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TWO-DIMENSIONAL MONAD THEORY

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We consider a 2-monad T with rank on a complete and cocomplete 2-category, and write T-Alg for the 2-category given the T-algebras, the morphisms preserving the structure to within coherent isomorphisms, and the appropriate 2-cells; T-Alg_s is the sub-2-category obtained by taking the *strict* morphisms. We show that T-Alg admits pseudo-limits and certain other limits, and that the inclusion 2-functor T-$\mathrm{Alg}_s \to T$-Alg has a left adjoint. We introduce the notion of *flexible* algebra, and use it to prove that T-Alg admits all bicolimits and that the 2-functor T-Alg $\to S$-Alg induced by a monad-map $S \to T$ admits a left biadjoint.

1. Introduction

1.1. This is the first of a series of articles reporting the work of Kelly and various of his colleagues on what we may call *two-dimensional universal algebra*: the study of structures borne not by a set but by a category, or by a family of categories, and so on; in the context, however, where the morphisms of primary interest are not the *strict* ones, which preserve the relevant structure on the nose, but those which preserve it only *to within coherent isomorphisms*.

This first article is concerned with the two-dimensional aspect of *monad theory*. Its results will be used in later articles on two-dimensional structures defined by finite-limit-theories; and they will be augmented by further articles on *presentations* of monads and related syntactic issues.

It is now very well known that the theory of monads and their algebras extends virtually unchanged from the case of ordinary categories to that of categories enriched over a (symmetric monoidal, locally-small, complete and cocomplete) closed category $\mathscr{V}$; see [7, 29, 32, 34]. The cases of interest to us are those where $\mathscr{V}$ is the Cartesian closed category **Cat** of small categories, or **Gpd** of small groupoids. (In fact, we do not have to consider the **Gpd** case separately; a **Cat**-category is the same thing as a (locally small) 2-category, and a **Gpd**-category is just such a 2-category in which every 2-cell is invertible.)

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Thus we have at hand the results of **Cat**-enriched monad theory; but this theory, as a special case of the $\mathscr{V}$-enriched one, knows of no morphisms but the strict ones. What we are calling the *two-dimensional* theory goes beyond the **Cat**-*enriched* one precisely in studying the non-strict morphisms which, when $\mathscr{V}$ is **Cat** or **Gpd**, are usually those of practical interest. There is, for example, a 2-monad T on **Cat** whose algebras are (small) categories with (assigned) finite limits; the morphisms of interest between these algebras are those functors that *preserve finite limits*, in the usual sense of this phrase – and not the strict morphisms, that take the assigned limits on the nose to the *assigned* limits of the codomain.

It is characteristic of our treatment (throughout the series) that we consider the strict along with the non-strict. At one level this enables us, by comparing strict and non-strict morphisms via adjoint 2-functors, to deduce two-dimensional results easily from the simple **Cat**-enriched results; here the strict morphisms, however irrelevant they may be in practice, play an essential ancillary role in the proofs of the theorems. At another level, it allows us (in later articles) to prove *coherence results*, to the effect that something non-strict may be replaced by an equivalent something that is strict.

We mention here just a few examples of 2-monads, to give the reader some notion of our scope, without making the introduction too long; we discuss these and other examples more fully in the final section. As we said above, categories with finite limits are the algebras for a 2-monad T on **Cat**. So are monoidal categories; these examples differ in that the structure is unique to within isomorphism (if it exists) in the first case, but not in the second. Symmetric monoidal *closed* categories are the algebras for a 2-monad T on the 2-category $\mathbf{Cat}_g$ of small categories, functors, and natural *isomorphisms*; it is not possible to extend this T to a 2-monad on the 2-category **Cat** – the enrichment here is really over **Gpd**. Similarly, elementary toposes are the algebras for a 2-monad on $\mathbf{Cat}_g$, the algebra-morphisms being the logical maps. The structure given by two symmetric monoidal closed categories and a symmetric monoidal functor between them is an algebra for a 2-monad on $\mathbf{Cat}_g^2$. If $\mathbf{Cat}_f$ denotes the full sub-2-category of **Cat** given by the finitely-presentable categories, there is a 2-monad on the functor-2-category $[\mathbf{Cat}_f, \mathbf{Cat}]$ whose algebras are the finitary (that is, filtered-colimit-preserving) 2-monads on **Cat**. For a small 2-category $\mathscr{P}$, the objects of the functor-2-category $[\mathscr{P}, \mathbf{Cat}]$ are the algebras for a 2-monad on $\mathbf{Cat}^X$, where X is the set of objects of $\mathscr{P}$.

1.2. For such general 2-categorical notions as are not explained below, see [28]. We consider a 2-monad T on a 2-category $\mathscr{K}$. Here T is to be a strict 2-monad; that is, a $\mathscr{V}$-monad where $\mathscr{V} = \mathbf{Cat}$. So the unit $i: 1 \to T$ and the multiplication $m: T^2 \to T$ are 2-natural transformations which satisfy on the nose the usual axioms of associativity and two-sided identity. We re-emphasize that there is no need to consider separately the case $\mathscr{V} = \mathbf{Gpd}$; a **Gpd**-monad on a **Gpd**-category $\mathscr{K}$ is just a 2-monad T, on $\mathscr{K}$ seen as a 2-category.

Throughout this article we take the notion of *T-algebra*, too, in the strict sense:

a T-algebra (A, a), or A for short, is an object A of $\mathscr{K}$ together with an *action* $a: TA \to A$; that is, an arrow in $\mathscr{K}$ satisfying on the nose the usual associativity and unit axioms. This is not to deny the importance of the *pseudo-T-algebras*, where the action satisfies the axioms only to within coherent isomorphisms; we shall study these in a later article, *using* the results of the present article to do so, and showing that (for reasonable $\mathscr{K}$ and T) a pseudo-T-algebra is just a (strict) T'-algebra for another 2-monad T'.

Where we depart from strictness is in the notion of morphism of algebras. If (A, a) and (B, b) are T-algebras, a *lax morphism* $f: A \to B$ of T-algebras is a pair $(f, \bar{f})$ where $f: A \to B$ is an arrow in $\mathscr{K}$ and $\bar{f}$ is a 2-cell

$$\begin{array}{ccc} TA & \xrightarrow{Tf} & TB \\ {\scriptstyle a}\downarrow & \Downarrow \bar{f} & \downarrow{\scriptstyle b} \\ A & \xrightarrow[f]{} & B \end{array} \tag{1.1}$$

satisfying the 'coherence axioms'

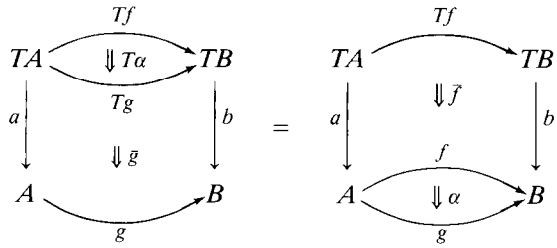
$$\begin{array}{ccc} T^2A & \xrightarrow{T^2f} & T^2B \\ {\scriptstyle mA}\downarrow & & \downarrow{\scriptstyle mB} \\ TA & \xrightarrow{Tf} & TB \\ {\scriptstyle a}\downarrow & \Downarrow \bar{f} & \downarrow{\scriptstyle b} \\ A & \xrightarrow[f]{} & B \end{array} \quad = \quad \begin{array}{ccc} T^2A & \xrightarrow{T^2f} & T^2B \\ {\scriptstyle Ta}\downarrow & \Downarrow T\bar{f} & \downarrow{\scriptstyle Tb} \\ TA & \xrightarrow{Tf} & TB \\ {\scriptstyle a}\downarrow & \Downarrow \bar{f} & \downarrow{\scriptstyle b} \\ A & \xrightarrow[f]{} & B \end{array} \tag{1.2}$$

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle iA}\downarrow & & \downarrow{\scriptstyle iB} \\ TA & \xrightarrow{Tf} & TB \\ {\scriptstyle a}\downarrow & \Downarrow \bar{f} & \downarrow{\scriptstyle b} \\ A & \xrightarrow[f]{} & B \end{array} \quad = \quad \text{identity} \tag{1.3}$$

The sense of the 2-cell $\bar{f}$ is purely conventional; reversing it gives the notion of a *colax morphism* $(f, \hat{f})$ – which is just a lax morphism of algebras for the 2-monad T^{co} on $\mathscr{K}^{\mathrm{co}}$. We call $f = (f, \bar{f})$ a *morphism* of T-algebras when $\bar{f}$ is invertible; and a *strict morphism* when $\bar{f}$ is the identity – in which case we often write f for f.

These various kinds of morphism were described in [28], but with a different terminology; there lax morphisms, morphisms, and strict morphisms were respectively called morphisms, strong morphisms, and strict morphisms – which, when T-algebras are monoidal categories, agrees with the nomenclature of [10] for monoidal functors (and would agree with the nomenclature of [2] for maps of bicategories if *strong morphism* were replaced by *homomorphism*). Another systematic terminology, which takes the strict things as the norm, is much used in related contexts, some of which – morphisms of 2-monads, for example – are special cases of our present context; had we adopted it here, our lax morphisms, morphisms, and strict morphisms would have been called, respectively, lax morphisms, pseudo-morphisms, and morphisms. Our present terminology seems best suited to the two-dimensional theory.

For lax morphisms $f, g : A \to B$ of T-algebras, we define a 2-cell $\alpha : f \to g$ as a 2-cell $\alpha : f \to g$ in $\mathscr{K}$ satisfying



(1.4)

With the evident laws of composition, we have a 2-category $T\text{-Alg}_l$ of T-algebras, lax morphisms, and 2-cells; restricting to morphisms or to strict morphisms, without changing the notion of 2-cell, gives the sub-2-categories T-Alg and $T\text{-Alg}_s$; thus the non-full inclusions

$$T\text{-Alg}_s \to T\text{-Alg} \to T\text{-Alg}_l \tag{1.5}$$

are locally fully faithful – in the sense that, for instance, the functor $T\text{-Alg}_s(A, B) \to T\text{-Alg}_l(A, B)$ is fully faithful. Note that $T\text{-Alg}_l$ coincides with T-Alg when $\mathscr{K}$ is only a **Gpd**-category, and that all algebra-2-cells are then invertible. There is an evident forgetful 2-functor $U_l : T\text{-Alg}_l \to \mathscr{K}$, restricting to $U : T\text{-Alg} \to \mathscr{K}$ and $U_s : T\text{-Alg}_s \to \mathscr{K}$. It is of course $T\text{-Alg}_s$ that is the Eilenberg-Moore 2-category $\mathscr{K}^T$ in the sense of **Cat**-enriched monad theory; so that the 2-functor U_s has a left adjoint F_s with $T = U_s F_s$.

1.3. We now outline our main results. We assume familiarity with the article [23] on 2-categorical limit notions, written as a preliminary to the present series. Note that, in all the examples given in the final paragraph of Subsection 1.1 above, the 2-category $\mathscr{K}$ bearing the 2-monad T is complete and cocomplete; that $\mathbf{Cat}_g$ is so follows from [23, Proposition 3.1]. Since our most central results require the cocompleteness of $\mathscr{K}$ and some completeness, we may as well suppose from the outset

that $\mathscr{K}$ is complete and cocomplete, which greatly simplifies some proofs; we note in an occasional aside places where less would suffice.

First, it is well known from the theory of $\mathscr{V}$-enriched monads that the 2-category $T\text{-Alg}_s$ is complete when $\mathscr{K}$ is so, (indexed) limits being formed as in $\mathscr{K}$ and given the evident algebra-structure. The 2-category T-Alg, however, is rarely complete; it is true that products and cotensor products in $T\text{-Alg}_s$ are also such in T-Alg, but T-Alg lacks equalizers in general. For instance, the 2-category **Lex** of small finitely-complete categories, left-exact functors, and natural transformations is (see Subsection 6.4 below) T-Alg for a finitary 2-monad T on **Cat**. If 1 is the unit category, if I is the category with objects 0 and 1 and mutually-inverse isomorphisms $0 \to 1$ and $1 \to 0$, and if $0, 1 : 1 \to I$ are the functors naming the two objects of I, there is no left-exact f with $0f = 1f$, since no object of **Lex** is empty. We show in Section 2 that T-Alg does, nevertheless, admit a large class of (indexed) limits, including all pseudo-limits and all lax limits. We show further that these have a striking and important property, which in the case of conical pseudo-limits reduces to the fact that the generators of the 'pseudo-limit-cone' are *strict* morphisms of algebras; we make essential use of this in Section 4. Observe that, since T-Alg admits all pseudo-limits, it *a fortiori* admits all bilimits by [23, Proposition 6.1].

Recall that T is said to *have a rank* if it preserves α-filtered colimits for some regular cardinal α; in most practical examples, T preserves *all* filtered colimits, and thus has rank ω, or equivalently is *finitary*. In Section 3 we show that the full inclusion 2-functor $T\text{-Alg}_s \to T/\mathscr{K}$ into the comma-2-category has a left adjoint (in the usual strict sense of **Cat**-enriched category theory) when, as we henceforth suppose, T has a rank; this is just an extension to 2-categories of a result of Kelly [20] for ordinary categories. We deduce that $T\text{-Alg}_s$ is cocomplete, and that the 2-functor $T\text{-Alg}_s \to S\text{-Alg}_s$ induced by a strict map $S \to T$ of 2-monads admits a left adjoint. We further deduce that the inclusion 2-functors $T\text{-Alg}_s \to T\text{-Alg}$ and $T\text{-Alg}_s \to T\text{-Alg}_l$ admit left adjoints.

Since the left adjoint $(\ )' : T\text{-Alg} \to T\text{-Alg}_s$ plays an essential role in deducing properties of T-Alg from the related properties of $T\text{-Alg}_s$, we examine it more closely in Section 4. Using the results of Section 2 on pseudo-limits, we show that the unit $p : A \to A'$ and the counit $q : A' \to A$ of this adjunction constitute an equivalence in T-Alg, and we distinguish the class of *flexible* T-algebras A: those for which q is a retraction (and then necessarily an equivalence) in $T\text{-Alg}_s$ itself. This notion of flexibility will play an important role, not only in Section 5 below, but also in subsequent articles; when $T\text{-Alg}_s$ is $[\mathscr{P}, \mathbf{Cat}]$ as in the last example of Subsection 1.1, we get the concept of *flexible indexing type* in connexion with indexed limits, which we shall explore further in [5]; in the penultimate example of Subsection 1.1, we get the concept of a *flexible finitary 2-monad on* **Cat** – we shall see in a later article that the 2-monad whose algebras are monoidal categories is flexible, while that whose algebras are *strict* monoidal categories is not.

In Section 5 we show that, if G is a 2-functor with domain T-Alg whose composite with the inclusion $J : T\text{-Alg}_s \to T\text{-Alg}$ has a left adjoint H, then the value of H at

any object is a flexible algebra; from which we conclude that G has JH as a *left biadjoint* (see [23, Section 6]). We use this to deduce, from the related properties of $T\text{-Alg}_s$ given in Section 3, that T-Alg admits all bicolimits when T has a rank, and that the 2-functor $T\text{-Alg} \to S\text{-Alg}$ induced by a strict map $S \to T$ of 2-monads admits a left biadjoint when both T and S have a rank. Note that T-Alg does *not* admit pseudo-colimits in general: a pseudo-initial-object is the same thing as an initial object, and **Lex** admits no initial object, the left-exact functors $A \to \boldsymbol{I}$ constant at 0 and at 1 being different for every A.

The final Section 6 is given over to a discussion of particular 2-monads, their algebras, and the morphisms of these, and includes a justification of some of our observations above on examples.

2. On limits in T-Alg

We observed in Subsection 1.3 that T-Alg is rarely complete, lacking equalizers in general; yet it does admit a large class of limits. For these we use the nomenclature and notation of [23]. We make the blanket assumption for this section that $\mathcal{K}$ *is complete*, although for many of the results less suffices.

Proposition 2.1. T-Alg *admits products, and these are preserved by* $U: T\text{-Alg} \to \mathcal{K}$. *The product-projections are strict morphisms of algebras, and the product in* T-Alg *is also the product in* $T\text{-Alg}_s$.

Proof. Given a family (A_i, a_i) of algebras for $i \in I$, let $(p_i: A \to A_i)$ be the product of the A_i in $\mathcal{K}$, and write $a: TA \to A$ for the unique map with $p_i a = a_i \cdot Tp_i$. Then a is an action, the necessary axioms following from those for the a_i since the p_i are jointly monomorphic; and each p_i is a strict morphism of algebras. To see that $(p_i: A \to A_i)$ is the product in T-Alg, consider a family $\boldsymbol{q}_i = (q_i, \bar{q}_i): D \to A_i$ of algebra-morphisms. There is a unique $h: D \to A$ in $\mathcal{K}$ satisfying $p_i h = q_i$. We have the 2-cells

$$p_i a \cdot Th = a_i \cdot Tp_i \cdot Th = a_i \cdot Tq_i \xrightarrow[\bar{q}_i]{} q_i d = p_i hd,$$

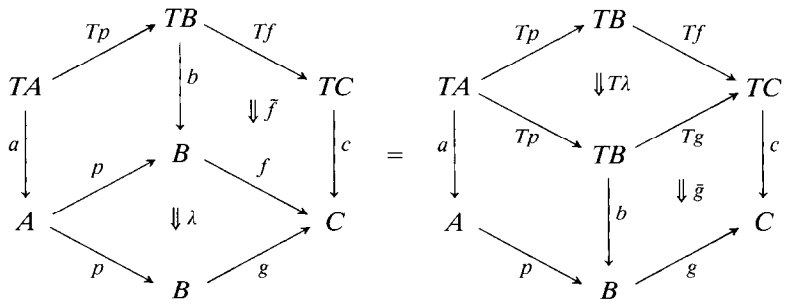
and so, by the two-dimensional aspect of the universal property of the product in $\mathcal{K}$, there is a unique 2-cell $\bar{h}: a \cdot Th \to hd$ satisfying $p_i \bar{h} = \bar{q}_i$. The axioms (1.2) and (1.3) for $(h, \bar{h})$ follow easily from those for the $(q_i, \bar{q}_i)$ and from the naturality of m and i, using the uniqueness clause in the two-dimensional universal property. Thus $(h, \bar{h})$ is the unique algebra-morphism $\boldsymbol{h}: D \to A$ satisfying $p_i \boldsymbol{h} = \boldsymbol{q}_i$. It remains to verify the two-dimensional aspect of the universal property in T-Alg. Let $\beta_i: p_i \boldsymbol{h} \to p_i \boldsymbol{k}$ be 2-cells in T-Alg. At the level of $\mathcal{K}$, there is a unique 2-cell $\alpha: h \to k$ with $p_i \alpha = \beta_i$; and the axiom (1.4) for α follows easily from that for the β_i, again using the uniqueness in the two-dimensional universal property. For the last assertion of the proposition, we have only to observe that, when each $\boldsymbol{q}_i$ above is strict, so is $\boldsymbol{h}$. □

Proposition 2.2. *Every parallel pair $f, g : B \to C$ in T-Alg admits an inserter $p : A \to B$, $\lambda : fp \to gp$, preserved by $U : T\text{-Alg} \to \mathcal{K}$. Moreover, p is a strict morphism of algebras; and any algebra-morphism $h : D \to A$ is strict if the composite ph is strict. Exactly the same is true with 'iso-inserter' in place of 'inserter'.*

Proof. We give the proof for inserters, that for iso-inserters being essentially identical. Let $p : A \to B$ and $\lambda : fp \to gp$ be the inserter of f and g in $\mathcal{K}$. We have the map $b \cdot Tp : TA \to B$ and the 2-cell

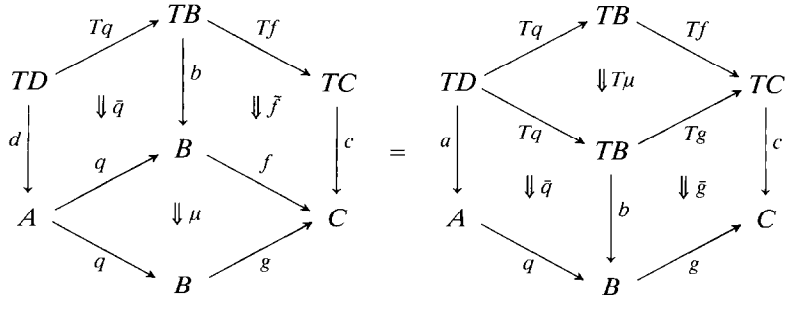
$$fb \cdot Tp \xrightarrow[\bar{f}^{-1} \cdot Tp]{} c \cdot Tf \cdot Tp \xrightarrow[c \cdot T\lambda]{} c \cdot Tg \cdot Tp \xrightarrow[\bar{g} \cdot Tp]{} gb \cdot Tp; \tag{2.1}$$

thus, by the one-dimensional universal property in $\mathcal{K}$, there is a unique $a : TA \to A$ for which $pa = b \cdot Tp$ and λa is the composite (2.1). That is to say, we have



(2.2)

From (2.2), the axioms (1.2) for f and for g, and the 2-naturality of m, we get $pa \cdot mA = pa \cdot Ta$ and $\lambda a \cdot mA = \lambda a \cdot Ta$; whence $a \cdot mA = a \cdot Ta$ by the uniqueness clause in the one-dimensional universal property. Similarly we have $a \cdot iA = 1$; so that a is an action and $A = (A, a)$ is an algebra. Clearly $p : A \to B$ is a strict algebra-morphism and – comparing (2.2) with (1.4) – λ is an algebra-2-cell $\lambda : fp \to gp$. To see that (p, λ) is the inserter of f and g in T-Alg, consider an algebra-morphism $q : D \to B$ and an algebra-2-cell $\mu : fq \to gq$; the axiom (1.4) for μ is



(2.3)

By the one-dimensional universal property in $\mathcal{K}$, there is a unique $h : D \to A$ satisfying $ph = q$ and $\lambda h = \mu$. We have the 2-cell

$$pa \cdot Th = b \cdot Tp \cdot Th = b \cdot Tq \xrightarrow[\bar{q}]{} qd = phd; \tag{2.4}$$

it satisfies (2.3) which, using $ph = q$ and $\lambda h = \mu$, may be written as

$$(\lambda hd)(f\bar{q})(\bar{f} \cdot Tp \cdot Th) = (g\bar{q})(\bar{g} \cdot Tp \cdot Th)(c \cdot T\lambda \cdot Th),$$

or equivalently, since (2.1) is λa, as $(\lambda hd)(f\bar{q}) = (g\bar{q})(\lambda a \cdot Th)$. Hence, by the two-dimensional aspect of the universal property in $\mathcal{K}$, there is a unique 2-cell $\bar{h} : a \cdot Th \to hd$ with $p\bar{h} = \bar{q}$. The axioms (1.2) and (1.3) for $(h, \bar{h})$ follow from those for $(q, \bar{q})$ using the uniqueness clause in the two-dimensional universal property. Thus $(h, \bar{h})$ is the unique algebra-morphism $\boldsymbol{h} : D \to A$ satisfying $p\boldsymbol{h} = \boldsymbol{q}$ and $\lambda \boldsymbol{h} = \mu$. Note that $\boldsymbol{h}$ is strict if $\boldsymbol{q}$ is strict, giving the penultimate assertion of the proposition. Finally we need the two-dimensional aspect of the universal property in T-Alg. Suppose then that we have algebra-morphisms $\boldsymbol{h}, \boldsymbol{k} : D \to A$ and an algebra-2-cell $\beta : p\boldsymbol{h} \to p\boldsymbol{k}$ satisfying $(\lambda \boldsymbol{k})(f\beta) = (g\beta)(\lambda \boldsymbol{h})$. By the two-dimensional universal property in $\mathcal{K}$, there is a unique 2-cell $\alpha : h \to k$ with $p\alpha = \beta$; it is in fact an algebra-2-cell, (1.4) for α following by the uniqueness clause from (1.4) for β. □

Proposition 2.3. *Every pair $\alpha, \beta : f \to g : B \to C$ of parallel 2-cells in T-Alg admits an equifier $p : A \to B$, preserved by $U : T\text{-Alg} \to \mathcal{K}$. Moreover, p is a strict morphism of algebras, and any algebra-morphism $\boldsymbol{h} : D \to A$ is strict if the composite $p\boldsymbol{h}$ is strict.*

Proof. Let $p : A \to B$ be the equifier of α and β in $\mathcal{K}$. Writing (1.4) for α as $(\alpha b)\bar{f} = \bar{g}(c \cdot T\alpha)$ and composing with the arrow Tp gives $(\alpha b \cdot Tp)(\bar{f} \cdot Tp) = (\bar{g} \cdot Tp)(c \cdot T\alpha \cdot Tp)$. Similarly $(\beta b \cdot Tp)(\bar{f} \cdot Tp) = (\bar{g} \cdot Tp)(c \cdot T\beta \cdot Tp)$. But $T\alpha \cdot Tp = T\beta \cdot Tp$ since $\alpha p = \beta p$, so that $(\alpha b \cdot Tp)(\bar{f} \cdot Tp) = (\beta b \cdot Tp)(\bar{f} \cdot Tp)$. Because $\bar{f}$ and hence $\bar{f} \cdot Tp$ is invertible, we have $\alpha b \cdot Tp = \beta b \cdot Tp$. By the one-dimensional universal property in $\mathcal{K}$, therefore, there is a unique $a : TA \to A$ with $pa = b \cdot Tp$. Moreover, a is an action, the necessary axioms following from those for b since p is monomorphic. Thus $A = (A, a)$ is an algebra, and $p : A \to B$ is a strict algebra-morphism with $\alpha p = \beta p$. To see that p is the equifier of α and β in T-Alg, consider an algebra-morphism $\boldsymbol{q} : D \to B$ with $\alpha \boldsymbol{q} = \beta \boldsymbol{q}$. By the one-dimensional universal property in $\mathcal{K}$, there is a unique $h : D \to A$ with $ph = q$. Just as in (2.4) above, we have the 2-cell $\bar{q} : pa \cdot Th \to phd$; by the two-dimensional universal property in $\mathcal{K}$, there is a unique 2-cell $\bar{h} : a \cdot Th \to hd$ with $p\bar{h} = \bar{q}$. The axioms (1.2) and (1.3) for $(h, \bar{h})$ follow from those for $(q, \bar{q})$ using the uniqueness clause in the two-dimensional universal property. Thus $(h, \bar{h})$ is the unique algebra-morphism $\boldsymbol{h} : D \to A$ with $p\boldsymbol{h} = \boldsymbol{q}$. Note that $\boldsymbol{h}$ is strict if $\boldsymbol{q}$ is strict, giving the final assertion of the proposition. Finally we need the two-dimensional aspect of the universal property in T-Alg. Suppose then that we have algebra-morphisms $\boldsymbol{h}, \boldsymbol{k} : D \to A$ and an algebra-2-cell $\mu : p\boldsymbol{h} \to p\boldsymbol{k}$. By the two-dimensional universal property in $\mathcal{K}$, there is a unique 2-cell $\lambda : h \to k$ with $p\lambda = \mu$; it is in fact an algebra-2-cell, (1.4) for λ following by the uniqueness clause from (1.4) for μ. □

Similarly we could construct *inverters* in T-Alg directly from inverters in $\mathcal{K}$. Since, however, we are supposing $\mathcal{K}$ to be complete, it is simpler to infer the

existence of further limits in T-Alg by combining the three propositions above with the results of [23].

Proposition 2.4. *Every 2-cell $\alpha : f \to g : B \to C$ in T-Alg admits an inverter $p : A \to B$, preserved by $U : T\text{-Alg} \to \mathscr{K}$. Moreover, p is a strict morphism of algebras, and any algebra-morphism $h : D \to A$ is strict if ph is strict.*

Proof. Since T-Alg admits inserters and equifiers by Propositions 2.2 and 2.3, it admits inverters by [23, Proposition 4.2]. For the final assertion we must look at the *construction* of the inverter in the proof in [23]: the inverter p appears there as uvw where (u, β) is an inserter and v and w are equifiers; and so the assertion follows from Propositions 2.2 and 2.3. □

Proposition 2.5. *T-Alg admits cotensor products $\{X, B\}$, and these are preserved by $U : T\text{-Alg} \to \mathscr{K}$. The unit $\xi : X \to T\text{-Alg}(\{X, B\}, B)$ takes its values in $T\text{-Alg}_s(\{X, B\}, B)$. Moreover, an algebra-morphism $h : D \to \{X, B\}$ is strict if the composite $T\text{-Alg}(h, 1)\xi : X \to T\text{-Alg}(D, B)$ takes its values in $T\text{-Alg}_s(D, B)$ (that is, if $\xi_x h$ is strict for each object x of X) – which is equally to say that $\{X, B\}$ is also the cotensor product in $T\text{-Alg}_s$.*

Proof. Since T-Alg admits products, inserters, and equifiers by Propositions 2.1–2.3, it admits cotensor products by [23, Proposition 4.4]. In the proof of this last, $\xi_x : \{X, B\} \to B$ is exhibited (changing some letters) as $p_x uv$, where u and v are equifiers and p_x is the x-component of a map $p : C \to B^{\mathrm{ob}\, X}$ forming part of an inserter (p, λ). The remaining assertions of the proposition now follow from Propositions 2.1–2.3. □

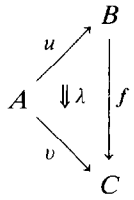
The most important result for our applications is the 'pseudo' case of the following:

Theorem 2.6. *For any $F : \mathscr{P} \to \mathbf{Cat}$ and $G : \mathscr{P} \to T$-Alg with $\mathscr{P}$ small, T-Alg admits the lax limit $\{F, G\}_l$, and this is preserved by $U : T\text{-Alg} \to \mathscr{K}$. The lax natural transformation $\zeta : F \to T\text{-Alg}(\{F, G\}_l, G-)$ forming the unit of the limit has the property that, for each object P of $\mathscr{P}$, the functor $\zeta_P : FP \to T\text{-Alg}(\{F, G\}_l, GP)$ takes its values in $T\text{-Alg}_s(\{F, G\}_l, GP)$. Moreover an algebra-morphism $h : D \to \{F, G\}_l$ is strict if, for each P, the composite $T\text{-Alg}(h, 1)\zeta_P : FP \to T\text{-Alg}(D, GP)$ takes its values in $T\text{-Alg}_s(D, GP)$. Exactly the same is true when lax limits are replaced by pseudo-limits $\{F, G\}_p$.*

Proof. We give the proof for lax limits, that for pseudo-limits being essentially identical. Given Propositions 2.1–2.3 and 2.5 above, the existence of the limit follows from [23, Proposition 5.1]. In the proof of this last, $\zeta_P : FP \to T\text{-Alg}(\{F, G\}_l, GP)$ is exhibited (changing some letters) as the image under adjunction of $p_P uv : \{F, G\}_l \to$

$\{FP, GP\}$, where $\{FP, GP\}$ is the cotensor product in T-Alg, while u and v are equifiers and p_P is the component of a map $p: C \to \prod_P \{FP, GP\}$ forming part of an inserter (p, λ). By Propositions 2.1–2.3, an algebra-morphism $h: D \to \{F, G\}_1$ is strict if and only if $p_P uvh: D \to \{FP, GP\}$ is strict for each P. By Proposition 2.5, this is so if and only if $T\text{-Alg}(p_P uvh, 1)\xi_P : FP \to T\text{-Alg}(D, GP)$ takes its values in $T\text{-Alg}_s(D, GP)$, where ξ_P is the unit for the cotensor product $\{FP, GP\}$. The remaining assertions of the theorem now follow from the facts that ζ_P, as the image under adjunction of $p_P uv$, is just $T\text{-Alg}(p_P uv, 1)\xi_P$, while $T\text{-Alg}(p_P uvh, 1)\xi_P$ is $T\text{-Alg}(h, 1)\zeta_P$. □

Remark 2.7. The content of this theorem is easier to grasp in the particular case of conical pseudo-limits. Any $G: \mathscr{P} \to T\text{-Alg}$ with $\mathscr{P}$ small has a pseudo-limit psd lim G, preserved by $U: T\text{-Alg} \to \mathscr{K}$. Its unit is a pseudo-cone ζ over G with vertex psd lim G, having components $\zeta_P : \text{psd lim } G \to GP$ for $P \in \mathscr{P}$ and components $\zeta_\varphi : G\varphi \cdot \zeta_P \cong \zeta_Q$ for $\varphi : P \to Q$ in $\mathscr{P}$. The components ζ_P are strict algebra-morphisms, and any algebra-morphism $h: D \to \text{psd lim } G$ is strict if each $\zeta_P h$ is strict. We apply this in Section 4 below when $\mathscr{P}$ is the arrow-category **2**, of the form $0 \to 1$; so that we are dealing (see [23]) with the pseudo-limit of a single algebra-morphism $f: B \to C$. This is the universal diagram in T-Alg of the form



(2.5)

with λ invertible. What the theorem asserts here, besides the existence of the limit and its preservation by U, is that the 'generators' u and v of the pseudo-cone are strict algebra-morphisms, and that any algebra-morphism $h: D \to A$ is strict if uh and vh are strict.

Remark 2.8. Although we are supposing $\mathscr{K}$ complete because this suffices for our applications, it is clear from the proofs of Propositions 2.1–2.3 that the existence in T-Alg of I-fold products needs only the existence of such products in $\mathscr{K}$, and similarly for inserters and equifiers. Had we given the corresponding direct proofs of Propositions 2.4 and 2.5 and Theorem 2.6, instead of deducing them from Propositions 2.1–2.3 and the results of [23], we should see that the same is true of inverters, of any given cotensor product $\{X, B\}$, and of any given lax- or pseudo-limit. It follows that, when $\mathscr{K}$ is *finitely complete* in the sense of [23, Section 3], T-Alg admits finite products, inserters, iso-inserters, equifiers, inverters, those cotensor products $\{X, B\}$ where the category X is finitely presentable, and those $\{F, G\}_1$ and $\{F, G\}_p$ for which ob $\mathscr{P}$ is finite and each category $\mathscr{P}(P, Q)$ and each category FP is finitely presentable – this last by an analysis of the proof of [23, Proposition 5.1].

Remark 2.9. Our interest in $T\text{-Alg}_l$ is very much secondary, although the left adjoint of Section 3 below to the inclusion $T\text{-Alg}_s \to T\text{-Alg}_l$, in the special case of the last example of Subsection 1.1 where $T\text{-Alg}_s = [\mathscr{P}, \mathbf{Cat}]$, is of importance (see [5]) in the study of lax limits and even of pseudo-limits. What we observe here is that $T\text{-Alg}_l$ is, even in comparison with the non-complete T-Alg, very poorly endowed with limits. It does admit products, which are those in $T\text{-Alg}_s$, by the same proof as that of Proposition 2.1. It also admits cotensor products, which are those in $T\text{-Alg}_s$ – not by our present proof of Proposition 2.5, but by a direct one. Finally, it admits (again by a direct proof) the *lax* limit of a single morphism. Now, for counter-examples, let T be the 2-monad on **Cat** whose algebras are (small) categories with initial objects. By Subsection 6.5 below, a lax morphism $f: B \to C$ of algebras may be identified with an *arbitrary* functor $f: B \to C$ between the underlying categories. The functors $1, 0 : 1 \to \mathbf{2}$ (that is, the names of the objects 1 and 0 of $\mathbf{2}$ – the context makes it clear that $1 : 1 \to \mathbf{2}$ does not denote an identity functor) admit no inserter in $T\text{-Alg}_l$, since no T-algebra is empty. The unique 2-cell $0 \to 1 : 1 \to \mathbf{2}$ admits no inverter. Consider the categories B and C with initial object 0, given respectively by

$$0 \longrightarrow x \underset{\beta}{\overset{\alpha}{\rightrightarrows}} y, \qquad u \leftarrow 0 \to v;$$

the 2-cells $\alpha, \beta : x \to y : 1 \to B$ admit no equifier, and the diagram given by the parallel pair $u, v : 1 \to C$ admits neither a conical pseudo-limit nor a conical lax limit.

3. The left adjoints of $T\text{-Alg}_s \to T/\mathscr{K}$, $T\text{-Alg}_s \to T\text{-Alg}$, and $T\text{-Alg}_s \to T\text{-Alg}_l$

To exhibit a left adjoint H of a 2-functor $G : \mathscr{K} \to \mathscr{L}$, we must provide a unit $1 \to GH$ and show that it induces an isomorphism $\mathscr{K}(HB, A) \cong \mathscr{L}(B, GA)$ of *categories*; that is, we must verify not only the usual one-dimensional universal property but a two-dimensional one as well. If $\mathscr{K}$ admits the cotensor products $\{\mathbf{2}, A\}$, however, we can replace this latter verification by the much simpler one that G preserves these cotensor products. Write $\mathscr{K}_0$ for the underlying ordinary category of a 2-category $\mathscr{K}$, obtained by forgetting the 2-cells, and write $G_0 : \mathscr{K}_0 \to \mathscr{L}_0$ for the underlying functor of the 2-functor $G : \mathscr{K} \to \mathscr{L}$. The following is essentially an adaptation to the case $\mathscr{V} = \mathbf{Cat}$ of [21, Theorem 4.85], but we give a direct proof.

Proposition 3.1. *Let $\mathscr{K}$ admit cotensor products of the form $\{\mathbf{2}, A\}$ and let $G : \mathscr{K} \to \mathscr{L}$ preserve them. Then G admits a left adjoint if $G_0 : \mathscr{K}_0 \to \mathscr{L}_0$ does so.*

Proof. Let $\eta B : B \to GHB$ be the unit of the adjunction $H_0 \dashv G_0$, so that every $p : B \to GA$ is $Gf \cdot \eta B$ for a unique $f : HB \to A$. Consider now a 2-cell $\varrho : p \to q : B \to GA$ where $p = Gf \cdot \eta B$ and $q = Gg \cdot \eta B$; we are to show that $\varrho = G\alpha \cdot \eta B$ for a unique $\alpha : f \to g$. Let C, with its unit $\lambda : u \to v : C \to A$, be the cotensor product $\{\mathbf{2}, A\}$; since

this is preserved by G, there is a unique $r: B \to GC$ such that $Gu \cdot r = p$, $Gv \cdot r = q$, and $G\lambda \cdot r = \varrho$. There is a unique $h: HB \to C$ with $Gh \cdot \eta B = r$. Because $Gu \cdot Gh \cdot \eta B = Gu \cdot r = p$, we have $uh = f$; similarly $vh = g$. The 2-cell $\lambda h: f = uh \to vh = g$ satisfies $G(\lambda h) \cdot \eta B = G\lambda \cdot Gh \cdot \eta B = G\lambda \cdot r = \varrho$, as required. It remains to show uniqueness; let $\alpha: f \to g$ be any 2-cell such that $G\alpha \cdot \eta B = \varrho$. There is a unique $k: HB \to C$ satisfying $uk = f$, $vk = g$, and $\lambda k = \alpha$. Now $Gu \cdot Gk \cdot \eta B = Gf \cdot \eta B = p$, and similarly $Gv \cdot Gk \cdot \eta B = q$; while $G\lambda \cdot Gk \cdot \eta B = G\alpha \cdot \eta B = \varrho$. By the uniqueness clause for the cotensor product GC, we have $Gk \cdot \eta B = r$; this gives $k = h$, so that $\alpha = \lambda h$. □

Remark 3.2. The observation in [23, Section 3], that the two-dimensional universal property of a putative limit in a 2-category $\mathscr{K}$ follows from the one-dimensional one if $\mathscr{K}$ admits tensor products of the form $\mathbf{2} * A$, is a special case of the dual of the above – or more precisely of its 'representation' rather than its 'adjunction' form. In the further results of this section, no completeness of $\mathscr{K}$, as distinct from its cocompleteness, is really necessary; but because $\mathscr{K}$ is complete in all of our examples, we add such unnecessary hypotheses to shorten our arguments by using Proposition 3.1.

We now consider, for any 2-category $\mathscr{K}$ (on which we make no blanket hypotheses for the moment), and for any endo-2-functor T of $\mathscr{K}$, the comma-2-category $T/\mathscr{K}$. An object of $T/\mathscr{K}$ is a triple (A, a, X) where A and X are objects of $\mathscr{K}$ and $a: TA \to X$; a morphism $(A, a, X) \to (B, b, Y)$ is a pair $(f: A \to B, p: X \to Y)$ such that $pa = b \cdot Tf$; and a 2-cell $(f, p) \to (g, q)$ is a pair of 2-cells $(\alpha: f \to g, \varrho: p \to q)$ such that $\varrho a = b \cdot T\alpha$. In other words, the hom-category in $T/\mathscr{K}$ is given by the following pullback in **Cat**:

$$\begin{array}{ccc} (T/\mathscr{K})((A,a,X),(B,b,Y)) & \longrightarrow & \mathscr{K}(X,Y) \\ \downarrow & & \downarrow{\scriptstyle \mathscr{K}(a,1)} \\ \mathscr{K}(A,B) \xrightarrow{T} \mathscr{K}(TA,TB) & \xrightarrow[\mathscr{K}(1,b)]{} & \mathscr{K}(TA,Y) \end{array} \tag{3.1}$$

We leave to the reader the very simple proof of

Proposition 3.3. *Given an object (C, c, Z) of $T/\mathscr{K}$, let $\lambda: u \to v: B \to C$ and $\mu: s \to t: Y \to Z$ be the cotensor products $\{\mathbf{2}, C\}$ and $\{\mathbf{2}, Z\}$ in $\mathscr{K}$. Write $b: TB \to Y$ for the unique map satisfying $sb = c \cdot Tu$, $tb = c \cdot Tv$, and $\mu b = c \cdot T\lambda$. Then $(\lambda, \mu): (u, s) \to (v, t): (B, b, Y) \to (C, c, Z)$ is the cotensor product $\{\mathbf{2}, (C, c, Z)\}$ in $T/\mathscr{K}$.* □

Proposition 3.4. *The 2-category $T/\mathscr{K}$ is cocomplete if $\mathscr{K}$ is cocomplete and admits the cotensor products $\{\mathbf{2}, C\}$.*

Proof. Let $F: \mathscr{P}^{\mathrm{op}} \to \mathbf{Cat}$ be an indexing type with $\mathscr{P}$ small. To give a 2-functor

$G: \mathscr{P} \to T/\mathscr{K}$ is clearly to give 2-functors $M, N: \mathscr{P} \to \mathscr{K}$ and a 2-natural $\alpha: TM \to N$; then $GP = (MP, \alpha P, NP)$, and similarly for morphisms and 2-cells. Consider what is to give a 2-natural $\gamma: F \to (T/\mathscr{K})(G-, (B, b, Y))$; in view of the pullback (3.1) it is to give 2-natural transformations $\varrho: F \to \mathscr{K}(M-, B)$ and $\sigma: F \to \mathscr{K}(N-, Y)$ rendering commutative

$$\begin{array}{ccccc} F & & \xrightarrow{\quad\sigma\quad} & & \mathscr{K}(N-, Y) \\ {\scriptstyle\varrho}\downarrow & & & & \downarrow{\scriptstyle\mathscr{K}(\alpha-, 1)} \\ \mathscr{K}(M-, B) & \xrightarrow[T]{} & \mathscr{K}(TM-, TB) & \xrightarrow[\mathscr{K}(1,b)]{} & \mathscr{K}(TM-, Y) \end{array} \tag{3.2}$$

To give ϱ and σ is equally to give (using the notation of [23] for colimits) maps $f: F * M \to B$ and $q: F * N \to Y$ in $\mathscr{K}$; and the translation of (3.2) is

$$\begin{array}{ccccc} F * TM & \xrightarrow{\tilde{T}} & T(F * M) & \xrightarrow{Tf} & TB \\ {\scriptstyle F*\alpha}\downarrow & & & & \downarrow{\scriptstyle b} \\ F * N & & \xrightarrow[q]{\quad} & & Y \end{array} \tag{3.3}$$

where $\tilde{T}$ is the canonical comparison map. Form the pushout

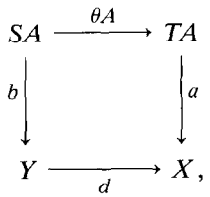
$$\begin{array}{ccc} F * TM & \xrightarrow{\tilde{T}} & T(F * M) \\ {\scriptstyle F*\alpha}\downarrow & & \downarrow{\scriptstyle a} \\ F * N & \xrightarrow[c]{} & X \end{array}$$

in $\mathscr{K}$; to give f and q as above satisfying (3.3) is to give $f: F * M \to B$ and $p: X \to Y$ satisfying $pa = b \cdot Tf$, which is to give a map $(f, p): (F * M, a, X) \to (B, b, Y)$ in $T/\mathscr{K}$. So $(F * M, a, X)$ is the colimit $F * G$ in $T/\mathscr{K}$, as far as the one-dimensional universal property goes; that it has the two-dimensional universal property as well follows from Remark 3.2 and Proposition 3.3. □

Consider now any 2-natural $\theta: S \to T$ between endo-2-functors of $\mathscr{K}$. There is an evident induced 2-functor $\theta^*: T/\mathscr{K} \to S/\mathscr{K}$ sending (A, a, X) to $(A, a \cdot \theta A, X)$.

Proposition 3.5. *If $\mathscr{K}$ admits pushouts and the cotensor products $\{2, C\}$, the 2-functor $\theta^*: T/\mathscr{K} \to S/\mathscr{K}$ induced by $\theta: S \to T$ has a left adjoint.*

Proof. It was observed in [20, Section 14.1] that $\theta^*: (T/\mathscr{K})_0 = T_0/\mathscr{K}_0 \to S_0/\mathscr{K}_0 = (S/\mathscr{K})_0$ has a left adjoint sending $(A, b, Y) \in S/\mathscr{K}$ to $(A, a, X) \in T/\mathscr{K}$ where a and X are defined by the pushout



the unit of the adjunction being $(1, d) : (A, b, Y) \to (A, a \cdot \theta A, X)$. Since $T/\mathscr{K}$ has the cotensor products $\{\mathbf{2}, (C, c, Z)\}$ by Proposition 3.3 and θ^* clearly preserves these, the result follows from Proposition 3.1. □

We now return to the situation where $T = (T, i, m)$ is a 2-monad on $\mathscr{K}$. *Let us identify a T-algebra* $A = (A, a)$ *with the object* (A, a, A) *of* $T/\mathscr{K}$.

Lemma 3.6. *Let* (A, a, X) *be a general object of* $T/\mathscr{K}$ *and let* $B = (B, b) = (B, b, B)$ *be a T-algebra. If* $(f, p) : (A, a, X) \to (B, b, B)$ *is a morphism in* $T/\mathscr{K}$ *we have* $f = pa \cdot iA$; *if* $(\alpha, \varrho) : (f, p) \to (g, q) : (A, a, X) \to (B, b, B)$ *is a 2-cell, we have* $\alpha = \varrho a \cdot iA$. *The full sub-2-category of* $T/\mathscr{K}$ *determined by the T-algebras is precisely* $T\text{-Alg}_s$.

Proof. By the naturality of i, we have $Tf \cdot iA = iB \cdot f$; since $b \cdot Tf = pa$ and $b \cdot iB = 1$, we have $f = pa \cdot iA$. If (A, a, X) too is a T-algebra (A, a, A), this gives $f = p$ since then $a \cdot iA = 1$; thus f is a strict morphism of T-algebras. The arguments at the level of 2-cells are identical, using the 2-naturality of i. □

Theorem 3.7. *When* $\mathscr{K}$ *is complete and cocomplete and T has a rank, the full inclusion 2-functor* $T\text{-Alg}_s \to T/\mathscr{K}$ *has a left adjoint.*

Proof. Consider a T-algebra $C = (C, c)$. If the cotensor product $\{\mathbf{2}, C\}$ in $\mathscr{K}$ is $\lambda : u \to v : B \to C$, the cotensor product $\{\mathbf{2}, C\}$ in $T\text{-Alg}_s$ has by Proposition 2.5 the same form (since it is the cotensor product in T-Alg, which is preserved by $U : T\text{-Alg} \to \mathscr{K}$, and for which u and v are strict algebra-morphisms). The action b of B must therefore be the unique $b : TB \to B$ for which $ub = c \cdot Tu$, $vb = c \cdot Tv$, and $\lambda b = c \cdot T\lambda$. It follows from Proposition 3.3 that the inclusion $T\text{-Alg}_s \to T/\mathscr{K}$ preserves the cotensor products $\{\mathbf{2}, C\}$. The underlying category $(T\text{-Alg}_s)_0$ is just the classical Eilenberg-Moore category T_0-Alg for the monad T_0 on $\mathscr{K}_0$. Of course (see [23, Section 3]) $\mathscr{K}_0$ is cocomplete when $\mathscr{K}$ is so; and to say that T has rank α is the same thing as to say that T_0 has rank α. Accordingly, the full inclusion $(T\text{-Alg}_s)_0 = T_0\text{-Alg} \to T_0/\mathscr{K}_0 = (T/\mathscr{K})_0$ has a left adjoint by [20, Theorem 25.2]. The result now follows from Proposition 3.1. □

From Proposition 3.4 and Theorem 3.7 we conclude (see [23, Section 3]) that:

Theorem 3.8. *When* $\mathscr{K}$ *is complete and cocomplete and T has a rank,* $T\text{-Alg}_s$ *is cocomplete.* □

Now consider a second 2-monad $S=(S,j,n)$ on $\mathscr{K}$ and a strict map $\theta: S \to T$ of 2-monads; for this notion, see [28, Section 3.2]. The 2-functor $\theta^*: T/\mathscr{K} \to S/\mathscr{K}$ above clearly restricts to a 2-functor $\theta^*: T\text{-Alg}_s \to S\text{-Alg}_s$ sending (A,a) to $(A, a \cdot \theta A)$. Of course, θ^* commutes with the forgetful 2-functors to $\mathscr{K}$; and it is well known (see [28, Section 3.6]) that any 2-functor $T\text{-Alg}_s \to S\text{-Alg}_s$ commuting with the forgetful 2-functors is θ^* for a unique such θ.

Theorem 3.9. *Let $\mathscr{K}$ be complete and cocomplete, let $\theta: S \to T$ be a strict map of 2-monads, and let T have a rank. Then the 2-functor $\theta^*: T\text{-Alg}_s \to S\text{-Alg}_s$ has a left adjoint.*

Proof. By Proposition 3.5 and Theorem 3.7, the composite of the inclusion $T\text{-Alg}_s \to T/\mathscr{K}$ and $\theta^*: T/\mathscr{K} \to S/\mathscr{K}$ has a left adjoint. Since this composite is equally the composite of $\theta^*: T\text{-Alg}_s \to S\text{-Alg}_s$ and the inclusion $S\text{-Alg}_s \to S/\mathscr{K}$, and since this latter inclusion is a full one, the result follows. □

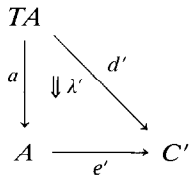
We saw in Lemma 3.6 that a morphism $(f,p): (A,a,X) \to (B,b,B)$ in $T/\mathscr{K}$ into a T-algebra is entirely determined by the map $p: X \to B$ in $\mathscr{K}$, as f must be $pa \cdot iA$. Clearly

Lemma 3.10. *If $B=(B,b)$ is a T-algebra, a map $p: X \to B$ in $\mathscr{K}$ corresponds as above to a morphism $(A,a,X) \to (B,b,B)$ in $T/\mathscr{K}$ if and only if $pa = b \cdot Tp \cdot Ta \cdot TiA$.* □

Lemma 3.11. *Consider a T-algebra $B=(B,b)$ and an object C of $\mathscr{K}$. Any map $q: C \to B$ in $\mathscr{K}$ determines a map $k: TC \to B$ by $k = b \cdot Tq$. Here q is fully determined by k, since the naturality of i and $b \cdot iB = 1$ give $q = k \cdot iC$. A given $k: TC \to B$ is of the form $b \cdot Tq$ if and only if $b \cdot Tk \cdot TiC = k$.* □

Proposition 3.12. *Let $\mathscr{K}$ be cocomplete and let $A=(A,a)$ be a T-algebra. We can find an object (C,c,Z) of $T/\mathscr{K}$ such that, for any T-algebra $B=(B,b)$, there is a bijection, natural in the T-algebra B, between morphisms $A \to B$ of T-algebras and morphisms $(C,c,Z) \to (B,b,B)$ in $T/\mathscr{K}$. There is a corresponding result with morphisms of T-algebras replaced by lax morphisms.*

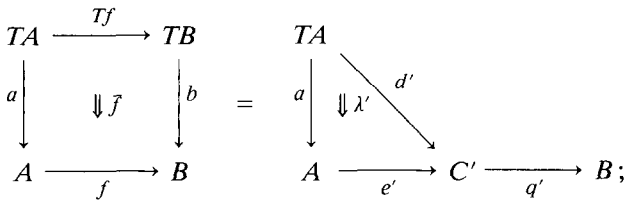
Proof. First form the pseudo-colimit



(3.4)

in $\mathscr{K}$ of the arrow a. To give the data $(f,\bar{f})$ as in (1.1) for an algebra-morphism

$f: A \to B$ is, by the universal property of (3.4), to give a map $q': C' \to B$ as in



(3.5)

that is to say, we have

$$f = q'e', \qquad \bar{f} = q'\lambda', \tag{3.6}$$

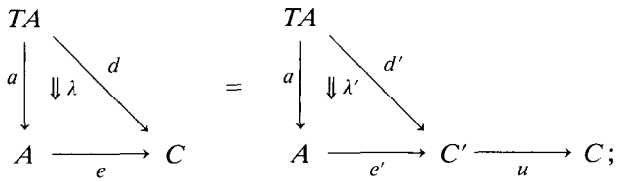
and we must of course require of q' that $q'd' = b \cdot Tf$, or

$$q'd' = b \cdot Tq' \cdot Te'. \tag{3.7}$$

The axiom (1.3) for $(f, \bar{f})$ becomes

$$q'\lambda' \cdot iA = \text{identity}. \tag{3.8}$$

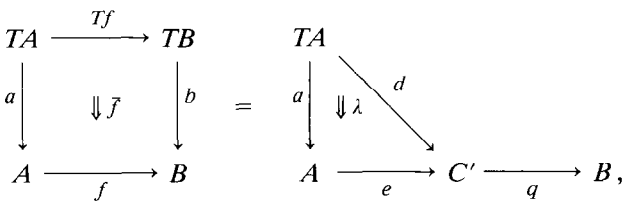
Accordingly, let $u: C' \to C$ be the co-identifier of $\lambda' \cdot iA$, and define d, e, λ by



note that, since $ud' \cdot iA = ue'a \cdot iA = ue'$ because $u\lambda' \cdot iA = \mathrm{id}$, we have

$$d \cdot iA = e. \tag{3.9}$$

To give a q' satisfying (3.8) is to give a map $q: C \to B$, whereupon $q' = qu$. Now (3.5) becomes



(3.10)

and (3.7) becomes

$$qd = b \cdot Tq \cdot Te. \tag{3.11}$$

Thus to give the data $(f, \bar{f})$ satisfying (1.3) is to give $q: C \to B$ satisfying (3.11). By Lemma 3.11, to give $q: C \to B$ is equally to give $k: TC \to B$ satisfying

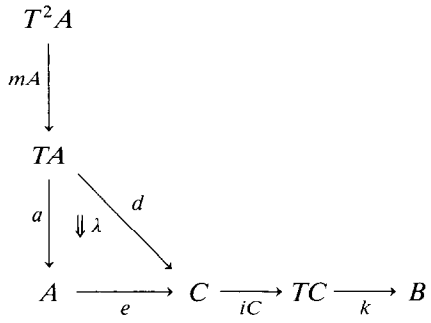
$$b \cdot Tk \cdot TiC = k; \tag{3.12}$$

whereupon $q = k \cdot iC$ and $k = b \cdot Tq$, so that (3.11) becomes

$$k \cdot iC \cdot d = k \cdot Te. \tag{3.13}$$

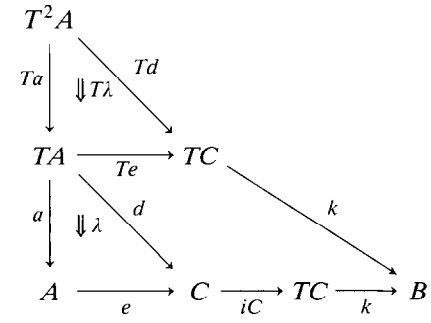
So to give the data $(f, \bar{f})$ satisfying (1.3) is equally to give $k : TC \to B$ satisfying (3.12) and (3.13).

It remains to satisfy the axiom (1.2) for $(f, \bar{f})$. Expressing f and $\bar{f}$ in terms of k using (3.10) and $q = k \cdot iC$, this axiom becomes the equality of the 2-cells



(3.14)

and



(3.15)

note that we have used (3.12) to reduce the right side of (1.2) to (3.15), and that the pentagon in (3.15) commutes by (3.13).

That the 1-cells forming the upper legs of (3.14) and (3.15) are equal, given (3.12) and (3.13), is clear from our construction, since they are equal in (1.2). Now, however, we cease for the moment to impose (3.12); so there is no guarantee that these upper legs are equal. But if they *are* equal, (3.13) is a consequence; we have only to compose each of them with $TiA : TA \to T^2A$ and use $mA \cdot TiA = 1$ and (3.9). Accordingly we consider the cocqualizer $v : TC \to D$ of $iC \cdot d \cdot mA$ and Td, and the coequifier $w : D \to Z$ of the pair of 2-cells obtained from (3.14) and (3.15) on replacing therein k by v. If we write $c : TC \to Z$ for wv, to give a map $k : TC \to B$ satisfying (3.13) and making (3.14) and (3.15) equal (at the levels of 1-cells and of 2-cells) is equally to give a map $p : Z \to B$; whereupon $k = pc$. The remaining condition (3.12) which we must now impose upon k if it is to correspond to an algebra-morphism $f = (f, \bar{f}) : A \to B$ becomes $b \cdot Tp \cdot Tc \cdot TiC = pc$; which by Lemma 3.10 is exactly the

condition for p to correspond to a morphism $(C, c, Z) \to (B, b, B)$ in $T/\mathcal{K}$. This completes the proof in the 'morphisms of algebras' case. The proof in the 'lax morphisms' case differs in only one point: we replace the pseudo-colimit (3.4) by the op-lax-colimit in which the 2-cell λ' is not required to be invertible (it is 'op-lax' and not 'lax' because colimits in $\mathcal{K}$ are limits in $\mathcal{K}^{\mathrm{op}}$, and the passage from $\mathcal{K}$ to $\mathcal{K}^{\mathrm{op}}$ reverses 1-cells but not 2-cells). □

Theorem 3.13. *If $\mathcal{K}$ is complete and cocomplete and T has a rank, the inclusion 2-functors $T\text{-Alg}_s \to T\text{-Alg}$ and $T\text{-Alg}_s \to T\text{-Alg}_l$ have left adjoints.*

Proof. $T\text{-Alg}_s$ has cotensor products, and the inclusions above preserve them, by Proposition 2.5 and Remark 2.9; by Proposition 3.1, therefore, it suffices to show that the inclusion functors $(T\text{-Alg}_s)_0 \to (T\text{-Alg})_0$ and $(T\text{-Alg}_s)_0 \to (T\text{-Alg}_l)_0$ have left adjoints. This is so, since we have isomorphisms $(T\text{-Alg})_0(A, B) \cong (T/\mathcal{K})_0((C, c, Z), (B, b, B)) \cong (T\text{-Alg}_s)_0((C, c, Z)^\circ, B)$ natural in the T-algebra B, by Proposition 3.12 and Theorem 3.7 respectively; here $(\ )^\circ$ denotes the left adjoint of the latter theorem to the inclusion $T\text{-Alg}_s \to T/\mathcal{K}$. Similarly, by the final assertion of Proposition 3.12, with $T\text{-Alg}_l$ in place of T-Alg. □

Remark 3.14. As we said in Remark 3.2, the theorems of this section remain true without any completeness hypothesis on $\mathcal{K}$, so long as it is cocomplete; the proofs, which are then longer, go as follows. We omit Propositions 3.1 and 3.3, and verify directly the two-dimensional universal properties of the colimit constructed in Proposition 3.4 and of the adjoint constructed in Proposition 3.5. Lemma 3.6 stays as it is. For Theorem 3.7, one must go back to [20] and trace through the stages of the proof of its Theorem 25.2, starting with the first results on algebras for a well-pointed endofunctor, and observing that everything carries over to the 2-categorical situation. We extend Lemmas 3.10 and 3.11 to include the corresponding results on 2-cells; and we use the two-dimensional universal property of the colimits in the proof of Proposition 3.12 to deduce an *isomorphism of categories* $T\text{-Alg}(A, B) \cong (T/\mathcal{K})((C, c, Z), (B, b, B))$. Now Theorem 3.13 follows.

Remark 3.15. Let $\mathcal{P}$ be a small 2-category and $\mathcal{L}$ a cocomplete one. We show in Subsection 6.6 below that, if X is the set of objects of $\mathcal{P}$, there is a finitary 2-monad T on the power $\mathcal{L}^X$, whose algebras are the 2-functors $\mathcal{P} \to \mathcal{L}$; the strict morphisms, the morphisms, and the lax morphisms being respectively the 2-natural transformations, the pseudo-natural transformations, and the lax natural transformations, while the algebra-2-cells are the modifications. Thus here the inclusions $T\text{-Alg}_s \to T\text{-Alg} \to T\text{-Alg}_l$ become $[\mathcal{P}, \mathcal{L}] \to \mathrm{Psd}[\mathcal{P}, \mathcal{L}] \to \mathrm{Lax}[\mathcal{P}, \mathcal{L}]$ in the sense of [23, Section 5]. Accordingly, by Theorem 3.13 if $\mathcal{L}$ is also complete, and in particular if $\mathcal{L} = \mathbf{Cat}$ (which is the only case used below), but by Remark 3.14 if $\mathcal{L}$ is only cocomplete, we have

Theorem 3.16. *For a small 2-category $\mathscr{P}$ and a cocomplete 2-category $\mathscr{L}$, the inclusion 2-functors $[\mathscr{P}, \mathscr{L}] \to \mathrm{Psd}[\mathscr{P}, \mathscr{L}]$ and $[\mathscr{P}, \mathscr{L}] \to \mathrm{Lax}[\mathscr{P}, \mathscr{L}]$ have left adjoints.* □

4. Flexibility

Still supposing that $\mathscr{K}$ is complete and cocomplete and that T has a rank, we devote this section to a closer examination of the first adjunction of Theorem 3.13. Write $J: T\text{-Alg}_s \to T\text{-Alg}$ for the locally-fully-faithful inclusion 2-functor; although we often suppress J and write A, f, α for $JA, Jf, J\alpha$, we must sometimes refer to it to avoid confusion. Write $(\ )': T\text{-Alg} \to T\text{-Alg}_s$ for the 2-functor left adjoint to J given by Theorem 3.13, and write $\pi_{AB}: T\text{-Alg}_s(A', B) \cong T\text{-Alg}(A, JB) = T\text{-Alg}(A, B)$ for the adjunction-isomorphism. Note carefully the extent of the 2-naturality of π_{AB}; it is 2-natural in A for $A \in T\text{-Alg}$, but is 2-natural in B only for $B \in T\text{-Alg}_s$. Let the unit and the counit of the adjunction be $p: 1 \to J(\ )': T\text{-Alg} \to T\text{-Alg}$ and $q: (\ )'J \to 1: T\text{-Alg}_s \to T\text{-Alg}_s$, observing that the components $q_A: A' \to A$, unlike the components $p_A: A \to A'$, are strict morphisms of algebras. In the same vein, p_A is 2-natural in A for $A \in T\text{-Alg}$, while q_A is 2-natural in A only for $A \in T\text{-Alg}_s$. In diagram form we have

$$\begin{array}{ccc} A & \xrightarrow{p_A} & A' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ B & \xrightarrow[p_B]{} & B' \end{array} \qquad \begin{array}{ccc} A' & \xrightarrow{q_A} & A \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ B' & \xrightarrow[q_B]{} & B \end{array} \tag{4.1}$$

where g in the right-hand square is to be strict; there is of course similar commutativity for 2-cells $f \to f^*$ and $g \to g^*$. The triangular equations for the unit and the counit are

$$\begin{array}{ccc} A & \xrightarrow{p_A} & A' \\ & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle q_A} \\ & & A \end{array} \qquad \begin{array}{ccc} A' & \xrightarrow{(p_A)'} & A'' \\ & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle q_{A'}} \\ & & A' \end{array} \tag{4.2}$$

As in [21, Section 1.11], we have commutativity in

$$\begin{array}{ccc} T\text{-Alg}_s(A', B) & \xrightarrow[\cong]{\pi_{AB}} & T\text{-Alg}(A, B) \\ & {\scriptstyle J_{A',B}}\searrow \quad \nearrow{\scriptstyle T\text{-Alg}(p_A, B)} & \\ & T\text{-Alg}(A', B) & \end{array} \tag{4.3}$$

and in

$$\begin{array}{ccc} T\text{-Alg}_s(A,B) & \xrightarrow{J_{AB}} & T\text{-Alg}(A,B) \\ & {\scriptstyle T\text{-Alg}_s(q_A,B)}\searrow \quad \cong \quad \nearrow{\scriptstyle \pi_{AB}} & \\ & T\text{-Alg}_s(A',B) & \end{array} \tag{4.4}$$

In elementary terms, the universal property of p_A expressed by (4.3) is the following: its one-dimensional aspect asserts that any morphism $f: A \to B$ of algebras has the form

$$\begin{array}{ccc} A & \xrightarrow{p_A} & A' \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle g} \\ & & B \end{array} \tag{4.5}$$

for a unique strict morphism g, while its 2-dimensional aspect asserts that, for strict g and g^*, any 2-cell $\beta: gp_A \to g^*p_A$ is αp_A for a unique $\alpha: g \to g^*$. Note that the left diagram of (4.2) gives $g = q_A$ when $f = 1_A$.

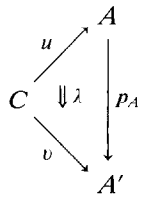
Remarks 4.1. We break off now to say a few words about equivalences in a 2-category; for more details, see the forthcoming [24]. An *adjoint equivalence* is an adjunction $\eta, \varepsilon: f \dashv u: A \to B$ for which the unit $\eta: 1 \to uf$ and the counit $\varepsilon: fu \to 1$ are invertible. For the general theory of adjunctions in a 2-category, see [28]; it follows therefrom that the adjoint equivalences in any 2-category themselves form a 2-category. Note that, for any adjunction whatsoever, ε is uniquely determined by u, f, and η; and that, given any $\eta: 1 \to uf$ and $\varepsilon: fu \to 1$, the triangular equation $(\varepsilon f)(f\eta) = \mathrm{id}$ is a consequence of the other triangular equation $(u\varepsilon)(\eta u) = \mathrm{id}$ if η is invertible. A map $u: A \to B$ is an *equivalence* if there is some adjoint equivalence $\eta, \varepsilon: f \dashv u$. For this it suffices that there be an $f: B \to A$ and invertible 2-cells $\eta: 1 \cong uf$ and $\varrho: fu \cong 1$; we have only to set $\varepsilon = \varrho(f\eta^{-1}u)(\varrho^{-1}fu)$. If u is an equivalence, its *equivalence-inverse* f is determined to within an isomorphism. An equivalence u is said to be *surjective* if it is a retraction – that is, if $uf = 1$ for some f; then any equivalence-inverse of u is necessarily isomorphic to f, and consequently there is an adjoint equivalence $\eta, \varepsilon: f \dashv u$ with η the identity. For 2-categories $\mathscr{P}$ and $\mathscr{K}$, a map $u: G \to H$ in $\mathrm{Hom}[\mathscr{P}, \mathscr{K}]$ (see [23, Section 6]) is an equivalence if and only if each $u_P: GP \to HP$ is an equivalence in $\mathscr{K}$; in fact, given u and adjoint equivalences $\eta_P, \varepsilon_P: f_P \dashv u_P: GP \to HP$, it follows from the discussion in [28] of *mates* that there is a unique extension of the f_P to an $f: H \to G$ in $\mathrm{Hom}[\mathscr{P}, \mathscr{K}]$ with $\eta, \varepsilon: f \dashv u$. Here, even when G and H are 2-functors and u is a 2-natural transformation – that is, a map in $[\mathscr{P}, \mathscr{K}]$ – it is not in general the case that u is an equivalence in $[\mathscr{P}, \mathscr{K}]$. A functor $u: A \to B$ in **Cat** is an equivalence if and only if it is fully faithful and there is a *function* assigning to each $b \in B$ an $fb \in A$ and an isomorphism $\eta_b: b \cong ufb$; it is not enough to require that every $b \in B$ be isomorphic to some ua, unless

our category **Set** of small sets satisfies the axiom of choice – which, in order that our results may continue to hold in wider contexts, we do not impose.

Recall that, by the left diagram in (4.2), we have $q_A p_A = 1$. In fact, q_A is a surjective equivalence in T-Alg:

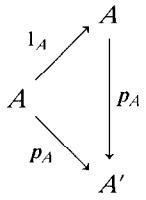
Theorem 4.2. *For each T-algebra A, there is a unique invertible algebra-2-cell $\varrho_A : p_A q_A \cong 1$ with $\varrho_A p_A = \mathrm{id}$ and $q_A \varrho_A = \mathrm{id}$; thus q_A is a surjective equivalence in T-Alg, and $\mathrm{id}, \varrho_A : p_A \dashv q_A$ is an adjoint equivalence.*

Proof. By Remarks 4.1, ϱ_A is unique if it exists, and an invertible $\varrho_A : p_A q_A \cong 1$ automatically satisfies the triangular equation $q_A \varrho_A = \mathrm{id}$ if it satisfies the triangular equation $\varrho_A p_A = \mathrm{id}$. By Remark 2.7, the pseudo-limit



(4.6)

exists in T-Alg, and u, υ are strict morphisms of algebras. Since we have the commutative diagram



there is by the one-dimensional universal property of (4.6) a unique algebra-morphism $f : A \to C$ satisfying

$$uf = 1_A, \qquad \upsilon f = p_A, \qquad \lambda f = \mathrm{id}. \tag{4.7}$$

Let $f = g p_A$ with g strict as in (4.5). Since $1_A = q_A p_A$ by (4.2), the first two equations of (4.7) may be written as $ug p_A = q_A p_A$ and $\upsilon g p_A = p_A$; whence $ug = q_A$ and $\upsilon g = 1$ by the uniqueness of g in (4.5). From the invertible $\lambda : p_A u \cong \upsilon$ we get the invertible $\varrho_A = \lambda g : p_A q_A = p_A u g \cong \upsilon g = 1$; and by the third equation of (4.7) we have $\varrho_A p_A = \lambda g p_A = \lambda f = \mathrm{id}$. □

Remark 4.3. The triangular equation represented by the left diagram of (4.2) is, of course, the A-component of the equation $Jq \cdot pJ = 1 : J \to J$. Since we do not use it below, we leave the reader to verify that the ϱ_A satisfy $g' \varrho_A = \varrho_B g'$ for a strict

morphism $g: A \to B$, and hence constitute a modification $\varrho: pJ \cdot Jq \to 1: J(\ )'J \to J(\ )'J$.

Returning to the context of Theorem 4.2, we consider those algebras A for which q_A is a surjective equivalence not only in T-Alg but in T-Alg_s.

Theorem 4.4. *For a T-algebra A, the following assertions are equivalent:*
(a) $q_A: A' \to A$ *is a surjective equivalence in* T-Alg_s.
(b) $q_A: A' \to A$ *is a retraction in* T-Alg_s.
(c) A *is a retract of* B' *in* T-Alg_s *for some T-algebra B.*

Proof. Since q_A is an equivalence in T-Alg by Theorem 4.2, and since the inclusion of T-Alg_s in T-Alg is locally fully faithful, (a) and (b) are equivalent by Remarks 4.1; moreover (b) implies (c) trivially. Suppose now that $r: B' \to A$ is a retraction in T-Alg_s. By the second diagrams of (4.1) and (4.2) we have in T-Alg_s the equation $q_A r'(p_B)' = r q_{B'}(p_B)' = r$; whence q_A, like r, is a retraction in T-Alg_s. □

Remark 4.5. We call a T-algebra A *flexible* if it satisfies the equivalent conditions of Theorem 4.4; as we said in Subsection 1.3, the notion of flexible algebra is important in a number of contexts going beyond its use below in the present article. The term 'flexible' was first used, in the special case where the T-algebras are themselves 2-monads, in [17, Section 3.3] and in [18, Section 8.9]. Of course, every algebra A of the form B' is flexible; but we shall see in Example 4.10 below that not every flexible algebra has this form. When $A = B'$, we have by the second diagram of (4.2) an *explicit* equivalence-inverse for q_A in T-Alg_s, namely $(p_B)'$. So alongside the surjective equivalence $p_{B'} \dashv q_{B'}: B'' \to B'$ in T-Alg we have a surjective equivalence $(p_B)' \dashv q_{B'}: B'' \to B'$ in T-Alg_s. In fact there is a further surjective equivalence $(p_B)' \dashv (q_B)': B'' \to B'$ in T-Alg_s; these three, although of course isomorphic, are in general distinct, as the following example shows:

Example 4.6. Take for T the finitary 2-monad on **Cat** for which TA is A provided freely with a terminal object, and the rest of the structure is evident. Then a T-algebra is a category A with an assigned terminal object t_A, a morphism $f: A \to B$ is a functor $f: A \to B$ such that the map $ft_A \to t_B$ is invertible, and a strict morphism is such a functor with $ft_A = t_B$; see Subsection 6.5 below. It is clear that A' is the category A with one new object $t_{A'}$, with a unique and invertible map $t_A \to t_{A'}$, and with all that flows freely from this; while $p_A: A \to A'$ is the inclusion, and q_A is the identity on A and sends $t_{A'}$ to t_A. In turn, A'' has yet a new object $t_{A''}$ isomorphic to $t_{A'}$ and to t_A. A simple calculation shows that $p_{A'}(t_{A'}) = t_{A'}$ while $(p_A)'(t_{A'}) = t_{A''}$, so that $p_{A'} \neq (p_A)'$; and that $q_{A'}(t_{A'}) = t_{A'}$, while $(q_A)'(t_{A'}) = t_A$, so that $q_{A'} \neq (q_A)'$.

We now consider the still larger class of those algebras A for which q_A is an equivalence in T-Alg_s but not necessarily a surjective one. It is not yet clear to us

whether this class is important enough to deserve a special name; for the moment we compromise by calling such algebras *semi-flexible*. However that may be, the following theorem *is* important; for various of the conditions equivalent to semi-flexibility are properties of flexible algebras needed below. We distinguish (e) from (f) in the theorem because we are avoiding any appeal to the axiom of choice.

Theorem 4.7. *For a T-algebra A, the following assertions are equivalent*:

(a) *$q_A : A' \to A$ is an equivalence in $T\text{-Alg}_s$.*

(b) *A is equivalent to B' in $T\text{-Alg}_s$ for some T-algebra B.*

(c) *A is equivalent in $T\text{-Alg}_s$ to some flexible algebra.*

(d) *$p_A : A \to A'$ is isomorphic to some strict morphism $k : A \to A'$.*

(e) *There is a function assigning to each morphism $f : A \to B$ with domain A a strict morphism $h : A \to B$ and an isomorphism $f \cong h$.*

(f) *Each morphism $f : A \to B$ with domain A is isomorphic to some strict morphism $h : A \to B$.*

(g) *The 2-natural transformation $J_{A-} : T\text{-Alg}_s(A, -) \to T\text{-Alg}(JA, J-)$ is an equivalence in $[T\text{-Alg}_s, \mathbf{Cat}]$.*

Proof. (a) implies (b) trivially, and (b) implies (c) trivially since B' is flexible. To see that (c) implies (a), let $g : A \to B$ be an equivalence in $T\text{-Alg}_s$ with B flexible; since the 2-functor $(\ )'J : T\text{-Alg}_s \to T\text{-Alg}_s$ preserves equivalences, $g' : A' \to B'$ is an equivalence in $T\text{-Alg}_s$; since B is flexible, q_B is an equivalence in $T\text{-Alg}_s$; it follows from the second diagram of (4.1) that q_A is an equivalence in $T\text{-Alg}_s$. Because q_A and p_A are equivalence-inverses in T-Alg by Theorem 4.2, it is immediate that (a) and (d) are equivalent. To see that (d) implies (e), let $\sigma : p_A \cong k$ with k strict; then with $f = gp_A$ as in (4.5) we have $g\sigma : f = gp_A \cong gk$, and we set $h = gk$. It is trivial that (e) implies (f) and (f) implies (d). It remains to prove the equivalence of (a) and (g). Since $\pi_{A-} = \pi_{JA,-} : T\text{-Alg}_s((JA)', -) \cong T\text{-Alg}(JA, J-)$ is an isomorphism in $[T\text{-Alg}_s, \mathbf{Cat}]$, it follows from (4.4) that (g) is equally the assertion that $T\text{-Alg}_s(q_A, -) : T\text{-Alg}_s(A, -) \to T\text{-Alg}_s((JA)', -)$ is an equivalence in $[T\text{-Alg}_s, \mathbf{Cat}]$. Because the Yoneda embedding $(T\text{-Alg}_s)^{\mathrm{op}} \to [T\text{-Alg}_s, \mathbf{Cat}]$ is fully faithful, this last assertion is indeed equivalent to (a). □

Corollary 4.8. *Given morphisms $f, f^* : A' \to B$ and a 2-cell $\beta : fp_A \to f^*p_A$, there is a unique 2-cell $\alpha : f \to f^*$ with $\alpha p_A = \beta$; and α is invertible when β is.*

Proof. That it is so when f and f^* are strict morphisms is the two-dimensional aspect of the universal property of p_A, as discussed following (4.5). However, since A' is flexible, it follows from Theorem 4.7(f) that f and f^* are isomorphic to strict morphisms $h, h^* : A' \to B$. The result follows. □

Remark 4.9. Any equivalence-inverse $k : A \to A'$ in $T\text{-Alg}_s$ to $q_A : A' \to A$ gives an explicit equivalence-inverse to the J_{A-} in (g) of Theorem 4.7, namely $T\text{-Alg}_s(k, -) \cdot \pi_{A-}^{-1}$.

The B-component of this sends $f: A \to B$ to $gk: A \to B$ where g is determined from f by $gp_A = f$ as in (4.5), and sends $\beta: f \to f^*$ to αk where α is determined by $\alpha p_A = \beta$. (Note that, when A is flexible and $q_A k = 1$, the equivalence $T\text{-Alg}_s(k, -) \cdot \pi_{A-}^{-1}$ is surjective; we may express this by saying that J_{A-} is an *op-surjective* equivalence.) In particular, $J_{A'-}: T\text{-Alg}_s(A', -) \to T\text{-Alg}(JA', J-)$ has by Remark 4.5 the equivalence-inverse sending $f: A' \to B$ to $g(p_A)'$, where $g: A'' \to A'$ is the strict morphism with $gp_{A'} = f$. On the other hand, q_A being by Theorem 4.2 an equivalence-inverse of p_A in T-Alg, it follows from (4.3) that $J_{A'-}$ is an op-surjective equivalence with equivalence-inverse $\pi_{A-}^{-1} \cdot T\text{-Alg}(p_A, -)$; the B-component of this last sends $f: A' \to B$ to $h: A' \to B$, where h is the strict morphism with $hp_A = fp_A$. These two equivalence-inverses of $J_{A'-}$, which are necessarily isomorphic, are in fact equal. To see this, it suffices by the uniqueness in (4.5) to show that $g(p_A)'p_A = hp_A$; but the left diagram of (4.1) gives $g(p_A)'p_A = gp_{A'}p_A = fp_A = hp_A$.

Example 4.10. It is easy to see that, in Example 4.6, although not every algebra A is of the form B' (since B' has at least two objects), yet every algebra is flexible. This is far from being the case in general. Let T be the finitary 2-monad on **Cat**, referred to in Subsection 1.3, whose algebras are small categories with assigned finite limits, so that T-Alg = **Lex**; here the strict morphisms (see Subsection 6.5 below) are the functors that preserve the assigned limits on the nose. Now the unit category 1 is not even semi-flexible. To see this, let $\boldsymbol{I}$ be the category with two objects 0 and 1 and mutually-inverse isomorphisms $0 \to 1$ and $1 \to 0$; and suppose the finite limits in $\boldsymbol{I}$ to be so assigned that 1 is the terminal object, while $1 \times 1 = 0$. Since $T\text{-Alg}_s(1, \boldsymbol{I})$ is empty while $T\text{-Alg}(1, \boldsymbol{I}) \cong \boldsymbol{I}$, it follows from (g) of Theorem 4.7 that 1 is not semi-flexible.

Example 4.11. A simpler example to illustrate the possibilities is the following. There is by Remark 3.15 a finitary 2-monad T on **Cat** $\times$ **Cat**, given on objects by $T(X, Y) = (X, X + Y)$, with $T\text{-Alg}_s = [\mathbf{2}, \mathbf{Cat}]$ and $T\text{-Alg} = \text{Psd}[\mathbf{2}, \mathbf{Cat}]$. In elementary terms, a T-algebra A is a functor $a: X \to Y$; a morphism from $a: X \to Y$ to $b: V \to W$ consists of functors $u: X \to V$ and $v: Y \to W$ and an isomorphism $\alpha: bu \cong va$; and a strict morphism is one for which α is the identity. It is easy to describe A' when A is the algebra $a: X \to 1$; it is the algebra $j: X \to \bar{X}$ where ob $\bar{X} =$ ob $X + \{*\}$ and $\bar{X}$ is chaotic (that is, every hom-set in $\bar{X}$ is a singleton). The morphism $p_A: A \to A'$ is given by the functors $1_X: X \to X$ and $*: 1 \to \bar{X}$, with the unique isomorphism $j1_X \cong *a$. (To verify these assertions, just *define* A' and p_A as above and check the universal property (4.5).) It follows that $q_A: A' \to A$ is given by $1_X: X \to X$ and the unique $\bar{a}: \bar{X} \to 1$. Any strict $k = (u, v): A \to A'$ must have u a constant functor; if k is to be an equivalence in $[\mathbf{2}, \mathbf{Cat}]$, u must be an equivalence in **Cat**; a constant functor $u: X \to X$ is an equivalence only if X is chaotic; so *not every algebra is semi-flexible.* For this A to be flexible, there must be an equivalence $k = (u, v): A \to A'$ in $[\mathbf{2}, \mathbf{Cat}]$ with $q_A k = 1$, and hence with $u = 1_X$; since X must be chaotic and u constant, this is impossible unless $X = 1$. Accordingly, $\boldsymbol{I} \to 1$, with $\boldsymbol{I}$ as in Example 4.10,

is not flexible. Yet it is easily seen to be equivalent in **[2, Cat]** to $1 \to 1$, and $1 \to 1$ is easily seen to be flexible; accordingly, by Theorem 4.7(b), *not every semi-flexible algebra is flexible.* Finally, *not every flexible A is B' for some B*; if the flexible $1 \to 1$ were B' where B was the algebra $b : V \to W$, the first diagram of (4.2) would force $V = W = 1$; but $(1 \to 1)'$ is not $1 \to 1$.

Proposition 4.12. *There is a unique pseudo-natural transformation $r : J(\ )' \to 1 : T\text{-Alg} \to T\text{-Alg}$ such that $rp = 1$ and $rJ = Jq$.*

Proof. The requirement $rJ = Jq$ forces the component $r_A : A' \to A$ to be q_A. The requirement $rp = 1$ forces the composite 2-cell

$$\begin{array}{ccccc} A & \xrightarrow{p_A} & A' & \xrightarrow{q_A} & A \\ {\scriptstyle f}\downarrow & & {\scriptstyle f'}\downarrow & \Downarrow r_f & \downarrow{\scriptstyle f} \\ B & \xrightarrow[p_B]{} & B' & \xrightarrow[q_B]{} & B \end{array} \tag{4.8}$$

to be the identity. Since $q_B f' p_A = q_B p_B f = 1f = f1 = f q_A p_A$, there is by Corollary 4.8 a unique r_f in (4.8) with $r_f p_A = \mathrm{id}$; moreover, r_f is invertible. That r so defined is a pseudo-natural transformation follows from the uniqueness in Corollary 4.8; and that $rJ = Jq$ follows from the second diagram of (4.1). □

5. On various biadjunctions

Recall from [23, Section 6] that, given homomorphisms $P : \mathscr{M} \to \mathscr{L}$ and $Q : \mathscr{L} \to \mathscr{M}$ of bicategories, an equivalence $t : \mathscr{M}(Q?, -) \to \mathscr{L}(?, P-)$ in the 2-category $\mathrm{Hom}[\mathscr{L}^{\mathrm{op}}, \mathrm{Hom}[\mathscr{M}, \mathbf{Cat}]]$ is said to exhibit Q as a *left biadjoint* of P. This definition is not in fact unsymmetrical, for Street points out in [36, (1.35)] that the 2-category $\mathrm{Hom}[\mathscr{M}, \mathrm{Hom}[\mathscr{L}^{\mathrm{op}}, \mathbf{Cat}]]$ is *isomorphic* to that above. The forthcoming [24] will give a fuller treatment of biadjunctions; but those that occur below are so special and so simple that the general theory is unnecessary for a precise understanding of their content. Accordingly, we confine ourselves here to a few remarks without proofs, chiefly to fix the nomenclature.

In our applications the bicategories $\mathscr{M}$ and $\mathscr{L}$ are 2-categories. The component $t_{L,QL}$ of t sends the identity of QL to a map $s_L : L \to PQL$; and t further determines components s_φ for $\varphi : L \to L^*$ which make of s a map $1 \to PQ$ in $\mathrm{Hom}[\mathscr{L}, \mathscr{L}]$, called the *unit* of the biadjunction. The equivalence of categories $t_{LM} : \mathscr{M}(QL, M) \to \mathscr{L}(L, PM)$ is *isomorphic* to the functor $\mathscr{L}(s_L, PM) P_{QL,M}$, which is therefore itself an equivalence for each M. This last fact is usually expressed by saying that s_L is the *unit* for a *birepresentation* $t_{L-} : \mathscr{M}(QL, -) \simeq \mathscr{L}(L, P-)$ of the homomorphism $\mathscr{L}(L, P-) : \mathscr{M} \to \mathbf{Cat}$; recall from [23] that $\simeq$ denotes an equivalence while $\cong$ denotes an isomorphism. Conversely, a homomorphism $P : \mathscr{M} \to \mathscr{L}$ admits a left biadjoint

if and only if $\mathscr{L}(L, P-)$ is birepresentable for each L; whereupon the left biadjoint Q is determined to within equivalence in $\mathrm{Hom}[\mathscr{L}, \mathscr{M}]$. We often write $Q \dashv_b P$ to denote that Q is a left biadjoint of P. (We repeat for emphasis that, when we say of a 2-functor $P: \mathscr{M} \to \mathscr{L}$ that it has a *left adjoint* – as we have already done many times in this article – we always mean an honest left adjoint in the sense of **Cat**-enriched category theory, given by an isomorphism $\mathscr{M}(Q?, -) \cong \mathscr{L}(?, P-)$ in $[\mathscr{L}^{\mathrm{op}}, [\mathscr{M}, \mathbf{Cat}]]$.)

We now return to our situation where T is a 2-monad with rank on a complete and cocomplete $\mathscr{K}$. Recall again from [28] the concept of *mates*.

Theorem 5.1. *Let $\mathscr{L}$ be a 2-category and $G: T\text{-Alg} \to \mathscr{L}$ a 2-functor such that the composite GJ of G with the inclusion 2-functor $J: T\text{-Alg}_s \to T\text{-Alg}$ has a left adjoint $H: \mathscr{L} \to T\text{-Alg}_s$ with unit $s: 1 \to GJH$. Then:*

(a) *If $k: H \to (\ )'JH$ is the mate, under the adjunction $H \dashv GJ$, of the composite*

$$1 \xrightarrow[s]{} GJH \xrightarrow[GpJH]{} GJ(\ )'JH,$$

we have $qH \cdot k = 1: H \to H$; thus, since $q_{HL} k_L = 1$ for every $L \in \mathscr{L}$, each algebra HL is flexible.

(b) *If $t: T\text{-Alg}(JH?, -) \to \mathscr{L}(?, G-)$ is the composite*

$$T\text{-Alg}(JH?, -) \xrightarrow[G_{JH?,-}]{} \mathscr{L}(GJH?, G-) \xrightarrow[\mathscr{L}_{(s?, G-)}]{} \mathscr{L}(?, G-) \qquad (5.1)$$

in $[\mathscr{L}^{\mathrm{op}}, [T\text{-Alg}, \mathbf{Cat}]]$, so that the component $t_{LA}: T\text{-Alg}(JHL, A) \to \mathscr{L}(L, GA)$ is the functor sending $\alpha: f \to f^*: JHL \to A$ to $G\alpha \cdot sL: Gf \cdot sL \to Gf^* \cdot sL: L \to GA$, then each t_{LA} is a surjective equivalence in **Cat**; *accordingly t is a surjective equivalence in $\mathrm{Hom}[\mathscr{L}^{\mathrm{op}}, \mathrm{Hom}[T\text{-Alg}, \mathbf{Cat}]]$, and exhibits JH as a left biadjoint of G, with $s: 1 \to GJH$ as unit.*

Proof. The k of (a) is determined by the equation $GJk \cdot s = GpJH \cdot s$; composing this with $GJqH$ on the left and using $Jq \cdot pJ = 1$ (see Remark 4.3) gives $GJ(qH \cdot k) \cdot s = 1 \cdot s = GJ1 \cdot s$, whence $qH \cdot k = 1$ by the one-dimensional universal property of s as the unit of the adjunction $H \dashv GJ$. We turn now to (b): by this same universal property of s, every $h: L \to GA$ is $Gg \cdot sL$ for a unique *strict* $g: JHL \to A$; writing $u_{LA}h$ for this g, we have $h = t_{LA} u_{LA} h$. Moreover t_{LA} is fully faithful: each $\beta: Gf \cdot sL \to Gf^* \cdot sL$ is $G\alpha \cdot sL$ for a unique $\alpha: f \to f^*$. For this is true, by the two-dimensional universal property of s as the unit of $H \dashv GJ$, when f and f^* are strict; and every $f: JHL \to A$ is isomorphic to a strict morphism, by Theorem 4.7 and the flexibility of HL given by (a). It follows from Remarks 4.1, first that each t_{LA} is a surjective equivalence in **Cat**, and then that t is a surjective equivalence in $\mathrm{Hom}[\mathscr{L}^{\mathrm{op}}, \mathrm{Hom}[T\text{-Alg}, \mathbf{Cat}]]$. □

Remark 5.2. The biadjunction of Theorem 5.1 is special in many ways. First, a left biadjoint, even of a 2-functor, is in general only a homomorphism of bicategories, and cannot be chosen to be a 2-functor; yet here G has the 2-functor JH as a left biadjoint. Secondly, the unit of a biadjunction is in general only pseudo-natural; but here it is the 2-natural s. Thirdly, the equivalence t here is a surjective one, and even more is true: not only is every $h: L \to GA$ of the form $Gg \cdot sL$ for some $g: JHL \to A$ (and in general for many such g), but among these g there is exactly one that is strict. Finally, although the general theory of Remarks 4.1 tells us only that t is a surjective equivalence in $\mathrm{Hom}[\mathscr{L}^{\mathrm{op}}, \mathrm{Hom}[T\text{-Alg}, \mathbf{Cat}]]$, we in fact have

Proposition 5.3. *The t of Theorem 5.1 is a surjective equivalence in* $[\mathscr{L}^{\mathrm{op}}, \mathrm{Hom}[T\text{-Alg}, \mathbf{Cat}]]$.

Proof. Write υ for the composite

$$\begin{array}{ccccc} \mathscr{L}(?, G-) & \xrightarrow{\mathscr{L}(?, Gp-)} & \mathscr{L}(?, GJ(\ )'-) & \xrightarrow{v_{?,(\ )'-}} & T\text{-Alg}_s(H?, (\ )'-) \\ & & & & \Big\downarrow J_{H?,(\ )'} \\ T\text{-Alg}(JH?, -) & \xleftarrow[T\text{-Alg}(JH?, r-)]{} & & & T\text{-Alg}(JH?, J(\ )'-) \end{array}$$

where $v: \mathscr{L}(?, GJ-) \cong T\text{-Alg}_s(H?, -)$ is the adjunction-isomorphism and $r: J(\ )' \to 1$ is the pseudo-natural transformation of Proposition 4.12. The first three factors lie in $[\mathscr{L}^{\mathrm{op}}, [T\text{-Alg}, \mathbf{Cat}]]$, but the fourth only in $[\mathscr{L}^{\mathrm{op}}, \mathrm{Hom}[T\text{-Alg}, \mathbf{Cat}]]$, since r lies only in $\mathrm{Hom}[T\text{-Alg}, T\text{-Alg}]$. Because t lies in $[\mathscr{L}^{\mathrm{op}}, [T\text{-Alg}, \mathbf{Cat}]]$ and is an equivalence in $\mathrm{Hom}[\mathscr{L}^{\mathrm{op}}, \mathrm{Hom}[T\text{-Alg}, \mathbf{Cat}]]$, and because the inclusions of the various 2-categories in question are *locally* fully faithful, the proposition will be proved if we show that $t\upsilon = 1$. We have the equations

$$\begin{aligned} t \cdot T\text{-Alg}(JH?, r-) &= \mathscr{L}(?, Gr-) \cdot t_{?, J(\ )'} \\ &= \mathscr{L}(?, Gr-) \cdot \mathscr{L}(s?, GJ(\ )'-) \cdot G_{JH?, J(\ )'-}, \end{aligned}$$

the first because t is 2-natural in both variables, and the second by the definition (5.1) of t. But the composite

$$\mathscr{L}(s?, GJ(\ )'-) \cdot G_{JH?, J(\ )'-} \cdot J_{H?,(\ )'-} \cdot v_{?,(\ )'-}$$

is the identity, by the analogue for the present adjunction of (4.3). It follows that $t\upsilon = \mathscr{L}(?, Gr-) \cdot \mathscr{L}(?, Gp-)$, which is indeed the identity by Proposition 4.12. □

Applying Theorem 5.1 to the case where G is the inclusion 2-functor $T\text{-Alg} \to T\text{-Alg}_l$, and recalling from Theorem 3.13 that the inclusion 2-functor $GJ: T\text{-Alg}_s \to T\text{-Alg}_l$ too has a left adjoint, we get

Corollary 5.4. *If* $(\ )^\dagger$ *is the left adjoint to the inclusion* $T\text{-Alg}_s \to T\text{-Alg}_l$, *the inclusion* $T\text{-Alg} \to T\text{-Alg}_l$ *has* $J(\ )^\dagger$ *as a left biadjoint; moreover every* $A^\dagger$ *is a flexible algebra.* □

Remark 5.5. In the special case (see Remark 3.15) where $T\text{-Alg}_s$ is $[\mathscr{P}, \mathbf{Cat}]$, the fact that each $A^\dagger$ is flexible is otherwise proved in Bird's thesis [4]. Bird further shows that the inclusion $T\text{-Alg} \to T\text{-Alg}_l$ (or $\mathrm{Psd}[\mathscr{P}, \mathbf{Cat}] \to \mathrm{Lax}[\mathscr{P}, \mathbf{Cat}]$ in his case) has in general no left adjoint. If it had a left adjoint S, we should have $A^\dagger \cong (SA)'$; but in fact a general $A^\dagger$ is not of the form B' – in particular, using the language of [23, Section 5], a lax limit is not in general a pseudo-limit. For a counter-example we take $\mathscr{P} = \mathbf{2}$, so that T is the 2-monad of Example 4.11 above. For $A = (1 \to 1)$, we see at once that $A^\dagger = (0 : 1 \to \mathbf{2})$. If this were $B' = (b : V \to W)'$, the first equation of (4.2), which forces V and W to be retracts of 1 and **2** respectively, leaves $(1 \to 1)$ and $(0 : 1 \to \mathbf{2})$ as the only possibilities for B; and neither of these satisfies $B' \cong A^\dagger$.

Applying Theorem 5.1 to the case where G is the forgetful 2-functor $U : T\text{-Alg} \to \mathscr{K}$, and recalling that $UJ = U_s : T\text{-Alg}_s \to \mathscr{K}$ has the left adjoint F_s with unit $i : 1 \to U_s F_s = T$, we get:

Corollary 5.6. *The forgetful* $U : T\text{-Alg} \to \mathscr{K}$ *has* JF_s *as a left biadjoint, with unit* $i : 1 \to UJF_s = T$; *moreover every free T-algebra* $TA = (TA, mA)$ *is flexible.* □

Remark 5.7. It follows from Theorem 4.7 and the flexibility of free algebras that every algebra-morphism $TA \to B$ is isomorphic to a strict one; this was proved otherwise in [18, Section 2.2].

We turn now to bicolimits in T-Alg, and refer the reader again to [23] in general and its Section 6 in particular. For a small 2-category $\mathscr{P}$, write $K : [\mathscr{P}, \mathbf{Cat}] \to \mathrm{Psd}[\mathscr{P}, \mathbf{Cat}]$ for the inclusion, $(\ )^\circ$ for its left adjoint given by Theorem 3.16, and $\boldsymbol{u} : 1 \to K(\ )^\circ$ for the unit – to avoid confusion with the adjunction $(\ )' \dashv J$ and its unit $\boldsymbol{p}$. (It is convenient to use bold-face letters not only, as we have done, to distinguish maps in T-Alg from those in $T\text{-Alg}_s$, but now too to distinguish maps in $\mathrm{Psd}[\mathscr{P}, \mathbf{Cat}]$ from those in $[\mathscr{P}, \mathbf{Cat}]$.) Recall from Theorem 3.8 that $T\text{-Alg}_s$ is cocomplete; we write as usual $F * G$ for the colimit of $G : \mathscr{P} \to T\text{-Alg}_s$ indexed by $F : \mathscr{P}^{\mathrm{op}} \to \mathbf{Cat}$, and we write $\upsilon_{F,G} : F \to T\text{-Alg}_s(G-, F * G)$ for its unit.

Theorem 5.8. *Let* $F : \mathscr{P} \to \mathbf{Cat}$ *and* $G : \mathscr{P} \to T\text{-Alg}$ *be 2-functors with* $\mathscr{P}$ *small, and write* $G' : \mathscr{P} \to T\text{-Alg}_s$ *for the composite* $(\ )'G$. *Then* $F^\circ * G'$, *together with the unit* $w_{F,G}$ *given by the composite*

$$F \xrightarrow[\boldsymbol{u}_F]{} F^\circ \xrightarrow[\upsilon_{F^\circ, G'}]{} T\text{-Alg}_s(G'-, F^\circ * G') \xrightarrow[\pi]{} T\text{-Alg}(G-, F^\circ * G') \quad (5.2)$$

where π *is the adjunction isomorphism of* (4.3), *provides an F-indexed bicolimit*

$F *_b G$ *of* G *in* T-Alg. *Accordingly* T-Alg *admits all bicolimits* $F *_b G$, *even those where* $\mathscr{P}$ *is a small bicategory and* F *and* G *merely homomorphisms of bicategories.*

Proof. The second assertion follows from the first by [23, Proposition 6.1]. By [23, Proposition 6.2], we establish the first assertion by producing a left biadjoint $- *_b G$ to the 2-functor $K\tilde{G}: T\text{-Alg} \to \mathrm{Psd}[\mathscr{P}^{\mathrm{op}}, \mathbf{Cat}]$, where $\tilde{G}: T\text{-Alg} \to [\mathscr{P}^{\mathrm{op}}, \mathbf{Cat}]$ is given by $\tilde{G}A = T\text{-Alg}(G-, A)$. However, the isomorphism $\pi: T\text{-Alg}_s((\ )'G-, A) \cong T\text{-Alg}(G-, JA)$ gives an isomorphism $(G')\tilde{} \cong \tilde{G}J: T\text{-Alg}_s \to [\mathscr{P}^{\mathrm{op}}, \mathbf{Cat}]$, so that $K\tilde{G}J \cong K(G')\tilde{}$ has the left adjoint $(- * G')(\ )^\circ = (-)^\circ * G'$; whence by Theorem 5.1 we have $J((-)^\circ * G')$ as a left biadjoint to $K\tilde{G}$, with the unit w given by (5.2). □

Remark 5.9. Recall from [23, Section 5] that $F^\circ * G'$ is the *pseudo-colimit* $F *_p G'$ in T-Alg$_s$; so that we can express the value of the bicolimit in T-Alg, when F and G are 2-functors, by

$$F *_b G = J(F *_p G'). \tag{5.3}$$

Remark 5.10. As the biadjunction of Theorem 5.1 has various special qualities, noted in Remark 5.2, so too has the bicolimit above. Not only is every map $k: F \to T\text{-Alg}(G-, A)$ isomorphic – as is required by the definition of bicolimit – to $T\text{-Alg}(G-, g)w$ for some $g: F *_b G \to A$ in T-Alg; it is actually *equal* to $T\text{-Alg}(G-, g)w$ for various g, exactly one of which is a strict morphism of algebras.

Remark 5.11. A little more can be said. Since $(G')\tilde{} \cong \tilde{G}J: T\text{-Alg}_s \to [\mathscr{P}^{\mathrm{op}}, \mathbf{Cat}]$ already has the left adjoint $- * G'$, so $\tilde{G}: T\text{-Alg} \to [\mathscr{P}^{\mathrm{op}}, \mathbf{Cat}]$ already has, by Theorem 5.1, the left biadjoint $J(- * G')$. What this implies, among other things, is that if the k in Remark 5.10 is 2-natural and not merely pseudo-natural, the unique strict $g: F *_b G = F *_p G' \to A$ there factorizes further through the canonical $F *_p G' \to F * G'$ in T-Alg$_s$ to give a unique strict $h: F * G' \to A$. The reader may find it interesting to work through this in the simple case $F = \Delta 1$ of conical bilimits.

We pass on now to the study of two 2-monads $T = (T, i, m)$ and $S = (S, j, n)$ on $\mathscr{K}$ and a strict map $\theta: S \to T$ of 2-monads. As we pointed out in the remarks preceding Theorem 3.9, there is a bijection $\theta \mapsto \theta^*$ between such maps θ and those 2-functors $T\text{-Alg}_s \to S\text{-Alg}_s$ which commute with the forgetful 2-functors to $\mathscr{K}$. Generalizing the nomenclature of Lawvere [31], where T and S are in effect finitary monads on **Set**, we may call such 2-functors *algebraic*. More often than not, θ is not given explicitly – indeed T and S may not be explicitly known – but its existence is inferred from that of an evident algebraic functor. Thus T-algebras and S-algebras might be, respectively, categories with finite limits and categories with finite products; or categories with finite products and symmetric monoidal categories; or strict monoidal categories and monoidal categories.

Still denoting by J the inclusion $T\text{-Alg}_s \to T\text{-Alg}$, we now write K for the inclu-

sion $S\text{-Alg}_s \to S\text{-Alg}$. The 2-functor θ^* extends to a 2-functor $\theta^\# : T\text{-Alg} \to S\text{-Alg}$, sending the algebra (A, a) to $(A, a \cdot \theta A)$, the morphism $f = (f, \bar{f}) : A \to B$ to $(f, \bar{f} \cdot \theta A)$, and the 2-cell $\alpha : f \to g$ to α. So $\theta^\# J = K\theta^*$, and clearly $\theta^\#$ commutes with the forgetful 2-functors from T-Alg and from S-Alg to $\mathcal{K}$.

Supposing now T and S each to have a rank, we retain $(\ )'$ for the left adjoint of J, with unit $p : 1 \to J(\ )'$, and write $(\ )^\circ$ for the left adjoint of K, with unit $u : 1 \to K(\ )^\circ$. We write $\theta_* : S\text{-Alg}_s \to T\text{-Alg}_s$ for the left adjoint of θ^* given by Theorem 3.9, with $z : 1 \to \theta^*\theta_*$ for its unit.

Theorem 5.12. *Let T and S be 2-monads with rank on a complete and cocomplete 2-category $\mathcal{K}$, and let $\theta : S \to T$ be a strict map of 2-monads, the rest of the notation being as above. Then $\theta^\# : T\text{-Alg} \to S\text{-Alg}$ has $\theta_\# = J\theta_*(\ )^\circ$ as a left biadjoint, with unit $w : 1 \to \theta^\#\theta_\#$ given by the composite*

$$1 \xrightarrow[u]{} K(\ )^\circ \xrightarrow[Kz(\ )^\circ]{} K\theta^*\theta_*(\ )^\circ = \theta^\# J\theta_*(\ )^\circ = \theta^\#\theta_\#,$$

whose component w_B at $B \in S\text{-Alg}$ is

$$B \xrightarrow[u_B]{} B^\circ \xrightarrow[z_{B^\circ}]{} \theta^*\theta_* B^\circ. \tag{5.4}$$

Proof. Immediate from Theorem 5.1 since $\theta^\# J = K\theta^*$ has the left adjoint $\theta_*(\ )^\circ$. □

Remark 5.13. If 1 is the identity 2-monad on $\mathcal{K}$, we have $1\text{-Alg}_s = 1\text{-Alg} = \mathcal{K}$. The forgetful $U_s : T\text{-Alg}_s \to \mathcal{K}$ is itself an algebraic functor, and the corresponding θ is $i : 1 \to T$, while $i^\#$ is $U : T\text{-Alg} \to \mathcal{K}$. Accordingly, Corollary 5.6 is a special case of Theorem 5.12.

Remark 5.14. In this special case where $S = 1$, the u_B of (5.4) is the identity, so that the biadjunction $F \dashv_b U$ and the adjunction $F_s \dashv U_s$ have the same unit i. There being flexibility in the choice of a unit for a birepresentation, the reader may well wonder whether $z_B : B \to \theta^*\theta_* B = \theta^\#\theta_* B$ is not itself a unit for a birepresentation of $S\text{-Alg}(B, \theta^\# -) : T\text{-Alg} \to \mathbf{Cat}$, in which case it could replace the more complicated w_B of (5.4). By [36, (1.11)] (see also [24]), this is so if and only if, for some equivalence $g : \theta_* B^\circ \to \theta_* B$ in T-Alg, the composite $\theta^\# g \cdot w_B$ is isomorphic to z_B. Now if $v : (\ )^\circ K \to 1$ is the counit of the adjunction $(\ )^\circ \dashv K$, the naturality of z gives $z_B v_B = \theta^*\theta_* v_B \cdot z_{B^\circ}$; and since $v_B u_B = 1$ by (4.2), we have $z_B = \theta^*\theta_* v_B \cdot w_B = \theta^\#\theta_* v_B \cdot w_B$. Accordingly, z_B is indeed a unit for the birepresentation whenever the map $\theta_* v_B$ of $T\text{-Alg}_s$ is an equivalence in T-Alg. This is certainly so when B is a semi-flexible S-algebra, so that v_B is already an equivalence in $S\text{-Alg}_s$; and we saw in Example 4.10 that there are non-trivial S – such as the 2-monad of Example 4.6 – for which every algebra is semi-flexible, although this is false in general. In fact it is not in general the case that z_B is a unit for the birepresentation, as the following considerations will show.

Let T-algebras and S-algebras be respectively strict monoidal categories and monoidal categories, θ^* being the inclusion. Here S-Alg is (see Subsection 6.1) the 2-category of monoidal categories, strong monoidal functors, and monoidal natural transformations; and T-Alg is the *full* (not just locally full) sub-2-category determined by the strict monoidal categories, $\theta^\#$ being the inclusion.

Proposition 5.15. *In this case of monoidal categories and strict monoidal categories, any unit* $\boldsymbol{x}: B \to \theta^* C$ *for a birepresentation of* $S\text{-Alg}(B, \theta^\# -): T\text{-Alg} \to \mathbf{Cat}$ *is an equivalence in* S-Alg.

Proof. It has long been known in the folklore that every monoidal category is equivalent in S-Alg to a strict one; there is a proof in [33]. Let $\boldsymbol{f}: B \to \theta^\# A$ be such an equivalence, with equivalence-inverse $\boldsymbol{g}$. Because $\boldsymbol{x}$ is a unit for the birepresentation, there is some $\boldsymbol{h}: C \to A$ in T-Alg with $\theta^\# \boldsymbol{h} \cdot \boldsymbol{x} \cong \boldsymbol{f}$. Since this gives $\boldsymbol{g} \cdot \theta^\# \boldsymbol{h} \cdot \boldsymbol{x} \cong \boldsymbol{g}\boldsymbol{f} \cong 1$, to prove $\boldsymbol{x}$ an equivalence it remains to show that $\boldsymbol{x}\boldsymbol{g} \cdot \theta^\# \boldsymbol{h} \cong 1$. Because $\theta^\#$ is full, $\boldsymbol{x}\boldsymbol{g}: \theta^\# A \to \theta^\# C$ has the form $\theta^\# \boldsymbol{k}$; and $\theta^\#(\boldsymbol{k}\boldsymbol{h}) \cdot \boldsymbol{x} = \boldsymbol{x}\boldsymbol{g} \cdot \theta^\# \boldsymbol{h} \cdot \boldsymbol{x} \cong \boldsymbol{x} = \theta^\# 1 \cdot \boldsymbol{x}$. The two-dimensional universal property of $\boldsymbol{x}$ now gives $\boldsymbol{k}\boldsymbol{h} \cong 1$, whence $\boldsymbol{x}\boldsymbol{g} \cdot \theta^\# \boldsymbol{h} = \theta^\#(\boldsymbol{k}\boldsymbol{h}) \cong 1$. □

Proposition 5.16. *It is not the case that to each monoidal category* B *we can assign a strict monoidal category* $\bar{B}$ *and a strict monoidal functor* $x_B: B \to \bar{B}$ *which is an equivalence of categories.*

Proof. Suppose the contrary. Let $B = (B, \otimes, I, a, l, r)$ be a monoidal category which admits a second monoidal structure $B_1 = (B, \otimes, I, a^1, l^1, r^1)$ with the same underlying category B, with the same $\otimes$ and I, but with say $a^1 \neq a$. Let $x_B: B \to \bar{B}$ be as in the statement of the proposition, so that $\bar{B} = (\bar{B}, \bar{\otimes}, \bar{I}, 1, 1, 1)$ is a strict monoidal category. Use the equivalence x to transport a^1, l^1, r^1 to $\bar{B}$, getting a new monoidal category $C = (\bar{B}, \bar{\otimes}, \bar{I}, a^2, l^2, r^2)$. (That is, if y is an equivalence-inverse of x with $\eta: 1 \cong xy$, we define the component $a^2_{b,c,d}: b \otimes c \otimes d \to b \otimes c \otimes d$ as $(\eta_b \otimes \eta_c \otimes \eta_d)^{-1}(xa^1_{yb,yc,yd})(\eta_b \otimes \eta_c \otimes \eta_d)$, and so on.) Now $a^2 \neq 1$ since $a^1 \neq a$. Yet the equivalence $x_C: C \to \bar{C}$ has $x(a^2) = 1$; and has $x(1) = 1$ simply because x is a functor. Since x, being an equivalence, is a faithful functor, this contradicts $a^2 \neq 1$. A suitable B for the argument is the category of graded abelian groups with its usual monoidal structure, and with the unusual structure given by $a^1((x \otimes y) \otimes z) = (-1)^{\dim x} x \otimes (y \otimes z)$, $l^1(1 \otimes y) = y$, and $r^1(x \otimes 1) = (-1)^{\dim x} x$; a simple check shows that the unusual structure does indeed satisfy the coherence axioms. □

So we have

Proposition 5.17. *In the situation of Theorem* 5.12, *there is in general no unit* $B \to \theta^\# C$ *for a birepresentation of* $S\text{-Alg}(B, \theta^\# -)$ *which is a strict map of S-algebras. Consequently,* $z_B: B \to \theta^* \theta_* B = \theta^\# \theta_* B$ *is not in general such a unit.* □

6. Some examples of 2-monads

6.1. We devote this final section to a largely-informal discussion of some representative examples of 2-monads, along with their algebras and the various morphisms of these, at greater length than would have been appropriate in the introduction, where a very few examples were mentioned in the last paragraph of Subsection 1.1; the reader should recall from that paragraph the 2-category $\mathbf{Cat}_g$ of small categories, functors, and natural *isomorphisms*, which is complete and cocomplete by [23, Proposition 3.1].

In practice one is seldom presented with a 2-monad and invited to consider its algebras; more commonly one contemplates some structure borne by a category, or by a family of categories, or by an object of a functor-category $[\mathscr{P}, \mathbf{Cat}]$, and so on, and one concludes in certain cases that the structure is given by an action of a 2-monad on whatever bears it.

Perhaps the simplest kind of a structure on a category A to recognize as 2-monadic is one given by a number of endomorphisms of A (say $t, s : A \to A$), a number of natural transformations between composites of these (say $\alpha : ts \to t$ and $\beta : ss \to s$), and a number of equations between composites of the latter (say $\alpha(t\beta) = \alpha(\alpha s) : tss \to t$). The most classic example is that of a monad on A; but we could equally well have a mere endofunctor, or a pointed endofunctor, or two monads with a distributive law between them, and so on. It is also possible to have equations at the level of the endofunctors: as for an idempotent monad, given by $t : A \to A$ with $t^2 = t$ and by $\eta : 1 \to t$ with $\eta t = t\eta = \mathrm{id}$. In each case there is a strict monoidal category M – which is the same thing as a monoid in **Cat** – generated by the *names* of the endofunctors and the natural transformations, subject to the equations. To give a structure of the prescribed type on a category A is to give a monoid-map $M \to [A, A]$, or equivalently an action $M \times A \to A$. Here the finitary 2-monad T on **Cat** is $M \times -$, and this of course determines what strict morphisms, morphisms, and lax morphisms of algebras should be, as well as the algebra-2-cells. In the case of a monad on A, the strict monoidal M is the simplicial category $\mathbf{\Delta}$; here the lax morphisms are precisely the *monad-functors* of Street [34], while the algebra-2-cells are his *monad-functor-transformations*.

Scarcely more complicated are such structures as a strict monoidal category, a monoidal category, or a symmetric monoidal category, where instead of endofunctors we now have functors of several variables such as $\otimes : A^2 \to A$ and $I : A^0 \to A$, but with natural transformations, like the associativity $\alpha : (a \otimes b) \otimes c \to a \otimes (b \otimes c)$ and the symmetry $\gamma : a \otimes b \to b \otimes a$, that do nothing more than permute the variables. Again there are equations – such as the coherence conditions – between composites of derived natural transformations; and possibly equations – as for a strict monoidal category – between iterates of the functors themselves. In [13] and [14], Kelly analyzed such situations in the following terms.

From categories A and B we form a category $\{A, B\}$, an object of which is a natural number n together with a functor $t : A^n \to B$; there are morphisms $(n, t) \to$

(m, s) only when $m = n$, and then a morphism is a permutation ξ of n together with a natural transformation $\alpha : tA^{\xi} \to s$. (The reader is unlikely to confuse this use of $\{\ ,\ \}$ with our earlier use for indexed limits in general and cotensor products in particular.) There is an evident *arities* functor $\{A, B\} \to \boldsymbol{P}$ where $\boldsymbol{P}$ is the category of natural numbers and permutations. The 2-functor $\{A, -\} : \mathbf{Cat} \to \mathbf{Cat}/\boldsymbol{P}$ has a left adjoint $- \circ A$; for an object $\Gamma : C \to \boldsymbol{P}$, or C for short, of $\mathbf{Cat}/\boldsymbol{P}$, an object of $C \circ A$ is an expression $c\langle a_1, \ldots, a_n\rangle$ where $c \in C$ with $\Gamma c = n$ and $a_i \in A$, while a map $c\langle a_1, \ldots, a_n\rangle \to d\langle b_1, \ldots, b_n\rangle$ is an expression $f\langle g_1, \ldots, g_n\rangle$ where $f : c \to d$ in C with $\Gamma f = \xi$ and $g_i : a_i \to b_{\xi i}$ in A. We may embed $\mathbf{Cat}$ in $\mathbf{Cat}/\boldsymbol{P}$ by identifying $A \in \mathbf{Cat}$ with the functor $A \to \boldsymbol{P}$ constant at 0; then the 2-functor $\circ : \mathbf{Cat}/\boldsymbol{P} \times \mathbf{Cat} \to \mathbf{Cat}$ extends in a simple way to a 2-functor $\circ : \mathbf{Cat}/\boldsymbol{P} \times \mathbf{Cat}/\boldsymbol{P} \to \mathbf{Cat}/\boldsymbol{P}$, which is immediately seen to be associative and to have as unit the functor $1 \to \boldsymbol{P}$ naming $1 \in \boldsymbol{P}$. Consequently $\mathbf{Cat}/\boldsymbol{P}$ is a monoidal 2-category which acts on $\mathbf{Cat}$, the tensor product and the action both being denoted by $\circ$. A $\circ$-monoid M in $\mathbf{Cat}/\boldsymbol{P}$ is called a *club*; each category A gives an endo-club $\{A, A\}$.

Consider now a structure on a category A of the kind discussed in the penultimate paragraph. It is shown in [14] that the names of the structural functors and the structural natural transformations, along with the equational axioms, provide generators and relations for a club M, and that to give such a structure on A is to give a monoid-map $M \to \{A, A\}$, or equivalently an action $M \circ A \to A$. Accordingly such A are the algebras for the finitary 2-monad $T = M \circ -$ on $\mathbf{Cat}$. It is now easy to see what the various algebra-morphisms are: for monoidal (symmetric monoidal) categories, strict or not, the lax morphisms are the monoidal functors (symmetric monoidal functors) of [10], and the morphisms are the strong such, while the algebra-2-cells are the monoidal natural transformations; for the details see [14, Section 7] and [16, Section 10.8].

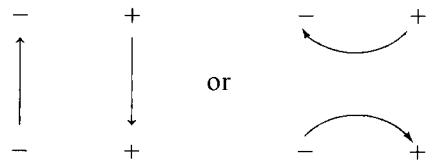
This theory of clubs is further extended in [13] and [14] to structures borne by a family of categories; so that for instance the structure given by two symmetric monoidal categories A and B and a symmetric monoidal functor $\Phi : A \to B$ is exhibited as an algebra for a finitary 2-monad on $\mathbf{Cat}^2$.

A different extension is possible, where $\boldsymbol{P}$ is replaced by the category $\boldsymbol{B}$ of natural numbers and braids. Thus Joyal and Street [12] exhibit *braid monoidal categories* as algebras for a 2-monad $M \circ -$ on $\mathbf{Cat}$, where M is a club in $\mathbf{Cat}/\boldsymbol{B}$.

6.2. Still in [13] and [14], augmented by [15], the notion of club was extended to cover certain cases where the structure involved functors, like an internal-hom, of mixed variances, although the natural transformations, now of the generalized kind introduced by Eilenberg and Kelly in [9], still linked the variables in pairs. $\boldsymbol{P}$ was replaced by $\boldsymbol{G}$, whose objects are finite strings of $+$'s and $-$'s indicating the variances, and whose morphisms are 'graphs' showing which variables are linked in the natural transformation. Because of the impossibility of composing incompatible natural transformations, $\mathbf{Cat}/\boldsymbol{P}$ was replaced, not by $\mathbf{Cat}/\boldsymbol{G}$, but by a full subcategory $\boldsymbol{L}$ of the latter. Here $\boldsymbol{L}$ was defined only as a monoidal category, not

a monoidal 2-category; and its action as a mere functor $\circ : L \times \mathbf{Cat}_\circ \to \mathbf{Cat}_\circ$, where $\mathbf{Cat}_\circ$ is the mere category underlying **Cat**. A closer examination shows that we can in fact take L to be **Gpd**-enriched, and $\circ$ to be a 2-functor $L \times \mathbf{Cat}_g \to \mathbf{Cat}_g$. With this apparatus, it is easy to show that monoidal right-closed categories, or monoidal biclosed categories, or symmetric monoidal closed categories, are the algebras for a finitary 2-monad $T = M \circ -$ on $\mathbf{Cat}_g$, where M – the corresponding club – is a monoid in L. Again there is an extension to structures borne by a family of categories; and the structure given by two symmetric monoidal closed categories A and B and a symmetric monoidal functor $\Phi : A \to B$ is exhibited as an algebra for a finitary 2-monad T on $\mathbf{Cat}_g^2$.

Our failure to exhibit symmetric monoidal closed categories as the algebras for a 2-monad on **Cat**, as distinct from one on $\mathbf{Cat}_g$, represents not just a lack of wit, but a mathematical fact. If M is the corresponding club, the underlying functor $\mathbf{Cat}_\circ \to \mathbf{Cat}_\circ$ of $M \circ -$ admits of *no* enrichment to a 2-functor $T : \mathbf{Cat} \to \mathbf{Cat}$; we cannot define T even on the unique 2-cell $\alpha : 0 \to 1 : 1 \to \mathbf{2}$. The proof depends on the partial coherence result for symmetric monoidal closed categories given by Kelly and Mac Lane [26], which – as is shown in [14] – is equally a partial determination of the club M, sufficient for our purposes. If we identify the category $M \circ 1$ with M, the functors $M \circ 0, M \circ 1 : M \to M \circ \mathbf{2}$ send $m \in M$ to $m\langle 0, \ldots, 0\rangle$ and $m\langle 1, \ldots, 1\rangle$ respectively; if $T\alpha$ could be defined, its m-component would be a map $f : m\langle 0, \ldots, 0\rangle \to m\langle 1, \ldots, 1\rangle$ in $M \circ \mathbf{2}$. Yet if we take $m = [\mathbf{1}, \mathbf{1}]$ where $\mathbf{1}$ is the formal identity of the club M and $[\ ,\ ]$ is the internal hom, there is just no map $f : [\mathbf{1}, \mathbf{1}]\langle 0, 0\rangle \to [\mathbf{1}, \mathbf{1}]\langle 1, 1\rangle$ in $M \circ \mathbf{2}$. For such a map is necessarily of the form $g\langle h, k\rangle$ where $g : [\mathbf{1}, \mathbf{1}] \to [\mathbf{1}, \mathbf{1}]$ in M and h, k are maps in $\mathbf{2}$. Since the arity of $[\mathbf{1}, \mathbf{1}]$ is $- +$, the 'graph' of g must be either



The first of these must be discarded, since there is no map $h : 1 \to 0$ in $\mathbf{2}$. By the algorithm given in [26, p. 129], under the heading "Proof of Theorem 2.1", there is no map g in M with the second graph above.

Algebra-morphisms between symmetric monoidal closed categories are not things that arise much in practice; they can be identified with strong monoidal functors f for which the $\hat{f} : f[a, b] \to [fa, fb]$, *induced* by the isomorphism $\tilde{f} : fa \otimes fb \to f(a \otimes b)$, is itself invertible. An algebra-2-cell is just an *invertible* monoidal natural transformation $\alpha : f \to g$; the axiom it must satisfy in relation to $\hat{f}$ and $\hat{g}$ is a *consequence* of the one it satisfies in relation to $\tilde{f}$ and $\tilde{g}$. Whatever their occurrence in practice, such algebra-morphisms are necessary to the general theory; and of course they, and in particular the strict ones, are essential to the coherence question "which diagrams commute?", which really comes to an explicit determination of the free

algebra $F_S A = TA$, and thus of the 2-monad T, or equivalently, in the present case, of the club M.

6.3. The club idea is not in fact a central one in the study of finitary 2-monads. For the simple kinds of structure to which it is applicable, it provides a concrete description $M \circ -$ of the 2-monad T, and makes it easy to see, in terms of the generators of M, just what the T-algebra-morphisms are. It does in fact extend just a little further, as observed in [13] and [14]; we can replace the $\mathbf{Cat}/\boldsymbol{P}$ of Subsection 6.1 by $\mathbf{Cat}/S$ or by $\mathbf{Cat}/S^{\mathrm{op}}$, where S is the category of natural numbers and functions (a skeleton of the category of finite sets). The first of these allows us to deal with such a structure as a category with finite coproducts where, in the structural natural transformations $a \to a+b$, $b \to a+b$, $a+a \to a$, $0 \to a$, the variables are no longer linked in pairs, but rather by a function from the variables of the domain to those of the codomain; the 'graphs' of the four natural transformations above are the two functions $1 \to 2$ in S and the unique functions $2 \to 1$ and $0 \to 1$. Another structure describable by a club in $\mathbf{Cat}/S$ is that of a category A with two symmetric monoidal structures $\otimes$ and $\oplus$ and a distributive law $\delta : (a \otimes b) \oplus (a \otimes c) \to a \otimes (b \oplus c)$, so long as we do not ask δ to be invertible: the graph of an inverse to δ would no longer be a function. (In spite of this restriction, the idea has proved useful – see [18] – in getting partial coherence results even when δ is invertible.) Categories with finite *products*, on the other hand, are algebras for a 2-monad on **Cat** given by a club not in $\mathbf{Cat}/S$ but in $\mathbf{Cat}/S^{\mathrm{op}}$.

When we come to a category with finite products *and* finite coproducts, or to a category with two symmetric monoidal structures and a distributive law that *is* an isomorphism, or to a category with finite *limits*, however, we can no longer express the 2-monad T as $M \circ -$ for some kind of club M; it is just no longer true that TA is fully determined by $M = T1$ and some functor Γ from M to some category of arities. (This will be pursued more technically in [25].) To recognize as monadic such structures as these, we first need a general analysis of 2-monads on **Cat** or on $\mathbf{Cat}_g$, and so on; especially of the finitary ones, which are the most important.

6.4. Such an analysis must await a later article [27] in this series, but the idea is the following. Let $\mathscr{V}$ be a locally-finitely-presentable closed category in the sense of [22], such as **Set**, **Gpd**, or **Cat**, and let $\mathscr{K}$ be a locally-finitely-presentable $\mathscr{V}$-category – important examples are $\mathbf{Cat}_\circ$, $\mathbf{Cat}_g$, and **Cat** corresponding to the closed categories above, but $\mathscr{K}$ might equally well be $\mathbf{Cat}^X$ or $[\mathscr{P}, \mathbf{Cat}]$. Using 'functor' to mean '$\mathscr{V}$-functor' and so on, we have the category $[\mathscr{K}, \mathscr{K}]$ of endofunctors of $\mathscr{K}$, which is a strict monoidal category with composition as its tensor product; the full subcategory given by the finitary endofunctors (those that preserve filtered colimits), being closed under composition, is again monoidal. But this subcategory is equivalent to $\mathscr{L} = [\mathscr{K}_f, \mathscr{K}]$, where $\mathscr{K}_f$ is the small full subcategory of $\mathscr{K}$ given by the finitely-presentable objects; for an endofunctor of $\mathscr{K}$ is finitary precisely when it is the left Kan extension of its restriction to $\mathscr{K}_f$. So $\mathscr{L}$ too is a monoidal category –

in fact a right-closed one – and the category $\mathscr{M}$ of monoids in $\mathscr{L}$ is in effect the category of finitary $\mathscr{V}$-monads on $\mathscr{K}$ and the strict maps of these. The forgetful functor $\mathscr{M} \to \mathscr{L}$ is itself finitary and monadic, so that $\mathscr{M} \cong R\text{-Alg}_s$ for some finitary $\mathscr{V}$-monad R on $\mathscr{L}$.

In a still later article, we shall apply the results of the present one, when $\mathscr{V} = \mathbf{Cat}$, to the 2-monad R; the 2-category R-Alg is that of finitary 2-monads and *pseudo-maps*; we have the left adjoint $(\ )' : R\text{-Alg} \to R\text{-Alg}_s$, so that pseudo-maps $T \to S$ correspond to strict maps $T' \to S$; in particular, pseudo-actions of T on A correspond to strict actions of T', so that pseudo-T-algebras are T'-algebras; there are the *flexible* 2-monads T for which the counit $T \to T'$ is a surjective equivalence in R-Alg$_s$, and sufficient conditions for such flexibility; and so on.

For our present purposes, however, we return to the previous paragraph and consider the composite forgetful functor $V : \mathscr{M} \to \mathscr{L} = [\mathscr{K}_f, \mathscr{K}] \to \mathscr{K}^X$, where X is the set of (isomorphism classes of) objects of $\mathscr{K}_f$ – that is, the set of finitely-presentable objects of $\mathscr{K}$. The conservative functor V has a left adjoint G, given by a simple explicit inductive process; and although V is not monadic, the counit of the adjunction is a regular epimorphism – whence every $T \in \mathscr{M}$ has a *presentation* as the coequalizer of a parallel pair of maps $GQ \to GP$ for some $Q, P \in \mathscr{K}^X$.

Given any such presentation of T, we have the following situation. It turns out that to give an action of the $\mathscr{V}$-monad GP on $A \in \mathscr{K}$ is to give, for each $x \in X$, a map $\theta_x : P_x \to \{\mathscr{K}(x, A), A\}$, where the codomain is the cotensor product of $\mathscr{K}(x, A) \in \mathscr{V}$ and $A \in \mathscr{K}$. We call P_x the $\mathscr{K}$-object of *basic operations of arity* x, while $(GP)_x$ is the $\mathscr{K}$-object of *derived operations of arity* x; in practice P_x – although not of course $(GP)_x$ – is often empty except for a few values of x. We call Q_x the $\mathscr{K}$-object of *equations of arity* x; the parallel pair $GQ \to GP$, or $Q \to VGP$, has (suppressing V) the components $\varphi_x, \psi_x : Q_x \to (GP)_x$; again Q_x is often empty for all but a few values of x. Now to give an action of T on A is to give maps θ_x as above whose extensions $\bar{\theta}_x : (GP)_x \to \{\mathscr{K}(x, A), A\}$ to the derived operations satisfy the equations $\bar{\theta}_x \varphi_x = \bar{\theta}_x \psi_x$. The upshot is that a structure on A is given by the action of a finitary $\mathscr{V}$-monad if and only if it can be presented in terms of basic operations as above, subjected to equations as above between derived operations.

The meaning of this last statement is explained more concretely by Dubuc and Kelly in [8], in the simple case where $\mathscr{V} = \mathbf{Set}$ and $\mathscr{K} = \mathbf{Cat}_o$. So armed, they were able to prove the monadicity of the forgetful functor to $\mathbf{Cat}_o$ from the *mere category* of structures and strict morphisms in a variety of cases: categories with finite limits or finite colimits or both; Cartesian-closed categories; locally-Cartesian-closed categories; elementary toposes or quasi-toposes; elementary toposes with a natural numbers object; and so on. A result of this kind does not suffice, of course, for our present purposes, in that it produces only a monad, not a 2-monad; but it is a first step – certainly a 2-functor $U : \mathscr{A} \to \mathbf{Cat}$ cannot be monadic unless its underlying functor $U_o : \mathscr{A}_o \to \mathbf{Cat}_o$ is so: the enriched situation will be addressed in [27]. By the way, *disproving* monadicity in a particular case may be difficult; Kelly gives in [19] some examples of structures borne by a category or a family of categories where

$U_\circ : \mathscr{A}_\circ \to \mathbf{Cat}_\circ^X$ is definitely not monadic, although the structures are certainly *essentially* algebraic – that is, definable by a finite-limit theory. (Note that the ideas adumbrated by Bénabou in [3] exhibit as essentially algebraic many structures, such as that of a regular category, which seem unlikely to be monadic.)

In [27] we shall give the details of the analysis above, for a general $\mathscr{V}$, but especially for $\mathscr{V} = \mathbf{Cat}$ or **Gpd**, distinguishing the kinds of operation that give a 2-monad on **Cat** or $\mathbf{Cat}^X$ (as is the case for a category with finite limits or colimits or both) from those that give only a 2-monad on $\mathbf{Cat}_g$ or $\mathbf{Cat}_g^X$ (as is the case for a Cartesian closed category), and describing the various algebra-morphisms in terms of the presentation of the theory.

We shall then have a further list of examples of algebras for a 2-monad on **Cat**, including the following: a category with two symmetric monoidal structures and an invertible distributive law (whose 2-monadicity, asserted on heuristic grounds in [18], was already established as part of a much wider result in Blackwell's unpublished thesis [6]); a pointed category (either with a zero object, or without – the two cases correspond to different 2-monads on **Cat**); an additive category (either with finite direct sums, or without); an abelian category.

6.5. The reader will have noticed that in many of our examples – such as a category with finite limits – a T-algebra structure on $A \in \mathscr{K}$, if any such exists, is unique to within isomorphism; while in many other examples – such as a symmetric monoidal category – nothing of the sort is true. In the first case, to admit such a structure is but a *property* of A; yet this is of course no reason to exclude such examples from the general theory. Such of our results as the existence in T-Alg of various limits and of bicolimits are equally valid and equally important for both kinds of example, and are trivial for neither. When it comes to the biadjunction $\theta_\# \dashv_b \theta^\# : T\text{-Alg} \to S\text{-Alg}$ associated to an algebraic functor $\theta^* : T\text{-Alg}_s \to S\text{-Alg}_s$ and hence to a strict map $\theta : S \to T$ of 2-monads, the two kinds become inextricably mixed. For instance, if T-algebras are categories with finite products and S-algebras are symmetric monoidal categories, there is an evident algebraic functor θ^* sending A to $(A, \times, 1)$, and consequently a map $\theta : S \to T$ and a left biadjoint $\theta_\#$ to $\theta^\#$. (The reader should note that this $\theta_\#$ admits of no simple explicit description, and observe – perhaps with some surprise – that what Fox provides in [11] is a *right* biadjoint to this $\theta^\#$, sending a symmetric monoidal A to the category of its commutative coalgebras.) Perhaps the earliest explicit description in the literature of such a left biadjoint $\theta_\#$ is that of Adelman [1], in the case where T-algebras are abelian categories and S-algebras are additive categories with finite direct sums.

For the moment, at least, we do not know how to distinguish, in terms of a presentation of T by operations and equations, those cases in which a T-algebra structure is essentially unique; this may well be a hard problem. Accordingly we have no precise theorem about algebra-morphisms in such cases. It would seem, however, from practical experience with a number of examples, that in these cases an algebra-morphism $f : (A, a) \to (B, b)$ is not a map $f : A \to B$ in $\mathscr{K}$ along with *extra*

data determining the $\bar{f}$ of Subsection 1.2 – such as the $\bar{f}: fa \bar{\otimes} fb \to f(a \otimes b)$ and the $f^\circ : \bar{I} \to fI$ for monoidal categories – but rather just a map $f: A \to B$ in $\mathscr{K}$ with certain properties; and that an algebra-2-cell $\alpha: f \to g$ is then any 2-cell $\alpha: f \to g$ in $\mathscr{K}$. We now expand a little on this.

There is a notion of *quasi-idempotent* 2-monad, first recognized and studied by Kock [30], with later contributions from Street [35, 36] and Zöberlein [37, 38]; the motivating example for Kock was that where the algebras are categories with limits of some specified size; Zöberlein's first ideas thereon in his thesis [37] seem to have been independent of Kock's; Street has called such 2-monads *Kock–Zöberlein monads*. The 2-monad T on $\mathscr{K}$ is *quasi-idempotent* if $m: T^2 \to T$ is right adjoint in the functor-2-category $[\mathscr{K}, \mathscr{K}]$ to iT, with identity unit. For such a T, any action $a: TA \to A$ is right adjoint to $iA: A \to TA$, again with identity unit. Street shows in [35, Proposition 4] that, for T-algebras A and B, any $f: A \to B$ in $\mathscr{K}$ extends to a unique colax morphism $(f, \hat{f}): A \to B$ of algebras, where $\hat{f}: fa \to b \cdot Tf$ is the mate under the adjunctions of the identity $iB \cdot f = Tf \cdot iA$. If now $f = (f, \bar{f}): A \to B$ is a morphism of algebras, $(f, \bar{f}^{-1})$ is a colax morphism, so that $\bar{f}^{-1}$ must be the unique $\hat{f}$ above. Accordingly an algebra-morphism $f: A \to B$ may be identified with a map $f: A \to B$ in $\mathscr{K}$ whose corresponding $\hat{f}$ is invertible. It further follows that if $f, g: A \to B$ are two such algebra-morphisms, every 2-cell $\alpha: f \to g$ in $\mathscr{K}$ is an algebra-2-cell.

When a T-algebra is a category with limits of some given small class, T is a quasi-idempotent 2-monad on **Cat**; and for a functor $f: A \to B$ between T-algebras, the unique $\hat{f}$ above has for its components the canonical comparison-maps $f(\lim x_i) \to \lim f(x_i)$ for the various limits involved: so that a T-algebra morphism is a functor f that preserves these limits, in the usual sense that the canonical comparison is invertible; while all 2-cells between such morphisms are algebra-2-cells. The proof of this in the general case must of course await the analysis in [27]; but when the given class of limits consists only of finite products, or perhaps only of a terminal object, we have so concrete a description of T, in terms of a club in $\mathbf{Cat}/S^{\mathrm{op}}$ in the language of Subsection 6.3 above, that the verification of these assertions is trivial. The extremely simple terminal-object case serves as well as **Lex** for the counter-examples in Subsection 1.3, which show T-Alg to lack equalizers and an initial object in general.

The 2-monad T on **Cat** whose algebras are categories with *colimits* of a given class is *co-quasi-idempotent*; that is, T^{co} is a quasi-idempotent 2-monad on $\mathbf{Cat}^{\mathrm{co}}$. Now the comparison $\operatorname{colim} f(x_i) \to f(\operatorname{colim} x_i)$ has the opposite sense; for T-algebras A and B, every $f: A \to B$ in $\mathscr{K}$ underlies a unique lax morphism $f: A \to B$, which is a morphism precisely when the comparison above is invertible.

When a T-algebra is a category with *both* finite limits and finite colimits, the 2-monad T on **Cat** is neither quasi-idempotent nor co-quasi-idempotent. Of course T is the coproduct, in the 2-category $\mathscr{M}$ of finitary 2-monads on **Cat**, of the 2-monad P for finite limits and the 2-monad Q for finite colimits; whence it follows easily that an algebra-morphism is a functor preserving both the limits and the colimits.

Cartesian closed categories, like the symmetric monoidal closed categories in Subsection 6.2, are the algebras for a 2-monad T not on **Cat** but only on $\mathbf{Cat}_g$. The algebra-morphisms turn out to be those product-preserving functors for which the induced map $f[a,b] \to [fa,fb]$ of internal homs is itself invertible; and any 2-cell in $\mathbf{Cat}_g$ between algebra-morphisms – which is to say, any *invertible* 2-cell between them in **Cat** – is an algebra-2-cell. We have not yet checked the details for locally-Cartesian-closed categories and for elementary toposes; but we have no doubt that the algebra-morphisms are once again just those functors that, in the natural sense, preserve the structure.

6.6. Our final example is that foreshadowed in Remark 3.15. Write X for the set of objects of the small 2-category $\mathscr{P}$ and $H: X \to \mathscr{P}$ for the inclusion, treating X as a discrete 2-category; and let $\mathscr{L}$ be a cocomplete 2-category. The 2-functor $U_s = [H, \mathscr{L}]: [\mathscr{P}, \mathscr{L}] \to [X, \mathscr{L}] = \mathscr{L}^X$ sends a 2-functor $A: \mathscr{P} \to \mathscr{L}$ to the family $(AP \mid P \in X)$ of objects of $\mathscr{L}$, sends a 2-natural $f: A \to B$ to the family (f_P) of its components, and sends a modification $\alpha: f \to g$ to the family (α_P). Since $\mathscr{L}$ is cocomplete, the left Kan extension Lan_H provides a left adjoint F_s to U_s, and so a 2-monad T on $\mathscr{L}^X$. Colimits in $[\mathscr{P}, \mathscr{L}]$ being formed pointwise, they are created by U_s; hence U_s is monadic, and $T\text{-Alg}_s$ is isomorphic to, and may be identified with, $[\mathscr{P}, \mathscr{L}]$. Since U_s preserves all colimits, so does T; in particular, the 2-monad T is finitary. We claim that T-Alg and $T\text{-Alg}_l$ are respectively $\mathrm{Psd}[\mathscr{P}, \mathscr{L}]$ and $\mathrm{Lax}[\mathscr{P}, \mathscr{L}]$ in the sense of [23, Section 5]; it suffices of course to deal with the lax case, the pseudo case then following by restricting to invertible $\bar{f}$.

X being discrete as a 2-category, Lan_H is particularly simple to describe, and we find at once that $T: \mathscr{L}^X \to \mathscr{L}^X$ is given by

$$(TA)Q = \sum_{P \in X} \mathscr{P}(P, Q) * AP \tag{6.1}$$

for an object $A = (AP \mid P \in X)$ of $\mathscr{L}^X$, with a corresponding formula for arrows and 2-cells; here $*$ is the tensor product of the category $\mathscr{P}(P, Q)$ and the object AP of $\mathscr{L}$. Observe that

$$\begin{aligned} (T^2A)R &= \sum_Q \mathscr{P}(Q, R) * (TA)Q \\ &= \sum_Q \mathscr{P}(Q, R) * \left(\sum_P \mathscr{P}(P, Q) * AP \right) \\ &\cong \sum_{P,Q} (\mathscr{P}(Q, R) \times \mathscr{P}(P, Q)) * AP. \end{aligned} \tag{6.2}$$

It follows easily that the Q-component of $iA: A \to TA$ arises from (6.1) and the identity-map $1 \to \mathscr{P}(Q, Q)$, while the R-component of $mA: T^2A \to TA$ arises from (6.2) and (6.1) and the composition-map $\mathscr{P}(Q, R) \times \mathscr{P}(P, Q) \to \mathscr{P}(P, R)$.

To give an $a: TA \to A$ is to give components $a_Q: (TA)Q = \sum_P \mathscr{P}(P, Q) * AP \to AQ$, and hence components $a_{PQ}: \mathscr{P}(P, Q) * AP \to AQ$, which correspond to components $A_{PQ}: \mathscr{P}(P, Q) \to \mathscr{L}(AP, AQ)$ in **Cat**. The associativity and identity axioms for an

action a assert precisely that the AP and the A_{PQ} constitute a 2-functor $A: \mathscr{P} \to \mathscr{L}$; in agreement with our observation above that $T\text{-Alg}_s = [\mathscr{P}, \mathscr{L}]$.

Consider now a lax morphism $(f, \bar{f}): A \to B$ of T-algebras as in (1.1). To give f is just to give a family $(f_P: AP \to BP)$ of arrows in $\mathscr{L}$; to give $\bar{f}$ is to give a family $(\bar{f}_P)$ of 2-cells in $\mathscr{L}$ of the form

$$\begin{array}{ccc} \sum\limits_P \mathscr{P}(P,Q) * AP & \xrightarrow{\sum\limits_P \mathscr{P}(P,Q) * f_P} & \sum\limits_P \mathscr{P}(P,Q) * BP \\ \Big\downarrow{\scriptstyle a_Q} & \Downarrow \bar{f}_P & \Big\downarrow{\scriptstyle b_Q} \\ AQ & \xrightarrow[f_Q]{} & BQ \end{array}$$

These in turn have components $\bar{f}_{PQ}: b_{PQ}(\mathscr{P}(P,Q) * f_P) \to f_Q a_{PQ}$, which correspond to natural transformations $\tilde{f}_{PQ}$ as in

$$\begin{array}{ccc} \mathscr{P}(P,Q) & \xrightarrow{B_{PQ}} & \mathscr{L}(BP, BQ) \\ \Big\downarrow{\scriptstyle A_{PQ}} & \Downarrow \tilde{f}_{PQ} & \Big\downarrow{\scriptstyle \mathscr{L}(f_P, BQ)} \\ \mathscr{L}(AP, AQ) & \xrightarrow[\mathscr{L}(AP, f_Q)]{} & \mathscr{L}(AP, BQ) \end{array}$$

If we now write $f_\varphi: B\varphi \cdot f_P \to f_Q \cdot A\varphi$ for the component of $\tilde{f}_{PQ}$ at $\varphi \in \mathscr{P}(P,Q)$, the f_P and the f_φ constitute the data for a lax natural transformation $f: A \to B$. One of the axioms for a lax natural transformation – the compatibility of f_φ and f_ψ with $A\beta$ and $B\beta$ for a 2-cell $\beta: \varphi \to \psi$ – is exactly the naturality of the $\bar{f}_{PQ}$; the other two axioms, that $f_{\theta\varphi}$ is the pasting-composite of f_θ and f_φ, and that f_φ is the identity when $\varphi = 1_A$, are exactly (1.2) and (1.3). Finally, the axiom (1.4) for an algebra-2-cell $\alpha: f \to g$ is exactly the assertion that α is a modification.

References

[1] M. Adelman, Abelian categories over additive ones, J. Pure Appl. Algebra 3 (1973) 103–117.

[2] J. Bénabou, Introduction to bicategories, Lecture Notes in Mathematics 47 (Springer, Berlin, 1967) 1–77.

[3] J. Bénabou, Théories relatives à un corpus, C.R. Acad. Sci. Paris (Sér. A) 281 (1975) 831–834.

[4] G.J. Bird, Limits of locally-presentable categories, Ph.D. Thesis, University of Sydney, 1984.

[5] G.J. Bird, G.M. Kelly, A.J. Power and R. Street, A good class of limits for 2-categories, J. Pure Appl. Algebra, to appear.

[6] R. Blackwell, Some existence theorems in the theory of doctrines, Ph.D. Thesis, University of New South Wales, 1976.

[7] E.J. Dubuc, Kan Extensions in Enriched Category-Theory, Lecture Notes in Mathematics 145 (Springer, Berlin, 1970).

[8] E.J. Dubuc and G.M. Kelly, A presentation of topoi as algebraic relative to categories or graphs, J. Algebra 81 (1983) 420–433.

[9] S. Eilenberg and G.M. Kelly, A generalization of the functorial calculus, J. Algebra 3 (1966) 366–375.

[10] S. Eilenberg and G.M. Kelly, Closed categories, in: Proc. Conf. on Categorical Algebra, La Jolla, 1965 (Springer, Berlin, 1966) 421–562.
[11] T. Fox, Coalgebras and cartesian categories, Comm. Algebra 4 (1976) 665–667.
[12] A. Joyal and R. Street, Braided tensor categories, in preparation.
[13] G.M. Kelly, Many-variable functorial calculus I, Lecture Notes in Mathematics 281 (Springer, Berlin, 1972) 66–105.
[14] G.M. Kelly, An abstract approach to coherence, Lecture Notes in Mathematics 281 (Springer, Berlin, 1972) 106–147.
[15] G.M. Kelly, A cut-elimination theorem, Lecture Notes in Mathematics 281 (Springer, Berlin, 1972) 196–213.
[16] G.M. Kelly, On clubs and doctrines, Lecture Notes in Mathematics 420 (Springer, Berlin, 1974) 181–256.
[17] G.M. Kelly, Doctrinal adjunction, Lecture Notes in Mathematics 420 (Springer, Berlin, 1974) 257–280.
[18] G.M. Kelly, Coherence conditions for lax algebras and for distributive laws, Lecture Notes in Mathematics 420 (Springer, Berlin, 1974) 281–375.
[19] G.M. Kelly, Examples of non-monadic structures on categories, J. Pure Appl. Algebra 18 (1980) 59–66.
[20] G.M. Kelly, A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on, Bull. Austral. Math. Soc. 22 (1980) 1–83.
[21] G.M. Kelly, Basic Concepts of Enriched Category Theory, London Mathematical Society Lecture Notes Series 64 (Cambridge University Press, Cambridge, 1982).
[22] G.M. Kelly, Structures defined by finite limits in the enriched context I, Cahiers Topologie Géom. Différentielle 23 (1982) 3–42.
[23] G.M. Kelly, Elementary observations on 2-categorical limits, Bull. Austral. Math. Soc. 39 (1989) 301–317.
[24] G.M. Kelly, Equivalences in 2-categories, birepresentations, and biadjoints, in preparation.
[25] G.M. Kelly, On the abstract notion of club, in preparation.
[26] G.M. Kelly and S. Mac Lane, Coherence in closed categories, J. Pure Appl. Algebra 1 (1971) 97–140; erratum, *ibid.* 1 (1971) 219.
[27] G.M. Kelly and J. Power, On finitary enriched monads and their presentations, in preparation.
[28] G.M. Kelly and R. Street, Review of the elements of 2-categories, Lecture Notes in Mathematics 420 (Springer, Berlin, 1974) 75–103.
[29] A. Kock, Closed categories generated by commutative monads, J. Austral. Math. Soc. 12 (1971) 405–424.
[30] A. Kock, Monads for which structures are adjoint to units, Aarhus Univ. Preprint Series 1972/73, No. 35.
[31] F.W. Lawvere, Functorial semantics of algebraic theories, Proc. Nat. Acad. Sci. U.S.A. 50 (1963) 869–872.
[32] F.E.J. Linton, Relative functorial semantics: adjointness results, Lecture Notes in Mathematics 99 (Springer, Berlin, 1969) 384–418.
[33] S. Mac Lane and R. Paré, Coherence for bicategories and indexed categories, J. Pure Appl. Algebra 37 (1985) 59–80.
[34] R. Street, The formal theory of monads, J. Pure Appl. Algebra 2 (1972) 149–168.
[35] R. Street, Fibrations and Yoneda's lemma in a 2-category, Lecture Notes in Mathematics 420 (Springer, Berlin, 1974) 104–133.
[36] R. Street, Fibrations in bicategories, Cahiers Topologie Géom. Différentielle 21 (1980) 111–160.
[37] V. Zöberlein, Doktrinen auf 2-Kategorien, Inaugural-Dissertation, Universität Düsseldorf, 1974.
[38] V. Zöberlein, Doctrines on 2-categories, Math. Z. 148 (1976) 267–279.