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Infinitary Addition, Real Numbers, and Taut Monads

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Abstract We make various observations on infinitary addition in the context of the *series monoids* introduced in our previous paper on *real sets*. In particular, we explore additional conditions on such monoids suggested by Tarski's *Arithmetic of Cardinal Algebras*, and present a monad-theoretic construction that generalizes our construction of *paradoxical real numbers*.

Keywords Infinitary addition · Series monoid · Commutative monoid · Summation · Positive reals · Cardinal algebra · Taut monad · Lextensive category · Monoidal category

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The original version of this article was revised: In the original publication of the article, Eq. 3.24 was published incorrectly. The equation was corrected in the article.

Dedicated to Bob Lowen, creator of APCS.

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Introduction

The main purpose of this paper is to initiate establishing two links:

- Between *series monoids* in the sense of [7] and *cardinal algebras* in the sense of Tarski [12];
- Between what is called *paradoxical reals* in [7] and taut monads in the sense of Manes [10] (which are the same as Alexandrov monads in the sense of Möbus [11]); more precisely, we show that addition for the paradoxical reals can be obtained as a special case of a construction on algebras over a taut monad.

However, the paper is presented as an attempt to give a brief general overview of infinitary addition, in the form of elementary-algebraic remarks to be added to the authors' previous paper [7], not touching its main part, which is category-theoretic.

In fact we begin (Sects. 1 and 2) with a very brief review of the analytic viewpoint, thinking that although it will not be seriously used below, it is good to say a word about it before entering our more abstract context. In Sect. 3 we recall our definition of series monoid from [7] and carefully analyze the relationship between various possible additional conditions on a series monoid suggested by the results of Sect. 1 of Tarski's book [12]. This analysis, which is the most important part of the paper, shows six levels of generality, from series monoids satisfying Tarski's Remainder Postulate to general series monoids. Section 4 makes a further comparison, with Tarski's Fundamental Law of Infinite Addition, introduces what we call analytic monoids, which is suggested by that comparison and, in a sense, by what we say in Sect. 2, and then gives some examples. Section 5 recalls the presentation of (non-negative) reals based on a result of Higgs [5], and then recalls the paradoxical reals. Section 6 presents a general construction for algebras over (what we call) an injection taut monad on a lextensive category (in the sense of [2]) that can be used to define the addition of paradoxical reals. Section 7 briefly suggests studying the series monoids of isomorphism classes of objects in a series monoidal category in the sense of [7].

1 Summation of Series of Real Numbers

This section is devoted to the classical notion of summation of series in real analysis. Its purpose is not to say anything new, but only to make reformulations of first basic definitions convenient for our purposes.

Let $\mathbb{R}$ be the abelian group of real numbers, and, accordingly, $\mathbb{R}^{\{0,1,2,\ldots\}}$ the set of sequences $(a_0, a_1, a_2, \ldots)$ of real numbers. The *classical summation* (of real series) is the partial map $\Sigma_{cl} : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ written as

$$(a_i)_{i\in\mathbb{N}} \mapsto \Sigma_{i\in\mathbb{N}} a_i = a_0 + a_1 + a_2 + \cdots \tag{1.1}$$

and defined by

$$a_0 + a_1 + a_2 + \cdots = \lim_{n\to\infty} (a_0 + \cdots + a_n). \tag{1.2}$$

Note that since the codomain of Σ_{cl} is $\mathbb{R}$, the "lim" here is understood as a limit that is a real number.

A *regular summation* is any partial map $\Sigma : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ that extends Σ_{cl} to a larger domain; "larger" is understood here as "larger or equal", that is, the classical summation is a special case of a regular one. What is a *summation* in general? Usually one does not define it except saying that it should be a "reasonable" partial map $\Sigma : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$; we, however, will use the following:

Definition 1.1 A summation (of real series) is a partial map $\Sigma : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ written as in (1.1) and satisfying

$$a_0 + a_1 + a_2 + \cdots = a_0 + \cdots + a_n \tag{1.3}$$

whenever $a_m = 0$ for all $m > n$. The domain of Σ will be denoted by $dom(\Sigma)$.

Apart from naive motivations like "(1.3) is obviously reasonable", Definition 1.1 is motivated by the fact that (1.3) can essentially be deduced from each of the three standard conditions, namely, *regularity* (mentioned above), *linearity*, and *stability*, that are usually required. More precisely, the following is obvious:

Proposition 1.2 *A partial map $\Sigma : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ written as in 1.1 is a summation (in the sense Definition 1.1) whenever at least one of the following three conditions holds:*

(a) *Σ is a regular summation (as defined above);*
(b) *$0 + \cdots + 0 + a + 0 + 0 + \cdots = a$ for each real number a, and Σ is linear in the sense that $dom(\Sigma)$ is a subspace of the real vector space $\mathbb{R}^{\mathbb{N}}$, and Σ determines an $\mathbb{R}$-linear map $\Sigma : dom(\Sigma) \to \mathbb{R}$;*
(c) *$0 + 0 + 0 + \cdots = 0$, and Σ is stable in sense that we always have*

$$a_0 + a_1 + a_2 + \cdots = a_0 + (a_1 + a_2 + a_3 + \cdots), \tag{1.4}$$

which includes the requirement that $(a_0, a_1, a_2, \ldots)$ is in $dom(\Sigma)$ if and only if so is $(a_1, a_2, a_3, \ldots)$.

Remark 1.3 (a) It is interesting to compare mathematical texts from different centuries... Hardy writes on Page 6 of [4]: "...There are reservations to be made ... but it is broadly true to say that mathematicians before Cauchy asked not 'How shall we *define* $1-1+1-\cdots$?' but 'What *is* $1 - 1 + 1 - \cdots$?', and that this habit of mind led them into unnecessary perplexities and controversies which were often really verbal..." On the other hand, explaining that one should *define* each particular method of summation he does not say what exactly *is* a summmation *in general*. For instance, in spite of what we said after formulating Definition 1.1, we see no way to find out whether or not he would agree with that definition. We can only repeat another sentence from the same page of [4], which says: " Actually, all definitions which we shall use satisfy (A) and (B), and most, though not all, satisfy (C)." Recall that Hardy's (A) and (B) together mean linearity (defined as in 1.2(b)), while (C) means stability (as defined in 1.2(c)).

(b) In Kangro's survey [8], a *summation method* on a set X is a pair $A = (\bar{A}, A^*)$, in which A^* is a set of sequences of elements of X and $\bar{A} : A^* \to X$ is an arbitrary map (thus, Kangro's $\bar{A}$ and A^* play the roles of our $Dom(\Sigma)$ and Σ, respectively). It is not clear there whether X is equipped with any finitary addition, and if not, then our condition (1.3) cannot even be stated. However, the next page (Page 7 in [8]) says that in what follows only linear methods will be considered.

(c) Condition (1.3) is satisfied in abstract contexts considered in [5–7,12,13], and in the following two sections of the present paper.

2 Involving Infinity and Removing Negatives

Replacing the abelian group $\mathbb{R}$ of real numbers with the commutative monoid $[0, \infty]$ of non-negative real numbers with ∞ included, we can apply classical theory of absolute convergence, which gives:

- The classical summation Σ_{cl} restricted to the set of convergent sequences of finite elements of $[0, \infty]$ and then extended, in the usual way, to the set of sequences of all elements of $[0, \infty]$, becomes a map (not merely a partial map) from $[0, \infty]^{\mathbb{N}}$ to $[0, \infty]$. We shall denote this map by $\bar{\Sigma}_{cl}$.
- Moreover, the system $([0, \infty], 0, \bar{\Sigma}_{cl})$ is a series monoid in the sense of [7], which appears as Example 2.5 in that paper. We postpone recalling the definition of series monoid to Sect. 3, where it will be used.
- Let us define an *extended non-negative summation* $\Sigma : [0, \infty]^{\mathbb{N}} \to [0, \infty]$ simply by replacing $\mathbb{R}$ with $[0, \infty]$ in Definition 1.1. It is then natural to call such a summation Σ *regular* if the restrictions of Σ and Σ_{cl} on the set of convergent sequences of finite elements of $[0, \infty]$ coincide with each other. On the other hand, adapting the notion of *totally regular* (as defined in [4]) in this context is useless since the total regularity of Σ will become nothing but $\Sigma = \bar{\Sigma}_{cl}$.
- We could try to require an extended non-negative summation to be continuous, but again, this requirement will make it nothing but equal to $\bar{\Sigma}_{cl}$.

There are many examples of regular extended non-negative summations different from Σ_{cl}, obtained by simple modifications of regular summations of real series. Unfortunately, those simple modification will almost never preserve linearity and stability. Therefore it is not clear to us whether such extended non-negative summations could be useful in mathematical analysis. However, they can be useful in a general theory of series (magmas and) monoids, as Example 3.5(a) used in the proof of Theorem 3.9 will show.

3 Tarski's Elementary Arithmetic in Series Monoids

A series monoid was defined in [7] as a triple $A = (A, 0, \Sigma)$, in which A is a set, 0 an element in A, and $\Sigma : A^{\mathbb{N}} \to A$ a map written as

$$(a_i)_{i\in\mathbb{N}} \mapsto \Sigma_{i\in\mathbb{N}} a_i = a_0 + a_1 + a_2 + \cdots \tag{3.1}$$

and satisfying the identities

$$0 + \cdots + 0 + a + 0 + 0 + \cdots = a \tag{3.2}$$

(that is, $\Sigma_{i\in\mathbb{N}} a_i = a$ whenever there exists $n \in \mathbb{N}$ with $a_n = a$ and $a_i = 0$ for all $i \neq n$) and

$$\Sigma_{m\in\mathbb{N}} \Sigma_{n\in\mathbb{N}} a_{m,n} = \Sigma_{n\in\mathbb{N}} \Sigma_{m\in\mathbb{N}} a_{m,n}, \tag{3.3}$$

in obvious notation.

Observation 3.1 (a) A magnitude module in the sense of Higgs [5] can be defined as a series monoid equipped with an additional structure: see Definition 4.1 in [7]. However, obviously, not every series monoid admits such a structure.

(b) Every cardinal algebra in the sense of Tarski [12] is a series monoid. Indeed, for any cardinal algebra, (3.2) follows, for example, from Definition 1.1.III and Theorems 1.7 and 1.38 of [12], while (3.3) is essentially the same as Theorem 1.44 of [12].

(c) It is easy to find a series monoid that is not a cardinal algebra. Moreover, as Higgs shows [5], not every magnitude module is a cardinal algebra, and not every cardinal algebra admits a magnitude module structure.

In this section we will consider various possible additional conditions on series monoids suggested by the properties of cardinal algebras established in the first section of [12].

Recall that there are two equivalent definitions of cardinal algebras in [12]. The first of them uses finitary and infinitary addition introduced as two separate components of the structure, while the second one uses only the infinitary addition, allowing us to define, say, $a + b$ as

$$a + b = a + b + 0 + 0 + \cdots ; \tag{3.4}$$

the fact that (3.4) also holds under the first definition is presented as the third assertion of Theorem 1.8 of [12]. Having this in mind we can actually rephrase Observation 3.1(b) as:

Proposition 3.2 *A triple $A = (A, 0, \Sigma)$ is a cardinal algebra if and only if it is a series monoid satisfying the following conditions:*

(a) *(Refinement Postulate of* [12]*) For a, b, c_0, c_1, ... in A, if*

$$a + b = \Sigma_{i\in\mathbb{N}} c_i, \tag{3.5}$$

then there exist a_0, a_1, ..., b_0, b_1, ... in A in A with

$$a = \Sigma_{i\in\mathbb{N}} a_i, \ b = \Sigma_{i\in\mathbb{N}} b_i, \text{ and } c_n = a_n + b_n \tag{3.6}$$

for $n = 0, 1, 2, \ldots$.

(b) *(Remainder Postulate of* [12]*) For $a_0, a_1, \ldots, b_0, b_1, \ldots$ in A, if $a_n = b_n + a_{n+1}$ for $n = 0, 1, 2, \ldots$, then there exists c in A with*

$$a_n = c + \Sigma_{i\in\mathbb{N}} b_{n+i} \tag{3.7}$$

for $n = 0, 1, 2, \ldots$.

Remark 3.3 Nether of the conditions (a) and (b) of Proposition 3.2 can be omitted even if $A = (A, 0, \Sigma)$ is merely a complete lattice with Σ being its countable join operation; simple counter-examples showing this are given in [5].

It is natural to ask, what part of Tarski's Arithmetic of cardinal algebras developed in the book [12] is valid for series monoids? In this section we shall, however, only consider what Tarski calls "elementary consequences of postulates", that is, only the properties given in the first section of [12], as mentioned above. Accordingly, we shall never use 3.2(a) since it is not used in any proof in the first section of [12]. Our first several answers, collected in Observation 3.4 below, are rather trivial.

Observation 3.4 (a) All results of Sect. 1 of [12] whose proofs do not involve (Refinement and) Remainder Postulates trivially hold in any series monoid. This includes Theorems and Corollaries 1.6–1.28, some of which use the preorder relation $\leq$ on a given series monoid A defined by:

$$a \leq b \ \Leftrightarrow \ a + x = b \ \text{ for some } \ x \in A$$

(b) Four Corollaries 1.32–1.35 of [12] extend to series monoids, as easily follows from Remark 7.2 in [7], which is exactly the second of them stated in a more general context.

(c) Theorem 1.38 of [12] is proved there using several previous results some of which depend on the Remainder Postulate. However, it is also a special case of Proposition 2.8 in [7], and so it extends to series monoids. Theorems and Corollaries 1.39–1.45 (of [12]) are essentially deduced from it, and so they also extend to series monoids. This includes Theorem 1.44, which was already mentioned in Observation 3.1(b).

In order to give some more answers we will later use:

Example 3.5 Let $A = (A, 0, +, \infty)$ be an additive (commutative) monoid with infinity, in which no non-zero element has the additive inverse. We can form a series monoid $(A, 0, \Sigma)$, in which Σ is defined by

$$\Sigma_{i\in\mathbb{N}} a_i = \begin{cases} a_0 + \cdots + a_n \text{ in } A, \text{ if } i > n \Rightarrow a_i = 0 \text{ (for a given natural } n); \\ \infty, \text{ otherwise.} \end{cases} \tag{3.8}$$

We will use the following three special cases:

(a) $A = [0, \infty]$ is the monoid of non-negative real numbers with ∞ included.
(b) A is a complete lattice (or, more generally, an ordered set with countable joints that has a largest element), considered as an additive monoid whose addition is the join operation.
(c) $A = (\mathbb{N}/E) \cup \infty$, where E has three congruence classes: $[0] = \{0\}$, $[1] = \{1, 3, 5, \ldots\}$, and $[2] = \{2, 4, 6, \ldots\}$.

From now on, in this and the next section, $A = (A, 0, \Sigma)$ will denote a fixed series monoid, and a, b, c, possibly with indices, will denote elements of A. Note that $\infty \cdot a$ is defined as the infinite sum $a + a + \cdots$.

There are seven theorems on cardinal algebras, namely 1.29–1.31, 1.36, 1.37, 1.46, and 1.47, of [12], not mentioned in Observation 3.4. In order to analyze them, consider the following conditions on elements of A, where " (T1.xy)" indicates that the condition corresponds to Theorem 1.xy in [12]:

$$a + b = b \Leftrightarrow \infty \cdot a \leq b, \quad \text{(T1.29)}; \tag{3.9}$$

$$a + b = b \Rightarrow \infty \cdot a \leq b; \tag{3.10}$$

$$(a \leq b \;\&\; b + c = c) \Rightarrow a + c = c; \quad \text{(T1.30)} \tag{3.11}$$

$$(a \leq b \;\&\; b \leq a) \Rightarrow a = b, \quad \text{(T1.31)}; \tag{3.12}$$

$$\begin{aligned} a_0 = \Sigma_{i\in\mathbb{N}} b_i \Leftrightarrow ((\exists_{a_1,a_2,\ldots}\, \forall_{n\in\mathbb{N}}\; a_n = b_n + a_{n+1}) \\ \;\&\; \forall_{c_0,c_1,\ldots}\, ((\forall_{n\in\mathbb{N}}\; c_n = b_n + c_{n+1}) \Rightarrow a_0 \leq c_0)); \quad \text{(T1.36)} \end{aligned} \tag{3.13}$$

$$(\forall_{n\in\mathbb{N}}\; a_n + b_{n+1} = b_n) \Rightarrow \Sigma_{i\in\mathbb{N}} a_i \leq b_0; \tag{3.14}$$

$$\infty \cdot a = b \Leftrightarrow (a + b = b \;\&\; \forall_c\, (a + c = c \Rightarrow b \leq c)); \quad \text{(T1.37)} \tag{3.15}$$

$$\forall_{n\in\mathbb{N}\cup\{\infty\}}\; (a + b = b \Leftrightarrow n \cdot a + b = b); \quad \text{(T1.46)} \tag{3.16}$$

$$\forall_{n\in\mathbb{N}\cup\{\infty\}}\; (a + b = b \Rightarrow n \cdot a + b = b); \tag{3.17}$$

$$a + b = b \Leftrightarrow \infty \cdot a + b = b; \tag{3.18}$$

$$a + b = b \Rightarrow \infty \cdot a + b = b; \tag{3.19}$$

$$\forall_{n\in\mathbb{N}\cup\{\infty\}}\ ((\forall_{i<n}\ a_i + b = b) \Leftrightarrow \Sigma_{i<n}a_i + b = b); \quad \text{(T1.47)} \tag{3.20}$$

$$\forall_{n\in\mathbb{N}\cup\{\infty\}}\ ((\forall_{i<n}\ a_i + b = b) \Rightarrow \Sigma_{i<n}a_i + b = b); \tag{3.21}$$

$$(\forall_{i\in\mathbb{N}}\ a_i + b = b) \Leftrightarrow \Sigma_{i\in\mathbb{N}}a_i + b = b; \tag{3.22}$$

$$(\forall_{i\in\mathbb{N}}\ a_i + b = b) \Rightarrow \Sigma_{i\in\mathbb{N}}a_i + b = b. \tag{3.23}$$

Remark 3.6 Note the following trivial implications obtained by replacing $\Leftrightarrow$ with $\Rightarrow$ [in (a)–(e)], by taking constant sequences [in (f)–(h)], by removing the finite sequences [in (i)–l)], and by definition of $\leq$ [in (m) and (n)]:

(a) (3.9)⇒(3.10);
(b) (3.16)⇒(3.17);
(c) (3.18)⇒(3.19);
(d) (3.20)⇒(3.21);
(e) (3.22)⇒(3.23);
(f) (3.14)⇒(3.10);
(g) (3.22)⇒(3.18);
(h) (3.23)⇒(3.19);
(i) (3.16)⇒(3.18);
(j) (3.17)⇒(3.19);
(k) (3.20)⇒(3.22);
(l) (3.21)⇒(3.23);
(m) (Remainder Postulate 3.2(b))⇒(3.14);
(n) (3.19)⇒(3.10);

Lemma 3.7 (a) *Replacing* $\Leftrightarrow$ *in* (3.9) *with* $\Leftarrow$, *we obtain an implication that always holds.*
(b) *Replacing* $\Leftrightarrow$ *in* (3.13) *with* $\Rightarrow$, *we obtain a condition equivalent to* (3.14).
(c) *Replacing* $\Leftrightarrow$ *in* (3.15) *with* $\Rightarrow$, *we obtain a condition equivalent to* (3.10).
(d) *Replacing* $\Leftrightarrow$ *in any of the conditions* (3.16), (3.18), (3.20), *and* (3.22) *with* $\Leftarrow$, *we obtain an implication that follows from* (3.12).

Proof (a) $\infty \cdot a \leq b$ gives $\infty \cdot a + c = b$ for some c, and then $a + b = a + \infty \cdot a + c = \infty \cdot a + c = b$.
(b) The desired implication becomes:

$$a_0 = \Sigma_{i\in\mathbb{N}}b_i \Rightarrow$$
$$((\exists_{a_1,a_2,\ldots}\ \forall_{n\in\mathbb{N}}\ a_n = b_n + a_{n+1})\ \&\ \forall_{c_0,c_1,\ldots}\ ((\forall_{n\in\mathbb{N}}\ c_n = b_n + c_{n+1}) \Rightarrow a_0 \leq c_0)), \tag{3.24}$$

which is of course the conjunction of

$$a_0 = \Sigma_{i\in\mathbb{N}}b_i \Rightarrow (\exists_{a_1,a_2,\ldots}\ \forall_{n\in\mathbb{N}}\ a_n = b_n + a_{n+1}), \tag{3.25}$$

and

$$a_0 = \Sigma_{i\in\mathbb{N}} b_i \Rightarrow \forall_{c_0,c_1,\ldots}\,((\forall_{n\in\mathbb{N}}\; c_n = b_n + c_{n+1}) \Rightarrow a_0 \leq c_0), \tag{3.26}$$

and we need to prove that this conjunction is equivalent to (3.14). However, (3.25) always holds since can take $a_n = \Sigma_{i\in\mathbb{N}} b_{n+i}$ (for each $n > 0$), while (3.25) is nothing but (3.14).

(c) is nothing but a simplified version of (b), where all sequences are supposed to be constant.

(d) Our four assertions here follow from obvious inequalities: for example for the first one, assuming $n \cdot a + b = b$, we have $a + b \leq n \cdot a + b = b$ and then $a + b = b$ follows from $b \leq a + b$ and (3.12).

□

Lemma 3.8 (3.10) ⇒ (3.9) ⇒ (3.11) ⇒ (3.12).

Proof (3.10)⇒(3.9) follows from Lemma 3.7(a).

(3.9)⇒(3.11): Assuming (3.9) we can rewrite (3.11) as

$$(a \leq b \;\&\; \infty \cdot b \leq c) \Rightarrow \infty \cdot a \leq c, \tag{3.27}$$

which is an obvious implication.

(3.11)⇒(3.12): By definition of $\leq$, (3.12) is the same as

$$(a + c = b \;\&\; b + d = a) \Rightarrow a = b. \tag{3.28}$$

Suppose (3.11) holds. Then $(a + c = b \;\&\; b + d = a)$ gives $(c + d) + a = a$, which, together with the trivial inequality $c \leq c + d$, gives $c + a = a$ and then $b = a$. □

Theorem 3.9 (a) (3.13) ⇔ (3.14).

(b) (3.20) ⇔ (3.21) ⇔ (3.22) ⇔ (3.23).

(c) (3.9) ⇔ (3.10) ⇔ (3.15) ⇔ (3.16) ⇔ (3.17) ⇔ (3.18) ⇔ (3.19).

(d) (3.11) ⇔ (3.12).

(e) *Remainder Postulate* 3.2(b) *is strictly stronger than all the listed conditions; each, out of* (a), (b), (c), *group of conditions is strictly stronger than the next one, and not every series monoid satisfies the equivalent conditions of* (d).

Proof (a) According to Lemma 3.7(b) we only need to prove that the implication

$$((\exists_{a_1,a_2,\ldots}\; \forall_{n\in\mathbb{N}}\; a_n = b_n + a_{n+1}) \;\&\; \forall_{c_0,c_1,\ldots}\,((\forall_{n\in\mathbb{N}}\; c_n = b_n + c_{n+1}) \Rightarrow a_0 \leq c_0))$$

$$\Rightarrow a_0 = \Sigma_{i\in\mathbb{N}} b_i, \tag{3.29}$$

obtained from (3.13) by replacing ⇔ with ⇐, follows from (3.14). Let us rewrite this implication as $(\alpha \;\&\; \beta) \Rightarrow a_0 = \Sigma_{i\in\mathbb{N}} b_i$ for short. Then α gives $\Sigma_{i\in\mathbb{N}} b_i \leq a_0$ by (3.14), while β applied to the sequence $c_0, c_1, \ldots$ defined by $c_n = \Sigma_{i\in\mathbb{N}} b_{n+i}$ (for all n) gives $a_0 \leq \Sigma_{i\in\mathbb{N}} b_i$. After that we only need to note that (3.14) also implies (3.12), which follows from Remark 3.6(f) and Lemma 3.8.

(b) Thanks to 3.6(d), 3.6(e), 3.6(k), and 3.6(l), we only need to prove that (3.23) implies (3.20). Moreover, our proof can use (3.12) since it follows from (3.23) (by 3.6(h), 3.6(n), and Lemma 3.8). First note that replacing ⇔ with ⇐ in (3.20) we obtain a statement that follows from (3.12); indeed, assuming $\Sigma_{i<n} a_i + b = b$ we will have $a_i + b \leq \Sigma_{i<n} a_i + b = b$ together with the trivial $b \leq a_i + b$. Therefore it suffices to prove (3.21). Next, the implication inside (3.21) becomes obvious when n is a natural number, while after excluding that (3.21) becomes nothing but (3.23).

(c) Conditions (3.9) and (3.10) are equivalent to each other by Lemma 3.7(a) (or Lemma 3.8). We also have (3.16)⇒(3.17)⇒(3.19)⇒(3.10), (3.16)⇒(3.18)⇒(3.19), and (3.15)⇒(3.10) by 3.6(b), 3.6(j), 3.6(n), 3.6(i), 3.6(c), and 3.8(c), respectively. Therefore we only need to prove that (3.10) implies (3.15) and (3.16), and again, our proofs can use (3.12) since it follows from (3.10) (by Lemma 3.8). The implication (3.10)⇒(3.15) can be proved similarly to (a) using 3.7(c) instead of 3.7(b), and it remains to prove (3.10)⇒(3.16). Again (cf. proof of (3.21) in the proof of (b)), the equivalence inside (3.16) becomes obvious when n is a natural number, while after excluding that (3.16) becomes the same as (3.18). Moreover, as follows from 3.7(d), we can replace it further with (3.19). Assuming (3.10) and $a + b = b$, we have $\infty \cdot (\infty \cdot a) = \infty \cdot a \leq b$, which gives the desired $\infty \cdot a = b$ by (3.9) (which we can use since it is equivalent to (3.10)).

(d) (3.11) implies (3.12) by Lemma 3.8, and so we only need to prove the implication (3.12)⇒(3.11). Assuming (3.12) and having in mind the trivial inequality $c \leq a + c$, we can rewrite (3.11) as

$$(a \leq b \;\&\; b + c = c) \Rightarrow a + c \leq c. \tag{3.30}$$

But this implication always holds since $a \leq b$ gives $a + d = b$ for some d, and so $a \leq b \;\&\; b + c = c$ gives $c = a + c + d$.

(e) Let us begin by proving the implication (3.14)⇒(3.23). Assuming (3.14) and $\forall_{i\in\mathbb{N}}\, a_i + b = b$, we observe:

(i) $\infty \cdot a_i + b = b$ for all i, by (3.19) (which we can use since it is equivalent to (3.10) by (c), while (3.10) follows from (3.14) by 3.6(f));

(ii) $\infty \cdot \Sigma_{i\in\mathbb{N}} a_i = \Sigma_{i\in\mathbb{N}} \infty \cdot a_i \leq b$, where the inequality follows from (i) and (3.14) in which a_i is replaced with $\infty \cdot a_i$ and b_i is replaced with b (for all i).

(iii) $\Sigma_{i\in\mathbb{N}} a_i + b = b$, by (ii) and (3.9) (and we can indeed use (3.9) since it follows from (3.14), as we see from (c) and 3.6(f)).

This proves the desired implication. □

Now we have 3.2(b)⇒(3.14)⇒(3.23)⇒(3.19)⇔(3.9)⇒(3.11), where:

- 3.2(b)⇒(3.14) is 3.6(m);
- (3.14)⇒(3.23) was just proved;
- (3.23)⇒(3.19) is 3.6(h);
- (3.19)⇔(3.9) is a part of (c);
- (3.9)⇒(3.11) is a part of Lemma 3.8.

That is, we have all the desired implications, and we only need to find five examples of series monoids, having the following properties, respectively:

(I) (3.14) but not the Remainder Postulate 3.2(b);
(II) (3.23) but not (3.14);
(III) (3.10) but not (3.23);
(IV) (3.12) but not (3.10);
(V) not (3.12).

We have:

(I) Every complete lattice (considered as a series monoid) obviously satisfies (3.14), but, as already mentioned in Remark 3.3, an example of a complete lattice that does not satisfy the Remainder Postulate 3.2(b) is constructed by Higgs in [5].

(II) In the series monoid of Example 3.5(a), $\forall_{i\in\mathbb{N}}\ a_i + b = b$ with at least one a_i being non-zero would make $b = \infty$. Therefore it satisfies (3.23). Now choose $a_0, a_1, \ldots$ to be the sequence $1/2, 1/4, 1/8, \ldots$ and $b_0, b_1, \ldots$ to be the sequence $1, 1/2, 1/4, \ldots$. Then $\forall_{n\in\mathbb{N}}\ a_n + b_{n+1} = b_n$ but $\Sigma_{i\in\mathbb{N}} a_i = \infty > b_0$. Therefore this series monoid does not satisfy (3.14).

(III) Let $A = (((\mathbb{N}\times\{0,1\})\cup\{(\infty,1)\})\setminus\{(0,1)\}), (0,0), \Sigma)$ be the series monoid in which Σ is defined by

$$\Sigma_{i\in\mathbb{N}}(x_i, k_i) = \begin{cases} (\Sigma_{k_i=1} x_i, 1), & \text{if } \exists_i\ k_i = 1 \text{ and } \{x_i \mid k_i = 0\} \text{ is a bounded set;} \\ (x_j, 0), & \text{if } \forall_i\ k_i = 0 \text{ and } x_j \le x_i \Rightarrow x_j = x_i; \\ (\infty, 1), & \text{otherwise.} \end{cases} \tag{3.31}$$

We omit a long but routine verification of the fact that this indeed determines a series monoid. Let us only mention that the corresponding binary addition becomes:

$$(x,k) + (y,l) = \begin{cases} (\max\{x,y\}, 0), & \text{if } k = l = 0; \\ (x, 1), & \text{if } k = 1 \text{ and } l = 0; \\ (y, 1), & \text{if } k = 0 \text{ and } l = 1; \\ (x + y, 1), & \text{if } k = l = 1. \end{cases} \tag{3.32}$$

Note also that

$$\infty \cdot (x,k) = \begin{cases} (x,0) = (x,k), & \text{if } k = 0; \\ (\infty,1) = (\infty,k), & \text{if } k = 1. \end{cases} \tag{3.33}$$

Using (3.33) it is easy to show that A satisfies (3.10). However, it does not satisfy (3.23), since we have $\forall_{i\in\mathbb{N}}\ (i,0) + (1,1) = (1,1)$ but $\Sigma_{i\in\mathbb{N}}(i,0) + (1,1) = (\infty,1) \neq (1,1)$.

(IV) The series monoid of Example 3.5(b) obviously satisfies (3.12), but, as soon as it has more than one element, it does not satisfy (3.10).

(V) The series monoid of Example 3.5(c) does not satisfy (3.12), since $[1] \le [2]$ and $[2] \le [3] = [1]$ there.

4 Tarski's Fundamental Law of Infinite Addition

According to [12], the *Fundamental Law of Infinite Addition* is formulated as

$$(\forall_{n\in\mathbb{N}}\ \Sigma_{i<n} a_i \le b) \Rightarrow \Sigma_{n\in\mathbb{N}} a_i \le b. \tag{4.1}$$

Since we are considering only series monoids (with $\le$ defined as in Sect. 3), here $\Rightarrow$ can equivalently be replaced with $\Leftrightarrow$.

Theorem 4.1 *In any series monoid $A = (A, 0, \Sigma)$, we have:*

(a) (4.1) $\Rightarrow$ (3.14).

(b) *If* (3.14) *holds and A admits countable joins with respect to the order $\le$ defined as in Sect. 3, then* (4.1) *also holds.*

Proof (a) Let $a_0, a_1, \ldots$ and $b_0, b_1, \ldots$ be sequences satisfying $(\forall_{n\in\mathbb{N}}\ a_n + b_{n+1} = b_n)$. We have $b_0 = a_0 + b_1 = a_0 + a_1 + b_2 = \cdots = \Sigma_{i<n} a_i + b_n \ge \Sigma_{i<n} a_i$, which gives the desired $\Sigma_{i<n} a_i \le b_0$ by (4.1).

(b) Let A' be the series monoid with the same underlying set A and infinitary addition Σ' defined by

$$\Sigma'_{i\in\mathbb{N}} a_i = \vee_{n\in\mathbb{N}} \Sigma_{i<n} a_i, \tag{4.2}$$

where $\vee$ is the countable join operation whose existence is required in (b). Assuming that A satisfies (3.14), and having in mind that (3.14) implies (3.12) (for example, by Theorem 3.9), we observe:

- A' obviously satisfies (4.1);
- since A' satisfies (4.1), it also satisfies (3.14) by (a);
- since A and A' both satisfy (3.14), they also satisfy (3.13) by Theorem 3.9(a);
- since A and A' both satisfy (3.13) and (obviously) determine the same finitary addition, we obtain $A = A'$ (as series monoids, that is, $\Sigma = \Sigma'$);
- since $A = A'$ and A' satisfies (4.1), A also satisfies (4.1), as desired.

□

Theorem 4.1 suggests to modify the notion of series monoid by involving an independently defined order, and to consider a number of new additional conditions on such a structure. We will only formulate these conditions and give some examples. In particular, in order to restrict the size of this paper, we shall not discuss the connection with the material of Sect. 3 of [12], although certainly it would be important to do so.

Definition 4.2 An analytic monoid is a system $A = (A, 0, \Sigma, \preceq)$ in which:

(a) $(A, 0, \Sigma)$ is a series monoid and $(A, \preceq)$ is an ordered set whose smallest element is 0;
(b) $(\forall_{n\in\mathbb{N}}\ \Sigma_{i<n} a_i \preceq b) \Rightarrow \Sigma_{n\in\mathbb{N}} a_i \preceq b$ holds for every $(a_i)_{i\in\mathbb{N}}$ in $A^{\mathbb{N}}$.

Condition 4.3 For an analytic monoid $A = (A, 0, \Sigma, \preceq)$, one might require one or more of the following conditions:

(a) The ordered set $(A, \preceq)$ has directed countable joins;
(b) The ordered set $(A, \preceq)$ has countable joins;
(c) $a \preceq a + b$ for every a and b in A;
(d) $a \preceq b \Rightarrow a + c \preceq b + c$ for every a, b, and c in A;
(e) Condition (3.12) holds, and the ordered set $(A, \leq)$, in which $\leq$ is defined as in Sect. 3, has directed countable joins;
(f) Condition (3.12) holds, and the ordered set $(A, \leq)$, in which $\leq$ is defined as in Sect. 3, has countable joins;
(g) Condition (3.12) holds and the order $\leq$ defined as in Sect. 3 coincides with the order $\preceq$.

Observation 4.4 (a) Note the trivial implications (b) $\Rightarrow$ (a), (d) $\Rightarrow$ (c), (f) $\Rightarrow$ (e), and (g) $\Rightarrow$ (d) among the parts of Condition 4.3.

(b) If 4.3(c) holds, then $\Rightarrow$ in 4.2(b) can be replaced with $\Leftrightarrow$; if so, then 4.2(b) becomes

$$\Sigma_{i\in\mathbb{N}} a_i = \vee_{n\in\mathbb{N}} \Sigma_{i<n} a_i, \tag{4.3}$$

for all $(a_i)_{i\in\mathbb{N}}$ in $A^{\mathbb{N}}$, where $\vee$ is the join with respect to the order $\preceq$; in particular, all such joins must exist. In particular, 4.2(b) and 4.3(g) imply (4.3), and, by Theorems 3.9 and 4.1, also imply (3.9)–(3.23).

(c) An analytic monoid satisfying 4.3(g) can equivalently be defined as a series monoid satisfying (4.1), or as a commutative monoid $(A, 0, +)$, for which $a \preceq b \Leftrightarrow \exists_x\ a + x = b$ defines an order relation $\preceq$ on A such that the join $\vee_{n\in\mathbb{N}}(a_0 +$

$\cdots + a_n)$ does exist for each $(a_i)_{i\in\mathbb{N}}$ in $A^{\mathbb{N}}$. In the series monoid determined by such $(A, 0, +)$, we will have

$$\Sigma_{i\in\mathbb{N}} a_i = \vee_{n\in\mathbb{N}}(a_0 + \cdots + a_n), \tag{4.4}$$

which of course agrees with (4.3), and agrees, in a sense, with (1.2).

Example 4.5 The series monoid $(\mathbb{N} \cup \{\infty\}, 0, \Sigma)$, where Σ is "standard" being defined by (3.8) with the usual meaning of finitary $+$, is, as shown in [7], the free series monoid on $\{1\}$. It obviously satisfies (4.1), making $(\mathbb{N} \cup \{\infty\}, 0, \Sigma, \leq)$, with $\leq$ being the usual order of $\mathbb{N} \cup \{\infty\}$, an analytic monoid satisfying 4.3(g).

Example 4.6 Let $((\mathbb{N} \setminus \{1\}) \cup \{\infty\}, 0, \Sigma)$ be the series monoid in which Σ is defined as above. Let us consider three ways to make it an analytic monoid by choosing three different orders on the set $(\mathbb{N} \setminus \{1\})$:

(a) The standard order, which gives an analytic monoid satisfying 4.3(a)–4.3(e) but not 4.3(f)–4.3(g);
(b) The order defined as in Sect. 3, which gives an analytic monoid satisfying 4.3(a), 4.3(c)–4.3(e) but not 4.3(b) and not 4.3(f)–4.3(g);
(c) The order defined by

$$a \leq b \Leftrightarrow \begin{cases} \exists_x\ a + x = b, \text{ or} \\ a = 3 \text{ and } b \neq 0, \end{cases} \tag{4.5}$$

which gives an analytic monoid satisfying 4.3(a), 4.3(c), 4.3(e), but none of the other conditions from 4.3(a)–4.3(g).

Example 4.7 Let $M = (M, 0, +)$ be a commutative monoid and A be the set of all subsets of M containing 0. Putting

$$a + b = \{x + y \mid x \in a \ \&\ y \in b\} \tag{4.6}$$

makes $(A, \{0\}, +, M)$ a commutative monoid with infinity. After that:

(a) We can make $(A, \{0\}, \Sigma)$ a series monoid as in Example 3.5. However, neither the inclusion order, nor the order defined as in Sect. 3 will make it an analytic monoid.
(b) On the other hand, defining Σ by

$$\Sigma_{i\in\mathbb{N}} a_i = \cup_{n\in\mathbb{N}}(a_0 + \cdots + a_n) \tag{4.7}$$

will make $(A, \{0\}, \Sigma, \subseteq)$ an analytic monoid satisfying 4.3(a)–4.3(f) (but almost never 4.3(g)).

5 Back to Non-negative Real Numbers

This section is a slightly reformulated copy of a part of Sect. 4 of [7].

As mentioned in Sect. 2, the system $([0, \infty], 0, \bar{\Sigma}_{cl})$ is a series monoid presented as Example 2.5 in [7]; we might simply call it the series monoid of non-negative reals with infinity. How far is it from a free series monoid? Our answer, which is Theorem 5.1 below, corresponding to Remark 4.7 in [7], is an easy consequence of the main result of [5].

Theorem 5.1 *The series monoid of non-negative reals with infinity can be described as the series monoid freely generated by a countable set* $X = \{x_0, x_1, x_2, \ldots\}$ *with the identities*

$$x_n = x_{n+1} + x_{n+2} + x_{n+3} + \cdots, \tag{5.1}$$

required for each $n = 0, 1, 2, \ldots$.

Next, since the identities (5.1) immediately imply

$$x_n = 2x_{n+1}, \tag{5.2}$$

it is natural to ask, what is the series monoid freely generated by the same set X and the identities (5.2)? This series monoid is nothing but $(\mathrm{Z}[0, \infty], 0, \bar{\Sigma}_z)$, where (Z stands for *Zeno*, cf. Introduction of [7], and) $\mathrm{Z}[0, \infty]$ is what is called the *paradoxical positive reals* in Remark 4.9 of [7] together with ∞, with $\bar{\Sigma}_z$ to be the addition as defined there, extended in the standard way. Recall that the paradoxical positive (better to say "non-negative") reals can be naturally presented as infinite binary expansions, where, however, $0.111... \neq 1.000...$, etc.

Let us also recall that the construction of the paradoxical positive reals is based on the following general semigroup construction:

Given a semigroup homomorphism $k : S \to X$, and assuming the underlying sets of S and X to be disjoint, we can make the set-theoretic union $X \cup S$ a semigroup by defining its multiplication via

$$xy = \begin{cases} xy \text{ in } X, & \text{if } x \text{ and } y \text{ are in } X; \\ k(x)y \text{ in } X, & \text{if } x \text{ is in } S \text{ and } y \text{ is in } X; \\ xk(y) \text{ in } X, & \text{if } x \text{ is in } X \text{ and } y \text{ is in } S; \\ xy \text{ in } S, & \text{if } x \text{ and } y \text{ are in } S. \end{cases} \tag{5.3}$$

The construction (5.3) is itself paradoxical in a sense: just as having subtraction for paradoxical real numbers one would quickly destroy them by proving $0.111... = 1.000...$, it is easy to see that there are no counterparts of (5.3) for groups and many other algebraic structures. This brings our next question: is there any abstract-monadic generalization of the construction (5.3)? The answer is: yes, and it is presented in the next section (see Theorem 6.3 and Example 6.6). We shall use the term *Zeno algebra* there (see Definition 6.4), motivated by $0.111... \neq 1.000...$, seen as *Zeno paradox*.

6 A Monadic Generalization

A morphism $f : A \to C$ in a category $\mathcal{X}$ is called an *injection* when there exists a morphism $g : B \to C$ such that $A \xrightarrow{f} C \xleftarrow{g} B$ is a coproduct diagram.

Definition 6.1 A monad $T = (T, \eta, \mu)$ on a category $\mathcal{X}$ with finite coproducts is called *injection taut* when

(a) $0 \to T0$ is an isomorphism;
(b) T takes injections to injections;
(c) T preserves pullbacks of injections along arbitrary morphisms;
(d) the naturality squares for η and μ at injections are pullbacks.

When $\mathcal{X} = \mathrm{Set}$, this is the same as a taut monad in the sense of Manes [10], which preserves the initial object 0.

Let $T = (T, \eta, \mu)$ be an injection taut monad on a lextensive category $\mathcal{X}$, let $k : (S, \sigma) \to (X, \chi)$ be a morphism of T-algebras, and let

$$X \xrightarrow{\iota_1} X + S \xleftarrow{\iota_2} S \tag{6.1}$$

be the coproduct diagram for X and S in the category $\mathcal{X}$. Then $T(S)$ is complemented in $T(X + S)$ and $T^2(S)$ is complemented in $T^2(X + S)$, so there exist objects $T_1(X, S)$ and $T_2(X, S)$ in $\mathcal{X}$, and morphisms $i_1 : T_1(X, S) \to T(X, S)$ and $i_2 : T_2(X, S) \to T^2(X, S)$ such that the diagrams

$$T_1(X, S) \xrightarrow{i_1} T(X + S) \xleftarrow{T(\iota_2)} T(S) \tag{6.2}$$

$$T_2(X, S) \xrightarrow{i_2} T^2(X + S) \xleftarrow{T^2(\iota_2)} T^2(S) \tag{6.3}$$

are coproduct diagrams.

Let $\varphi = \chi T([1_X, k])i_1 + \sigma : T(X + S) \to X + S$ be the morphism defined via the commutative diagram

$$\begin{array}{ccccc} T_1(X, S) & \xrightarrow{i_1} & T(X + S) & \xleftarrow{T(\iota_2)} & T(S) \\ {\scriptstyle \chi T([1_X,k])i_1}\downarrow & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \sigma} \\ X & \xrightarrow[\iota_1]{} & X + S & \xleftarrow[\iota_2]{} & S\,. \end{array} \tag{6.4}$$

Lemma 6.2 (a) *There exists a (unique) morphism $j : T(X) \to T_1(X, S)$ with $i_1 j = T(\iota_1)$.*
(b) *There exists a (unique) morphism $m : T_2(X, S) \to T_1(X, S)$ with $i_1 m = \mu_{X+S} i_2$.*
(c) *There exists a (unique) morphism $t : T_2(X, S) \to T_1(X, S)$ with $i_1 t = T(\varphi) i_2$.*
(d) $[1_X, k]\varphi = \chi T([1_X, k])$.

Proof (a) Consider the commutative diagram

$$\begin{array}{ccccc} P & \xrightarrow{q} & T(X) & \xleftarrow{} & 0 \\ {\scriptstyle p}\downarrow & ① & \downarrow{\scriptstyle T(\iota_1)} & ② & \downarrow \\ T_1(X, S) & \xrightarrow[i_1]{} & T(X + S) & \xleftarrow[T(\iota_2)]{} & T(S) \end{array} \tag{6.5}$$

whose square ① is a pullback. Since $\mathcal{X}$ is lextensive, T is injection taut, the square ② also is a pullback. Since $\mathcal{X}$ is lextensive and both squares ① and ② are pullbacks, the top row of (6.5) is a coproduct diagram. Since the top row of (6.5) is a coproduct diagram, q is an isomorphism. Since q is an isomorphism, we can simply take $j = pq^{-1}$.

(b) and (c) can be proved similarly using the commutative diagrams

$$\begin{array}{ccccc} P' & \xrightarrow{q'} & T_2(X, S) & \xleftarrow{} & 0 \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle \mu_{X+S} i_2} & & \downarrow \\ T_1(X, S) & \xrightarrow[i_1]{} & T(X + S) & \xleftarrow[T(\iota_2)]{} & T(S) \end{array} \tag{6.6}$$

$$\begin{array}{ccccc} P'' & \xrightarrow{q''} & T_2(X,S) & \xleftarrow{} & 0 \\ {\scriptstyle p''}\downarrow & & \downarrow{\scriptstyle T(\varphi)i_2} & & \downarrow \\ T_1(X,S) & \xrightarrow[i_1]{} & T(X+S) & \xleftarrow[T(\iota_2)]{} & T(S) \end{array} \tag{6.7}$$

whose left-hand squares are pullbacks. For, we only need to show that the right-hand squares of these diagrams are pullbacks too. Since T is injection taut, the diagrams

$$\begin{array}{ccc} T^2(X+S) & \xleftarrow{T^2(\iota_2)} & T^2(S) \\ {\scriptstyle \mu_{X+S}}\downarrow & & \downarrow{\scriptstyle \mu_S} \\ T(X+S) & \xleftarrow[T(\iota_2)]{} & T(S) \end{array} \qquad \begin{array}{ccc} T^2(X+S) & \xleftarrow{T^2(\iota_2)} & T^2(S) \\ {\scriptstyle T(\varphi)}\downarrow & & \downarrow{\scriptstyle T(\sigma)} \\ T(X+S) & \xleftarrow[T(\iota_2)]{} & T(S) \end{array} \tag{6.8}$$

are pullbacks (for the second one this also uses the fact that the right-hand square of (6.4) is a pullback). Therefore to prove that the right-hand squares of diagrams (6.6) and (6.7) are pullbacks is to prove that so is the diagram

$$\begin{array}{ccc} T_2(X+S) & \xleftarrow{} & 0 \\ {\scriptstyle i_2}\downarrow & & \downarrow \\ T^2(X+S) & \xleftarrow[T^2(\iota_2)]{} & T^2(S) \end{array} \tag{6.9}$$

But this follows from the definition of i_2, since $\mathcal{X}$ is lextensive.
(d) We have:
$[1_X, k]\varphi i_1 = [1_X, k]i_1\chi T([1_X, k])i_1$ (by the commutativity of (6.4))
$= \chi T([1_X, k])i_1$
and
$[1_X, k]\varphi T(\iota_2) = [1_X, k]\iota_2\sigma$ (by the commutativity of (6.4))
$= k\sigma = \chi T(k)$ (since k is a T-algebra morphism from (S, σ) to (X, χ))
$= \chi T([1_X, k])T(\iota_2)$.

Since (6.2) is a coproduct diagram, this gives $[1_X, k]\varphi = \chi T([1_X, k])$, as desired.

□

Theorem 6.3 *For every morphism $k : (S, \sigma) \to (X, \chi)$ of T-algebras, the pair $(X + S, \chi T([1_X, k])i_1 + \sigma)$, in which i_1 is as in the coproduct diagram (6.2), is a T-algebra.*

Proof. Denoting again $\chi T([1_X, k])i_1 + \sigma$ by φ, consider the commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\iota_1} & X+S & \xleftarrow{\iota_2} & S \\ {\scriptstyle j\eta_X}\downarrow & & \downarrow{\scriptstyle \eta_{X+S}} & & \downarrow{\scriptstyle \eta_S} \\ T_1(X,S) & \xrightarrow[i_1]{} & T(X+S) & \xleftarrow[T(\iota_2)]{} & T(S) \\ {\scriptstyle \chi T([1_X,k])i_1}\downarrow & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \sigma} \\ X & \xrightarrow[\iota_1]{} & X+S & \xleftarrow[\iota_2]{} & S \end{array} \tag{6.10}$$

where j is as in Lemma 6.2(a). For its left-hand and right-hand composites of vertical arrows, we have:

$$\begin{aligned}
&\chi T([1_X, k])i_1 j\eta_X = T([1_X, k])T(\iota_1)\eta_X \quad (\text{since } i_1 j = T(\iota_1) \text{ in Lemma 6.2(a)})\\
&= \chi\eta_X \quad (\text{since } [1_X, k]\iota_1 = 1_X)\\
&= 1_X \quad (\text{since } (X, \chi) \text{ is a } T\text{-algebra}),
\end{aligned}$$

and $\sigma\eta_S = 1_S$ (since (S, σ) is a T-algebra). Therefore $\varphi\eta_{X+S} = 1_{X+S}$.

It remains to prove the equality $\varphi\mu_{X+S} = \varphi T(\varphi)$. Consider the diagram

$$\begin{array}{ccccc}
T_2(X,S) & \xrightarrow{i_2} & T^2(X+S) & \xleftarrow{T^2(\iota_2)} & T^2(S)\\
{\scriptstyle m}\downarrow\downarrow{\scriptstyle t} & & {\scriptstyle \mu_{X+S}}\downarrow\downarrow{\scriptstyle T(\varphi)} & & {\scriptstyle \mu_S}\downarrow\downarrow{\scriptstyle T(\sigma)}\\
T_1(X,S) & \xrightarrow[i_1]{} & T(X+S) & \xleftarrow[T(\iota_2)]{} & T(S)\\
{\scriptstyle \chi T([1_X,k])i_1}\downarrow & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \sigma}\\
X & \xrightarrow[\iota_1]{} & X+S & \xleftarrow[\iota_2]{} & S
\end{array} \tag{6.11}$$

where m and t are as in Lemma 6.2. This diagram is "reasonably" commutative in the sense that its triples (i_2, i_1, ι_1) and $(T^2(\iota_2), T(\iota_2), \iota_2)$ are morphisms of forks. More precisely, we have:

$$\begin{aligned}
&\chi T([1_X, k])i_1 m = \chi T([1_X, k])\mu_{X+S} i_2 \quad (\text{since } i_1 m = \mu_{X+S} i_2 \text{ in Lemma 6.2(b)})\\
&= \chi\mu_X T^2([1_X, k])i_2 \quad (\text{by the naturality of } \mu)\\
&= \chi T(\chi)T^2([1_X, k])i_2 \quad (\text{since } (X, \chi) \text{ is a } T\text{-algebra})\\
&= \chi T([1_X, k])T(\varphi)i_2 \quad (\text{since } \chi T([1_X, k]) = [1_X, k]\varphi, \text{ by Lemma 6.2(d)})\\
&= \chi T([1_X, k])i_1 t \quad (\text{since } T(\varphi)i_2 = i_1 t, \text{ in Lemma 6.2(c)}),
\end{aligned}$$

and $\sigma\mu_S = \sigma T(\sigma)$ (since (S, σ) is a T-algebra), which tells us that both left-hand and right-hand column of (6.11) are forks. Since the top row of (6.11) is a coproduct diagram this implies that the middle column of (6.11) also is a fork; that is, $\varphi\mu_{X+S} = \varphi T(\varphi)$, as desired.

Definition 6.4 For a morphism $k : (S, \sigma) \to (X, \chi)$ of T-algebras, the T-algebra $(X + S, \chi T([1_X, k])i_1 + \sigma)$ constructed above will be called the Zeno algebra of k and denoted by $Z(k)$.

Example 6.5 Suppose T preserves finite coproducts. Then we need neither lextensivity of $\mathcal{X}$ nor injection tautness of T to observe:

(a) We can assume that $(T_1(X, S), i_1) = (T(X), T(\iota_1))$ and $(T_2(X, S), i_2) = (T^2(X), T^2(\iota_1))$.
(b) The assumption (a) makes $T([1_X, k])i_1 = 1_X$, and then makes $\chi T([1_X, k])i_1 + \sigma = \chi + \sigma$.
(c) As follows from (b), $Z(k)$ becomes nothing but the coproduct $X + S$ of X and S in the category of T-algebras.

Example 6.6 Here we take (for simplicity) $\mathcal{X}$ to be the category of sets and T to be the semigroup monad, which is not just taut but even cartesian, as follows, for example, from the results of [1] (see also Sect. 3 in [3] and references therein). We have:

(a) $T(X) = X + X^2 + X^3 + \cdots$; we shall assume (for simplicity again) that the sets X, X^2, X^3, ... are pairwise disjoint and identify their coproduct (in the category of sets) with their union.

(b) We can assume that $T_1(X + S)$ consists of all elements $(t_1, \ldots t_n)$ in $T(X + S)$ such that at least one of the elements t_1, …, t_n is in X, and that i_1 is the inclusion map $T_1(X, S) \to T(X + S)$.
(c) It is then easy to check that $Z(k)$ becomes the semigroup whose (underlying set is $X + S$ and whose) multiplication is defined by (5.3).
(d) In particular, if $S = \{s\}$ has only one element with $k(s)$ playing the role of identity element in X, then $Z(k)$ becomes X with the freely added (multiplicative) identity element.
(e) On the other hand, if X has only one element, then $Z(k)$ becomes S with the freely added multiplicative zero element.

Remark 6.7 According to Theorem 4.14 in [9], a finitary monad T on the category of sets is taut if and only if the corresponding algebraic theory is *balanced* in the sense that it is axiomatizable by identities having the same sets of variables on both sides. This generalizes the cases considered in Examples 6.5 and 6.6, and suggests, for example, the following modifications of Example 6.6:

(a) One can replace the semigroup monad with the commutative semigroup monad. For commutative X and S, the resulting $Z(k)$ will be the same, however, as in Example 6.6. The same applies to the band (= idempotent semigroup) monad and to the commutative band (= semilattice without 1) monad.
(b) One can also replace the semigroup monad with a monad whose algebras have several such structures and are defined by a list of balanced identities. Among them we have the semiring-without-zero monad, using which we can make the set of paradoxical positive reals of [7], also mentioned in Sect. 5, a commutative semiring (=*rig* in Lawvere's terminology).

7 Remaining Questions

Every series monoidal category $\mathcal{A} = (\mathcal{A}, \mathbf{0}, \Sigma, \gamma, \lambda)$, in the sense of [7], has its associated series monoid $A = (A, 0, \Sigma)$ of isomorphism classes of objects of $\mathcal{A}$ (whose construction does not depend on γ and λ of course). In particular, such series monoids occur in cartesian and cocartesian cases, where Σ is the countable product functor or the countable coproduct functor, respectively. Finding out, which of Tarski's properties [12], and especially which of the properties considered in Theorems 3.9 and 4.1 of the present paper, hold in such series monoids in various concrete cases, is an interesting open question. However, this question is very general, and one should carefully choose first special cases to begin with. The category-theoretic viewpoint would probably suggest to begin with cocartesian and cartesian monoidal structures on toposes and abelian categories, and then maybe extend results to extensive (cf. Lawvere's idea of studying rigs corresponding to extensive categories) and semi-abelian contexts. Or, following Tarski [12], one could begin by considering the category formed by series monoids themselves—and indeed, Tarski's Fundamental Theorem of the Algebra of Isomorphism Types (Theorem 12.17 in [12]) can be considered as one of his remarkable achievements in this direction.

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