DOCTRINAL ADJUNCTION

by

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This paper deals with adjunctions $\eta, \varepsilon\colon f \dashv u\colon A \to A'$ where A, A' are categories with structure; its purpose is to unify and simplify various isolated observations in the literature, at the same time extending them widely.

(a) First, and trivially, there is the classical result that a left adjoint f preserves colimits.

(b) If A and A' are monoidal categories, perhaps closed, and if $f \dashv u$ is merely an adjunction in *CAT*, it is not in general the case that f "preserves tensor products" in the sense that $f(X \otimes' Y) \cong fX \otimes fY$ and $fI' \cong I$. For instance, let $A = A' = \mathcal{S}et$, and let $f(X) = X + X$, so that $u(A) = A \times A$. Yet it is striking how often in natural examples f <u>does</u> preserve tensor products: a whole class of cartesian closed examples is provided by geometric morphisms of topoi, but there are innumerable others, of which perhaps the best known is that where u is the forgetful functor from $(\mathcal{A}b, \otimes)$ to $(\mathcal{S}et, \times)$.

One can dig out from the proof of an old theorem of mine ([3] §§5.1 and 5.2), which was not itself directed to this question, a <u>sufficient</u> condition for f to preserve tensor products; namely that A and A' be symmetric monoidal closed, that the adjunction $f \dashv u$ lie not merely in *CAT* but in *Sym Mon CAT*, and finally that u be <u>normal</u> (i.e. commute with the canonical symmetric monoidal functors $A \to \mathcal{S}et$ and $A' \to \mathcal{S}et$).

(c) Two theorems of Street deal with adjunctions for categories with structure: Theorem 9 of [8] asserts that the left adjoint of a

monad-functor is an op-monad-functor, and Theorem 1 of [9] asserts that the left adjoint of a lax natural transformation is an op-lax natural transformation.

(d) Let V be a symmetric monoidal closed category and let T be a monoidal monad (= commutative monad) on V. Then provided V admits equalizers, the category V^T of T-algebras is closed, in the original "internal-hom" sense of [2] (cf. Kock [7]); it has been shown to have a tensor product, making it monoidal closed, only when it is cocomplete; and cocompleteness has only been demonstrated under highly restrictive hypotheses on V and T. In this context Wolff [11] has looked at certain adjunctions $V^T \to V^{T'}$ (arising in fact from a distributive law), and shown them by direct calculations to be closed adjunctions. Perhaps the right approach here is to change universes, recovering the missing tensor product, and then to use the easier "monoidal" methods; but it might also be of value to have a simple criterion, directly in terms of internal-hom, for an adjunction to be closed.

(e) Finally there is Day's result [1] giving sufficient conditions for a full reflective subcategory A of a biclosed monoidal category A' to admit itself a biclosed monoidal structure, in such a way that the adjunction $f \dashv u$, where u is now the inclusion, becomes a monoidal adjunction.

It turns out that there are some simple general results that illuminate all of the above situations. We start in §1 with a doctrine D, which could be on any 2-category at all, but which we take to be a doctrine on CAT, purely because the nomenclature is there more vivid. (For what we need here about doctrines we refer to [6] above in this volume, especially §3.5.) We suppose that A and A' are D-categories, and that we are given an adjunction $\eta,\varepsilon\colon\ f \dashv u\colon\ A \to A'$ in CAT. Our first result is that there is a bijection between enrichments of u to a D-functor $(u,\bar{u})$ and enrichments of f to an op-D-functor $(f,\bar{f}')$.

Our second result is that if the adjunction $f \dashv u$ admits enrichment to an adjunction $(f,\bar{f}) \dashv (u,\bar{u})$ in D-*CAT*, then $\bar{f}$ is the inverse of the above $\bar{f}'$, so that $(f,\bar{f})$ and $(f,\bar{f}')$ are <u>strong</u>; moreover that if the enrichment $\bar{u}$ is given, $(u,\bar{u})$ has a left adjoint in D-*CAT* precisely when $\bar{f}'$ is an isomorphism, the left adjoint then being $(f,\bar{f})$ where $\bar{f}$ and $\bar{f}'$ are inverse; and finally that if the enrichment $\bar{f}$ is given, $(f,\bar{f})$ has a right adjoint in D-*CAT* precisely when it is strong, the right adjoint being $(u,\bar{u})$ where $\bar{u}$ corresponds by our first result to the inverse $\bar{f}'$ of $\bar{f}$.

The first result encompasses the two theorems of Street in (c) above - for the second of these the relevant 2-category is not *CAT* but a suitable *CAT*/Λ. The second result gives a proper answer to the question raised in (b) above: to say that f preserves tensor products is to say that it is strong, so that this happens precisely when the adjunction is one in *Mon CAT*, which can also be expressed as a condition on $\bar{u}$, here represented by its generators $\tilde{u}$: $uA \otimes' uB \to u(A \otimes B)$ and u°: $I' \to uI$ (cf. §10.8 of [4] above in this volume). This shows how wide of the mark the sufficient condition given in (b) above really is: that A and A' should be closed, or even symmetric, is totally irrelevant; as for the normality of u, we shall see that it is a <u>consequence</u> of the adjunction's lying in *Mon CAT*, and not an independent condition at all. The observation (a) that left adjoints preserve colimits can also be seen as a trivial case of this second result, by taking D as the doctrine whose algebras are categories-with-colimits: <u>any</u> functor gives a comparison of colimits, and is hence canonically a D-functor, so that the adjunction is <u>always</u> in D-*CAT* and f is <u>always</u> strong, which here means colimit-preserving.

In §2 we look at the case where D is the doctrine whose algebras are monoidal categories, but supposing A,A' to be actually <u>monoidal closed</u>. Then the giving of $\tilde{u}$: $uA \otimes' uB \to u(A \otimes B)$ is equivalent to the giving of a certain $\hat{u}$: $u[A,B] \to [uA,uB]'$ where the square brackets

are the internal-homs; and the conditions on $\tilde{u}$, u^o for the adjunction to be monoidal translate into conditions on $\hat{u}$ and u^o. That on u^o is equivalent to the normality of u (whether the categories are closed or not); that on $\hat{u}$ is independent (showing again how mixed-up is the "sufficient condition" given in (b) above). Similarly the conditions on $\tilde{f}$, f^o (namely that they be isomorphisms) translate into conditions on $\hat{f}$ and f^o. Now, however, that the conditions are expressed entirely on the internal-hom level, we can ask whether they are still the conditions for the adjunction to be a closed one, even in the absence of the tensor products, so as to provide the "simple criterion" desired in (d) above. (Recall that for monoidal closed categories, monoidal functor = monoidal closed functor = closed functor, and similarly for natural transformations.) The answer is that they are indeed still the precise conditions. I suppose an ideal proof would be in terms of pro-D-structures; but the theory of these has not been developed, and in view of the marginal interest of non-monoidal closed categories I merely give an independent ad hoc proof and leave it at that - the role here of the general results of §1 is merely to suggest the right conditions.

Finally in §3 we return to the case of a general doctrine D and turn to the analogue of Day's result in (e) above. So we now suppose A to be a full reflective subcategory of A', or equivalently we suppose the counit ε: $fu \Rightarrow 1$ to be the identity. Now only A' is given as a D-category; we seek conditions under which it is possible to give a D-structure to A too, at the same time enriching the adjunction to one in D-CAT. Our results in §1 show that f must be strong; this at once shows that the enrichment is essentially unique if it exists, and gives a necessary condition; we show that the condition is roughly sufficient. By "roughly" I mean that we may have to make do with a pseudo-D-structure on A (axioms satisfied only to within isomorphism). We can get an honest D-structure in two cases: for any D, if we actually

ask f to be strict (accepting the correspondingly stronger necessary condition); and for certain "flexible" D, where a pseudo-D-algebra is "the same thing" as an honest one. The D for monoidal categories is "flexible"; that for strict monoidal categories is not. In the monoidal case, then, our condition is necessary and sufficient, and it reduces to Day's. This shows two things about Day's result. First, biclosedness is irrelevant; the result is essentially a "monoidal" one; we just get as a bonus that A is closed or biclosed if A' is. Secondly, since u must be normal by our §2, the restriction to normal u, which Day imposes upon himself in the first paragraph of the introduction of [1], is in fact no restriction at all.

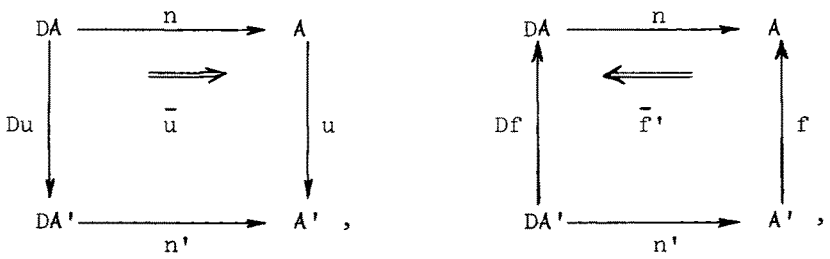
1. The main results

1.1 We refer heavily to [6] above in this volume, both for its results on adjunction and for all terminology not otherwise explained.

We suppose given a doctrine D = (D,m,j) on some 2-category. Everything we say is quite independent of the 2-category in question, and so, merely for notational simplicity, we write as if the 2-category were CAT. We suppose that A and A' are D-categories, with actions $n: DA \to A$ and $n': DA' \to A'$. We further suppose given an adjunction $\eta,\varepsilon: f \dashv u: A \to A'$ in CAT.

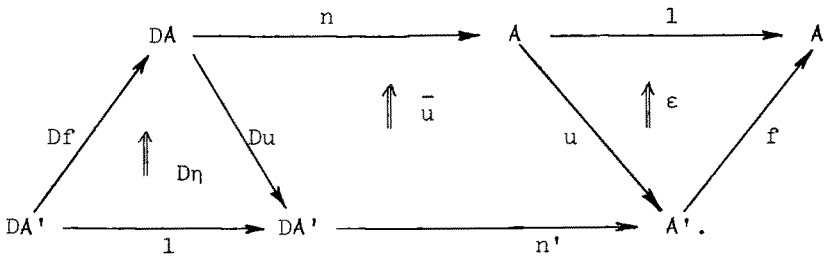
We consider pairs of natural transformations $\bar{u},\bar{f}'$ as in

(1.1)



which are mates ([6] §2.2) under the adjunctions $Df \dashv Du$ and $f \dashv u$. We record from [6] Proposition 2.1 the value of $\bar{f}'$ in terms of $\bar{u}$, to wit

(1.2)



Denote by I, II the axioms for $(u,\bar{u})$ to be a D-functor, namely (3.17) and (3.18) of [6] with f replaced by u. Denote by I', II' the corresponding axioms for $(f,\bar{f}')$ to be an op-D-functor — the same (3.17) and (3.18) of [6] with $\bar{f}$ reversed in sense and re-named $\bar{f}'$, and with A,A' interchanged.

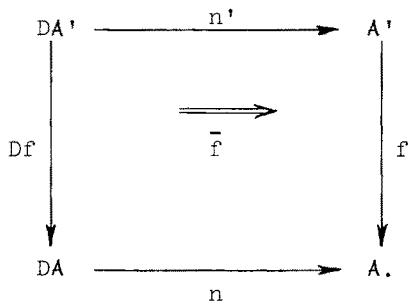
Lemma 1.1 Axiom I is equivalent to I' and II to II'.

Proof. Proposition 2.5 of [6] shows that the mates of the left squares of the left sides of I and II are identities; Proposition 2.4 of [6] shows that the mate of the left square of the right side of I is $D\bar{f}'$; Proposition 2.2 of [6] completes the proof. □

Theorem 1.2 There is a bijection between enrichments of u to a D-functor $U = (u,\bar{u})$: $A \to A'$ and enrichments of f to an op-D-functor $F' = (f,\bar{f}')$: $A' \to A$. This bijection is given by taking $\bar{u}$ and $\bar{f}'$ to be mates under the adjunctions $Df \dashv Du$ and $f \dashv u$. □

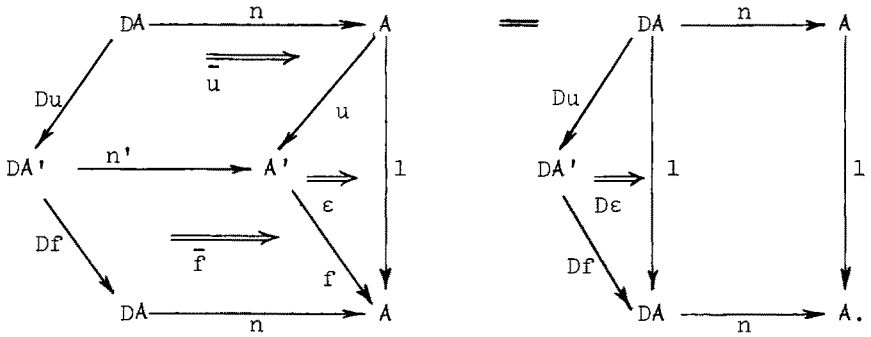
1.2 Now consider a further natural transformation $\bar{f}$ as in

(1.3)

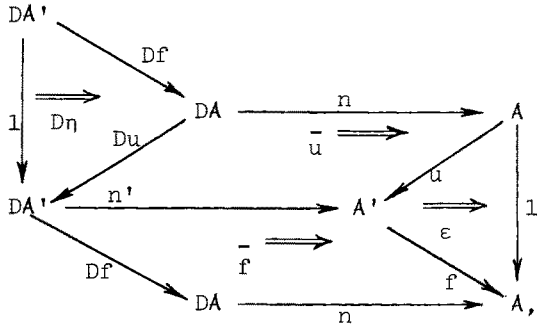


Let us call I", II" the axioms for $(f,\bar{f})$ to be a D-functor F. If these are satisfied, the condition for ε: $fu \Rightarrow 1$ to be a D-natural transformation ε: $FU \Rightarrow 1$ is, by (3.19) of [6],

(1.4)

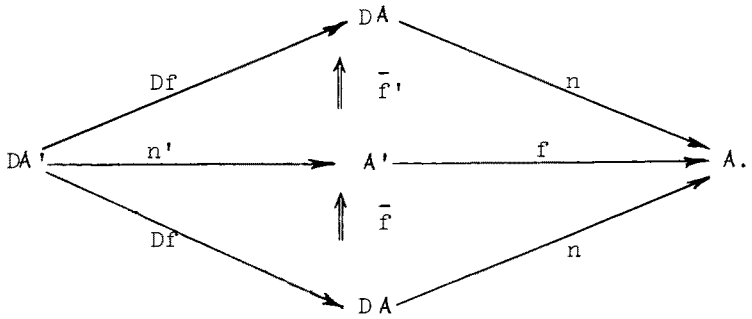


We get an equivalent condition by pasting $D\eta$ to each side along the edge Du, which is an invertible process by (2.1) and (2.2) of [6]: it comes to the same thing to say that we pass to mates under the adjunctions $Df \dashv Du$ and $1 \dashv 1$. The mate of the right side of (1.4) is then the identity natural transformation of n.Df; the mate of the left side is



which in view of (1.2) above can be rewritten as

(1.5)



We conclude that, if $F = (f,\bar{f})$ is a D-functor, then ε is D-natural if and only if $\bar{f}'.\bar{f} = 1$, the identity of n.Df. An exactly similar argument shows that $\eta\colon 1 \Rightarrow UF$ is D-natural if and only if $\bar{f}.\bar{f}' = 1$. Since I", II" are equivalent to I',II' when $\bar{f}$ and $\bar{f}'$ are inverse, we have:

Proposition 1.3 Given $\bar{f}$ and $\bar{f}'$, and hence the mate $\bar{u}$ of the latter, η and ε constitute an adjunction in D-CAT between D-functors $U = (u,\bar{u})$ and $F = (f,\bar{f})$ if and only if

(i) $(u,\bar{u})$ is a D-functor;

(ii) $(f,\bar{f})$ is a D-functor; and

(iii) $\bar{f}$ and $\bar{f}'$ are mutually inverse.

Moreover (i) is equivalent to

(i') $(f,\bar{f}')$ is an op-D-functor;

and (i') and (ii) are equivalent in the presence of (iii). □

Immediate consequences are:

Theorem 1.4 In order that a D-functor $U = (u,\bar{u})\colon A \to A'$ have a left adjoint $F = (f,\bar{f})$ in D-CAT, it is necessary and sufficient that u have the left adjoint f in CAT, and that the $\bar{f}'$ given by (1.2) be an isomorphism. Then $\bar{f} = \bar{f}'^{-1}$, and F is necessarily strong. □

Theorem 1.5 In order that a D-functor $F = (f,\bar{f})\colon A' \to A$ have a right adjoint $U = (u,\bar{u})$ in D-CAT, it is necessary and sufficient that f have the right adjoint u in CAT, and that F be strong. Then $\bar{u}$ is the mate of $\bar{f}' = \bar{f}^{-1}$, in the sense of (1.1). □

1.3 We leave the reader to formulate the obvious dual theorems obtained by replacing the doctrine D by the opposite doctrine D*, where $D^*A = (DA^{op})^{op}$; he will get in this way theorems about adjunction in the 2-category of D-categories and op-D-functors; here it is the right adjoint that must be strong; and so on.

2. The monoidal and closed cases

2.1 Now let D be either the doctrine whose algebras are monoidal categories, or else that whose algebras are symmetric monoidal categories. These doctrines arise from clubs in $CAT/\underline{P}$, and it follows from §10.8 of [4] above in this volume that a D-functor is the same thing as a monoidal functor or a symmetric monoidal functor, as the case may be. In both cases $\bar{u}$ is determined by its components $\tilde{u}$: $uA \otimes' uB \to u(A\otimes B)$ and u^o: $I' \to uI$, subject to the usual axioms ([2] page 473 in the monoidal case, with the extra axiom of [2] page 513 in the symmetric monoidal case). It further follows from §10.8 of [4] that $\bar{u}$ is an isomorphism precisely when $\tilde{u},u^o$ are isomorphisms. Similarly a D-natural transformation is, in both cases, a monoidal natural transformation.

Theorems 1.4 and 1.5 therefore apply to an adjunction $(f,\tilde{f},f^o)\dashv(u,\tilde{u},u^o)$ either in *Mon CAT* or in *Sym Mon CAT*, via the intermediary of an op-monoidal functor $(f,\tilde{f}',f^{o\prime})$. The condition in Theorem 1.4 is now that $\tilde{f}'$ and $f^{o\prime}$ be isomorphisms; that in Theorem 1.5 is now that $\tilde{f}$ and f^o be isomorphisms. There is no difference between the non-symmetric and symmetric cases; it is just automatic that $\tilde{u}$ satisfies the extra axiom in the symmetric case if and only if $\tilde{f}$ or $\tilde{f}'$ does. We get the explicit expressions for $\tilde{f}'$: $f(X\otimes'Y) \to fX \otimes fY$ and for $f^{o\prime}$: $fI' \to I$ in terms of $\tilde{u}$ and u^o from (1.2), giving D its value in terms of the appropriate club. We find for $f^{o\prime}$ and $\tilde{f}'$ respectively the composites

$$(2.1)\qquad fI' \xrightarrow[fu^o]{} fuI \xrightarrow[\varepsilon]{} I;$$

$$(2.2)\quad f(X\otimes'Y) \xrightarrow[f(\eta\otimes\eta)]{} f(ufX\otimes'ufY) \xrightarrow[f\tilde{u}]{} fu(fX\otimes fY) \xrightarrow[\varepsilon]{} fX\otimes fY.$$

Given a monoidal functor $U = (u,\tilde{u},u^o)$: $A \to A'$, write V for $A(I,-)$: $A \to Set$ and V' for $A'(I',-)$: $A' \to Set$ with their canonical monoidal-functor structures ([2] page 504). There is a canonical monoidal natural transformation $V \Rightarrow V'U$ with A-component

$$A(I,A) \xrightarrow[u_{IA}]{} A'(uI,uA) \xrightarrow[A'(u^o,1)]{} A'(I',uA),$$

as shown in [2] page 510; we call U <u>normal</u> if this is a natural <u>isomorphism</u> $V \cong V'U$ (we do not require it, as we did in [2], to be actually an <u>equality</u>). If we use the adjunction $f \dashv u$ we find at once:

<u>Proposition 2.1</u> U <u>is normal precisely when</u> (2.1) <u>is an isomorphism. In particular normality of</u> U <u>is a necessary condition for the existence of an adjunction</u> $F \dashv U$ <u>in</u> *Mon CAT*. □

One can similarly transform the condition that (2.2) be an isomorphism, but I see no special meaning in the thus-transformed condition, and omit it.

<u>2.2</u> Now suppose that the monoidal categories A and A' are <u>closed</u>, so that $-\otimes B$ and $-\otimes' Y$ have right adjoints $[B,-]$ and $[Y,-]'$. Then instead of giving $\tilde{u}$: $UA \otimes' uB \to u(A\otimes B)$ we may equally well give its mate under the adjunctions $-\otimes B \dashv [B,-]$ and $-\otimes' uB \dashv [uB,-]'$; which is a natural transformation

$$(2.3) \qquad \hat{u}\colon\ u[B,C] \longrightarrow [uB,uC]'.$$

It is shown on page 487 of [2] that the monoidal-functor axioms for $\tilde{u},u^o$ on page 473 of [2] translate into the closed-functor axioms for $\hat{u},u^o$ on page 434 of [2]; and similarly for monoidal natural transformations and closed natural transformations.

We leave the reader to verify that the condition that (2.2) be an isomorphism translates into the condition that

$$(2.4) \qquad u[fY,C] \xrightarrow[\hat{u}]{} [ufY,uC]' \xrightarrow[[\eta,1]]{} [Y,uC]'$$

be an isomorphism. Moreover, if $\tilde{f}$ is given, and corresponds to $\hat{f}$, then the condition that $\tilde{f}$ be an isomorphism translates into the condition that

$$(2.5)\qquad [Y,uC]' \xrightarrow[\eta]{} uf[Y,uC]' \xrightarrow[u\hat{f}]{} u[fY,fuC] \xrightarrow[u[1,\varepsilon]]{} u[fY,C]$$

be an isomorphism. So $(u,\hat{u},u^{\circ})$ has a left adjoint when u does if and only if (2.1) and (2.4) are isomorphisms; and $(f,\hat{f},f^{\circ})$ has a right adjoint when f does if and only if f° and (2.5) are isomorphisms.

We omit the proofs of these reductions precisely because we are now going to give direct proofs of the last two statements that apply even to non-monoidal closed categories. Our motive for doing this was given in the introduction. As the reader will see, the proofs that follow largely parallel those of §1, and would doubtless be best seen in the context of pro-D-structures, if the theory of these had been worked out.

2.3 Our definition of (non-monoidal) closed category is an inessential modification of that of [2]; a category A, a functor $[\ ,\ ]: A^{op} \times A \to A$, an object I of A, and natural transformations $L: [B,C] \to [[A,B],[A,C]]$, $j: I \to [A,A]$, $i: A \cong [I,A]$, this last being an isomorphism; subjected to the axioms CC1 - CC4 of [2] page 429, together with the axiom that the $A(A,B) \to A(I,[A,B])$ induced by j be an isomorphism.

We suppose given as before an adjunction $\eta,\varepsilon: f \dashv u: A \to A'$ in CAT, but now we suppose A and A' to be closed, and not necessarily monoidal. It is still the case that there is a bijection between morphisms $u^{\circ}: I' \to uI$ and morphisms $f^{\circ}{}': fI' \to I$, where $f^{\circ}{}'$ is given in terms of u° by (2.1): this is of course immediate by the adjunction $f \dashv u$. But in place of the bijection between natural transformations $\tilde{u}$ and natural transformations $\tilde{f}'$ that we had in the monoidal case, we get something more complicated. Precisely, natural transformations $\hat{u}$ as in (2.3) are in bijection with natural transformations

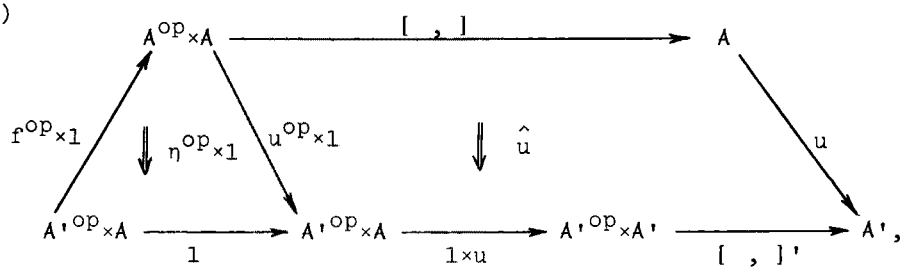
$$u_1 : \; u[fY,C] \longrightarrow [Y,uC]',$$

u_1 being given in terms of $\hat{u}$ as the composite (2.4); and natural transformations $\hat{f}$: $f[Y,Z]' \to [fY,fZ]$ are in bijection with natural transformations

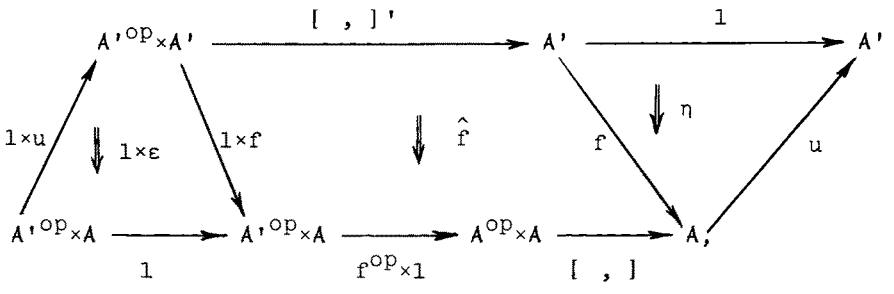
$$f_1 : \; [Y,uC]' \longrightarrow u[fY,C],$$

f_1 being given in terms of $\hat{f}$ as the composite (2.5). To see that $\hat{u} \mapsto u_1$ and $\hat{f} \mapsto f_1$ are indeed bijections, we have only to write (2.4) and (2.5) in 2-dimensional form as, respectively,

(2.6)



(2.7)



exhibiting u_1 as the mate of $\hat{u}$ under certain adjunctions, and f_1 as the mate of $\hat{f}$ under others, in the sense of [6] §2.2.

A closed functor is of course said to have an adjoint if it has one in the 2-category of closed categories, closed functors, and closed natural transformations. The main result of this section, suggested by and extending the assertions of §2.2 above, is:

Theorem 2.2 A closed functor $U = (u,\hat{u},u^o) : A \to A'$ has a left adjoint $F = (f,\hat{f},f^o)$ if and only if u has the left adjoint f and moreover $f^o{}'$ and u_1, given by (2.1) and (2.4) respectively, are

isomorphisms. Then f^o is the inverse of $f^{o\prime}$ and $\hat{f}$ corresponds by (2.5) to $f_1 = u_1^{-1}$.

A closed functor $F = (f,\hat{f},f^o)\colon A' \to A$ has a right adjoint $U = (u,\hat{u},u^o)$ if and only if f has the right adjoint u and moreover f^o and f_1, the latter given by (2.5), are isomorphisms. Then u^o corresponds by (2.1) to $f^{o\prime} = f^{o-1}$, and $\hat{u}$ corresponds by (2.4) to $u_1 = f_1^{-1}$.

Proof Take all the data $\hat{u},u^o,\hat{f},f^o$ as given, with the mates $u_1,f^{o\prime},f_1$ of the first three, without supposing yet that the closed-functor axioms CF1-CF3 of [2] page 434 are necessarily satisfied by U or by F. This last in no way prevents us from mechanically forming the composites $FU = \Phi = (\phi,\hat{\phi},\phi^o)$ and $UF = \Psi = (\psi,\hat{\psi},\psi^o)$ as on [2] page 434.

Our first observation is that $\eta\colon 1 \Rightarrow \Psi$ and $\varepsilon\colon \Phi \Rightarrow 1$ satisfy the axioms CN1 and CN2 for closed natural transformations, on page 441 of [2], if and only if f^o is inverse to $f^{o\prime}$ and f_1 to u_1. In fact ε satisfies CN2 if and only if $f_1u_1 = 1$; η satisfies CN2 if and only if $u_1f_1 = 1$; ε satisfies CN1 if and only if $f^{o\prime}f^o = 1$; and η satisfies CN1 if and only if $f^of^{o\prime} = 1$.

We prove only the first of these assertions, leaving the others to the reader. Express $f_1u_1 = 1$ by pasting (2.6) on top of (2.7). In the resulting diagram the 2-cells $\hat{f}$ and $\hat{u}$ have a common edge, and the result of pasting them along this edge is just $\hat{\phi}$. The other three 2-cells are the triangles $\eta^{op} \times 1$, η, $1 \times \varepsilon$. Get rid of the first two of these by pasting on an $\varepsilon^{op} \times 1$ and an ε to cancel them out, as in (2.1) and (2.2) of [6]; the expression of $f_1u_1 = 1$ is now that $\hat{\phi}$ with a $1 \times \varepsilon$ pasted on is equal to the identity of $u[\ ,\ ](f^{op} \times 1)$ with an $\varepsilon^{op} \times 1$ and an ε pasted on in suitable places. But this is just CN2 for ε in 2-dimensional form.

This establishes the "only if" part of both the assertions of the theorem. For the "if" part of the first assertion, we use the prescription of the theorem to define $\hat{f}$ and f^o; and it remains to show that $F = (f,\hat{f},f^o)$ satisfies the closed-functor axioms; similarly for the "if" part of the second assertion.

We shall show here that F (resp. U) satisfies the axiom CF3 of [2] page 434 when U (resp. F) does, and when the hypothesis is verified that u_1 and f_1 are inverse; we leave the easier axioms CF1, CF2 to the reader. Write the two legs of CF3, say for Ψ, as

$$p_1,p_2:\ \psi[Y,Z] \longrightarrow [\psi[X,Y],[\psi X,\psi Z]];$$

we are now for simplicity writing $[Y,Z]$ for $[Y,Z]'$. We similarly use p_1,p_2 for the two legs of CF3 for U or for F.

We do not, a priori, know CF3 for Ψ; but we do know that $\eta: 1 \Rightarrow \Psi$ satisfies CN2, and by an easy diagram-filling-in argument we get that

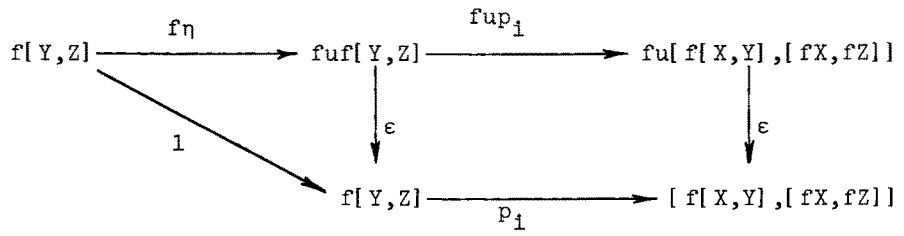
$$(2.8)\quad [Y,Z] \xrightarrow[\eta]{} \psi[Y,Z] \xrightarrow[p_i]{} [\psi[X,Y],\ [\psi X,\psi Z]]$$

is independent of i. Now give ψ its value uf, and express the p_i for Ψ in terms of those for U and for F, as is done in the big diagram on page 435 of [2], which we shall call Δ.

First suppose we know CF3 for U. Then (2.8) and Δ give that

$$(2.9)\qquad \begin{array}{ccccc} [Y,Z] & \xrightarrow{\eta} & uf[Y,Z] & \xrightarrow{up_i} & u[f[X,Y],[fX,fZ]] \\ & & & & \hat{u}\downarrow \\ & & & & [uf[X,Y],\ u[fX,fZ]] \xrightarrow[[1,\hat{u}]]{} [uf[X,Y],[ufX,ufZ]] \end{array}$$

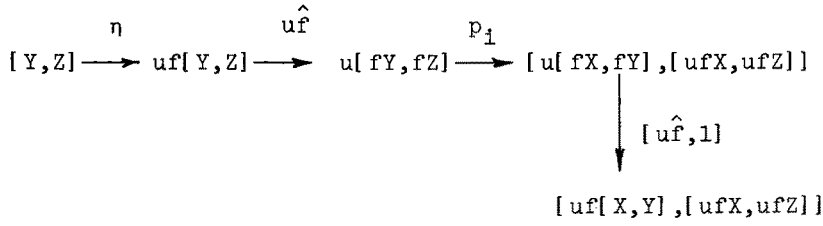
is independent of i. But since (2.4) is the isomorphism u_1, the morphisms $\hat{u}$ and $[1,\hat{u}]$ in (2.9) are both coretractions and can be cancelled: the composite of the first two morphisms in (2.9) is already independent of i. This implies that the top leg of the diagram



is independent of i. But the square commutes by naturality, and the triangle by one of the adjunction-equations; hence p_i is independent of i, which is CF3 for F.

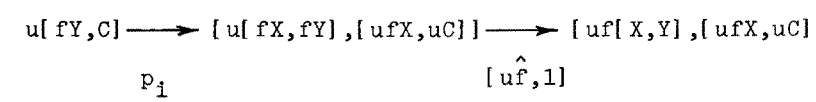
Now suppose instead that we know CF3 for F and seek it for U. We conclude from (2.8) and Δ that

(2.10)



is independent of i. Write this for Z = uC, compose on the end with $[1,[1,u\varepsilon]] : [uf[X,Y], [ufX,ufuC]] \to [uf[X,Y],[ufX,uC]]$, and use naturality to move this last morphism back through $[\hat{uf},1]$ and p_i; in the process it becomes $u[1,\varepsilon] : u[fY,fuC] \to u[fY,C]$; so that the first three morphisms in the thus-transformed composite (2.10) form the isomorphism f_1 of (2.5) and can be removed. What we now have is that

(2.11)



is independent of i. Now set Y = uB, compose with $u[\varepsilon,1] : u[B,C] \to u[fuB,C]$ on the front, and with $[\eta,1] : [uf[X,uB],[ufX,uC]] \to [[X,uB],[ufX,uC]]$ on the end. Using naturality to move the first morphism through the second, and observing that the last three morphisms then give an instance of the isomorphism $[f_1,1]$, we have that

(2.12) $u[B,C] \xrightarrow[p_i]{} [u[fX,B],[ufX,uC]]$

is independent of i. The "extraordinary" naturality of p_i in A gives the commutativity of

$$\begin{array}{ccc} u[B,C] & \xrightarrow{p_i} & [u[fuA,B],[ufuA,uC]] \\ \downarrow{\scriptstyle p_i} & & \downarrow{\scriptstyle [u[\varepsilon,1],1]} \\ [u[A,B],[uA,uC]] & \xrightarrow[[1,[u\varepsilon,1]]]{} & [u[A,B],[ufuA,uC]] \end{array} ;$$

since by (2.12) the top leg is independent of i, so is the bottom leg; but $[1,[u\varepsilon,1]]$ is a coretraction and hence cancellable, $u\varepsilon$ being a retraction by one of the adjunction equations. Thus the p_i on the left is independent of i, which is CF 3 for U. □

3. Reflective subcategories

3.1 We return now to the case of an arbitrary doctrine D, and use the results of §1 to generalize Day's result in [1].

So once again we are given an adjunction $\eta,\varepsilon\colon\ f \dashv u\colon A \to A'$ in the 2-category on which D acts, but now we suppose that we are in the special case where

(3.1) $fu = 1$ and $\varepsilon = 1\colon\ fu \Rightarrow 1$.

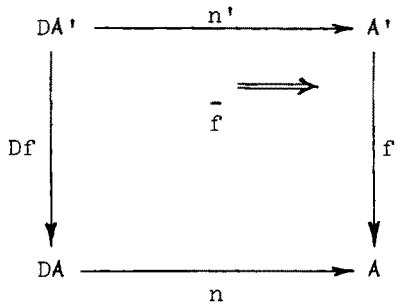
The results being independent of the 2-category, we again write as if this were CAT; then (3.1) may be expressed by calling A a full reflective subcategory of A', with inclusion u and reflexion f. The adjunction equations ((2.1) and (2.2) of [6] above) now become

(3.2) $f\eta = 1\colon\ f \Rightarrow fuf$; $\eta u = 1\colon\ u \Rightarrow ufu$.

We suppose A', but not A, to be given with a D-structure, with action n': $DA' \to A'$. By an enrichment of this situation we mean the giving of a D-structure to A, with action n: $DA \to A$ say, and the enriching of u and f to D-functors $U = (u,\bar{u})$ and $F = (f,\bar{f})$ in such a way that η,ε provide an adjunction $F \dashv U$ in D-CAT.

By Theorem 1.5 to give an enrichment is to give a D-structure to A and to give an isomorphism $\bar{f}$ as in

(3.3)

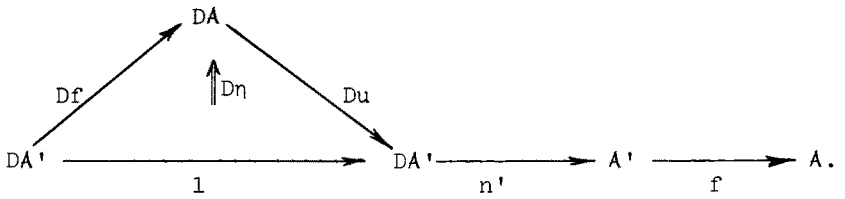


that makes of $(f,\bar{f})$ a strong D-functor. If such an enrichment exists at all, it is unique to within a suitable isomorphism; for (3.3) gives $n.Df \cong fn'$, while (3.1) gives $Df.Du = 1$, so that n is effectively determined by

(3.4) $n \cong f.n'.Du.$

Now consider the composite

(3.5)



If an enrichment exists, (3.5) is isomorphic by (3.3) to $n.Df.D\eta$, which is the identity by (3.2). Thus a necessary condition for an enrichment to exist is that (3.5) be an isomorphism. In the case where D is the doctrine for monoidal categories, this clearly reduces to the single condition

(3.6) $f(\eta\otimes'\eta)\colon f(X\otimes'Y) \longrightarrow f(ufX\otimes'ufY)$ is an isomorphism;

and Day [1] has shown that in this case (3.6) is also sufficient. However the invertibility of (3.5) can hardly be sufficient for a general D; if D were the doctrine for strict monoidal categories, the invertibility of (3.5) would still reduce to (3.6), but the monoidal structure constructed by Day on A, given by $A\otimes B = f(uA\otimes'uB)$, would not in general be strict even though that on A' was strict. As we said in the Introduction, the doctrine for monoidal categories has a certain "flexibility" - an ability to absorb isomorphisms - which the doctrine for strict monoidal categories lacks.

It turns out that the invertibility of (3.5) is necessary even for the existence of a pseudo-enrichment; and that for this it is also sufficient. To be precise, let D' be the doctrine whose algebras are the normalized pseudo D-algebras in the sense of Street [10] §2 in this volume. That is to say, a D'-category differs from a D-category in that the associativity axiom for the action is satisfied only to within a (prescribed) isomorphism, subjected to suitable axioms; the unitary axiom however (this is what "normalized" means) is satisfied on the nose. Then a D'-functor [resp. strict D'-functor] is what Street calls a lax homomorphism [resp. strict homomorphism] of pseudo-D-algebras. Since every D-category is a fortiori a D'-category, and every strict D-functor is a fortiori a strict D'-functor, there is a doctrine-map $p\colon D' \Rightarrow D$; and it is shown in my paper [5] below in this volume, which also justifies the other remarks above, that p is a retraction.

3.2 We can now give the main result of this section.

Theorem 3.1 Let the reflexion $\eta,\varepsilon\colon f \dashv u\colon A \to A'$ be given as above, and let A' be a D-category, hence also a D'-category. Then the reflexion admits a D'-enrichment if and only if (3.5) is an isomorphism.

Proof First, the necessity. Consider the composite of (3.5) with pA': $D'A' \to DA'$. By the 2-naturality of p, the morphism pA' can be moved past the triangle $D\eta$ in (3.5), turning the latter into $D'\eta$. But then pA' composed with n': $DA' \to A'$ is just the action of D' on A'. Thus the composite of (3.5) with pA' is just the analogue of (3.5) with D' replacing D. We already know that this must be invertible if a D'-enrichment is to exist. But then (3.5) itself must be invertible, since pA' is a retraction.

We turn to the sufficiency. Guided by (3.4), we define $n: DA \to A$ by

$$n = f.n'.Du. \tag{3.7}$$

We can then write (3.5) as an isomorphism

$$\phi = f.n'.D\eta : fn' \Rightarrow n.Df, \tag{3.8}$$

and we define the $\bar{f}$ of (3.3) to be the inverse of ϕ.

We check that n satisfies the unitary law for an action. We have

$$\begin{aligned} n.jA &= f.n'.Du.jA && \text{by (3.7)} \\ &= f.n'.jA'.u && \text{by the naturality of j} \\ &= fu && \text{by the unitary law for n'} \\ &= 1 && \text{by (3.1).} \end{aligned}$$

As regards the associative law, we have

$$\begin{aligned} n.mA &= f.n'.Du.mA && \text{by (3.7)} \\ &= f.n'.mA'.D^2u && \text{by the naturality of m} \\ &= f.n'.Dn'.D^2u && \text{by the associative law for n';} \end{aligned}$$

while

$$n.Dn = n.Df.Dn'.D^2u \qquad \text{by (3.7).}$$

Therefore the isomorphism

(3.9) $\mu = \phi.Dn'.D^2u$: $f.n'.Dn'.D^2u \Rightarrow n.Df.Dn'.D^2u$
is an isomorphism

(3.10) μ: $n.mA \Rightarrow n.Dn$,
and we define ν to be μ^{-1}.

It remains to verify that n and ν constitute a normalized pseudo-action of D on A, and that f and $\bar{f}$ constitute a "lax homomorphism of pseudo-D-algebras" in Street's language. Then in terms of D', we have a D'-action on A and a strong D'-functor $(f,\bar{f})$: $A' \to A$, whence the desired D'-enrichment follows by Theorem 1.5.

The axioms to be verified are the axioms (1), (2), (3) of [10] §2 for ν, and the axioms (5), (6) of [10] §2 for $\bar{f}$. The notations are rather different: Street's i is our j, his c is both our m and our n, as well as our n'; his ζ is 1 in our case (normality); his θ is our ν, and also our ν' which is 1 (A' being a honest D-category); and his θ_f is our $\bar{f}$. In verifying these axioms we replace ν and $\bar{f}$ by their inverses, μ and ϕ respectively, inverting the arrows accordingly.

Axiom (1) reads $\mu.jDA = 1$. But

$$\begin{aligned} \mu.jDA &= \phi.\ Dn'.D^2u.\ jDA && \text{by (3.9)} \\ &= \phi.\ jA'.\ n'.Du && \text{by the naturality of j.} \end{aligned}$$

This will therefore follow from axiom (5), which reads $\phi.jA' = 1$. But

$$\begin{aligned} \phi.jA' &= f.n'.\ D\eta.jA' && \text{by (3.8)} \\ &= f.n'.jA'.\eta && \text{by the 2-naturality of j} \\ &= f\eta && \text{by the unitary axiom for n'} \\ &= 1 && \text{by (3.2).} \end{aligned}$$

Axiom (2) reads $\mu.DjA = 1$. But

$$\begin{aligned} \mu.DjA &= \phi.Dn'.D^2u.DjA && \text{by (3.9)} \\ &= \phi.Dn'.DjA'.Du && \text{by the naturality of j} \\ &= \phi.Du && \text{by the unitary axiom for n'} \\ &= f.n'.D\eta.Du && \text{by (3.8)} \end{aligned}$$

$= 1$ since $\eta u = 1$ by (3.2).

This leaves us with axioms (3) and (6). The first of these reduces to the second, via (3.9). For the left side of axiom (3) is the vertical composite of $n.D\mu$ with $\mu.Dm_A$, and the right side is the vertical composite of $\mu.D^2n$ with $\mu.mD_A$. Now using (3.9)

$n.D\mu = n.D\phi.D^2n'.D^3u;$

$\mu.Dm_A = \phi.Dn'.D^2u.Dm_A$

$= \phi.Dn'.Dm_{A'}.D^3u$ by the naturality of m

$= \phi.Dn'.D^2n'.D^3u$ by the associativity axiom for n';

$\mu.D^2n = \mu.D^2f.D^2n'.D^3u$ by (3.7);

$\mu.mD_A = \phi.Dn'.D^2u.mD_A$

$= \phi.m_{A'}.D^2n'.D^3u$ by the naturality of m.

But the left side of axiom (6) is the vertical composite of $n.D\phi$ with $\phi.Dn'$, and the right side is the vertical composite of $\mu.D^2f$ with $\phi.m_{A'}$. So axiom (3) follows from axiom (6) on composing with $D^2n'.D^3u$.

It remains then to verify axiom (6). Now

$n.D\phi = f.n'.Du.Df.Dn'.D^2\eta$ by (3.7) and (3.8);

$\phi.Dn' = f.n'.D\eta.Dn'$ by (3.8);

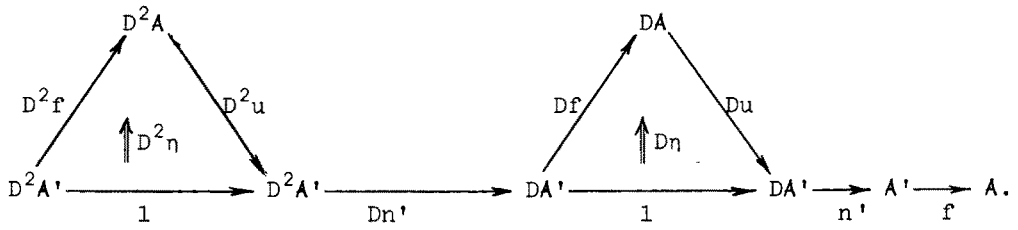
$\mu.D^2f = f.n'.D\eta.Dn'.D^2u.D^2f$ by (3.9) and (3.8);

$\phi.m_{A'} = f.n'.D\eta.m_{A'}$ by (3.8)

$= f.n'.m_{A'}.D^2\eta$ by the 2-naturality of m

$= f.n'.Dn'.D^2\eta$ by the associativity axiom for n'.

The vertical composite of the first two of these is indeed equal to that of the second two, both being



This completes the proof. □

3.3 As we have said in the Introduction, there are two cases in which we can actually get a D-enrichment. The first is that where (3.5) is not merely an isomorphism but an identity; which corresponds to asking $F = (f,\bar{f})$ to be not merely strong (as it must be by Theorem 1.5) but strict. This may seem rather an artificial case; but it often arises in nature, as the paper [5] below will show, and I have found it extremely useful in various (unpublished) considerations related to coherence. To state it formally:

Theorem 3.2 Let the reflexion $\eta,\varepsilon\colon f \dashv u\colon A \to A'$ be given as above, and let A' be a D-category. Then the reflexion admits a D-enrichment with F strict if and only if (3.5) is the identity (which includes the assertion that $f.n'.Du.Df = f.n'$).

Proof For the necessity, we are to have (3.3) with $\bar{f} = 1$, so that we must have $f.n' = n.Df$. So (3.5) is $n.Df.D\eta$, which is the identity since $f\eta = 1$ by (3.2).

The sufficiency is contained in the proof of Theorem 3.1, since in the present case we have $\phi = 1$ and $\mu = 1$, or equally $\bar{f} = 1$ and $\nu = 1$. □

We said towards the end of §3.1 that the doctrine-map $p\colon D' \Rightarrow D$ is a retraction; that is, there is a 2-natural transformation $q\colon D \Rightarrow D'$ with $pq = 1$. In general we cannot suppose q to be a doctrine-map. If we can find a doctrine-map q with $pq = 1$, we call the doctrine D flexible. In the paper [5] below, we show that for any doctrine D the doctrine D' is flexible; and we also show that D is flexible if D is of the form K°-, where K is a club in $CAT/\underline{S}$ (or even in $CAT/\mathcal{S}et$) whose discrete club $|K|$ of objects is a free discrete club in the sense of [4] §10.3 above. Thus the club for monoidal categories, or that for symmetric monoidal categories, is flexible, since the functor-operations are generated freely by $\otimes$ and I; the club for strict

monoidal categories escapes the result since $\otimes$ and I are subjected to axioms like $\otimes(\otimes,\underline{1}) = \otimes(\underline{1},\otimes)$.

By applying the theorem below in the case where D is replaced by D', we see that Theorem 3.1 remains true even when A' is originally given as a D'-category rather than a D-category.

Theorem 3.3 Let the reflexion $\eta,\varepsilon\colon f \dashv u\colon A \to A'$ be given as above, and let A' be a D-category, where D is flexible. Then the reflexion admits a D-enrichment if and only if (3.5) is an isomorphism.

Proof The necessity was shown in §3.1. For the sufficiency, first use Theorem 3.1 to get a D'-enrichment. Then apply the 2-functor q-CAT: D'-CAT → D-CAT (cf. [6] §3.6 above) to get a D-enrichment. The only point at issue is whether the D-structure A' now has is that it started with; this is ensured by pq = 1. □

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