REVIEW OF THE ELEMENTS OF 2-CATEGORIES

by

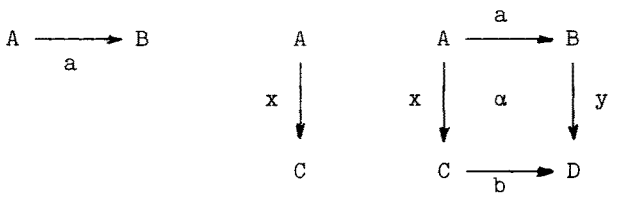
G.M. Kelly and Ross Street

The purpose of this review is to serve as a common introduction to the authors' papers in this volume, by collecting together some basic notions needed by each of us, and establishing some notation, to avoid later duplication. Nothing of substance here is original, but we could find no connected account in the literature that exactly satisfied our needs. In §1 we rehearse the most elementary facts about 2-categories, partly in the hope of making our papers below self-contained for such beginners as may care to read them, but chiefly to introduce our notation and especially the operation of pasting that we use constantly. In §2 we use the pasting operation to give a treatment, which seems to us simpler and more complete than any we have seen, of the isomorphism $(bu,u'a) = (f'b,af)$ arising from adjunctions $f \dashv u$ and $f' \dashv u'$ in any 2-category, and of its naturality. In §3 we recall the basic properties of monads in a 2-category, and then mention some enrichments of these that become available in the 2-category of 2-categories (because it is really a 3-category).

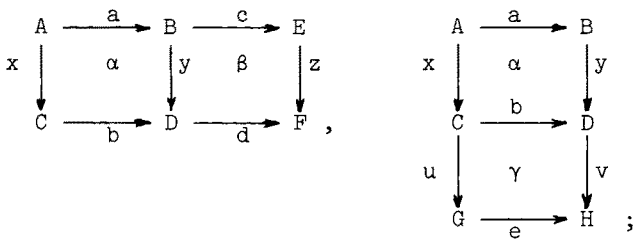
§1 DOUBLE CATEGORIES AND 2-CATEGORIES

1.1 Both notions are originally due to Ehresmann; see [6] and [7]. We recall first the notion of double category. We denote by $\mathcal{CAT}$, $\mathcal{SET}$, $\mathcal{Cat}$, $\mathcal{Set}$, respectively the categories of all categories, of all sets, of small categories, and of small sets. Conceptually, a double category is a category object in $\mathcal{CAT}$; but it admits the following elementary description.

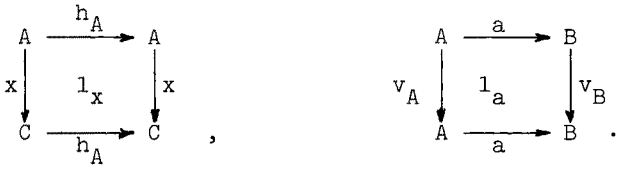
It has objects A etc.; horizontal arrows a etc.; vertical arrows x etc.; and squares α etc.; there are various domain and codomain functions sufficiently indicated by the diagrams



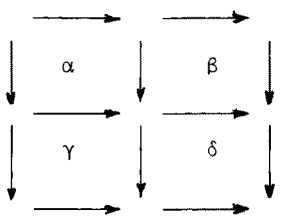
The objects and the horizontal arrows form a category, with identities $h_A : A \to A$; the objects and the vertical arrows form a category, with identities $\begin{matrix} A \\ \downarrow v_A \\ A \end{matrix}$. The squares have horizontal and vertical laws of composition, represented by



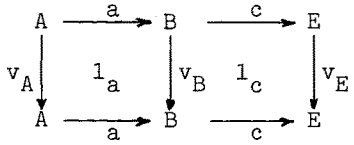
under each of these laws they form a category, with respective identities



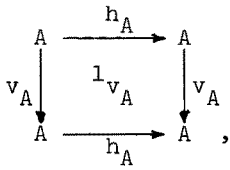
In the situation

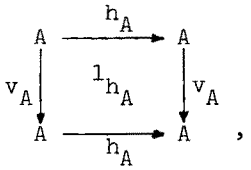


the result of composing first horizontally and then vertically is to be the same as the result of composing first vertically and then horizontally. The composite



is to be 1_{ca}, and similarly for vertical composition of horizontal identities. Finally the horizontal and vertical identities



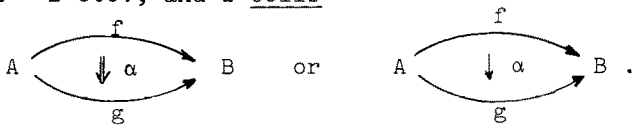


are to coincide.

Examples of double categories may be found in Palmquist [21], and in §2.2 below.

1.2. A 2-category K may be thought of as a double category in which all the vertical arrows are identities. A more direct description is as follows.

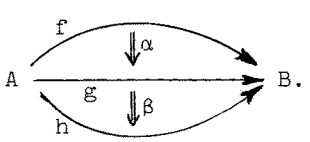
K has objects or 0-cells A etc., arrows or morphisms or 1-cells $f: A \to B$ etc., and 2-cells



(Kelly tends to use double arrows for 2-cells, and Street single ones - it is purely a matter of taste.)

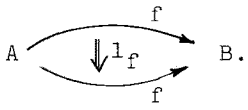
The objects and the arrows form a category K_0, called the underlying category of K, with identities $1_A: A \to A$; when the context shows what is meant, we sometimes write K for K_0.

For fixed A and B, the arrows $A \to B$ and the 2-cells between them form a category $K(A,B)$ under the operation known as vertical composition:



The vertical composite $f \Rightarrow h$ above is denoted by $\beta \cdot \alpha$ or $\beta.\alpha$, or rarely

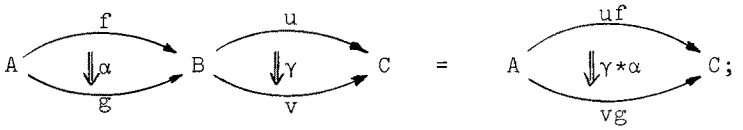
by $\beta\alpha$ when no confusion is likely with the horizontal composite to be introduced below; its identities are denoted by



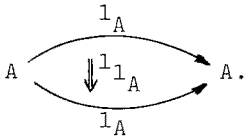
There is also a law of horizontal composition of 2-cells, whereby from 2-cells



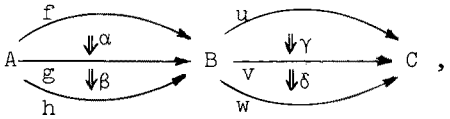
we get a 2-cell



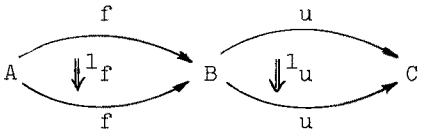
this composite $\gamma*\alpha$ is also denoted by $\gamma\alpha$: uf ⇒ vg. Under this law the 2-cells are to form a category, with identities



We require finally that, in the situation

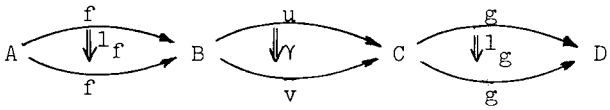


the composites $(\delta*\beta)\cdot(\gamma*\alpha)$ and $(\delta\cdot\gamma)*(\beta\cdot\alpha)$ coincide; and that in the situation

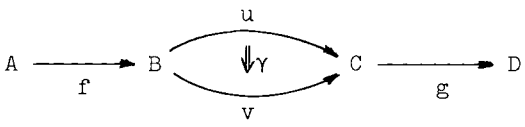


we have $1_u*1_f = 1_{uf}$.

We also freely use the convention whereby the name of an object A or of an arrow f is also used as the name of its identity arrow 1_A or its identity 2-cell 1_f. In particular the horizontal composite

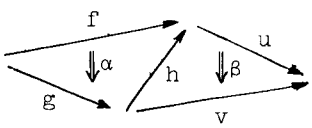


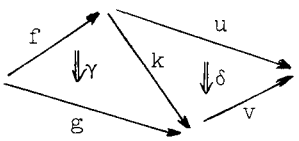
is also written as



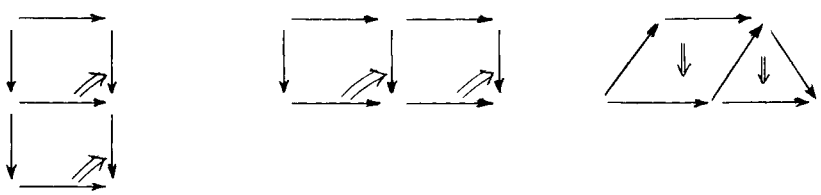
and denoted by gγf.

The above operations on 2-cells can be combined to give the more general operation of <u>pasting</u>, introduced by Benabou [1]. The two basic situations are

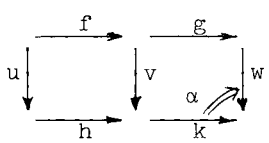




The first of these is meant to indicate the 2-cell $\beta g \cdot u\alpha$: $uf \Rightarrow uhg \Rightarrow vg$, and the second to indicate the 2-cell $v\gamma \cdot \delta f$: $uf \Rightarrow vkf \Rightarrow vg$. Thus we give meaning to such composites as

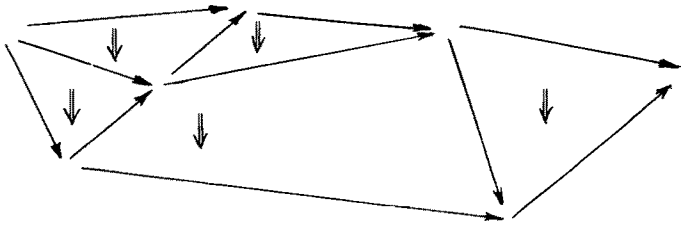


If in a diagram such as

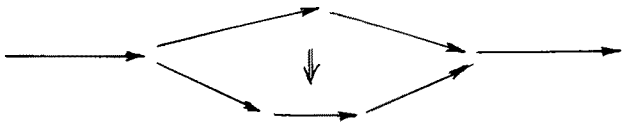


one region has no 2-cell marked in it, it is to be understood that the identity 2-cell is meant, which implies that $vf = hu$.

One can generalize the pasting operation further still, so as to give meaning to such multiple composites as



This is meant to indicate a vertical composite of horizontal composites of the form



;

there is usually a choice of the order in which the composites are taken, but the result is independent of this choice; this is clear in simple cases, and can be proved inductively in the general case after an appropriate formalization in terms of polygonal decompositions of the disk.

1.3. As for examples of 2-categories, the paradigmatic one is *CAT*, just as the paradigmatic category is *SET*. The objects are categories, the arrows are functors, and the 2-cells are natural transformations. The context will show when *CAT* is considered as a 2-category and when merely the underlying category is meant. There is the sub-2-category *Cat* of small categories.

For a monoidal category V, we have the 2-category V-*CAT* of V-categories, V-functors, and V-natural transformations, in the sense of [8]; again with the sub-2-category V-*Cat* of small ones. Monoidal categories themselves, with monoidal functors and monoidal natural transformations in the sense of [8], form a 2-category *Mon CAT*.

The category K of ordered objects in any category A becomes a 2-category when we observe that the hom-set $K(A,B)$ has a natural

order, and can therefore be regarded as a category.

For a category A, the comma category CAT/A, an object of which is a category $\mathcal{B}$ together with a functor $B: \mathcal{B} \to A$, and an arrow of which from $(\mathcal{B},B)$ to (C,C) is a functor $T: \mathcal{B} \to C$ with $CT = B$, becomes a 2-category when we take the 2-cells $\alpha: T \Rightarrow S: \mathcal{B} \to C$ to be the natural transformations $\alpha: T \Rightarrow S$ for which $C\alpha = id$. Many other examples will arise in the papers below.

1.4. Besides the elementary definition of 2-category in §1.2 above, there is an equivalent but non-elementary one: the category CAT is cartesian closed, and a 2-category K is just a CAT-category, its CAT-valued hom being $K(A,B)$. This definition determines what we shall mean by 2-functor and by 2-natural transformation: namely CAT-functor and CAT-natural transformation. Similarly for 2-adjoint, = CAT-adjoint; see [14] for a general treatment of V-adjunction. Note that we do not use "2-natural" or "2-adjoint" in the more general senses given to them by Gray in [10].

In elementary terms, a 2-functor $D: K \to L$ sends objects of K to objects of L, arrows of K to arrows of L, and 2-cells of K to 2-cells of L, preserving domains and codomains and all types of identity and composition. A 2-natural transformation $\eta: D \Rightarrow E: K \to L$ assigns to each object A of K an arrow η_A or $\eta A: DA \to EA$ in L, which is not only natural in the ordinary sense that, for $f: A \to B$, we have $\eta B.Df = Ef.\eta A$, but also 2-natural in the sense that, for each 2-cell $\alpha: f \Rightarrow g$ in K, we have

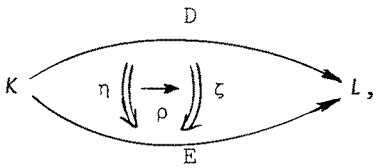
(1.1)
$$\begin{array}{ccccccc} & & Df & & & & Ef \\ DA & \Downarrow D\alpha & DB & \xrightarrow[\eta B]{} & EB = DA & \xrightarrow[\eta A]{} & EA \quad \Downarrow E\alpha \quad EB. \\ & & Dg & & & & Eg \end{array}$$

As in general the V-natural transformations from the V-functor D to the V-functor E form not only a set but an object of V, so here where $V = CAT$ the 2-natural transformations $D \Rightarrow E$ form a category; in other words 2-CAT is really a 3-category, i.e. a 2-CAT-

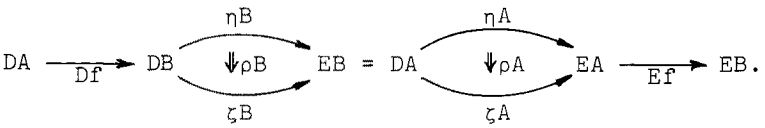
category. We follow Benabou [1] in calling morphisms of 2-natural transformations modifications. A modification

$$\rho\colon \eta \to \zeta\colon D \Rightarrow E\colon K \to L,$$

also written



assigns to each object A of K a 2-cell $\rho A\colon \eta A \Rightarrow \zeta A$ such that, for $f\colon A \to B$, we have

(1.2)


In particular, taking a fixed 2-category K, we get a 2-category End K of endomorphisms of K; its objects are endo-2-functors $D\colon K \to K$ of K; its arrows are 2-natural transformations $\eta\colon D \Rightarrow E$; and its 2-cells are modifications $\rho\colon \eta \to \zeta$. When working totally within this 2-category Kelly reserves the right to write $\eta\colon D \to E$ and $\rho\colon \eta \Rightarrow \zeta$ if he feels like it.

The non-elementary definition of a 2-category K also determines what K^{op} shall mean: we have $K^{op}(A,B) = K(B,A)$, so that we reverse the 1-cells but not the 2-cells. We write K^{co} for the other dual, induced by the ordinary duality on CAT, so that $K^{co}(A,B) = K(A,B)^{op}$; thus we reverse the 2-cells but not the 1-cells. Observe that $K^{coop} = K^{opco}$.

To say that 2-functors $D\colon K \to L$ and $E\colon L \to K$ are 2-adjoint is to say that they are CAT-adjoint; that is, that there is an isomorphism of categories $K(EB,A) \cong L(B,DA)$ which is 2-natural in A and B. Specializing the result of [14] on V-adjunction to the case $V = CAT$, we see that it comes to the same thing to have 2-natural transformations $\eta\colon 1 \Rightarrow DE$ and $\varepsilon\colon ED \Rightarrow 1$ satisfying the usual conditions; so that 2-adjunction is just adjunction in the 2-category 2-CAT, in the sense

of §2.1 below.

1.5. A great many notions definable in V-CAT for any V admit in 2-CAT generalizations that would make no sense for an arbitrary V; and this because 2-CAT is a 3-category. For where a diagram was to commute in the basic notion, we can now allow it to commute only to within a given isomorphism, or even go further and replace this isomorphism by a given morphism (in a given sense); the given isomorphisms or morphisms will then usually be required to satisfy some "coherence axioms". Moreover experience shows that these "relaxed notions" do indeed occur in nature; in some cases they seem in fact to be the more normal thing, that we must willy-nilly consider; in others we can prove a "coherence theorem" allowing them to be replaced by the simpler "strict" notion.

It is not our intention to go into this systematically here; such things are treated in detail in the forthcoming book (doubtless to appear before this one) of Gray [12]. We should however like to indicate a systematic nomenclature that we shall use.

Where the original "strict" notion N is taken as the norm, we say "pseudo-N" for the relaxed notion with equality replaced by an isomorphism, and "lax N" for the still more relaxed notion with equality replaced by a mere morphism: we have no general principle to offer for our various choices of the sense of this morphism, but having chosen a sense by some criterion of usefulness, we call the same notion with the sense reversed an "op-lax N".

An example is the concept of lax functor $D: K \to L$ between 2-categories, which has 2-cells $Dg.Df \Rightarrow D(gf)$ and $1_{DA} \Rightarrow D1_A$ instead of equalities; these have been considered by Street [24] and Roberts [22], and are a special case of Benabou's morphisms of bicategories [1]. Indeed a bicategory in Benabou's sense is itself a pseudo-2-category, and this illustrates a general point: when one relaxes some "morphism-like" notion, the definition continues to make sense when

the domain and the codomain are themselves "lax". Lax functors as such do not occur in the present book, but pseudo-functors occur in the paper [25] of Street below, and go back to Grothendieck [13].

A second example is the notion of lax natural transformation in place of 2-natural transformation; these do occur below in the paper of Kelly [15], and are the things called "2-natural" by Gray in [10] and "quasi-natural" (now adopted by Gray) in Bunge [2]. Another is the notion of lax monad (relaxed 2-monad) of Bunge [3]; the corresponding pseudo-monads are considered by Zöberlein [27]. Then there are the lax algebras or the pseudo-algebras over these, and their lax morphisms...

This brings us to the second aspect of our nomenclature. Even with strict 2-monads, and strict algebras for them, the lax morphisms of algebras are the normal thing, as is shown by the special case of monoidal functors, where strict ones (preserving the tensor product and identity on the nose) are quite rare in nature. In these cases we take the lax notion as our basic notion N; we then call the pseudo--notion "strong N" and the strict notion "strict N"; thus "monoidal functor" (with a comparison $\phi A \otimes \phi B \to \phi(A \otimes B)$), "strong monoidal functor" (with an isomorphism $\phi A \otimes \phi B \cong \phi(A \otimes B)$), and "strict monoidal functor" (with equality $\phi A \otimes \phi B = \phi(A \otimes B)$).

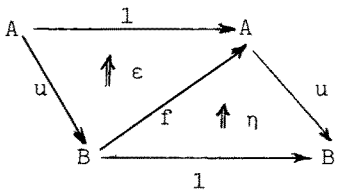
We say no more here; such lax or pseudo notions as we need are introduced in the individual papers below.

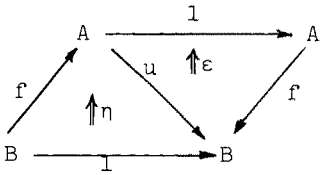
§2. ADJUNCTION IN A 2-CATEGORY

2.1. Other accounts of the matter below, of varying degrees of completeness, can be found in [19], [14], and [21]. The utility of pasting for the neat expression of the adjunction equations (2.1) and (2.2) below we learnt from R.F.C. Walters.

An adjunction $\eta, \varepsilon\colon f \dashv u\colon A \to B$ in a 2-category K consists of arrows $u\colon A \to B$ and $f\colon B \to A$ together with 2-cells $\eta\colon 1 \Rightarrow uf$ and

ε: fu $\Rightarrow$ 1 satisfying the axioms

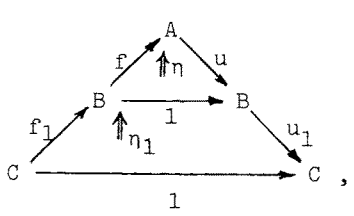
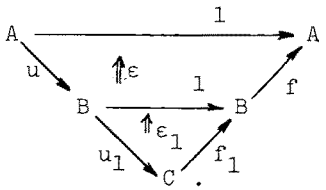
(2.1)  equals identity,

(2.2)  equals identity.

We say that f is <u>left adjoint</u> to u, and that u is <u>right adjoint</u> to f; we call η the <u>unit</u>, and ε the <u>counit</u>, of the adjunction.

When K is V-CAT for a symmetric monoidal closed V, it has been shown by Kelly in [14] that adjunctions $f \dashv u$ are in bijection with V-natural isomorphisms $A(fb,a) \cong B(b,ua)$; in particular we get the usual notion of adjunction when $K = CAT$.

If η_1,ε_1: $f_1 \dashv u_1$: B $\to$ C is a second adjunction, we clearly get a composite adjunction η_2,ε_2: $ff_1 \dashv u_1u$: A $\to$ C if we define η_2,ε_2 as the composites

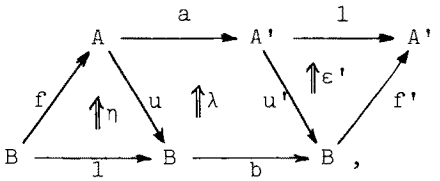
(2.3)  ,  .

Thus adjunctions in K form a category, with 1,1: $1 \dashv 1$: A $\to$ A as identities.

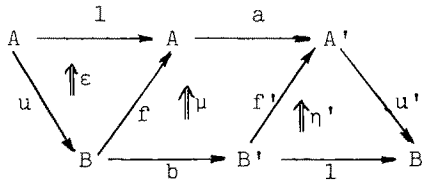
<u>2.2</u>. If we look upon (2.1) and (2.2) as asserting that the 2-cells η and ε are mutually inverse under the indicated pasting operations, the following proposition becomes evident:

Proposition 2.1. Let $\eta,\varepsilon\colon f \dashv u\colon A \to B$ and $\eta',\varepsilon'\colon f' \dashv u'\colon A' \to B'$. Let $a\colon A \to A'$ and $b\colon B \to B'$. Then there is a bijection between 2-cells $\lambda\colon bu \Rightarrow u'a$ and 2-cells $\mu\colon f'b \Rightarrow af$, where

(2.4) μ is the composite



(2.5) λ is the composite



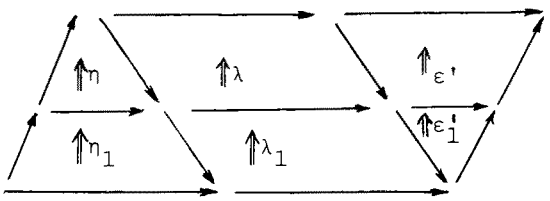
□

The naturality of this bijection between λ and μ may be expressed as follows. Consider two double categories. In each the objects are those of K, the horizontal arrows are the arrows of K, and the vertical arrows are the adjunctions in K. Composition of horizontal arrows is just composition in K, while composition of vertical arrows is the composition of adjunctions of §2.1 above. (In our diagrams "vertical" is conveniently shown as "oblique".) In the first double category, a square with sides $a\colon A \to A'$, $b\colon B \to B'$, $\eta,\varepsilon\colon f \dashv u\colon A \to B$, and $\eta',\varepsilon'\colon f' \dashv u'\colon A' \to B'$, is a 2-cell λ in K such as appears in (2.4). In the second double category, a square with the same sides is a 2-cell μ in K such as appears in (2.5). In both double categories, horizontal and vertical composition of squares are given by pasting, horizontally or vertically, the corresponding 2-cells of K. The "naturality" in question is now expressed by:

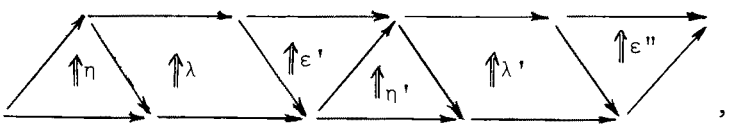
Proposition 2.2. The above bijection between λ and μ is an isomorphism between the two double categories we have just described. That is to say, the bijection respects composition and identities, vertical and horizontal.

Proof. As regards vertical identities: if $f \dashv u$ and $f' \dashv u'$ are respectively $1,1\colon 1 \dashv 1\colon A \to A$ and $1,1\colon 1 \dashv 1\colon A' \to A'$, we have by (2.4) that $\mu = \lambda$ and in particular that, when $a = b$, $\mu = 1$ if and only if $\lambda = 1$. For horizontal identities, we have a and b both identities, and $f \dashv u$ coinciding with $f' \dashv u'$; then if $\lambda = 1$ we have $\mu = 1$ by (2.4) and (2.2), and similarly if $\mu = 1$ then $\lambda = 1$.

For vertical composition we have only to write



and to look at (2.3) and (2.4). For horizontal composition we paste together two diagrams like (2.4) to get



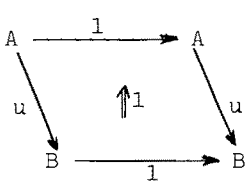
,

and observe that the central triangles ε' and η' cancel out by (2.1).□

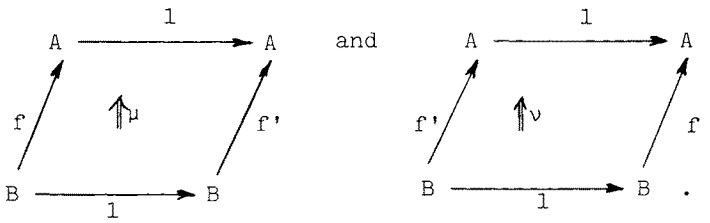
It will usually, in context, be unambiguous if we call λ and μ mates under the adjunctions $f \dashv u$ and $f' \dashv u'$, without explicit mention of a and b.

Proposition 2.3. If $f \dashv u$ and $f' \dashv u$ then f and f' are canonically isomorphic.

Proof. Let the mates of



under the adjunctions $f \dashv u$ and $f' \dashv u$, and under the adjunctions $f' \dashv u$ and $f \dashv u$, be respectively



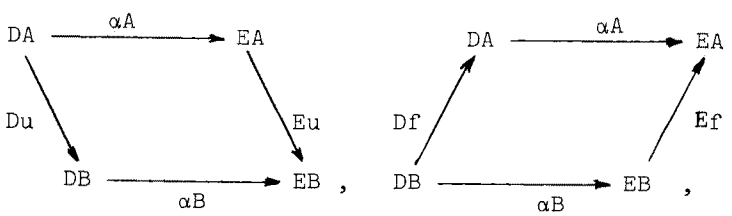
Then by the "horizontal" part of Proposition 2.2, μ and ν are mutually inverse. □

2.3. If $D: K \to L$ is a 2-functor, an adjunction $\eta,\varepsilon: f \dashv u: A \to B$ in K clearly gives an adjunction $D\eta, D\varepsilon: Df \dashv Du: DA \to DB$ in L. By applying D to (2.4) and (2.5) we get:

Proposition 2.4. *If λ and μ are mates under the adjunctions $f \dashv u$ and $f' \dashv u'$ in K, and if $D: K \to L$ is a 2-functor, then $D\lambda$ and $D\mu$ are mates under the adjunctions $Df \dashv Du$ and $Df' \dashv Du'$ in L.* □

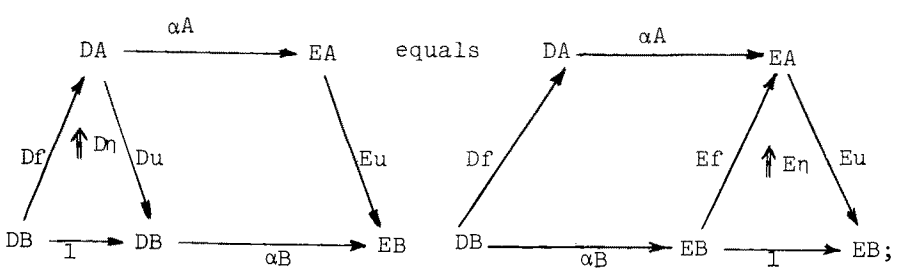
Of course it is not in general true that μ is the identity 2-cell when λ is; but we have:

Proposition 2.5. *Let $D,E: K \to L$ be 2-functors and let $\alpha: D \Rightarrow E$ be a 2-natural transformation. Then the identity 2-cells*



are mates under the adjunctions $Df \dashv Du$ and $Ef \dashv Eu$ in L.

Proof. The 2-naturality of α expressed in the form (1.1) above gives



pasting $E\varepsilon$ on the right of each and using (2.2) gives what we want in the form (2.4). □

§3 MONADS IN A 2-CATEGORY

3.1 The notion of monad (= triple) makes sense in any 2-category K; the classical case is that where $K = CAT$. A monad in K, on the object B of K, is an endomorphism $t: B \to B$ together with 2-cells $\eta: 1 \Rightarrow t$, $\mu: t^2 \Rightarrow t$ called respectively the unit and the multiplication of the monad t; these are to satisfy the usual equations

$$\mu \cdot t\eta = 1, \quad \mu \cdot \eta t = 1, \quad \mu \cdot t\mu = \mu \cdot \mu t. \tag{3.1}$$

A detailed treatment of monads in this generality has been given by Street in [23]; here we recall some of the simpler aspects and then pass to the special case where $K = 2\text{-}CAT$.

An action of the monad t above on an arrow $s: A \to B$ is a 2-cell $\nu: ts \Rightarrow s$ satisfying

$$\nu \cdot \eta s = 1, \quad \nu \cdot t\nu = \nu \cdot \mu s. \tag{3.2}$$

An arrow s together with such an action ν is called a t-algebra (with domain A). A morphism of t-algebras (with common domain A) is a $\sigma: s \Rightarrow s'$ such that

$$\nu' \cdot t\sigma = \sigma \cdot \nu. \tag{3.3}$$

Thus the t-algebras with domain A form a category $A\ell g(A,t)$, with a forgetful functor U_A to $K(A,B)$ sending (s,ν) to s and σ to itself.

For any $r: A \to B$ it is clear from (3.1) that tr, with action $\mu r: t^2 r \Rightarrow tr$, is a t-algebra; and that for any $\rho: r \Rightarrow r'$, $t\rho: tr \Rightarrow tr'$ is a morphism of t-algebras. This gives a functor $F_A: K(A,B) \to A\ell g(A,t)$. It is further clear from (3.2) that, for a t-algebra s, the action $\nu: ts \Rightarrow s$ is also a morphism of t-algebras, when ts is taken with the action μs. From these remarks we easily verify the following proposition, in the light of which we call the t-algebras tr free t-algebras:

Proposition 3.1 F_A is left adjoint to U_A. In detail, let $s: A \to B$ be a t-algebra and $r: A \to B$ any arrow. Then there is a bijection between 2-cells $\alpha: r \Rightarrow s$ and t-algebra morphisms $\sigma: tr \Rightarrow s$, given by

$$\sigma = \nu \cdot t\alpha, \qquad \alpha = \sigma \cdot \eta r. \tag{3.4}$$

In the classical case $K = CAT$ the phrase "t-algebra" is more commonly restricted to those with domain the unit category $\underline{1}$. The t-algebra $s: \underline{1} \to B$ is then identified with the corresponding object s of B, and $\nu: ts \Rightarrow s$, $\sigma: s \Rightarrow s'$ are morphisms in B. The category $A\ell g(\underline{1},t)$ of t-algebras in this primary sense is classically denoted by B^t. Identifying $K(\underline{1},B)$ with B, we write the adjoint functors $U_{\underline{1}}$ and $F_{\underline{1}}$ in this classical case as $u^t: B^t \to B$ and $f^t: B \to B^t$. Of course in the general case the monad t on B induces a classical monad $K(A,t)$ on the category $K(A,B)$, and $A\ell g(A,t)$ is just the category $K(A,B)^{K(A,t)}$ of $K(A,t)$-algebras.

3.2 If t' is another monad on the same B, a map of monads is a 2-cell $\tau: t \Rightarrow t'$ such that

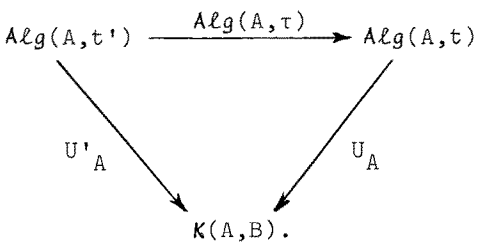
$$\mu' \cdot \tau^2 = \tau \cdot \mu, \quad \eta' = \tau \cdot \eta; \tag{3.5}$$

here τ^2 denotes $\tau\tau: tt \Rightarrow t't'$. If (s,ν') is a t'-algebra with domain A, then clearly (s,ν) is a t-algebra where

$$\nu = \nu' \cdot \tau s; \tag{3.6}$$

and a morphism $\sigma: s \Rightarrow s'$ of t'-algebras is also a morphism of t-algebras. This gives a functor $A\ell g(A,\tau)$ rendering commutative

(3.7)



In particular the t'-algebra t' itself is a t-algebra under the action

$$(3.8) \qquad \theta\colon tt' \underset{\tau t'}{\Rightarrow} t't' \overset{}{\underset{\mu'}{\Rightarrow}} t';$$

and τ can be recovered from θ in the form

$$(3.9) \qquad \tau\colon t \underset{t\eta'}{\Longrightarrow} tt' \overset{}{\underset{\theta}{\Rightarrow}} t',$$

since $\mu' \cdot t'\eta' = 1$. If an arbitrary action $\theta\colon tt' \Rightarrow t'$ of t on t' is given, the necessary and sufficient condition for it to arise as in (3.8) from a map (3.9) of monads is that $\mu'\colon t't' \Rightarrow t'$ be a t-algebra morphism, when t't' is taken with the t-action $\theta t'$; that is, that

$$(3.10) \qquad \theta \cdot t\mu' = \mu' \cdot \theta t'.$$

If $\eta', \varepsilon'\colon f' \dashv u'\colon C \to B$ is any adjunction in K, it is immediate that (t', η', μ') is a monad on B where $t' = u'f'$ and $\mu' = f'\varepsilon'u'$. We call this the monad generated by the adjunction $f' \dashv u'$. Observe that u' is a t'-algebra with action

$$(3.11) \qquad u'\varepsilon'\colon t'u' = u'f'u' \Rightarrow u'.$$

If t' is this monad and if t is any monad on B, there is by Proposition 2.1 a bijection between 2-cells $\tau\colon t \Rightarrow t' = u'f'$ and 2-cells $\nu\colon tu' \Rightarrow u'$, given by

$$(3.12) \qquad \nu = u'\varepsilon' \cdot \tau u', \quad \tau = \nu f' \cdot t\eta'.$$

Proposition 3.2. *τ is a map of monads if and only if ν is an action of t on u'.*

Proof. If τ is a map of monads, $u'\varepsilon'$ is an action of t' on u' by (3.11) and thus ν is an action of t on u' by (3.6). If ν is an action of t on u', then $\theta = \nu f'$ is an action of t on $u'f' = t'$, which trivially satisfies (3.10); whence by (3.9) τ is a map of monads. □

3.3 We say that K admits the construction of algebras if, for every monad t in K, the notion of t-algebra can be "internalized" in

the sense that the 2-functor $A \mapsto A\ell g(A,t)$ from K^{op} to CAT is representable, so that

$$A\ell g(A,t) \cong K(A,B^t) \tag{3.13}$$

(2-naturally in A) for some B^t in K, called the object of t-algebras. When this is so, the adjunction of Proposition 3.1 becomes an adjunction $K(A,B^t) \to K(A,B)$ which, because of its 2-naturality in A, arises from an adjunction $\eta^t, \varepsilon^t : f^t \dashv u^t : B^t \to B$. As the notation suggests, CAT does admit the construction of algebras, and in that case the B^t, f^t, u^t are those at the end of §3.1.

When $K = V\text{-}CAT$ for a symmetric monoidal closed category V, K admits the construction of algebras provided that V has equalizers (at least of pairs with a common left inverse); this case is treated in detail by Dubuc [5]. Here again the primary meaning of t-algebra is one with domain $\underline{1}$, which now denotes not the unit category but the unit V-category; so in this sense a t-algebra is an object s of the V-category B with an action of t on it. The category $A\ell g(\underline{1},t)$ of these t-algebras admits a canonical enrichment to a V-category, which is B^t. In particular, taking $V = CAT$, this applies to the case $K = 2\text{-}CAT$.

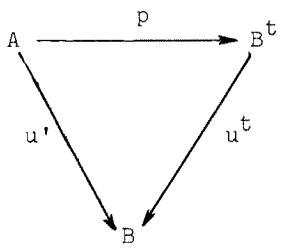
The best general result we know of — it is easy to prove and a more general result is contained in Gray [12] — is that K admits the construction of algebras if it is finitely complete as a 2-category. In accordance with the general definition of completeness for V-categories in [4], this means that K has all finite limits, that these are preserved by the representable functors $K(A,-) : K \to CAT$, and that K admits cotensor products $[X,B]$ for each finite category X and each B in K. It turns out to be sufficient to demand the existence of the cotensor product $[\underline{2},B]$ where $\underline{2}$ is the arrow category $0 \to 1$; the existence of the other $[X,B]$ then follows. If we replace "all finite limits" above by "all pullbacks" we get the representable

2-categories of Gray [11] or of Street's paper [25] in this volume; so a finitely complete 2-category is a representable one with a terminal object preserved by the $K(A,-)$.

3.4 We henceforth suppose that K admits the construction of algebras. We can express (3.13) as follows: there is a t-algebra $u^t: B^t \to B$; and the functor sending an arrow $p: A \to B^t$ to the t-algebra $u^t p$, and sending the 2-cell $\pi: p \Rightarrow p'$ to the algebra morphism $u^t\pi$, is an isomorphism $K(A,B^t) \to Alg(A,t)$. It is easy to check that the t-algebra $t: B \to B$ arises thus from the arrow $f^t: B \to B^t$, and that the monad $u^t f^t$ generated by $f^t \dashv u^t$ is t itself.

Proposition 3.3. *Let* $f' \dashv u': A \to B$. *Then the arrows* $p: A \to B^t$ *rendering commutative*

(3.14)



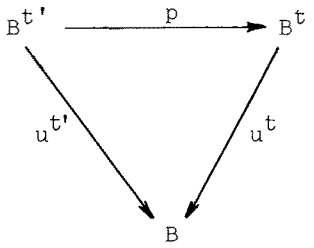
are in bijection with t-*actions on* u', *and hence by Proposition 3.2 with monad-maps* $\tau: t \Rightarrow u'f'$. □

If the monad u'f' is t itself and $\tau = 1$, the p in (3.14) is called the *canonical comparison arrow*; the adjunction $f' \dashv u'$ is said to be *monadic* (*weakly monadic*) if p is an isomorphism (an equivalence). In fact it is easily seen that monadicity is a property of u' itself, and does not depend on the choice of the left adjoint f' to u'.

If in Proposition 3.3 we let t' be a second monad on B and take $f' \dashv u'$ to be the adjunction $f^{t'} \dashv u^{t'}: B^{t'} \to B$, we get:

Proposition 3.4. *There is a bijection between arrows* $p: B^{t'} \to B^t$ *rendering commutative*

(3.15)



and monad maps $\tau: t \Rightarrow t'$. □

We write B^τ for the p in (3.15) corresponding to τ, and call it an algebraic map between the algebra objects $B^{t'}$ and B^t. It is clear that (3.15) internalizes (3.7).

We can regard Proposition 3.4 as asserting that $t \mapsto u^t$, $\tau \mapsto B^\tau$ embeds the dual of the category of monads on B fully in the category K_0/B of objects over B; the canonical comparison arrow provides a reflexion into this subcategory of those u in K_0/B which have left adjoints. This is one form of the "semantics-structure" duality. For further details on the general theory of monads, and in particular on distributive laws, we refer again to Street [23].

3.5 Lawvere in [18] used the name "equational doctrine" for a monad in 2-*CAT* on *CAT*. Here we use the name doctrine (or 2-monad) for any monad in 2-*CAT*.

For us, then, a doctrine on a 2-category K is a 2-functor $D: K \to K$ with, for its unit and its multiplication, 2-natural transformations $j: 1 \to D$ and $m: D^2 \to D$, satisfying on the nose the equations (3.1) (with D,j,m for t,η,μ).

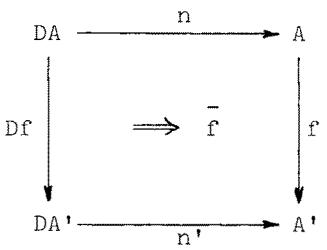
Various relaxations are possible. Some things we should like to be doctrines (cf. [15] below) have the equalities in (3.1) replaced by isomorphisms. What Zöberlein [27] calls "doctrines" not only have isomorphisms for equalities in (3.1), but have j and m only lax natural transformations. The lax monads of Bunge [3] are weaker still: D only a lax 2-functor and mere morphisms in (3.1). To avoid complications we stick to the strict doctrines, and hope that there is a nice

coherence theorem that will allow the results in our papers below to be applied at least to the "pseudo" case.

We also take D-algebra here in the strict sense: an object A of K (or more generally a 2-functor A with codomain K) with an action $n: DA \to A$ satisfying (3.2) (with A,n for s,ν). However Street considers lax algebras in [25] below, and defines them there; and Kelly considers them in relation to strict algebras in [16] and [17] below. When $K = CAT$, we also use "D-category" for "D-algebra".

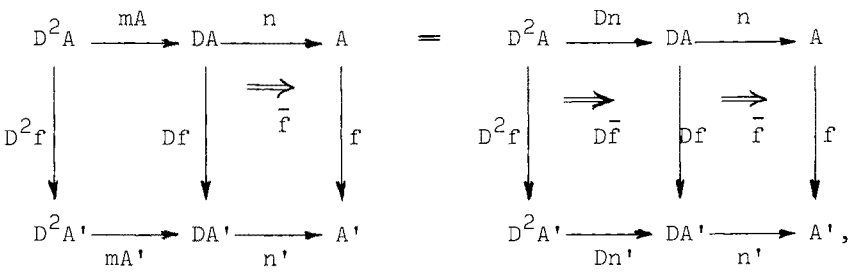
For morphisms of D-algebras, on the other hand, the lax ones are the usual ones in nature, as we said in §1.5. We therefore depart from the nomenclature of §3.1, and define, for D-algebras A,A', a D-morphism $F: A \to A'$ to be a pair $(f,\bar{f})$ where $f: A \to A'$ is an arrow in K and $\bar{f}$ is a 2-cell

(3.16)

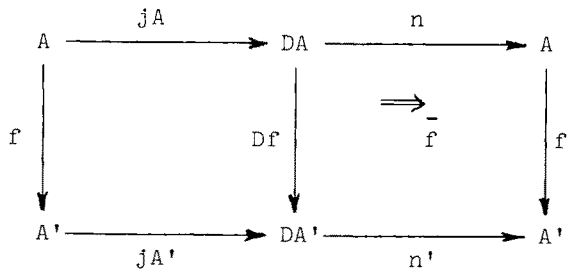


satisfying the axioms

(3.17)



(3.18)



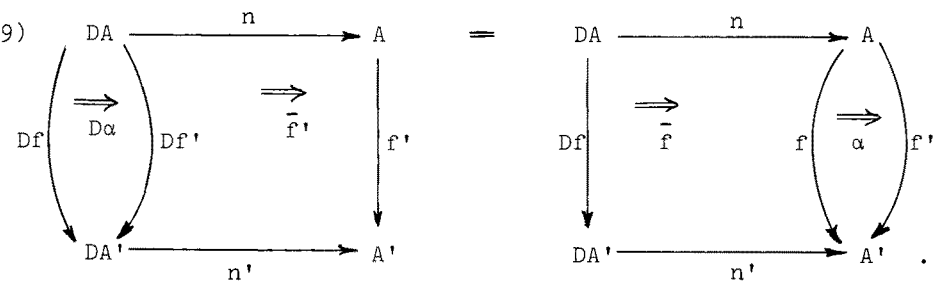
In the case $K = CAT$ we also say "D-functor" for "D-morphism". We call the D-morphism F <u>strong</u> if $\bar{f}$ is an isomorphism; we call it <u>strict</u> when $\bar{f} = 1$, so that $fn = n'.Df$. The strict D-morphisms, then, are the "morphisms of D-algebras" of §3.1; we also write f for the strict D-morphism $F = (f,1)$.

If we reverse the sense of $\bar{f}$ in (3.16), and also in the axioms (3.17) and (3.18), we get what we call an <u>op</u>-D-<u>morphism</u>. Clearly if $F = (f,\bar{f})$ is a strong D-morphism, then $(f,\bar{f}^{-1})$ is a strong op-D-morphism.

D-algebras and D-morphisms form a category under the operation of vertical pasting of diagrams like (3.16). We get subcategories with the same objects by restricting to strong or to strict D-morphisms. We now make these into 2-categories.

For D-morphisms $F,G: A \to A'$ we define a D-2-cell $\alpha: F \Rightarrow G$ to be a 2-cell $\alpha: f \Rightarrow g$ satisfying

(3.19)



In the case $K = CAT$ we also say "D-natural transformation" for "D-2-cell". With the obvious laws of composition, D-algebras, D-morphisms, and D-2-cells form a 2-category D-Alg, which we also write as D-CAT in the case $K = CAT$ or as D-Cat when $K = Cat$.

We denote by D-Alg_* the sub-2-category in which only the strict D-morphisms are considered; it is this that is the object of algebras (here 2-category of algebras) K^D in the sense of §3.3. We say of K^D that it is doctrinal, or 2-monadic, over K (to emphasise that we mean monadic in 2-CAT, and not just in CAT).

It is not our intention here to go beyond definitions: some more substantive relations between D-morphisms and strict D-morphisms are examined below by Kelly in [16].

Examples of D-CAT are: monoidal categories, monoidal functors, and monoidal natural transformations; strict monoidal categories with arbitrary monoidal functors and monoidal natural transformations; symmetric monoidal categories, symmetric monoidal functors, and monoidal natural transformations; categories bearing a monad, with the monad functors and monad functor transformations of Street [23]; categories with assigned finite coproducts, arbitrary functors, and arbitrary natural transformations; categories bearing two monoidal structures $\otimes$ and $\oplus$ and having a distributivity of $\otimes$ over $\oplus$, with appropriate morphisms and 2-cells. With $K = CAT^2$ the objects of D-Alg may be pairs of monoidal categories with a monoidal functor between them; with $K = CAT^{|A|}$ for a category A, the objects of D-Alg may be lax functors $A \to CAT$, and the morphisms lax natural transformations (cf. Street [24]).

On the other hand the category of symmetric monoidal closed categories, and of morphisms preserving all the structure on the nose - the internal-hom as well as $\otimes$ and I - is not doctrinal over CAT. Indeed it is only a category, not a 2-category: there seems to be no natural definition of 2-cell. It is monadic over CAT, but the

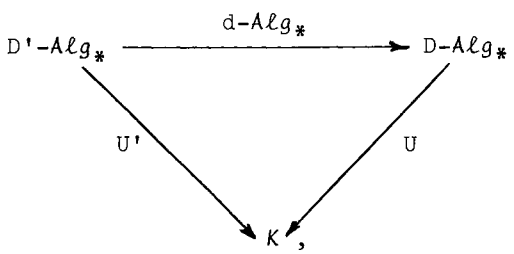
monad on CAT is only a functor, not a 2-functor.

3.6 Because in the doctrine case we have D-$A\ell g$ as well as D-$A\ell g_*$ = K^D, some new questions arise about the matter in §3.2 and §3.4 above.

We are using D-$A\ell g$ to denote the D-algebras in the primary sense: actual objects of K acted on by D, and not 2-functors $A \to K$ acted on by D. Write U: D-$A\ell g \to K$ for the forgetful 2-functor. Then it is not true in the sense of §3.5 that D acts on U: there is not a 2-natural transformation $DU \to U$, but rather an op-lax-natural-transformation $DU \rightsquigarrow U$. So a 2-functor $E: A \to$ D-$A\ell g$ corresponds to a functor $G = UE: A \to K$ together with an action of D on G only in the weak sense that the action $DG \rightsquigarrow G$ is only op-lax-natural. Those $G: A \to K$ with honest actions of D correspond of course to the 2-functors $A \to$ D-$A\ell g_*$, or to those 2-functors $A \to$ D-$A\ell g$ that factorize through D-$A\ell g_*$. Perhaps, therefore, the right thing to do is to allow both the n and the f in (3.16), when A is not an object of K but a 2-functor $A \to K$, to be op-lax; certainly the definitions work equally well if we do so, and reduce to the given ones when $A = \underline{1}$; but we want to stay as far from laxity as we can, and we don't think we shall need this extra generality in our papers below. So we shall pursue it no further here.

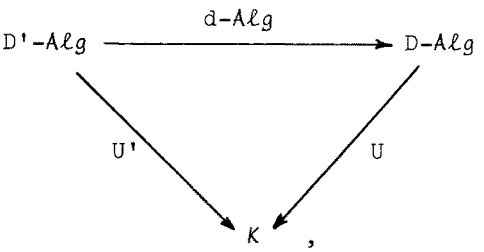
The same kind of observation, of course, applies to Proposition 3.4. A map $d: D \to D'$ of doctrines, in the sense of §3.2, not only gives a 2-functor d-$A\ell g_*$ = K^d satisfying

(3.20)



but also an evident 2-functor d-$A\ell g$ satisfying

(3.21)



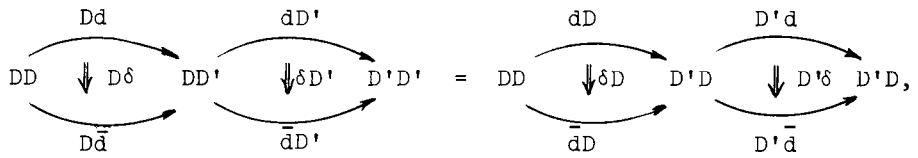
where of course we are using U in an extended as well as a restricted sense. While, however, the only P: D'-$A\ell g_*$ → D-$A\ell g_*$ with UP = U' are the d-$A\ell g_*$ for a 2-natural d: D → D' that is a doctrine map, there are many more P: D'-$A\ell g$ → D-$A\ell g$ with UP = U'; they correspond to op-lax-natural d: D ⇝ D' satisfying the appropriate axioms. Again we hope to avoid these.

While we are at this level, we recall that the algebraic 2-functor d-$A\ell g_*$ in (3.20) has a left adjoint in suitable cases. If K is 2-complete, so is D'-$A\ell g_*$ = $K^{D'}$, and d-$A\ell g_*$ preserves not only 2-limits but also cotensor products. It follows from Proposition 4.1 of Kelly [14] that the 2-functor d-$A\ell g_*$ has a 2-left-adjoint if the underlying functor has a left adjoint. By Proposition 1.5.1 of Manes [20], this is so if the category D'-$A\ell g_*$ has coequalizers of reflexive pairs. By Satz 10.3 of Gabriel-Ulmer [9], this is so when the underlying category K_0 of K is locally presentable (e.g. $K = Cat$, $K = Cat^2$, $K = Cat/\Lambda$, etc.) and when D' has a <u>rank</u> in their sense - moreover they show that for a locally presentable K_0 the various reasonable definitions of having a rank coincide. We have just not thought out the question of whether d-$A\ell g$ has a left adjoint in this situation; for example, j-$A\ell g$: D-$A\ell g$ → 1-$A\ell g$ is just U: D-$A\ell g$ → K; does it have a left adjoint?

We have in the present case the notion of a <u>modification of doctrine maps</u> δ: d ⇒ $\bar{d}$: D → D'; namely a modification δ: d ⇒ $\bar{d}$ satisfying

(3.22) $j' = \delta j$, $m'.\delta\delta = \delta.m$.

Here $\delta\delta$ denotes neither the vertical composite $\delta\cdot\delta$ nor the horizontal composite $\delta*\delta$ of modifications, but the common value of

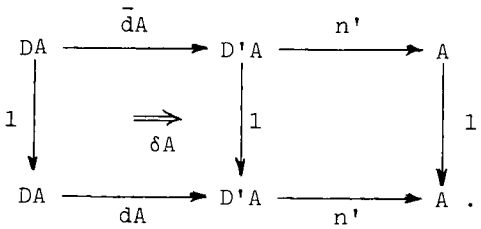
(3.23)


which coincide by (1.1) and (1.2); notation in a 3-category presents frightful problems.

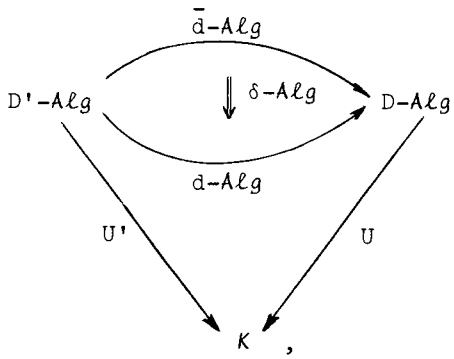
If δ is such a modification of doctrine maps, and if A is a D'-algebra with action n': $D'A \to A$, then A has two D-algebra structures

$$n\colon DA \xrightarrow[dA]{} D'A \xrightarrow[n']{} A, \qquad \bar{n}\colon DA \xrightarrow[dA]{} D'A \xrightarrow[n']{} A.$$

It is easily verified that $(1, n'.\delta A)$ is a D-morphism $(A,\bar{n}) \to (A,n)$:

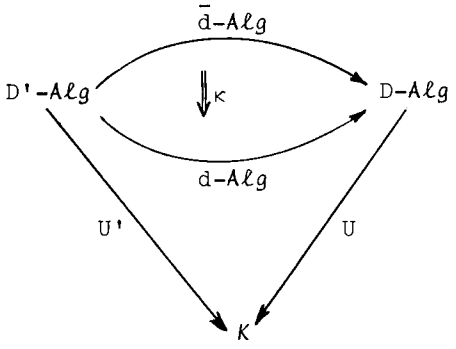
(3.24)


It follows that δ induces a 2-natural transformation δ-$A\ell g$ rendering commutative

(3.25)


in the sense that U.δ-$A\ell g$ = identity. We leave the reader to prove:

Proposition 3.5. Any 2-natural transformation κ rendering commutative



is δ-$A\ell g$ for a unique modification δ: d $\Rightarrow$ $\bar{d}$ of doctrine maps. □

It follows that equivalent doctrines D,D', in the sense that there are doctrine maps d: D $\to$ D', e: D' $\to$ D with d.e $\cong$ 1 and e.d $\cong$ 1 by modifications of doctrine maps, give equivalent 2-categories D-$A\ell g$ and D'-$A\ell g$; since the modifications here are isomorphisms, so that the D-morphism (3.24) is strong, even D-$A\ell g_{**}$ and D'-$A\ell g_{**}$ are equivalent, where D-$A\ell g_{**}$ is the sub-2-category of D-$A\ell g$ where only the strong D-morphisms are taken. However it is by no means the case that D-$A\ell g_{*}$ and D'-$A\ell g_{*}$ are equivalent.

As an example, let a D-algebra be a monoidal category A, with $\otimes$: A×A $\to$ A and I: $\underline{1}$ $\to$ A; let a D'-category be a category A with, for each n $\geq$ 0, functors $\overset{n}{\otimes}$: $A^n \to A$, and with coherent isomorphisms $\overset{n}{\otimes}(\overset{m_1}{\otimes}, \ldots, \overset{m_n}{\otimes}) \cong \overset{m}{\otimes}$, where $m = m_1 + \ldots + m_n$. A D'-algebra A gives a D-algebra A when we set $\otimes = \overset{2}{\otimes}$, I = $\overset{0}{\otimes}$; strict morphisms go into strict morphisms, so this comes from a doctrine map d: D $\to$ D'. A D-algebra A gives a D'-algebra A on setting $\overset{0}{\otimes} = I$, $\overset{1}{\otimes} = 1$, $\overset{n}{\otimes} = \otimes(1, \overset{n-1}{\otimes})$; again strict morphisms go to strict morphisms, so this comes from a doctrine map e: D' $\to$ D. The composite D-$A\ell g$ $\to$ D'-$A\ell g$ $\to$ D-$A\ell g$ is the identity, whence e.d = 1; the composite D'-$A\ell g$ $\to$ D-$A\ell g$ $\to$ D'-$A\ell g$ is clearly

isomorphic to 1, whence $d.e \cong 1$, and D and D' are equivalent. However the composite $D'\text{-}A\ell g_* \to D\text{-}A\ell g_* \to D'\text{-}A\ell g_*$ is not isomorphic to 1; the original and final D'-algebra structures on A are such that $1: A \to A$ is a strong, but not a strict, isomorphism between them.

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