10. CLUBS IN $CAT/\underline{\underline{S}}$

10.1 We now return to this most important case, referred to in §1.4 above as arising in coherence problems, along with its subcases $CAT/\underline{\underline{P}}$ and $CAT/\underline{\underline{N}}$ (and, for that matter, $CAT/I = CAT$ itself; here $\circ$ reduces to cartesian product $\times$). As a matter of notation, we write $A[X] \in A \circ B$ in the form $A[X_1, \ldots, X_n]$ when $\Gamma A = n$; and we write the $a[x]$ of (5.2), when $\Gamma a = \phi: n \to n'$, as

$$a[x_1, \ldots, x_n]: A[X_1, \ldots, X_n] \to A'[X'_1, \ldots, X'_{n'}]; \tag{10.1}$$

here

$$x_i: X_i \to X'_{\phi i}\ . \tag{10.2}$$

We replace the square brackets by round ones to denote the image of $A[X]$ under an action $K \circ A \to A$, and in particular under a multiplication $K \circ K \to K$; so the image of $A[X]$ becomes $A(X)$, or in more detail that of $A[X_1, \ldots, X_n]$ becomes $A(X_1, \ldots, X_n)$, and similarly for morphisms. This agrees with our "parenthesis notation" for Θ; since $n(m)$ or

$n(m_1,\dots, m_n) = m_1+\dots+ m_n$ is indeed the image of $n[m]$ under the multiplication Θ: $\underline{\underline{S}}\circ\underline{\underline{S}} \to \underline{\underline{S}}$. When we have a club K, we also write $\underline{\underline{1}}$ for the image of the unit η: $J \to K$; the $\underline{\underline{1}}$ of $\underline{\underline{S}}$ is just 1. Similarly we have the notation $\phi(\psi_1\dots \psi_n)$: $n(m_1\dots m_n) \to n'(m'_1\dots m'_{n'})$ for the image under Θ of morphisms, where ϕ: $n \to n'$ and ψ_i: $m_i \to m'_{\phi i}$.

In accordance with the conventions of §9.3 we abbreviate

(10.3) $$1_A[x_1\dots x_n]: A[X_1\dots X_n] \to A[X'_1\dots X'_n]$$

to

(10.4) $$A[x_1\dots x_n]: A[X_1\dots X_n] \to A[X'_1\dots X'_n]$$

where x_i: $X_i \to X'_i$; and we abbreviate

(10.5) $$a[1_{X'_{\phi 1}}\dots 1_{X'_{\phi n}}]: A[X'_{\phi 1}\dots X'_{\phi n}] \to A'[X'_1\dots X'_{n'}]$$

to

(10.6) $$a[X'_1\dots X'_{n'}]: A[X'_{\phi 1}\dots X'_{\phi n}] \to A'[X'_1\dots X'_{n'}].$$

We make also the corresponding abbreviations with round brackets, as we did in §9.4. As in (9.22), the typical morphism (10.1) then appears as the composite

(10.7) $$A[X_1\dots X_n] \xrightarrow[A[x_1\dots x_n]]{} A[X'_{\phi 1}\dots X'_{\phi n}] \xrightarrow[a[X'_1\dots X'_{n'}]]{} A'[X'_1\dots X'_{n'}].$$

The reader will have no trouble expressing (9.24) and (9.26) in this expanded notation.

Since the special case $CAT/\underline{\underline{P}}$ has been treated in detail in [5] and [6], and since there is not much to change, I shall give only an outline in this section. There are some mild notational differences from [5] and [6]; there ϕ was always an isomorphism and (10.2) was replaced by x_i: $X_{\phi^{-1}i} \to X'_i$, which is impossible here. That, however, had the effect of making (10.6) conformable with the convention that X' stands for $1_{X'}$; here we have to sacrifice something and we choose it to be this.

10.2 What makes the present case specially simple is the discreteness of the images ΓA under the augmentation; in fact much of what we say below would apply equally well to $CAT/\mathcal{set}$, but we stick to the most practical case of finitary operations.

Let us agree that for this section "club" means "club in $CAT/\underline{\underline{S}}$". From the above discreteness it follows that if K is a club, so is its set of objects $|K|$, with of course the same "multiplication" $T[S_1,\ldots, S_n] \mapsto T(S_1,\ldots, S_n)$ for objects and the same identity $\underline{\underline{1}}$; and it follows also that the inclusion $|K| \to K$ is a club map in $CAT/\underline{\underline{S}}$.

A club which is discrete as a category is called a discrete club; it may be regarded as a $\circ$-monoid in $SET/\underline{\underline{N}}$. (Of course there are clubs in $CAT/\underline{\underline{N}}$ that are not discrete). The further consequence of the above is that, in constructing a club K from generators and relations, we can construct $|K|$ first and settle the objects before starting on the morphisms. It is this that is false for $CAT/\mathcal{cat}_0$, and it is this that forms the basis of all that we do in this section.

10.3 Let any object (P,Γ) of $SET/\underline{\underline{N}}$ be given. It is easy to construct the free discrete club (Q,Γ) it generates - which is also automatically the free club it generates. We construct the set Q inductively by

(10.8) $\underline{\underline{1}} \in Q$;

(10.9) if $P \in P$ with $\Gamma P = n$ and if $Q_1,\ldots, Q_n \in Q$ then $P(Q_1 \ldots Q_n) \in Q$.

We embed P in Q by identifying P with $P(\underline{\underline{1}} \ldots \underline{\underline{1}})$, and extend Γ to Q by $\Gamma\underline{\underline{1}} = 1$, $\Gamma P(Q_1 \ldots Q_n) = n(\Gamma Q_1,\ldots, \Gamma Q_n)$. To save repetition, let us agree that until further notice "club map" means one in $CAT/\underline{\underline{S}}$ (i.e. over $\underline{\underline{S}}$), and not a general one in $CAT\int^*\underline{\underline{S}}$. Let us also agree that all augmentations are denoted by Γ if no other name is given. It is clear that for any club L and any morphism $P \to L$ over $\underline{\underline{S}}$, there is a unique extension to a club map $Q \to L$.

Now let ρ be a relation on $\mathcal{Q}$ such that $T\rho S$ implies $\Gamma T = \Gamma S$, and consider club-maps $M : \mathcal{Q} \to L$ such that $T\rho S$ implies $MT = MS$. It is clear that these factorize uniquely into club maps $\mathcal{Q} \to S \to L$, where S is the quotient discrete club of $\mathcal{Q}$ obtained by first enlarging ρ to a congruence $\bar{\rho}$ with respect to the operation $T(S_1,\ldots,S_n)$ of substitution, and then setting $S = \mathcal{Q}/\bar{\rho}$. We can call S the discrete club with generators P and relations ρ.

A model of P is a category A together with, for each $P \in P$, a functor $|P| : A^n \to A$, where $n = \Gamma P$. This is clearly the same thing as a morphism $P \to \{A, A\}$ over $\underline{\underline{S}}$; the endo-internal-hom $\{A, A\}$ is of course always canonically a club; and so a model of P is the same thing as a club-map $\mathcal{Q} \to \{A, A\}$, or again as an action $\mathcal{Q} \circ A \to A$; so it is just a $\mathcal{Q}$-algebra or $\mathcal{Q}$-category.

A model of P,ρ is such a model A of P satisfying $|T| = |S| : A^n \to A$ whenever $T\rho S$. It is therefore an S-category for the discrete club S.

The most obvious example is that of a strict monoidal category A; here P has objects $\otimes$ and I with $\Gamma\otimes = 2$ and $\Gamma I = 0$, and $\mathcal{Q}$ is subjected to relations ρ identifying $\otimes(\otimes,\underline{\underline{1}})$ with $\otimes(\underline{\underline{1}}, \otimes)$, $\otimes(I,\underline{\underline{1}})$ with $\underline{\underline{1}}$, and $\otimes(\underline{\underline{1}},I)$ with $\underline{\underline{1}}$; the corresponding discrete club S is $\underline{\underline{\mathbb{N}}}$ itself, with identity augmentation.

10.4 Given a discrete club S, whether by generators and relations or directly, let us be given a graph $\mathcal{D}$ (in the usual sense of graph - objects and arrows - not that of [5] and [6]) with object-set $|\mathcal{D}| = S$, and let us be given a morphism of graphs $\Gamma : \mathcal{D} \to \underline{\underline{S}}$ extending $\Gamma : S \to \underline{\underline{S}}$. We might as well now replace S by $|\mathcal{D}|$. Consider for a club L those graph-morphisms $\mathcal{D} \to L$ over $\underline{\underline{S}}$ such that the composite $|\mathcal{D}| \to \mathcal{D} \to L$ is a club-map. I assert that there is a universal such map $\mathcal{D} \to M$ into a club M, so that any such $\mathcal{D} \to L$ factors through a unique club-map $M \to L$; we can call M the club generated by S and $\mathcal{D}$. In particular

a model of $(S,\mathcal{D})$ means an S-category A together with, for each $f : T \to S$ in $\mathcal{D}$, a natural transformation $|f| : |T| \to |S|$ of type Γf, in the sense of (9.20). So such a model A is just an M-category for the club M.

To construct M we begin by enlarging $\mathcal{D}$ twice, without changing its objects; first we define an instance of an $f : T \to S$ in $\mathcal{D}$ to be a formal expression

$$(10.10) \qquad f(R_1 \ldots R_{n'}) : T(R_{\phi 1} \ldots R_{\phi n}) \to S(R_1 \ldots R_{n'})$$

where $\phi = \Gamma f$, and where the objects $T(R_{\phi 1} \ldots R_{\phi n})$ etc. are of course not formal but are the objects of $|\mathcal{D}|$ so denoted. We identify f with $f(\underline{1} \ldots \underline{1})$, and so the instances form a bigger graph $\mathcal{D}'$ extending $\mathcal{D}$, to which we also extend Γ.

We next, for an instance $g : T \to S$ in $\mathcal{D}'$, define an expansion of g, i.e. an expanded instance of some f, to be an expression

$$(10.11) \qquad P(1 \ldots g \ldots 1) : P(\underline{1} \ldots T \ldots \underline{1}) \to P(\underline{1} \ldots S \ldots \underline{1})$$

for some $P \in |\mathcal{D}|$. We identify g itself with $\underline{1}(g)$, and now have a still bigger graph $\mathcal{D}''$, to which we again extend Γ.

We then pass to $\mathcal{D}'''$, the free category generated by the graph $\mathcal{D}''$, to which we again extend Γ. The category $\mathcal{D}'''$ is not yet a club, but a partial structure of $\circ$-monoid can be defined on it; and it is clear that any morphism $\mathcal{D} \to L$ over $\underline{S}$ into a club L, whose restriction $|\mathcal{D}| \to L$ is a club-map, extends uniquely to a functor $\mathcal{D}''' \to L$ which respects the partial club-structure on $\mathcal{D}'''$. There are two sorts of diagrams in $\mathcal{D}'''$ which are necessarily sent into commutative diagrams by $\mathcal{D}''' \longrightarrow L$; they are typified by

(10.12)
$$\begin{array}{ccc} T(S_1 \ldots S_i \ldots S_j \ldots S_n) & \xrightarrow{T(1\ldots h\ldots 1\ldots 1)} & T(S_1 \ldots S_i' \ldots S_j \ldots S_n) \\ \downarrow{\scriptstyle T(1\ldots 1\ldots g\ldots 1)} & & \downarrow{\scriptstyle T(1\ldots 1\ldots g\ldots 1)} \\ T(S_1 \ldots S_i \ldots S_j' \ldots S_n) & \xrightarrow[T(1\ldots h\ldots 1\ldots 1)]{} & T(S_1 \ldots S_i' \ldots S_j' \ldots S_n) \end{array}$$

and by

(10.13)
$$\begin{array}{ccc} T(S_{\phi 1} \ldots \ldots \ldots S_{\phi n}) & \xrightarrow{f(S_1 \ldots S_i \ldots S_{n'})} & T'(S_1 \ldots S_i \ldots S_{n'}) \\ \downarrow{\scriptstyle T(1\ldots k\ldots k\ldots 1)} & & \downarrow{\scriptstyle T'(1\ldots k\ldots 1)} \\ T(S_{\phi 1} \ldots \ldots \ldots S_{\phi n}) & \xrightarrow[f(S_1 \ldots S_i' \ldots S_{n'})]{} & T'(S_1 \ldots S_i' \ldots S_{n'}). \end{array}$$

Here (10.13) needs some explanation. The left vertical arrow stands for either leg of a diagram like (10.12); or the corresponding thing when there are more than two maps k in it; and these k's occur at all those indices j for which $\phi j = i$. The diagram (10.12) says in effect that T is "functorial", and (10.13) that f is "natural". It suffices in fact to impose these diagrams not for all $f,g,h,k \in \mathcal{D}'''$ but for $f \in \mathcal{D}''$, $g,h,k \in \mathcal{D}'$; the more general cases then follow automatically. All this is rather more complicated than in the $CAT/\underline{P}$ case of [6].

If we write *Funct* for the class of diagrams of the form (10.12) and *Nat* for those of the form (10.13), and $\mathcal{D}'''/(Funct + Nat)$ for the quotient category with the same objects got by imposing these relations, it turns out that this last is the desired club M; for it indeed admits a club-structure in which, in analogy with (10.7), $f(g_1 \ldots g_n)$ is defined to be the composite

$$(10.14)\quad T(S_1\ldots S_n) \xrightarrow{T(g_1\ldots g_n)} T(S'_{\phi 1}\ldots S'_{\phi n}) \xrightarrow{f(S'_1\ldots S'_{n'})} T'(S'_1\ldots S'_{n'}).$$

10.5 Now suppose that, besides the above, we are given a set σ of diagrams $f,g : T \to S$ in M with $\Gamma f = \Gamma g$; or, what is perhaps more usual, a set σ of such diagrams in $\mathcal{D}'''$; and we consider only those graph-morphisms $\mathcal{D} \to L$ over $\underline{S}$ into a club L, still requiring $|\mathcal{D}| \to \mathcal{D} \to L$ to be a club-map, for which, in the induced $\mathcal{D}''' \to L$ or $M \to L$, the images of f and g coincide for each $f,g: T \to S$ in σ. There is again a universal such L, say K, a quotient-category of M, and itself a club, such that any such map factorizes as $\mathcal{D} \to M \to K \to L$ for a unique club-map $K \to L$. In fact K is obtained from M by imposing as extra relations all the <u>expanded instances</u> of the relations σ; or better, is obtained as

$$(10.15)\quad K = \mathcal{D}'''/(Funct + Nat + Imp),$$

where *Imp*, standing for the <u>imposed</u> relations, consists of all the <u>expanded instances</u> of the relations σ.

We may in fine call K <u>the club generated by the function-generators</u> P, <u>with relations</u> ρ, <u>and the natural-transformation-generators</u> $\mathcal{D}$, <u>with the relations</u> σ.

A model A of $(P,\rho,\mathcal{D},\sigma)$ is of course a model of $(P,\rho,\mathcal{D})$ satisfying the relations σ; it is the same thing as a K-category A for the club K.

A typical example is that of symmetric monoidal categories, where P consists of $\otimes$ and I with $\Gamma\otimes = 2$ and $\Gamma I = 0$; where ρ is vacuous; where $\mathcal{D}$ consists of

$a : \otimes(\otimes,\underline{1}) \to \otimes(\underline{1},\otimes)$, $\quad \bar{a} : \otimes(\underline{1},\otimes) \to \otimes(\otimes,\underline{1})$,

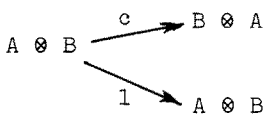
$r : \otimes(\underline{1},I) \to \underline{1}$, $\quad \bar{r} : \underline{1} \to \otimes(\underline{1},I)$, and

$c : \otimes \to \otimes$, with Γa, $\Gamma\bar{a}$, Γr, $\Gamma\bar{r}$ identities and with Γc the non-identity permutation of 2; and where σ consists of the usual coherence axioms, together with $a\bar{a} = 1$, $\bar{a}a = 1$, $r\bar{r} = 1$, $\bar{r}r = 1$, establishing $\bar{a},\bar{r}$ as inverses of a and r. Here the club K is in fact in $CAT/\underline{P}$;

other examples leading to clubs in $CAT/\underline{S}$ are suggested in §1.4 above.

10.6 It should now be clear that a complete solution to the "coherence problem" for such a structure $(P,\rho,\mathcal{D},\sigma)$ consists in the explicit determination of the club K. To find the objects of K is already a word-problem, but usually easy, and trivial if ρ is vacuous. To find the morphisms is a word-problem that is usually much harder. We know the generators, and it is a question of knowing when $f,g : T \to S$ in $\mathcal{D}'''$ coincide in K because of the relations in (10.15); that is, of knowing "which diagrams commute". A necessary condition is of course $\Gamma f = \Gamma g$; this is the condition for the writability of a diagram, in terms of generic components of the "natural transformations" f and g. For example, in the symmetric-monoidal-category case, we have $c,1 : \otimes \to \otimes$ with $\Gamma c \neq \Gamma 1$; and the generic components

(10.16)



do not form a closed diagram. The "all diagrams commute" case is that where $\Gamma f = \Gamma g$ is also sufficient for $f = g$, i.e. where Γ is faithful. It is of course a rare case.

In (10.16) the special components $c,1 : A \otimes A \to A \otimes A$ form a closed diagram, but this does not commute in a typical model, and makes no sense in the club K. If one wants to know which such specialized diagrams commute, one has only to look at the free model $K\bullet A$ on A, which of course we also know completely when we know K.

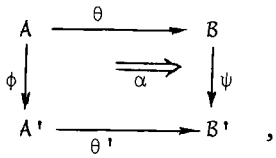
10.7 The considerations of §§10.3 - 10.5 assert that a "map" $(P,\rho,\mathcal{D},\sigma) \to L$ of the kind described there leads to a club map $K \to L$ in our current sense of the word, that is to say, in $CAT/\underline{S}$. This situation may be recognized even when L itself is given only by generators and relations $(P',\rho',\mathcal{D}',\sigma')$. Thus we are able without calculation to conclude for example the existence of a club map $K \to L$

over $\underline{\underline{S}}$ when K is the club for monoidal categories and L that for symmetric monoidal categories; or when K is that for symmetric monoidal categories and L is that for two symmetric monoidal structures together with a distributive law, as in (1.1) above. This is often very handy, in view of the observation in §9.5 that not <u>all</u> club-maps $K \to L$ for clubs K,L in $CAT/\underline{\underline{S}}$ themselves lie in $CAT/\underline{\underline{S}}$. We may also observe in [8] below that we have to leave $CAT/\underline{\underline{S}}$ for $CAT\int^{*}\underline{\underline{S}}$ when considering a <u>distributive law</u> $K\circ L \to L\circ K$ between clubs that themselves lie in $CAT/\underline{\underline{S}}$.

<u>10.8</u> A further application of the considerations of §§10.3 - 10.5 is in connection with K-morphisms $(\phi,\bar{\phi}) : A \to A'$ between K-categories A and A'; that is, for (lax) D-morphisms, in the sense of [12]§3.5 in this volume, where D is the doctrine $K\circ -$ for a club K in $CAT/\underline{\underline{S}}$.

Denote by FUN the 2-category which is like $[\![\underline{\underline{2}}, CAT]\!]$ except that the morphisms are not lax n.t.'s but op-lax n.t.'s; if we took $[\![\underline{\underline{2}}, CAT]\!]$ itself we should get op-K-morphisms instead of K-morphisms. So an object of FUN is a functor $\phi : A \to A'$, and a morphism is θ,θ',α of the form

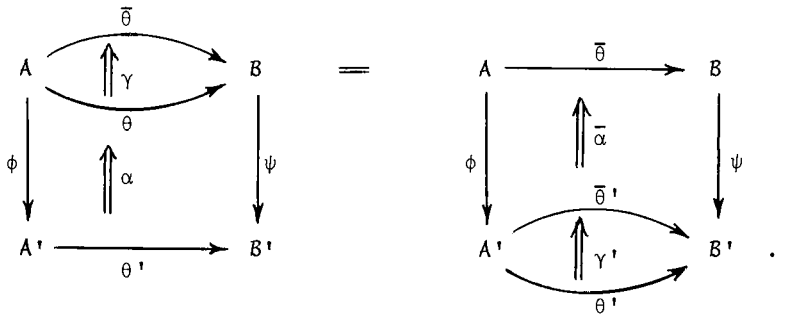
(10.17)



,

while a 2-cell is γ,γ' where $\gamma : \theta \Rightarrow \bar{\theta}$ and $\gamma' : \theta' \Rightarrow \bar{\theta}'$ satisfy

(10.18)



.

There is an evident 2-functor

(10.19) $\hat{\circ} : CAT/\underline{S} \times FUN \to FUN$

sending K, ϕ to $K\circ\phi$, sending $T : K \to K'$ and $(\theta, \theta', \alpha) : \phi \to \phi'$ to $(T\circ\theta, T\circ\theta', T\circ\alpha)$, and sending $\eta : T \Rightarrow \bar{T}$ and $(\gamma,\gamma') : (\theta, \theta', \alpha) \Rightarrow (\bar{\theta},\bar{\theta}',\bar{\alpha})$ to $(\eta\circ\gamma, \eta\circ\gamma')$. So there is really no need to distinguish notationally $\hat{\circ}$ from $\circ$. It is moreover clear that (10.19) provides an action of $CAT/\underline{S}$ on FUN, in the sense that $(K\circ L)\hat{\circ}\,\phi \cong K\hat{\circ}(L\hat{\circ}\phi)$, etc. It is finally clear that we have a 2-adjunction

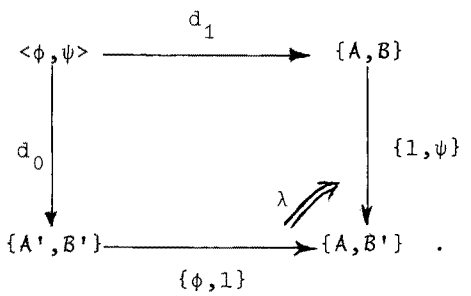
(10.20) $FUN(K\hat{\circ}\phi,\psi) \cong CAT/\underline{S}\,(K,<\phi,\psi>)$,

where $<\phi,\psi>$ denotes the comma object

(10.21) $<\phi,\psi> = \{\phi,1\}/\{1,\psi\}$

in $CAT/\underline{S}$, as in

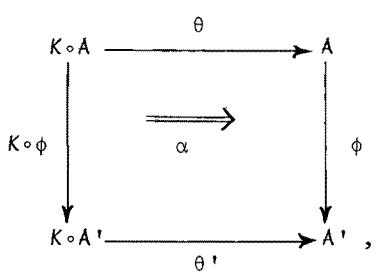
(10.22)



Thus FUN is exhibited as a tensored $CAT/\underline{S}$ - category.

An action (θ,θ',α): $K\hat{\circ}\phi \to \phi$ of the club K in $CAT/\underline{S}$ on $\phi \in FUN$ is clearly the same thing, by §3.5 of [12], as actions $\theta : K\circ A \to A$ and $\theta' : K\circ A' \to A'$, making A and A' into K-categories, together with an α,

(10.23)



making of (ϕ,α) a K-functor $A \to A'$. Such an action then is a club-map $K \to <\phi,\phi>$ over $\underline{S}$ where $<\phi,\phi>$ has its evident club-structure.

To give α when θ,θ' are already given is to give such a $K \to \langle\phi,\phi\rangle$ whose composites with d_0 and d_1 are the given club-maps $K \to \{A',A'\}$ and $K \to \{A,A\}$.

Such an α has components

(10.24)
$$\alpha_{T[A_1 \ldots A_n]} : T(\phi A_1 \ldots \phi A_n) \to \phi\, T(A_1 \ldots A_n)$$

which are to be natural in T and in the A_i, and the axioms (3.17), (3.18) of [12] become

(10.25)
$$\alpha_{T(S_1 \ldots S_n)[A_1 \ldots A_m]} =$$

$$\alpha_{T[S_1(A_1 \ldots A_{m_1}) \ldots S_n(\ldots A_m)]} \cdot T(\alpha_{S_1[A_1 \ldots A_{m_1}]} \cdots \alpha_{S_n[\ldots A_m]});$$

(10.26)
$$\alpha_{\underline{1}[A]} = 1.$$

If now K is given by generators and relations $(P,\rho,\mathcal{D},\alpha)$, the description of the structure as a club-map $K \to \langle\phi,\phi\rangle$ shows how to give α in terms of these generators and relations. First, one has only to give $\alpha_T = \alpha_{T[A_1 \ldots A_n]}$ for $T \in P$, and it is of course to be natural in the A_i. One then gets the general α_T by (10.25) and (10.26). To a relation $T\rho S$ one must impose the axiom $\alpha_T = \alpha_S$. It remains to impose as axioms the naturality of α_T in T only for morphisms $f : T \to S$ in $\mathcal{D}$, and one is finished.

In this way one sees for instance that, when K is the club either for monoidal categories or for strict monoidal categories, a K-functor is a monoidal functor in the usual sense (<u>not</u> a strict one in the strict monoidal category case; the axioms from the identifications $\otimes(\otimes,\underline{1}) = \otimes(\underline{1},\otimes)$ in this case are the same as the axioms expressing naturality in a: $\otimes(\otimes,\underline{1}) \to \otimes(\underline{1},\otimes)$ in the non-strict case, etc.) Similarly for <u>symmetric</u> monoidal functors, monad functors,

and other well-known cases.

In the same way we see that for a K-natural-transformation $\eta\colon (\phi,\alpha) \Rightarrow (\psi,\beta)\colon A \to A'$, we need impose on the natural transformation $\eta\colon \phi \Rightarrow \psi$ the axiom

(10.27)

$$\begin{array}{ccc} T(\phi A_1 \ldots \phi A_n) & \xrightarrow{\alpha_{T[A_1 \ldots A_n]}} & \phi T(A_1 \ldots A_n) \\ \Big\downarrow{\scriptstyle T(\eta_{A_1} \ldots \eta_{A_n})} & & \Big\downarrow{\scriptstyle \eta_{T(A_1 \ldots A_n)}} \\ T(\psi A_1 \ldots \psi A_n) & \xrightarrow[\beta_{T[A_1 \ldots A_n]}]{} & \psi T(A_1 \ldots A_n) \end{array} ,$$

representing the axiom (3.19) of [12], only for those objects T in P; as we commonly do for monoidal natural transformations.

10.9 We conclude by noticing some other properties that seem to be special to the case of clubs in $CAT/\underline{S}$, or perhaps $CAT/\mathcal{S}et_0$.

Let K then be a club in $CAT/\underline{S}$. By a closed K-category we mean a K-category A such that, for each $T \in K$, with $\Gamma T = n$ say, and for each i, the functor $T(A_1 \ldots A_{i-1} - A_{i+1} \ldots A_n)\colon A \to A$ has a right adjoint for each choice of the $A_j \in A$. Thus symmetric monoidal closed category, monoidal biclosed category, cartesian closed category, etc. The reader may like to check what such a thing is when a K-category is a category A bearing a monad, or a comonad.

I showed in [7] that, when K actually lies in $CAT/\underline{P}$, the closed K-categories were the algebras for a club L of the mixed-variance kind of §1.2 above; in particular they are monadic over CAT. I left open there the question whether the canonical $K \to L$ in this case was faithful.

This question can be answered by using a Yoneda embedding, as announced in [11]. In fact, it is to clubs K in $CAT/\underline{S}$ that the work of Day in [1] seems to extend.

For if A is a K-category for a K in $CAT/\underline{S}$, we can extend the K-structure on A to one on $B = [A^{op}, Set]$, and indeed uniquely (to within isomorphism) if we ask B to be a closed K-category; for then the realization $|T|: B^n \to B$ of $T \in K$ must be the left Kan extension of its realization $|T|: A^n \to A$. This solves the above question when K lies in $CAT/\underline{P}$; for if we take B to be $[K^{op}, Set]$ there is a unique strict L-map $L \to B$ taking $\underline{1} \in L$ to the image under Yoneda of the $\underline{1}$ of K; the strict K-map $K \to L \to B$ then coincides with the Yoneda embedding, since they have the same effect on $\underline{1}$; whence $K \to L$ is faithful since the Yoneda embedding is so. (We are using of course that $K \cong K \circ I$ is the free K-category on I, i.e. on $\underline{1} \in K$).

Note that the above Yoneda extension does not work for an arbitrary doctrine D, or even for a club K in say CAT/cat_0. For I is always a D-category, and yet $[I^{op}, Set] \cong Set$ need not be. But Set, or at any rate set, is certainly a K-category for a club K in CAT/set, since there is a club-map $\Gamma: K \to set$.

Of course the above is a little loose, since the lack of strict associativity for colimits in Set causes $B = [A^{op}, Set]$ to be only a pseudo-K-algebra in general; either we take account of this by comparing pseudo-algebras with algebras, as we shall to some extent do in [8] below; or we replace Set by an equivalent category admitting strictly associative colimits - I have not thought out the question of whether set has these.

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