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FLEXIBLE LIMITS FOR 2-CATEGORIES

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Many important 2-categories - such as **Lex**, **Fib**/$\mathscr{B}$, elementary toposes and logical morphisms, the dual of Grothendieck toposes and geometric morphisms, locally-presentable categories and left adjoints, the dual of this last, and the Makkai-Paré 2-category of accessible categories and accessible functors - fail to be complete, lacking even equalizers. These examples do in fact admit all *bilimits* - those weakenings of the limit notion that represent not by an isomorphism but only by an equivalence - but much more is true: they admit important classes of honest limits, including products, cotensor products, comma objects, Eilenberg-Moore objects, descent objects, inserters, equifiers, inverters, lax limits, pseudo limits, and idempotent-splitting. We introduce the class of *flexible* limits, which includes all of the above and is, in the technical sense, a *closed* class. Note that such honest limits, when they exist, have many advantages over bilimits: they are unique to within isomorphism, and their universal properties are both stronger and more convenient to use, a whole level of coherent families of invertible 2-cells being avoided.

1. Revision of elementary limit-notions for 2-categories

It has long been recognised that the classical 'conical' limits do not suffice for the purposes of enriched category theory: we need the *indexed* limits described in [11, Chapter 3] which, besides the conical ones, include cotensor products, ends, and so on; and we call an enriched category *complete* only when it admits all small indexed limits. (It has been suggested to us that the epithet *indexed* is an unfortunate choice, being used in a different sense in the context of 'indexed categories', and that *weighted* would be better; we accept the point and use the new term from now on.)

These remarks apply in particular to 2-categories, which (if they are locally small) are just categories enriched over the Cartesian-closed category **Cat** of small categories. (If they are not locally small, they are enriched over the **CAT** of some higher universe. In fact questions of size play no overt role below, except that we elect for simplicity to consider only *small* limits.)

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The elementary facts about weighted limits for 2-categories are set forth in [13], which the interested reader will doubtless wish to consult. The Editor, however, requests us to recall briefly here enough of these to make this article readable as it stands, referring to [13] and other sources for proofs but not for basic definitions or statements of central results.

When only 2-categories are in question, we often use 'functor' for '2-functor' and 'natural transformation' for '2-natural transformation'; there is no danger of confusion so long as we clearly distinguish a 2-category $\mathscr{A}$ from its underlying category $\mathscr{A}_0$ obtained by discarding the 2-cells. We sometimes say '2-functor' or '2-natural' for emphasis or clarity; if we speak of a 2-functor $\mathscr{A} \to \mathscr{B}$, we imply that $\mathscr{A}$ and $\mathscr{B}$ are 2-categories. We regard a category as a locally-discrete 2-category, and a set as a discrete category. For 2-categories $\mathscr{A}$ and $\mathscr{B}$, we write $[\mathscr{A}, \mathscr{B}]$ for the 2-category of functors, natural transformations, and modifications.

A functor $F: \mathscr{A} \to \mathbf{Cat}$ whose domain is a *small* 2-category $\mathscr{A}$ may be called a *weight*. The *F-weighted limit* $\{F, G\}$ of $G: A \to \mathscr{B}$, if it exists, is the object of $\mathscr{B}$ providing a representation – that is, an isomorphism of categories natural in $B \in \mathscr{B}$ –

$$\mathscr{B}(B, \{F, G\}) \cong [\mathscr{A}, \mathbf{Cat}](F, \mathscr{B}(B, G-)), \tag{1.1}$$

with unit say

$$\xi: F \to \mathscr{B}(\{F, G\}, G-). \tag{1.2}$$

Note that, (1.1) being an isomorphism of categories and not just of sets, the universal property of $(\{F, G\}, \xi)$ has a two-dimensional as well as a one-dimensional aspect: not only are natural transformations $\alpha: F \to \mathscr{B}(B, G-)$ to correspond via ξ to morphisms $f: B \to \{F, G\}$, but modifications $\theta: \alpha \to \alpha'$ are to correspond to 2-cells $\phi: f \to f'$. See [13] for a deeper discussion of this point, showing that the one-dimensional property does not imply the two-dimensional one in general, but does so if $\mathscr{B}$ admits certain tensor products.

An F-weighted *colimit* in $\mathscr{B}$ is just an F-weighted limit in $\mathscr{B}^{\mathrm{op}}$; it is, however, convenient to adjust the notation somewhat. Think of classical limits in ordinary categories: a pullback in $\mathscr{B}$ is the limit of a certain $H: \mathscr{A} \to \mathscr{B}$, so that a pushout is the limit of a certain $H: \mathscr{A} \to \mathscr{B}^{\mathrm{op}}$; but we call it rather the *colimit* of $H^{\mathrm{op}}: \mathscr{A}^{\mathrm{op}} \to \mathscr{B}$. Accordingly, for weighted colimits, it is usual to write the weight as $F: \mathscr{A}^{\mathrm{op}} \to \mathbf{Cat}$, and to define the *F-weighted colimit* $F * G$ of $G: \mathscr{A} \to \mathscr{B}$ by

$$\mathscr{B}(F * G, B) \cong [\mathscr{A}^{\mathrm{op}}, \mathbf{Cat}](F, \mathscr{B}(G-, B)). \tag{1.3}$$

Other notations for $\{F, G\}$ and $F * G$ are $\lim(F, G)$ and $\mathrm{colim}(F, G)$; the present notation has the virtue of emphasising the strong analogy with 'hom' and 'tensor', seen to advantage in the formulas (3.18)–(3.23) of [11]. Indeed, when $\mathscr{B} = \mathbf{Cat}$, it is immediate that $\{F, G\}$ is precisely the hom-category $[\mathscr{A}, \mathbf{Cat}](F, G)$; so that the definition (1.1) is a 'representable' one, defining limits in $\mathscr{B}$ via those in **Cat**. In particular **Cat** is complete. That it is also cocomplete is shown in [11, Section 3].

We now recall some special cases of limits. When the weight F is the representable

$\mathscr{A}(A, -): \mathscr{A} \to \mathbf{Cat}$, the Yoneda lemma applied to the right side of (1.1) yields the *Yoneda isomorphism*

$$\{\mathscr{A}(A, -), G\} \cong GA. \tag{1.4}$$

When the weight F is the functor $\Delta 1 : \mathscr{A} \to \mathbf{Cat}$ constant at the terminal category $\mathbf{1}$, the right side of (1.1) is isomorphic to the category $[\mathscr{A}, \mathscr{B}](\Delta B, G)$ of *cones* over G with vertex B and modifications of these. We write $\lim G$ for $\{\Delta 1, G\}$, and call such limits *conical*; now (1.1) becomes

$$\mathscr{B}(B, \lim G) \cong [\mathscr{A}, \mathscr{B}](\Delta B, G). \tag{1.5}$$

When the 2-category $\mathscr{A}$ is merely a category, the one-dimensional aspect of (1.5) asserts that $\lim G$ is the classical limit $\lim G_0$ of the ordinary functor $G_0 : \mathscr{A} = \mathscr{A}_0 \to \mathscr{B}_0$; but even here – indeed even for binary products, where $\mathscr{A}$ is the mere set 2 – the existence of $\lim G_0$ does not imply that of $\lim G$, with its two-dimensional aspect; see [11, Section 3.8]. Among such conical limits are the various familiar ones – products, equalizers, pullbacks, and so on – along with less-familiar ones where the 2-category $\mathscr{A}$ is not a mere category. We mention two more conical limits needed below.

We wish to express *the splitting of idempotents* as a conical limit. By an *idempotent* $e : B \to B$ in the 2-category $\mathscr{B}$ we always mean an idempotent in the strict sense, with $e^2 = e$ rather than $e^2 \cong e$. The idempotent is said to *split* if we can express it as $e = ir$ with $ri = 1$. To give such an idempotent in $\mathscr{B}$ is to give a functor $G : \mathbf{E} \to \mathscr{B}$ where $\mathbf{E}$ is the category with one object $*$, presented by the graph $\varepsilon : * \to *$ with the relation $\varepsilon^2 = \varepsilon$. It is easy to see that e splits if and only if $\lim G$ exists. We further wish to express *the splitting of idempotent equivalences* as a conical limit. It follows easily that an idempotent $e : B \to B$ is an equivalence in $\mathscr{B}$ if and only if there is an isomorphism $\beta : \varepsilon \cong 1_B$. If we replace β by $\alpha = \beta(\beta e)^{-1}$, we get an isomorphism $\alpha : e \cong 1_B$ with $\alpha e = \mathrm{id}$, which implies $e\alpha = \mathrm{id}$ since $(e\alpha)\alpha = (\alpha e)\alpha$. So to give an idempotent equivalence in $\mathscr{B}$ is in effect to give $H : \mathbf{F} \to \mathscr{B}$ where $\mathbf{F}$ is the 2-category with underlying category $\mathbf{E}$ and otherwise freely generated by an invertible 2-cell $\gamma : \varepsilon \to 1_*$ with $\gamma\varepsilon = \varepsilon\gamma = \mathrm{id}$. It is now easy to see that the idempotent equivalence e, seen merely as an idempotent, splits if and only if $\lim H$ exists. The reason for our interest in the splitting of idempotent equivalences, alongside that of general idempotents, is made clear in Remark 7.6 below.

When $\mathscr{A}$ is the terminal 2-category $\mathbf{1}$, the F and G of (1.1) are in effect objects of $\mathbf{Cat}$ and of $\mathscr{B}$, and now $\{F, G\}$ is called the *cotensor product* of F and G, defined by $\mathscr{B}(B, \{F, G\}) \cong [F, \mathscr{B}(B, G)]$ where $[\ ,\ ]$ is the internal-hom of $\mathbf{Cat}$. It is shown in [13] that $\mathscr{B}$ is complete if and only if it admits products, equalizers, and the cotensor products $\{\mathbf{2}, G\}$ where $\mathbf{2}$ is the arrow-category $0 \to 1$; for all limits can be constructed from these.

We write $0, 1 : 1 \to \mathbf{2}$ for the *names* of the objects 0 and 1 of $\mathbf{2}$; it is evident that 1 does not here denote an identity functor. When $\mathscr{A}$ and its images under F and G are

$$\bullet \rightrightarrows \bullet \qquad 1 \underset{1}{\overset{0}{\rightrightarrows}} \mathbf{2} \qquad B \underset{g}{\overset{f}{\rightrightarrows}} C, \tag{1.6}$$

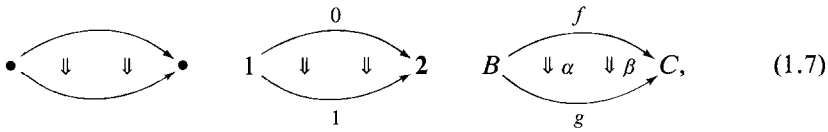
the limit $\{F, G\}$ with its unit ξ amount to an arrow $p : \{F, G\} \to B$ and a 2-cell $\lambda : fp \to gp$ with a univeral property, whose one-dimensional aspect is the assertion that, given $q : D \to B$ and $\mu : fq \to gq$, there is a unique $h : D \to \{F, G\}$ with $ph = q$ and $\lambda h = \mu$; the two-dimensional aspect is easy to express and is given explicitly in [13, Section 4]. We call $(\{F, G\}, p, \lambda)$ the *inserter* of f and g. Replacing **2** here by **I**, the category with two objects 0 and 1 and inverse isomorphisms $0 \to 1$ and $1 \to 0$, gives the *iso-inserter*; now λ, and the μ in the universal property, are invertible. (Of course, replacing **2** by 1, so that $F = \Delta 1$, gives the *equalizer*.)

We pass quickly over the *comma-object* f/g, which is $\{F, G\}$ where $\mathscr{A}$ and its images under F and G are

$$\bullet \longrightarrow \bullet \longleftarrow \bullet \qquad 1 \underset{0}{\longrightarrow} \mathbf{2} \underset{1}{\longleftarrow} 1 \qquad B \underset{f}{\longrightarrow} D \underset{g}{\longleftarrow} C;$$

it gives a universal $u : f/g \to B$, $v : f/g \to C$, and $\lambda : fu \to gv$. Replacing **2** here by **I** gives the *iso-comma-object*; now λ is invertible.

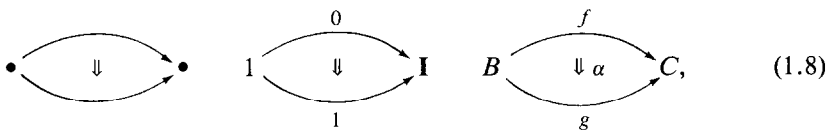
When $\mathscr{A}$ and its images under F and G are



(1.7)

$\{F, G\}$ and ξ amount to an arrow $p : \{F, G\} \to B$ universal with the property that $\alpha p = \beta p$; this is called the *equifier* of α and β.

When $\mathscr{A}$ and its images under F and G are



(1.8)

$\{F, G\}$ and ξ amount to an arrow $p : \{F, G\} \to B$ universal with the property that αp is invertible; this is called the *inverter* of α. Replacing **I** here by 1 gives the *identifier* of α: the universal $p : \{F, G\} \to B$ with $fp = gp$ and $\alpha p = \mathrm{id}$.

For the expression as limits of the Eilenberg-Moore object of a monad in $\mathscr{B}$, and the descent object of a truncated cosimplicial object in $\mathscr{B}$, see Street's articles [21] and [24] - modifying the latter to replace his weighted bilimits by weighted limits.

Many of the limits above can be constructed from a small number of them, if these latter exist. The proofs of the following can be found in [13, Propositions 4.2 and 4.4] and in the two articles of Street just mentioned:

Proposition 1.1. *If $\mathscr{B}$ admits inserters and equifiers, from these we can construct iso-inverters, inverters, Eilenberg-Moore objects, and descent objects. If it also admits*

finite products, we can construct comma objects and iso-comma-objects; and if it further admits arbitrary small products, we can construct cotensor products. □

We emphasize here " we can construct" rather than saying "$\mathscr{B}$ also admits", because it follows from the constructions that a functor $\mathscr{B} \to \mathscr{C}$ which *preserves* inserters, equifiers, and products, also preserves all the other limits in the proposition; see Section 3 below.

2. Lax limits and pseudo limits

The basic facts about these, too, are given in [13]; we recall them briefly. Given 2-functors $F, G : \mathscr{A} \to \mathscr{B}$, a *lax-natural transformation* $\alpha : F \to G$ assigns to each $A \in \mathscr{A}$ an arrow $\alpha_A : FA \to GA$ and to each arrow $f : A \to B$ in $\mathscr{A}$ a 2-cell α_f as in

$$\begin{array}{ccc} FA & \xrightarrow{\alpha_A} & GA \\ {\scriptstyle Ff}\downarrow & \Downarrow \alpha_f & \downarrow{\scriptstyle Gf} \\ FB & \xrightarrow[\alpha_B]{} & GB, \end{array} \tag{2.1}$$

these data being subject to three axioms: α_{gf} is to be the pasting composite of α_g and α_f; when $f = 1_A$ we are to have $\alpha_f = \mathrm{id}$; and α_f and α_h are to be compatible with $F\phi$ and $G\phi$ for a 2-cell $\phi : f \to h$. The lax-natural α is *pseudo-natural* if each α_f is invertible; and it is just a 2-natural transformation if each α_f is an identity. For lax $\alpha, \beta : F \to G$, a *modification* $\theta : \alpha \to \beta$ is a family $\theta_A : \alpha_A \to \beta_A$ of 2-cells satisfying the evident condition. The 2-functors $F : \mathscr{A} \to \mathscr{B}$, the lax-natural $\alpha : F \to G$, and the modifications $\theta : \alpha \to \beta$, form a 2-category $\mathrm{Lax}[\mathscr{A}, \mathscr{B}]$; restricting to pseudo-natural α gives a sub-2-category $\mathrm{Psd}[\mathscr{A}, \mathscr{B}]$, and we have inclusion 2-functors (not full, although locally so)

$$[\mathscr{A}, \mathscr{B}] \xrightarrow[J]{} \mathrm{Psd}[\mathscr{A}, \mathscr{B}] \xrightarrow[K]{} \mathrm{Lax}[\mathscr{A}, \mathscr{B}]. \tag{2.2}$$

Reversing the sense of α in (2.1) gives the notion of *oplax-natural transformation*, and we have equally an inclusion

$$L : \mathrm{Psd}[\mathscr{A}, \mathscr{B}] \to \mathrm{Oplax}[\mathscr{A}, \mathscr{B}]; \tag{2.3}$$

in fact $\mathrm{Oplax}[\mathscr{A}, \mathscr{B}]$ can be seen either as $\mathrm{Lax}[\mathscr{A}^{\mathrm{op}}, \mathscr{B}^{\mathrm{op}}]^{\mathrm{op}}$ or as $\mathrm{Lax}[\mathscr{A}^{\mathrm{co}}, \mathscr{B}^{\mathrm{co}}]^{\mathrm{co}}$. (We re-emphasize that the objects of the 2-categories in (2.2) and (2.3) are still the 2-functors $\mathscr{A} \to \mathscr{B}$; if we extend $\mathrm{Psd}[\mathscr{A}, \mathscr{B}]$ by allowing all *pseudo-functors* (that is, *homomorphisms of bicategories*) as objects, we get a bigger 2-category $\mathrm{Hom}[\mathscr{A}, \mathscr{B}]$; and this, unlike the others, continues to make sense even if $\mathscr{A}$ and $\mathscr{B}$ are only bicategories. Our present concern, however, is with 2-categories.)

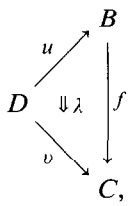
Given $G : \mathscr{A} \to \mathscr{B}$ and a weight $F : \mathscr{A} \to \mathbf{Cat}$, we define the *lax limit* $\{F, G\}_l$, if it

exists, to be the object of $\mathscr{B}$ providing a representation (still, as in (1.1), an *isomorphism* of categories)

$$\mathscr{B}(B, \{F, G\}_{\mathrm{l}}) \cong \mathrm{Lax}[\mathscr{A}, \mathbf{Cat}](F, \mathscr{B}(B, G-)); \tag{2.4}$$

now, of course, the unit $F \to \mathscr{B}(\{F, G\}_{\mathrm{l}}, G-)$ of this representation is only a *lax-natural* transformation. Similarly we define the *pseudo limit* $\{F, G\}_{\mathrm{p}}$ and the *oplax limit* $\{F, G\}_{\mathrm{ol}}$, replacing $\mathrm{Lax}[\mathscr{A}, \mathbf{Cat}]$ in (2.4) by $\mathrm{Psd}[\mathscr{A}, \mathbf{Cat}]$ and $\mathrm{Oplax}[\mathscr{A}, \mathbf{Cat}]$. Once again the definitions are 'representable', in that we have $\{F, G\}_{\mathrm{l}} \cong \mathrm{Lax}[\mathscr{A}, \mathbf{Cat}](F, G)$ when $\mathscr{B} = \mathbf{Cat}$. The notation for the *lax colimit* of $G : \mathscr{A} \to \mathscr{B}$ weighted by $F : \mathscr{A}^{\mathrm{op}} \to \mathbf{Cat}$ is $F *_{\mathrm{l}} G$, and so on.

The only special case worth noting is that where $F = \Delta 1 : \mathscr{A} \to \mathbf{Cat}$; we write $\mathrm{lax\,lim}\, G$ for $\{\Delta 1, G\}_{\mathrm{l}}$ and call it the *conical lax limit* of $G : \mathscr{A} \to \mathscr{B}$; similar for pseudo and oplax. These conical ones seem to have been first introduced by Gray [7]. A very simple case is that where $\mathscr{A} = \mathbf{2}$, so that to give G is to give an arrow $f : B \to C$ in $\mathscr{B}$; the (conical) lax limit of f is the universal diagram of the form



(2.5)

the universal property still having a two-dimensional aspect as well as a one-dimensional one. Note the pseudo limit where λ is invertible, and the oplax limit where its sense is reversed.

We mention here in passing, partly for information but largely to emphasize the contrast between the *isomorphism* of categories in the pseudo-analogue of (2.4) above and the *equivalence* in (2.6) below, the notion of (weighted) *bilimit*. This can be defined (see Street [22]) even when $\mathscr{A}$ and $\mathscr{B}$ are only bicategories and $F : \mathscr{A} \to \mathbf{Cat}$ and $G : \mathscr{A} \to \mathscr{B}$ only homomorphisms of bicategories; it is an object $\{F, G\}_{\mathrm{b}}$ of $\mathscr{B}$ and a pseudo-natural $F \to \mathscr{B}(\{F, G\}_{\mathrm{b}}, G-)$ inducing merely an *equivalence* of categories

$$\mathscr{B}(B, \{F, G\}_{\mathrm{b}}) \simeq \mathrm{Hom}[\mathscr{A}, \mathbf{Cat}](F, \mathscr{B}(B, G-)). \tag{2.6}$$

Clearly, when $\mathscr{A}$ and $\mathscr{B}$ are 2-categories and F and G are 2-functors, $\{F, G\}_{\mathrm{p}}$ is, if it exists, one possible value of $\{F, G\}_{\mathrm{b}}$ (which is determined only to within equivalence) - since then $\mathrm{Psd}[\mathscr{A}, \mathbf{Cat}]$ is a full sub-2-category of $\mathrm{Hom}[\mathscr{A}, \mathbf{Cat}]$. In fact - see [13, Proposition 6.1] - a 2-category $\mathscr{B}$ that admits all pseudo limits also admits all bilimits, even those where $\mathscr{A}$ is merely a bicategory and F, G homomorphisms. When - as in all the examples in our abstract - pseudo limits do exist, we have something strictly stronger than the corresponding bilimits, and more amenable to calculation. Even for a *conical* bilimit, the universal property, with all its 'coherence

conditions', is (as we said in the abstract) far more complicated to describe than that for a pseudo limit, and some authors have slurred over the fine points here; moreover a bilimit may involve *axiom-of-choice* considerations if 'equivalence' is taken in its weak sense, rather than that of 'adjoint equivalence'; see Remark 7.9 below. Of course bilimits may exist when pseudo limits do not, as in $\mathbf{Lex}^{\mathrm{op}}$; the category 1 is a bi-initial object in **Lex**, but a pseudo initial object would be an initial one, and no such exists.

We now observe that lax, oplax, and pseudo limits are not new classes of limits extending the weighted limits of Section 1 above, but rather special cases of these. It is shown in [4, Theorem 3.16] that the 2-functors J and KJ of (2.2) admit left adjoints $(\ )'$ and $(\ )^{\dagger}$ whenever $\mathscr{A}$ is small and $\mathscr{B}$ is cocomplete, and in particular – the only case that concerns us here – when $\mathscr{B}=\mathbf{Cat}$. By an identical argument, inverting the sense of the 2-cells, LJ has a left adjoint $(\ )^{\diamond}$, where L is as in (2.3). By 'adjoint' here we mean '2-adjoint', or '**Cat**-enriched adjoint'; for instance, $(\ )' : \mathrm{Psd}[\mathscr{A}, \mathbf{Cat}] \to [\mathscr{A}, \mathbf{Cat}]$ is a 2-functor, and we have an isomorphism of categories

$$[\mathscr{A}, \mathbf{Cat}](F', G) \cong \mathrm{Psd}[\mathscr{A}, \mathbf{Cat}](F, G) \tag{2.7}$$

2-natural in $G \in [\mathscr{A}, \mathbf{Cat}]$ and in $F \in \mathrm{Psd}[\mathscr{A}, \mathbf{Cat}]$. In a later article [3] we shall give explicit formulas for $F', F^{\dagger}$, and $F^{\diamond}$ in the case $\mathscr{B}=\mathbf{Cat}$, in terms of the Grothendieck construction of fibrations; but we do not really need these for our present considerations.

Comparison of (2.4) with (1.3) in the light of (2.7) shows that we have isomorphisms

$$\{F, G\}_{\mathrm{p}} \cong \{F', G\}, \qquad \{F, G\}_{\mathrm{l}} \cong \{F^{\dagger}, G\}, \qquad \{F, G\}_{\mathrm{ol}} \cong \{F^{\diamond}, G\}, \tag{2.8}$$

either side existing if the other does. Of course, limits being determined only to within isomorphism, we can if we wish take these isomorphisms to be equalities. (Similarly for colimits: $F *_{\mathrm{p}} G = F' * G$, and so on.) Accordingly pseudo limits and lax limits are, as we said above, only special cases of (weighted) limits. Note, however, that *conical* pseudo-limits are not thereby reduced to *conical* limits: for $(\Delta 1)'$ and $(\Delta 1)^{\dagger}$ are quite different from $\Delta 1$ – see [21] or [3] – and indeed the full sub-2-category of **Cat** given by those categories with at most one object admits all conical limits but not the conical lax or pseudo limits of the form (2.5).

There is a direct proof in [13, Proposition 5.2] of the following (except the oplax part, which requires only trivial changes):

Proposition 2.1. *If $\mathscr{B}$ admits inserters, equifiers, and products, we can construct from these all pseudo limits, lax limits, and oplax limits.* □

3. Classes of limits

When we speak of a 2-category $\mathscr{B}$ as admitting all limits of a certain class, we

are really referring to *a class Φ of weights*, and saying that $\mathscr{B}$ admits all *Φ-limits*; that is, all limits $\{F, G\}$ with $F \in \Phi$; whereupon we say that $\mathscr{B}$ is *Φ-complete*. Thus to admit conical limits, or pseudo limits, or lax limits, or inserters, is to be Φ-complete where Φ consists of all weights of the form $\Delta 1$, or all of the form H' for some H, or all of the form $H^{\dagger}$, or the single weight given in (1.6). From this point of view, what is 'pseudo' when F is of the form H' is not the limit $\{F, G\}$ but the weight F; we might call such an *F a weight of pseudo type*, since 'a pseudo weight' would be puzzling.

For a class Φ of weights and a small 2-category $\mathscr{A}$, we write $\Phi[\mathscr{A}]$ for the full sub-2-category (possible empty) of $[\mathscr{A}, \mathbf{Cat}]$ determined by those elements of Φ with domain $\mathscr{A}$; to give Φ is of course equally to give $\Phi[\mathscr{A}]$ for each small $\mathscr{A}$.

This language of classes of weights allows us to attach a precise meaning to such a question as "Is an inserter a pseudo limit?"; this is just to ask whether the weight given in (1.6) of the form H' – and we shall see in Section 6 below that it is not.

There is a closure operation on classes of weights, studied by Albert and Kelly in [1] for enriched categories in general. A weight F is in the *closure* Φ^* of Φ when every Φ-complete 2-category is F-complete *and* every functor between Φ-complete 2-categories that preserves Φ-limits also preserves F-limits. (Whether the latter clause adds anything is unknown; in some cases – see [1] and [15] – it does not.) The main result of [1] is

Proposition 3.1. *$F \in [\mathscr{A}, \mathbf{Cat}]$ is in Φ^* if and only if it lies in the closure of the representables $\mathscr{A}^{\mathrm{op}}$ under Φ-colimits in $[\mathscr{A}, \mathbf{Cat}]$.* □

An almost immediate corollary is [1, Proposition 6.1], namely

Proposition 3.2. *Φ is a closed class, in the sense that $\Phi^* = \Phi$, if and only if, for each small $\mathscr{A}$, $\Phi[\mathscr{A}]$ contains the representables and is closed in $[\mathscr{A}, \mathbf{Cat}]$ under Φ-colimits.* □

If (*PIE*) denotes the class of products, inserters, and equifiers (or more properly the corresponding weights – we reserve the right to use the looser but simpler terminology where appropriate), then by Propositions 1.1 and 2.1 we have that (*PIE*)* includes all the limits mentioned in Proposition 1.1, as well as all pseudo limits, lax limits, and oplax limits. We should be happy to have an explicit description of (*PIE*)*, since it was shown in [4] that the 2-category T-Alg of algebras for a 2-monad T on a complete 2-category $\mathscr{K}$, with the morphisms that preserve the structure (only) to within coherent isomorphisms, is (*PIE*)-complete; however we lack such a description.

The class (Psd) of weights of pseudo type is certainly not a closed class; we shall see in Section 6 that it does not, in the sense of Proposition 3.2, contain the representables. It surely contains any $F: \mathscr{A} \to \mathbf{Cat}$ where $\mathscr{A}$ is a mere set, since then $[\mathscr{A}, \mathbf{Cat}] = \mathrm{Psd}[\mathscr{A}, \mathbf{Cat}]$; thus it contains all products and cotensor products. We

mentioned a few paragraphs back that it does not contain the inserters; nor does it contain, as we shall see in Section 6, the equifiers, or the iso-inserters, or the splitting of idempotents or of idempotent equivalences. These last two weights do not belong even to (Psd)*, since we shall show in Section 6 that they do not belong to (*PIE*)*, which includes (Psd)* as we have seen. Whether inserters, equifiers, or iso-inserters lie in (Psd)* is unknown to us; it is hard to apply Proposition 3.1 to settle this, in the absence of a manageably-small subclass Φ of (Psd) for which we can assert that $\Phi^* = (\text{Psd})^*$; we do not know what to add to products and cotensor products to get such a Φ; certainly (as we shall see) products and cotensor products alone do not suffice.

What we are going to do is to exhibit a closed class (Flex), containing (*PIE*) and hence (Psd), (Lax), and (Oplax), such that all the 2-categories of our abstract are (Flex)-complete; as is T-Alg, not for *all* T, but for those of the greatest interest in practice. We also say *flexibly complete* for (Flex)-complete.

4. Flexible weights and flexible limits

By an *equivalence* $u : A \to B$ in a 2-category we always mean an *adjoint equivalence*, in the sense that we have an adjunction $\eta, \varepsilon : f \dashv u$ with the unit $\eta : 1 \to uf$ and the counit $\varepsilon : fu \to 1$ invertible; for which it suffices that there be some $f : B \to A$ with isomorphisms $1 \cong uf$ and $fu \cong 1$. We call the equivalence u *surjective* if the adjunction can be so chosen that $\eta = \mathrm{id}$; an equivalence u is surjective if and only if it is a *retraction*, meaning that $uh = 1$ for some $h : B \to A$.

The notion of flexibility was introduced in [4] in the context of the adjunction $(\ )' \dashv J : T\text{-Alg}_s \to T\text{-Alg}$, of which (2.7) above is a special case: here T-Alg$_s$ is the 2-category of T-algebras and *strict* morphisms, which in (2.7) is $[\mathcal{A}, \mathbf{Cat}]$, and J as in (2.2) is the inclusion. We recall from [4] the essential facts in our special case; as an aid to clarity we often use bold-face letters for pseudo-natural transformations and ordinary letters for 2-natural (now also called *strict*) ones.

We write $\boldsymbol{p}_F : F \to F'$ for the unit of the adjunction (2.7); its universal property is that any $\boldsymbol{m} : F \to G$ in Psd$[\mathcal{A}, \mathbf{Cat}]$ is $n\boldsymbol{p}_F$ for a unique strict $n : F' \to G$, along with the two-dimensional aspect of this making (2.7) an isomorphism of categories. In particular there is a unique strict $q_F : F' \to F$ with

$$q_F \boldsymbol{p}_F = 1; \tag{4.1}$$

in fact q_F is the counit of the adjunction (2.7), while (4.1) is one of the triangular equations, the other being

$$q_{F'}(\boldsymbol{p}_F)' = 1. \tag{4.2}$$

It is shown in [4, Theorem 4.2] that

Proposition 4.1. *There is a unique invertible modification* $\varrho_F : \boldsymbol{p}_F q_F \cong 1$ *giving an*

adjoint equivalence id, $\varrho_F : p_F \dashv q_F$; *so that* $q_F : F' \to F$ *is a surjective equivalence in* Psd[$\mathscr{A}$, **Cat**]. □

Given this, the proof of the following result, which is [4, Theorem 4.4], is easy using (4.2) and the naturality of q:

Proposition 4.2. *For* $F : \mathscr{A} \to$ **Cat**, *the following are equivalent*:
(a) $q_F : F' \to F$ *is a surjective equivalence in* [$\mathscr{A}$, **Cat**];
(b) $q_F : F' \to F$ *is a retraction in* [$\mathscr{A}$, **Cat**];
(c) F *is retract of* G' *in* [$\mathscr{A}$, **Cat**] *for some* $G \in$ [$\mathscr{A}$, **Cat**]. □

The weights F satisfying these conditions are said to be *flexible*, as are the corresponding limits. It is trivial from (c) above that (Psd) $\subset$ (Flex), the objects of (Flex) being just the retracts of those of (Psd). Of course much more is true, since we promised at the end of Section 3 to show that (*PIE*) $\subset$ (Flex).

Remark 4.3. There is a still larger class of weights, which we shall not emphasize because we are unsure of its practical value, but which we certainly should mention: call F *semi-flexible* if q_F is an equivalence in [$\mathscr{A}$, **Cat**], but not necessarily a surjective one; by [4, Theorem 4.7] it comes to the same thing to say that F is equivalent in [$\mathscr{A}$, **Cat**] to some G', or to some flexible weight. A series of counter-examples is given in [4, Example 4.11] in the case $\mathscr{A} = \mathbf{2}$. An object F of [**2**, **Cat**] is just a functor $a : X \to Y$, and it is easy to calculate F' directly (without recourse to the explicit formulas to be given in [3]) in the simple case $Y = 1$. To wit, F' is the inclusion $j : X \to \bar{X}$ where ob $\bar{X} =$ ob $X + \{*\}$ and $\bar{X}$ is a chaotic category - that is, every hom-set is a singleton. The pseudo-natural $p_F : F \to F'$ consists of the functors $1_X : X \to X$ and $* : 1 \to \bar{X}$, with the unique isomorphism $j 1_X \cong *a$; while $q_F : F' \to F$ consists of the functors $1_X : X \to X$ and $\bar{X} \to 1$. It is now easily verified (*loc. cit.*) that $(X \to 1)$ cannot be semi-flexible unless X is chaotic, so that not every weight is semi-flexible; that $(\mathbf{I} \to 1)$ is semi-flexible but not flexible; and that $\Delta 1 = (1 \to 1)$ is flexible but not in (Psd). It may interest the reader to consider the limit $\{F, G\}$ where F is the semi-flexible but not flexible $(\mathbf{I} \to 1)$ and $G : \mathbf{2} \to \mathscr{B}$ is $(f : A \to B)$; it is an object C, arrows $u, v : C \to A$, and an invertible 2-cell $\alpha : u \cong v$, universal with the property that $fu = fv$ and $f\alpha = \mathrm{id}$.

We further recall part (a) of [4, Theorem 5.1] in the special case of our adjunction (2.7):

Proposition 4.4. *If, for a* 2*-category* $\mathscr{C}$ *and a* 2*-functor* $T : \mathrm{Psd}[\mathscr{A}, \mathbf{Cat}] \to \mathscr{C}$, *the composite* $TJ : [\mathscr{A}, \mathbf{Cat}] \to \mathscr{C}$ *has a left* 2*-adjoint* $S : \mathscr{C} \to [\mathscr{A}, \mathbf{Cat}]$, *each* SC *is flexible.* □

Since, as we said in Section 2, [4, Theorem 3.16] provides left adjoints to KJ and

to LJ, in the notation of (2.2) and (2.3), we have at once

Proposition 4.5. *Lax and oplax limits are flexible.* □

We shall of course have an alternative proof of this, by Proposition 2.1, when we prove that $(PIE) \subset$ (Flex) and that (Flex) is a closed class. By Proposition 3.2, to show that (Flex) is closed we must in particular show that

Proposition 4.6. *Every representable $\mathscr{A}(A, -)$ is flexible.*

Proof. We apply Proposition 4.4 with $T: \mathrm{Psd}[\mathscr{A}, \mathbf{Cat}] \to \mathbf{Cat}$ being evaluation at A: this is the 2-functor sending $F: \mathscr{A} \to \mathbf{Cat}$ to FA, sending the pseudo-natural $\alpha: F \to G$ to $\alpha_A: FA \to GA$, and sending the modification $\theta: \alpha \to \beta$ to θ_A. The composite TJ has the left adjoint S sending the category X to the tensor product $X * \mathscr{A}(A, -)$. Taking $X = 1$ we see that $\mathscr{A}(A, -)$ is flexible. □

Proposition 4.7. *Products, inserters, and equifiers are flexible limits; so are the splitting of idempotents (in the strict sense of Section 1 above) and the splitting of idempotent equivalences.*

Proof. In the second-last paragraph of Section 3, we observed that products are pseudo limits; *a fortiori* they are flexible ones. For the other limits above we calculate F' directly: in these simple cases this is easy, without reference to the explicit formulas to be given in [3]. Note that the one-dimensional aspect of the universal property for the adjunction (2.7) suffices to *determine* F', p_F, and q_F; see [13, Section 2].

Write $\mathbf{P}$ for the domain of the weight F of (1.6) for inserters; a general $G: \mathbf{P} \to \mathbf{Cat}$ is a pair $f, g: B \to C$ as in (1.6). To give a pseudo-natural $m: F \to G$ is to give an object b of B, a morphism $\phi: c \to d$ in C, and invertible morphisms $\varrho: fb \to c$ and $\sigma: gb \to d$; here m is 2-natural when ϱ and σ are identities. Accordingly F' is $u, v: 1 \to \mathbf{R}$ where $\mathbf{R}$ is the category $u \cong 0 \to 1 \cong v$, while $p_F: F \to F'$ corresponds to the unique object of 1, the morphism $0 \to 1$ in $\mathbf{R}$, and the invertible morphisms $u \cong 0$ and $v \cong 1$. It follows that $q_F: F' \to F$ is given by the functor $1 \to 1$ and the functor $k: \mathbf{R} \to \mathbf{2}$ with $k(u) = k(0) = 0$ and $k(v) = k(1) = 1$. A 2-natural right inverse for q_F is given by the functor $1 \to 1$ and the functor $h: \mathbf{2} \to \mathbf{R}$ corresponding to $u \to v$. So q_F is a retraction in $[\mathbf{P}, \mathbf{Cat}]$, and inserters are flexible limits.

Now write $\mathbf{Q}$ for the domain of the weight F of (1.7) for equifiers; a general $G: \mathbf{Q} \to \mathbf{Cat}$ corresponds to $\alpha, \beta: f \to g: B \to C$ as in (1.7). Once again, to give a pseudo-natural $m: F \to G$ is to give some $b \in B$, some $\phi: c \to d$ in C, and invertible morphisms $\varrho: fb \to c$ and $\sigma: gb \to d$; but now these are to satisfy $\sigma^{-1} \phi \varrho = \alpha_b = \beta_b$ (the b-components of α and β). It follows that, with the same $\mathbf{R}$ as in the last paragraph, F' is λ, $\lambda: u \to v: 1 \to \mathbf{R}$, where $\lambda: u \to v$ is the unique morphism; while $p_F: F \to F'$ again corresponds to the unique object of 1, the morphism $0 \to 1$ in $\mathbf{R}$,

and the invertible morphisms $u \cong 0$ and $v \cong 1$. So q_F is again given by the functors $1 \to 1$ and $k : \mathbf{R} \to \mathbf{2}$ of the last paragraph, and it still has the 2-natural right inverse given by $1 \to 1$ and $h : \mathbf{2} \to \mathbf{R}$; showing that equifiers are flexible limits.

We turn to the splitting of idempotents; here (see Section 1) the domain $\mathbf{E}$ has one object $*$ and is generated by $\varepsilon : * \to *$ with $\varepsilon^2 = \varepsilon$, and the weight F is $\Delta 1 : \mathbf{E} \to \mathbf{Cat}$. To give a general $G : \mathbf{E} \to \mathbf{Cat}$ is to give a category B with an idempotent $e : B \to B$. To give a pseudo-natural $F \to G$ is to give $b \in B$ and an invertible $\varrho : eb \to b$ with $\varrho \cdot e\varrho = \varrho$, or equivalently with $e\varrho : e^2 b \to eb$ being the identity of eb. It follows at once that F' is $\mathbf{I}$ with the idempotent $\Delta 0 : \mathbf{I} \to \mathbf{I}$ constant at $0 \in \mathbf{I}$, and that $p_F : F \to F'$ corresponds to $b = 1 \in \mathbf{I}$ with $\varrho : (\Delta 0)b \to b$ being the unique $0 \to 1$ in $\mathbf{I}$. Of course the 2-natural $q_F : F' \to F$ is given by the unique functor $\mathbf{I} \to 1$; it has a 2-natural right inverse given by the functor $0 : 1 \to \mathbf{I}$; whence idempotent-splitting is a flexible limit. We leave the reader to check that, for the splitting of idempotent equivalences, there is no essential change to this last argument; the extra invertible 2-cell automatically satisfies all that is required of it. □

Proposition 4.8. *If $\mathscr{B}$ admits either* (a) *all pseudo limits and the splitting of idempotent equivalences, or* (b) *products, inserters, equifiers, and the splitting of idempotent equivalences, we can construct from the given limits all flexible limits.*

Proof. By Proposition 2.1, it suffices to prove (a). Let $F : \mathscr{A} \to \mathbf{Cat}$ be a flexible weight, and write $q : F' \to F$ for q_F. By Proposition 4.2(a) there is some $h : F \to F'$ in $[\mathscr{A}, \mathbf{Cat}]$ with $qh = 1$ and $hq \cong 1$; so $hq : F' \to F'$ is an idempotent equivalence. Since the isomorphism $\pi : \mathscr{B}(B, \{F', G\}) \to [\mathscr{A}, \mathbf{Cat}](F', \mathscr{B}(B, G-))$ is 2-natural in B, there is a unique $e : \{F', G\} \to \{F', G\}$ with $[\mathscr{A}, \mathbf{Cat}](hq, 1)\pi = \pi \mathscr{B}(1, e)$; and e too is an idempotent equivalence. Let it split as $e = ir$, where $i : C \to \{F', G\}$ and $r : \{F', G\} \to C$ with $ri = 1$. Define functors $\theta : \mathscr{B}(B, C) \to [\mathscr{A}, \mathbf{Cat}](F, \mathscr{B}(B, G-))$ and $\phi : [\mathscr{A}, \mathbf{Cat}](F, \mathscr{B}(B, G-)) \to \mathscr{B}(B, C)$ by $\theta = [\mathscr{A}, \mathbf{Cat}](h, 1) \cdot \pi \cdot \mathscr{B}(1, i)$ and $\phi = \mathscr{B}(1, r) \cdot \pi^{-1} \cdot [\mathscr{A}, \mathbf{Cat}](q, 1)$; clearly θ and ϕ are 2-natural in B. The composite $\phi\theta$ is $\mathscr{B}(1, r)\mathscr{B}(1, e)\mathscr{B}(1, i)$, which is the identity because $rei = riri = 1$; while $\theta\phi = 1$ for precisely similar reasons. Thus $\{F, G\}$ exists, being given by C. □

Write S for the weight for splitting idempotents, and S_e for the weight for splitting idempotent equivalences; by Section 1, these are $\Delta 1 : \mathbf{E} \to \mathbf{Cat}$ and $\Delta 1 : \mathbf{F} \to \mathbf{Cat}$.

Theorem 4.9. *The class* (Flex) *of (the weights for) flexible limits is closed. It is equal to each of the closures* $(PIES)^*$, $(PIES_e)^*$, $(\mathrm{Psd}, S)^*$, *and* $(\mathrm{Psd}, S_e)^*$; *and it contains all the limits mentioned in Propositions* 1.1 *and* 2.1.

Proof. By Propositions 4.8, 4.7, 1.1, and 2.1, we need prove only the first assertion. By Propositions 3.2 and 4.6, we have only to prove that, for each small $\mathscr{A}$, the full sub-2-category $(\mathrm{Flex})[\mathscr{A}]$ of flexible weights is closed in $[\mathscr{A}, \mathbf{Cat}]$ under flexible colimits. By Proposition 4.8, it suffices to prove it closed under $(PIES)$ colimits.

Write $\mathscr{F}$ for $[\mathscr{A}, \mathbf{Cat}]$ to abbreviate, and recall that non-bold-face letters denote arrows in $\mathscr{F}$, not in $\mathrm{Psd}[\mathscr{A}, \mathbf{Cat}]$.

That the flexible weights are closed under idempotent-splitting is trivial, since retracts of retracts are retracts. Now let $j_n : F_n \to F$ be the coprojections of a coproduct in $\mathscr{F}$, where each F_n is flexible. By Proposition 4.2, we have arrows $h_n : F_n \to F'_n$ in $\mathscr{F}$ with $q_{F_n} h_n = 1$. We also have the arrows $j'_n : F'_n \to F'$ obtained by applying $(\ )'J$ to the j_n. By the universal property of the coproduct there is a unique $h : F \to F'$ with $hj_n = j'_n h_n$. By the naturality of $q : (\ )'J \to 1$ we have $q_F j'_n = j_n q_{F_n}$. So $q_F h j_n = q_F j'_n h_n = j_n q_{F_n} h_n = j_n$, whence $q_F h = 1$ by the uniqueness clause. Thus F is flexible.

Now let the pair $f, g : G \to H$ in $\mathscr{F}$ have the coinserter $t : H \to F$, $\lambda : tf \to tg$. We show that F is flexible when H is, whether G is so or not. We have some $k : H \to H'$ with $q_H k = 1$. From Proposition 4.1 we have $\varrho_H : p_H q_H \cong 1$ with $q_H \varrho_H = \mathrm{id}$; multiplying on the right by k gives $\varrho_H k : p_H \cong k$, with $q_H \varrho_H k = \mathrm{id}$; that is, setting $\sigma = \varrho_H k$, we have $\sigma : p_H \cong k$ with $q_H \sigma = \mathrm{id}$. By the naturality of $p : 1 \to J(\ )'$, we have $p_F t = t' p_H$; so we have a modification τ given by

$$t'kf \xrightarrow[t'\sigma^{-1}f]{} t'p_H f = p_F tf \xrightarrow[p_F\lambda]{} p_F tg = t'p_H g \xrightarrow[t'\sigma g]{} t'kg.$$

By the universal property of the coinserter, there is a unique $h : F \to F'$ with $ht = t'k$ and $h\lambda = \tau$. Since $q_F t' = tq_H$ by the naturality of q, we have $q_F ht = tq_H k = t$ and $q_F h\lambda = q_F \tau = \lambda$ (for $q_F t'\sigma g = tq_H \sigma g = \mathrm{id}$, with $q_F t'\sigma^{-1} f = \mathrm{id}$ similarly, while $q_F p_F \lambda = \lambda$ by (4.1)). It follows by the uniqueness clause that $q_F h = 1$, so that F is flexible.

Finally let $\alpha, \beta : f \to g : G \to H$ have the coequifier $t : H \to F$; again we show that F is flexible when H is, whether G is so or not. Let $k : H \to H'$ and $\sigma : p_H \cong k$ be as in the last paragraph, and recall that $p_F t = t' p_H$ by naturality. This last gives $t' p_H \alpha = t' p_H \beta$ since $t\alpha = t\beta$; whence $t'k\alpha = t'k\beta$ since $p_H \cong k$. By the universal property of the coequifier, there is a unique $h : F \to F'$ with $ht = t'k$. Since once again we have $q_F t' = tq_H$ by the naturality of q, we have $q_F ht = t$; whence $q_F h = 1$ by the uniqueness clause, and F is flexible. □

We may note that the argument above works equally well when $J : [\mathscr{A}, \mathbf{Cat}] \to \mathrm{Psd}[\mathscr{A}, \mathbf{Cat}]$ is replaced by the more general $J : T\text{-Alg}_s \to T\text{-Alg}$ of [4]; so that we have

Proposition 4.10. *For a 2-monad T with rank on a cocomplete 2-category $\mathscr{K}$, the flexible algebras are closed under flexible colimits in T-Alg_s.* □

Now that we know (Flex) to be a closed class, we may call any class $\Phi \subset (\mathrm{Flex})$ with $\Phi^* = (\mathrm{Flex})$ a *basis* for it. Among such bases are the classes $(PIES_e)$ and so on of Theorem 4.9. Another basis is given in Street's article [24], which corrects an oversight in (1.25) of his [22]. Since the indications in [24], being but an appendage

to observations primarily concerned with bilimits, do not spell out the full details, we do so here. We denote the class of iso-inserters (see Section 1) by $(I_!)$, and that of cotensor products by (C); these are flexible limits by Theorem 4.9. Street's result is:

Theorem 4.11. *If $\mathscr{B}$ admits all $(PI_!CS)$-limits, we can construct from these all flexible limits*; *whence* $(PI_!CS)^* = (\text{Flex})$.

Proof. By Proposition 4.8, it suffices to construct all pseudo limits; by [13, Proposition 5.1] we can construct these from $(PI_!C)$ and from equifiers of those $\alpha, \beta : f \to g : B \to C$ for which α and β are invertible. Such an equifier is equally that of $\gamma, \mathrm{id} : f \to f : B \to C$ where γ is the automorphism $\alpha^{-1}\beta$; and we show how to construct this from $(I_!CS)$. Write $\mathbf{J}$ for the one-object category corresponding to the additive group of integers, so that a functor $\mathbf{J} \to X$ is an object x of X with an automorphism ξ of x. Form the cotensor product $\{\mathbf{J}, C\}$, its unit given by $d : \{\mathbf{J}, C\} \to C$ and the automorphism $\delta : d \to d$. Write $\bar{\gamma}, \bar{f} : B \to \{\mathbf{J}, C\}$ for the arrows given by $d\bar{\gamma} = f$, $\delta\bar{\gamma} = \gamma$ and $d\bar{f} = f$, $\delta\bar{f} = \mathrm{id}$. The equifier we seek is of course the equalizer of $\bar{\gamma}$ and $\bar{f}$; lacking equalizers in general, we form an approximation to it by taking the iso-inserter $h : D \to B$, $\lambda : \bar{\gamma}h \cong \bar{f}h$ of $\bar{\gamma}$ and $\bar{f}$. Since $\mathscr{B}$ is a 2-category, we have the equality

$$\left(d\bar{\gamma}h \xrightarrow[d\lambda]{} d\bar{f}h \xrightarrow[\delta\bar{f}h]{} d\bar{f}h\right) = \left(d\bar{\gamma}h \xrightarrow[\delta\bar{\gamma}h]{} d\bar{\gamma}h \xrightarrow[d\lambda]{} d\bar{f}h\right);$$

here $\delta\bar{f}h = \mathrm{id}$ since $\delta\bar{f} = \mathrm{id}$; it follows, $d\lambda$ being invertible, that $\delta\bar{\gamma}h = \mathrm{id}$. Having $d\bar{\gamma}h = fh = d\bar{f}h$ and $\delta\bar{\gamma}h = \mathrm{id} = \delta\bar{f}h$, we conclude that $\bar{\gamma}h = \bar{f}h$. By the universal property of the iso-inserter (h, λ), there is a unique $e : D \to D$ with $he = h$ and $\lambda e = \mathrm{id}$. Composing on the right with e gives $he^2 = he = h$ and $\lambda e^2 = \mathrm{id}$; so that $e^2 = e$ by the uniqueness clause. Let the idempotent e (which, by the way, is *not* an idempotent equivalence) split as $e = ir$, where $i : A \to D$ and $r : D \to A$ with $ri = 1$. Then $hi : A \to B$ is the desired equalizer of $\bar{\gamma}$ and $\bar{f}$. To see this, first note that $\lambda ir = \lambda e = \mathrm{id}$ gives, on right multiplication by i, that $\lambda i = \mathrm{id}$. Now consider any $k : E \to B$ with $\bar{\gamma}k = \bar{f}k$; by the universal property of the iso-inserter, there is a unique $t : E \to D$ with $ht = k$ and $\lambda t = \mathrm{id}$; all we want of this is the existence of some $t : E \to D$ with $ht = k$. Setting $s = rt : E \to A$, we have $his = hirt = het = ht = k$. We next need the uniqueness property of the equalizer: let $s, s' : E \to A$ have $his = his'$. Since $\lambda i = \mathrm{id}$, we also have $\lambda is = \lambda is'$; by the uniqueness-clause for the iso-inserter, we have $is = is'$, giving $s = s'$ on left multiplication by r. We leave to the reader the easy verification of the two-dimensional aspect of the universal property of the equalizer. □

Remark 4.12. We have not investigated semi-flexible limits any further, and in particular have not considered whether they form a closed class.

5. Finite flexible limits

The concept of *finite flexible limit* will be important in the forthcoming article [14] on the two-dimensional analogue of 'finite-limit theories', which will both use and extend the results of [4] on algebras for a 2-monad. Accordingly we recall the notion of *finite* (*weighted*) *limit* for 2-categories introduced (in a wider context of enriched category theory) by Kelly in [12].

To this end we first revise some facts about finite presentability. An object K of a category $\mathscr{K}$ with filtered colimits is said to be *finitely presentable* when the representable $\mathscr{K}(K, -) : \mathscr{K} \to \mathbf{Set}$ preserves filtered colimits. This definition, due to Gabriel-Ulmer [6], is justified by the fact that it coincides with the classical one when $\mathscr{K}$ is the category of algebras for a one-sorted finitary algebraic theory (see [6, Article 7.7]), or even a many-sorted one (see [12, (8.12)]). It also coincides with the usual one when $\mathscr{K}$ is the (mere) category $\mathbf{Cat}_0$ of small categories and functors - the case of present interest to us, which we now look at more closely.

Let Ψ be the left adjoint of the forgetful functor $\mathbf{Cat}_0 \to \mathbf{Gph}$ to the category of graphs. It was shown in [12, (8.12)], using an adaptation and modification of the original proof in [6, Article 7.7(e)], that a category A is finitely presentable precisely when there are finite graphs P and Q such that A is the coequalizer in $\mathbf{Cat}_0$ of two morphisms $\Psi P \to \Psi Q$. We need a refinement of this which, although surely well known, we cannot find explicitly in print - it is really implicit in the proof in [12].

Proposition 5.1. *A category A is finitely presentable precisely when there are a finite graph Q and a finite set X (which we treat as a discrete category) such that A is the coequifier in* **Cat** *of a pair of natural transformations $\varrho, \sigma : u \to v : X \to \Psi Q$.*

Proof. It is shown in [12, (8.12)] that, for a finitely-presentable category A, we can find a finite graph Q and a regular epimorphism $q : \Psi Q \to A$ in $\mathbf{Cat}_0$ *which is the identity on objects.* Since $\mathbf{2}$ is a strong generator of $\mathbf{Cat}_0$, it follows from [12, (8.3)] that q is the coequalizer of two functors $r, s : X * \mathbf{2} \to \Psi Q$ where X is a finite set and $X * \mathbf{2}$ the copower of $\mathbf{2}$. Because q is the identity on objects, the components $r_x, s_x : \mathbf{2} \to \Psi Q$, seen as morphisms ϱ_x and σ_x in ΨQ, have for each $x \in X$ the same domain ux and the same codomain vx. It is clear that q, besides being the coequalizer in $\mathbf{Cat}_0$ and hence in **Cat** of r and s, is equally the coequifier in **Cat** of $\varrho, \sigma : u \to v : X \to \Psi Q$. The converse is evident: $X * \mathbf{2}$ and ΨQ are finitely presentable, and the finitely presentables are closed in $\mathbf{Cat}_0$ under finite colimits - see [12]. □

Specializing Kelly's definition in [12, (4.1)] to the case of 2-categories, we say that a weight $F : \mathscr{A} \to \mathbf{Cat}$ is *finite* when (i) the set of isomorphism classes of objects of $\mathscr{A}$ is finite, (ii) each category $\mathscr{A}(A, B)$ is finitely presentable, and (iii) each category FA is finitely presentable. The class (Fin) of finite weights is not closed, any more than what are commonly called finite limits in ordinary category theory form a closed class; we shall describe below its closure (Fin)*.

It is clear that among the 2-categorical finite limits are those conical limits, such as finite products and equalizers, where the domain $\mathscr{A}$ is a finite mere category, as well as those cotensor products $\{A, -\}$ where the category A is finitely presentable. It is shown in [12, (4.3)] that all finite limits can be constructed from finite products, equalizers, and the cotensor products above. Once again - see the corresponding statement in Section 1 about *all* limits - it suffices to have the cotensor products $\{\mathbf{2}, -\}$; for, $\mathbf{2}$ being a finitely-presentable strong generator of $\mathbf{Cat}_0$, every finitely-presentable category F is, by [12, (7.2)], in the closure of $\mathbf{2}$ in $\mathbf{Cat}_0$ under finite coproducts and coequalizers. This construction of finite limits is, however, unsuited for our present concerns, since equalizers - as we shall see in Section 6 - are not flexible limits.

We define the class $(\mathrm{Flex}_\mathrm{f})$ of *finite flexible weights* to be $(\mathrm{Flex}) \cap (\mathrm{Fin})$. By Theorem 4.9 and Proposition 5.1, it contains the class (P_f) of finite products, the class (C_f) of those cotensor products $\{A, -\}$ with A finitely presentable, the weights I, I_i, E, S, S_e for inserters, iso-inserters, equifiers, the splitting of idempotents, and the splitting of idempotent equivalences, and also the weights for inverters, comma objects, and iso-comma-objects.

Write $(\mathrm{Psd}_\mathrm{f})$ and $(\mathrm{Lax}_\mathrm{f})$ for the classes of pseudo limits $\{F, G\}_\mathrm{p}$ and lax limits $\{F, G\}_\mathrm{l}$ where the weight F is finite; we omit further reference to Oplax since it behaves exactly like Lax. More precisely, $(\mathrm{Psd}_\mathrm{f})$ and $(\mathrm{Lax}_\mathrm{f})$ consist of those weights F' and $F^\dagger$ for which F is finite. Note that we are *not* claiming here that $(\mathrm{Psd}_\mathrm{f}) \subset (\mathrm{Fin})$, still less that $(\mathrm{Psd}_\mathrm{f}) = (\mathrm{Psd}) \cap (\mathrm{Fin})$; we cannot address these questions, or the corresponding ones for Lax, without the explicit formulas for F' and $F^\dagger$ to be given in [3].

Since the set of morphisms of a finitely-presentable category may be infinite, the proof of [13, Proposition 4.4], which constructs cotensor products from products, inserters, and equifiers, does not adapt to give a proof of the first assertion of the following proposition; we appeal instead to Proposition 5.1.

Proposition 5.2. $(P_\mathrm{f}IE)^*$ *contains* (C_f), $(\mathrm{Psd}_\mathrm{f})$, *and* $(\mathrm{Lax}_\mathrm{f})$.

Proof. Let $\mathscr{B}$ have $(P_\mathrm{f}IE)$-limits and let the finitely-presentable category A be presented as in Proposition 5.1, where the finite graph Q has object-set Y, arrow-set Z, and domain and codomain functions $d, c : Z \to Y$. For $B \in \mathscr{B}$, it is easy to see that the inserter of the pair $B^d, B^c : B^Y \to B^Z$ is the cotensor product $\{\Psi Q, B\}$. Of course the cotensor product $\{X, B\}$ is just B^X. We now construct the cotensor product $\{A, B\}$ as the equifier of $\{\varrho, B\}, \{\sigma, B\} : \{u, B\} \to \{v, B\} : \{\Psi QB\} \to \{X, B\}$. The proof of [13, Proposition 5.1] then adapts at once to the finite case to give the remaining assertions, when we recall from Proposition 1.1 that iso-inserters can be constructed from inserters and equifiers, and from [12, (5.9)] that a finite product of finitely-presentable categories is finitely presentable. □

Now that we have Proposition 5.2, the proofs of Proposition 4.8 and Theorem

4.11 (the latter because the category **J** there is finitely presentable) adapt without change to the finite case to give

Theorem 5.3. *From each of the subclasses* (P_fIES_e), (P_fIES), $(P_fI_iC_fS)$ *of* (Flex_f) *we can construct all finite flexible limits; whence the closure of each of these subclasses coincides with* $(\mathrm{Flex}_f)^*$. □

We have no explicit description of $(\mathrm{Flex}_f)^*$; being $((\mathrm{Flex})\cap(\mathrm{Fin}))^*$, it is certainly *contained in* $(\mathrm{Flex})\cap(\mathrm{Fin})^*$, which – as the intersection of closed classes – is itself a closed class; but we have no proof of equality here. Let us first describe $(\mathrm{Fin})^*$ and then explain our difficulty.

Recall from [12, (2.1)] that an object B of (not now a category, but) a 2-category $\mathscr{B}$ admitting filtered conical colimits is called *finitely presentable* when the representable $\mathscr{B}(B,-):\mathscr{B}\to\mathbf{Cat}$ preserves filtered colimits. Since conical colimits in $\mathscr{B}$ are also colimits in $\mathscr{B}_0$, and since preservation refers to the invertibility of a morphism in $\mathbf{Cat}_0$, it comes to the same thing to say that the mere functor $\mathscr{B}(B,-)_0:\mathscr{B}_0\to\mathbf{Cat}_0$ preserves filtered colimits. Because the set-of-objects functor $\mathrm{ob}:\mathbf{Cat}_0\to\mathbf{Set}$ preserves filtered colimits, while the composite of ob with $\mathscr{B}(B,-)_0$ is $\mathscr{B}_0(B,-)$, it follows that a B which is finitely presentable in the 2-category $\mathscr{B}$ is also finitely presentable in the mere category $\mathscr{B}_0$. The converse is false in general. It is however true (happily, since otherwise there would be a real danger of confusion) when $\mathscr{B}$ is a mere category regarded as a locally-discrete 2-category; for then $\mathscr{B}(B,-)_0$ is $D\mathscr{B}_0(B,-)$, where $D:\mathbf{Set}\to\mathbf{Cat}_0$ sends a set to the corresponding discrete category – and D preserves all colimits. The converse is again true (see [12, (7.5)]) when $\mathscr{B}$ is locally finitely presentable as a 2-category, in the sense of [12, (3.2)]; and hence in particular (see [12, (3.4)]) when $\mathscr{B}=\mathbf{Cat}$, or when $\mathscr{B}=[\mathscr{A},\mathbf{Cat}]$ for some small $\mathscr{A}$. Since, by [12, (7.2)], the closure under finite colimits in $[\mathscr{A},\mathbf{Cat}]$ of the representables consists of the finitely presentables, Proposition 3.1 gives

Proposition 5.4. *A weight* $F:\mathscr{A}\to\mathbf{Cat}$ *lies in* $(\mathrm{Fin})^*$ *precisely when* F *is finitely presentable in* $[\mathscr{A},\mathbf{Cat}]$. □

Remark 5.5. Since finiteness of F involves heavy restrictions on $\mathscr{A}$, while the representable $\mathscr{A}(A,-)$ lies in $(\mathrm{Fin})^*$ for *any* $\mathscr{A}$, the class $(\mathrm{Fin})^*$ is strictly bigger than (Fin). The class of weights called 'finitary' by Street in his article [21], written long before the general question of closed classes was raised in [1], lies strictly between (Fin) and $(\mathrm{Fin})^*$.

Remark 5.6. Our difficulty in deciding whether $(\mathrm{Flex}_f)^*=(\mathrm{Flex})\cap(\mathrm{Fin})^*$ comes to this: a weight in the latter lies in the closure of the representables under finite colimits; why should it lie in their closure under finite *flexible* colimits? Given that $(\mathrm{Flex})=(PIES)^*$ and $(\mathrm{Flex}_f)\subset(P_fIES)^*$, while I,E,S are both flexible and finite, one is tempted to believe that the transfinite construction of an F in $(\mathrm{Flex})\cap(\mathrm{Fin})^*$,

starting from the representables and closing up repeatedly under $(P), (I), (E), (S)$, never requires infinite coproducts; and to set about proving this inductively. It is true that, if an F in (Fin)* is a coproduct $\sum_{i \in I} G_i$ in $[\mathscr{A}, \mathbf{Cat}]$, it is the filtered colimit of the finite sub-coproducts, and hence – because it is finitely presentable – a retract of one of these. There is, however, no reason to suppose that the G_i are finitely presentable because $\sum G_i$ is so; in the category of commutative rings, non-finitely-presentable A and B have $A + B = A \otimes B = 1$ (the terminal ring, usually called 0) if $pA = qB = 0$ for different primes p and q.

Remark 5.7. For any regular cardinal α, we have the notion of a weight F of size α, the finite case above being that where $\alpha = \omega$. We just replace "finite set" by "set of cardinality less than α", and "finitely presentable" by "α-presentable" (the representable functor is now to preserve α-filtered colimits). All of the results of this section continue to hold with α in place of ω, for essentially trivial reasons; they in fact become easier for $\alpha > \omega$, since then an α-presentable category is just one with fewer than α morphisms.

6. Various counter-examples

Example 6.1. This article would be devoid of content if all limits were flexible; however Remark 4.3 shows that this is not so. Since all limits can be constructed from products, cotensor products, and equalizers, as we remarked in Section 1, and since products and cotensor products are flexible (being in (Psd) by the penultimate paragraph of Section 3), it follows that equalizers are not flexible limits. There are many other ways of seeing this. If T is the finitary 2-monad on **Cat** for which T-algebras are categories with a terminal object, the morphisms $0, 1 : 1 \to \mathbf{I}$ clearly lack an equalizer in T-Alg; yet T-Alg is (PIE)-complete by [4, Section 2], and admits the splitting of idempotents by a simple calculation, so that it admits all flexible limits by Proposition 4.8. More directly, if $\mathbf{P}$ again denotes the domain category of (1.6), and $F = \Delta 1 : \mathbf{P} \to \mathbf{Cat}$ is the weight for equalizers, we see at once that F' is $u, v : 1 \to \mathbf{S}$ where $\mathbf{S}$ is the category $u \cong w \cong v$; and q_F, which is the unique map $F' \to F = \Delta 1$, is clearly not a retraction in $[\mathbf{P}, \mathbf{Cat}]$.

We now turn to showing that none of the bases for the closed classes (Flex) and $(\mathrm{Flex}_f)^*$ given in Theorems 4.9, 4.11, and 5.3 is redundant. Doing so is simplified on recalling that $(\mathrm{Psd}, C, I_!) \subset (PIE)^*$ by Propositions 1.1 and 2.1, that $(C_f) \subset (C)^*$ and $(P_f) \subset (P)^*$ trivially, and that $S_e \in (S)^*$ by the arguments in the relevant paragraph of Section 1.

Example 6.2. We begin by showing that $S \notin (PIE)^*$. We do so, not by a direct use of Proposition 3.1, but rather by showing (since we want to refer to this in Section 7) that idempotents do not in general split in $\mathrm{Psd}[\mathscr{A}, \mathbf{Cat}]$, which admits (PIE)-limits by [4, Propositions 2.1–2.3 and Section 6.6]. We take for $\mathscr{A}$ the category presented

by two objects u, v and two arrows $\iota : u \to v$ and $\varrho : v \to u$, subject to the relation $\varrho\iota = 1_u$. Writing as usual 0 and 1 for the objects of $\mathbf{I}$, and writing $*$ for the unique object of 1, consider $F : \mathscr{A} \to \mathbf{Cat}$ where $Fu = \mathbf{I}$ and $Fv = \mathbf{I} + 1$, while $F\iota$ is the injection $i : \mathbf{I} \to \mathbf{I} + 1$ and $F\varrho$ is the functor $r : \mathbf{I} + 1 \to \mathbf{I}$ which is the identity on $\mathbf{I}$ and has $r* = 0$. Define the pseudo-natural $e : F \to F$ by $e_u = 1 : \mathbf{I} \to \mathbf{I}$, $e_v = \Delta 0 : \mathbf{I} + 1 \to \mathbf{I} + 1$ (the functor constant at 0), with e_ι the unique isomorphism $i \cdot 1 \to \Delta 0 \cdot i$ and e_ϱ the unique isomorphism $r \cdot \Delta 0 \to 1 \cdot r$; the conditions for pseudo-naturality are immediate, since all diagrams in $\mathbf{I}$ commute. Moreover e is easily seen to be idempotent. If it had a splitting $e = kt$ where $t : F \to G$ and $k : G \to F$ with $tk = 1$, we should have in particular splittings of the idempotents e_u and e_v, so that G (replaced by an isomorph if necessary) must have $G_u = \mathbf{I}$ and $G_v = 1$, with $G\iota = j : \mathbf{I} \to 1$ and $G\varrho = s : 1 \to \mathbf{I}$ satisfying $sj = 1$. This last being impossible, e does not split in $\mathrm{Psd}[\mathscr{A}, \mathbf{Cat}]$. {Note that it *does* split in $\mathrm{Hom}[\mathscr{A}, \mathbf{Cat}]$, where G need only be a pseudo-functor, since $sj \cong 1$.} It further follows, since $(PIES_e)^* = (\mathrm{Flex}) = (PIES)^*$ by Theorem 4.9, that $S_e \notin (PIE)^*$. Again, since $(\mathrm{Psd}, C, I_i) \subset (PIE)^*$ as we recalled in the last paragraph, we have that S and S_e do not belong to $(\mathrm{Psd})^*$ or to $(PI_iC)^*$. Accordingly we cannot omit S (or S_e, as the case may be) from any of the bases in the three theorems mentioned above.

Example 6.3. To show that (P), or (P_f) where appropriate, cannot be omitted from any of these bases, it suffices to show that $(P_f) \not\subset (\Pi_i ECS)^*$. To see that not even binary products are in the latter class, we apply Proposition 3.1 to $[2, \mathbf{Cat}]$, where 2 (in contrast to **2**) is the *discrete* category with two objects 0 and 1; an object of $[2, \mathbf{Cat}]$ is of course a pair (A, B) of categories. The full sub-2-category of $[2, \mathbf{Cat}]$ determined by those pairs (A, B) at least one of which is empty contains the representables and is clearly closed under $(\Pi_i ECS)$-colimits, but does not contain the weight $\Delta 1 = (1, 1)$ for binary products.

Example 6.4. We next show that $I \notin (PES)^*$, applying Proposition 3.1 to $[\mathbf{P}, \mathbf{Cat}]$ where $\mathbf{P}$ again denotes the domain category in (1.6); a general object of $[\mathbf{P}, \mathbf{Cat}]$ is $f, g : B \to C$, and I is $0, 1 : 1 \to \mathbf{2}$. The representables are $0, 1 : 1 \to 2$ and the unique pair of functors $0 \to 1$. The full sub-2-category given by those $f, g : B \to C$ with B and C discrete contains the representables and is closed under (PES)-colimits, but does not contain I.

Example 6.5. The argument that $E \notin (PIS)^*$ is a little harder. We prove it indirectly by showing that $(C_f) \not\subset (PIS)^*$; this suffices since $(C) \subset (PIE)^*$ by Proposition 1.1. More precisely, we show that $(PIS)^*$ does not contain the cotensor product $\{\mathbf{I}, -\}$; that is, that $\mathbf{I}$ does not belong to the closure of 1 in **Cat** under (PIS)-colimits. Consider the full sub-2-category $\mathscr{B}$ of **Cat** determined by those categories A in which a composite xy is never an identity unless x and y are identities; we have $\mathbf{I} \notin \mathscr{B}$, but $1 \in \mathscr{B}$ and $\mathscr{B}$ is closed in **Cat** under coproducts and the splitting of idempotents. It remains to show that it is also closed under coinserters. A little thought shows

that the coinserter $h : B \to C$ in **Cat** of $f, g : A \to B$ can be presented as follows. Let G be the graph whose objects are those of B and whose arrows are the morphisms of B along with one arrow $\lambda_a : fa \to ga$ for each object $a \in A$. Then C is the quotient of the free category ΨG on G by the congruence generated by the following 'equations' (between pairs with common domain and common codomain): (i) for composable morphisms x, y of B we are to identify the composite xy in B with the two-letter word $x \cdot y$ in ΨG; (ii) for each object $b \in B$ we are to identify the identity morphism 1_b in B with the empty word in ΨG; (iii) for each morphism $z : a \to d$ in A we are to identify the words $\lambda_d \cdot fz$ and $gz \cdot \lambda_a$ in ΨG. The functor $h : B \to C$ is the composite of the graph-morphism $B \to \Psi G$ and the quotient functor $k : \Psi G \to C$. We show that $C \in \mathscr{B}$ whenever $B \in \mathscr{B}$. Suppose that u, v are composable words in ΨG with $k(u)k(v) = 1$ in C. This is to say that the word $u \cdot v$ can be reduced to the appropriate 1_b by a finite number of expanded instances of the identifications above. This is clearly impossible if either u or v contains some λ_a as a letter, since none of the identifications changes the number of λ's in a word. So we may as well, using (i) and (ii), take u and v to be morphisms of B, with $k(uv) = k(1_b)$. Since the identifications (i) and (ii), applied to words containing only morphisms in B, do not alter the composite in B of the letters, we have $uv = 1_b$. Because $B \in \mathscr{B}$, each of u, v is an identity; so each of $k(u), k(v)$ is an identity.

Example 6.6. The last four examples show that there is no redundancy in the bases $(PIES)$ and $(PIES_e)$ of Theorem 4.9 or the bases (P_fIES) and (P_fIES_e) of Theorem 5.3. To show the same of the bases (Psd, S) and (Psd, S_e) of Theorem 4.9 it suffices, given the results of Example 6.2, to show that $(\mathrm{Psd}) \not\subset (S)^*$. In **[2, Cat]** the representables are the functors $1 \to 1$ and $0 \to 1$; the full sub-2-category consisting of these and their isomorphs is already closed under the splitting of idempotents, but does not contain $(\Delta 1)'$, which by Remark 4.3 is $1 \to Y$ where Y is the chaotic two-object category.

Example 6.7. It remains to show the non-redundancy of the basis (PI_iCS) in Theorem 4.11 and of its finite version $(P_fI_iC_fS)$ in Theorem 5.3. We saw in Examples 6.2 and 6.3 that we cannot omit S or P (or P_f in the second case). We now show that C (or C_f) cannot be omitted, by showing that $(C_f) \not\subset (PI_iS)^*$. We use Proposition 3.1 with $\mathscr{A} = 1$. The full sub-2-category of **Cat** determined by the groupoids contains the representable 1, and is clearly closed under coproducts and the splitting of idempotents; it is also closed under iso-coinserters, whose construction in **Cat** is exactly like that of coinserters in Example 6.5, except that the λ_a are now invertible; but it does not contain the finitely-presentable **2**.

Example 6.8. Finally we use Proposition 3.1 to show that $I_i \notin (PCS)^*$. We are again considering **[P, Cat]** as in Example 6.4, where the representables are given; and we recall that I_i is $0, 1 : 1 \to \mathbf{I}$. Let $\mathscr{B}$ be the full sub-2-category of **[P, Cat]** given by the objects $i, j : A \to A + A + B$ where i and j are the first and second coprojections; it

contains the representables, is clearly closed under coproducts and tensor products, and is easily seen to be closed under the splitting of idempotents; yet it does not contain I_i.

Example 6.9. Various minor independence results which complete Proposition 1.1, the discussion of idempotent-spitting, and so on, we leave to the reader - who will have no trouble in using Proposition 3.1 to show that I_i and Inv (the latter being the weight for inverters), which belong to $(IE)^*$, belong to neither $(I)^*$ nor $(E)^*$; that $I \notin (I_i)^*$; and that $S \notin (S_e)^*$.

We move on to some observations about pseudo limits, recalling from Section 1 the importance of these as providing, when they exist, particular instances of bilimits, but having better properties than the general bilimit. We include a few remarks too about *lax* limits, which are perhaps less central, but are by no means without importance: the *collages* of [23] and [5] are nothing but lax colimits.

Example 6.10. We said in Section 3 that (Psd) is not a closed class, as it does not even contain the representables. We can see this by taking $\mathscr{A} = \mathbf{2}$; the representable $\mathbf{2}(0, -) = (1 \to 1)$, by Remark 4.3, is not in (Psd).

Example 6.11. By Proposition 2.1, we can construct pseudo limits and lax limits from (PIE)-limits. None of P, I, E can be omitted here. Since cotensor products are pseudo limits, as we remarked in Section 3, and since they are also lax limits for the same reason, it suffices to show that (C_f) is contained in none of $(PI)^*$, $(PE)^*$, or $(IE)^*$. That it is not contained in $(PI)^*$, or even in $(PIS)^*$, was shown in Example 6.5. By Proposition 3.1 applied to **Cat**, $(C_f) \not\subset (PES)^*$, since the closure of 1 in **Cat** under (PES)-colimits consists of the discrete categories; this provides an alternative proof to that given in Example 6.4 that $I \notin (PES)^*$. Again, $(C_f) \not\subset (IES)^*$, since the closure of 1 in **Cat** under (IES)-colimits consists only of one-object categories - as is easily seen from the description in Example 6.5 of coinserters in **Cat**, and the observation that coequifiers are epimorphic in any 2-category and hence surjective-on-objects in **Cat**. {It is not hard to show that this last closure consists *precisely* of the finitely-presentable monoids.}

Example 6.12. We observed in Section 3 that products and cotensor products are pseudo limits. As would be expected, $(\mathrm{Psd}) \not\subset (PC)^*$; the closure of the representables in $[\mathbf{2}, \mathbf{Cat}]$ under (PC)-colimits consists of the injections $A \to A + B$, and does not contain $(\mathbf{2} \to 1)'$ which, by Remark 4.3, is a functor from $\mathbf{2}$ to the chaotic three-object category.

Example 6.13. We verify our assertion in Section 3 that inserters are not pseudo limits. Write $(\phi, \psi : u \to v)$ for the domain category $\mathbf{P}$ in (1.6); supposing $I = (0, 1 : 1 \to \mathbf{2})$ to be of the form F' for some $F = (f, g : B \to C)$ in $[\mathbf{P}, \mathbf{Cat}]$, we derive a

contradiction using (4.1). Let the (u, v, ϕ, ψ)-components of p_F be $i: B \to 1$, $j: C \to \mathbf{2}$, $\alpha: 0i \cong jf$, and $\beta: 1i \cong jg$; and let the (u, v)-components of the strict q_F be $r: 1 \to B$ and $s: \mathbf{2} \to C$. From (4.1) we have $ri = 1_B$ and $sj = 1_C$; whence B must be 1, and C either 1 or $\mathbf{2}$ (to within isomorphisms, which we can ignore). Moreover, if $C = \mathbf{2}$, the functor s must be the identity, whence $F = I$ by the strict naturality of q_F. So F is either $\Delta 1$ or I. It follows by elementary considerations that $(\Delta 1)' = (0, 1 : 1 \to X)$ where X is the chaotic category $\{0, 1, 2\}$, while I' is $(a, b : 1 \to Y)$ where Y is the category $(a \cong c \to d \cong b)$; so that in neither case is F' isomorphic to I. Of course, I being flexible, I and I' are equivalent in $[\mathbf{P}, \mathbf{Cat}]$, so that the inserter $\{I, G\}$ and the *pseudo-inserter* $\{I, G\}_{\mathrm{p}} = \{I', G\}$ are *equivalent* by the 2-functoriality of $\{-, -\}$; either one is a possible value of the *bi-inserter* $\{I, G\}_{\mathrm{b}}$; but this is not the point here, where we are emphasizing the virtues of true limits, where they exist, over mere bilimits.

Example 6.14. A wholly similar argument shows that inserters are not *lax* limits; for (4.1) still holds when we replace p_F and q_F by the unit and the counit for the adjunction $(\ )^{\dagger} \dashv KJ: [\mathscr{A}, \mathbf{Cat}] \to \mathrm{Lax}[\mathscr{A}, \mathbf{Cat}]$ (*in the language of* (2.2)), and it is easy to calculate $(\Delta 1)^{\dagger}$ and $I^{\dagger}$ and observe that neither is isomorphic to I.

Remark 6.15. As we said in Section 3, we do not know what to add to (P) and (C) to get a handy basis for (Psd), and accordingly have no easy way to decide whether $I \in (\mathrm{Psd})^*$. This is probably in fact manageable by a direct consideration of pseudo limits in $[\mathbf{P}, \mathbf{Cat}]$, which we leave to others; our guess is that $I \notin (\mathrm{Psd})^*$.

Example 6.16. Again, arguments entirely similar to those of Examples 6.13 and 6.14 show that equifiers and iso-inserters are neither pseudo limits nor lax limits. This brings up a point we should like to make: some authors, dealing with particular concrete cases of iso-inserters, have called them 'pseudo-equalizers'. Such a name is not consistent with the systematic nomenclature of this article, according to which the pseudo-equalizer of $F = (f, g : B \to C)$ is $\{\Delta 1, F\}_{\mathrm{p}} = \{(\Delta 1)', F\}$ where $\Delta 1 : \mathbf{P} \to \mathbf{Cat}$; that is, the conical pseudo limit of F. When we are looking at limits in **Cat**, an object of the iso-inserter is an object $b \in B$ together with an isomorphism $fb \cong gb$; while an object of the pseudo-equalizer is an object $b \in B$, an object $c \in C$, and isomorphisms $fb \cong c$ and $gb \cong c$. The morphisms in these two categories being the obvious ones, it is clear that the iso-inserter and the pseudo-equalizer are equivalent - but not isomorphic - categories. Going back to the weights, I_{i}, $(I_{\mathrm{i}})'$, and $(\Delta 1)'$ (which is described in Example 6.13) represent three different isomorphism-types, although all are equivalent and hence indistinguishable at the level of bilimits. A point of the same kind is that it is improper, in our nomenclature, to use the term 'pseudo-pullback' for an iso-comma-object.

Example 6.17. We end our counter-examples by observing that lax limits are not in general pseudo limits, nor *vice versa*. Consider the lax limit and the pseudo limit

of an arrow, as illustrated in (2.5). Here the domain category is **2**, and the corresponding weights $(\Delta 1)^\dagger$, $(\Delta 1)' : \mathbf{2} \to \mathbf{Cat}$ for these two limits are easily seen to be $0 : 1 \to \mathbf{2}$ and $0 : 1 \to \mathbf{I}$. By an argument very like that in Example 6.13, $(\Delta 1)^\dagger$ is not of the form F'; while by a similar argument along the lines of Example 6.14, $(\Delta 1)'$ is not of the form $F^\dagger$.

Remark 6.18. In our proof of Theorem 4.11, we had to split an idempotent that was not an equivalence. This suggests the likelihood that we cannot replace S by S_e in that theorem, or equivalently that $S \notin (PI_iCS_e)^*$. An attempt to show this using Proposition 3.1 leads, however, at first glance, to such complications that we have not considered it worth while to pursue it. It may amuse some reader to do so.

7. Final remarks and examples of flexibly-complete 2-categories

Remark 7.1. We recall from [4] that, for a 2-monad T on a 2-category $\mathscr{K}$, the 2-category T-Alg consists of the (strict) T-algebras, those morphisms that preserve the structure to within coherent isomorphisms, and the appropriate 2-cells. It was shown in [4, Propositions 2.1–2.4 and Remark 2.8] that T-Alg admits (PIE)-limits if $\mathscr{K}$ does so; by Theorem 4.9, therefore, it then admits all flexible limits if it admits the splitting of idempotents. However idempotents need not split in T-Alg even when $\mathscr{K}$ is complete; we saw in Example 6.2 that idempotents need not split in $\mathrm{Psd}[\mathscr{A}, \mathbf{Cat}]$, which by [4, Section 6.6] is T-Alg for a finitary 2-monad T on $[X, \mathbf{Cat}]$ where X is the set of objects of $\mathscr{A}$. Yet idempotents do indeed split in T-Alg for those T of the greatest importance in practice, as we now observe.

Remark 7.2. Consider for simplicity finitary 2-monads T (see [4]) on a 2-category $\mathscr{K}$ which, like **Cat** and related 2-categories, is locally finitely presentable in the sense of [12], and write $\mathscr{K}_f$ for the full sub-2-category of $\mathscr{K}$ determined by the finitely-presentables. It will be shown in [16] (in a wider context of enriched category theory) that such 2-monads T are themselves the algebras for a finitary 2-monad R on $[\mathscr{K}_f, \mathscr{K}]$, the strict morphisms of R-algebras being the usual strict morphisms of 2-monads; and that each $A \in \mathscr{K}$ gives rise to an endo-2-monad $\langle A, A\rangle$ such that strict 2-monad-morphisms $T \to \langle A, A\rangle$ correspond to (strict) actions $TA \to A$, making A into a T-algebra. It will be further shown in [17], specializing now to the case of 2-categories, that *general* morphisms of R-algebras are the usual *pseudo-morphisms* of 2-monads, as defined for example in [10, Section 4]. A pseudo-morphism $T \to \langle A, A\rangle$ corresponds to a *pseudo-action* $TA \to A$, making A into a *pseudo-T-algebra*; these constitute a 2-category Psd-T-Alg. It is easy to see that, when we take an idempotent e in T-Alg and split it in the underlying 2-category $\mathscr{K}$, the splitting object admits a pseudo-action of T, so that e splits in Psd-T-Alg. (We saw an instance of this in Example 6.2, where an idempotent that fails to split in $\mathrm{Psd}[\mathscr{A}, \mathbf{Cat}]$ does so in $\mathrm{Hom}[\mathscr{A}, \mathbf{Cat}]$; the latter 2-category is in fact Psd-T-Alg for the 2-monad

T on $[X, \mathbf{Cat}]$ of Remark 7.1 for which T-Alg $=$ Psd$[\mathscr{A}, \mathbf{Cat}]$.) Accordingly idempotents split in T-Alg if every pseudo-T-algebra is isomorphic to a strict one. Now by [4, Theorem 3.13] we have a left adjoint $(\ )'$ to the inclusion $R\text{-Alg}_s \to R\text{-Alg}$, where as usual the subscript s indicates strict morphisms; thus a pseudo-morphism $T \to \langle A, A\rangle$ corresponds to a strict morphism $T' \to \langle A, A\rangle$, and it follows that the 2-category Psd-T-Alg is isomorphic to T'-Alg. In accordance with the general definition of flexibility in [4, Theorem 4.4 and Remark 4.5], of which our Proposition 4.2 is a special case, the 2-monad T is *flexible* whenever the counit $T' \to T$ is a surjective equivalence in $R\text{-Alg}_s$. When this is so, the 2-categories T-Alg and T'-Alg are equivalent, so that every T'-algebra is indeed isomorphic to a T-algebra. Thus T-Alg admits all flexible limits whenever the 2-monad T is flexible.

Remark 7.3. It will be shown in [17] that a 2-monad T is flexible when it can be so presented by operations and equations that there are no equations between objects, only between morphisms. This cuts out *strict* monoidal categories, where we have an equation $(A \otimes B) \otimes C = A \otimes (B \otimes C)$; but includes (for suitable $\mathscr{K}$ – see [4, Section 6] and [17]) monoidal categories, symmetric monoidal categories, symmetric monoidal closed categories, categories with finite limits or finite colimits or both, Cartesian closed categories, locally-Cartesian-closed categories, elementary toposes, elementary toposes with a natural-numbers object, and so on.

Remark 7.4. Since every 2-monad of the form T' is trivially flexible, it follows from Remark 7.2 that Hom$[\mathscr{A}, \mathbf{Cat}]$ admits all flexible limits. So the 2-category $\mathbf{Fib}/\mathscr{B}$ of fibrations over $\mathscr{B}$, which is equivalent to Hom$[\mathscr{B}^{\mathrm{op}}, \mathbf{Cat}]$, also admits them. It is true that, classically, $\mathbf{Fib}/\mathscr{B}$ is defined only when $\mathscr{B}$ is a mere category; but it is easy so to define it when $\mathscr{B}$ is a 2-category that we still have the equivalence above: see Hakim [8].

Remark 7.5. This list of algebra-categories with flexible limits will be considerably extended when, in later articles such as [14], we move on from the monadic case to that of algebras – such as pretoposes or regular categories – defined by a two-dimensional projective sketch involving only finite flexible limits.

Remark 7.6. There is another way, related to the notion of transport of structure along an equivalence, of inferring in many cases that idempotents split in a 2-category $\mathscr{B}$. Suppose that $\mathscr{B}$ is a sub-2-category, not necessarily full, of a 2-category $\mathscr{C}$ in which idempotents – or at least idempotent equivalences – split; suppose that every object of $\mathscr{C}$ equivalent to one in $\mathscr{B}$ itself lies in $\mathscr{B}$, that every equivalence in $\mathscr{C}$ between objects of $\mathscr{B}$ is an arrow in $\mathscr{B}$, and that every invertible 2-cell in $\mathscr{C}$ between equivalences in $\mathscr{B}$ is a 2-cell in $\mathscr{B}$. Then, when an idempotent equivalence e in $\mathscr{B}$ splits in $\mathscr{C}$ as $e = ir$ with $ri = 1$, both i and r are equivalences since $ir = e \cong 1$, so that in fact e splits in $\mathscr{B}$. Accordingly, if $\mathscr{B}$ admits (PIE)-limits, or even pseudo limits, it admits all flexible limits by Theorem 4.9. (Note that our desire to

use this argument was the reason for our including S_e, and not merely S, in that theorem.) This argument can be used, in place of the flexibility of T, to prove the existence of flexible limits in those examples of T-Alg in Remark 7.3 where T-algebras are just categories with certain *properties* – that is, all of the examples after symmetric monoidal closed categories; the 2-category $\mathscr{C}$ here is of course **Cat**.

Remark 7.7. The objects of the flexibly-complete 2-categories given as examples in Remark 7.3 are, at least in the first instance, *small* categories with an algebraic structure of some kind; since they are T-algebras for a 2-monad T either on the 2-category **Cat** of small categories or else (more commonly) on the 2-category $\mathbf{Cat}_g$ of small categories, functors, and natural *isomorphisms* – see [4, Section 6.2]. The smallness here is ultimately irrelevant; we can always replace **Cat** by the **CAT** of some larger universe. We should, however, like to know that the 2-category of *locally-small* T-algebras is closed under flexible limits; this is indeed so, since the same is true of the base 2-category $\mathscr{K}$, and since, by [4, Section 2], the underlying objects of the limits in question are formed as in $\mathscr{K}$.

Remark 7.8. The situation is quite other in such cases as the 2-category **Ladj** of locally-presentable categories, left-adjoint functors, and natural transformations, where the objects are never small categories unless they are trivial (in the case of **Ladj**, complete lattices). In his unpublished thesis [2], Bird adapts unpublished arguments of Ulmer [25] to show that **Ladj** is flexibly complete, with flexible limits formed as in **CAT**; Bird further shows, by arguments of a different kind, that the same is true of $\mathbf{Ladj}^{op}$, whose morphisms can be seen as right-adjoint functors, so that it too is a sub-2-category of **CAT**; and that it is even true, for instance, of the sub-2-category of $\mathbf{Ladj}^{op}$ formed by the locally-finitely-presentable categories and the *finitary* right-adjoint functors between them – that is, those that preserve filtered colimits. The hard part of the proofs here is not, of course, showing that the limit in **CAT** is cocomplete, but in producing the right kind of dense subcategory to prove it locally presentable; this would seem to be typical of the situation where the objects are large categories, having various limits and colimits and certain 'exactness' relations between these, but further required to have small subcategories which generate in some sense or are dense. It accordingly follows that the dual $\mathbf{GrTop}^{op}$ of the 2-category of Grothendieck toposes and geometric morphisms is flexibly complete; for the limit in **CAT** has the required 'exactness', while it has a generator as a special case of Bird's results on **Ladj**. This is a stronger result than that given in [18, Theorem 5.2.2], and attributed there originally to Moerdijk, that $\mathbf{GrTop}^{op}$ has all bilimits. Again, the proof of [18, Theorem 5.1.4], asserting that the 2-category **Acc** of accessible categories and accessible functors has all bilimits, really shows (as the authors point out, although their language differs from ours) that it has all pseudo limits; since idempotent equivalences split in it by Remark 7.6, **Acc** is in fact flexibly complete.

Remark 7.9. In the article [9] of Hyland and Moerdijk, the elegance of the functoriality results and the simplicity of the proofs in their Section 2 is a testimony to the value of having and using - in this case in $\mathbf{GrTop}^{\mathrm{op}}$ - flexible limits rather than mere bilimits, as suggested to them by Power after analysis of their first draft.

Remark 7.10. The central results of the article [4] on two-dimensional monad theory will be extended in [14] to essentially-algebraic theories in the following sense: a *theory* is a small 2-category (resp. category enriched over the closed category **Gpd** of groupoids) $\mathscr{T}$, possessing all finite flexible limits of some class Φ; a *model* is a functor $\mathscr{T} \to \mathbf{Cat}$ (resp. $\mathscr{T} \to \mathbf{Gpd}$) preserving Φ-limits; a *morphism of models* is a pseudo-natural transformation; and a *strict morphism* is a 2-natural transformation. The flexibility of the limits here is essential to the arguments. One could of course study the analogue of this with bicategories and finite bilimits; but our approach is simpler and, as in [4], yields finer results (such as the existence of flexible limits rather than mere bilimits in the 2-category of models). That the results at the bicategorical level will indeed follow from ours is a consequence of two forthcoming articles of Power: [19], in which it is shown that any [small] bicategory with finite limits is biequivalent to a [small] 2-category with finite flexible limits; and [20], in which it is shown that, for a theory $\mathscr{T}$ as above with all finite flexible limits, there is a biequivalent theory $\mathscr{S}$ with these limits such that the 2-category of finite-bilimit-preserving functors $\mathscr{T} \to \mathbf{Cat}$ is biequivalent with the 2-category of finite-flexible-limit-preserving functors $\mathscr{S} \to \mathbf{Cat}$.

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