COHERENCE THEOREMS FOR LAX ALGEBRAS AND FOR DISTRIBUTIVE LAWS

by

G.M. Kelly

1.1 We prove two main results. First, let D be a doctrine, say on *Cat*. (For the most basic notions about doctrines, see [13] above, familiarity with which we assume.) Let D^* be that doctrine whose (honest) algebras are the lax algebras for D, in the sense of [24] above. Since a D-algebra is a fortiori a D^*-algebra, there is a doctrine-map $s: D^* \to D$. We prove the existence, in the 2-category $[Cat,Cat]$ of endo-2-functors, of a morphism $h: D \to D^*$ (not a doctrine-map) such that $sh = 1$, and of a 2-cell $\eta: 1 \Rightarrow hs$ such that $s\eta = 1$ and $\eta h = 1$. Borrowing the terminology usual in the 2-category *Cat*, we may say that this exhibits s as a "reflexion" of D^* onto the "full reflective subobject" D. Moreover, if we replace "lax" above by "pseudo", the same is still true, but now η is an isomorphism; so that in this case s is an equivalence of endo-2-functors. Part of this result, along with some refinements we shall also prove, has been applied in my paper [10] above.

For our second result, let $p: D'D \to DD'$ be a distributive law between two doctrines ——— or rather a pseudo-distributive-law; for strict ones in the original sense of Beck [2] are rare in natural examples at the doctrine level (just as $A \otimes B = B \otimes A$ is hopelessly rare for categories, while $A \otimes B \cong B \otimes A$ is common). A category provided with actions both of D and of D', and for which the D-action is given the structure of a (non-strict!) D'-morphism, is an algebra for some doctrine D^*. Were the distributive law a strict one,

DD' would itself be a doctrine, and its algebras would be categories with a D-action and a D'-action, the D-action being a strict D'-morphism; this is the classical situation of Beck [2]. In that case there would clearly be a doctrine-map $s: D^* \to DD'$. In our pseudo-distributive case, DD' is no longer a doctrine, but is at any rate a D^*-algebra; so we can define $s: D^* \to DD'$ as the unique strict D^*-morphism whose composition with $j^*: 1 \to D^*$ is $jj': 1 = 11 \to DD'$. Our result is once again that s is a reflexion of D^* onto a full reflective subobject DD'; and that it is actually an equivalence of endo-2-functors if we require the D-action to be a strong (although still not a strict) D'-morphism.

Kock [14] distinguished, among the enriched monads on a symmetric monoidal closed category, those that he called the commutative monads — now more often called monoidal monads. Once again, strictly commutative doctrines are rare, but pseudo-commutative ones are easily found. One such is the doctrine for symmetric monoidal categories; let us now call this D'. Then if D is any pseudo-commutative doctrine, there is a pseudo-distributive-law $p: D'D \to DD'$, and the above result applies. In particular it applies when D too is taken to be this particular D'. In that case our result contains that of Laplaza [19], who considers a category with two symmetric monoidal structures $\otimes$ and $\oplus$ together with a "distributivity" $A\otimes(B\oplus C) \to (A\otimes B)\oplus(A\otimes C)$.

The similarity of our two main results suggests that each is a special case of something very general, relating the doctrines for a more lax and a less lax situation. I have not attempted to formulate this: the present proofs have in common only some general principles collected in §2 and §3 below, and are otherwise distinct.

The rest of this introduction is devoted to general comments on coherence problems, including the relation between such results as we have stated and results about the commutativity of diagrams.

1.2 To be told, as in the above situations, what are the algebras for some doctrine D^* (along with the strict morphisms of these, and the 2-cells between the strict morphisms) is to be given D^* implicitly. By the "coherence problem" for D^* I understand primarily that of determining D^* explicitly from this information. A complete solution of this is often beyond our powers. By a "coherence theorem" I understand a complete or partial solution of this problem: a result which tells us something at least about D^* qua endo-2-functor of Cat. For instance, that it is equivalent to some known endo-2-functor D; or even that it contains a known D, in some definite way, as a full reflective subobject. Results about certain diagrams commuting may be part of such a theorem, or may be among its important consequences; but such results can, I believe, no longer be seen as constituting the essence of a coherence theorem.

I shall return later to the matter of commuting diagrams. First, I want to observe that the last paragraph has been over-simplified in various ways for the sake of its polemical point.

(a) The word "doctrine" (= 2-monad) may have to be replaced by "monad". Categories endowed with some equational structure are the algebras for a doctrine on Cat when the structural functors are all covariant; but they are the algebras for a mere monad when, as in the case of symmetric monoidal closed categories, functors of mixed variance, such as the internal-hom, are involved.

(b) One takes much too narrow a view if one considers only doctrines on Cat. An equational structure may be borne by a Λ-indexed family of categories, and these are algebras for a doctrine on the 2-category Cat/Λ. It is easy to conceive of doctrines on V-Cat for a symmetric monoidal closed V. But then too there are things like "lax doctrines"; an $E \in [Cat, Cat]$ with multiplication $E^2 \to E$ and unit $1 \to E$ satisfying the "doctrine axioms" only to within certain 2-cells, themselves subjected to so-called "coherence axioms";

such an E is an algebra for some doctrine D^* on the 2-category $[\mathcal{C}at,\mathcal{C}at]$. Clearly we should take D^* to be a doctrine (or possibly a mere monad) on an arbitrary 2-category K; although it may be reasonable to suppose that K has some of the good properties of $\mathcal{C}at$, such as being locally presentable.

(c) It is reasonable - and probably very common - to use the term "coherence theorem" for a result which, having found out something about D^* from a knowledge of its algebras, goes back to these algebras and deduces something useful about them. For instance, Mac Lane's original coherence theorem [23] may be interpreted as saying that the doctrine-map $s: D^* \to D$ is an equivalence of endo-2-functors, where D^* is the doctrine for symmetric monoidal categories and D that for <u>strict</u> symmetric monoidal categories. This in fact gives a complete determination of D^* (since D^* is P_o- for a club P which is wholly determined by Mac Lane's result). But it has the further consequence that every symmetric monoidal category is symmetric-monoidally equivalent to a strict one. [This is well known although I do not recall any precise statement in print. Beck [3] takes it for granted and proceeds to refine the equivalence in this case, and in one other, via categories of fractions. It is a special case of the assertion of Isbell in [7], but this assertion has been withdrawn by its author; it asserts more than is true, and he discovered a radical error in the proof (at the Oberwolfach conference "Kategorien 1973").] How generally the existence of such an equivalence $s: D^* \to D$ implies that every D^*-algebra is D^*-equivalent to a D-algebra has not to my knowledge been discussed; nor do we pursue it in this paper - but it cries out for investigation, and would be many people's ideal of what a coherence theorem should be.

1.3 We return now to the matter of commuting diagrams. In some cases we know at once from the description of the D^*-algebras that D^* is $C\circ-$ for some club C, and that C is given by specified generators and relations; see §1 and §10 of [9] above, and the further references given there. The word-problem involved in finding the objects of C is typically easy - often indeed they are freely generated - while that involved in finding the morphisms is typically much harder. It is a matter of finding the category generated by a certain graph subject to certain relations - a generalization of the word-problem for monoids, inasmuch as a category is a "monoid with many objects". Deciding when two words in the generators represent the same morphism of C is deciding which diagrams commute in C. This is the most common sense in which solving a coherence problem, completely or partially, may involve proving that certain diagrams commute. The typical example is that of symmetric monoidal closed categories, the club for which was determined in part by Kelly-Mac Lane [11], and then more fully by Voreadu ([27],[28],[29]), but is still not known completely.

(Other examples where the club is partially but not completely determined are: a symmetric monoidal closed category V, two V-categories, two V-functors, and a V-natural transformation (Kelly-Mac Lane [12]); two symmetric monoidal closed categories and a symmetric monoidal functor (Lewis [21],[22]); a category with two symmetric monoidal structures $\otimes$ and $\oplus$ along with a "distributivity" $A\otimes(B\oplus C) \to (A\otimes B)\oplus(A\otimes C)$ that is not required to be an isomorphism (Laplaza [18] and [19]). Some examples where the club is completely determined are: a monoidal or symmetric monoidal category (Mac Lane [23]); a category with a tensor-product and a non-isomorphic $a: (A\otimes B)\otimes C \to A\otimes(B\otimes C)$ (Laplaza [17]); a category with a monad (Lawvere [20]; Lambek [16]) - the doctrine is just $\underline{\Delta}\times-$; two symmetric monoidal categories and a symmetric monoidal functor (Lewis [21] and

[22]). I do not suggest that the above results are expressed in terms of clubs by their authors; but this is what they amount to. The last-mentioned result of Lewis is so expressed, and indeed must be: for it sets out to determine C completely in a case where the augmentation $\Gamma\colon C \to T$ is not faithful. Finally, this list is not meant to be exhaustive.)

One could also regard the determination of D^* as the problem of determining D^*A for every object A; that is, of determining the free D^*-algebras. This is the view taken by Lambek ([15] and [16]), who sets up generators and relations for D^*A, and who attacks the problem of which diagrams in D^*A commute when A is a discrete category. To this end he has introduced a brilliant adaptation of Gentzen's work on cut-elimination. Yet when D^* is $C\circ -$ for a club C, as it is in the examples considered by Lambek, the consideration of $D^*A = C\circ A$ in place of C is an unnecessary complication - for any knowledge of C immediately gives corresponding knowledge of $C\circ A$, and this for any A, discrete or not; while the cut-elimination techniques work perfectly well at the level of C. In fact, those results of Lambek of the form "equi-generality implies commutativity" turn out to be re-phrasings of "$\Gamma\colon C \to T$ is faithful"; and some of them are then seen to be false (cf. [8] §6). When, however, D^* does not come from a club, Lambek's idea of setting up generators and relations for D^*A, and studying commutativity of diagrams at this level, may well be the best way of getting at D^*. An example is that where a D^*-algebra is a cartesian closed category, which has been investigated by Szabo ([25] and [26]).

1.4 Thus proving diagrams commutative may be a tool in establishing a coherence theorem, or even a way of stating it; but it need not be the only way of getting some grip on D^*, as is shown by our results stated in §1.1. On the other hand I now point out that such results imply theorems about diagrams commuting, at least when

D,D' come from clubs: and imply them wholesale, in so far as these clubs are arbitrary. In this way I believe one clinches the argument for calling them "coherence theorems".

In the first of those results it is easy to see that if D is $\mathcal{D}\circ -$ for a club $\mathcal{D}$ in $CAT/\underline{S}$, then D^* is $\mathcal{D}^*\circ -$ for another such club $\mathcal{D}^*$, and that s,h,η all arise from things in $CAT/\underline{S}$; so that we may as well change notation and write s: $\mathcal{D}^* \to \mathcal{D}$, h: $\mathcal{D} \to \mathcal{D}^*$, η: $1 \Rightarrow hs$, $sh = 1$, $s\eta = 1$, $\eta h = 1$. If f,g: $T \to S$ is a diagram in $\mathcal{D}^*$, a necessary condition for its commutativity is $sf = sg$. This is not in general sufficient, for it implies only that $\eta_S f = \eta_S g$, where η_S: $S \to hsS$. It is however sufficient if η_S is an isomorphism, as it always is in the "pseudo" case, and as it is in the "lax" case when S is in the image of h, i.e. when S belongs to the full subcategory $\mathcal{D}$ of $\mathcal{D}^*$. Since s is a club-map, sf can be calculated explicitly and directly; so we have an effective test for the commutativity of any diagram in the "pseudo" case, and of a large class in the "lax" case.

The same is true in the second result of §1.1, in the form in which it is first stated (the D-action an <u>arbitrary</u> D'-morphism). If f,g: $T \to S$ in $\mathcal{D}^*$ with S in the full reflective subcategory $\mathcal{D}\circ\mathcal{D}'$, then $f = g$ if and only if $sf = sg$. In the case where the D-action is to be a <u>strong</u> D'-morphism, it is no longer the case that D^* comes from a club in $CAT/\underline{S}$ when D and D' do. This is for the reasons indicated in §1.5 of [9]; in the special case corresponding to Laplaza's problem of two symmetric monoidal structures and a distributivity d: $A\otimes(B\oplus C) \to (A\otimes B)\oplus(A\otimes C)$, the <u>type</u> of d lies in $\underline{S}^{op}$, and we can take it to be in $\underline{S}$ by passing to the opposite doctrine. But if we ask d to be an isomorphism, its inverse d^{-1} <u>already</u> has its type in $\underline{S}$, and dd^{-1}: $(A\otimes B)\oplus(A\otimes C) \to (A\otimes B)\oplus(A\otimes C)$ has its type neither in $\underline{S}$ nor in $\underline{S}^{op}$; the doctrine does not come from a club in any sense we can at present give to that notion.

What we can do is to return to the club $\mathcal{D}^*$ in the non-strong case, and consider a model A (that is, a $\mathcal{D}^*$-algebra) in which the image of d happens to be an isomorphism (or more generally one in which the image of η happens to be a monomorphism). Then although $sf = sg$ does not imply $f = g$ in $\mathcal{D}^*$, it does imply the equality of their images in $\{A,A\}$, the "rich endo-functor category" of [9] §9, whose objects are functors $A^n \to A$ and whose morphisms are suitably general natural transformations. So for such a model A we have a criterion at the model-level for commutativity of any diagram $f,g: T \to S$ describable in $\mathcal{D}^*$ - which means (in Laplaza's case) a diagram involving d but not d^{-1}.

With this said, and with the observation that in Laplaza's case the doctrines $\mathcal{D}$ and $\mathcal{D}'$ are both the doctrine P for symmetric monoidal categories, which is equivalent to $\underline{\underline{P}}$ by Mac Lane's original result [23], so that s can be regarded as a map $\mathcal{D}^* \to \underline{\underline{P}} \circ \underline{\underline{P}}$, it is easy to see that s is Laplaza's "distortion" ([19] §2), and that our result includes his theorem in [19] §4. We give more details below.

1.5 We end this introduction by observing that the kind of coherence theorem exemplified by the results of §1.1 is pretty common.

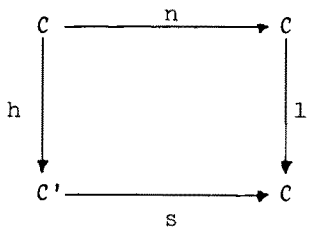
First, if we take Mac Lane's original result in [23], and look just at the associativity part, we see that the invertibility of $a: (A \otimes B) \otimes C \to A \otimes (B \otimes C)$ is not central to the main argument. What is really proved is that $s: \mathcal{D}^* \to \mathcal{D}$ is a reflexion, where $\mathcal{D}$ is the subcategory of the appropriate club $\mathcal{D}^*$, consisting of the objects bracketed wholly from the right. It is only because a is invertible that the reflexion is in fact an equivalence. It follows that, even when a is not an isomorphism, $sf = sg$ is necessary and sufficient for $f = g: T \to S$ whenever S lies in $\mathcal{D}$. This is a partial coherence result in that case. A total one requires more work, and is given by Laplaza in [17].

To get back to the partial one, however: I say that it is virtually predictable, and in a sense automatic. The method of proof of the present paper requires a $\mathcal{D}^*$-functor h: $\mathcal{D} \to \mathcal{D}^*$ with sh = 1. Identifying $\mathcal{D}$ with the set of natural numbers >0, we define h as a functor by h(1) = $\underline{\underline{1}}$, h(n) = $\underline{\underline{1}} \otimes$ h(n-1). Enriching it to a $\mathcal{D}^*$-functor means giving $\tilde{h}$: hn $\otimes$ hm $\to$ h(n+m); in particular we need $\tilde{h}$: h2 $\otimes$ h1 $\to$ h3, and this is of course a; the other components of $\tilde{h}$ are then given by an easy induction. But $(h,\tilde{h})$ has to satisfy an axiom if it is to be a $\mathcal{D}^*$-functor: and this reduces at once to the pentagonal axiom for a. This is in itself a lightweight observation; but note that similar considerations predict what the "coherence axioms" should be when we also have a constant object I and non-isomorphic maps ℓ: I$\otimes$A $\to$ A, r: A$\otimes$I $\to$ A; this is not an artificial situation, but arises precisely in the old, unsolved, coherence problem for a non-monoidal, non-symmetric closed category.

A final comment on Mac Lane's original coherence result, say for monoidal categories. If D = $\underline{\underline{N}}$∘- is the doctrine for strict monoidal categories, and if D' = N∘- is that for monoidal categories, Mac Lane's result gives a doctrine-map D' $\to$ D that is an equivalence of endo-2-functors. Our first result gives a similar equivalence $D^* \to D$. But Mac Lane's D' is not isomorphic to our D^*; although they are equivalent as doctrines, and not merely as endo-2-functors. This equivalence is discussed towards the end of §3 of [13] above.

As a last example, let C be a club in $CAT/\underline{\underline{S}}$ and consider pairs h: $A \to A'$ of C-categories connected by a C-functor h. Lewis ([21] and [22]) has determined the corresponding club completely when $C = P$ or N. But we can get a partial result for a general C with hardly any effort. Observe that the diagrams whose commutativity was needed by Eilenberg-Kelly [6] in the monoidal (N) and symmetric monoidal (P) cases were always rather special: they contained variables only from A, not from A'; they lay in A'; and they always had codomain of the form hA. A commutativity criterion for such

diagrams can be given at once, for a general C. Having variables only from A means looking not at the whole club, which is the free algebra on $A = I$ and $A' = I$ where I is the unit category, but at the free algebra on $A = I$ and A' = empty. Let this be $h: C \to C'$; it is immediate that its domain is C itself, while C' is the "unknown". Since h is free on the given generators and since $1: C \to C$ is a model, there are unique strict C-functors n,s rendering commutative



and having $n(\underline{\underline{1}}) = \underline{\underline{1}}$. But this last requirement, since C is the free C-category on $\underline{\underline{1}}$, gives $n = 1$; so that $sh = 1$. It is easy to show that s is in fact a reflexion of C' onto the full subcategory C, embedded by h; so that if S is in the image of h, the diagram $f,g: T \to S$ in C' commutes if and only if $sf = sg$. A lot less than Lewis proves for $C = P$ or N, admittedly; but much more general, very useful, and incredibly cheap.

2. A method of constructing D-reflexions

2.1 Since this section is purely formal, we take (D,m,j) to be a doctrine on an arbitrary 2-category K, and we understand "D-algebra" in the wide sense: an action $a: DA \to A$ of D on a 2-functor $A: L \to K$, not necessarily on an object A of K(the special case $L = I$). As far as possible we use a,b, etc., for actions of D on A,B, etc.

We assume acquaintance with the general facts about monads and doctrines given in [13] §3 above; we recall in particular that the

free D-algebra DA has action mA: $D^2A \to DA$, and that an action a: $DA \to A$ is a strict D-morphism. We add a few enrichments appropriate to the doctrine case.

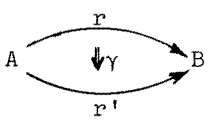
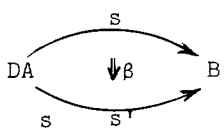
First, if $\gamma: r \Rightarrow r': A \to B$ is any 2-cell, then in

(2.1) $D\gamma: Dr \Rightarrow Dr': DA \to DB$

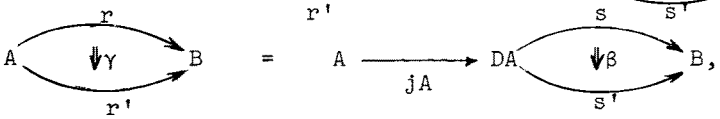
not only are Dr, Dr' strict D-morphisms but $D\gamma$ is a D-2-cell. Next, for a free D-algebra DA and a D-algebra B, there is an isomorphism not only of sets but of categories

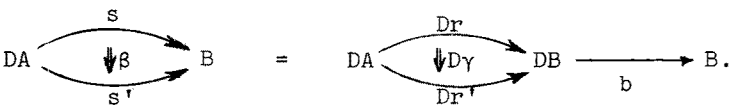
(2.2) $\Phi: [\mathcal{L},\mathcal{K}](A,B) \to D\text{-}\mathcal{A}\ell g_*(DA,B)$.

(Here $D\text{-}\mathcal{A}\ell g_*$ has as objects the D-algebras of domain $\mathcal{L}$; its morphisms are strict D-morphisms, and its 2-cells are D-2-cells. We recall that we replace $D\text{-}\mathcal{A}\ell g_*$ by $D\text{-}\mathcal{A}\ell g$ when we allow all D-morphisms as 1-cells.) In detail,

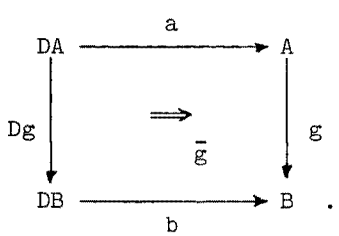
(2.3) Φ sends  to 

where

(2.4) 

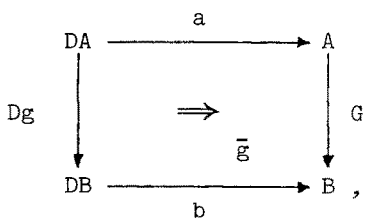
(2.5) 

Now consider an arbitrary (i.e. not necessarily strict) D-morphism $G = (g,\bar{g}): A \to B$, so that $\bar{g}$ is a 2-cell

(2.6) 

Theorem 2.1 $\bar{g}$ is a D-2-cell

(2.7)



all the edges now being regarded as D-morphisms, of which Dg,a,b are strict.

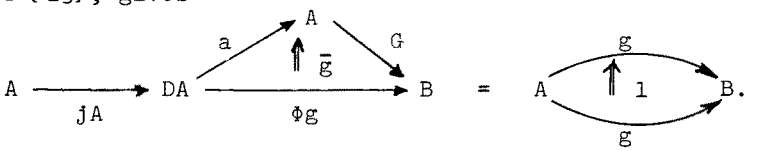
Proof The D-naturality axiom (3.19) of [13] for $\bar{g}$ reduces to the associativity axiom (3.18) of [13] for a D-morphism $(g,\bar{g})$. □

Observe that, since $b.Dg = \Phi g$ by (2.5), we can also write (2.7) as a D-2-cell

(2.8) $\bar{g}$: $\Phi g \Rightarrow Ga$.

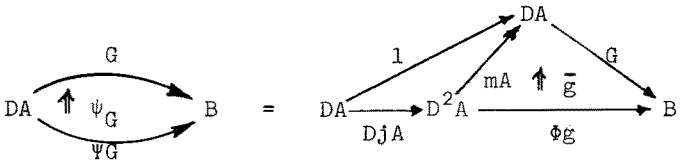
Observe further that the other axiom for a D-morphism, the identity axiom (3.17) of [13], gives

(2.9)

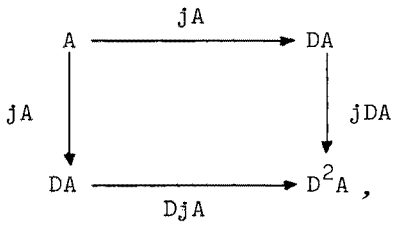


2.2 Now, in the situation of Theorem 2.1, replace A by a free D-algebra DA. Define the strict D-morphism ΨG and the D-2-cell ψ_G by:

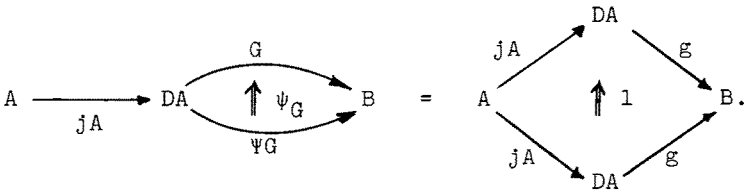
(2.10)



Had we replaced DjA by jDA on the right side of this definition, we should by (2.9) have got merely the identity. Since, however, we have



we can conclude from (2.9), not that $\psi_G = 1$, but that

(2.11) 

Note in particular that, by (2.4),

(2.12) $\Psi G = \Phi(g.jA)$.

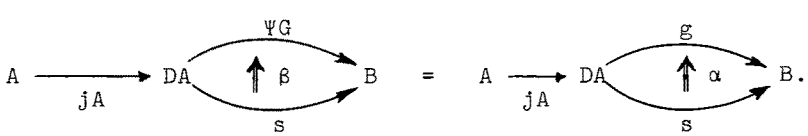
Theorem 2.2 ψ_G: $\Psi G \Rightarrow G$ is the coreflexion of G into the strict D-morphisms $DA \to B$. That is to say, if $s: DA \to B$ is a strict D-morphism and $\alpha: s \Rightarrow G$ is a D-2-cell, then

(2.13) $\alpha = s \underset{\beta}{\Longrightarrow} \Psi G \underset{\psi_G}{\Longrightarrow} G$

for a unique D-2-cell β. Explicitly,

(2.14) $\beta = \Phi(\alpha.jA)$,

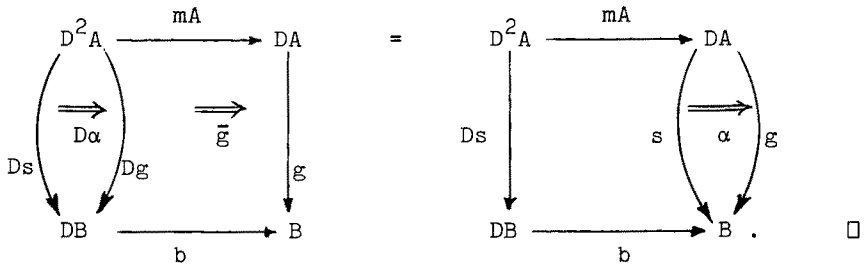
and β is the unique D-2-cell satisfying

(2.15) 

Proof Since $\psi_G.jA = 1$ by (2.11), (2.13) implies (2.15), which in turn gives (2.14) by (2.4), proving the uniqueness of β satisfying (2.13).

Define β therefore by (2.14), so that by (2.5) the explicit value of β is $b.D(\alpha.jA)$. Using this, the definition (2.10), and the fact that $mA.DjA = 1$, we get (2.13) if we compose with DjA the

diagram expressing the D-naturality of α, to wit

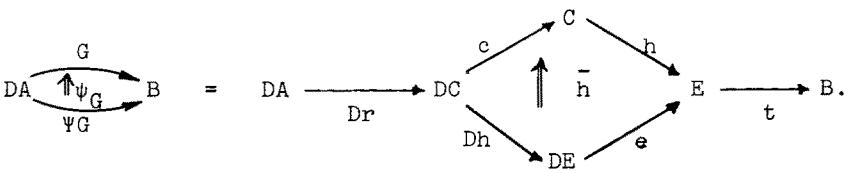


The naturality of Ψ,ψ is summed up in:

Proposition 2.3 Let G be the composite

(2.16) $$DA \xrightarrow[s]{} C \xrightarrow[H]{} E \xrightarrow[t]{} B$$

where $H = (h,\bar{h})$, t and s are strict, and $s = \Phi r$ for $r: A \to C$. Then

(2.17) 

Proof $\psi_G = \bar{g}.\ DjA$ by (2.10)

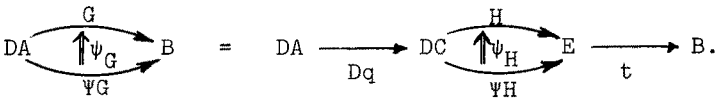
$= t.\bar{h}.Ds.DjA$ by (2.16)

$= t.\bar{h}.\ Dr$ by (2.4). □

Corollary 2.4 Let G be the composite

(2.18) $$DA \xrightarrow[Dq]{} DC \xrightarrow[H]{} E \xrightarrow[t]{} B;$$

then

(2.19) 

Proof In this case the r of Proposition 2.3 is jC.q. □

2.3 In [10] §3 above we considered the problem of enriching an adjunction $s \dashv h: B \to B'$ to a D-adjunction, in the case $sh = 1$, given B' as a D-algebra. A condition must be satisfied for the enrichment to exist; when it does, s necessarily becomes a strong D-morphism; and we considered in particular the special case where s is to be strict.

Here we approach the same situation from a different starting point, in the case where B' is a free D-algebra DA.

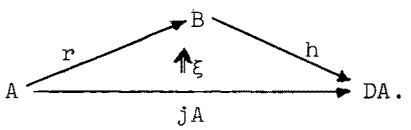
For this section, we suppose that B is given as a D-algebra; that we are given a strict D-morphism $s: DA \to B$, say $s = \Phi r$ for $r: A \to B$; that we are given a D-morphism $H = (h, \bar{h}): B \to DA$; and that we have

(2.20) $sH = 1$

as D-morphisms. We ask what else we need in order to get a D-2-cell $\eta: 1 \Rightarrow Hs$ satisfying $s\eta = 1$ and $\eta h = 1$, so as to complete the data for a D-reflexion of DA onto B.

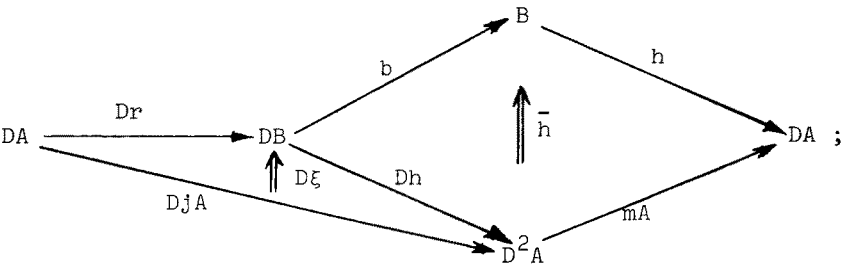
Write G for the composite Hs; then by (2.12) $\Psi G = \Phi(h.s.jA)$, which is $\Phi(hr)$ by (2.4). On the other hand $1: DA \to DA$ is a strict D-morphism, and is $\Phi(jA)$. By Theorem 2.2, there is a bijection between D-2-cells $\eta: 1 \Rightarrow Hs = G$ and D-2-cells $\zeta: 1 \Rightarrow \Psi G$; such D-2-cells $\zeta: \Phi(jA) \Rightarrow \Phi(hr)$ are in turn in bijection by (2.2) with (mere) 2-cells $\xi: jA \Rightarrow hr$, as in

(2.21)



Using (2.5) to write $\zeta = \Phi\xi$, and using (2.17) (with $t = 1$) to write ΨG and ψ_G, we see by Theorem 2.2 that η is the composite

(2.22)



of course the top leg of this is hs since b.Dr = s by (2.5), and the bottom leg is 1 since mA.DjA = 1.

Now consider sη. By (2.20), $s\bar{h} = 1$, so

$$\begin{aligned} s\eta &= s.mA.D\xi \\ &= b.\ Ds.\ D\xi \qquad \text{since s is strict} \\ &= b.D(s\xi) \\ &= \Phi(s\xi) \quad \text{by (2.5)} . \end{aligned}$$

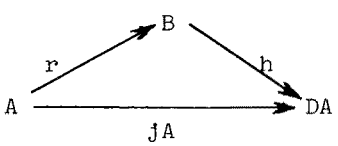
Since Φ is an isomorphism of categories, sη = 1 if and only if sξ = 1. Summing all thus up:

Theorem 2.5 Given D-morphisms s: DA → B and H: B → DA with s = Φr strict and with sH = 1, there is a bijection as in (2.22) between D-2-cells η: 1 ⇒ Hs and 2-cells ξ: jA ⇒ hr; and sη = 1 if and only if sξ = 1. □

To get our desired reflexion, we also need ηh = 1; I see no general way of expressing this, and have it verify it by an ad hoc method in each case.

There is an important special case which, while it occurs in only one of the two problems studied in this paper, does occur widely in similar problems. It is that where ΨG = 1. Since ΨG = Φ(hr) and 1 = Φ(jA), this is the case where

(2.23)



commutes. The condition $s\xi = 1$ is of course automatically satisfied if we now choose $\xi = 1$. On the other hand this choice is *forced* on us if we want $\eta h = 1$; for then $\eta hr = 1$ and (2.23) gives $\eta.jA = 1$; since $\psi_G.jA = 1$ by (2.11), this gives $\zeta.jA = 1$; which is $\xi = 1$ by (2.4). (Even in this case, $\xi = 1$ is not *sufficient* for $\eta h = 1$; this still requires *ad hoc* verification.) Summing up this special case gives:

Theorem 2.6 *If the data are as in Theorem 2.5, and if moreover* $hr = jA$, *there is at most one D-2-cell* $\eta: 1 \Rightarrow Hs$ *with* $s\eta = 1$ *and* $\eta h = 1$. *It must be the D-2-cell* $\eta = \bar{h}.Dr$ *obtained by setting* $\xi = 1$ *in* (2.22), *and this automatically satisfies* $s\eta = 1$. *It satisfies* $\eta h = 1$ *if and only if* $\bar{h}.Dr.h = 1$. □

3. D-algebras and D-morphisms as doctrine-maps

3.1 We are going to obtain our results announced in §1.1 by applying Theorem 2.5 and its special case Theorem 2.6. So we are seemingly going to get more than we asserted in §1.1; the adjunction $s \dashv h$ is actually going to be a D^*-adjunction. Once given, however, that s is a strict D^*-morphism, as it is in both the results of §1.1, it is automatic by Theorem 1.5 of [10] above that *any* such adjunction enriches to a D^*-adjunction. So our working with D^*-adjunctions is a requirement not of our results but of our method of proof.

What this method of proof *does* require, however, is that we can recognize and deal with non-strict D^*-morphisms, when all we are told originally about the unknown doctrine D^* is the 2-category $D^*\text{-}Alg_*$ of D^*-algebras, *strict* D^*-morphisms, and D^*-2-cells. Moreover I here mean $D^*\text{-}Alg_*$ in the *narrow* sense: a D^*-algebra is an action of D^* on an object of K, not on a 2-functor $L \to K$. However in saying that we are given $D^*\text{-}Alg_*$, I do mean to imply that we are also given the forgetful functor from $D^*\text{-}Alg_*$ to K; and it is

classical that all this determines D^* to within isomorphism. The purpose of this section is to introduce the simple machinery that allows us to recognize D^*-morphisms from the implicit knowledge of D^*.

3.2 We now assume once for all that the 2-category K has "small hom-categories" - that is, that each $K(A,B)$ is a small category; and that K is 2-complete, i.e. Cat-complete. We recall from [4] that this means (a) that K admits all small limits; (b) that these limits are preserved by the representables $K(A,-): K \to Cat$; and (c) that K admits cotensor products $[n,A]$ for $n \in Cat$, $A \in K$. As is well known, we can then construct comma objects in K, using pullbacks and the cotensor product $[\underline{2},A]$ where $\underline{2}$ is the arrow category $0 \to 1$; and these are then 2-comma-objects (the universal property extends to 2-cells). The 2-category Cat itself has these properties, as does Cat/A for a set A.

It is true that many of the common doctrines on Cat extend canonically to doctrines on CAT; certainly all those of the form $\mathcal{D}\circ -$ for a small club $\mathcal{D}$ do so - one can form $\mathcal{D}\circ A$ independently of the size of A. For that matter we can always extend D on Cat to CAT by right Kan extension. However we do not pursue here the properties of these extensions, which add nothing new in the club case. What is important for us is that D takes small categories to small categories.

We write E for the 2-category $[K,K]$ of endo-2-functors. It is a strict monoidal category with composition as tensor product and 1 as identity. It fails to be a closed category; only for certain S,R in E is it the case that $E(TS,R) \cong E(T,V)$ for some V, namely when the right Kan extension of R along S exists; then this right Kan extension is V.

However E acts on K, in the sense that there is a 2-functor $E\times K \to K$, sending (D,A) to DA, and satisfying the associativity and identity laws $(ED)A = E(DA)$, $1A = A$(along with the corresponding things for 1-cells and 2-cells). Moreover we have 2-naturally

(3.1) $K(DA,B) \cong E(D,\{A,B\})$,

where $\{A,B\}$ is the right Kan extension of $B: I \to K$ along $A: I \to K$. We know explicitly what this is, namely for C in K

(3.2) $\{A,B\}C = [K(C,A),B]$,

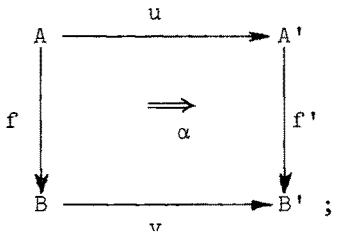
the cotensor product of $K(C,A) \in Cat$ with $B \in K$. The counit of the adjunction (3.1) is a 2-natural evaluation $e: \{A,B\}A \to B$, and there is similarly a 2-natural unit $d: D \to \{A,DA\}$. We have formally the same kinds of things as we have in a closed category, in particular a 2-natural multiplication $\mu: \{B,C\}\{A,B\} \to \{A,C\}$ and a unit $\iota: 1 \to \{A,A\}$, which make K into an E-category. (That the underlying 2-category of this E-category really is K itself is immediate, since $K(1,\{A,B\}) \cong K(A,B)$ by (3.1).)

It is moreover easy to check that the E-valued endomorphism object $\{A,A\}$, with its multiplication $\mu: \{A,A\}\{A,A\} \to \{A,A\}$ and its unit $\iota: 1 \to \{A,A\}$, is a doctrine; and that for any doctrine D, an action $a: DA \to A$ corresponds under (3.1) precisely to a doctrine-map $D \to \{A,A\}$.

3.3 We now write K' for the functor 2-category $[\underline{2},K]$ and K'' for the 2-category which is like the $[\![\underline{2},K]\!]$ of [9] §2.2 above except that its morphisms are not lax natural transformations but op-lax ones; it is what we called *FUN* in [9] §10.8 above in the case $K = CAT$, and it is described there (in the context of just such considerations as follow, but there specialized to the club case).

So K' and K'' have the same objects, namely morphisms $f: A \to B$ in K; a typical morphism in K'' is a triple (u,v,α) of the form

(3.3)



;

a typical 2-cell $(u,v,\alpha) \Rightarrow (\bar{u},\bar{v},\bar{\alpha})$ is a pair of 2-cells $\rho\colon\ u \Rightarrow \bar{u}$, $\sigma\colon\ v \Rightarrow \bar{v}$ in K satisfying the obvious conditions with respect to α and $\bar{\alpha}$; and the various compositions are by pasting. The morphisms of K' are those of K'' in which $\alpha = 1$, and are thus just pairs (u,v) making the outside of (3.3) commute; its 2-cells are those in K'' between these morphisms.

There are evident actions of E on K' and on K'', the first being the restriction of the second; namely (D,f) goes to Df, $(D,(u,v,\alpha))$ goes to $(Du,Dv,D\alpha)$, and in general D is applied to everything in sight, with appropriate arrangements also for the 1-cells and 2-cells of E. Just as in the case of K, these actions have right adjoints; we have 2-natural isomorphisms

(3.4) $\quad K'(Df,f') \cong E(D,[f,f'])$,

(3.5) $\quad K''(Df,f') \cong E(D,\langle f,f'\rangle)$,

where $[f,f']$ is the pullback

(3.6)
$$\begin{array}{ccc} [f,f'] & \xrightarrow{\delta_0} & \{A,A'\} \\ {\scriptstyle \delta_1}\downarrow & & \downarrow{\scriptstyle \{1,f'\}} \\ \{B,B'\} & \xrightarrow[\{f,1\}]{} & \{A,B'\} \end{array}$$

and $\langle f,f'\rangle$ is the comma object

(3.7)
$$\begin{array}{ccc} \langle f,f'\rangle & \xrightarrow{\partial_0} & \{A,A'\} \\ {\scriptstyle \partial_1}\downarrow & \overset{\lambda}{\nearrow} & \downarrow{\scriptstyle \{1,f'\}} \\ \{B,B'\} & \xrightarrow[\{f,1\}]{} & \{A,B'\} \end{array} \quad ;$$

of course there is a canonical map

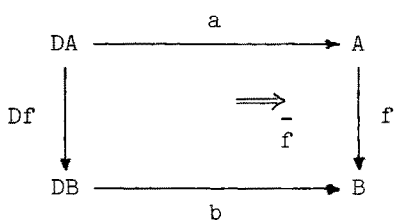
(3.8) $\varepsilon\colon [f,f'] \to \langle f,f' \rangle$

with $\partial_0\varepsilon = \delta_0$, $\partial_1\varepsilon = \delta_1$, $\lambda\varepsilon = 1$; it is a monomorphism since δ_0 and δ_1 are jointly monomorphic.

Thus K' and K'' become E-categories, just like K, with the same kind of formal properties. In particular, taking $f' = f$, we observe that $[f,f]$ and $\langle f,f \rangle$ are doctrines; and it is easy to see in this case that δ_0, δ_1, ∂_0, ∂_1, ε are all doctrine-maps.

Moreover for any doctrine D in E, a doctrine-map $k\colon D \to \langle f,f \rangle$ corresponds under (3.5) to an action $(a,b,\bar{f})\colon Df \to f$ of D on f in K'', which when written out as

(3.9)



is easily seen to amount to actions $a\colon DA \to A$ and $b\colon DB \to B$ of D on A and B, together with an enrichment $\bar{f}$ of f to a D-morphism $F = (f,\bar{f})\colon A \to B$. The action $a\colon DA \to A$ itself corresponds to the doctrine-map

(3.10) $$D \xrightarrow[k]{} \langle f,f \rangle \xrightarrow[\partial_0]{} \{A,A\},$$

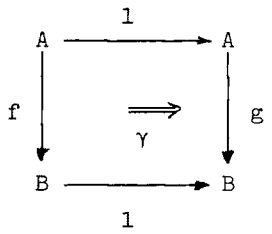
and similarly for b.

Similarly a doctrine-map $D \to [f,f]$ or an action of D on f in K' amounts to actions of D on A and on B such that f is a <u>strict</u> D-map; the situation of (3.9) with $\bar{f} = 1$. Since ε is a monomorphism, there is no difference between a doctrine-map $D \to [f,f]$ and a doctrine-map $D \to \langle f,f \rangle$ that happens to factorize through $[f,f]$.

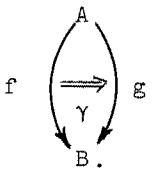
<u>3.4</u> Finally, in order to describe D-2-cells as doctrine-maps, we go one step further. Consider $(K'')' = [\underline{2},K'']$, the 2-category of morphisms of K''; so an object of $(K'')'$ is a triple (u,v,α) as in (3.3). Now write $K^{\#}$ for the <u>full</u> sub-2-category of $(K'')'$ determined by those

objects $(1,1,\gamma)$; that is, those of the form

(3.11)

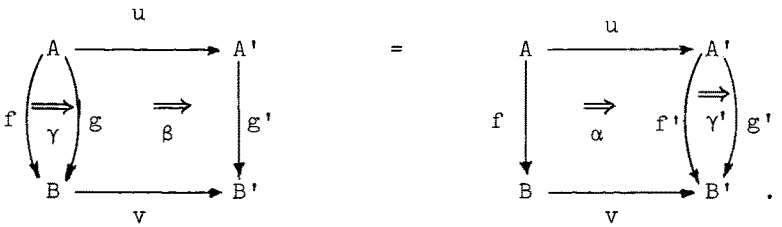


or



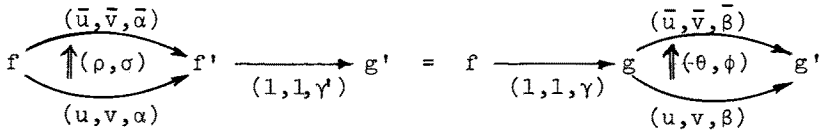
Calling this object γ for short, a morphism from γ to γ' in $K^{\#}$ consists of pairs (u,v,α): $f \to f'$ and (x,y,β): $g \to g'$ in K'' such that $(x,y,\beta)(1,1,\gamma) = (1,1,\gamma')(u,v,\alpha)$. This last implies that $x = u$, $y = v$; so finally a morphism $\gamma \to \gamma'$ in $K^{\#}$ consists of (u,v,α,β) satisfying

(3.12)



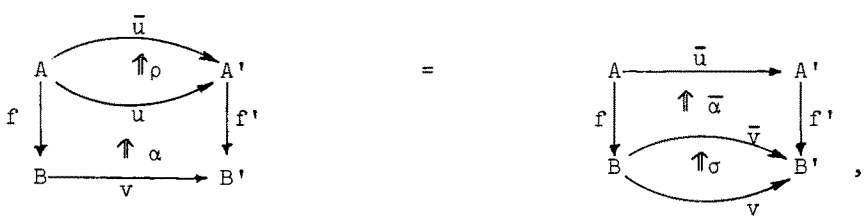
Then a 2-cell $(u,v,\alpha,\beta) \Rightarrow (\bar{u},\bar{v},\bar{\alpha},\bar{\beta})$ in $K^{\#}$ consists of a pair of 2-cells in K'', say (ρ,σ): $(u,v,\alpha) \Rightarrow (\bar{u},\bar{v},\bar{\alpha})$ and (θ,ϕ): $(u,v,\beta) \Rightarrow (\bar{u},\bar{v},\bar{\beta})$, satisfying the equation

(3.13)



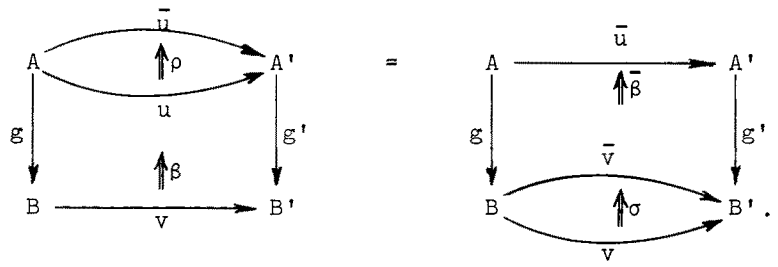
in K''. But this last equation gives $\theta = \rho$, $\phi = \sigma$; so in the end a 2-cell $(u,v,\alpha,\beta) \Rightarrow (\bar{u},\bar{v},\bar{\alpha},\bar{\beta})$ in $K^{\#}$ consists of a pair ρ: $u \Rightarrow \bar{u}$, σ: $v \Rightarrow \bar{v}$ satisfying the conditions that (ρ,σ): $(u,v,\alpha) \Rightarrow (\bar{u},\bar{v},\bar{\alpha})$ and that (ρ,σ): $(u,v,\beta) \Rightarrow (\bar{u},\bar{v},\bar{\beta})$ are 2-cells in K'', namely

(3.14)



,

(3.15)

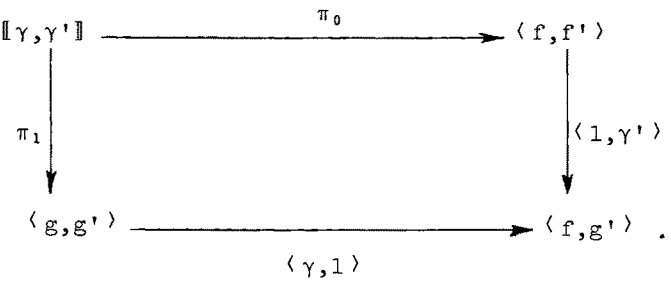


Now since E acts on K'' it also acts on $(K'')'$, and this action restricts to one on $K^{\#}$; and we have an evident 2-natural isomorphism

(3.16) $\quad K^{\#}(D\gamma,\gamma') \cong E(D,[\![\gamma,\gamma']\!])$

where $[\![\gamma,\gamma']\!]$ is the pullback

(3.17)



Once again when $\gamma' = \gamma$ we get a doctrine $[\![\gamma,\gamma]\!]$, and π_0,π_1 are then doctrine-maps. It is clear that a doctrine-map $D \to [\![\gamma,\gamma]\!]$, or equally an action of D on γ in $K^{\#}$, is just a D-2-cell $\gamma\colon F \Rightarrow G$ between the D-morphisms $F = (f,\bar{f})\colon A \to B$ and $G = (g,\bar{g})\colon A \to B$, as defined in (3.19) of [13].

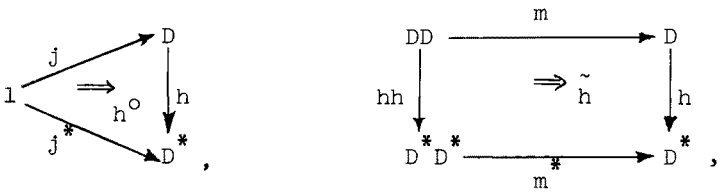
3.5 There is an analogue of (3.7) in which λ is an isomorphism: it can be called either the strong comma object or the pseudo pullback. Each of the main results stated in §1.1 has both a "lax" and a "pseudo" case. In order to make our treatment of these cases formally identical, we agree to use $\langle f,f\rangle$ in the strong sense, without introducing a new name for it, when handling the "pseudo" case. Then the $\bar{f}$ of (3.9) is invertible, etc.

4. Lax algebras

4.1 We first define a lax doctrine map $H = (h,\tilde{h},h^{\circ}): D \to D^{*}$ between two doctrines (D,m,j) and (D^{*},m^{*},j^{*}) on the 2-category K. The most elegant way of doing it would be to take the 2-category $E = [K,K]$, and to form E'' from it as we formed K'' from K in §3.2; then to observe that E'' was a strict monoidal category, the tensor product of the objects $f: D \to E$ and $f': D' \to E'$ being $ff': DD' \to EE'$; and finally to define a lax doctrine map as an object $h: D \to D^{*}$ of E'' with the structure of a monoid for this tensor product. However we say the same thing in elementary terms.

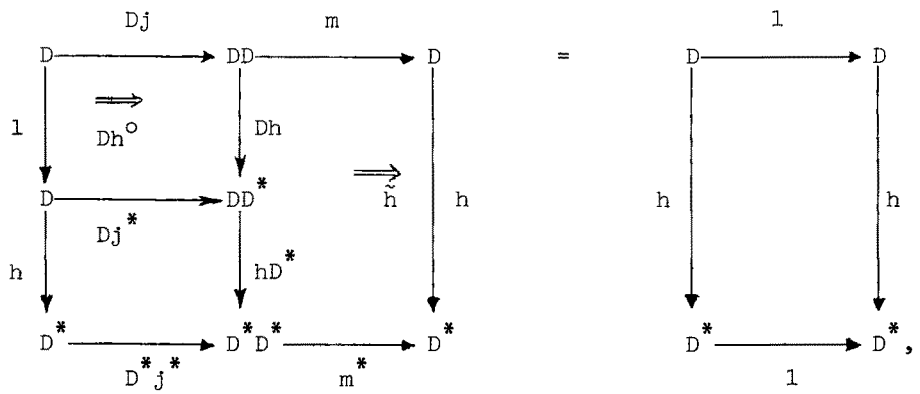
A lax doctrine map, then, consists of a 2-natural transformation $h: D \to D^{*}$ together with modifications $h^{\circ},\tilde{h}$ as in

(4.1)

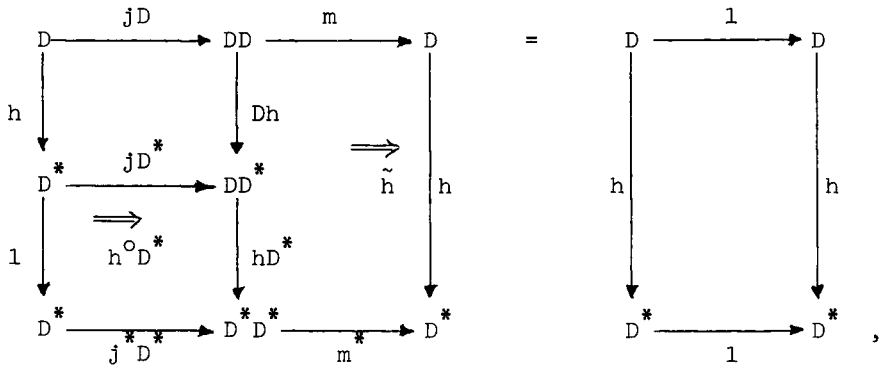


satisfying the axioms

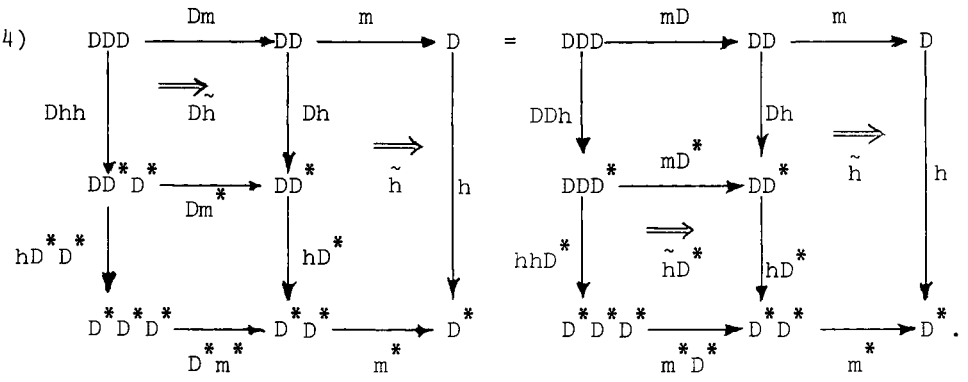
(4.2)



(4.3)



(4.4)



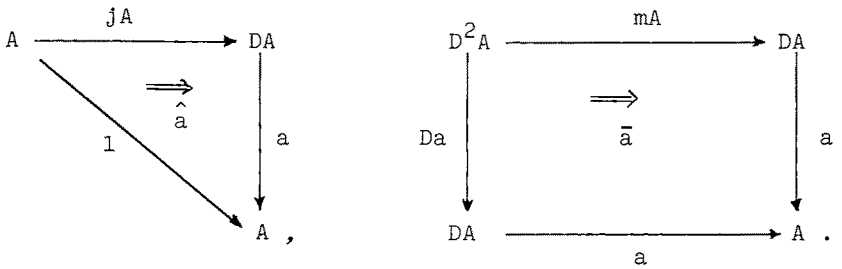
Lax doctrine maps compose by vertical pasting of diagrams (4.1); that the composite satisfies the axioms follows easily from the 2-naturality of h etc. Thus we get a category of doctrines and lax doctrine maps, which becomes a 2-category *Lax Doct* when we define a 2-cell to be a modification ρ: $h \Rightarrow k$ satisfying the obvious axioms with respect to $h^\circ, \tilde{h}, k^\circ, \tilde{k}$. It has the sub-2-category *Doct* with the same objects and the same 2-cells, but whose 1-cells are those with $\tilde{h}$ and h° identities; these are the ordinary (strict) doctrine maps, and the 2-cells are the modifications of these, as defined in [13] §3.6.

There are lots of 2-categories intermediate between *Lax Doct* and *Doct*. We call the lax doctrine map $H = (h, \tilde{h}, h^\circ)$ a pseudo doctrine map if $\tilde{h}$ and h° are isomorphisms; we call it normal or normalized if

h^o is an identity. Our arguments below are expressed in terms of the relation between *Lax Doct* and *Doct*, but they go over absolutely unchanged if we replace "lax" by "pseudo" or if we require normality.

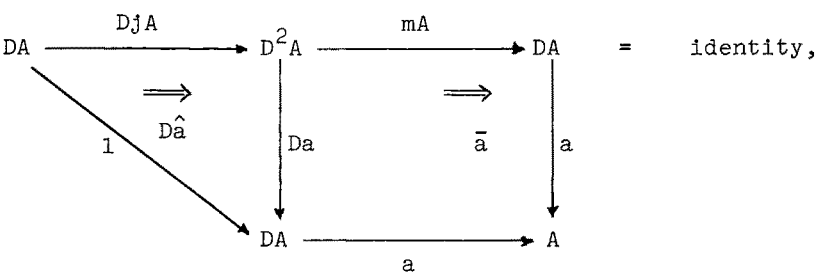
4.2 For $A \in K$, we define a lax action of the doctrine D on A to be a lax doctrine map $K: D \to \{A,A\}$ where $\{A,A\}$ is the doctrine defined in §3.2. If $k: D \to \{A,A\}$ corresponds under the isomorphism (3.1) to $a: DA \to A$, then the kind of argument familiar in the context of closed categories shows that $k^o, \tilde{k}$ correspond respectively to $\hat{a}$, $\bar{a}$ in

(4.5)

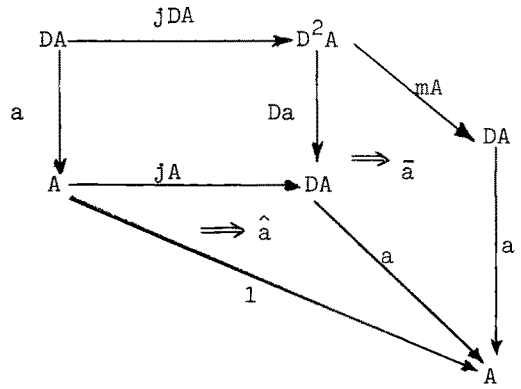


(In detail, we go from (4.1) to (4.5) by multiplying (4.1) on the right by A and composing on the tail end with the evaluation $e: \{A,A\}A \to A$. We go in the other direction by applying $\{A,-\}$ to (4.5) and composing on the front with the unit $E \to \{A,EA\}$ where $E = 1$ or D^2.) Equally simple calculations show that the axioms (4.2)-(4.4) are equivalent to the axioms (4.6)-(4.8) below:

(4.6)

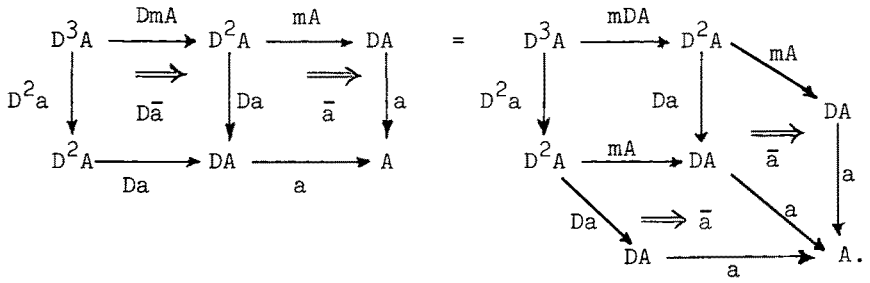


(4.7)



= identity,

(4.8)

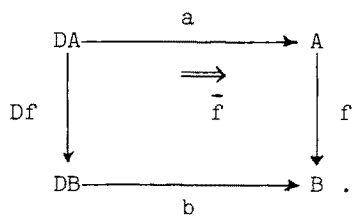


We more commonly use the term lax action of D on A not for the lax doctrine map $K = (k,\tilde{k},k^{o})$: $D \to \{A,A\}$ but for the above triple $(a,\bar{a},\hat{a})$ satisfying the axioms (4.6)-(4.8). This latter definition can be used even when A is not an object of K but a morphism $L \to K$, in which case the first definition fails because the right Kan extension $\{A,A\}$ of A along itself may not exist; and we do use it in this extended sense. The A with such a lax action of D is called a lax D-algebra; note that this definition agrees precisely with that of Street in [24] §2 above. Moreover $\bar{a}$, for instance, is an isomorphism or an identity precisely when $\tilde{k}$ is ; so we carry over the words "pseudo", "normal" from the lax doctrine maps to the lax algebras. Finally the strict doctrine maps correspond of course to the honest D-algebras, where both $\bar{a}$ and $\hat{a}$ are identities.

4.3 For $f: A \to B$ in K, and a doctrine D on K, consider a lax doctrine map $P: D \to \langle f,f \rangle$ where $\langle f,f \rangle$ is the doctrine defined in §3.3. Because the passage from (4.1)-(4.4) to (4.5)-(4.8) is purely formal, depending only on the action of E on K satisfying (3.1), we can repeat it all at the level of the action of E on K'' satisfying (3.5). We conclude that P corresponds to a lax action of D on f in K'', given by precisely the data (4.5) and the axioms (4.6)-(4.8), but with A replaced by f and with the 1-cell a and the 2-cells $\hat{a},\bar{a}$ of K replaced by 1-cells and 2-cells of K''.

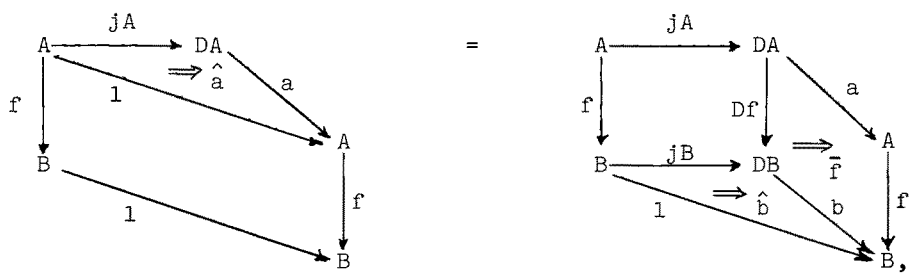
Now consider what this means. Let the morphism $Df \to f$ in K'' replacing $a: DA \to A$ in K be $(a,b,\bar{f})$ as in

(4.9)

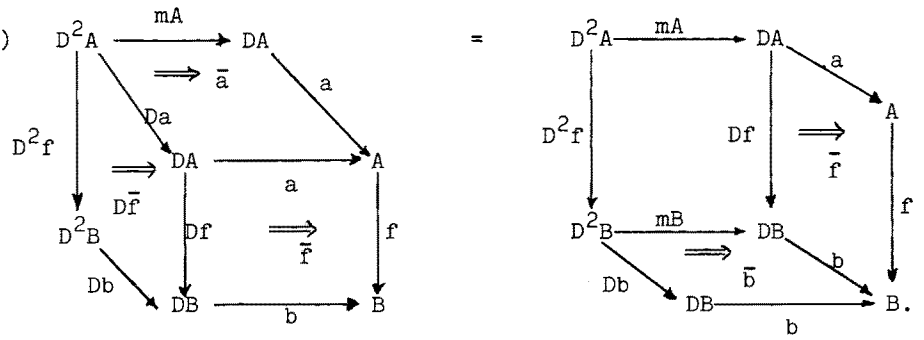


Let the 2-cells in K'' replacing the 2-cells $\hat{a},\bar{a}$ of K be the respective pairs $(\hat{a},\hat{b})$, $(\bar{a},\bar{b})$. By the definition of what a 2-cell is in K'', these have to satisfy

(4.10)



(4.11)



When it comes to the axioms (4.6)-(4.8), we observe that these are essentially about 2-cells in K'', which are only pairs of 2-cells in K - satisfying conditions indeed, but these latter automatic once given (4.10) and (4.11). Thus (4.6)-(4.8) for the lax action of D on f reduce to the original (4.6)-(4.8) for $\hat{a},\bar{a}$ and for $\hat{b},\bar{b}$. Putting all this together, we see that a lax action of D on f in K'' is the same thing as a lax action $(a,\bar{a},\hat{a})$ of D on A (corresponding of course to the lax doctrine map

$$D \xrightarrow[P]{} \langle f,f\rangle \xrightarrow[\partial_0]{} \{A,A\}),$$

a corresponding lax action $(b,\bar{b},\hat{b})$ of D on B, and an $\bar{f}$ as in (4.9) satisfying (4.10) and (4.11).

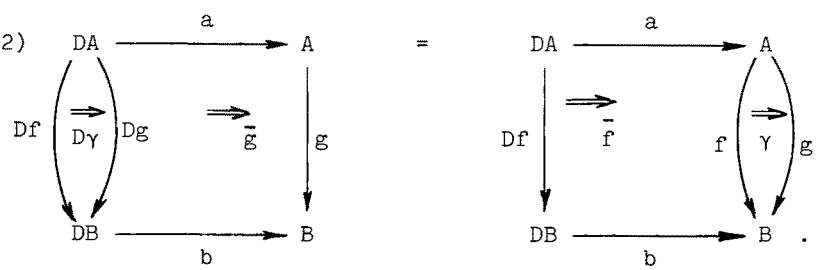
We call the pair $F = (f,\bar{f})$ a morphism of lax D-algebras $F\colon A \to B$; it is just what Street calls a "lax homomorphism" in [24] §2 above. We call it a strong morphism if $\bar{f}$ is an isomorphism, and a strict one if $\bar{f}$ is an identity. Of course an argument precisely similar to that above identifies a strict morphism $f\colon A \to B$ of lax D-algebras as a lax action of D on f in K', and hence as a lax doctrine map $Q\colon D \to [f,f]$. This shows that, for lax doctrine maps just as for strict ones, there is no difference between one into $[f,f]$, and one into $\langle f,f\rangle$ that happens to factorize through $\varepsilon\colon [f,f] \to \langle f,f\rangle$; which could alternatively have been seen by going

back to the definitions of $[f,f]$ and $\langle f,f\rangle$ as a pullback and a comma object. This point needs to be kept in mind for our argument below. (Note that we get the strong morphisms by using the different $\langle f,f\rangle$ of §3.5.)

Finally, just as we agreed to extend the definition of lax algebra given by (4.5)-(4.8) to the case where A is not an object of K but a 2-functor $L \to K$, so we extend the definition of morphism of lax algebras to this case, by (4.9)-(4.11), where f is now 2-natural and $\bar{f}$ is a modification.

4.4 Lastly in this hierarchy, we have to consider a lax doctrine map $R: D \to [\![\gamma,\gamma]\!]$, where $\gamma: f \Rightarrow g: A \to B$ in K as in (3.11), and $[\![\gamma,\gamma]\!]$ is the doctrine defined in §3.4. Arguing as above, we see that this is the same thing as a lax action of D on γ in $K^{\#}$. Let the analogue of the $a: DA \to A$ of §4.2 be $(a,b,\bar{f},\bar{g}): D\gamma \to \gamma$, so that as in (3.12) we have

(4.12)



Let the analogues of the 2-cells $\hat{a},\bar{a}$ of §4.2 be the 2-cells $(\hat{a},\hat{b})$ and $(\bar{a},\bar{b})$ of $K^{\#}$. Then $\hat{a}$ and $\bar{a}$ are as in (4.5), and $(\hat{b},\bar{b})$ similarly. The axioms (3.14) and (3.15) that these must satisfy to be 2-cells in $K^{\#}$ reduce to (4.10) and (4.11), with their analogues for $g,\bar{g}$. Finally the lax-action axioms (4.6)-(4.8) for $(\hat{a},\hat{b})$ and $(\bar{a},\bar{b})$ reduce to (4.6)-(4.8) for $\hat{a},\bar{a}$ and the analogues for $\hat{b},\bar{b}$. Summing up, we have lax D-algebra structures on A and B, morphisms $F,G: A \to B$ of these where $F = (f,\bar{f})$ and $G = (g,\bar{g})$, and finally a 2-cell $\gamma: f \Rightarrow g$ satisfying (4.12) (which is identical with the (3.19) of [13], defining D-2-cells in the case of honest D-algebras).

We therefore call such a γ satisfying (4.12) a D-2-cell even in this lax case; and we extend this definition to the case where the lax D-algebras A and B are not objects of K but 2-functors $L \to K$, in which case γ is of course a modification.

Now that we have lax D-algebras, morphisms of these, and D-2-cells, we make a 2-category $Lax\text{-}D\text{-}Alg$ of these elements, defining the various compositions by the evident pasting operations; if we restrict to the strict morphisms of lax D-algebras, we get the sub-2-category $Lax\text{-}D\text{-}Alg_*$. There is a similar restriction to the strong morphisms.

4.5 It is easy to see, although tedious to write out, that the inclusion of $Doct$ in $Lax\ Doct$ is 2-continuous: it preserves limits and cotensor products. It follows that a doctrine D, regarded as an object of $Lax\ Doct$, admits a 2-reflexion $H = (h,\tilde{h},h^\circ)$: $D \to D^*$ into $Doct$, provided that the appropriate solution-set condition is satisfied for the given D. That is to say, there is a doctrine D^* and a lax doctrine map H: $D \to D^*$ such that any lax doctrine map K: $D \to E$ is of the form $K = tH$ for a unique strict doctrine map t: $D^* \to E$. We get the same 2-continuity if we replace "lax" by "pseudo", or if we require normality; and exactly the same considerations apply.

I do not intend to study in this paper the conditions under which, for a given D, the solution-set condition is satisfied and the reflexion D^* exists. Certainly when D is $\mathcal{D}_\circ$- for a small club $\mathcal{D}$ in $Cat/\underline{S}$ it is immediate that D^* exists, as we see below in §4.10. I suppose the methods of Barr [1] or of Dubuc [5] will show D^* to exist if say K is locally presentable and D has a rank. In part I leave the question aside because I haven't thought out the details; but in part too because I suspect that it doesn't really matter: it seems to me likely that D^* always exists as a doctrine on CAT when D is one on Cat, or that D^* exists on a suitable completion of K in a bigger universe when D is given on K. The question is connected with that of extensions of D adumbrated in the second paragraph of §3.2, and

really deserves a fuller treatment in its own right at another time. Our concern in this paper is with the consequences of the existence of D^*, and the club case is sufficient to show that our considerations are not vacuous.

A final point in this vein: I assumed in [10] above, for the proof of Theorem 3.1 of that paper, that D^* (there called D') does exist. If it does so only in some bigger universe, that in no way affects its utility for the proof of that theorem. And if it does not, there is no doubt that that theorem remains true, replacing " D'-category" by "normalized pseudo D-category" and so on; we should merely be denied the use of the results of [10] §1 as they stand, and should have to prove them over again in the pseudo-algebra case, which would be an awful nuisance. But no-one would doubt that such purely formal diagram-arguments, which clearly remain such when translated from D'-terms to pseudo-D-terms, would be valid independently of the "size" of D - even though the metatheorem involved is felt rather than stated.

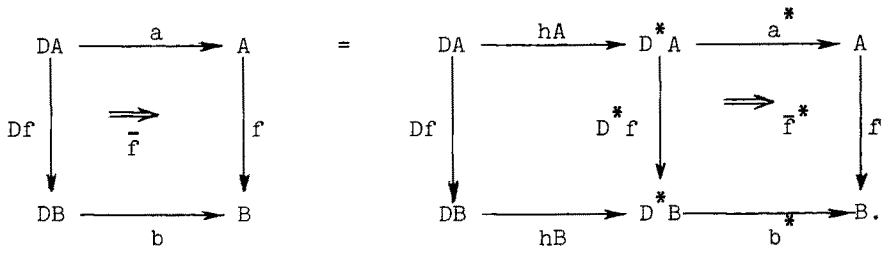
4.6 Coming back to the main point, we identify the D^*-algebras A in K - something we have anticipated in the last paragraph. Such an algebra is given by a strict doctrine map $t: D^* \to \{A,A\}$; and these are in bijection under $K = tH$ with lax doctrine maps $K: D \to \{A,A\}$. Thus the D^*-algebras in K are just the lax D-algebras. Moreover if K corresponds as before to the lax action $(a,\bar{a},\hat{a})$, while t corresponds to the honest action $a^*: D^*A \to A$, the relation $K = tH$ translates into

$$(4.13) \qquad a = a^*.hA, \quad \bar{a} = a^*.\tilde{hA}, \quad \hat{a} = a^*.h^{\circ}A.$$

Otherwise put, every lax D-action is of the form (4.13) for a unique D^*-action a^*.

Similarly a morphism $F^* = (f,\bar{f}^*): A \to B$ of D^*-algebras is given by a strict doctrine map $D^* \to \langle f,f \rangle$ and hence by a lax doctrine map $D \to \langle f,f \rangle$, which corresponds to a morphism $F = (f,\bar{f}): A \to B$ of lax D-algebras. The connexion between F and F^* is given by

(4.14)



Again, we can say that any F is of the form (4.14) for a unique F^*. Since moreover ε: $[f,f] \to \langle f,f \rangle$ is a strict doctrine map, $D^* \to \langle f,f \rangle$ factorizes through $[f,f]$ if and only if $D \to \langle f,f \rangle$ does so; whence

(4.15) $\bar{f} = 1$ if and only if $\bar{f}^* = 1$,

that is, F is strict if and only if F^* is so.

Finally doctrine maps $D^* \to [\![\gamma,\gamma]\!]$ correspond to lax doctrine maps $D \to [\![\gamma,\gamma]\!]$, and we conclude that

(4.16) the D^*-2-cells γ: $F^* \Rightarrow G^*$ are just the D-2-cells γ: $F \Rightarrow G$.

Moreover the correspondences (4.14) and (4.15) respect the various kinds of composition; the only case that is not absolutely trivial is that of vertical pasting of diagrams (4.14), which clearly produces another diagram of the same kind by the naturality of h. Thus we have identified the 2-category D^*-$A\ell g$ [resp. D^*-$A\ell g_*$], namely as essentially the 2-category Lax-D-$A\ell g$ [resp. Lax-D-$A\ell g_*$]. Since we also know the forgetful 2-functor to K, we do indeed have an implicit determination of D^*. (This whole section remains valid for the stronger $\langle f,f \rangle$ of §3.5.)

4.7 For the purposes of our proof, we need the results of §4.6 not only for algebras A that are objects of K but for algebras that are 2-functors A: $L \to K$ (L being K itself in our applications). Since for such an A the right Kan extension $\{A,A\}$ of A along itself need not exist, we cannot argue directly as in §4.6. Instead we have to argue object-wise, using the results of §4.6.

Let $A: L \to K$ then be a lax D-algebra, with lax action $(a,\bar{a},\hat{a})$. For each $X \in L$ we get a lax D-action $(aX,\bar{a}X,\hat{a}X)$ on AX, and hence as in (4.13) a unique D^*-action a^*X on AX such that $aX = a^*X.hAX$, $\bar{a}X = a^*X.\tilde{h}AX$, and $\hat{a}X = a^*X.h^{\circ}AX$. What has to proved is that a^* is 2-natural, so that a^* is indeed a D^*-action on A, and of course the unique one satisfying (4.13).

To show the naturality of a^*, consider $\phi: X \to Y$ in L. Because a is natural, we have commutativity in

(4.17)
$$\begin{array}{ccc} DAX & \xrightarrow{aX} & AX \\ {\scriptstyle DA\phi}\downarrow & & \downarrow{\scriptstyle A\phi} \\ DAY & \xrightarrow[aY]{} & AY \end{array} ;$$

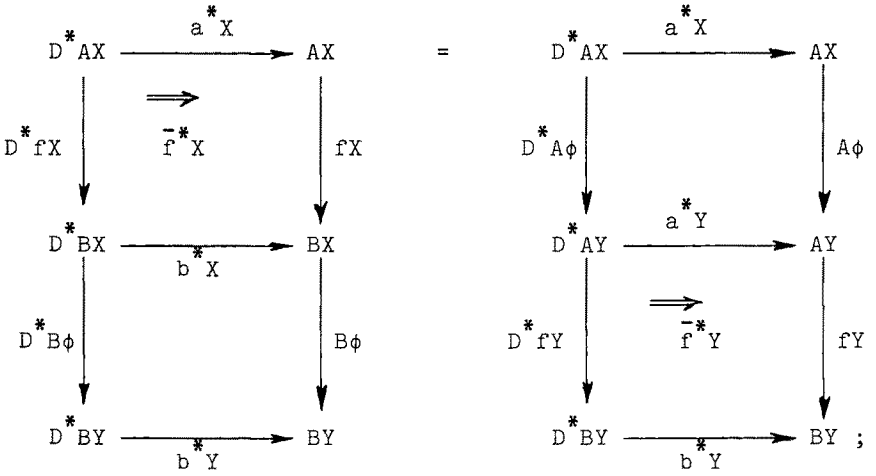
and $A\phi$ is in fact a strict morphism of lax D-algebras, the axioms (4.10) and (4.11) being satisfied because $\hat{a}$ and $\bar{a}$ are modifications. Hence by (4.14) and (4.15) we have commutativity in

(4.18)
$$\begin{array}{ccc} D^*AX & \xrightarrow{a^*X} & AX \\ {\scriptstyle D^*A\phi}\downarrow & & \downarrow{\scriptstyle A\phi} \\ D^*AY & \xrightarrow[a^*Y]{} & AY \end{array} ,$$

showing the naturality of a^*. For its 2-naturality, let $\alpha: \phi \Rightarrow \psi: X \to Y$ in L. Then $A\alpha: A\phi \Rightarrow A\psi$ is a D-2-cell by the 2-naturality of a, whence it is a D^*-2-cell by (4.16), giving the 2-naturality of a^*.

Next, let $A,B: L \to K$ be lax D-algebras and $F = (f,\bar{f}): A \to B$ a morphism of these. Then $FX = (fX,\bar{f}X): AX \to BX$ is a morphism of lax D-algebras in K, which therefore corresponds as in (4.14) to a unique morphism $F^*X = (fX,\bar{f}^*X)$ of D^*-algebras in K, where $\bar{f}X = \bar{f}^*X.hAX$. If

F is strict, so is F^*, and there is no more to prove; but in the general case we must show that $\bar{f}^*$ is a modification. This means the equality



which we have because the correspondence (4.14) respects composition, because under this correpondence $\bar{f}^*X$ corresponds to $\bar{f}X$ and (4.18) to (4.17), and because $\bar{f}$ is a modification by hypothesis.

At the next level, that of D-2-cells, there is by (4.16) nothing to prove.

We conclude that the results of §4.6, expressed in the form that every lax D-action $(a,\bar{a},\hat{a})$ is of the form (4.13) for a unique D^*-action a^*, etc., remain true when we take the algebras in the present, wider, sense.

4.8 We can now proceed rapidly to the proof of our first main result.

Theorem 4.1 Let D be a doctrine on the 2-complete 2-category K. Suppose that, considered as an object of *Lax Doct*, D admits a reflexion $H = (h,\tilde{h},h^o)$: $D \to D^*$ into *Doct*. Let s: $D^* \to D$ be the unique strict doctrine map satisfying

(4.19) $sH = 1$,

which exists because $1\colon D \to D$ is a strict doctrine map and a fortiori a lax one. Then there is a modification $\eta\colon 1 \Rightarrow hs$ such that $s\eta = 1$ and $\eta h = 1$. The same is true if we require normality, or if we replace "lax" by "pseudo"; in the last case η is an isomorphism.

Proof. The last sentence of the theorem is immediate from the comments at the end of §4.1, from those in the first paragraph of §4.5, and from the details of the following proof, in the light of §3.5.

Observe that, by the definition of composition of lax doctrine maps, (4.19) may be written as

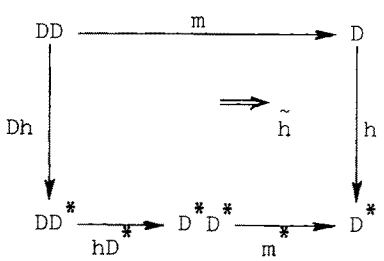
$$(4.20)\qquad sh = 1, \quad s\tilde{h} = 1, \quad sh^{\circ} = 1.$$

Because s is a doctrine map, $m.sD\colon D^*D \to D$ is a D^*-action on D, as in (3.8) of [13]. The lax D-action on D which corresponds to this under (4.13) is, by (4.20), the lax action (m,1,1); that is to say, the strict action $m\colon D^2 \to D$ of D on itself.

The D^*-action $m^*\colon D^*D^* \to D^*$ of D^* on itself corresponds by (4.13) to the lax action $(m^*.hD^*, m^*.\tilde{h}D^*, m^*.h^{\circ}D^*)$ of D on D^*.

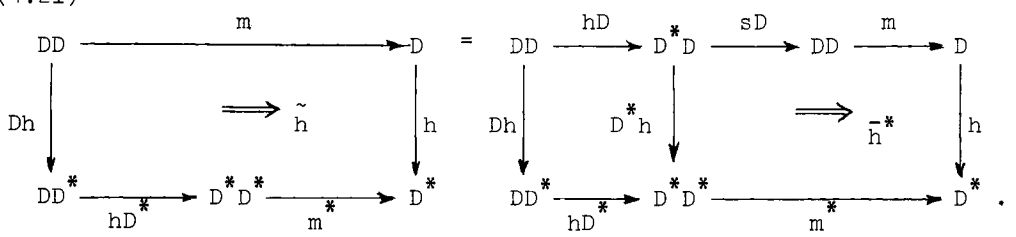
The second diagram of (4.1) may be written as

(4.20)



I assert that $\tilde{H} = (h,\tilde{h})$ is a morphism $D \to D^*$ of lax D-algebras. For the axiom (4.10) in this case reduces to (4.3), and the axiom (4.11) reduces to (4.4). It follows as in (4.14) that there is a unique morphism $H^* = (h,\bar{h}^*)\colon D \to D^*$ of D^*-algebras satisfying

(4.21)



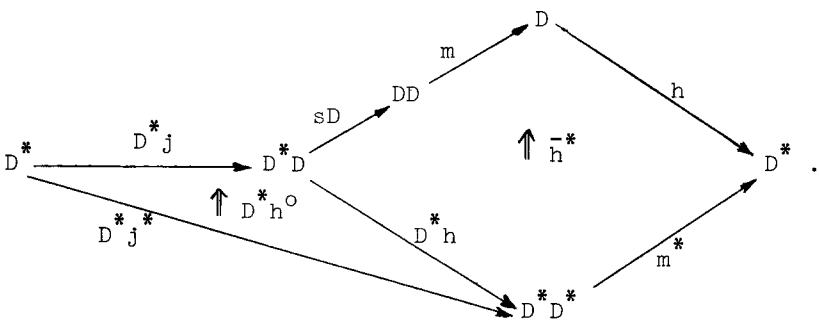
On the other hand $s\colon D^* \to D$ is a strict morphism of D^*-algebras, and hence a strict morphism of lax D-algebras. Since $s\tilde{H} = 1$ by (4.20) and since the correspondence (4.14) respects composition, we have

(4.22) $\quad sH^* = 1.$

We are now in a position to apply Theorem 2.5; with of course D replaced by D^*, DA replaced by $D^*1 = D^*$, B replaced by D, and H replaced by H^*. The r of Theorem 2.5 now becomes, by (2.4), the composite $sj^*\colon 1 \to D$, which is $j\colon 1 \to D$ since s is a doctrine map. The ξ of Theorem 2.5 is therefore to be a modification $j^* \Rightarrow hj$; and we take it to be the h° of (4.1). Since $sh^\circ = 1$ by (4.20), we have $s\xi = 1$, so that by Theorem 2.5 again we have $s\eta = 1$.

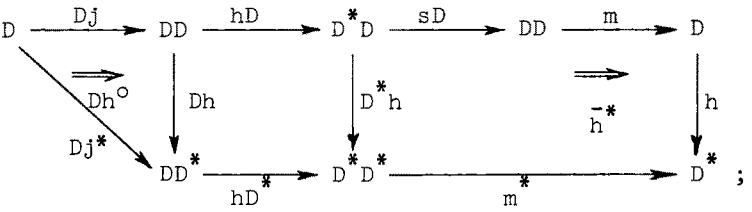
According to (2.22), the $\eta\colon 1 \Rightarrow H^*s$ produced by Theorem 2.5 is the composite

(4.23)



It remains only to give a proof that $\eta h = 1$. But by the 2-naturality of h we have $D^*h^\circ.h = hD^*.Dh^\circ$, so that (4.23) composed with h becomes

(4.24)



;

however the composite of the two rectangles on the right is $\tilde{h}$ as in (4.21), whence (4.24) is the identity by (4.2). This completes the proof. □

4.9 In my paper [10] above in this volume, I stated at the end of §3.1, and used in the proof of Theorem 3.1, only one very small part of the above theorem: namely that s: $D^* \to D$ had a right inverse h, as in (4.20). The D^* in question here is that for the normalized pseudo case, which is the case that arises in [10].

In §3.3 of [10] I called the doctrine D flexible if s: $D^* \to D$ had a right inverse t: $D \to D^*$ that was a (strict) doctrine map; this notion was used in Theorem 3.3 of [10]. It was stated there that the doctrine D^* is always flexible: we now justify this.

The proof is elementary, not depending on Theorem 4.1, and applies equally in the lax case as in the pseudo case, with or without normality.

Proposition 4.2 Let H: $D \to D^*$ be the reflexion of D, considered as an object of *Lax Doct*, into *Doct*; and let K: $D^* \to D^{**}$ be the reflexion of D^*, considered as an object of *Lax Doct*, into *Doct*. Let q: $D^{**} \to D^*$ be the unique strict doctrine map such that $qK = 1$. Then there is a strict doctrine map p: $D^* \to D^{**}$ with $qp = 1$.

Proof Since KH: $D \to D^{**}$ is a lax doctrine map, and since H: $D \to D^*$ is the reflexion, we have $KH = pH$ for a unique strict doctrine map p: $D^* \to D^{**}$. Then $qKH = qpH$; but again $qKH = H = 1H$ since $qK = 1$. Since qp and 1 are both strict, the uniqueness property for the reflexion H gives $qp = 1$. □

In the pseudo case, since q is an equivalence of endo-2-functors by Theorem 4.1, it follows that $pq \cong 1$. I suspect, but have not proved, that the isomorphism $pq \cong 1$ can be chosen to be a doctrine-modification, so that D^* and D^{**} are equivalent doctrines. This would be very convenient if it were true, and the question should be looked into.

4.10 We consider finally the special but important case when $D = \mathcal{D}\circ -$ for some club $\mathcal{D}$ in $Cat/\underline{S}$. We again refer the reader to §1 and §10 of [9] above for general facts about clubs of this kind.

To give a lax D-algebra structure, which we shall also call a lax $\mathcal{D}$-algebra structure, on a category A, we have first to give a functor $a: \mathcal{D}\circ A \to A$, or equally a morphism $\mathcal{D} \to \{A,A\}$ in $Cat/\underline{S}$; here $\{A,A\}$ is now the "rich endo-functor category" of [9] §9, and not the right Kan extension of (3.2) above. So for $T \in \mathcal{D}$ we have to give $|T|: A^n \to A$ where $\Gamma T = n$; and for $f: T \to S$ in $\mathcal{D}$ we have to give the generalized natural transformation $|f|: |T| \Rightarrow |S|$ of type Γf. Since this is to be a functor, we must require

$$(4.25) \qquad |fg| = |f|\,|g| \text{ and } |1| = 1.$$

Next we have to give $\hat{a}$ as in (4.5), a natural transformation with components

$$\hat{a}_A: A \to |\underline{1}|(A),$$

that is, a natural transformation

$$\hat{a}: 1 \to |\underline{1}|.$$

In the normal case we require $\hat{a} = 1$, which involves demanding that $|\underline{1}| = 1$. In the pseudo case we have to provide $\hat{a}$ with an inverse, so we must also give a natural transformation

$$\hat{a}': |\underline{1}| \to 1,$$

and demand that

$$\hat{a}\ \hat{a}' = 1 \quad \text{and} \quad \hat{a}'\hat{a} = 1.$$

Similarly for the $\bar{a}$ of (4.5); it is to have components

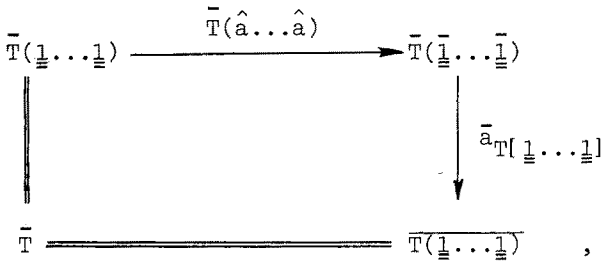
$$\bar{a}_{T[S_1 \ldots S_n][A_1 \ldots A_m]} : |T|(|S_1|(A_1 \ldots), \ldots, |S_n|(\ldots A_m)) \rightarrow |T(S_1 \ldots S_n)|(A_1 \ldots A_m),$$

with inverses $\bar{a}'$ provided in the pseudo case. All these data are then to satisfy the further axioms corresponding to (4.6)-(4.8).

It is clear from [9] §10 that such a lax $\mathcal{D}$-algebra is itself an algebra for a club $\mathcal{D}^*$ in $Cat/\underline{\underline{S}}$, which we have in effect just described by its generators and relations. Explicitly, the objects of $\mathcal{D}^*$ are generated by objects $\bar{T}$ in bijection with the objects T of $\mathcal{D}$, and with $\Gamma\bar{T} = \Gamma T$. These are subjected to no relations except in the normal case, where we impose the relation $\bar{\underline{\underline{1}}} = \underline{\underline{1}}$. The generators for the morphisms of $\mathcal{D}^*$ consist of an $\bar{f}: \bar{T} \rightarrow \bar{S}$ with $\Gamma\bar{f} = \Gamma f$ for each $f: T \rightarrow S$ in $\mathcal{D}$; of an $\hat{a}: \underline{\underline{1}} \rightarrow \bar{\underline{\underline{1}}}$, with $\Gamma\hat{a} = 1$, which is to be omitted in the normal case; and of an

$$\bar{a}_{T[S_1 \ldots S_n]} : \bar{T}(\bar{S}_1, \ldots, \bar{S}_n) \rightarrow \overline{T(S_1, \ldots, S_n)},$$

with $\Gamma\bar{a} = 1$, for each object $T[S_1, \ldots, S_n]$ of $\mathcal{D} \circ \mathcal{D}$. In the pseudo case, these are to be augmented by further generators $\hat{a}'$ and $\bar{a}'_{T[S_1 \ldots S_n]}$ in the reversed senses. The relations between expanded instances of these generators are $\bar{f}\bar{g} = \overline{fg}$ and $\bar{1}_T = 1_{\bar{T}}$ corresponding to (4.25); the relations $\hat{a}\hat{a}' = 1$, $\hat{a}'\hat{a} = 1$, $\bar{a}\bar{a}' = 1$, $\bar{a}'\bar{a} = 1$ in the pseudo case only; and finally the relations corresponding to (4.6)-(4.8). The first of these for instance asserts the commutativity of



and the other two are equally easily written down.

It is further easy to check that a $\mathcal{V}^*$-morphism is the same thing as a D^*-morphism, and a $\mathcal{V}^*$-2-cell as a D^*-2-cell; so that D^* is indeed the doctrine $\mathcal{V}^*\circ-$ for the club $\mathcal{V}^*$.

We can regard h and s as maps h: $\mathcal{V} \to \mathcal{V}^*$ and s: $\mathcal{V}^* \to \mathcal{V}$; they might a priori lie in $Cat/^*\underline{S}$, by the general principles of [9], but it is easy to see that in fact they lie in $Cat/\underline{S}$. That this is so for s follows from the considerations of [9] §10.7; s is clearly defined on generators and relations by $s\bar{T} = T$, $s\bar{f} = f$, $s\,\hat{a} = 1$, $s\,\bar{a} = 1$. As for h, we have only to recall just how a $\mathcal{V}^*$-algebra gives a lax $\mathcal{V}$-algebra: clearly $hT = \bar{T}$, $hf = \bar{f}$, while the components of h° and of $\tilde{h}$ are given by $\hat{a}$ and $\bar{a}$ respectively.

It follows that h embeds $\mathcal{V}$ as a full subcategory of $\mathcal{V}^*$, onto which s reflects $\mathcal{V}^*$; with the consequences for commutativity of diagrams described in §1.4 above.

4.11 It remains only to prove the further assertion of [10] §3.3, namely that D is flexible if it is $\mathcal{V}\circ-$ for a club $\mathcal{V}$ in $Cat/\underline{S}$ whose objects are freely generated. I recall that the definition of flexibility in [10] was concerned with the D^* for the normalized pseudo case; the result we shall prove is independent of normality, but does require that we be pseudo rather than lax.

Proposition 4.3. Let $\mathcal{D}$ be a club in $Cat/\underline{S}$ whose discrete club $|\mathcal{D}|$ of objects is freely generated, and let $\mathcal{D}^*$ be the club for pseudo $\mathcal{D}$-algebras, normalized or not, with $s: \mathcal{D}^* \to \mathcal{D}$ the canonical map of clubs. Then $st = 1$ for some club-map $t: \mathcal{D} \to \mathcal{D}^*$ in $Cat/\underline{S}$.

Proof. As in §4.10 we already have $h: \mathcal{D} \to \mathcal{D}^*$ with $sh = 1$. Let G be the set of free generators of $|\mathcal{D}|$. For $G \in G$, set $tG = hG$. Take $t: |\mathcal{D}| \to \mathcal{D}^*$ to be the unique club-map extending $t: G \to \mathcal{D}^*$ (by club-map in this proof we always mean club-map in $Cat/\underline{S}$). Then for any $T \in \mathcal{D}$ we have an isomorphism $\alpha_T: tT \to hT$ which is a composite of expanded instances of $\bar{a}$ and $\hat{a}$. Define $t: \mathcal{D} \to \mathcal{D}^*$ to be the conjugate of h under α. Since s is a club-map and since $s\bar{a} = 1$ and $s\hat{a} = 1$, we have $st = 1$. Note that t is in $Cat/\underline{S}$ since h is and since $\Gamma\bar{a} = 1$, $\Gamma\hat{a} = 1$. It remains to prove that t is a club-map, that is, that we have commutativity in

(4.26)
$$\begin{array}{ccc} \mathcal{D}\circ\mathcal{D} & \xrightarrow{m} & \mathcal{D} \\ {\scriptstyle t\circ t}\downarrow & & \downarrow{\scriptstyle t} \\ \mathcal{D}^*\circ\mathcal{D}^* & \xrightarrow[m^*]{} & \mathcal{D}^* \end{array} .$$

But we have forcibly made (4.26) commute on objects, by taking $t: |\mathcal{D}| \to \mathcal{D}^*$ to be a club-map. So we need only check that it commutes on morphisms. But the composite of (4.26) with $s: \mathcal{D}^* \to \mathcal{D}$ commutes, since s is a club-map with $st = 1$. Hence (4.26) itself commutes, since s is an equivalence of categories and so in particular a faithful functor. □

5. Pseudo distributive laws

5.1 Let (D,m,j) and (D',m',j') be doctrines on a 2-category K. To give a 2-functor $\tilde{D}_*$ rendering commutative

(5.1)
$$\begin{array}{ccc} D'\text{-}Alg_* & \xrightarrow{\tilde{D}_*} & D'\text{-}Alg_* \\ {\scriptstyle U'_*}\downarrow & & \downarrow{\scriptstyle U'_*} \\ K & \xrightarrow[D]{} & K \end{array},$$

where U'_* is the forgetful 2-functor, and to give 2-natural $\tilde{j}_*\colon 1 \to \tilde{D}_*$ and $\tilde{m}_*\colon \tilde{D}_*^2 \to \tilde{D}_*$ such that $U'_*\tilde{j}_* = j\,U'_*$ and $U'_*\tilde{m}_* = m\,U'_*$, is to give a distributive law of D' over D in the classical sense of Beck [2]. Because U'_* is faithful, $\tilde{j}_*$ and $\tilde{m}_*$ are unique if they exist, and automatically make of $\tilde{D}_*$ a doctrine on $D'\text{-}Alg_*$.

We get a much more lax notion of distributive law if we ask the doctrine D to lift not through $U'_*\colon D'\text{-}Alg_* \to K$ but through its extension $U'\colon D'\text{-}Alg \to K$. The pseudo distributive laws of this paper occupy an intermediate place, being only a mild extension of the classical ones.

For a pseudo distributive law we suppose that the 2-functor D is given a lifting $\tilde{D}_*$ as in (5.1). Then, as we shall show shortly, $\tilde{D}_*$ has a canonical extension to a 2-functor $\tilde{D}$ satisfying

(5.2)
$$\begin{array}{ccc} D'\text{-}Alg & \xrightarrow{\tilde{D}} & D'\text{-}Alg \\ {\scriptstyle U'}\downarrow & & \downarrow{\scriptstyle U'} \\ K & \xrightarrow[D]{} & K \end{array}.$$

We suppose that j lifts as above to a $\tilde{j}_*\colon 1 \to \tilde{D}_*$, and we observe that the components of $\tilde{j}_*$ are also those of a 2-natural $\tilde{j}\colon 1 \to \tilde{D}$, satisfying $U'\tilde{j} = jU'$. What we do not suppose is that m lifts to an $\tilde{m}_*$. Instead, we suppose given an $\tilde{m}\colon \tilde{D}^2 \to \tilde{D}$ satisfying $U'\tilde{m} = mU'$; since U' is not faithful, $\tilde{m}$ is not uniquely determined by this - the morphism part of each component is so, but not the 2-cell part - so more data is involved. We further suppose that each component of $\tilde{m}$ is a strong D'-morphism. (If we were to replace strong here by strict, we should be back in the classical case: this then is what our relaxation consists in.) Finally we suppose that $(\tilde{D},\tilde{m},\tilde{j})$ satisfy the axioms for a doctrine; these are no longer automatic as in the classical case.

Examples of such pseudo distributive laws will be given in §6.

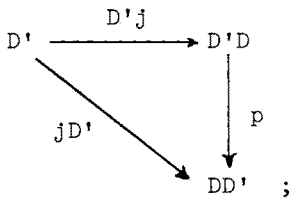
5.2 Beck showed in [2] that to give a classical distributive law amounted to giving a $p\colon D'D \to DD'$ satisfying four axioms. We now carry out a corresponding analysis of a pseudo distributive law.

It turns out that to give a pseudo distributive law we must first give a 2-natural $p\colon D'D \to DD'$ satisfying three of Beck's axioms, to wit

(5.3)
$$\begin{array}{ccc} D & \xrightarrow{j'D} & D'D \\ & \searrow^{Dj'} & \downarrow p \\ & & DD', \end{array}$$

(5.4)
$$\begin{array}{ccccc} D'D'D & \xrightarrow{D'p} & D'DD' & \xrightarrow{pD'} & DD'D' \\ \downarrow m'D & & & & \downarrow Dm' \\ D'D & & \xrightarrow{\quad p \quad} & & DD', \end{array}$$

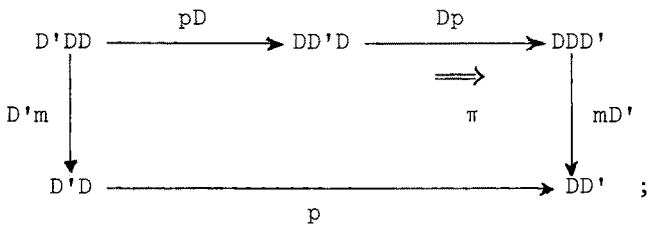
(5.5)



;

then we must give an invertible modification π sitting in what, if π were 1, would be Beck's fourth axiom:

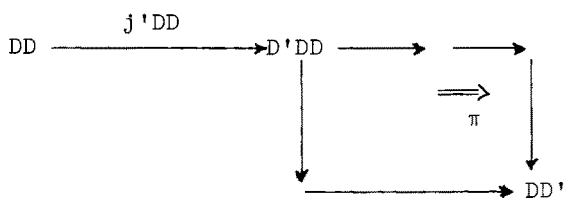
(5.6)



;

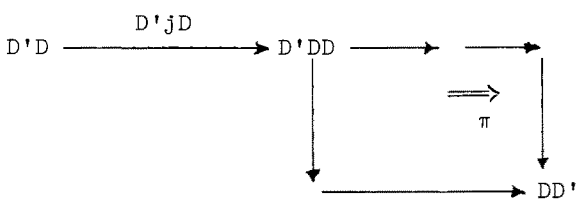
and finally we must subject π to five axioms; the reader will note that in each of these equality at the level of 1-cells is automatic by (5.3)-(5.5) and naturality:

(5.7)



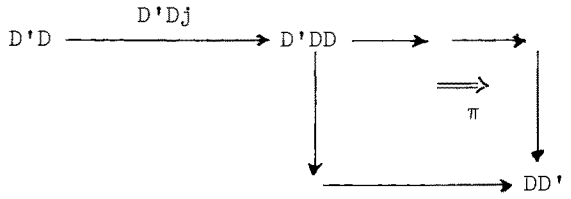
= identity,

(5.8)



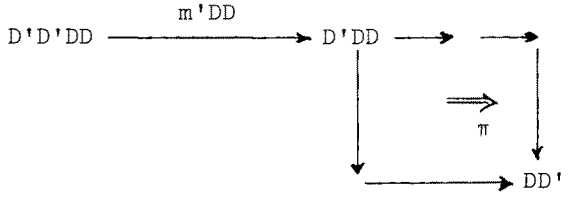
= identity,

(5.9)

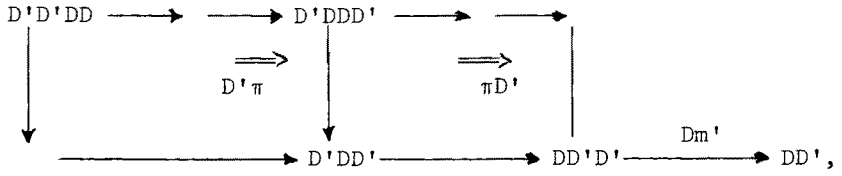


= identity,

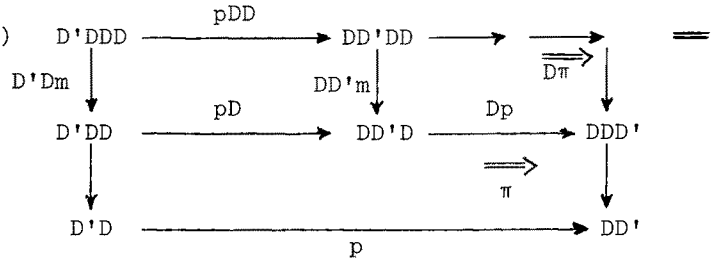
(5.10)



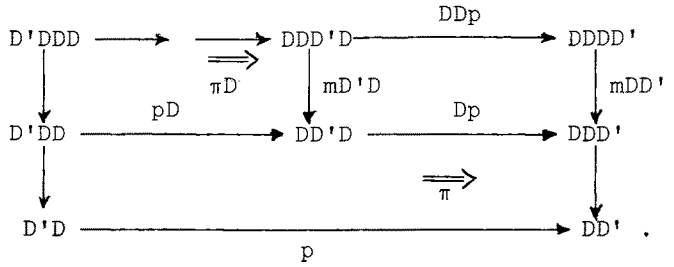
=



(5.11)



=



.

We now justify the above assertion. First, to give $\tilde{D}_*$ as in (5.1) is by [13] §3.3 to give an action q: $D'DU'_* \to DU'_*$ of D' on DU'_*. By [13] Proposition 2.1 to give a 2-natural q as above is to give a 2-natural p: $D'D \to D'U'_*F'_*$ where F'_* is the left adjoint of U'_*; that is, to give a 2-natural p: $D'D \to DD'$. The conditions for q to be an action translate at once into (5.3) and (5.4).

We give the explicit form of $\tilde{D}_*$ in terms of p; but we may as well give at the same time its canonical extension $\tilde{D}$, and write $\tilde{D}A$ rather than $\tilde{D}_*A$, etc. In doing so we use A both for an object of K and for a D'-algebra (A,a') consisting of the object A together with the D'-action a': $D'A \to A$. We then have

(5.12) $\quad \tilde{D}A = (DA,\ Da'.pA),$

the D'-algebra with object DA and with D'-action

(5.13) $\quad D'DA \xrightarrow[pA]{} DD'A \xrightarrow[Da']{} DA.$

For a morphism $F' = (f,\bar{f}')$: $A \to B$ of D'-algebras, we have a D'-morphism

(5.14) $\quad \tilde{D}F' \ = \ (Df,\ D\bar{f}'.pA)$

as in

(5.15)
$$\begin{array}{ccccc} D'DA & \xrightarrow{pA} & DD'A & \xrightarrow{Da'} & DA \\ {\scriptstyle D'Df}\downarrow & & {\scriptstyle DD'f}\downarrow & \Rightarrow\ {\scriptstyle D\bar{f}'} & \downarrow{\scriptstyle Df} \\ D'DB & \xrightarrow[pB]{} & DD'B & \xrightarrow[Db']{} & DB \end{array} .$$

Finally a D'-2-cell γ: $F' \Rightarrow G'$ gives a D'-2-cell $\tilde{D}\gamma$: $\tilde{D}F' \Rightarrow \tilde{D}G'$ directly by

(5.16) $\tilde{D}\gamma = D\gamma$.

The 2-functor $\tilde{D}_*$ is what we get by restricting $\tilde{D}$ to strict D'-morphisms; observe that if f: $A \to B$ is a strict D'-morphism then $\tilde{D}_*f = \tilde{D}f$ is just the strict D'-morphism Df: $DA \to DB$. Since what we have said about $\tilde{D}_*$ is all justified by our general remarks above, all that needs separate verification is that (5.15) really is a D'-morphism when F' is not strict, that (5.16) really is a D'-2-cell when F',G' are not strict, and that $\tilde{D}$ really is a 2-functor: all this is easy.

The next thing we wanted was $\tilde{j}_*$: $1 \to \tilde{D}_*$ with $U'_*\tilde{j}_* = jU'_*$. The last requirement forces the component $\tilde{j}_*A$: $A \to \tilde{D}_*A$ to be jA: $A \to DA$; the desired $\tilde{j}_*$ exists precisely when jA: $A \to DA$ is indeed a strict D'-morphism, and this condition reduces to (5.5). Once we have this, it is immediate that jA: $A \to DA$ is 2-natural not only for <u>strict</u> D'-morphisms $A \to B$ but for all; so we in fact get a 2-natural $\tilde{j}$: $1 \to \tilde{D}$ with the same components:

(5.17) $\tilde{j}A = \tilde{j}_*A = jA$;

of course we have $U'\tilde{j} = jU'$.

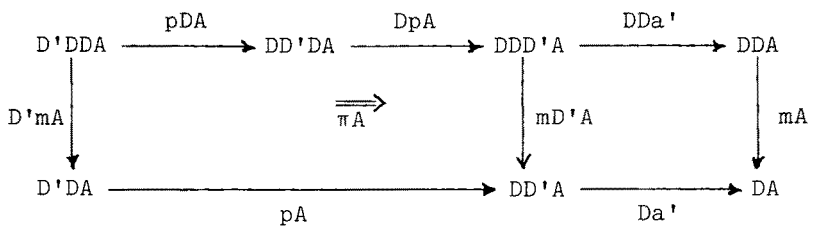
Then we wanted $\tilde{m}$: $\tilde{D}^2 \to \tilde{D}$ with $U'\tilde{m} = mU'$. The last requirement forces the morphism-part of the component $\tilde{m}A$: $D^2A \to DA$ to be mA. Thus

(5.18) $\tilde{m}A = (mA, \bar{m}'A)$

for some invertible 2-cell $\bar{m}'A$ (since $\tilde{m}A$ is to be <u>strong</u>). The form of $\bar{m}'A$ is determined by the requirement that $\tilde{m}$ be 2-natural. Indeed,

mere naturality in A of $\tilde{m}A$, and that merely for strict D'-morphisms $f: A \to B$, suffices to fix the form of $\bar{m}'A$. First, taking $f: A \to B$ to be $a': D'A \to A$, we easily see that $\bar{m}'A$ is of the form

(5.19)



for an invertible πA; next, taking $f: A \to B$ to be $D'f: D'A \to D'B$, we easily see that π must be a modification, as in (5.6). Then no more is needed to make $\tilde{m}$ 2-natural: the verification is simple.

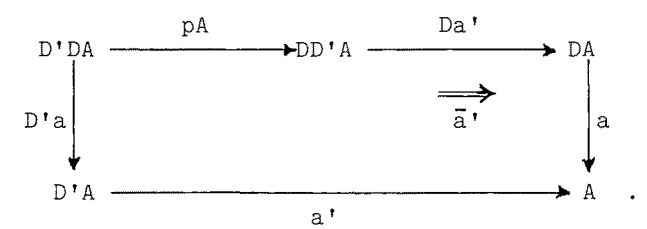
The further things we do need are that $\tilde{m}A$ satisfy the axioms for a D'-morphism - these reduce to (5.7) and (5.10) - and that $(\tilde{D},\tilde{m},\tilde{j})$ satisfy the axioms for a doctrine; these reduce to (5.8), (5.9), and (5.11) (automatic in the classical case $\pi = 1$, but not here).

This completes our justification of the above analysis of a pseudo distributive law in terms of p and π.

5.3 We now look at the 2-category $\tilde{D}$-$A\ell g$ and describe its elements in elementary terms involving D,D',p,π and elements of K.

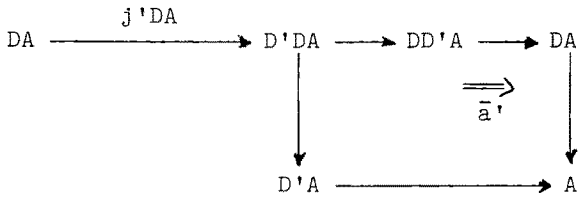
A $\tilde{D}$-algebra A is a D'-algebra A (with D'-action $a': D'A \to A$) together with a $\tilde{D}$-action $\tilde{a}: \tilde{D}A \to A$. The latter is to be a D'-morphism $\tilde{a} = (a,\bar{a}')$: $DA \to A$, which in view of (5.13) has the form

(5.20)

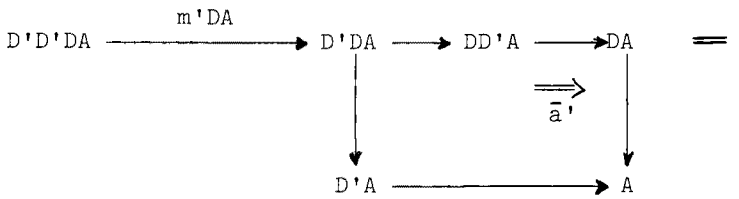


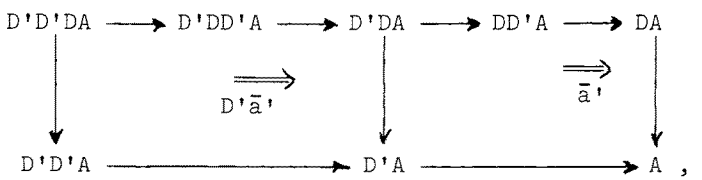
There are four axioms to be satisfied: the first two say that $\tilde{a}$ really is a D'-morphism, and the second two say that it is a D-action:

(5.21)

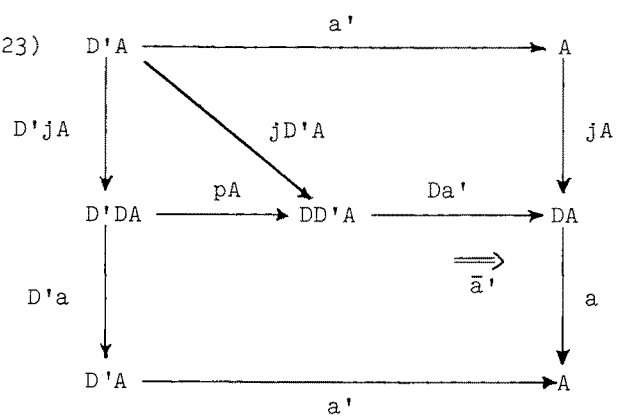


(5.22)

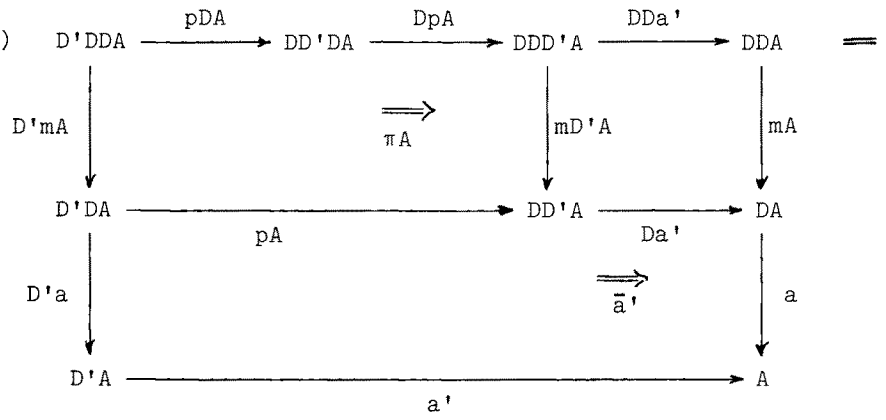


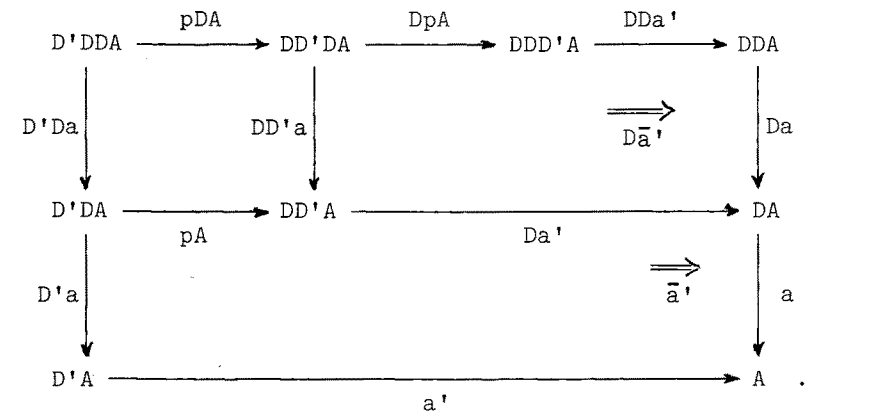


(5.23)



(5.24)



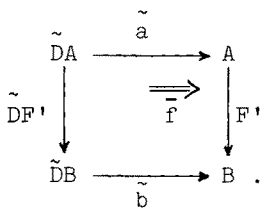


Of these, (5.21) and (5.22) are straightforward; (5.23) expresses $\tilde{a}.\tilde{j}A = 1$ using (5.20) and (5.17); (5.24) expresses $\tilde{a}.\tilde{m}A = \tilde{a}.D\tilde{a}$, using (5.20), (5.19), and (5.15). Note that at the level of 1-cells the equalities (5.21), (5.22) are automatic, since a' was given as a D'-action. The same is not true of (5.23) and (5.24), where equality at the 1-cell level already gives new information; in fact precisely the information that a: DA → A is a D-action. (Clearly this implies 1-cell equality in (5.23) and (5.24); and it is implied by them, as we see by composing (5.23) with j'A and (5.24) with $j'D^2A$.) It is very convenient to separate the 1-cell from the 2-cell information in such equalities, for example in the kind of reasonings used in §4.3 above.

Thus in fine to give a $\tilde{D}$-algebra A is to give the object A of K, with a D-action A and a D'-action a', and to give a 2-cell $\bar{a}'$ as in (5.20) satisfying the four axioms (5.21)-(5.24). We may write the $\tilde{D}$-algebra A as $(A,a,a',\bar{a}')$. (In §1.1 of the Introduction we described such an algebra, in rough terms, as one provided with both a D-action and a D'-action, the D-action being given the structure of a D'-morphism. We have now made this precise.)

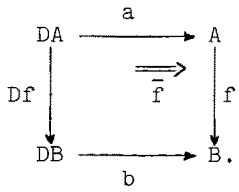
Now we consider what it is to give a $\tilde{D}$-morphism $\tilde{F}$: $A \to B$. It is a pair $\tilde{F} = (F',\bar{f})$ where F' is a D'-morphism $A \to B$ and $\bar{f}$ is a D'-2-cell as in

(5.25)



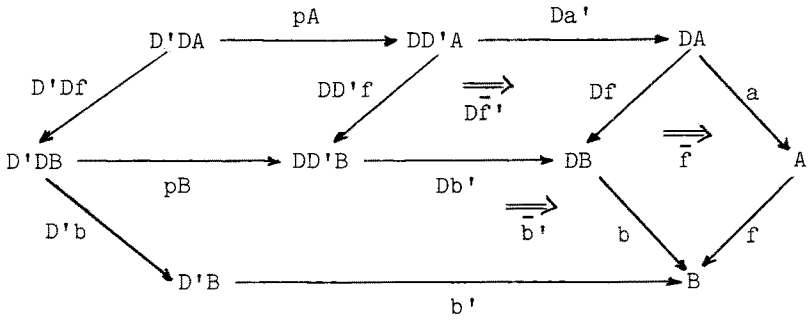
If F' is itself the pair $(f,\bar{f}')$: $A \to B$ then as a 2-cell $\bar{f}$ is of the form

(5.26)

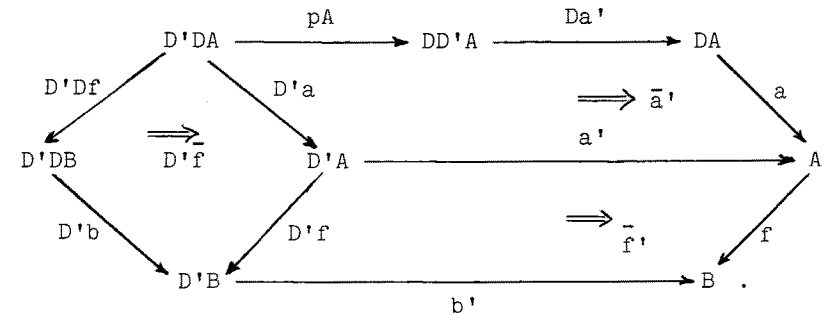


The requirement that $\bar{f}$ actually be a D'-2-cell is (cf. [13](3.19))

(5.27)



=



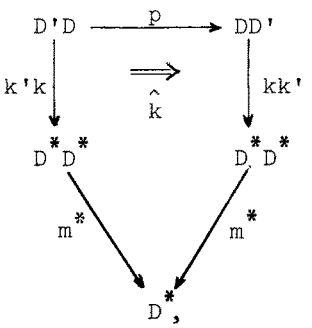
Finally the axioms for $\tilde{F} = (F', \bar{f})$ to be in fact a $\tilde{D}$-morphism, since they assert only an equality at the level of 2-cells, reduce to the axioms for $F = (f, \bar{f})$ to be a D-morphism. Thus in fine to give a $\tilde{D}$-morphism $\tilde{F}$: $(A,a,a',\bar{a}') \to (B,b,b',\bar{b}')$ is to give a triple $\tilde{F} = (f,\bar{f},\bar{f}')$ such that $F = (f,\bar{f})$: $A \to B$ is a D-morphism, $F' = (f,\bar{f}')$: $A \to B$ is a D'-morphism, and (5.27) is satisfied.

Lastly we consider what a $\tilde{D}$-2-cell γ: $\tilde{F} \Rightarrow \tilde{G}$: $A \to B$ is. It has first of all to be a D'-2-cell γ: $F' \Rightarrow G'$; but then the condition for it to be a $\tilde{D}$-2-cell is, since it is a pure equality of 2-cells, just the condition for it to be a D-2-cell γ: $F \Rightarrow G$. So a $\tilde{D}$-2-cell γ: $(f,\bar{f},\bar{f}') \Rightarrow (g,\bar{g},\bar{g}')$ is just a 2-cell γ: $f \Rightarrow g$ that is at once a D-2-cell γ: $F \Rightarrow G$ and a D'-2-cell γ: $F' \Rightarrow G'$.

5.4 We intend to exhibit $\tilde{D}$-algebras as the algebras for a doctrine on K. The first step in doing this is to transform (5.20) so that it becomes a diagram with leading vertex D'D and with terminal vertex the $\{A,A\}$ of §3.2; it will then involve not the actions $a: DA \to A$ and $a': D'A \to A$ but the corresponding doctrine maps $D \to \{A,A\}$ and $D' \to \{A,A\}$, together with something corresponding to $\bar{a}'$. The axioms (5.21)-(5.24) will transform similarly. However we abstract the transform we get in this way, replacing $\{A,A\}$ by an arbitrary doctrine. If we compare with §4, then §5.3 above is analogons to §§4.2-4.4, while the present §5.4 is analogons to §4.1. Of course we now suppose K to have small homs and to be 2-complete.

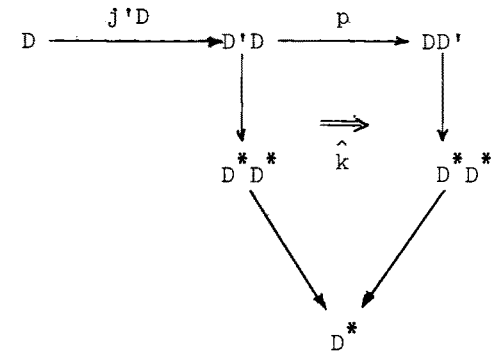
This leads us to the following notion. For any doctrine (D^*, m^*, j^*) on K, we define a map K from (D,D',p,π) to D^* to consist of doctrine-maps $k: D \to D^*$ and $k': D' \to D^*$, together with a modification $\hat{k}$ as in

(5.28)



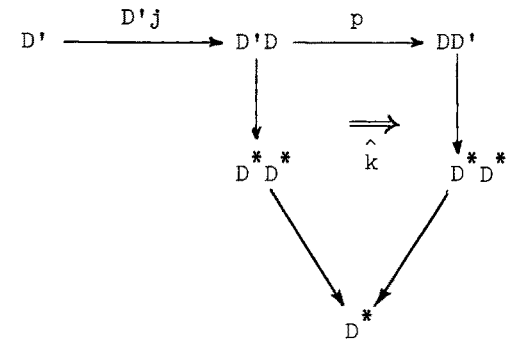
satisfying the following four axioms (which are all automatic at the 1-cell level):

(5.29)



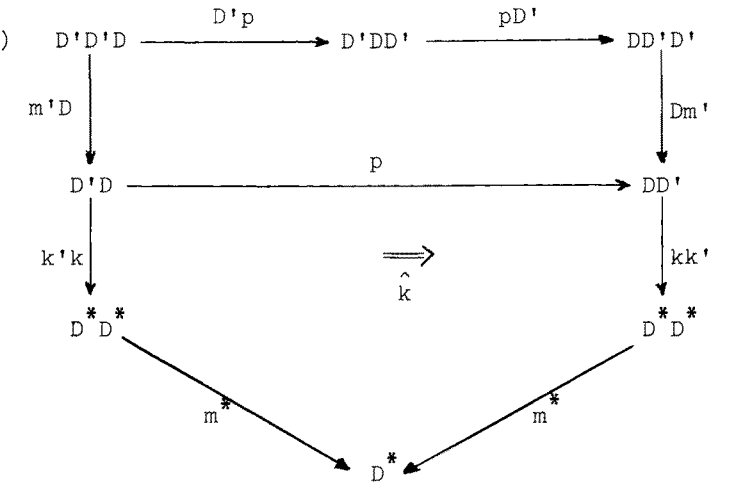
= identity

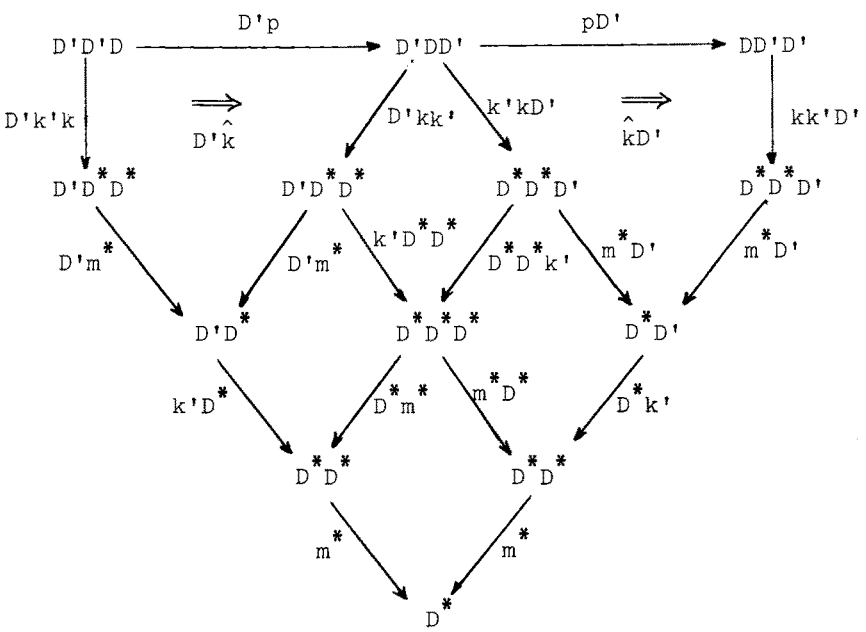
(5.30)



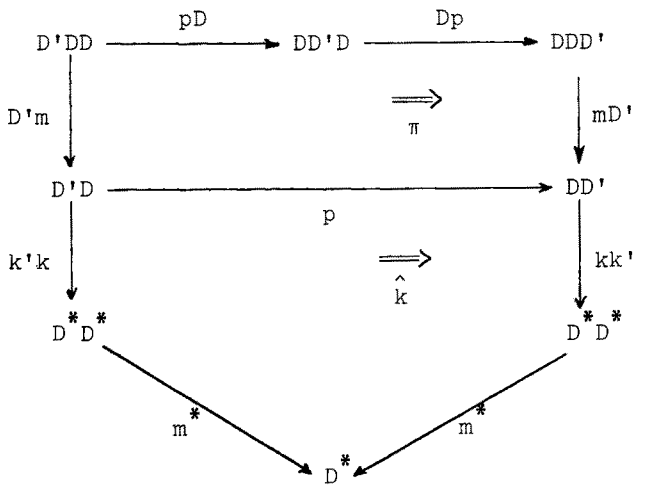
= identity

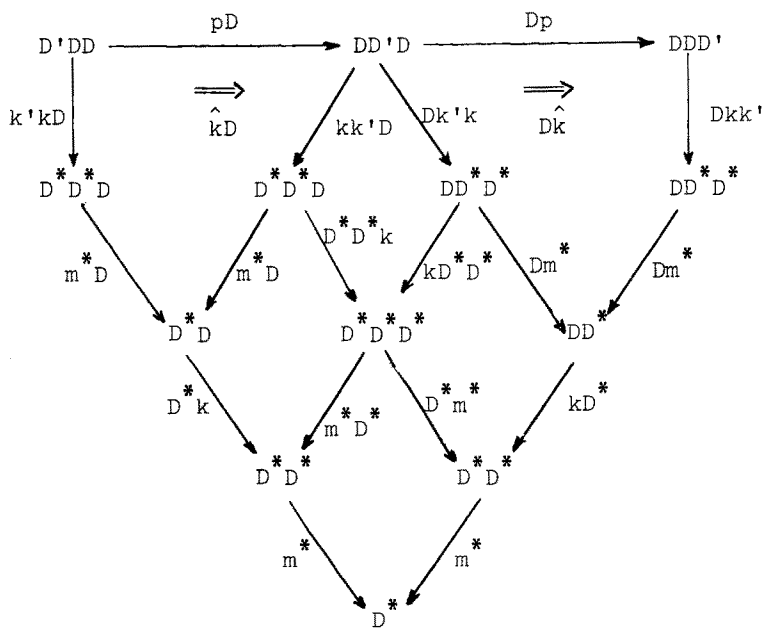
(5.31)


D'p
pD'
D'D'D
D'DD'
DD'D'
m'D
Dm'
p
D'D
DD'
k'k
kk'
k̂
D*D*
D*D*
m*
m*
D*


D'p
pD'
D'D'D
D'DD'
DD'D'
D'k'k
D'k̂
D'kk'
k'kD'
k̂D'
kk'D'
D'D*D*
D'D*D*
D*D*D'
D*D*D'
D'm*
D'm*
k'D*D*
D*D*k'
m*D'
m*D'
D'D*
D*D*D*
D*D'
k'D*
D*m*
m*D*
D*k'
D*D*
D*D*
m*
m*
D*

(5.32)


D'DD
pD
DD'D
Dp
DDD'
=
D'm
π
mD'
D'D
p
DD'
k'k
k̂
kk'
D*D*
D*D*
m*
m*
D*


D'DD
pD
DD'D
Dp
DDD'
k'kD
k̂D
kk'D
Dk'k
Dk̂
Dkk'
D*D*D
D*D*D
DD*D*
DD*D*
D*D*k
m*D
m*D
kD*D*
Dm*
Dm*
D*D
D*D*D
DD*
D*k
m*D*
D*m*
kD*
D*D*
D*D*
m*
m*
D*

If $K = (k,k',\hat{k})$ and $H = (h,h',\hat{h})$ are maps from (D,D',p,π) to D^*, we define a 2-cell $H \Rightarrow K$ to consist of doctrine-modifications $\rho\colon\ h \Rightarrow k$ and $\rho'\colon\ h' \Rightarrow k'$, satisfying the evident axioms with respect to $\hat{h}$ and $\hat{k}$. Thus we get a category $Map((D,D',p,\pi),D^*)$, or $M(D^*)$ for short.

A doctrine map $t\colon\ D^* \to E$ induces a functor $M(t)\colon\ M(D^*) \to M(E)$, sending $(k,k',\hat{k})$ to $(tk,tk',t\hat{k})$. A doctrine modification $\tau\colon\ t \Rightarrow t_1$ induces a natural transformation $M(t) \Rightarrow M(t_1)$ whose $(k,k',\hat{k})$-component is $(\tau k,\tau k')$. Thus M constitutes a 2-functor $\mathcal{D}oct \to CAT$.

Now it is easy to see that a map $K\colon\ (D,D',p,\pi) \to \{A,A\}$ corresponds exactly to a $\tilde{D}$-algebra structure on A as described in (5.20)-(5.24); we have only to replace D^* by $\{A,A\}$ in (5.28), multiply on the right by A, and compose with the evalutation $e\colon\ \{A,A\}A \to A$, to get the situation of (5.20). The axioms (5.29), (5.30), (5.31), (5.32) easily reduce to (5.21), (5.23), (5.22), (5.24) respectively; of course they were set up to do just this.

But now a map from (D,D',p,π) to the doctrine $\langle f,f\rangle$ of §3.3 gives the analogue of (5.20) with f replacing A and the diagram now living in K''. So a,a' get replaced by actions of D,D' on f in K'', that is to say, by enrichments of f to a D-morphism $(f,\bar{f})$ and to a D'-morphism $(f,\bar{f}')$, and $\bar{a}'$ gets replaced by a 2-cell $(\bar{a}',\bar{b}')$ in K''; the condition for this to be a 2-cell in K'' is precisely (5.27). The axioms (5.21)-(5.24) reduce of course to those for $\bar{a}',\bar{b}'$ separately. Hence to give a map $(D,D',p,\pi) \to \langle f,f\rangle$ is to give $\tilde{D}$-algebra structures to A and to B and to enrich f to a $\tilde{D}$-morphism $(f,\bar{f},\bar{f}')\colon\ A \to B$.

Finally, for $\gamma\colon\ f \Rightarrow g\colon\ A \to B$, a map from (D,D',p,π) to the doctrine $[\![\gamma,\gamma]\!]$ of §3.4 is equally easily seen to be just what makes γ a D-2-cell.

Note that a map from (D,D',p,π) to the doctrine $[f,f]$ of §3.3, or equally a map to $\langle f,f\rangle$ that happens to factorize through $[f,f]$, corresponds to a $\tilde{D}$-morphism in which both $\bar{f}$ and $\bar{f}'$ are identities. This is not the same as a strict $\tilde{D}$-morphism, which clearly has $\bar{f} = 1$ but $\bar{f}'$ arbitrary.

5.5 We now pass on to the analogues of §§4.5-4.7. It is clear that the 2-functor M: $\mathcal{D}oct \to CAT$ is 2-continuous; it is therefore 2-representable if the appropriate solution-set condition is satisfied. The remarks in §4.5 concerning the conditions under which this is likely to be so apply equally here. It is certainly so in our applications in §6, where D comes from a club in $\mathcal{C}at/\underline{S}$ and D' from the particular club for symmetric monoidal categories. We henceforth suppose it to be so, and we henceforth take K: $(D,D',p,\pi) \to D^*$ to be the representing map. Thus any map H: $(D,D',p,\pi) \to E$ is tK for a unique doctrine map t: $D^* \to E$.

We want in fact to consider two cases simultaneously. The case we have considered so far may be called the lax case. However, because we have supposed $\tilde{m}A$ to be strong, the doctrine $\tilde{D}$ restricts to one on the sub-2-category $D'\text{-}\mathcal{A}\ell g_{**}$ of $D'\text{-}\mathcal{A}\ell g$ where we retain only the strong D'-morphisms. This corresponds to requiring $\bar{a}'$ in (5.20) to be an isomorphism, and hence to requiring $\hat{k}$ in (5.28) to be an isomorphism. We call this the pseudo case. There is no formal difference between the two cases; M is 2-continuous in the latter as in the former; we treat them together with identical notation, although of course it is a different D^* in the latter case. However, in our prime application, D^* will not in the pseudo case come from a club, as it does in the lax case: and I have therefore not the same direct proof of its existence (cf. §1.4 of the Introduction). Of course in the pseudo case we give $\langle f,f\rangle$ its stronger meaning of §3.5, and all our 2-cells are isomorphisms.

In view of §5.4, then, we have an isomorphism between $\tilde{D}$-Alg and D^*-Alg. Not, as we saw in the last paragraph of §5.4, between $\tilde{D}$-Alg_* and D^*-Alg_*; the strict D^*-morphisms are only some of the strict $\tilde{D}$-morphisms; but we know exactly which ones they are (those with $\bar{f} = 1$ and $\bar{f}' = 1$), and hence we do know D^* implicitly.

In analogy with §4.6, we give the isomorphism explicitly. The D^*-algebra A with action $a^*: D^*A \to A$ corresponds to the $\tilde{D}$-algebra $(A,a,a',\bar{a}')$ where

$$(5.33) \quad a = a^*.kA, \quad a' = a^*.k'A, \quad \bar{a}' = a^*.\hat{k}A.$$

The D^*-morphism $F^* = (f,\bar{f}^*): A \to B$ corresponds to the $\tilde{D}$-morphism $\tilde{F} = (f,\bar{f},\bar{f}')$ where

$$(5.34) \quad \bar{f} = \bar{f}^*.kA, \quad \bar{f}' = \bar{f}^*.k'A;$$

and we have

$$(5.35) \quad \bar{f}^* = 1 \text{ if and only if } \bar{f} = 1 \text{ and } \bar{f}' = 1.$$

Finally,

$$(5.36) \quad \gamma: f \Rightarrow g \text{ is a } D^*\text{-2-cell if and only if it is a } \tilde{D}\text{-2-cell.}$$

We can make these statements in the alternative form that every $\tilde{D}$-algebra $(A,a,a',\bar{a}')$ is of the form (5.33) for a unique D^*-action a^*, and so on. The same techniques as in §4.7 allow us to extend this to the more general case where the algebras are not objects A of K but 2-functors $A: L \to K$, which we need to complete our proof; we leave the easy details to the reader, and suppose this done.

5.6 In this section (analogous to §4.8) we proceed to state and prove our second main theorem. The proof itself is not in detail analogous to that of Theorem 4.1; doubtless one could make it so by proceeding as there is a more direct manner, but as we give it we avoid large diagrams by being less direct.

In this section the "underlying objects" of all our algebras are endo-2-functors of K.

Theorem 5.1 Let (D,D',p,π) be a pseudo distributive law on the 2-complete 2-category K, and suppose that maps from (D,D',p,π) into doctrines are represented by $K = (k,k',\hat{k})$: $(D,D',p,\pi) \to D^*$. Then DD' is a D^*-algebra. Let s: $D^* \to DD'$ be the unique strict D^*-morphism such that

$$(5.37) \qquad sj^* = jj'.$$

Define h: $DD' \to D^*$ as the composite

$$(5.38) \qquad DD' \xrightarrow[kk']{} D^*D^* \xrightarrow[m^*]{} D^*.$$

Then $sh = 1$, and there is a modification η: $1 \Rightarrow hs$ such that $s\eta = 1$ and $\eta h = 1$. This applies to both the lax and the pseudo cases - the latter being that where the third element of a map from (D,D',p,π) to a doctrine is required to be an isomorphism; in the latter case $\hat{k}$ is an isomorphism, and so is η.

Proof As in the proof of Theorem 4.1, the differences in the pseudo case take care of themselves: $\hat{k}$ is an isomorphism, whence the $\bar{a}'$ below is invertible by (5.39) below, whence $\bar{h}$ and $\bar{h}'$ below are invertible; thus $\bar{h}^*$ below is invertible because we use the stronger $\langle h,h\rangle$ of §3.5 in the pseudo case; so finally η is invertible from the details of Theorem 2.6.

For the purpose of naming the various actions on D^* and on DD', we treat these as if their names were A and B respectively.

D^* is a D^*-algebra with action m^*, and is therefore a $\tilde{D}$-algebra $(D^*,a,a',\bar{a}')$ where by (5.33) we have

$$(5.39)\qquad a = m^*.kD^*, \quad a' = m^*.k'D^*, \quad \bar{a}' = m^*.\hat{k}D^*.$$

D' is a D'-algebra with action m', so we can form the free $\tilde{D}$-algebra on it, namely $\tilde{D}D'$ with $\tilde{D}$-action $\tilde{m}D'$(also called $\tilde{b} = (b,\bar{b}')$). By (5.12) the underlying object of $\tilde{D}D'$ is DD'. Thus DD' is a $\tilde{D}$-algebra $(DD', b, b', \bar{b}')$; hence it is also a D^*-algebra with action b^*.

Since D^* is the free D^*-algebra on 1, there is as in the statement of the theorem a unique strict D^*-morphism $s: D^* \to DD'$ satisfying (5.37).

We shall need below the value of the composite

$$(5.40)\qquad D' \xrightarrow[k']{} D^* \xrightarrow[s]{} DD'.$$

As a strict D^*-morphism, s is a fortiori a strict D'-morphism. Moreover the doctrine map k' is a strict D'-morphism; note that the D'-action a' on D^* given by (5.39) is just that arising in the usual way from the doctrine map k'. Thus the composite (5.40) is a strict D'-morphism; whence as in (2.4) and (2.5) it is determined by its composite with $j': 1 \to D'$. But $k'j' = j^*$ since k' is a

doctrine map, so that using (5.37) we have $sk'j' = sj^* = jj'$, which may be written as

$$1 \xrightarrow[j']{} D' \xrightarrow[jD']{} DD'.$$

Since jD' too is a strict D'-morphism by §5.2, we conclude that the composite (5.40) is given by

(5.41) $\quad sk' = jD'.$

Since D^* is a $\tilde{D}$-algebra and $\tilde{D}D'$ is a free one, there is by (2.4) and (2.5) a unique strict $\tilde{D}$-morphism $\tilde{H}\colon \tilde{D}D' \to D^*$ satisfying

(5.42) $\quad \tilde{H}.\tilde{j}D' = k'\colon D' \to D^*$

and given explicitly as the composite

(5.43) $\quad \tilde{D}D' \xrightarrow[\tilde{D}k']{} \tilde{D}D^* \xrightarrow[\tilde{a}]{} D^*.$

As in §5.3, we write $\tilde{H}$ in the extended form $\tilde{H} = (h,\bar{h},\bar{h}')$. Because it is a strict $\tilde{D}$-morphism, $\bar{h} = 1$ by (5.25). As a D'-morphism, $\tilde{a} = (a,\bar{a}')$ as in §5.3; and $\tilde{D}k'$ is $(Dk',1)$ by (5.14), k' already being strict as a D'-morphism. We conclude that

(5.44) $\quad h = a.Dk', \quad \bar{h} = 1, \quad \bar{h}' = \bar{a}'.D'Dk'.$

Substituting in this value for h the value of a from (5.39), we see that h has indeed the value (5.38) in the statement of the theorem.

Being a strict D^*-morphism, s is *a fortiori* a strict $\tilde{D}$-morphism. Consider the composite strict $\tilde{D}$-morphism

$$\tilde{D}D' \xrightarrow[\tilde{H}]{} D^* \xrightarrow[s]{} \tilde{D}D'.$$

As in (2.4) and (2.5), it is determined by its composite with $\tilde{j}D'$. By (5.42) and (5.41), this composite is jD'; which by (5.17) is another name for $\tilde{j}D'$. Hence

(5.45) $\quad s\tilde{H} = 1.$

Now let $H^* = (h, \bar{h}^*)$: $DD' \to D^*$ be the D^*-morphism corresponding as in (5.34) to the $\tilde{D}$-morphism $\tilde{H}$. Since this correspondence preserves composition, and since s corresponds to itself by (5.35), we have

(5.46) $\quad sH^* = 1.$

So we are now in the situation of §2.3, with $D^*, 1$ replacing D, A respectively. By (5.37) the $r: 1 \to DD'$ corresponding to $s: D^* \to DD'$ is jj'. So

$$\begin{aligned} hr &= h.jj' \\ &= m^*.kk'.jj' \quad \text{by (5.38)} \\ &= m^*.j^*j^* \quad \text{since } k,k' \text{ are doctrine maps} \\ &= m^*.D^*j^*.j^* \\ &= j^* \quad \text{since } m^*.D^*j^* = 1 \quad \text{(doctrine axiom).} \end{aligned}$$

Therefore we are in the special case of §2.3 in which (2.23) commutes. Our desired reflexion result now follows from Theorem 2.6, once we prove that

(5.47) $\quad \bar{h}^*.D^*jj'.h = 1.$

The rest of the proof consists in verifying this.

We do this, by hindsight, in several small steps. First, by (5.23),

(5.48) $\quad \bar{a}'.D'jD^* \;=\; 1.$

Compose this with D'k': $D'D' \to D'D^*$, and use the naturality of j to replace $D'jD^*.D'k'$ by $D'Dk'.D'jD'$. Using the third equation of (5.44), we now have

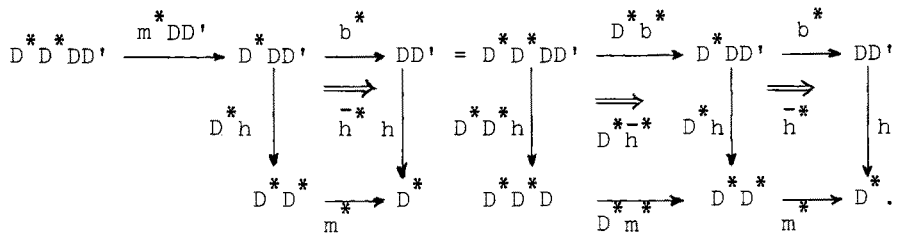
(5.49) $\quad \bar{h}'.D'jD' \;=\; 1.$

Use the second equation of (5.34) to express $\bar{h}'$ in terms of $\bar{h}^*$ (mentally replacing f: A → B by h: DD' → D^*); we have $\bar{h}' = \bar{h}^*.k'DD'$, so that (5.49) simplifies to

(5.50) $\quad \bar{h}^*.k'jD' \;=\; 1.$

Now one of the axioms satisfied by the D^*-morphism H^* is

(5.51)



Compose both sides on the front end with $D^*k'jD'$: $D^*D'D' \to D^*D^*DD'$, and use D^* of (5.50) to simplify the right side. We end up with

(5.52) $\quad \bar{h}^*.m^*DD'.D^*k'jD' \;=\; \bar{h}^*.D^*b^*.D^*k'jD'.$

(That is, we have used (5.50) to kill the $D^*\bar{h}^*$ on the right of (5.51), and then reverted to linear rather than diagrammatic notation).

Now compose (5.52) with kD'j': $DD' \to D^*D'D'$, which can also be written as

$$DD' \xrightarrow[kD']{} D^*D' \xrightarrow[D^*D'j']{} D^*D'D'.$$

What we then have on the right side is

$$\bar{h}^*.D^*b^*.D^*k'jD'.D^*D'j'.kD',$$

or $\bar{h}^*.D^*y.kD'$ say, where $y = b^*.k'jD'.D'j'$. By the naturality of k, this is equally $\bar{h}^*.kDD'.Dy$. But by (5.34), $\bar{h}^*.kDD' = \bar{h}$, and by (5.44) this is 1. So after composing (5.52) with kD'j', we get 1 on the right side. Simplifying a little what we get on the left side, we now have

(5.53) $\quad \bar{h}^*.m^*DD'.kk'jj' = 1.$

Write kk'jj' as the composite

$$DD' \xrightarrow[kk']{} D^*D^* \xrightarrow[D^*D^*jj']{} D^*D^*DD'$$

and the use the naturality of m^* to replace $m^*DD'.D^*D^*jj'$ by $D^*jj'.m^*$. Then (5.53) becomes

(5.54) $\quad \bar{h}^*.D^*jj'.m^*.kk' = 1,$

which by (5.38) is the (5.47) that we seek. □

6. Pseudo-commutative doctrines and clubs

6.1 An endofunctor D of $\mathcal{C}at$ is made into an endo-2-functor by giving a natural transformation $[A,B] \to [DA,DB]$ (the "strength" of D) subject to two axioms (multiplicative and unitary). From this strength we derive a natural (and in fact 2-natural) transformation

(6.1) $\quad t: \ A\times DB \to D(A\times B)$

as the image under adjunction of the composite

(6.2) $\quad A \to [B,A\times B] \to [DB,D(A\times B)]$.

It is well known that giving the strength of D is equivalent to giving t; in fact t and the strength are mates, in the sense of [13] §2.2, under the adjunctions $-\times A \dashv [A,-]$ and $-\times DA \dashv [DA,-]$; so that t is often called the "monoidal strength" of D: of course t must satisfy two axioms corresponding to those for the strength.

If now (D,m,j) is a doctrine on $\mathcal{C}at$, it is easy to express the monoidal strength of D^2 in terms of t, while that of 1 is the identity. The 2-naturality, as distinct from the naturality, of j and of m can be expressed in terms of the monoidal strengths, in the form of a commutative diagram involving j and t and another involving m and t. So all told, to make a mere monad (D,m,j) on $\mathcal{C}at$ into a doctrine is to give a natural t as in (6.1) satisfying four axioms.

We use the symbol $^{\#}$ in a general way to denote conjugation under the symmetry c: $A\times B \to B\times A$ of $\mathcal{C}at$. So alongside t we also have

(6.3) $\quad t^{\#}: \ DA\times B \to D(A\times B)$,

namely the composite

$$(6.4)\qquad DA\times B \xrightarrow[c]{} B\times DA \xrightarrow[t]{} D(B\times A) \xrightarrow[Dc]{} D(A\times B).$$

We can then form the 2-natural

$$(6.5)\qquad \tilde{d}\colon\ DA\times DB \to D(A\times B)$$

as the composite

$$(6.6)\qquad DA\times DB \xrightarrow[t^{\#}]{} D(A\times DB) \xrightarrow[Dt]{} D^2(A\times B) \xrightarrow[m]{} D(A\times B),$$

as well as its conjugate

$$(6.7)\qquad \tilde{d}^{\#}\colon\ DA\times DB \to D(A\times B),$$

namely the composite

$$(6.8)\qquad DA\times DB \xrightarrow[t]{} D(DA\times B) \xrightarrow[Dt^{\#}]{} D^2(A\times B) \xrightarrow[m]{} D(A\times B).$$

Moreover, using I for the unit category, we can set

$$(6.9)\qquad d^{\circ} = jI\colon\ I \to DI.$$

Kock [14] shows that $\tilde{d}$ and d° enrich D to a <u>monoidal</u> 2-functor $(D,\tilde{d},d^{\circ})\colon\ \mathcal{C}at \to \mathcal{C}at$. This is not in general <u>symmetric monoidal</u>; the extra condition needed for this is precisely

$$(6.10)\qquad \tilde{d} = \tilde{d}^{\#}.$$

The unit $j\colon\ 1 \to D$ is always <u>monoidally</u> 2-natural; recall that for natural transformations there is no difference between "monoidally 2-natural" and "symmetric monoidally 2-natural". The multiplication

m: $D^2 \to D$ is not in general monoidally 2-natural, but is so <u>if and only if</u> (6.10) is satisfied.

Kock called a doctrine - or more generally a V-monad on a symmetric monoidal closed V - <u>commutative</u> if it satisfied (6.10). Because of the last remark, it has now become rather more common to call them <u>monoidal</u> V-monads; they are the monads on V in the 2-category of monoidal V-categories, and necessarily lie in the 2-category of symmetric monoidal V-categories. For commutative D, under the mildest completeness assumptions on V, the V-category of algebras V^D is symmetric closed, and indeed symmetric monoidal closed if it admits coequalizers; and the forgetful and free V-functors $U: V^D \to V$, $F: V \to V^D$ are symmetric monoidal. Conversely, of course, if V^D,U,F are monoidal, so is $D = UF$. Thus, modulo the always-troublesome matter of the existence of coequalizers in V^D, commutativity of D is the necessary and sufficient condition for V^D,U,F to be symmetric monoidal closed: it generalizes Linton's criterion (cf. [6] p.549) when $V = \mathit{Sets}$ and D comes from a finitary theory.

A final remark at this level. Suppose D is a commutative monad on Sets. Then, being symmetric monoidal, D takes a commutative monoid A to a commutative monoid DA, and in fact lifts to a monad $\tilde{D}$ on the category of commutative monoids. So if D' is the monad on Sets whose algebras are commutative monoids, we get an <u>honest</u> distributive law $p: D'D \to DD'$.

<u>6.2</u> In the doctrine case, commutativity as expressed by (6.10) seems to be rare in natural examples. We call a doctrine <u>pseudo-commutative</u> if there is instead an isomorphism (invertible modification)

$$(6.11) \qquad \gamma: \tilde{d} \Rightarrow \tilde{d}^{\#}$$

satisfying suitable axioms (one of which is that γ be involutary; $\gamma^{\#}\gamma: \tilde{d} \Rightarrow \tilde{d}^{\#} \Rightarrow \tilde{d}$ is the identity). Then we get a <u>pseudo-distributive</u>

law (p,π): $D'D \to DD'$ as in §§5.1, 5.2, where now D' is not (as in the last paragraph) the monad for commutative monoids, but rather the doctrine for symmetric monoidal categories.

I shall do no more than sketch this in the above generality, for in all my examples D is $\mathcal{V}\circ-$ for a club $\mathcal{V}$ in $Cat/\underline{S}$, and then everything becomes much easier because all the diagrams come down by one dimension. The extra generality would be pleasant, but unless I succeed in finding a more compact treatment of it, my applications do not justify the extra complication.

Let then A be a D'-category, that is, a symmetric monoidal category, and let D be pseudo-commutative with γ as in (6.11) satisfying axioms to be determined. We give a symmetric monoidal structure to DA. Its tensor product and identity object are given by

$$(6.12)\qquad DA\times DA \underset{d}{\overset{\sim}{\to}} D(A\times A) \underset{D\otimes}{\to} DA,$$

$$(6.13)\qquad I \underset{d^\circ}{\to} DI \underset{DI}{\to} DA;$$

its associativity and right identity isomorphisms are obtained from those for A (called a and r) by composing Da with

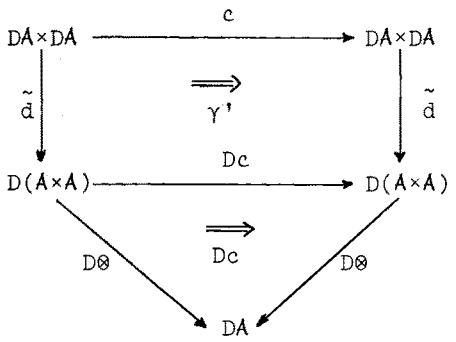
$$DA\times DA\times DA \underset{\tilde{d}\times 1}{\to} D(A\times A)\times DA \underset{\tilde{d}}{\overset{\sim}{\to}} D(A\times A\times A)$$

and Dr with

$$DA\times I \underset{1\times d^\circ}{\to} DA\times DI \underset{\tilde{d}}{\overset{\sim}{\to}} D(A\times I).$$

Its commutativity isomorphism is obtained from that (called c) for A as the composite

(6.14)



where γ' denotes $Dc.\gamma$. The monoidal-category axioms are immediate because $(D,\tilde{d},d^\circ)$ is monoidal; the symmetric-monoidal-category axioms ($c^2 = 1$ and the hexagonal axiom) require respectively that γ be involutory and that it satisfy a kind of hexagonal axiom (relating $\gamma_{A\times B,C}$ to $\gamma_{A,C}$ and $\gamma_{B,C}$).

The above passage from A to DA respects strict symmetric monoidal functors and the corresponding natural transformations, and therefore gives us a lifting of D as in (5.1), whence also a lifting as in (5.2). The lifting of j at the (5.1) level is automatic, because $(D,\tilde{d},d^\circ)$ is monoidal, except as regards the symmetry: for this it suffices to impose the axiom that the composite of γ with $j\times 1$: $A\times DB \to DA\times DB$ is the identity.

It remains to lift m at the (5.2) level, that is, to enrich $m: D^2A \to DA$ to a strong (but not strict) symmetric monoidal functor $(m,\hat{m},m^\circ)$; here I use $\hat{m}$ in place of the more usual $\tilde{m}$, which unfortunately has another sense in §5, being in fact the name of the triple $(m,\hat{m},m^\circ)$. We can take m° to be the identity. To get an $\hat{m}$, it turns out that we need a modification $\mu: \tilde{d}(m\times m) \Rightarrow m.D\tilde{d}.\tilde{d}: D^2A\times D^2B \to D(A\times B)$. The four axioms satisfied by t,m,j allow us to write this as $\mu: m.Dt.m.D\tilde{d}.t^\# \Rightarrow m.Dt.m.D\tilde{d}^\#.t^\#$, and we set $\mu = m.Dt.m.D\gamma.t^\#$. Of the three axioms saying that $(m,\hat{m},m^\circ)$ is a symmetric monoidal functor, the one involving m° is satisfied in virtue

of our axiom $\gamma(j\times 1) = 1$, leaving two. Of the final three axioms stating that $(\tilde{D},\tilde{m},\tilde{j})$ is a doctrine, the two involving $\tilde{j}$ are satisfied in virtue of the same axiom $\gamma(j\times 1) = 1$, leaving one. So three more axioms for γ ensure that we get our desired pseudo distributive law.

I suspect that the six axioms we now have for γ reduce to one or two, or at most three, by analogy with the club case that follows: but I have not yet succeeded in so reducing them, and am reluctant to delay this volume longer while trying to do so. So I pass on to the club case without giving a definitive set of axioms for the γ of a "pseudo commutative doctrine". In the club case we establish the pseudo distributive law ab initio.

6.3 We now suppose that $D = \mathcal{D}\circ -$ for a club $\mathcal{D}$ in $Cat/\underline{\underline{S}}$, and we use the notation of [9] §10. In particular the augmentation of $\mathcal{D}$ is $\Gamma\colon \mathcal{D} \to \underline{\underline{S}}$; we typically write τ for ΓT, σ for ΓS, and so on; in fact we are running so short of symbols that we shall quite generally use Greek letters for natural numbers, as well as for permutations or functions. It should cause no confusion if we use $m\colon \mathcal{D}\circ\mathcal{D} \to \mathcal{D}$ for the multiplication of $\mathcal{D}$, although to date m has denoted the 2-natural $\mathcal{D}\circ\mathcal{D}\circ - \to \mathcal{D}\circ -$; then m sends $T[S_1 \ldots S_\tau]$ to $T(S_1 \ldots S_\tau)$ and similarly for morphisms. Again as in [9] §10, the image under $j\colon \mathcal{J} \to \mathcal{D}$ of the unique object of $\mathcal{J}$ is $\underline{\underline{1}} \in \mathcal{D}$. The associative law for m expresses the equality of $T(S_1(R_{11} \ldots R_{1\sigma_1}) \ldots S_\tau(R_{\tau 1} \ldots R_{\tau\sigma_\tau}))$ with $T(S_1 \ldots S_\tau)(R_{11} \ldots R_{\tau\sigma_\tau})$, and similarly for morphisms; the two unitary laws for m and j express the equalities $T(\underline{\underline{1}} \ldots \underline{\underline{1}}) = T$ and $\underline{\underline{1}}(T) = T$, and similarly for morphisms. The ordered pairs that are the objects and the morphisms of $A\times B$ are written $\langle A,B\rangle$ and $\langle f,g\rangle$.

The functor $t\colon A \times \mathcal{D}\circ B \to \mathcal{D}\circ(A\times B)$ of (6.1) is easily seen to be given by

$$t\langle A,\ S[B_1 \ldots B_\sigma]\rangle = S[\langle A,B_1\rangle \ldots \langle A,B_\sigma\rangle],$$

$$t\langle f,\ h[g_1 \ldots g_\sigma]\rangle = h[\langle f,g_1\rangle \ldots \langle f,g_\sigma\rangle];$$

with analogous formulae for $t^{\#}$. Hence we calculate $\tilde{d},\tilde{d}^{\#}$: $\mathcal{D}\circ\mathcal{A} \times \mathcal{D}\circ\mathcal{B} \to \mathcal{D}\circ(\mathcal{A}\times\mathcal{B})$; they send the object $\langle T[A_1 \ldots A_\tau], S[B_1 \ldots B_\sigma]\rangle$ respectively to

$$(6.15)\quad T(S \ldots S)[\langle A_1,B_1\rangle \ldots \langle A_1,B_\sigma\rangle \ldots \langle A_\tau,B_1\rangle \ldots \langle A_\tau,B_\sigma\rangle],$$

$$(6.16)\quad S(T \ldots T)[\langle A_1,B_1\rangle \ldots \langle A_\tau,B_1\rangle \ldots \langle A_1,B_\sigma\rangle \ldots \langle A_\tau,B_\sigma\rangle],$$

with similar effects on morphisms. Here S occurs τ times in (6.15) and T occurs σ times in (6.16); the order of the $\langle A_\alpha,B_\beta\rangle$ is in (6.15) the lexicographical order of the $\langle\alpha,\beta\rangle$, and in (6.16) the lexicographical order of the $\langle\beta,\alpha\rangle$.

To give an isomorphism γ as in (6.11) is to give for each $T,S \in \mathcal{D}$ a natural isomorphism

$$(6.17)\quad z_{TS}:\ T(S \ldots S) \to S(T \ldots T)$$

whose type $\Gamma z_{TS} = \zeta_{\tau,\sigma}$ is the permutation of $\tau\sigma$ demanded by a comparison of (6.15) and (6.16); then the $\langle T[A_1 \ldots A_\tau],S[B_1 \ldots B_\sigma]\rangle$-component of $\gamma = \gamma_{\mathcal{A},\mathcal{B}}$ is the morphism $z_{TS}[1,1,\ldots,1]$ from (6.15) to (6.16). By the <u>naturality</u> of z_{TS} we mean of course that we have commutativity in

(6.18)
$$\begin{array}{ccc} T(S\ldots S) & \xrightarrow{z_{TS}} & S(T\ldots T) \\ {\scriptstyle f(g\ldots g)}\downarrow & & \downarrow{\scriptstyle g(f\ldots f)} \\ T'(S'\ldots S') & \xrightarrow[z_{T'S'}]{} & S'(T'\ldots T'). \end{array}$$

(This diagram is more complicated than it looks: in general $\tau' \neq \tau$, and $\Gamma f = \phi$ is a function from τ to τ'; the α-th g in $f(g\ldots g)$ is a morphism from the α-th S in $T(S\ldots S)$ to the $\phi\alpha$-th S' in $T'(S'\ldots S')$, in accordance with the conventions of [9] §10; however it does make good sense, both legs having the same type.)

Thus does the club case fit into the general situation sketched in §6.2; it remains only to put such axioms on z_{TS} as ensure those we require for γ. However we deal henceforth only with the club case, forgetting γ and referring only to z_{TS}, and carrying out in detail what we barely outlined in §6.2.

6.4 We define then a pseudo-commutative club to be a club $\mathcal{D}$ in $Cat/\underline{S}$ together with an isomorphism z_{TS} as in (6.17), of the type $\zeta_{\tau,\sigma}$ described above, natural in the sense of (6.18), and satisfying in addition one axiom: namely the commutativity of

(6.19)
$$\begin{array}{ccc} T(S(R_1\ldots R_\sigma)\ldots S(R_1\ldots R_\sigma)) & \xrightarrow{z_{T,S(R_1,\ldots R_\sigma)}} & S(R_1\ldots R_\sigma)(T\ldots T) \\ \| & & \| \\ T(S\ldots S)(R_1\ldots R_\sigma\ldots R_1\ldots R_\sigma) & & S(R_1(T\ldots T)\ldots R_\sigma(T\ldots T)) \\ {\scriptstyle z_{TS}(1,1,\ldots,1)}\downarrow & & \downarrow{\scriptstyle S(z_{R_1T},\ldots,z_{R_\sigma T})} \\ S(T\ldots T)(R_1\ldots R_1\ldots R_\sigma\ldots R_\sigma) & = & S(T(R_1\ldots R_1)\ldots T(R_\sigma\ldots R_\sigma)). \end{array}$$

We derive some immediate consequences of (6.19). First, put $S = \underline{\underline{1}}$ (so that $\sigma = 1$) and $R_1 = \underline{\underline{1}}$. The top and the left edges are then both the isomorphism $z_{T\underline{\underline{1}}}$ while the right edge is $z_{\underline{\underline{1}}T}$, whence

$$z_{\underline{\underline{1}}T} = 1. \tag{6.20}$$

Now in (6.19) put $T = \underline{\underline{1}}$, $S = \underline{\underline{1}}$, $R_1 = R$. Using (6.20) we get

$$z_{R\underline{\underline{1}}} = 1. \tag{6.21}$$

Finally just put $S = \underline{\underline{1}}$, $R_1 = R$, T arbitrary. Using (6.21) we get

$$z_{RT}z_{TR} = 1. \tag{6.22}$$

Hence z is involutary and satisfies (6.20).

Before giving examples we make the following remark. Any club $\mathcal{D}$ becomes a strict monoidal category, with $\underline{\underline{1}}$ as identity, if we set $T \otimes S = T(S \ldots S)$. Moreover $\Gamma\colon \mathcal{D} \to \underline{\underline{S}}$ is a strict morphism of strict monoidal categories when $\underline{\underline{S}}$ is given the cartesian monoidal structure $\tau \otimes \sigma = \tau \times \sigma = \tau\sigma$. For a pseudo-commutative club, $z_{TS}\colon T \otimes S \to S \otimes T$ makes $\mathcal{D}$ symmetric monoidal; the hexagon axiom is got by setting $R_1 = \ldots = R_\sigma = R$ in (6.19), and the other axiom is (6.22). Moreover $\Gamma\colon \mathcal{D} \to \underline{\underline{S}}$ is then a strict symmetric monoidal functor, as $\zeta_{\tau,\sigma}$ is the classical symmetry on $\underline{\underline{S}}$. This observation is useful in limiting our search for examples.

Example 6.1 The club $\underline{\underline{S}}$ itself. We recall that $\tau(\sigma_1, \ldots, \sigma_\tau)$ is $\sigma_1 + \ldots + \sigma_\tau$ while $\underline{\underline{1}}$ is 1. We take $z_{\tau\sigma}\colon \tau(\sigma \ldots \sigma) \to \sigma(\tau \ldots \tau)$ to be the permutation $\zeta_{\tau,\sigma}\colon \tau\sigma \to \sigma\tau$ of §6.3. This is easily seen to satisfy the axiom (6.19); in fact if it did not the axiom would not make sense, its two legs being of different types. Note that an $\underline{\underline{S}}$-algebra is just a category with strictly-associative finite coproducts.

Example 6.2 The club S whose algebras are categories-with-finite-coproducts. The augmentation $\Gamma\colon\ S \to \underline{\underline{S}}$ is an equivalence, and we define z_{TS} (as we must) to be the unique morphism such that $\Gamma z_{TS} = \zeta_{\tau,\sigma}$.

Example 6.3 The sub-club $\underline{\underline{P}}$ of $\underline{\underline{S}}$ with the natural numbers as objects but with permutations as its only morphisms; the augmentation $\underline{\underline{P}} \to \underline{\underline{S}}$ is of course the inclusion. Again $z_{\tau\sigma} = \zeta_{\tau,\sigma}$. The $\underline{\underline{P}}$-algebras are the strict symmetric monoidal categories.

Example 6.4 The club P whose algebras are the symmetric monoidal categories. By Mac Lane's original coherence result [23], the augmentation Γ is an equivalence of P with $\Gamma(P) = \underline{\underline{P}} \subset \underline{\underline{S}}$. Again z_{TS} is the unique morphism with $\Gamma z_{TS} = \zeta_{\tau,\sigma}$.

Example 6.5 The full sub-club of $\underline{\underline{S}}$ determined by the objects 0 and 1; it is the arrow category $\underline{\underline{2}}$. Its algebras are categories-with-an-initial-object.

Example 6.6 The full sub-club of $\underline{\underline{P}}$ determined by the objects 0 and 1; it is the discrete category 2. Its algebras are categories-with-a-distinguished-object.

Example 6.7 Our remaining examples are all of the kind where the augmentation $\Gamma\colon\ \mathcal{D} \to \underline{\underline{S}}$ is the constant functor at $1 \in \underline{\underline{S}}$. A club of this kind is nothing but a strict monoidal category $\mathcal{D}$, with $T(S) = T \otimes S$ and with $\underline{1}$ as the identity for $\otimes$. It is pseudo-commutative if and only if it is symmetric monoidal, in which case we set $z_{TS}\colon\ T(S) \to S(T)$ equal to the symmetry $c\colon\ T \otimes S \to S \otimes T$. Since $\mathcal{D} \circ A$ is just $\mathcal{D} \times A$, a $\mathcal{D}$-algebra is a category A with a strictly associative and unitary action $\mathcal{D} \times A \to A$ (as for example E acts on K in §3.2 above). Note that the club whose algebras are categories-bearing-a-monad, namely $\underline{\underline{\Delta}}$, is not an example, for the monoidal structure on the simplicial category $\underline{\underline{\Delta}}$ is not symmetric. We pass to some particular cases of this

example.

Example 6.7.1 Let C be a symmetric monoidal category and consider the club $\mathcal{D}$ whose algebras are categories A together with a coherently (but not strictly) associative and unitary functor $\otimes$: $C\times A \to A$. For instance, any tensored C-category A is such an algebra (where by C-category I mean "category enriched over C " - a remark necessitated by the fact that I sometimes, for a club $\mathcal{D}$, call a $\mathcal{D}$-algebra a $\mathcal{D}$-category). Since all the operations on A are unary, the augmentation of $\mathcal{D}$ is constant at 1. We have to show that $\mathcal{D}$ is symmetric, and is thus a pseudo-commutative club. But we know just what $\mathcal{D}$ is: it is the canonical strict monoidal category equivalent to C; its objects are n-ads $\langle C_1,\ldots,C_n\rangle$ of objects of C, its tensor product is given on objects by concatenation, and its morphisms $\langle C_1\ldots,C_n\rangle \to \langle B_1,\ldots,B_m\rangle$ are the morphisms $C_1\otimes(C_2\otimes\ldots(C_{n-1}\otimes C_n)) \to B_1\otimes(B_2\otimes\ldots(B_{m-1}\otimes B_m))$ in C. It is now clear that $\mathcal{D}$ is symmetric when C is.

Example 6.7.2 Let $\mathcal{D}$ be the discrete category $\underline{\mathbb{N}}$ of natural numbers, which is symmetric monoidal with + as its tensor product. A $\mathcal{D}$-algebra is a category A with an endofunctor E: $A \to A$.

Example 6.7.3 Let $\mathcal{D}$ be the discrete category 2 with the symmetric monoidal structure having the usual multiplication of its objects 0 and 1 as tensor product. This is not the same as Example 6.6, for the augmentation is now constant at 1, whereas there it was $\Gamma 0 = 0$, $\Gamma 1 = 1$. An algebra is a category A with an endofunctor E such that $E^2 = E$.

Example 6.7.4 Let $\mathcal{D}$ be the arrow category $\underline{\underline{2}}$ with the symmetric monoidal structure given on objects as in the last example. This is again different from Example 6.5, as the augmentation is again constant at 1. An algebra is a category A together with an indempotent comonad. Replacing $\underline{\underline{2}}$ by $\underline{\underline{2}}^{op}$ gives another example, where an algebra is a category bearing an indempotent monad.

6.5 Let $\mathcal{V}$ be a pseudo-commutative club as in §6.4, giving the doctrine $D = \mathcal{V}\circ -$; let D' be the doctrine $\mathcal{P}\circ -$ whose algebras are symmetric monoidal (s.m.) categories. We exhibit a pseudo distributive law of D' over D, as described in §§5.1, 5.2. (We could equally produce a pseudo distributive law if we took D' to be the doctrine $\underline{\underline{P}}\circ -$ for strict s.m. categories; this requires only the easy observation that if the s.m. structure on A is strict, so is that we construct below on $\mathcal{V}\circ A$; we shall not refer further to this case.)

The first step is to produce the lifting (5.1) of D, and its extension (5.2). Let $(A,\otimes,I,a,r,c)$ be a s.m. category; we give $DA = \mathcal{V}\circ A$ a s.m. structure to get $\tilde{D} = (\mathcal{V}\circ A,\bar{\otimes},\bar{I},\bar{a},\bar{r},\bar{c})$. We set

$$(6.23)\quad T[A_1 \dots A_\tau]\,\bar{\otimes}\,S[B_1 \dots B_\sigma] = T(S\dots S)[A_1\otimes B_1,\dots,A_1\otimes B_\sigma,\dots,A_\tau\otimes B_1,\dots,A_\tau\otimes B_\sigma],$$

and define $\bar{\otimes}$ similarly on morphisms. We define the rest of the structure by

$$(6.24)\quad \bar{I} = \underline{\underline{1}}[I],$$

$$(6.25)\quad \begin{array}{ccc} (T[A_1\dots]\,\bar{\otimes}\,S[B_1\dots])\,\bar{\otimes}\,R[C_1\dots]) & \xrightarrow{\bar{a}} & T[A_1\dots]\,\bar{\otimes}\,(S[B_1\dots]\,\bar{\otimes}\,R[C_1\dots]) \\ \| & & \| \\ T(S\dots S)(R\dots R)[(A_1\otimes B_1)\otimes C_1,\dots] & \xrightarrow[1[a,a,\dots,a]]{} & T(S\dots S)(R\dots R)[A_1\otimes(B_1\otimes C_1),\dots], \end{array}$$

$$(6.26)\quad \begin{array}{ccc} T[A_1\dots]\,\bar{\otimes}\,\bar{I} & \xrightarrow{\bar{r}} & T[A_1\dots] \\ \| & & \| \\ T[A_1\dots]\otimes \underline{\underline{1}}[I] & & \| \\ \| & & \| \\ T[A_1\otimes I,\dots] & \xrightarrow[1[r,r,\dots,r]]{} & T[A_1\dots], \end{array}$$

$$(6.27)\qquad \begin{array}{ccc} T[A_1\ldots]\bar{\otimes}S[B_1\ldots] & \xrightarrow{\bar{c}} & S[B_1\ldots]\bar{\otimes}T[A_1\ldots] \\ \| & & \| \\ T(S\ldots S)[A_1\otimes B_1,\ldots] & \xrightarrow[z_{TS}[c,c,\ldots,c]]{} & S(T\ldots T)[B_1\otimes A_1,\ldots]. \end{array}$$

The axioms for $\bar{a},\bar{r},\bar{c}$ follow from those for a,r,c together with (6.22) (needed for $\bar{c}^2 = 1$) and the special case of (6.19) where $R_1 = \ldots = R_\sigma = R$ (needed for the hexagonal axiom connecting $\bar{a}$ and $\bar{c}$). (Note that when A is the unit category I this is the s.m. structure on $\mathcal{D}\circ I$, $= \mathcal{D}$ itself, described just before Example 6.1 above.) If $\phi\colon A \to B$ is a strict s.m. functor so too is $\mathcal{D}\circ\phi\colon \mathcal{D}\circ A \to \mathcal{D}\circ B$ sending $T[A_1\ldots A_\tau]$ to $T[\phi A_1\ldots\phi A_\tau]$, and similarly for s.m. natural transformations. Thus we have the lifting (5.1) of D.

Its automatic extension (5.2) sends the non-strict s.m. functor $(\phi,\hat{\phi},\phi^\circ)$, where $\hat{\phi}\colon \phi A \otimes \phi B \to \phi(A\otimes B)$ and $\phi^\circ\colon I_B \to \phi I_A$, to $(\Phi,\hat{\Phi},\Phi^\circ)$ where $\Phi = \mathcal{D}\circ\phi$, where $\hat{\Phi}$ has components $T(S\ldots S)[\hat{\phi},\ldots,\hat{\phi}]$, and where $\Phi^\circ = \underline{\underline{1}}[\phi^\circ]$. (Recall that we are using $\hat{\phi}$ for the more familiar $\tilde{\phi}$ to avoid confusion with the $\tilde{m}$ etc. of §5.) Simiarly $\tilde{D}$ sends the s.m. natural transformation $\alpha\colon (\phi,\hat{\phi},\phi^\circ) \Rightarrow (\psi,\hat{\psi},\psi^\circ)$ to that with $T[A_1\ldots A_\tau]$-component $T[\alpha A_1,\ldots,\alpha A_\tau]$.

The functor $j\circ A\colon A \to \mathcal{D}\circ A$ is the functor sending A to $\underline{\underline{1}}[A]$, and is clearly a strict s.m. functor because $z_{\underline{\underline{1}}\,\underline{\underline{1}}} = 1$ by (6.20). This gives us the $\tilde{j}_*$ and the $\tilde{j}$ of §5.1.

The next thing is to enrich $m\circ A\colon \mathcal{D}\circ\mathcal{D}\circ A \to \mathcal{D}\circ A$ to a strong s.m. functor $\tilde{m}A = (m\circ A,\hat{m}A,m^\circ A)$, as in (5.18); we abbreviate $\tilde{m}A$, $\hat{m}A$, $m^\circ A$ to $\tilde{m},\hat{m},m^\circ$ where no confusion is likely. Use $\bar{\bar{\otimes}}$ etc. for the s.m. structure on $\mathcal{D}\circ\mathcal{D}\circ A$. Since $\bar{\bar{I}} = \underline{\underline{1}}[\bar{I}] = \underline{\underline{1}}[\underline{\underline{1}}[I]] = \underline{\underline{1}}[\underline{\underline{1}}][I]$, we have $(m\circ A)\bar{\bar{I}} = \underline{\underline{1}}[I] = \bar{I}$; so we take $m^\circ = 1$. Now let

$$X = T[P_1[A_{11}\ldots A_{1\pi_1}], \ldots, P_\tau[A_{\tau 1}\ldots A_{\tau\pi_\tau}]]$$

$$Y = S[R_1[B_{11}\ldots B_{1\rho_1}], \ldots, R_\sigma[B_{\sigma_1}\ldots B_{\sigma\rho_\sigma}]]$$

be objects of $\mathcal{D}\circ\mathcal{D}\circ A$. We need $\hat{m}$: $(m\circ A)X \bar{\otimes} (m\circ A)Y \to (m\circ A)(X\bar{\bar{\otimes}}Y)$. Now $(m\circ A)X = T(P_1\ldots P_\tau)[A_{11}\ldots]$ and

$$(6.28)\quad (m\circ A)X \bar{\otimes} (m\circ A)Y$$

$$= T(P_1\ldots P_\tau)(S(R_1\ldots R_\sigma)\ldots S(R_1\ldots R_\sigma))[A_{11}\otimes B_{11},\ldots]$$

$$= T(M_1\ldots M_\tau)[A_{11}\otimes B_{11},\ldots],$$

where for example

$$(6.29)\quad M_1 = P_1(S\ldots S)(R_1\ldots R_\sigma\ldots R_1\ldots R_\sigma).$$

On the other hand,

$$X\bar{\bar{\otimes}}Y = T(S\ldots S)[P_1[A_{11}\ldots]\bar{\otimes}R_1[B_{11}\ldots], \ldots]$$

$$= T(S\ldots S)[P_1(R_1\ldots R_1)[A_{11}\otimes B_{11},\ldots], \ldots],$$

so that

$$(6.30)\quad (m\circ A)(X\bar{\bar{\otimes}}Y)$$

$$= T(S\ldots S)(P_1(R_1\ldots R_1)\ldots)[A_{11}\otimes B_{11},\ldots]$$

$$= T(N_1\ldots N_\tau)[A_{11}\otimes B_{11},\ldots]$$

where for example

$$(6.31)\quad N_1 = S(P_1\ldots P_1)(R_1\ldots R_1\ldots R_\sigma\ldots R_\sigma).$$

We now take the X,Y-component of $\hat{m}$ to be

$$(6.32)\quad T(z_{P_1S}(1,1,\ldots,1),\ \ldots,\ z_{P_\tau S}(1,1,\ldots,1))[1,1,\ldots,1],$$

where for example $z_{P_1S}(1,1,\ldots,1)\colon\ M_1 \to N_1$.

Using the properties (6.18)-(6.22) of z, it is trivial if somewhat tedious to verify that $\tilde{m} = (m\circ A,\hat{m},m^\circ)$ satisfies the axioms for a s.m. functor. It is clearly 2-natural in A by its construction. Finally, using the same properties of z and a large sheet of paper, we easily verify that $(\tilde{D},\tilde{m},\tilde{j})$ satisfies the axioms for a doctrine. Thus we have our distributive law p,π as in §5.2.

We shall not need the explicit values of p and π. One point, however, should be noted. As a 2-natural transformation $P\circ\mathcal{D}\circ- \to \mathcal{D}\circ P\circ-$, p comes from a map $P\circ\mathcal{D} \to \mathcal{D}\circ P$ in $Cat\int^{*}\underline{\underline{S}}$, that we may as well still call p; see for example Theorem 8.1 of [9]. However, although P and $\mathcal{D}$ are both clubs in $Cat/\underline{\underline{S}}$, the map p: $P\circ\mathcal{D} \to \mathcal{D}\circ P$ does not lie in $Cat/\underline{\underline{S}}$. For instance, it sends $\otimes[T,S]$ to $T(S\ldots S)[\otimes,\otimes,\ldots,\otimes]$; the type of the former is $\tau + \sigma$, while that of the latter is $2\tau\sigma$; so p does not commute with the augmentation Γ. This is in contrast with say §4.10 above, where we did not have to move out of $Cat/\underline{\underline{S}}$; and it is perhaps the first example to show that, even if we are only concerned with clubs in $Cat/\underline{\underline{S}}$, we need the fuller theory of [9]. (This was adumbrated in §10.7 of [9].)

We sum up the result of this section in:

Theorem 6.8 If $\mathcal{D}$ is a pseudo-commutative club, the construction above defines a pseudo distributive law of D' over $D = \mathcal{D}\circ-$, where $D' = P\circ-$ is the doctrine for symmetric monoidal categories. Similarly if D' is the doctrine $\underline{\underline{P}}\circ-$ for strict symmetric monoidal categories.

The examples in §6.4 of pseudo-commutative clubs now give examples of pseudo distributive laws, as promised in §5.1.

6.6 For the special pseudo distributive laws obtained in this way, the $\tilde{D}$-algebras, and hence the D^*-algebras, admit a simpler description than that given in §5.3. We work out from first principles what they are.

To give such an algebra $\mathcal{B}$ is to give a s.m. category $\mathcal{B}$ together with an action $\tilde{b}\colon \tilde{D}\mathcal{B} \to \mathcal{B}$. So $\tilde{b}$ is to be a s.m. functor $\tilde{b} = (b, \hat{b}, b^\circ)\colon \mathcal{D}\circ\mathcal{B} \to \mathcal{B}$ that satisfies the axioms for a $\tilde{D}$-action. (we are using $\mathcal{B}, b$ rather than A, a to avoid confusion with the associativity a.) The s.m. structure on $\mathcal{D}\circ\mathcal{B}$ is of course that described in §6.4, which we employ without further comment.

At the level of ordinary functors, the action axioms tell us that $b\colon \mathcal{D}\circ\mathcal{B} \to \mathcal{B}$ is to be a $\mathcal{D}$-action on $\mathcal{B}$ in the ordinary sense. We write the image under b of $T[B_1 \ldots B_\tau] \in \mathcal{D}\circ\mathcal{B}$ in the usual way as $T(B_1 \ldots B_\tau) \in \mathcal{B}$, and similarly for morphisms. The action axioms now express the equality of $T(S_1(A_{11}\ldots) \ldots S_\tau(A_{\tau 1}\ldots))$ with $T(S_1 \ldots S_\tau)(A_{11}\ldots A_{\tau 1}\ldots)$ and of $\underline{\underline{1}}(A)$ with A, together with the similar equalities for morphisms.

Since $j\circ\mathcal{B}\colon \mathcal{B} \to \mathcal{D}\circ\mathcal{B}$ is the strict s.m. functor sending B to $\underline{\underline{1}}[B]$, the unitary axiom for an action immediately forces b° to be the identity:

$$(6.33)\quad b^\circ = 1\colon\ I \to \underline{\underline{1}}(I) = I.$$

Since $m^\circ = 1$ too, the associativity axiom for an action is satisfied at the level of $^\circ$-components.

The only extra piece of structure, then, that a D^*-algebra $\mathcal{B}$ has, over and above its symmetric monoidal structure and its $\mathcal{D}$-algebra structure, is the natural transformation

(6.34)

$$\begin{array}{c} T(A_1\ldots A_\tau)\otimes S(B_1\ldots B_\sigma) \\ \downarrow \hat{b}_{T[A_1\ldots A_\tau],S[B_1\ldots B_\sigma]} \\ T(S\ldots S)(A_1\otimes B_1,\ldots,A_1\otimes B_\sigma,\ldots,A_\tau\otimes B_1,\ldots,A_\tau\otimes B_\sigma), \end{array}$$

which we abbreviate to $\hat{b}_{TS}$ or just to $\hat{b}$.

The axioms for $(b,\hat{b},1)$ to be a s.m. functor reduce to the following three:

(6.35)

$$\begin{array}{ccc} T(A_1\ldots A_\tau)\otimes\underline{1}(I) & = & T(A_1\ldots A_\tau)\otimes I \\ \downarrow \hat{b} & & \\ T(\underline{1}\ldots\underline{1})(A_1\otimes I\ldots A_\tau\otimes I) & & \downarrow r \\ \| & & \\ T(A_1\otimes I\ldots A_\tau\otimes I) & \xrightarrow[1(r,r,\ldots,r)]{} & T(A_1\ldots A_\tau). \end{array}$$

(6.36)

$$\begin{array}{ccc} (T(A_1\ldots)\otimes S(B_1\ldots))\otimes R(C_1\ldots) & \xrightarrow{a} & T(A_1\ldots)\otimes(S(B_1\ldots)\otimes R(C_1\ldots)) \\ \downarrow \hat{b}\otimes 1 & & \downarrow 1\otimes\hat{b} \\ T(S\ldots S)(A_1\otimes B_1,\ldots)\otimes R(C_1\ldots) & & T(A_1\ldots)\otimes S(R\ldots R)(B_1\otimes C_1,\ldots) \\ \downarrow \hat{b} & & \downarrow \hat{b} \\ T(S\ldots S)(R\ldots R)((A_1\otimes B_1)\otimes C_1,\ldots) & \xrightarrow[1(a,a,\ldots,a)]{} & T(S\ldots S)(R\ldots R)((A_1\otimes(B_1\otimes C_1),\ldots \end{array}$$

(6.37)

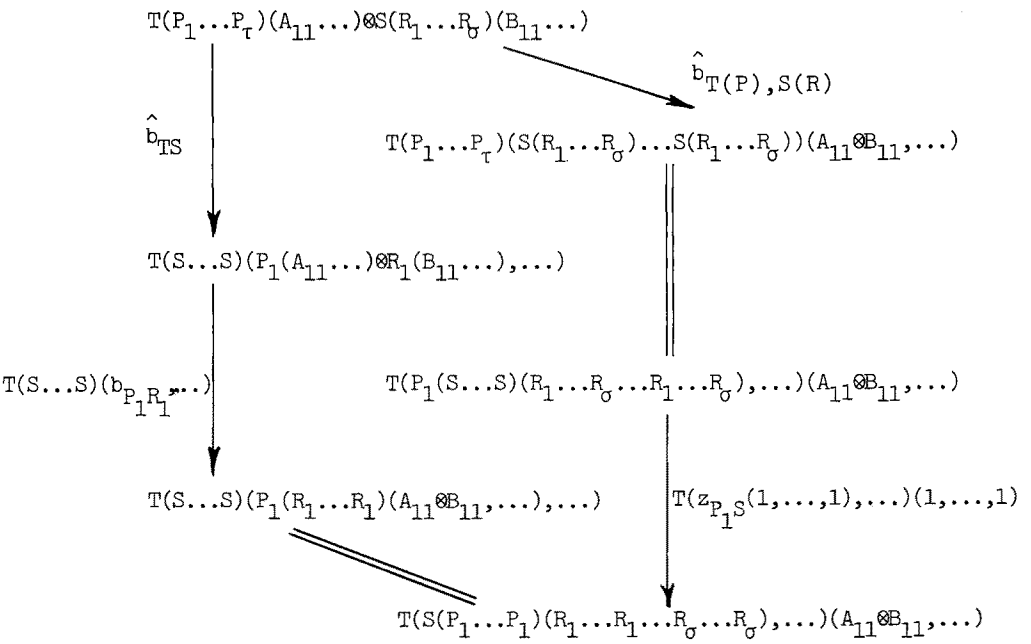
$$\begin{array}{ccc} T(A_1\ldots)\otimes S(B_1\ldots) & \xrightarrow{c} & S(B_1\ldots)\otimes T(A_1\ldots) \\ \downarrow \hat{b} & & \downarrow \hat{b} \\ T(S\ldots S)(A_1\otimes B_1,\ldots) & \xrightarrow[z_{TS}(c,c,\ldots,c)]{} & S(T\ldots T)(B_1\otimes A_1,\ldots). \end{array}$$

The $\hat{}$-component of the unitary axiom for an action reduces to

(6.38) $\hat{b}_{\underline{\underline{1}}[A],\underline{\underline{1}}[B]}$: $\underline{\underline{1}}(A)\otimes\underline{\underline{1}}(B) \to \underline{\underline{1}}(\underline{1})(A\otimes B)$ is 1: $A\otimes B \to A\otimes B$.

Finally the $\hat{}$-component of the associativity axiom for an action reduces, in view of (6.32), to

(6.39)



where in the topmost arrow T(P) stands for $T(P_1...P_\tau)$, etc.

Now write

(6.40) $e^B_{T[A_1...A_\tau]}$ for $\hat{b}_{T[A_1...A_\tau],\underline{\underline{1}}[B]}$,

abbreviating it where desirable to e^B_T or e_T or e, so that

(6.41) e: $T(A_1...A_\tau)\otimes B \to T(A_1\otimes B,...,A_\tau\otimes B)$.

We show that $\hat{b}$ can be given in terms of e alone, the axioms on $\hat{b}$ reducing to simple axioms on e.

Write

(6.42) $e^{\#}$: $B \otimes T(A_1 \ldots A_\tau) \to T(B \otimes A_1, \ldots, B \otimes A_\tau)$

for the conjugate

(6.43) $e^{\#} = T(c, c, \ldots c).e.c$

of e under c. Then (6.37) gives

(6.44) $\hat{b}_{\underline{\underline{1}}T} = e^{\#}$.

Now in (6.39) set each P_α equal to $\underline{\underline{1}}$ and set $S = \underline{\underline{1}}$. Since $z_{\underline{\underline{1}}\underline{\underline{1}}} = 1$ by (6.20), we find that $\hat{b}_{TR}$ is the composite

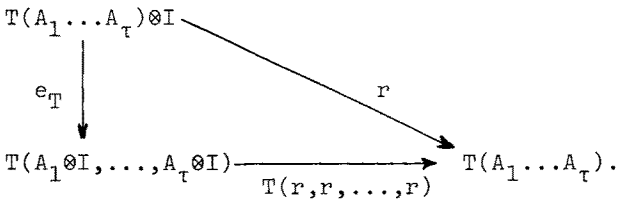
(6.45)
$$
\begin{array}{l}
T(A_1 \ldots A_\tau) \otimes R(B_1 \ldots B_\rho) \\
\qquad\qquad \big\downarrow e \\
T(A_1 \otimes R(B_1 \ldots B_\rho), \ldots, A_\tau \otimes R(B_1 \ldots B_\rho) \\
\qquad\qquad \big\downarrow T(e^{\#}, \ldots, e^{\#}) \\
T(R(A_1 \otimes B_1, \ldots, A_1 \otimes B_\rho), \ldots, R(A_\tau \otimes B_1, \ldots, A_\tau \otimes B_\rho)) \\
\qquad\qquad \big\| \\
T(R \ldots R)(A_1 \otimes B_1, \ldots, A_1 \otimes B_\rho, \ldots, A_\tau \otimes B_1, \ldots, A_\tau \otimes B_\rho).
\end{array}
$$

Consider the following five axioms on e:

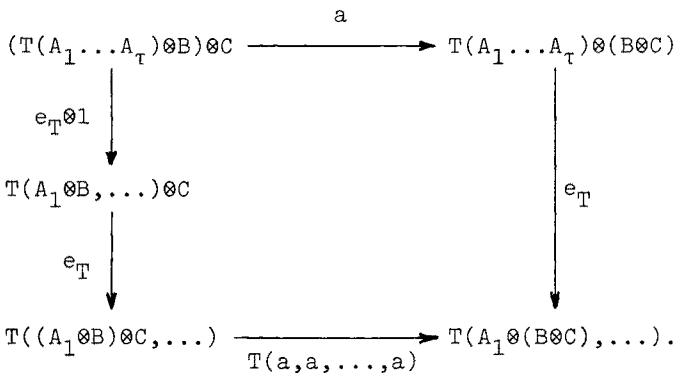
(6.46)
$$
\begin{array}{lcl}
T(S_1 \ldots S_\tau)(A_{11} \ldots A_{\tau\sigma_\tau}) \otimes B & = & T(S_1(A_{11} \ldots) \ldots S_\tau(A_{\tau 1} \ldots)) \otimes B \\
\big\downarrow e_{T(S_1 \ldots S_\tau)} & & \big\downarrow e_T \\
T(S_1 \ldots S_\tau)(A_{11} \otimes B, \ldots) & = & T(S_1(A_{11} \ldots) \otimes B, \ldots S_\tau(A_{\tau 1} \ldots) \otimes B) \\
 & & \big\downarrow T(e_{S_1}, \ldots, e_{S_\tau}) \\
 & = & T(S_1(A_{11} \otimes B, \ldots) \ldots S_\tau(A_{\tau 1} \otimes B, \ldots)).
\end{array}
$$

(6.47) $e_{\underline{\underline{1}}}: \ \underline{\underline{1}}(A)\otimes B \to \underline{\underline{1}}(A\otimes B)$ is 1: $A\otimes B \to A\otimes B$.

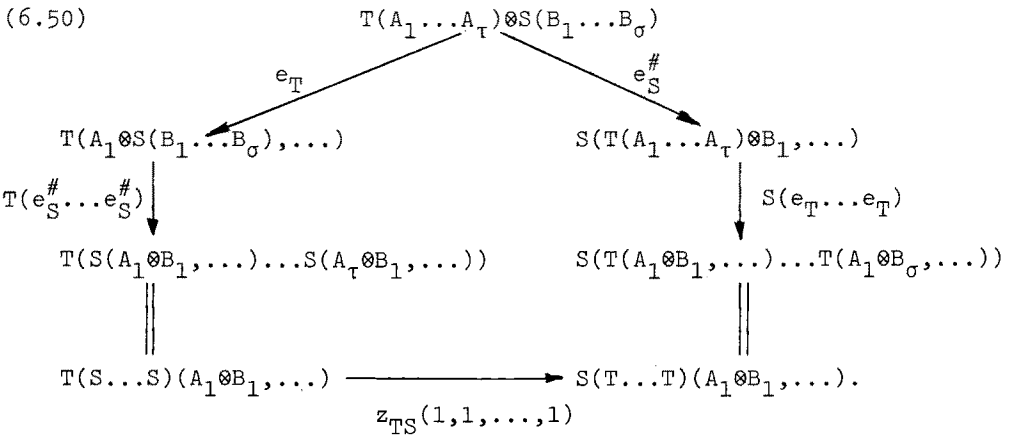
(6.48)



(6.49)



(6.50)



Proposition 6.9 Let $\mathcal{B}$ be a symmetric monoidal category and also a $\mathcal{D}$-algebra for the pseudo-commutative club $\mathcal{D}$. There is a bijection between natural transformations $\hat{b}$ as in (6.34) satisfying (6.35)-(6.39), and natural transformations e as in (6.41) satisfying (6.46)-(6.50) where $e^{\#}$ is defined by (6.43). We get e from $\hat{b}$ by (6.40) and $\hat{b}$ from e by (6.45). A $\mathcal{B}$ with all this structure is precisely an algebra for the doctrine D^* of §5.5 (lax case). We pass to the pseudo case by supposing $\hat{b}$, or equivalently e, to be an isomorphism.

Proof We have already shown that if we start with $\hat{b}$ and define e by (6.40) then (6.45) gives us back $\hat{b}$. If on the other hand we start with e and define $\hat{b}$ by (6.45), then (6.40) gives us back e; just put $S = \underline{1}$ in (6.50), use (the conjugate of) (6.47), and use (6.21).

The only other point to be verified is the equivalence of the $\hat{b}$-axioms with the e-axioms. Note that each of the e-axioms (6.x) implies its conjugate $(6.x)^{\#}$. First, then, (6.35) is identical with (6.48); (6.37) identical with (6.50); and (6.38) identical with (6.47). Next we get (6.46) from (6.39) by setting $S = R_1 = \underline{1}$ in the latter and using (6.21); and we get (6.49) from (6.36) by setting $S = R = \underline{1}$ and using (6.38).

It remains, given the e-axioms, to get (6.36) and (6.39). The latter is easy using the definition (6.45) of $\hat{b}$, using (6.46) and $(6.46)^{\#}$, and using (6.50). As for (6.36) we easily get the special case (6.36)' where $T = R = \underline{1}$ by using (6.49) and $(6.49)^{\#}$. From this together with (6.46) and (6.49) we next get the special case (6.36)" where $R = \underline{1}$. Finally from this, from $(6.46)^{\#}$, and from $(6.49)^{\#}$, we get the desired (6.36). The reader will have no trouble in supplying the details. □

The first four of the e-axioms admit conceptual interpretations, and we are led to the following definitive description of D^*-algebras.

Theorem 6.10 Let $\mathcal{V}$ be a pseudo-commutative club and let D^* be the doctrine arising as in §5.5 from the pseudo distributive law of Theorem 6.8. Then a D^*-algebra is a $\mathcal{V}$-algebra $\mathcal{B}$ that is also a symmetric monoidal category, together with for each $B \in \mathcal{B}$ an enrichment of the functor $-\otimes B\colon \mathcal{B} \to \mathcal{B}$ to an op-$\mathcal{V}$-functor $(-\otimes B, e^B)\colon \mathcal{B} \to \mathcal{B}$, satisfying axioms to be given below. We agree to abbreviate the $T[A_1 \ldots A_\tau]$-component

$$(6.51)\quad e^B_{T[A_1 \ldots A_\tau]}\colon\ T(A_1 \ldots A_\tau)\otimes B \to T(A_1\otimes B, \ldots, A_\tau \otimes B)$$

of e^B to e_T where convenient; and we observe that conjugation by the symmetry c of $\mathcal{B}$ produces from e^B an enrichment of $B\otimes -$ to an op-$\mathcal{V}$-functor $(B\otimes -, e^{\#B})$. The axioms now are:

(6.52) For $f\colon B \to C$, the natural transformation $-\otimes f\colon -\otimes B \to -\otimes C$ is op-$\mathcal{V}$-natural.

(6.53) $a\colon (-\otimes B)\otimes C \to -\otimes(B\otimes C)$ is op-$\mathcal{V}$-natural.

(6.54) $r\colon -\otimes I \to 1$ is op-$\mathcal{V}$-natural.

(6.55) e and $e^{\#}$ are related by the commutativity of (6.50), which it suffices to impose for T,S belonging to a set of generators of the discrete club $|\mathcal{V}|$.

The above is for the lax case; for the pseudo case we require the op-$\mathcal{V}$-functor $-\otimes B$ to be strong, that is, e to be an isomorphism.

Proof By [9] §10.8 to say that $(-\otimes B, e^B)$ is an op-$\mathcal{D}$-functor is to say that the e of (6.51) is natural in T and the A_α, and that it satisfies (6.46) and (6.47) (corresponding to (10.25) and (10.26) of [9]). The axiom (6.52) above just makes e natural in B as well as in T and the A_α. The axioms (6.53) and (6.54) reduce, by (10.27) of [9], to (6.49) and (6.48) respectively. The result now follows from Proposition 6.9, except for the last clause in (6.55).

The proof of that last clause is easy: using (6.46) and (6.19) we see that (6.50) for T,S, where $T = R(P_1 \ldots P_\rho)$, is a consequence of (6.50) for R,S and for P_α,S. □

When $\mathcal{D}$ is presented by a small number of generators and relations (cf. [9] §10) the above theorem gives a very compact description of D^*-algebras.

Example 6.11 Let $\mathcal{D}$ be itself the club $\mathcal{P}$ for symmetric monoidal categories, which is pseudo-commutative by Example 6.4. Then a D^*-algebra $\mathcal{B}$ has two symmetric monoidal structures: the $\mathcal{D}'$-structure denoted as above by $(\mathcal{B},\otimes,I,a,r,c)$, and the $\mathcal{D}$-structure denoted by $(\mathcal{B},\oplus,N,a',r',c')$. To give the enrichment e^B of $-\otimes B$ to a symmetric op-monoidal functor, we need as usual only to give its components

$$e_\oplus: \quad (A_1 \oplus A_2)\otimes B \to (A_1 \otimes B)\oplus(A_2 \otimes B),$$

$$e_N: \quad N \otimes B \to N,$$

corresponding to the generators for the objects of $\mathcal{D}$, subject to the usual three axioms. The axiom (6.52) merely asks these components to be natural in B as well as the A_α. The condition (6.53) that a be op-s.m.-natural reduces as usual to two diagrams, as does similarly (6.54) for r. Finally (6.55) requires us to impose (6.50) in the three cases $(T,S) = (\oplus,\oplus)$, $(\oplus,N)$, and (N,N) (that for $(N,\oplus)$ being the conjugate of that for $(\oplus,N)$). All told we have ten simple axioms

involving $e_{\oplus}$ and e_N; and it is easy to see that these are precisely the independent axioms among the 24 given for this situation by Laplaza [18].

6.7 If we take the description of D^*-algebras given by Theorem 6.10, and modify it by changing the sense of the arrow e in (6.51), at the same time changing its name to ${}^*e$, so that ${}^*e$ enriches $-\otimes B$ to a $\mathcal{V}$-functor rather than an op-$\mathcal{V}$-functor, we obtain a description of the algebras for some new doctrine, which we shall call ${}^*D$.

In the situation of §5 we should have got ${}^*D$ rather than D^* if from the pseudo distributive law we had constructed a doctrine $\tilde{D}$ not on D'-Alg but on the 2-category with D'-algebras as objects and op-D'-morphisms as 1-cells. Since π is invertible this ${}^*D$ case is exactly like the D^*-case; there is nothing to change except the sense of the 2-cells. In particular Theorem 5.1 in the ${}^*D$-case gives a coreflexion $s: {}^*D \to DD'$, with $h: DD' \to {}^*D$, $sh = 1$, $\eta: hs \Rightarrow 1$, $s\eta = 1$, $\eta h = 1$.

The point of our introducing ${}^*D$ is that, in the lax case, it is of the form ${}^*\mathcal{V}\circ-$ for a club ${}^*\mathcal{V}$ in $Cat/\underline{S}$. In the light of the general principles of [9] §10, this is clear from Theorem 6.10; for the type of the natural transformation ${}^*e$ lies in $\underline{S}$.

The ${}^*D$ for the pseudo case, and the D^* for the lax and the pseudo cases, sometimes arise from clubs in $Cat/\underline{S}$, namely when the type of e in (6.51) is in $\underline{S}$, which only happens when $\tau = 1$ in (6.51); since this is to be so for all $T \in \mathcal{V}$, it only happens when $\Gamma: \mathcal{V} \to \underline{S}$ is constant at 1.

We can bring Example 6.11 above, the one studied by Laplaza, under the ${}^*\mathcal{V}$-setting, just by replacing the model $\mathcal{B}$ by $\mathcal{B}^{op}$ (i.e. in effect by passing to the opposite doctrine). This works in this case because $\mathcal{B}^{op}$ is symmetric monoidal when $\mathcal{B}$ is.

Example 6.12 Another important example occurs naturally in the $^*\mathcal{D}$-setting. Let $\mathcal{D}$ be the club S of Example 6.2, whose algebras are categories with finite coproducts. Consider any S-algebra $\mathcal{B}$ that is also symmetric monoidal. For $T \in S$ with $\Gamma T = \tau$, and for $A_\alpha \in \mathcal{B}$, the object $T(A_1 \ldots A_\tau)$ is a coproduct in $\mathcal{B}$ of the A_α, the coprojections being $f_\alpha(1)$ for suitable $f_\alpha\colon \underline{1} \to T$ in S whose types are the various maps $1 \to \tau$ in $\underline{\mathbb{S}}$.

It follows that there is a unique $^*e$ turning $\mathcal{B}$ into a $^*S$-algebra. For the naturality in T of $^*e$ demands commutativity in

$$\begin{array}{ccc} T(A_1\otimes B,\ldots,A_\tau\otimes B) & \xrightarrow{\ ^*e_T\ } & T(A_1\ldots A_\tau)\otimes B \\ \uparrow{\scriptstyle f_\alpha(1)} & & \uparrow{\scriptstyle f_\alpha(1)\otimes 1} \\ \underline{1}(A_1\otimes B) & \xrightarrow[\ ^*e_{\underline{1}}\ ]{} & \underline{1}(A_1)\otimes B \ ; \end{array}$$

and $^*e_{\underline{1}}$ is to be 1 by (6.47). So the only possible $^*e$ is that whose α-th component is $f_\alpha(1)\otimes 1$. On the other hand it is easily seen that this does satisfy the axioms.

We are led to the conclusion that the doctrine $^*S\circ-$ is the coproduct, in the category of doctrines, of the doctrines $S\circ-$ and $P\circ-$; or equally that, in the category of clubs in $Cat/\underline{\mathbb{S}}$ and club-maps in $Cat/\underline{\mathbb{S}}$, $^*S$ is the coproduct of S and P.

6.8 We can apply Theorem 5.1 to the present situation, to obtain a reflexion $s\colon D^* \to \mathcal{D}\circ P\circ-$, or $sA\colon D^*A \to \mathcal{D}\circ P\circ A$, which is an equivalence in the pseudo case: or equally a coreflexion $s : {}^*DA \to \mathcal{D}\circ P\circ A$ which, in the lax case, taking $A = I$, can be written $s\colon {}^*\mathcal{D} \to \mathcal{D}\circ P$.

I shall just indicate briefly how s is calculated. First, however, observe that this special case of Theorem 5.1 can be proved independently of §5, taking Theorem 6.10 as the definition of

D^*-algebra, and applying Theorem 2.6 directly, using the techniques of the present §6 alone. This gives for instance quite a short proof of Laplaza's result; but it conceals the connexion between "distributivity of $-\otimes B$" in the sense of (6.51) and "(pseudo) distributive law" in the sense of Beck.

$P\circ A$ is a s.m. category, indeed a free one, whence $\mathcal{V}\circ P\circ A$ is also a s.m. category, with the structure given in §6.5. At the same time $\mathcal{V}\circ P\circ A$ is a $\mathcal{V}$-category, in fact a free one. For $T \in \mathcal{V}$ and for $A_1,\dots,A_\tau,B \in \mathcal{V}\circ P\circ A$ an easy calculation of $T(A_1\dots A_\tau)\otimes B$ and of $T(A_1\otimes B,\dots,A_\tau\otimes B)$ shows these to be equal. We can therefore set $e = 1$ in (6.51). The axioms (6.46)-(6.49) are trivially satisfied; and (6.50) is in fact satisfied (not trivially because $e^\#$ given by (6.43) is not 1 when $e = 1$; but easily in view of the value of c in $\mathcal{V}\circ P\circ A$ given by (6.27)).

Thus $\mathcal{V}\circ P\circ A$ (and in particular $\mathcal{V}\circ P$) is a D^*-category; even for the pseudo case, hence also for the lax case; and equally a ${}^*D$-category. Since s is a strict D^* (resp. ${}^*D$)-morphism, we can write at once the image $s(\Delta)$ of any diagram Δ, in the free D^*-category D^*A, made out of the elementary data; and in the pseudo case Δ commutes if and only if $s(\Delta)$ does so.

In the lax-${}^*D$-case we have the usual more flexible results expressed in terms of clubs. The objects of ${}^*\mathcal{V}$ are generated by $\otimes, I$, and the objects $T \in \mathcal{V}$; the morphisms by a,r,c, those of $\mathcal{V}$, and ${}^*e$; the functor s: ${}^*\mathcal{V} \to \mathcal{V}\circ P$ respects all this structure, and in particular sends ${}^*e$ to 1. In the other direction h: $\mathcal{V}\circ P \to {}^*\mathcal{V}$ sends $T[V_1\dots V_\tau]$ to $T(V_1\dots V_\tau)$. Then $sh = 1$ and η: $hs \Rightarrow 1$ for a suitable η with $s\eta = 1$, $\eta h = 1$. To find η directly, and to provide a proof independent of Theorem 5.1, we enrich h to an op-${}^*\mathcal{V}$-functor: we make it strict as a $\mathcal{V}$-functor, and as an op-monoidal-functor $(h,\hat{h},h^\circ)$ it has $h^\circ = 1$ and $\hat{h}$ formally the same as the $\hat{b}$ of (6.34) (whose sense is now reversed). The η is then formed as in Theorem 2.6

from the $\bar{h}$ of the resulting op-$^*\mathcal{V}$-functor $(h,\bar{h})$.

In the case $\mathcal{V} = \mathcal{P}$ studied by Laplaza, it is easy to see that the composite of s: $^*P \to P \circ P$ with the equivalence $\Gamma \circ \Gamma$: $P \circ P \to \underline{\underline{P}} \circ \underline{\underline{P}}$ gives a functor $^*P \to \underline{\underline{P}} \circ \underline{\underline{P}}$ that is precisely his distortion ([19] §2). So as indicated in §1.4 above, our coreflexion result implies his theorem on page 231 of [19]. (Note that Laplaza is considering diagrams writable in $^*\mathcal{V}$ - that is, involving $^*e$ but not its inverse - but studying their commutativity in a model in which $^*e$ happens to be an isomorphism. He actually says "monomorphism", which since we have dualized would be "epimorphism", but this is not enough as it stands, and is too rare in examples to be worth pursuing.)

The other main result of Laplaza, given in his earlier paper ([18] Proposition 10) follows from the same coreflexion theorem, if we observe that, for a discrete A, two morphisms f,g: $X \to Y$ in $\underline{\underline{P}} \circ \underline{\underline{P}} \circ A$ necessarily commute if $Y = n[m_1[A_{11}\ldots],\ldots,m_n[A_{n1}\ldots]]$ where all the $m_\alpha[A_{\alpha 1}\ldots]$ are different and, for each α, all the $A_{\alpha 1},\ldots,A_{\alpha m_\alpha}$ are different (for the permutations expressing f,g are then determined and equal).

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