JOURNAL OF ALGEBRA 49, 238–260 (1977)

Rings, Fields, and Spectra

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Communicated by Saunders MacLane

Received September 9, 1976

1. INTRODUCTION

Let **T** be a finitary algebraic theory [15] (or equivalently a variety of universal algebras [1]). It is well known that if $\mathscr{E}$ is any category with finite products, we may define the notion of "**T**-model in $\mathscr{E}$" by interpreting each m-ary operation of **T** as a morphism $A^m \to A$ (A being the underlying object of the **T**-model), and each equation of **T** as a commutative diagram. However, if we wish to impose additional (nonequational) first-order axioms on a **T**-model, we must demand that $\mathscr{E}$ have certain additional structure: for example, that of a regular category [22] or of a logical category [25]. Throughout this paper, we shall assume that $\mathscr{E}$ is a topos (in the sense of Lawvere and Tierney, see [9] or [27], though readers unfamiliar with this notion of topos may substitute that of "Grothendieck topos" [5] without serious damage). We shall also require that $\mathscr{E}$ have a natural number object; it is by now well known that this assumption implies the existence of a free **T**-model functor for any finitely presented algebraic theory **T** [10, 17].

The axioms of a topos are certainly sufficient to permit the interpretation of arbitrary first-order formulas (and indeed of higher-order formulas) (cf. [18, 19]). However, we shall find it convenient to distinguish certain formulas called *geometric formulas*, which are built up from the atomic formulas by the use of the connectives $\wedge$, $\vee$ and $\exists$ (but not $\Rightarrow$, $\neg$, or $\forall$). A *geometric sequent* is a formula of the form $(\varphi \Rightarrow \psi)$, where φ and ψ are geometric formulas; and a *geometric theory* is a pair $\mathbf{T} = (\mathbf{L}, A)$ consisting of a language[1] **L** and a set A of geometric sequents of **L**, called *axioms* of **T**. (By convention, we also include the vacuous sentences *true* and *false* as geometric formulas; this enables us to replace any geometric formula φ by the sequent $(\mathit{true} \Rightarrow \varphi)$, and its negation by the sequent $(\varphi \Rightarrow \mathit{false})$,

[1] By "language" we mean a (possibly) many-sorted first-order language with equality, having a number of primitive (finitary) function symbols and predicates. The special case of an algebraic theory occurs when the language has one sort and no primitive predicates except equality, and all the axioms have the form $(\mathit{true} \Rightarrow \varphi)$ where φ is atomic.

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ISSN 0021-8693

and hence to include geometric formulas and their negations amongst the axioms of a geometric theory.)

The reason for singling out these formulas is that any exact functor $T: \mathscr{E} \to \mathscr{F}$ between toposes (in particular the inverse image of any geometric morphism) preserves the interpretations of $\wedge$, $\vee$, and $\exists$, and hence of any geometric formula; so it preserves the truth of any geometric sequent. Moreover, if T is faithful then it reflects the truth of geometric sequents; thus if $\mathscr{E}$ is the topos of sheaves on a topological space X, a sheaf A satisfies a particular geometric sequent iff its stalk at each point of X satisfies the corresponding sequent in the topos $\mathscr{S}$ of constant sets.

(Strictly speaking, what we have defined here are *finitary* geometric formulas, sequents, and theories. Inverse image functors also preserve infinite colimits whenever they exist; so when we are working with toposes in which infinite colimits exist (e.g., Grothendieck toposes), we shall extend the notion of geometric formula by allowing infinite disjunctions of formulas, provided that the total number of *free* variables in any formula remains finite.)

The reader should be warned that a formula whose truth is preserved by arbitrary exact functors need not be equivalent to a geometric sequent. An important example is the axiom of *decidability* $((\mathbf{a} = \mathbf{a}') \vee \neg(\mathbf{a} = \mathbf{a}'))$ which says that the diagonal subobject $A \rightarrowtail A \times A$ has a complement; this is clearly preserved by exact functors, but it is not reflected by exact faithful functors. In order to make decidability geometric, we must introduce a new binary predicate R and the axioms $(\mathit{true} \Rightarrow ((\mathbf{a} = \mathbf{a}') \vee R(\mathbf{a}, \mathbf{a}')))$ and $(((\mathbf{a} = \mathbf{a}') \wedge R(\mathbf{a}, \mathbf{a}')) \Rightarrow \mathit{false})$; while this does not change the notion of model of the theory under discussion, it does change the notion of homomorphism, since these are now required to preserve the relation R and hence to be monomorphisms. Thus the theory of decidable gadgets is not normally a geometric quotient of the theory of gadgets.

We shall also be interested in the more restricted class of formulas which are preserved by arbitrary left exact functors (for example, the direct images of geometric morphisms). As usual, we define a *Horn formula* to be either *true* or a finite conjunction of atomic formulas; Horn sequents and Horn theories are similarly defined (compare [11]). However, it has been pointed out by Coste [4] that the class of formulas we want is slightly larger than this, in that it is permissible to use an existential quantifier provided we already know that the thing being quantified is unique. We thus define a $\varinjlim$-*theory* to be one defined by sequents constructed using $\wedge$ and $\exists$, with the proviso that whenever a formula of the form $\exists\, \mathbf{a} \cdot \varphi$ appears, the sequent $(\varphi \wedge \varphi[\mathbf{a}'/\mathbf{a}]) \Rightarrow (\mathbf{a} = \mathbf{a}')$ must be deducible from the previous axioms. We shall also talk about $\varinjlim$-formulas and $\varinjlim$-sequents relative to a particular (algebraic or Horn) theory; thus the formula $\exists \mathbf{a}'(\mathbf{a}\mathbf{a}' = 1)$, which expresses the fact that $\mathbf{a}$ is a unit, is a $\varinjlim$-formula in the theory of commutative rings, since the multiplicative inverse of an element is unique if it exists.

The use of the word "deducible" in the previous paragraph reminds us that we should ideally set up a formal deduction system for first-order formulas in a topos. For reasons of space, we refrain from doing so explicitly here; a full account will be found in [3] or [19], but all the reader really needs to know is that the deduction system is essentially that of the intuitionistic predicate calculus, with certain restrictions on free variables in the use of *modus ponens*, which derive from the fact that objects in a topos need not have global support (see [19, p. 313]). Since the types of the variables which we shall encounter (being the underlying objects of rings) will always have global support, we shall not have to worry about this restriction.

In this paper we shall be concerned with the algebraic theory of (commutative, unitary) rings, and with some of its geometric quotients. Our basic aim is to study the relationship between the syntax of these theories and their models in a general topos, by means of the (essentially syntactic) construction of *generic models* (cf. [6, 10, 22, 24]). In this sense the paper is a particular instance of the "functorial model theory" described by Lawvere in [16, Introduction]; the methods used are by now fairly familiar to most workers in topos theory, but I hope that this paper may serve to introduce them also to algebraists and logicians, who should find them worthy of further study. Many of the ideas in this paper have been developed in conversations with Julian Cole, John Kennison, Anders Kock, Chris Mulvey, and Myles Tierney; I owe them many thanks for their help.

2. Rings and Fields in a Topos

Let **ann** denote the theory of commutative rings with 1. We wish to consider the various additional axioms which we may impose on a model of **ann** to express the fact that it is a field, a local ring, or an integral ring (domain). It was first emphasized by Mulvey [18] that, because the internal logic of a topos is intuitionistic, each of these concepts has several (classically equivalent) definitions which have different interpretations in a general topos. In this section, we tabulate some of these definitions and discuss the implications between them.

All the rings that we consider here will satisfy the (geometric) axiom of *nontriviality* ($(0 = 1) \Rightarrow$ *false*). If A is a ring, we write $U(A) \rightarrowtail A$ for the *object of units*, i.e., the interpretation of the formula $\exists \mathbf{a}'(\mathbf{a}\mathbf{a}' = 1)$; we thus write $\mathbf{a} \in U(A)$ (or simply $\mathbf{a} \in U$) as shorthand for the above formula.

Mulvey [18] considered the following three forms of the field axiom:

$$\text{F1} \quad (\mathbf{a} = 0) \vee (\mathbf{a} \in U);$$

$$\text{F2} \quad \neg(\mathbf{a} = 0) \Rightarrow (\mathbf{a} \in U);$$

$$\text{F3} \quad \neg(\mathbf{a} \in U) \Rightarrow (\mathbf{a} = 0).$$

Of these, F1 is a geometric formula; we shall therefore call a nontrivial ring satisfying F1 a *geometric field*. However, it is found to be unpleasantly restrictive in practice; in particular, from F1 and $\neg(0 = 1)$ we may deduce $((\mathbf{a} = 0) \vee \neg(\mathbf{a} = 0))$, and hence (since $(\mathbf{a} = \mathbf{a}') \Leftrightarrow (\mathbf{a} - \mathbf{a}' = 0)$) any geometric field is decidable. Since many interesting "field-like" objects in a topos (in particular the object of Dedekind real numbers) are not normally decidable, it becomes interesting to examine the two weaker field axioms F2 and F3.

We shall say that a ring A is *U-decidable* if the subobject $U(A) \rightarrowtail A$ has a complement (i.e., if the axiom $((\mathbf{a} \in U) \vee \neg(\mathbf{a} \in U))$ is satisfied).

LEMMA 2.1. *Let A be a nontrivial ring. Then the following are equivalent:*

(i) *A satisfies F1.*

(ii) *A is decidable and satisfies F2.*

(iii) *A is U-decidable and satisfies F3.*

In particular, F1 implies both F2 and F3.

Proof. We have already observed that F1 implies decidability. To show that it implies F2, assume $\neg(\mathbf{a} = 0)$. Then we have $(\neg(\mathbf{a} = 0) \wedge (\mathbf{a} = 0)) \vee (\neg(\mathbf{a} = 0) \wedge (\mathbf{a} \in U))$; but the first half of this disjunction is contradictory, and so $\neg(\mathbf{a} = 0) \wedge (\mathbf{a} \in U)$, from which we deduce $\mathbf{a} \in U$. Conversely, from $((\mathbf{a} = 0) \vee \neg(\mathbf{a} = 0))$ and $(\neg(\mathbf{a} = 0) \Rightarrow (\mathbf{a} \in U))$ we may immediately deduce F1.

The equivalence of (i) and (iii) is similar. ∎

For the notion of integral ring, we again have three forms of the axiom:

$$\text{I1} \quad (\mathbf{a}\mathbf{a}' = 0) \Rightarrow ((\mathbf{a} = 0) \vee (\mathbf{a}' = 0));$$

$$\text{I2} \quad ((\mathbf{a}\mathbf{a}' = 0) \wedge \neg(\mathbf{a} = 0)) \Rightarrow (\mathbf{a}' = 0);$$

$$\text{I3} \quad (\neg(\mathbf{a} = 0) \wedge \neg(\mathbf{a}' = 0)) \Rightarrow \neg(\mathbf{a}\mathbf{a}' = 0).$$

LEMMA 2.2. *I1 implies I2 implies I3. F1 implies I1. Both F2 and F3 imply I2.*

Proof. The first two implications are tautologies of the Heyting propositional calculus. To prove the third, assume $\mathbf{a}\mathbf{a}' = 0$; then from F1 we deduce $(\mathbf{a} = 0) \vee ((\mathbf{a} \in U) \wedge (\mathbf{a}\mathbf{a}' = 0))$, and from the second half of this disjunction we deduce $\mathbf{a}' = 0$. So F1 implies I1.

Now from $((\mathbf{a}\mathbf{a}' = 0) \wedge \neg(\mathbf{a} = 0))$ and F2 we deduce $((\mathbf{a}\mathbf{a}' = 0) \wedge (\mathbf{a} \in U))$, from which $\mathbf{a}' = 0$ follows easily. Similarly, from $((\mathbf{a}\mathbf{a}' = 0) \wedge \neg(\mathbf{a} = 0))$ and $\mathbf{a}' \in U$ we obtain a contradiction, so $((\mathbf{a}\mathbf{a}' = 0) \wedge \neg(\mathbf{a} = 0))$ implies $\neg(\mathbf{a}' \in U)$, whence by F3 we deduce $\mathbf{a}' = 0$. So F3 implies I2. ∎

In [18], Mulvey took I2 as the definition of an integral domain, because he wished to make the assertion that the rings satisfying F2 are precisely the fields of

fractions of integral domains (cf. 2.4 below). However, it seems that in many ways the other two axioms are more important. The significance of I1 is, of course, that it is a geometric sequent, and we shall take it as the fundamental concept to which we attach the name of *integral ring*. If one is interested in constructing fields of fractions, then I3 (which says that the nonzero elements of A form a multiplicatively closed subset) is sufficient:

LEMMA 2.3. *Let A be a nontrivial ring satisfying I3, and let $S \rightarrowtail A$ be the interpretation of the formula $\neg(\mathbf{a} = 0)$. Then the ring of fractions $B = A[S^{-1}]$ satisfies F2.*

Proof. Full details of the construction of rings of fractions will be found in [14] or [23]; for the present, we need only note that if $A \xrightarrow{f} B$ is the canonical homomorphism, then the formulas $\exists \mathbf{a} \exists \mathbf{a}'((\mathbf{a}' \in S) \wedge (f(\mathbf{a}')\,\mathbf{b} = f(\mathbf{a})))$ and $(\mathbf{b} \in U(B)) \Leftrightarrow \exists \mathbf{a} \exists \mathbf{a}'((\mathbf{a} \in S) \wedge (\mathbf{a}' \in S) \wedge (f(\mathbf{a}')\,\mathbf{b} = f(\mathbf{a}))$ are valid. From the second, we deduce $(\mathbf{a}' \in S) \Rightarrow (f(\mathbf{a}') \in U(B))$; and hence $\neg(\mathbf{b} = 0)$ implies $\neg \exists \mathbf{a}'((\mathbf{a}' \in S) \wedge (f(\mathbf{a}')\,\mathbf{b} = 0))$. So if $\mathbf{a}$, $\mathbf{a}'$ are as in the first formula, we deduce $\neg(\mathbf{a} = 0)$, i.e., $\mathbf{a} \in S$. Hence by the second formula we have $\mathbf{b} \in U(B)$. ∎

We shall give the name *weak integral ring* to a nontrivial ring satisfying I3. Thus on combining 2.2 and 2.3 we obtain

COROLLARY 2.4. *Let A be a nontrivial ring. Then A satisfies F2 iff it is isomorphic to the field of fractions of a weak integral ring.*

Proof. If A satisfies F2, it is a weak integral ring by 2.2, and the object S of nonzero elements is just the object $U(A)$. So the canonical map $A \rightarrow A[S^{-1}]$ is an isomorphism. The converse implication is a restatement of 2.3. ∎

In view of this result, we call a nontrivial ring satisfying F2 a *field of fractions*.

Kennison [12] has proposed another integrality axiom, namely,

$$\text{I0} \qquad (\mathbf{a} = 0) \vee (\mathbf{a}' = 0) \vee \neg(\mathbf{a}\mathbf{a}' = 0).$$

It is not hard to see that this implies I1; however, by substituting 1 for $\mathbf{a}'$ we obtain the decidability axiom. Conversely, I1 plus decidability (or even I3 plus decidability) implies I0. We call a nontrivial ring satisfying I0 a *strong integral ring*.

In most previous work on the subject, a *local ring* in a topos has been defined as a nontrivial ring A satisfying $(\mathbf{a} \in U) \vee (1 - \mathbf{a} \in U)$ (cf. [6, 13, 18, 23]). We shall find it convenient to rewrite this axiom in the equivalent form

$$\text{L1} \qquad (\mathbf{a} + \mathbf{a}' \in U) \Rightarrow ((\mathbf{a} \in U) \vee (\mathbf{a}' \in U)).$$

The formal similarity between this axiom and I1 immediately inspires us to write down two weaker versions L2, L3, and a stronger version L0:

$$\text{L0} \quad (\mathbf{a} \in U) \vee (\mathbf{a}' \in U) \vee \neg(\mathbf{a} + \mathbf{a}' \in U);$$
$$\text{L2} \quad ((\mathbf{a} + \mathbf{a}' \in U) \wedge \neg(\mathbf{a} \in U)) \Rightarrow (\mathbf{a}' \in U);$$
$$\text{L3} \quad (\neg(\mathbf{a} \in U) \wedge \neg(\mathbf{a}' \in U)) \Rightarrow \neg(\mathbf{a} + \mathbf{a}' \in U).$$

A nontrivial ring satisfying L0 (resp. L3) will be called a *strong* (resp. *weak*) *local ring*. The following three results are proved by methods similar to those of 2.2–2.4.

LEMMA 2.5. *L0 implies L1 implies L2 implies L3. L3 plus U-decidable implies L0. F1 implies L0. F2 and F3 both imply L2.* ∎

LEMMA 2.6. *Let A be a weak local ring, $M \rightarrowtail A$ the interpretation of $\neg(a \in U)$. Then M is an ideal, and the quotient ring A/M satisfies F3.* ∎

COROLLARY 2.7. *A nontrivial ring satisfies F3 iff it is isomorphic to the residue field of a weak local ring.* ∎

In view of the last result, we call a nontrivial ring satisfying F3 a *residue field*.

It remains to introduce two further variants of the field axiom. In [13], Kock defines a field to be a nontrivial ring satisfying:

F2A For each (external) natural number n,

$$\neg \bigwedge_{i=1}^{n} (\mathbf{a}_i = 0) \Rightarrow \bigvee_{i=1}^{n} (\mathbf{a}_i \in U).$$

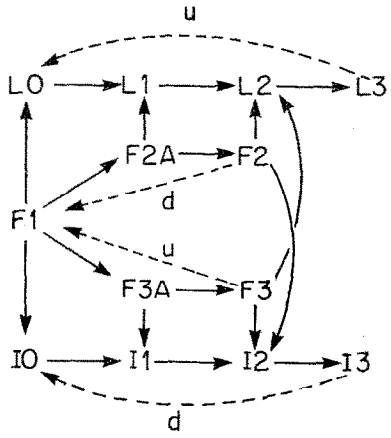
This clearly implies F2, and is implied by F1 since decidability allows us to replace $\neg \bigwedge_{i=1}^{n} (\mathbf{a}_i = 0)$ by $\bigvee_{i=1}^{n} \neg(\mathbf{a}_i = 0)$. Moreover, the case $n = 2$ of F2A implies the geometric form (L1) of the local-ring axiom, since from $\neg(0 = 1)$ we can deduce $(\mathbf{a} + \mathbf{a}' \in U) \Rightarrow \neg((\mathbf{a} = 0) \wedge (\mathbf{a}' = 0))$. Kock points out that the effect of replacing F2 by F2A is to give the relation of not equality good combinatorial properties (similar to those of the "apartness relation" in intuitionistic analysis), which enable one to prove standard theorems of affine and projective geometry with coordinates in A. There is a similar (though less obviously useful) strengthening of F3:

F3A For each natural number n,

$$\neg \bigwedge_{i=1}^{n} (\mathbf{a}_i \in U) \Rightarrow \bigvee_{i=1}^{n} (\mathbf{a}_i = 0).$$

Once again, this is implied by F1, and it implies the geometric form of the integral ring axiom.

Summarizing the implications established in this section, we have the diagram:



Here the implications marked *d* hold in the presence of decidability; those marked *u* hold in the presence of *U*-decidability.

We conclude this section with a number of counterexamples to the possibility of filling in further implications on the diagram. For reasons of space, we shall not give a complete list; however, a number of further examples will occur in the next two sections.

EXAMPLE 2.8. Let $\mathscr{E}$ be the topos of sheaves on a topological space X, and let A be the sheaf of (germs of) continuous real-valued functions on X (i.e., the Dedekind real number object in $\mathscr{E}$). In [18], Mulvey observes that A satisfies F3 and L1 (indeed, this is true of the Dedekind reals in any topos; see [9, Theorem 6.65]); but A does not normally satisfy I1 (since its stalks are not integral domains), nor does it satisfy F2.

EXAMPLE 2.9. Again, let $\mathscr{E}$ be the topos of sheaves on X, and let B be the sheaf of rings whose sections are equivalence classes of continuous real-valued functions defined on dense open subsets of X, modulo the equivalence relation of agreement on a dense open subset. Mulvey observes that this is simply the field of fractions of the ring in the last example; it therefore satisfies F2, and it is not hard to see that it also satisfies F3. But it does not satisfy either I1 or L1.

For the next two examples, we shall make use of the topos $\mathscr{S}^2$ whose objects are morphisms $A_0 \xrightarrow{f} A_1$ in $\mathscr{S}$, and whose morphisms are commutative squares. This topos is commonly called the *Sierpinski topos*.

EXAMPLE 2.10. We give a pair of examples to show that the properties of decidability and *U*-decidability are independent. Let A be a ring in $\mathscr{S}^2$ such that A_1 is a field K in $\mathscr{S}$, A_0 is the ring of "dual K-numbers" (i.e., the ring

$K[\epsilon]/(\epsilon^2)$), and f is defined by $\epsilon \mapsto 0$. Now an object $A_0 \to^f A_1$ of $\mathscr{S}^2$ is easily seen to be decidable iff f is a monomorphism; so the ring A is not decidable. But $U(A)$ does have a complement, namely the subobject $K\epsilon \to \{0\}$.

Now define B to be the ring $\mathbf{Z} \to^i \mathbf{Q}$, where i is the inclusion of the integers in the rational numbers. Since i is mono, B is decidable; but $U(B)$ is the subobject $\{\pm 1\} \to \mathbf{Q} - \{0\}$, which does not have a complement. It may also be shown that the ring A satisfies L0 and F2A, but not F3 or I1; whereas B satisfies I0 and F3A, but not L1 or F2.

EXAMPLE 2.11. Again in the topos $\mathscr{S}^2$, we give an example to show that I3 does not imply I2. Let A be the ring defined by $A_1 = \mathbb{Z}$, $A_0 = \mathbb{Z}[x, y]/(xy)$, $f(x) = 1, f(y) = 0$. Now it is not hard to see that a ring B in $\mathscr{S}^2$ satisfies I3 iff B_1 is an integral domain in $\mathscr{S}$; so A satisfies I3. But the two global elements of A defined by the elements x and y of A_0 satisfy $xy = 0$ and $\neg(x = 0)$, but not $y = 0$. A similar example can be given to separate L2 and L3.

3. SHEAFIFICATIONS OF THE GENERIC RING

Let $\mathbf{T}$ be a finitary algebraic theory. It was first pointed out by Hakim [6] that if C denotes the category of finitely presented T-models (in $\mathscr{S}$), the topos $\mathscr{S}^C$ is a *classifying topos* for $\mathbf{T}$-models in $\mathscr{S}$-toposes, in the following sense: There exists a *generic* $\mathbf{T}$-model M in $\mathscr{S}^C$ such that, for any $\mathscr{S}$-topos $\mathscr{E}$, the functor

$$\mathfrak{Top}/\mathscr{S}(\mathscr{E}, \mathscr{S}^C) \to \mathbf{T}\text{-mod}(\mathscr{E}); \qquad f \mapsto f^*M$$

is an equivalence of categories. (We are assuming here for convenience that $\mathscr{S}$ is the topos of constant sets, but in fact the result remains true for any topos with a natural number object; see [8].) Specifically, the underlying object of M is the forgetful functor $C \to \mathscr{S}$, i.e., the functor represented by the free $\mathbf{T}$-model $F(1)$ on one generator; its $\mathbf{T}$-model structure derives from the fact that $F(1)$ is a co-$\mathbf{T}$-model in $\mathbf{T}$-mod($\mathscr{S}$) and hence in C.

Moreover, if A is any set of geometric sequents in the language of $\mathbf{T}$, it is possible to impose a Grothendieck topology J_A on C^{op} such that the corresponding topos of sheaves classifies the quotient theory of $\mathbf{T}$ defined by A; i.e., a geometric morphism $\mathscr{E} \to^f \mathscr{S}^C$ over $\mathscr{S}$ factors through the inclusion $\mathrm{Shv}(C^{\mathrm{op}}, J_A) \to \mathscr{S}^C$ iff the corresponding $\mathbf{T}$-model f^*M satisfies the axioms in A. In this section we shall, as usual, take $\mathbf{T}$ to be **ann**; and we shall investigate the generic models of some of the geometric quotients of **ann**, by constructing the corresponding topologies on C^{op}. A curious feature of these generic models, first observed by Kock [13], is that they frequently satisfy additional (non-geometric) axioms which are not implied by the axioms of the geometric theory;

this means that the generic model is often more convenient to work with than an arbitrary model in an arbitrary topos.

The syntactic version of the last remark is the following. Let $\mathbf{T}$ be a geometric theory and let Φ be an additional (nongeometric) axiom satisfied by the generic model of $\mathbf{T}$. Then if Ψ is a geometric sequent deducible from $\mathbf{T} + \Phi$, it is satisfied by any $\mathbf{T}$-model in a topos defined over $\mathscr{S}$, and hence (by the completeness theorem for geometric theories) deducible from the axioms of $\mathbf{T}$ alone. Thus we may use Φ as an "auxiliary axiom" to simplify proofs in the theory $\mathbf{T}$.

To describe the relationship between geometric theories and Grothendieck topologies, let us consider a sequent of the form $\Phi = (\varphi \Rightarrow \bigvee_{i=1}^{n} \psi_i)$, where φ and each of the ψ_i is a finite conjunction of formulas of the form $(f(\mathbf{a}_1, \ldots, \mathbf{a}_m) = 0)$ or $(g(\mathbf{a}_1, \ldots, \mathbf{a}_m) \in U)$. (For the present, we shall regard these two types of formulas as being atomic.) Now to each such finite conjunction φ we assign a finitely presented ring $A(\varphi)$; specifically, if $\varphi = (\bigwedge_{j=1}^{p} (f_j(\mathbf{a}_1, \ldots, \mathbf{a}_m) = 0) \wedge \bigwedge_{k=1}^{q} (g_k(\mathbf{a}_1, \ldots, \mathbf{a}_m) \in U))$, $A(\varphi)$ is the ring $(\mathbf{Z}[t_1, \ldots, t_m]/(f_j(t_1, \ldots, t_m)))[g_k(t_1, \ldots, t_m)^{-1}]$. If φ' is a conjunction containing φ, we clearly have a ring homomorphism $A(\varphi) \to A(\varphi')$; and we now associate a cosieve $S(\Phi)$ on $A(\varphi)$ with the sequent Φ, namely, that generated by the family of morphisms $A(\varphi) \to A(\varphi \wedge \psi_i)$ $(1 \leqslant i \leqslant n)$. Finally if $\{\Phi_\alpha \mid \alpha \in A\}$ is the set of axioms for a theory, then J_A is the smallest topology on C^{op} for which each of the $S(\Phi_\alpha)$ is covering.

For example, consider the sequent $((0 = 1) \Rightarrow \mathit{false})$. The ring $A(0 = 1)$ is evidently the quotient $\mathbf{Z}/(1)$, i.e., the trivial ring; and if we regard *false* as the empty disjunction, we obtain the empty cosieve on this ring. But since the trivial ring is a strict terminal object of C, the Grothendieck topology corresponding to the theory of nontrivial rings has this sieve as its only nontrivial cover. Thus the sheaves for this topology are precisely those presheaves which take the value 1 (the one-point set) at the trivial ring; in particular, the generic ring M is alrcady a shcaf.

LEMMA 3.1. *The generic nontrivial ring satisfies F2.*

Proof. Using Kripke–Joyal semantics [13, 20], we can interpret the axiom F2 as follows. "Let $a \in M(A)$ (i.e., $a \in A$) for some object A of C. If every ring homomorphism $A \xrightarrow{f} B$ such that $f(a) = 0$ in B is such that B is covered by the empty sieve, then $a \in U(A)$." But the hypothesis implies that the ring $A/(a)$ is trivial, and hence that the ideal (a) is the whole of A, i.e., a is a unit. ∎

Consider next the theory of local rings. Expressing the axiom L1 in the form $(\mathit{true} \Rightarrow ((\mathbf{a} \in U) \vee (1 - \mathbf{a} \in U)))$, we deduce that the sieve on $\mathbf{Z}[t]$ generated by $\mathbf{Z}[t] \to \mathbf{Z}[t, t^{-1}]$ and $\mathbf{Z}[t] \to \mathbf{Z}[t, (1-t)^{-1}]$ is covering, and that this sieve (together with the empty sieve on the trivial ring) generates the topology.

LEMMA 3.2. *The topology corresponding to the theory of local rings is generated by the pretopology* [5, II 1.3] *whose covering families have the form* $(A \to A[a_i^{-1}]|$

$1 \leqslant i \leqslant n$), *where* $(a_1, \ldots, a_n)$ *is a finite family of elements generating the ideal* A.

Proof. One simply has to verify that these families satisfy the axioms for a pretopology, and that each of them can be obtained by composing families which are pushouts (i.e., pullbacks in C^{op}) of the particular family defined above. The details are straightforward. ∎

It follows easily from 3.2. that the local-ring topology is subcanonical, i.e., that the representable functors $C \to \mathscr{S}$ (and in particular the generic ring M) are sheaves for it. So we may immediately deduce

LEMMA 3.3 [13]. *The generic local ring satisfies F2A.*

Proof. The statement we have to prove reduces, as in 3.1, to the following: "If $a_1, \ldots, a_n \in A$ are such that any $A \to^f B$ with $f(a_i) = 0$ for all i has B covered by the empty sieve, then there exists a covering family $(g_i : A \to A_i \mid 1 \leqslant i \leqslant n)$ with $g_i(a_i) \in U(A_i)$ for all i." But the hypothesis implies that the ideal $(a_1, \ldots, a_n)$ is the whole of A, and so if we take $A_i = A[a_i^{-1}]$ we have the required covering family by 3.2. ∎

Next we consider the theory of integral rings. In this case the "generic" covering family, corresponding to the sequent I1, consists of the pair of morphisms $\mathbf{Z}[t, u]/(tu) \to \mathbf{Z}[u]$, $\mathbf{Z}[t, u]/(tu) \to \mathbf{Z}[t]$ which send $t \mapsto 0$ and $u \mapsto 0$, respectively. As in 3.2, we may easily identify the corresponding pretopology:

LEMMA 3.4. *The integral-ring topology is generated by the pretopology whose covering families have the form* $(A \to A/J_i \mid 1 \leqslant i \leqslant n)$, *where* $(J_1, \ldots, J_n)$ *is a finite family of (finitely generated) ideals of* A *such that* $\prod_{i=1}^{n} J_i = \{0\}$.

Proof. Similar to 3.2. ∎

Note that the condition "$\prod J_i = \{0\}$" in 3.4 could be replaced equivalently by "$\prod J_i$ is nilpotent," since the latter condition implies that we can enlarge $(J_1, \ldots, J_n)$ to a family whose product is zero, simply by repeating each J_i a sufficient number of times. Also, the words "finitely generated" are redundant since all the rings in C are finitely presented and hence are Noetherian.

However, the topology of 3.4 differs from that of 3.2 in that it is not subcanonical; in particular, the generic ring M is not a sheaf for it, and so the generic integral ring must be described as the associated sheaf of M for this topology. We shall not attempt to give an explicit description of this associated sheaf (though the description is implicit in the results of Section 5 below on domain representability); but even without such a description, we can still prove

LEMMA 3.5. *The generic integral ring satisfies F3A.*

Proof. If M were a sheaf, we would argue as follows. "Let $(a_1,..., a_n)$ be a finite family of elements of a ring A such that $A[a_1^{-1},..., a_n^{-1}]$ is covered by the empty sieve (i.e., is the trivial ring). Then 0 must be in the multiplicative submonoid of A generated by $(a_1,..., a_n)$, and so the product $a_1a_2 \cdots a_n$ is nilpotent. Hence by 3.4 the family of morphisms $(A \to A/(a_i) \mid 1 \leqslant i \leqslant n)$ is covering for the integral-ring topology."

But if we write $\hat{M}$ for the associated sheaf of M, then it follows from the iterated-colimit description of the associated sheaf functor [5, II 3.2] that each element of $\hat{M}(A)$ is determined by a family of elements $a_i \in M(A/J_i)$, for some covering family $(A \to A/J_i)$, with the property that for each pair (i, j), the elements a_i and a_j agree everywhere on a family of rings covering $A/(J_i + J_j)$. Moreover, two such families are equal as elements of $\hat{M}(A)$ iff they agree everywhere on some (finer) covering family of A.

This description enables us to reduce statements about elements of $\hat{M}(A)$ to statements about actual elements of rings; we may then apply the argument in inverted commas above. The combinatorial details of this argument are rather involved, so we shall not give it in full. ∎

The results of 3.1, 3.3, and 3.5 do not exhibit the complete symmetry between integral and local which we observed in the last section, since the generic nontrivial ring satisfies F2 but not F3. The reason for this is that the interpretation of $\neg(\mathbf{a} \in U)$ is the subsheaf of M which sends a ring A to the set of all those $a \in A$ such that $A[a^{-1}]$ is trivial, i.e., the nilradical of A. To make the generic ring satisfy F3, we must therefore force it to be *nilpotent-free*, i.e., to satisfy the axiom

$$\text{NF} \qquad ((\mathbf{a}^2 = 0) \Rightarrow (\mathbf{a} = 0)).$$

Note that NF is actually a Horn sequent, and that it is implied by I1 (substitute $\mathbf{a}$ for $\mathbf{a}'$ in I1) and also by F3 (since $(\mathbf{a}^2 = 0)$ implies $\neg(\mathbf{a} \in U)$). Note also that the corresponding axiom implied by L1 and F2 is the lim-sequent $((2\mathbf{a} \in U) \Rightarrow (\mathbf{a} \in U))$, which is true already in the theory of rings.

Now in the topology corresponding to NF, it is easily seen that each object A of C has a minimal covering sieve, namely, that generated by the single morphism $A \to A/N$, where N is the nilradical of A. It follows immediately that the associated sheaf of M for this topology is the presheaf which sends A to the ring A/N. (Note that this is simply the residue field of M regarded as a weak local ring in the topos of 3.1.) It is therefore easy to prove

LEMMA 3.6. *The generic (nontrivial) nilpotent-free ring satisfies F3.* ∎

In conclusion, note that we may replace any of the four topologies considered here by a larger topology, provided only that we do not allow any nontrivial ring to be covered by the empty sieve, and the associated sheaf of M will still

satisfy the appropriate nongeometric field axiom. For example, by taking the join of the topologies of 3.2 and 3.4, we obtain an example of a ring which satisfies F2A and F3A but not F1.

4. Three Examples of Spectra

It was Hakim [6] who first pointed out that the global spectrum construction for ringed toposes could be viewed as an adjoint functor; specifically, the assignment which sends a ringed topos to its spectrum is a right adjoint for the inclusion functor from local-ringed toposes to ringed toposes. More recently, Cole [2] has observed that, once one has the general machinery for constructing classifying toposes [24], the commutative algebra involved in constructing the spectrum may be reduced to the following "Factorization Lemma":

Lemma 4.1. *Let $\mathscr{E}$ be a topos, and let $A \to^f L$ be a ring homomorphism in $\mathscr{E}$ with L local. Then there exists a factorization $A \to^p A_f \to^{\hat{f}} L$ of f where A_f is a local ring and $\hat{f}$ is a local morphism, which is "best possible" in the sense that, for any other factorization $A \to^q B \to^g L$ with B and g local, there exists a unique (necessarily local) morphism $A_f \to^s B$ with $sq = p$ and $gs = \hat{f}$. Moreover, this best possible factorization is preserved by inverse image functors.*

Proof. Form the pullback

$$\begin{array}{ccc} S & \longrightarrow & U(L) \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & L \end{array},$$

and define $A_f = A[S^{-1}]$. It is then straightforward to show that A_f is local and that the unique factorization of f through $A \to A_f$ is local; and the universal property of A_f is immediate from the definition of a ring of fractions. Moreover, the construction of A_f involves only finite limits and colimits, and so it is preserved by inverse image functors. ∎

The general notion of a spectrum, then, has the following ingredients: a pair of geometric theories $\mathbf{S}$, $\mathbf{T}$ such that $\mathbf{T}$ is a quotient of $\mathbf{S}$, a class $\mathbf{A}$ of "admissible" morphisms of $\mathbf{T}$-models (i.e., the class of local morphisms in the example above), and a factorization lemma similar to 4.1 for morphisms from an $\mathbf{S}$-model to a $\mathbf{T}$-model. Given this data, we may then prove

Proposition 4.2 [2]. *Let $\mathbf{S}$, $\mathbf{T}$, and $\mathbf{A}$ be as above. Let $\mathbf{S}$-$\mathfrak{Top}$ denote the 2-category of $\mathbf{S}$-modeled toposes; i.e., its objects are pairs $(\mathscr{E}, A)$ where $\mathscr{E}$ is a topos (with natural number object) and A is an $\mathbf{S}$-model in $\mathscr{E}$, its 1-arrows $(\mathscr{F}, B) \to (\mathscr{E}, A)$ are pairs (p, f) where $\mathscr{F} \to^p \mathscr{E}$ is a geometric morphism and*

$p^*A \to^f B$ is an **S***-model morphism, and its 2-arrows $(p,f) \to (q,g)$ are natural transformations $\eta: p^* \to q^*$ such that $g \cdot \eta_A = f$. Let* **A**-$\mathfrak{Top}$ *denote the sub-2-category of* **S**-$\mathfrak{Top}$ *whose objects are pairs $(\mathscr{E}, A)$ such that A is a* **T***-model, and whose 1-arrows are pairs (p,f) such that f is admissible. Then the inclusion functor* **A**-$\mathfrak{Top} \to$ **S**-$\mathfrak{Top}$ *has a right adjoint (in the up-to-equivalence sense)* Spec: **S**-$\mathfrak{Top} \to$ **A**-$\mathfrak{Top}$. ∎

Specifically, Spec$(\mathscr{E}, A)$ is $(\mathscr{F}, \tilde{A})$ where $\mathscr{F}$ is the classifying topos for the theory of extremal morphisms with domain A (a morphism from an **S**-model to a **T**-model is said to be *extremal* if the second part of its factorization is an isomorphism), and $\tilde{A}$ is the codomain of the generic extremal morphism. The generic extremal morphism itself is the counit of the adjunction.

In this section we shall take **S** to be **ann**; and for **T** we shall take the three quotient theories **loc**, **int**, **fld** of local rings, integral rings, and (geometric) fields, respectively, with a suitable definition of admissible morphism in each case. The three spectrum functors which we obtain will be denoted LSpec, ISpec, and FSpec, respectively; while the first of these is very well known, the other two (though closely related to the first) are much less familiar.

In each of the three cases, if we take A to be a ring in $\mathscr{S}$, the underlying topos of Spec$(\mathscr{S}, A)$ turns out to be spatial, i.e., of the form Shv(X) for some topological space X. (It is not altogether clear why this should be so, but it appears to be a reflection of the fact that the category of models of the theory classified by Spec$(\mathscr{S}, A)$ is (equivalent to) a partially ordered set.) Moreover, we can give an explicit description of the space X in terms of the theory classified by Shv(X); this description is essentially due to Mulvey, though it may be considered as a particular case of the general syntactic construction of classifying toposes by Makkai and Reyes.

PROPOSITION 4.3. *Let X be a sober space* [5, IV 4.2.1], *and suppose* Shv(X) *is the classifying topos for a geometric theory* **T**. *Then the points of X correspond to isomorphism classes of* **T***-models in $\mathscr{S}$, and the open sets of X are the (sets of models satisfying) provable-equivalence classes of geometric sentences (i.e., formulas without free variables) in the language of* **T**. ∎

Note that since the language of **T** allows finite conjunctions and arbitrary disjunctions, the equivalence classes of sentences do indeed form a topology on the set of isomorphism classes of **T**-models.

We consider first the familiar case of LSpec. In this case, the extremal morphisms are those homomorphisms $A \to^f L$ for which $L \cong A[S^{-1}]$ in the notation of 4.1; but such a morphism is determined by the subobject $S \rightarrowtail A$, which is readily seen to be a *prime filter*, i.e., to satisfy the (geometric) axioms

$$\textit{true} \Rightarrow (1 \in S), \qquad (0 \in S) \Rightarrow \textit{false},$$
$$(\mathbf{a}\mathbf{a}' \in S) \Leftrightarrow ((\mathbf{a} \in S) \wedge (\mathbf{a}' \in S)),$$

and

$$(\mathbf{a} + \mathbf{a}' \in S) \Rightarrow ((\mathbf{a} \in S) \vee (\mathbf{a}' \in S)).$$

(In $\mathscr{S}$, these axioms say precisely that S is the complement of a prime ideal of A.) Moreover, if S is a prime filter then $A[S^{-1}]$ is easily seen to be local; so the local-spectrum of A is the classifying topos for the theory of prime filters of A.

Now let A be a ring in $\mathscr{S}$. Then by 4.3 the points of the underlying space of $\mathrm{LSpec}(\mathscr{S}, A)$ are the prime filters of A, or equivalently the prime ideals of A; we shall write spec A (with a small s) for the set of prime ideals of A. Now the language of the theory of prime filters of A may be considered to be that of the theory of rings, enriched with a constant (nullary operation) for each element of A and a new unary predicate $(-) \in S$. The sentences of the language thus include the atomic sentences $(a \in S)$ for every $a \in A$; and the corresponding open set is clearly the basic Zariski-open set $D(a) = \{P \in \mathrm{spec}\ A \mid a \notin P\}$. Moreover, it is not hard to see that *any* sentence in the theory is provably equivalent to a disjunction of finite conjunctions of these atomic sentences (note that $\exists \mathbf{a} \cdot \varphi$ is equivalent to $\bigvee_{a \in A} (\varphi[a/\mathbf{a}])$, and any atomic sentence $(f(a_1, \ldots, a_n) = g(a_1, \ldots, a_n))$ is provably either *true* or *false*); and so the sets $D(a)$ form a subbase for the topology. (In fact they form a base, since it is well known that $D(a) \cap D(b) = D(ab)$.) We have thus proved

PROPOSITION 4.4. *Let A be a ring in $\mathscr{S}$. Then* $\mathrm{LSpec}(\mathscr{S}, A) = (\mathrm{Shv}(\mathrm{spec}\ A_{\mathrm{zar}}), \tilde{A})$, *where* spec A_{zar} *denotes the space of prime ideals of A with the Zariski topology, and $\tilde{A}$ is the familiar structure sheaf on* spec A_{zar} *whose stalk at a point P is the local ring A_P.* ∎

To construct the integral-spectrum, we must first give a definition of "integral morphism" for which the factorization lemma is valid. Since a local morphism $A \to^{f} B$ is one which reflects the property of being a unit, i.e., satisfies $((f(\mathbf{a}) \in U(B)) \Rightarrow (\mathbf{a} \in U(A)))$, it is clear that an integral morphism should be one which reflects the property of being equal to zero, i.e., it should be a monomorphism. And indeed 4.1 remains valid if we substitute "integral ring" for "local ring" and "monomorphism" for "local morphism"; the ring A_f is simply the (topos-theoretic) image of $A \to^{f} L$. (The fact that A_f is integral follows easily from the fact that it is a subring of the integral ring L.)

So the extremal morphisms in this case are the surjections from A to an integral ring; but any such surjection is determined by its kernel, which is clearly required to be a *prime ideal* of A, i.e., a subobject $P \rightarrowtail A$ satisfying

$$\textit{true} \Rightarrow (0 \in P), \qquad (1 \in P) \Rightarrow \textit{false},$$

$$(\mathbf{a}\mathbf{a}' \in P) \Leftrightarrow ((\mathbf{a} \in P) \vee (\mathbf{a}' \in P)),$$

and

$$((\mathbf{a} \in P) \wedge (\mathbf{a}' \in P)) \Rightarrow (\mathbf{a} + \mathbf{a}' \in P).$$

Thus $\mathrm{ISpec}(\mathscr{E}, A)$ is the classifying topos for the theory of prime ideals of A.

In particular, taking A to be a ring in $\mathscr{S}$, we obtain a topological space whose points are the prime ideals of A, as before, but whose subbasic open sets have the form $V(a) = \{P \in \mathrm{spec}\, A \mid a \in P\}$; i.e., they are the complements of the basic Zariski-open sets. It seems reasonable to call this topology the *coZariski topology*; since the family of sets $\{V(a) \mid a \in A\}$ is not closed under finite intersections, we shall find it convenient to introduce the notation $V(a_1, \ldots, a_n)$ for $\bigcap_{i=1}^{n} V(a_i)$.

Consider the presheaf of rings on $\mathrm{spec}\, A_{\mathrm{cozar}}$ defined by the assignment $V(a_1, \ldots, a_n) \mapsto A/\sqrt{(a_1, \ldots, a_n)}$ (where $\sqrt{J}$ denotes the radical of an ideal J). It is not hard to verify that the stalk of this presheaf at a point P is the integral ring A/P, and so its associated sheaf $\tilde{A}$ is an integral ring in $\mathrm{Shv}(\mathrm{spec}\, A_{\mathrm{cozar}})$. (We shall consider the problem of describing the sections of this sheaf in the next section; in general its global sections, unlike those of $\tilde{A}$, are not simply the elements of A.)

PROPOSITION 4.5. *Let A be a ring in $\mathscr{S}$. Then*

$$\mathrm{ISpec}(\mathscr{S}, A) \simeq (\mathrm{Shv}(\mathrm{spec}\, A_{\mathrm{cozar}}), \tilde{A}). \quad \blacksquare$$

The coZariski topology on spec A has hitherto been studied very little; but in fact it enjoys all the formal properties of the Zariski topology. For example, we have

LEMMA 4.6. *The space* $\mathrm{spec}\, A_{\mathrm{cozar}}$ *is* coherent; *i.e., it is compact, and the topology has a base which is closed under finite intersections and consists of compact open sets. (This is equivalent to saying that* $\mathrm{Shv}(\mathrm{spec}\, A_{\mathrm{cozar}})$ *is a coherent topos; cf.* [5, VI 2.3].)

Proof. To prove that $\mathrm{spec}\, A_{\mathrm{cozar}}$ is compact, it suffices by the Alexander Subbase Theorem [26, p. 129] to consider a covering of the form $\{V(a_i) \mid i \in I\}$. Then since every prime ideal of A contains one of the a_i, the ring $A[a_i^{-1} \mid i \in I]$ has no prime ideals and is therefore trivial; hence 0 is in the monoid generated by the a_i. But now we must have an expression for 0 as a finite product of a_i's, and we obtain a corresponding finite subcover. The second part of the statement now follows from the fact that $V(a_1, \ldots, a_n)$ is homeomorphic to $\mathrm{spec}(A/(a_1, \ldots, a_n))_{\mathrm{cozar}}$. $\blacksquare$

It is clear, too, that $\mathrm{spec}\, A_{\mathrm{cozar}}$ is sober, since it contains one point for each isomorphism class of models of the theory of prime ideals of A; and so it is a *spectral space* in the sense of Hochster [7]. In fact 4.6 could have been deduced from the following lemma of Hochster [7, Proposition 8]:

LEMMA 4.7. *Let X be a spectral space, and let $\mathcal{V}$ be the set of complements of compact open subsets of X. Then the topology on X with $\mathcal{V}$ as a base is again spectral.* ∎

It follows also from 4.7 that the main theorem of [7] also applies to the coZariski spectrum; i.e., every spectral space is homeomorphic to the coZariski spectrum of some ring. In fact it is intriguing to note that this is what Hochster actually proves, in that he works with integral rings (the stalks of $\hat{A}$) rather than with the stalks of $\tilde{A}$.

Finally, we consider the field-spectrum of A. In this case there is no possible ambiguity about what we should take for the class **A** of admissible morphisms; for it is easily verified that the concepts "ring homomorphism," "monomorphism," and "local morphism" all coincide for morphisms between (geometric) fields. The factorization lemma is again easy to verify; the extremal morphisms are those homomorphisms $A \to F$ which are epimorphisms in $\mathbf{ann}(\mathscr{E})$, or equivalently those for which F is isomorphic to the field of fractions of the image. Now such a morphism is determined by its kernel; but since a geometric field is decidable, this kernel must be a *complemented* prime ideal of A. Conversely, if P is a complemented prime ideal of A, then A/P is a strong integral ring, from which it follows easily that its field of fractions is a geometric field.

To describe the theory of complemented prime ideals, we must adjoin two unary predicates $(-) \in P$, $(-) \in S$ to the theory of rings, together with axioms which say that P and S are complementary and P is a prime ideal (or equivalently, S is a prime filter). For a ring A in $\mathscr{S}$, the field-spectrum thus has the same points as the other two spectra, but its topology is the join of the Zariski and coZariski topologies, since both $(a \in P)$ and $(a \in S)$ are sentences in the language. This topology is called the *patch topology* by Hochster [7], and is reasonably well known to algebraic geometers as the *constructible topology*. It is easily seen to be a Stone space (i.e., a Hausdorff spectral space, or equivalently a totally disconnected compact Hausdorff space). However, the existence of a sheaf of rings on spec A_{cons} whose stalk at P is the field of fractions of A/P (equivalently, the residue field of A_P) is rather less well known; though it does make a fleeting appearance in Section 10 of Hochster's paper [7], where its ring of global sections is described.

5. DOMAIN REPRESENTABILITY

In [12], Kennison asks the question: What can one say about those rings which are representable as the ring of global sections of a sheaf of integral rings? For reasons which will appear below, this is tantamount to asking for a syntactic description of those rings which occur as images of integral rings under left exact functors, or for a lim-theory which implies all those lim-

sequents which are true in the theory of integral rings. Yet another way of posing the question is to ask for a description of the sections (rather than the stalks) of the sheaf of integral rings which we constructed on spec A_{cozar} in the last section.

We have already observed that the integral-ring axiom (I1) implies the nilpotent-free axiom (NF) and that the latter is a Horn sequent; so if $\mathscr{F} \to^{T} \mathscr{E}$ is any left exact functor (e.g., the direct image of a geometric morphism) and A is an integral ring in $\mathscr{F}$, $T(A)$ is nilpotent-free. However, Kennison showed that this condition alone is not sufficient to characterize $T(A)$; specifically, he introduced a sequence of axioms (DRn), $n > 0$, as follows:

$$\mathrm{DR}n \qquad \Big((\mathbf{b}(\mathbf{b}-\mathbf{a}_1^2)(\mathbf{b}-\mathbf{a}_2^2)\cdots(\mathbf{b}-\mathbf{a}_n^2)=0) \wedge \bigwedge_{i=1}^{n}\bigwedge_{j=1}^{n}(\mathbf{a}_i\mathbf{a}_j\mathbf{b}=\mathbf{c}_i\mathbf{c}_j)\Big)$$
$$\Rightarrow \exists \mathbf{r}\Big((\mathbf{r}^2=\mathbf{b}) \wedge \bigwedge_{i=1}^{n}(\mathbf{a}_i\mathbf{r}=\mathbf{c}_i)\Big),$$

where $\mathbf{a}_1,\ldots,\mathbf{a}_n$, $\mathbf{b}$, $\mathbf{c}_1,\ldots,\mathbf{c}_n$ are $(2n+1)$ free variables of type A.

Following Kennison, we shall say that a nilpotent-free ring satisfying (DRn) for all n is *domain representable.*

LEMMA 5.1 [12]. (i) *DRn is a* $\varinjlim$*-sequent relative to the theory of nilpotent-free rings.*

(ii) *Any integral ring satisfies DRn.*

Proof. (i) For simplicity, we consider DR1. Suppose we are given $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{r}$, $\mathbf{s}$ such that $\mathbf{b}^2=\mathbf{a}^2\mathbf{b}=\mathbf{c}^2$, $\mathbf{r}^2=\mathbf{b}=\mathbf{s}^2$, and $\mathbf{ar}=\mathbf{c}=\mathbf{as}$. Then

$$\begin{aligned}(\mathbf{r}-\mathbf{s})^5 &= \mathbf{r}^5-5\mathbf{r}^4\mathbf{s}+10\mathbf{r}^3\mathbf{s}^2-10\mathbf{r}^2\mathbf{s}^3+5\mathbf{r}\mathbf{s}^4-\mathbf{s}^5\\ &= \mathbf{b}^2(\mathbf{r}-5\mathbf{s}+10\mathbf{r}-10\mathbf{s}+5\mathbf{r}-\mathbf{s})\\ &= 16\mathbf{b}^2(\mathbf{r}-\mathbf{s})\\ &= 16\mathbf{b}\mathbf{a}^2(\mathbf{r}-\mathbf{s})\\ &= 16\mathbf{b}\mathbf{a}(\mathbf{c}-\mathbf{c})=0.\end{aligned}$$

So by (NF) we deduce $\mathbf{r}-\mathbf{s}=0$. (In the corresponding proof for DRn, we have to consider $(\mathbf{r}-\mathbf{s})^{2n+3}$, in order to extract a factor of $\mathbf{b}^{n+1}$ from the binomial expansion.)

(ii) Suppose A satisfies I1. Then from $\mathbf{b}(\mathbf{b}-\mathbf{a}_1^2)\cdots(\mathbf{b}-\mathbf{a}_n^2)=0$ we deduce $(\mathbf{b}=0) \vee \bigvee_{i=1}^{n}(\mathbf{b}=\mathbf{a}_i^2)$. From $\mathbf{b}=0$, we deduce that $\mathbf{r}=0$ satisfies the equations; from $\mathbf{b}=\mathbf{a}_i^2$, we deduce $\mathbf{c}_i^2=\mathbf{a}_i^4$, whence $(\mathbf{c}_i=\mathbf{a}_i^2) \vee (\mathbf{c}_i=-\mathbf{a}_i^2)$ and we take $\mathbf{r}=\mathbf{a}_i$ or $\mathbf{r}=-\mathbf{a}_i$ accordingly. ∎

Kennison has shown that there exist nilpotent-free rings which do not satisfy

the conditions DRn; in particular, the ring C^k of k times ontinuously differentiable functions $\mathbf{R} \to \mathbf{R}$ satisfies DR1 iff $k = 0$ or ∞. The justification for the name "domain representable" is contained in the next proposition, which is also due to Kennison, though our argument differs slightly from his.

PROPOSITION 5.2. *Let A be a ring in $\mathscr{S}$. The following conditions are equivalent:*

(i) *A is domain representable.*

(ii) *A is the image of an integral ring under a left exact functor.*

(iii) *The canonical map $A \to \Gamma(\hat{A})$ is an isomorphism, where $\hat{A}$ is the sheaf of integral rings on the coZariski spectrum of A which was constructed in the last section.*

Proof. (ii) $\Rightarrow$ (i) follows from 5.1; and (iii) $\Rightarrow$ (ii) is trivial. Suppose A is domain representable; then since A is nilpotent free, the canonical map $A \to \Gamma(\hat{A})$ is certainly mono, for its kernel is the intersection of all prime ideals of A. Let s be a global section of $\hat{A}$; then s assigns an element $s_P \in A/P$ to each prime ideal P of A. Moreover, since $\hat{A}$ is the associated sheaf of a presheaf whose values are quotient rings of A, s is "locally representable" by elements of A, i.e., we can find an open cover of spec A_{cozar} by subsets U_i, and elements $a_i \in A$, such that $a_i P = s_P$ for all $P \in U_i$. Furthermore, we can take this cover to be finite, and each U_i to be a basic open set of the form $V(J_i)$, where J_i is a finitely generated ideal of A. We shall now prove by induction on the number of sets in the cover that s is globally representable by an element of A.

Consider first a covering by two sets $V(J_0)$, $V(J_1)$. The fact that this is a cover is expressed by the fact that $J_0 J_1$ is contained in every prime ideal of A, and is therefore zero. Now we may clearly assume $a_0 = 0$, by subtracting the global section which it represents from the given section s, and so we have $a_1 P = 0$ for all $P \in V(J_0) \cap V(J_1)$. Hence a_1 is in the radical of $J_1 + J_2$, so we can write $a_1^{2^m} = b_0 + b_1$ for some m and some $b_i \in J_i$. But $b_0 b_1 = b_0(b_0 - a_1^{2^m}) = 0$, so by DR1 we can find r with $r^2 = b_0$ and $r a_1^{2^{m-1}} = b_0$. Now we have $(r - a_1^{2^{m-1}})^2 = b_1$, and hence $r(r - a_1^{2^{m-1}}) = 0$ since A is nilpotent-free. Continuing inductively, we arrive at an element t with $t^{2^m} = b_0$ and $(t - a_1)^{2^m} = b_1$, which implies that $tP = s_P$ for all $P \in \text{spec } A$. So t represents the section s.

Now suppose we have proved the result for n-element covers, and consider an $(n+1)$-element cover $\{V(J_0),\ldots, V(J_n)\}$. As before, we may assume $a_0 = 0$, and hence we can find m such that we can write $a_i^{2^m} = b_i + d_i$ for each $i \in \{1,\ldots, n\}$, with $b_i \in J_0$ and $d_i \in J_i$. (For convenience, we shall assume $m = 1$; the general case is treated by an induction on m similar to that already given.) Now $b_i - b_j = a_i^2 - a_j^2 - d_i + d_j$, which is congruent to zero modulo every prime in $V(J_i) \cap V(J_j)$; so by the inductive hypothesis we can find a single element $b \in A$ with $bP = b_i P$ for all $P \in V(J_0) \cup V(J_i)$. Now $b - b_i \in \sqrt{J_i}$,

so we can rewrite our equations in the form $a_i^2 = b + d_i$ for all i, $b \in \sqrt{J_0}$, $d_i \in \sqrt{J_i}$. But since the $V(J_i)$ cover spec A, we have $\prod_{i=0}^{n} J_i = \{0\}$, and therefore $b \cdot \prod_{i=1}^{n} d_i = b(b - a_1^2) \cdots (b - a_n^2) = 0$.

Now consider the section t_i of A defined by the element b over $V(J_0) \cup V(J_i)$ and by $a_i a_j$ over $V(J_j)$ for $j \neq i$. (It is easily checked that these elements agree modulo every prime ideal in the overlaps.) By the inductive hypothesis, t_i is represented by an element c_i of A; and we have $a_i a_j b = c_i c_j$ for all i, j, since the equation holds modulo every prime ideal. So by DRn we have an element r of A with $r^2 = b$ and $a_i r = c_i$ for all i; but the latter equation implies that $(r - a_i)^2 = 2b - 2c_i + d_i$, which is in the radical of J_i. So $rP = s_P$ for all $P \in \operatorname{spec} A$. ∎

There is also a syntactic version of 5.2, which may be expressed as follows.

COROLLARY 5.3. *Let Φ be a* $\varinjlim$*-sequent relative to the theory of domain representable rings, and suppose Φ is provable in the theory of integral rings (or even in the theory of rings satisfying F3A). Then Φ is provable in the theory of domain representable rings.*

Proof. If Φ is provable in the theory of integral rings, then 5.2 ensures that it is satisfied in every domain representable ring in $\mathscr{S}$. Hence by the completeness theorem it is provable. The fact that "integral rings" may be replaced by "rings satisfying F3A" follows from 3.5. ∎

Note that the ring $\Gamma(\tilde{A})$ is domain representable for any A; in fact it is not hard to see that $\Gamma(\tilde{A})$ is the reflection of A in the category of domain representable rings (cf. [12]). So the proof of 5.2 also shows that any nilpotent-free ring is a subring of a domain representable ring. This has two syntactic consequences: first, that any Horn sequent provable in domain representable rings is provable in nilpotent-free rings; and second, that any $\varinjlim$-sequent relative to domain representable rings is already a $\varinjlim$-sequent relative to nilpotent-free rings.

If we substitute the words "geometric field" for "integral ring," the answer to the questions posed at the beginning of this section is rather better known; but it is of interest to observe that it can be obtained by precisely the same methods we have just used. We define a commutative ring to be *regular* if it satisfies the axiom

$$\text{Reg} \qquad \exists \mathbf{b}((\mathbf{a}^2\mathbf{b} = \mathbf{a}) \wedge (\mathbf{b}^2\mathbf{a} = \mathbf{b})).$$

(The usual definition of a regular ring requires only that $\mathbf{a}^2\mathbf{b} = \mathbf{a}$, but this equation is not sufficient to determine $\mathbf{b}$ uniquely. In any ring, if we can find b satisfying $a^2 b = a$, then the element $b' = b^2 a$ satisfies both $a^2 b' = a$ and $b'^2 a = b'$.)

LEMMA 5.4. (i) *(Reg) is a* $\varinjlim$*-formula relative to the theory of nilpotent-free rings. (Note that (Reg) itself implies (NF).)*

(ii) *Any geometric field satisfies (Reg).*

Proof. (i) Suppose $\mathbf{a}^2\mathbf{b} = \mathbf{a} = \mathbf{a}^2\mathbf{c}$, $\mathbf{b}^2\mathbf{a} = \mathbf{b}$, $\mathbf{c}^2\mathbf{a} = \mathbf{c}$. Then $\mathbf{a}^2(\mathbf{b} - \mathbf{c})^2 = (\mathbf{a} - \mathbf{a})(\mathbf{b} - \mathbf{c}) = 0$, so by (NF) $\mathbf{a}(\mathbf{b} - \mathbf{c}) = 0$. Now $\mathbf{b} - \mathbf{c} = (\mathbf{b}^2 - \mathbf{c}^2)\,\mathbf{a} = \mathbf{a}(\mathbf{b} - \mathbf{c})(\mathbf{b} + \mathbf{c}) = 0$.

(ii) If $\mathbf{a} = 0$, then $\mathbf{b} = 0$ satisfies the given equations; if $\mathbf{a} \in U(A)$, then we can take $\mathbf{b} = \mathbf{a}^{-1}$. ∎

LEMMA 5.5. *Let A be a regular ring (in $\mathscr{S}$). Then the three spectra of A defined in the last section all coincide with the Pierce representation* [21] *of A.*

Proof. Regularity implies that every element of A is simultaneously a divisor and a multiple of some idempotent; so every prime ideal and every prime filter of A is generated by the idempotents which it contains, and these form a prime ideal (respectively, a prime filter) in the Boolean algebra of idempotents of A. But every prime ideal in a Boolean algebra has a complementary prime filter; so every prime ideal of A is complemented. Moreover, every basic Zariski-open set can be written in the form $D(e)$ for some idempotent e, and is thus equal to $V(1 - e)$; so the Zariski, coZariski, and constructible topologies on spec A coincide.

Finally, if e is idempotent, adjoining an inverse for e has the same effect as factoring out the ideal $(1 - e)$; so if P is a prime ideal of A with complement S, the rings A/P and $A[S^{-1}]$ are isomorphic, and both are fields. So the three sheaves on spec A are isomorphic; and the description of the Pierce representation on [21, p. 16] makes it clear that this too is isomorphic to the other three. ∎

PROPOSITION 5.6. *Let A be a ring in $\mathscr{S}$. The following conditions are equivalent:*

(i) *A is regular.*

(ii) *A is the image of a geometric field under a left exact functor.*

(iii) *A is isomorphic to the ring of global sections of its field spectrum.*

Proof. As in 5.2, the only nontrivial implication is (i) ⇒ (iii). But it is well known that any ring is isomorphic to the ring of global sections of its Pierce representation (or indeed of its local-spectrum), so this follows from 5.5. ∎

In passing, let us remark that the Pierce representation of a general ring is yet another example of a spectrum, in which we compare the theory of rings with the theory of *indecomposable* rings, i.e., those satisfying $\neg(0 = 1)$ and $((\mathbf{a}^2 = \mathbf{a}) \Rightarrow ((\mathbf{a} = 1) \vee (\mathbf{a} = 0)))$. If we consider an arbitrary morphism of indecomposable rings to be admissible, then we obtain a factorization lemma in which extremal morphisms with domain A correspond to ideals J which are generated by idempotents and satisfy $((\mathbf{a}^2 - \mathbf{a} \in J) \Rightarrow ((\mathbf{a} \in J) \vee (1 - \mathbf{a} \in J)))$. But these in turn correspond to prime ideals of the Boolean algebra of idempotents of A, and hence to points of the Pierce representation.

Finally, we should comment on the fact that, in both 5.2 and 5.6, we required

the ring A to be in the particular topos $\mathscr{S}$ of constant sets, rather than an arbitrary topos. This was because, in both cases, we made use of the explicit description of the spectrum as a spatial topos which we established in the last section, and in particular of the interplay between points and open sets (i.e., stalks and sections) which exists in such a topos. Such an argument depends heavily on the fact that $\mathscr{S}$ satisfies the axiom of choice; if we try simply to translate it into the internal language of a general topos $\mathscr{E}$, we encounter the problem that the spectrum may fail to have enough "$\mathscr{E}$-valued points." (For a dramatic instance of this failure, in the case of the generic ring, see [23].)

However, although we cannot talk about the points of the spectrum in this generality, we can still talk about the open sets. In other words, we can construct the (internal) partially ordered set of basic open subsets of the spectrum, and the Grothendieck topology on this partially ordered set which expresses the notion of finite open cover, without ever referring to the points. And that part of 4.3 which refers to open sets rather than points is still valid in this case; i.e., the underlying topos of Spec($\mathscr{E}$, A) is precisely the topos of $\mathscr{E}$-valued sheaves on this internal site.

For example, in the situation of 5.5 the space spec A is simply the Stone space of the Boolean algebra $B(A)$ of idempotents of A; and it is well known that specifying a sheaf on such a Stone space is equivalent to specifying a sheaf for the finite cover topology [22] on the ordered set $B(A)$ itself. The site used in 5.2 has a more complicated description, but it may still be defined without any reference to points.

Now if we similarly remove all mention of points from the proof of 5.2 (e.g., if we replace the statement "a and b are congruent modulo every prime ideal in $V(J)$" by "there exists m such that $(a-b)^m \in J$"—where, for a ring in a general topos, both the natural number m and the existential quantifier will have to be interpreted in the internal sense of that particular topos), then we can regard it as a proof that a certain explicitly constructed presheaf, whose global sections are known to be isomorphic to A, satisfies the sheaf axiom for the given topology and is therefore the integral ring $\hat{A}$. And a similar interpretation may be given to the proofs of 5.5 and 5.6; so here too the requirement that the base topos be $\mathscr{S}$ can be eliminated.

The principal reason why we did not in fact write out the proofs of 5.2 and 5.5 in this "pointless" form was that to do so would in our opinion, at least in the case of 5.2, render an already complicated proof totally incomprehensible. The traditional interplay between stalks and sections in classical sheaf theory may be logically indefensible here, in that its use involves a totally unnecessary invocation of the axiom of choice, but it is still a very powerful aid to the workings of "geometrical intuition."

In conclusion, we should point out that there are many other examples of spectra, even amongst the quotient theories of rings, which may profitably be investigated by the methods used in this paper. We have already mentioned the

indecomposable-spectrum, and Kennison [12] gives a number of applications to the theory of ordered rings; further examples will doubtless occur to the reader. It would also be interesting to consider the analogs of our results for noncommutative rings; it is fairly well known that in this case we must replace "regular" by "strongly regular" to obtain the analog of 5.6, but I do not know how to express the noncommutative version of domain representability. For noncommutative rings, representability by a local ring is also known to be a nontrivial condition; it would be interesting to have a syntactic characterization of this.

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