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Fundamental Study

Comprehension categories and the semantics of type dependency

Bart Jacobs

Department of Mathematics, University of Utrecht, P.O. Box 80.010, 3508 TA Utrecht, The Netherlands

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Abstract

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A *comprehension category* is defined as a functor $\mathscr{P}: \mathbf{E} \to \mathbf{B}^{\to}$ satisfying (a) $cod \circ \mathscr{P}$ is a fibration, and (b) f is cartesian in $\mathbf{E}$ implies that $\mathscr{P}f$ is a pullback in $\mathbf{B}$. This notion captures many structures which are used to describe type dependency (like display-map categories (Taylor (1986), Hyland and Pitts (1989) and Lamarche (1988)), categories with attributes (Cartmell (1978) and Moggi (1991)), D-categories (Ehrhard (1988)) and comprehensive fibrations (Pavlović (1990)). It also captures comprehension as occurring in topos theory and as described by Lawvere's (1970) hyperdoctrines. This paper is meant as an introduction to these comprehension categories.

A comprehension category will be called *closed* if it has appropriate dependent products and sums. A few examples of closed comprehension categories will be described here; more of them may be found in Jacobs (1991); applications occur in Jacobs (1991) and Jacobs et al. (1991).

Contents

1. Introduction 170
2. Fibrations 171
3. Category theory over a basis 175
4. Comprehension categories 181
5. Quantification 196
Acknowledgment 206
References 206

Correspondence to: B. Jacobs, Department of Mathematics, University of Utrecht, P.O. Box 80.010, 3508 TA Utrecht, The Netherlands. Email: bjacobs@math.ruu.nl.

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1. Introduction

The expression "type dependency" denotes the possibility in a calculus of types and terms to have types depending on term variables. Such calculi were first studied by de Bruijn [4] and Martin-Löf [23] as formal systems for (constructive) mathematics. Also in computer science it is quite convenient to have type dependency, e.g. to use $List(n)$ as the type of lists with length n. What makes this type dependency complicated is the fact that in the languages having such features the distinction between compile time and run time (which one does have for polymorphic calculi) disappears. In this paper we are primarily concerned with the categorical semantics of type dependency and refer to [23, 32] for more information on syntax.

The main question in the categorical description of type dependency is how to understand contexts. These cannot simply be cartesian products of the constituent types, because certain dependencies may occur among these types. In more concrete form, the question becomes how to understand *context extension*, i.e. the passage from the statement $\Gamma \vdash \sigma : Type$ to the extended context $\Gamma, x{:}\sigma$.

From categorical logic it is known that one should view statements $\Gamma \vdash \sigma : Type$ as objects which are fibred over contexts Γ. Hence, one needs at least a fibration $p : \mathbf{E} \to \mathbf{B}$. Context extension will be captured by a functor $\mathscr{P}_0 : \mathbf{E} \to \mathbf{B}$, which comes equipped with a natural transformation $\mathscr{P}_0 \dot{\to} p$. Components of the latter are understood as *projections* $\Gamma, x{:}\sigma \to \Gamma$. Two functors $\mathbf{E} \rightrightarrows \mathbf{B}$ and a natural transformation between them correspond to a functor $\mathbf{E} \to \mathbf{B}^{\to}$, where $\mathbf{B}^{\to}$ is the category of arrows of $\mathbf{B}$. By adding the more technical requirement that projections are "stable under substitution" (see also Lemma 4.4) one arrives at the notion of a comprehension category.

Over the past 15 years, various categorical structures have been proposed to describe type dependency, see e.g. [5, 29–31, 8, 24, 27]. There are differences in details, but the above aspects of context extension can be found in all of these structures. In this way a comprehension category can be understood as a "minimal" notion. An additional strong point of comprehension categories is that they give rise to a clean categorical theory. Not all of the relevant aspects can be described here in this introduction, but more can be found in [18, 17]. There, the theory is further developed. In [17], comprehension categories are used as building blocks to construct suitable categorical structures describing arbitrary type systems.

The structures used here are called "comprehension categories" because they involve a weak form of comprehension; it can be described by disjoint unions, see after Lemma 4.4 The context extension mentioned above is handled by this kind of comprehension. Thus, we sometimes speak of "context comprehension" in $\Gamma, x{:}\sigma$. Other notions of comprehension (by Pavlović, Ehrhard and Lawvere) fit in our general scheme.

This work is about type theory and category theory. We use a metaphor from computer science to describe our view on their relation: we often think of the language of category theory as an assembly language in which much "programming" – e.g. about substitution or isomorphisms – has to be done in detail. Type theory, on the

other hand, may be seen as a higher-level language for (certain parts of) category theory and, if one is willing to push the comparison even further, interpretation may be seen as compilation. In this way, category theory provides a (variable-free) alternative formalism for logic and type theory. This forms a key aspect of categorical abstract machines, see [6, 7], for an overview and further references.

The paper starts with two sections about fibred category theory. All of the material there (except perhaps Definition 3.2) is standard and is mostly due to Grothendieck and Bénabou. Fibrations form the "backbones" of comprehension categories and fibred adjunctions are essential for products and sums. This use of fibred adjunctions elegantly provides the validity of substitution properties like $(\lambda x{:}\sigma.P)[x:=M] \equiv \lambda x{:}\sigma[x:=M].(P[x:=M])$. These matters are investigated in Section 3, relating fibred adjunctions and the Beck–Chevalley condition. We believe that fibred category theory provides the proper mathematical framework to study categories varying over others (categorically, a context is an index for the category of types and terms derivable in that context). Nevertheless, there are "indexed categories" [26], which may seem closer to the intuition. The description in Proposition 3.4 of fibred adjunctions in terms of collections of "fibrewise" adjunctions satisfying the Beck–Chevalley condition forms a practical compromise between formulations used for fibred and indexed categories, see also Remark 12.3 in [2].

In Section 4 comprehension categories are introduced. Our main concern there is to show how specific examples and alternative notions fit in. Section 5 is about quantification for comprehension categories.

2. Fibrations

The basic facts about fibrations are presented in this section; more information can be obtained from [2, 12–14]. In order to increase the readability, we often spare on parentheses.

Definition 2.1 (Grothendieck [14]). Let $p:\mathbf{E}\to\mathbf{B}$ be a functor.

(i) A morphism $f:D\to E$ in $\mathbf{E}$ is called *cartesian* over $u:A\to B$ in $\mathbf{B}$ if
  (a) $pf=u$
  (b) for every $f':D'\to E$ with $pf'=u$, there is a unique $\phi:D'\to D$ with $p\phi=id_A$ and $f'=f\circ\phi$.

(ii) Dually, $g:D\to E$ is called *cocartesian* over u if g in $\mathbf{E}^{\mathbf{op}}$ is cartesian over u in $\mathbf{B}^{\mathbf{op}}$, i.e.
  (a) $pg=u$
  (b) for every $g':D\to E'$ with $pg'=u$, there is a unique $\psi:E\to E'$ with $p\psi=id_B$ and $g'=\psi\circ g$.

This is diagrammatically shown in Fig. 1. From this figure it is clear why such an f is sometimes called a terminal lifting and g an initial lifting of u.

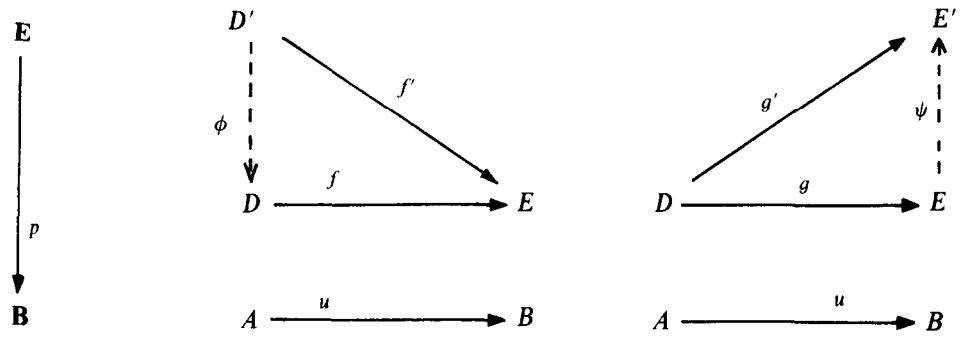


Fig. 1.

(iii) The functor $p: \mathbf{E} \to \mathbf{B}$ is called a *fibration* or a *fibred category* if both
(a) for every $E \in \mathbf{E}$ and $u: A \to pE$ in $\mathbf{B}$ there is a cartesian $f: D \to E$ over u in $\mathbf{E}$;
(b) the composition of two cartesian morphisms is cartesian again.

One often calls $\mathbf{B}$ the base category and $\mathbf{E}$ the total category.

Dually, p is a *cofibration* if $p^{\mathrm{op}}: \mathbf{E}^{\mathrm{op}} \to \mathbf{B}^{\mathrm{op}}$ is a fibration, i.e. if above every $u: pD \to B$ there is a cocartesian arrow with domain D, and cocartesianity is closed under composition. Finally, p is a *bifibration* if p is both a fibration and a cofibration.

The "arrow category" $\mathbf{B}^{\to}$ of $\mathbf{B}$ has arrows of $\mathbf{B}$ as objects and pairs of arrows yielding a commuting square as morphisms. The functor $dom: \mathbf{B}^{\to} \to \mathbf{B}$ is an example of a fibration. If $\mathbf{B}$ has pullbacks, the functor $cod: \mathbf{B}^{\to} \to \mathbf{B}$ is also a fibration, with cartesian morphisms in $\mathbf{B}^{\to}$ given by pullback squares. It is even a bifibration. Another example of a bifibration is given by modules over rings, see [13, 1.10] (there, a cofibration is called an opfibration).

The cartesian (cocartesian) morphisms $f, g: D \to E$ from Definition 2.4, are often denoted by $\bar{u}(E): u^*(E) \to E$ ($\underline{u}(D): D \to u_*(D)$), because these $u^*(E)$ and $u_*(D)$ are determined up-to-isomorphism. In order to have the fibration explicit, we sometimes write $u^{*p}(E)$.

One may call a morphism $f: D \to E$ *strong cartesian* (Gray) or *hyper cartesian* (Bénabou) over $u: A \to B$ if $pf = u$ and for any $f': D' \to E$ such that $pf' = u \circ v$ in $\mathbf{B}$, there is a unique $\phi: D' \to D$ in $\mathbf{E}$ with $p\phi = v$ and $f' = f \circ \phi$. Obviously, a strong cartesian morphism is cartesian; it is not hard to verify that if p is a fibration, then a cartesian morphism is also strong cartesian. Hence, when working with fibrations – which will be done throughout this paper – these notions coincide (sometimes cartesian morphisms are defined as strong cartesian ones; fibrations can then be defined without the requirement that the composition of cartesian arrows should be cartesian again).

Definition 2.2. Let $p: \mathbf{E} \to \mathbf{B}$ be a functor. For $B \in \mathbf{B}$ the *fibre* $\mathbf{E}_B$ is the category having objects $E \in \mathbf{E}$ with $pE = B$ and arrows f in $\mathbf{E}$ with $pf = id_B$. That is, $\mathbf{E}_B$ has objects *above* B with *vertical* morphisms (as in the diagram in Definition 2.1).

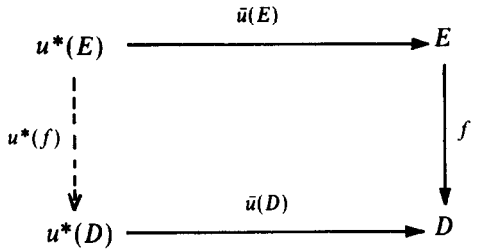


Fig. 2.

For $E, D \in \mathbf{E}$ and $u: pE \to pD$ in $\mathbf{B}$, one writes $\mathbf{E}_u(D, E) = \{f \in \mathbf{E}(D, E) \mid pf = u\}$. Note that $\mathbf{E}_u(D, E) \cong \mathbf{E}_{pD}(D, u^*(E))$, if p is a fibration and that $\mathbf{E}_u(D, E) \cong \mathbf{E}_{pE}(u_*(D), E)$, if p is a cofibration.

Suppose $p: \mathbf{E} \to \mathbf{B}$ is fibration and u is a morphism $A \to B$ in $\mathbf{B}$. For an arrow $f: E \to D$ in the fibre $\mathbf{E}_B$ one can define an arrow, say $u^*(f)$, from (chosen) $u^*(E)$ to $u^*(D)$ in $\mathbf{E}_A$ as follows. The arrow $\bar{u}(D): u^*(D) \to D$ is by definition cartesian over u; furthermore, $f \circ \bar{u}(E): u^*(E) \to D$ is above u and so there is a unique arrow $u^*(f): u^*(E) \to u^*(D)$ in $\mathbf{E}_A$ with $\bar{u}(D) \circ u^*(f) = f \circ \bar{u}(E)$ (see Fig. 2). This construction yields a pullback in $\mathbf{E}$. It is easy to show that $u^*(g \circ f) = u^*(g) \circ u^*(f)$ and $u^*(id_E) = id_{u^*(E)}$, using uniqueness. Hence, for every $u: A \to B$ in $\mathbf{B}$ one can choose in this way a functor $u^*: \mathbf{E}_B \to \mathbf{E}_A$ which is called *inverse image* or *reindexing* functor (sometimes also change-of-base functor, but we prefer to reserve this expression for the special case described in Proposition 2.6). One easily verifies that there are natural isomorphisms between different choices of inverse image functors u^* for a given u in $\mathbf{B}$. A particular collection of choices $\{u^*, \bar{u}\}_{u \text{ in } \mathbf{B}}$ is called a *cleavage* of p.

One might expect to get a functor $\Psi: \mathbf{B}^{\mathrm{op}} \to \mathbf{Cat}$ (with $\Psi(B) = \mathbf{E}_B$ and $\Psi(u) = u^*$) in this way, but in general there are natural isomorphisms $(v \circ u)^* \cong u^* \circ v^*$ and $id^* \cong Id$ instead of identities (and, thus, a "pseudo-functor" is obtained, see [26]). In case there is a cleavage $\{u^*, \bar{u}\}$ such that $\overline{v \circ u}(E) = \bar{v}(E) \circ \bar{u}(v^*(E))$ and $\overline{id}(E) = id_E$, the fibration is called *split*; then such a functor Ψ can be constructed; this cleavage $\{u^*, \bar{u}\}$ is then called a splitting. Note that a cleavage can always be chosen in such a way that the latter condition $\overline{id}(E) = id_E$ holds.

The other way round, functors $\Psi: \mathbf{B}^{\mathrm{op}} \to \mathbf{Cat}$ give rise to split fibrations with base $\mathbf{B}$. This is established by the so-called Grothendieck construction, which goes as follows. Write $\Sigma(\Psi)$ for the category with pairs (A, X) such that $X \in \Psi A$ as objects. Morphisms $(A, X) \to (B, Y)$ in $\Sigma(\Psi)$ are pairs (u, f) with $u: A \to B$ in $\mathbf{B}$ and $f: X \to \Psi(u)(Y)$ in ΨA. The first projection $\Sigma(\Psi) \to \mathbf{B}$ is then a split fibration with (u, f) cartesian if and only if f is an isomorphism.

Proposition 2.3. *Let $p: \mathbf{E} \to \mathbf{B}$ be a fibration;*

(i) *p is a bifibration iff every inverse image functor u^* has a left adjoint Σ_u.*

(ii) *if $r: \mathbf{B} \to \mathbf{A}$ is a fibration then $rp: \mathbf{E} \to \mathbf{A}$ too.*

Proof. (i) *if*: Let $u: A \to B$ in $\mathbf{B}$ and $\Sigma_u \dashv u^*$ (with unit η and counit ε) be given; for D above A we have to construct a cocartesian arrow $\underline{u}(D): D \to u_*(D)$. Take $u_*(D) = \Sigma_u(D)$ and $\underline{u}(D) = \bar{u}(\Sigma_u(D)) \circ \eta_D$, which is above u. If also $g: D \to E$ is above u, then let $\phi: D \to u^*(E)$ be the unique arrow with $g = \bar{u}(E) \circ \phi$. Writing $\hat{\phi} = \eta_E \circ \Sigma_u(f): \Sigma_u(D) \to E$ for the (vertical) transpose of ϕ, one has $u^*(\hat{\phi}) \circ \eta_D = \tilde{\phi} = \phi$. Hence, we have

$$\begin{aligned}
\hat{\phi} \circ \underline{u}(D) &= \hat{\phi} \circ \bar{u}(\Sigma_u(D)) \circ \eta_D \\
&= \bar{u}(E) \circ u^*(\hat{\phi}) \circ \eta_D \quad \text{[by definition of } u^*(\hat{\phi})\text{]} \\
&= \bar{u}(E) \circ \phi \\
&= g.
\end{aligned}$$

It is left to the reader to show that $\hat{\phi}$ is uniquely determined by this property.

Only if: For $u: A \to B$ in $\mathbf{B}$, choose $\Sigma_u = u_*: \mathbf{E}_A \to \mathbf{E}_B$. Then $\mathbf{E}_B(\Sigma_u(D), E) \cong \mathbf{E}_u(D, E) \cong \mathbf{E}_A(D, u^*(E))$, see Definition 2.2. It is not difficult to show that the resulting isomorphism $\mathbf{E}_B(\Sigma_u(D), E) \cong \mathbf{E}_A(D, u^*(E))$ is natural in D and E.

(ii): Left to the reader. □

Definition 2.4. (i) Let $p: \mathbf{E} \to \mathbf{B}$ and $q: D \to \mathbf{B}$ be fibrations with the same basis $\mathbf{B}$. A functor $H: \mathbf{E} \to \mathbf{D}$ is called *cartesian* if $q \circ H = p$ and f is p-cartesian implies that Hf is q-cartesian. This determines a category $Fib(\mathbf{B})$.

More generally, a category Fib is defined by taking a pair (H, K) as a morphism $(p: \mathbf{E} \to \mathbf{B}) \to (q: \mathbf{D} \to \mathbf{A})$ if $H: \mathbf{E} \to \mathbf{D}$ and $K: \mathbf{B} \to \mathbf{A}$ are functors such that $q \circ H = K \circ p$ and f is p-cartesian implies that Hf is q-cartesian.

(ii) These categories $Fib(\mathbf{B})$ and Fib become 2-categories by stipulating, for morphisms $H, H': p \to q$ in $Fib(\mathbf{B})$ that $\sigma: H \dot{\to} H'$ is a 2-cell in $Fib(\mathbf{B})$ iff $\sigma: H \dot{\to} H'$ is a natural transformation with vertical components (i.e. $q\sigma = id_p$).

For morphisms $(H, K), (H', K'): p \to q$ in Fib, $(\sigma, \tau): (H, K) \dot{\to} (H', K')$ is a 2-cell in Fib iff $\sigma: H \dot{\to} H'$ and $\tau: K \dot{\to} K'$ are natural transformations satisfying $q\sigma = \tau p$.

Lemma 2.5. *Let $p: \mathbf{E} \to \mathbf{B}$ and $q: \mathbf{D} \to \mathbf{B}$ be fibrations and let $F: p \to q$ be a cartesian functor in $Fib(\mathbf{B})$.*

(i) *For every $A \in \mathbf{B}$, one obtains a "fibrewise" functor $F|_A: \mathbf{E}_A \to \mathbf{D}_A$ by restriction. Then*

$$F \text{ is } \textit{full (faithful)} \Leftrightarrow \textit{every } F|_A \textit{ is full (faithful)}.$$

(ii) *If F is full and faithful, then*

$$f \textit{ is } p\textit{-cartesian} \Leftrightarrow Ff \textit{ is } q\textit{-cartesian}.$$

Proposition 2.6. (i) (Change-of-base) *The pullback in* **Cat** *of a fibration* $p:\mathbf{E}\to\mathbf{B}$ *and an arbitrary functor* $K:\mathbf{A}\to\mathbf{B}$ *yields a fibration* $K^*(p):\mathbf{A}\underset{K,p}{\times}\mathbf{E}\to\mathbf{A}$ *and a morphism* $K^*(p)\to p$ *of fibrations.*

(ii) *The functor* $Fib\to\mathbf{Cat}$, *sending a fibration to its base, is a fibration itself with the categories* $Fib(\mathbf{B})$ *as fibres.*

(iii) *The categories* $Fib(\mathbf{B})$ *have finite products, which are preserved under change-of-base. (In the language of Definitions* 3.5 *and* 3.6, *the fibration* $Fib\to\mathbf{Cat}$ *admits a terminal object and cartesian products.)*

Proof. (i): Since a morphism (u,f) in $\mathbf{A}\underset{K,p}{\times}\mathbf{E}$ is cartesian iff f is cartesian in $\mathbf{E}$.

(ii): Obvious from (i).

(iii): The fibration $Id_{\mathbf{B}}:\mathbf{B}\to\mathbf{B}$ is terminal and for fibrations $p:\mathbf{E}\to\mathbf{B}$ and $q:\mathbf{D}\to\mathbf{B}$ one can take $p\times q=p\circ p^*(q):\mathbf{E}\underset{p,q}{\times}\mathbf{D}\to\mathbf{B}$, using (i) and Proposition 2.3(ii). □

3. Category theory over a basis

Since $Fib(\mathbf{B})$ is a 2-category, one has a notion of fibred adjunction.

Definition 3.1. Let $p:\mathbf{E}\to\mathbf{B}$ and $q:\mathbf{D}\to\mathbf{B}$ be fibrations and $F:p\to q$ and $G:q\to p$ be cartesian functors; F is called a *fibred left adjoint* of G if F is a left adjoint of G in the usual way and the unit η of this adjunction is vertical (or equivalently, the counit ε is vertical).

In the theory of fibred categories isomorphisms play an important role. What we need here is a generalization of the classical definition of maps between adjunctions (see [22, IV, 7]), which allows certain identities to be isomorphisms. This is achieved in Definition 3.2. As is shown in Lemma 3.3, the formulation we use contains some redundancy: one can equivalently use (a generalized version of) the Beck–Chevalley condition. But the formulation we start with in Definition 3.2 is intuitively clear and nicely symmetrical.

Definition 3.2. Suppose two (ordinary) adjunctions $F\dashv G$ and $F'\dashv G'$ are given – say with $F:\mathbf{E}\to\mathbf{D}$ and $F':\mathbf{E}'\to\mathbf{D}'$ and with η, ε and η', ε' as unit and counit, respectively. A *pseudo map* from $F\dashv G$ to $F'\dashv G'$ consists of a quadruple (K, L, φ, ψ) such that

$$K:\mathbf{E}\to\mathbf{E}', \qquad L:\mathbf{D}\to\mathbf{D}'$$

are functors commuting up to φ and ψ with the F's and G's, i.e.

$$\varphi:F'K\xrightarrow{\sim}LF, \qquad \psi:G'L\xrightarrow{\sim}KG$$

and also preserving up to φ and ψ the units and counits (see Fig. 3).

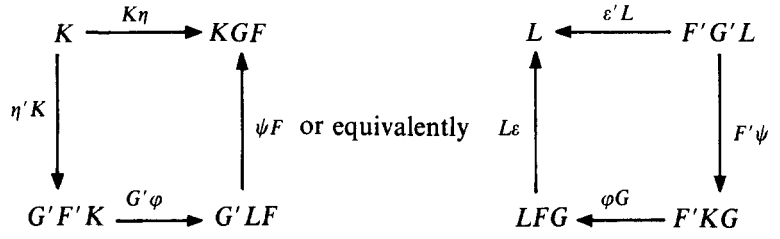


Fig. 3.

Mac Lane [22, IV, 7] speaks of a map from $F \dashv G$ to $F' \dashv G'$ in case both the φ and ψ above are identities. The equivalence mentioned in the definition (regarding commutation of the diagrams for unit and counit) can be proved similarly as Proposition 1, loc. cit. It is not hard to verify that pseudo maps between adjunctions can be composed: for the composition $(K', L', \varphi', \psi')$ after (K, L, φ, ψ) one takes $(K'K, L'L, L'\varphi \circ \varphi' K, K'\psi \circ \psi' L)$. Hence, pseudo maps between adjunctions can serve as morphisms in a category.

Lemma 3.3. *In the above situation, the natural isomorphisms φ and ψ determine each other in the following sense.*

An isomorphism $F'K \cong LF$ induces a pseudo map from $F \dashv G$ to $F' \dashv G'$ iff the canonical natural transformation $KG \dot{\to} G'L$ is an isomorphism.

An isomorphism $G'L \cong KG$ induces a pseudo map from $F \dashv G$ to $F' \dashv G'$ iff the canonical natural transformation $F'K \dot{\to} LF$ is an isomorphism.

Proof. We shall only prove the first statement; therefore, let $\varphi : F'K \xrightarrow{\sim} LF$ be given. Since $L\eta \circ \varphi G : F'KG \dot{\to} L$, one has a transpose

$$\chi = G'(L\varepsilon \circ \varphi G) \circ \eta' KG : KG \dot{\to} G'L.$$

Only if: Suppose $\psi : G'L \xrightarrow{\sim} KG$ as in Definition 3.2 is given; we show that $\chi = \psi^{-1}$ and, thus, that χ is an isomorphism.

$$\begin{aligned}\chi \circ \psi &= G'(L\varepsilon \circ \varphi G) \circ \eta' KG \circ \psi \\ &= G'L\varepsilon \circ G'\varphi G \circ G'F'\psi \circ \eta' G'L \\ &= G'\varepsilon' L \circ \eta' G'L \\ &= id_{G'L}.\end{aligned}$$

If: Suppose $\chi : KG \dot{\to} G'L$ as determined above in an iso; it is elementary to show that $\psi = \chi^{-1}$ makes (one of) the diagrams in Fig. 3 commute. □

Example. Let $F, F' : \mathbf{E} \to \mathbf{D}$ be functors such that $\varphi : F' \xrightarrow{\sim} F$ and both $F \dashv G$ and $F \dashv G'$. Since adjoints are determined up-to-isomorphism, there is also a natural iso $\psi : G' \xrightarrow{\sim} G$. Even more, $(Id_{\mathbf{E}}, Id_{\mathbf{D}}, \varphi, \psi)$ is a pseudo map from $F \dashv G$ to $F \dashv G'$.

A fibred adjunction is an elegant and at the same time powerful notion: suppose $F: p \to q$ in $Fib(\mathbf{B})$ has a fibred right adjoint G; for each $A \in \mathbf{B}$, one obtains "fibrewise" adjunctions $F|_A \dashv G|_A$, between the fibres, say $\mathbf{E}_A$ and $\mathbf{D}_A$, of p and q by restriction. These are preserved under inverse images. The other way round, given an arbitrary collection $\{H_A: \mathbf{E}_A \to \mathbf{D}_A\}$ of fibrewise functors, we say that $F: p \to q$ *underlies* this collection if $F|_A = H_A$ for each $A \in \mathbf{B}$. Obviously, every cartesian functor underlies its own collection of fibrewise functors. One can ask under which conditions "fibrewise" adjunctions have an underlying fibred one. Proposition 3.4 – which is folklore – gives an answer. The proof is not very important and left to the interested reader.

Proposition 3.4. *Let $F: p \to q$ in $Fib(\mathbf{B})$ be a cartesian functor such that every local functor $F|_A$ has a right adjoint G_A. The following statements are equivalent.*

(i) *F has a fibred right adjoint G underlying $\{G_A\}$.*

(ii) *for every morphism $u: A \to B$ in $\mathbf{B}$, reindexing functors u^{*p} and u^{*q} determine a pseudo map of adjunctions $F|_B \dashv G_B \to F|_A \dashv G_A$.*

(iii) *for every morphism $u: A \to B$ in $\mathbf{B}$ and for all reindexing functors u^{*p} and u^{*q}, the canonical natural transformation $u^{*p}G_B \dot{\to} G_A u^{*q}$ is an isomorphism.*

It is easy to obtain a dual version of this result. In the rest of this section standard categorical notions like terminal, products, sums, etc., will be described for fibred categories (i.e. over a base category). Examples occur mainly in Sections 4 and 5.

Definition 3.5. A fibration $p: \mathbf{E} \to \mathbf{B}$ admits a terminal object if the unique morphism from p to the terminal object in $Fib(\mathbf{B})$ has a fibred right adjoint (see Fig. 4).

Hence, a fibration $p: \mathbf{E} \to \mathbf{B}$ admits a terminal object iff for every $A \in \mathbf{B}$, there is a terminal object $1A$ in the fibre $\mathbf{E}_A$ and for every $u: A \to B$ in $\mathbf{B}$, the canonical map $u^*(1B) \to 1A$ is an iso.

Definition 3.6. A fibration $p: \mathbf{E} \to \mathbf{B}$ admits (cartesian) products if the morphism $\Delta: p \to p \times p$ in $Fib(\mathbf{B})$ has a fibred right adjoint (see Fig. 5).

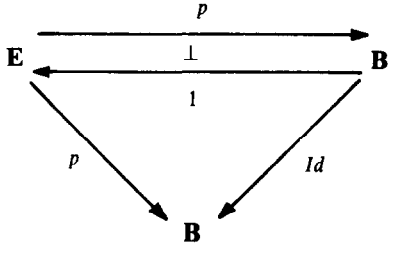


Fig. 4.

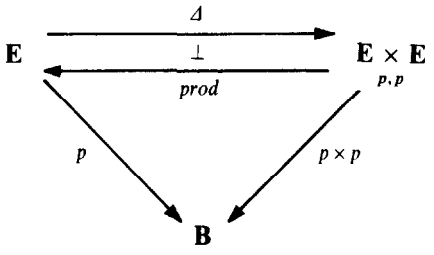


Fig. 5.

Hence, a fibration $p: \mathbf{E} \to \mathbf{B}$ admits cartesian products iff for every $A \in \mathbf{B}$, there is a product $(-) \times_A (-)$ in the fibre $\mathbf{E}_A$ and for every $u: A \to B$ in $\mathbf{B}$ and $E, D \in \mathbf{E}_B$ the canonical map

$$\langle u^*(\pi), u^*(\pi') \rangle : u^*(E \times_B D) \to u^*(E) \times_A u^*(D)$$

is an iso.

In order to define equalizers, let us write $\mathbf{2}+$ for the category shown in Fig. 6. Assume $p: \mathbf{E} \to \mathbf{B}$ is a fibration; it is easy to verify that for any category $\mathbf{A}$, the induced functor $\mathbf{E}^{\mathbf{A}} \to \mathbf{B}^{\mathbf{A}}$ is a fibration again. Hence, one can form the fibration p^{2+} by change of base, as shown in Fig. 7.

Definition 3.7. A fibration $p: \mathbf{E} \to \mathbf{B}$ admits equalizers if the morphism $\Delta: p \to p^{2+}$ in $Fib(\mathbf{B})$ has a fibred right adjoint.

Hence, a fibration $p: \mathbf{E} \to \mathbf{B}$ admits equalizers iff every fibre category has equalizers and for every $u: A \to B$ in $\mathbf{B}$ and parallel arrows f, g above B, the canonical arrow

$$u^*(Eq_B(f, g)) \to Eq_A(u^*(f), u^*(f))$$

is an iso.

In the 2-category **Cat**, exponents are defined via adjunctions with parameters. The following reformulation of such adjunctions is useful in *Fib* (and also in internal

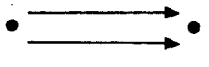


Fig. 6.

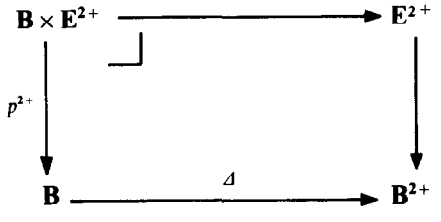


Fig. 7.

category theory). An improvement suggested by Moggi (with respect to an earlier version) is incorporated.

Lemma 3.8. *Let* $F: \mathbf{A} \times \mathbf{P} \to \mathbf{B}$ *be a bifunctor; the following three statements are equivalent.*

(i) (*MacLane* [22, *IV*, 7.3]) *For every object* $p \in \mathbf{P}$, *there is a right adjoint* $G(-, p): \mathbf{B} \to \mathbf{A}$ *to the functor* $F(-, p)$ *via an isomorphism*

$$\psi_{a,b}: \mathbf{B}(F(a, p), b) \cong \mathbf{A}(a, G(b, p))$$

natural in a *and* b.

(ii) *For every groupoid subcategory* $|\mathbf{P}|$ *of* $\mathbf{P}$ *with* $Obj|\mathbf{P}| = Obj\,\mathbf{P}$, *one has*

$$(*) \quad \begin{cases} \text{the functor } \tilde{F}: \mathbf{A} \times |\mathbf{P}| \to \mathbf{B} \times |\mathbf{P}| \text{ defined by } (a, p) \mapsto (F(a, p), p) \text{ and} \\ (f, g) \mapsto (F(f, g), g) \text{ has a right adjoint } \tilde{G}. \end{cases}$$

(iii) *There is a groupoid subcategory* $|\mathbf{P}|$ *of* $\mathbf{P}$ *with* $Obj|\mathbf{P}| = Obj\,\mathbf{P}$ *satisfying* $(*)$.

Proof. (i) $\Rightarrow$ (ii): The functor $G(-, p): \mathbf{B} \to \mathbf{A}$ induces a functor $B \times \mathbf{P}^{op} \to \mathbf{A}$ (see [22, IV, 7.3]), which can be turned into a functor $\tilde{G}: \mathbf{B} \times |\mathbf{P}| \to \mathbf{A} \times |\mathbf{P}|$ using the obvious inclusion $|\mathbf{P}| \subsetneq \mathbf{P}^{op}$.

(ii) $\Rightarrow$ (iii): Since there is at least one such groupoid, viz. the discrete category with objects from $\mathbf{P}$.

(iii) $\Rightarrow$ (i): Assume $\tilde{F} \dashv \tilde{G}$, with unit $\tilde{\eta}$ and counit $\tilde{\varepsilon}$. One defines a functor $G(-, p): \mathbf{B} \to \mathbf{A}$ by $b \mapsto fst(\tilde{G}(b, p))$ and $h \mapsto fst(\tilde{G}(h, id_p))$. Using that $snd(\tilde{\varepsilon}_{(b,p)})$: $snd(\tilde{G}(b, p)) \to p$ in $|\mathbf{P}|$ is an isomorphism, one obtains the required $\psi_{a,b}$. Indeed, one can take

$$\psi_{a,b}(h) = fst\{\tilde{G}(h, id_p) \circ \tilde{\eta}_{(a,p)}\}$$

$$\psi^{-1}_{a,b}(k) = fst\{\tilde{\varepsilon}_{(b,p)} \circ \tilde{F}(f, snd(\tilde{\varepsilon}_{(b,p)})^{-1})\}. \qquad \square$$

Let $p: \mathbf{E} \to \mathbf{B}$ be a fibration. We write $Cart(\mathbf{E})$ for the category obtained from $\mathbf{E}$ by taking all objects but only the cartesian arrows. By restriction one obtains a fibration $|p|: Cart(\mathbf{E}) \to \mathbf{B}$. Since an arrow which is both cartesian and vertical is an isomorphism one has that the fibre categories of $|p|$ are groupoids. Using Lemma 3.8, one comes to the following definition of fibred exponents.

Definition 3.9. A fibration $p: \mathbf{E} \to \mathbf{B}$ (with cartesian products) admits exponents if the functor $\widetilde{prod}: p \times |p| \to p \times |p|$ in $Fib(\mathbf{B})$ defined by

$$(E, D) \mapsto (prod(E, D), D)$$

$$(f, g) \mapsto (prod(f, g), g)$$

has a fibred right adjoint (see Fig. 8).

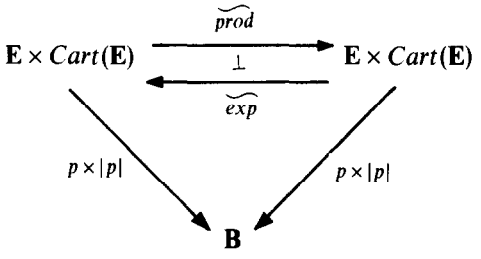


Fig. 8.

Using Lemma 3.8, one can verify that the fibration $p: \mathbf{E} \to \mathbf{B}$ (with products $\times_A$ in the fibres $\mathbf{E}_A$) admits exponents iff for every $A \in \mathbf{B}$, there are exponents $(-) \Rightarrow_A (-)$ in the fibres $\mathbf{E}_A$ and for every $u: A \to B$ in $\mathbf{B}$ and $E, D \in \mathbf{E}_B$, the canonical map

$$\Lambda(u^*(ev) \circ \alpha): u^*(E \Rightarrow_B D) \to u^*(E) \Rightarrow_A u^*(D)$$

is an iso, where $\alpha: u^*(E \Rightarrow_B D) \times_A u^*(E) \xrightarrow{\sim} u^*((E \Rightarrow_B D) \times_B E)$.

Using Definitions 3.5, 3.6 and 3.9 one can speak of a *fibred CCC* (cartesian-closed category); combining Definitions 3.5–3.7 one obtains the notion of a fibration with *fibred finite limits* (or a *fibred LEX category*).

Definition 3.10. Let $p: \mathbf{E} \to \mathbf{B}$ be a fibration, where $\mathbf{B}$ is a category with pullbacks.

(i) One says that p has *sums* if every reindexing functor u^* has a left adjoint Σ_u in such a way that Beck–Chevalley holds: if a pullback in $\mathbf{B}$ is given by Fig. 9 then the canonical natural transformation $\Sigma_u s^* \dot{\to} r^* \Sigma_v$ is an isomorphism.

(ii) Similarly, p has *products* if there are adjunctions $u^* \dashv \Pi_u$ such that for the pullback in Fig. 9, one has $r^* \Pi_v \cong \Pi_u s^*$ canonically.

It is worth mentioning that products and sums for fibrations can also be described via fibred adjunctions, see [17]. The above "fibrewise" description, however, is standard in the literature.

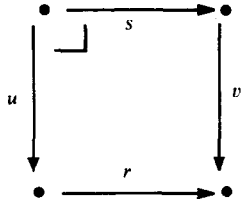


Fig. 9.

In order to have at least one example, let **B** be a category with finite limits. Then

(i) $cod: \mathbf{B}^{\to} \to \mathbf{B}$ has fibred finite limits;

(ii) $cod: \mathbf{B}^{\to} \to \mathbf{B}$ has sums (via composition);

(Conversely, one also has that if for an arbitrary **A** these sum functors have right adjoints, then $cod: \mathbf{A}^{\to} \to \mathbf{A}$ is a fibration.)

(iii) The following three statements are equivalent (see [11]).

(a) Every slice category $\mathbf{B}/A$ is a CCC.

(b) $cod: \mathbf{B}^{\to} \to \mathbf{B}$ is a fibred CCC.

(c) $cod: \mathbf{B}^{\to} \to \mathbf{B}$ has products (as defined above).

In case these last three conditions hold, **B** is called a *locally cartesian-closed category* (LCCC).

4. Comprehension categories

In Definition 4.1 comprehension categories are introduced. Only the so-called "full" comprehension categories are relevant for the description of type dependency and in Section 5 only those will be considered. Here we investigate the basic properties and a number of examples and constructions for comprehension categories in general. Especially, one may find comprehension categories constructed from display-map categories in Example 4.5, from categories with attributes in Example 4.10, from toposes in Example 4.7, from D-categories in Definition 4.12 (under the name comprehension category with unit), from Lawvere's comprehension in Example 4.18 and from comprehensive fibrations in Example 4.21.

Definition 4.1. A *comprehension category* is a functor $\mathscr{P}: \mathbf{E} \to \mathbf{B}^{\to}$ satisfying

(i) $cod \circ \mathscr{P}: \mathbf{E} \to \mathbf{B}$ is a fibration;

(ii) f is cartesian in $\mathbf{E} \Rightarrow \mathscr{P}f$ is a pullback in **B**.

This $\mathscr{P}$ is called a *full* comprehension category in case $\mathscr{P}$ is a full and faithful functor. It is called *cloven* or *split* whenever the fibration involved is cloven or split.

Note that we do not require that the base category **B** has *all* pullbacks. In case it does, $\mathscr{P}$ is a cartesian functor. It is easy to verify that $\mathscr{P}$ is a full comprehension category iff $\mathscr{P}$ is fibrewise a full and faithful functor, similarly as Lemma 2.5(i).

Notation 4.2. For a comprehension category $\mathscr{P}: \mathbf{E} \to \mathbf{B}^{\to}$ we standardly write $p = cod \circ \mathscr{P}$ and $\mathscr{P}_0 = dom \circ \mathscr{P}$. The object part of $\mathscr{P}$ then forms a natural transformation $\mathscr{P}: \mathscr{P}_0 \dot{\to} p$. Similarly, for $\mathscr{Q}: \mathbf{D} \to \mathbf{A}^{\to}$, we write $q = cod \circ \mathscr{Q}$ and $\mathscr{Q}_0 = dom \circ \mathscr{Q}$. The functors $(-)_0$ do the work of context extension (or comprehension) as can be seen clearly in Example 4.3 (term model). In Lawvere's [21] notation, one can denote for an object $E \in \mathbf{E}$ above $A \in \mathbf{B}$ the corresponding "extent" $\mathscr{P}_0 E$ by $\{A \mid E\}$.

The components $\mathscr{P}E$ will be called *projections*; reindexing functors of the form $\mathscr{P}E^*$ are called *weakening* functors. For an object $E \in \mathbf{E}$ we write $|E| = \{u : pE \to \mathscr{P}_0 E \mid \mathscr{P}E \circ u = id\}$; elements of $|E|$ may be called *terms of type E*. Motivation for this terminology may be found in Example 4.3.

Example 4.3 (*Term model*). In order to convey the type-theoretic intuition underlying Definition 4.1 we start with a term model construction. For a calculus with type dependency (see e.g. [23, 32]), one can form a full comprehension category $\mathscr{P} : \mathbf{E} \to \mathbf{B}^{\to}$ in the following way. We write the outer braces $[-]$ to denote the equivalence classes of conversion.

The objects of $\mathbf{B}$ are equivalence classes $[\Gamma]$ of contexts Γ. A morphism $[\Gamma] \to [\Delta]$, where $\Delta \equiv y_1 : \tau_1, \ldots, y_n : \tau_n$ consists of an n-tuple of equivalence classes of terms $\langle [M_1], \ldots, [M_n] \rangle$ satisfying $\Gamma \vdash M_i : \tau_i[x_1 := M_1, \ldots, x_{i-1} := M_{i-1}]$. Objects of the category $\mathbf{E}$ are of the form $[\Gamma \vdash \sigma : Type]$ and arrows $[\Gamma \vdash \sigma : Type] \to [\Delta \vdash \tau : Type]$ are pairs $([\vec{M}], [N])$ with $[\vec{M}] : [\Gamma] \to [\Delta]$ in $\mathbf{B}$ and $\Gamma, x : \sigma \vdash N : \tau[\vec{y} := \vec{M}]$. The functor $\mathscr{P}$ is then described by $[\Gamma \vdash \sigma : Type] \mapsto$ (the projection $[\Gamma, x : \sigma] \to [\Gamma]$). If Γ is of the form $x_1 : \sigma_1, \ldots, x_m : \sigma_m$, this projection is simply $\langle [x_1], \ldots, [x_m] \rangle$.

Note that the functor $\mathscr{P}_0$ performs "context comprehension" $[\Gamma \vdash \sigma : Type] \mapsto [\Gamma, x : \sigma]$. Similarly, other type-theoretic operations can be understood categorically using this specific comprehension category.

Lemma 4.4. *Let $\mathscr{P} : \mathbf{E} \to \mathbf{B}^{\to}$ be a comprehension category. For every $E \in \mathbf{E}$ and $u : A \to pE$ in $\mathbf{B}$ one can always choose a pullback of the form shown in Fig. 10. Hence, one can choose a pullback functor $\mathscr{P}E^{\#} : \mathbf{B}/pE \to \mathbf{B}/\mathscr{P}_0 E$ by $u \mapsto \mathscr{P}_0 \bar{u}(E)$.*

Proof. By requirement (ii) in Definition 4.1. □

Let $\mathscr{P} : \mathbf{E} \to \mathbf{B}^{\to}$ be a comprehension category. As a result of Lemma 4.4 one obtains for an object $E \in \mathbf{E}$ above $B \in \mathbf{B}$ and a morphism $u : A \to B$ in $\mathbf{B}$ an isomorphism

$$\mathbf{B}/B(u, \mathscr{P}E) \cong |u^*(E)|$$

by factorizing a map $w : A \to \mathscr{P}_0 E$ through $\mathscr{P}_0 \bar{u}(E)$ in Fig. 10. In a different but equivalent formulation one obtains an isomorphism

$$\mathbf{B}(A, \mathscr{P}_0 E) \cong \dot{\bigcup_{u : A \to pE}} |u^*(E)|.$$

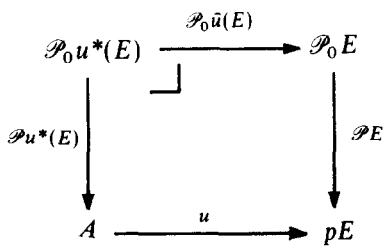


Fig. 10.

One finds a "disjoint union" encoded in a comprehension category. In this way one can understand the operation of context extension, which is performed by the functor $\mathscr{P}_0$, as explained in the introduction.

Example 4.5 (*Display-map categories*). If $\mathbf{B}$ is a category with pullbacks then the identity functor $\mathbf{B}^{\to} \to \mathbf{B}^{\to}$ is obviously a full comprehension category. A generalization of this example is given by the display-map categories from [31] (see also [16, 20]). One considers a category $\mathbf{B}$ together with a collection $\mathscr{D}$ of morphisms from $\mathbf{B}$ which are called "display maps". These should satisfy (at least) the condition that for every $f: X \to B$ in $\mathscr{D}$ and $u: A \to B$ in $\mathbf{B}$, there is a pullback of the form shown in Fig. 11 such that $u^*(f)$ is in $\mathscr{D}$ again. Let $\mathbf{B}^{\to}(\mathscr{D})$ be the full subcategory of $\mathbf{B}^{\to}$ with display maps from $\mathscr{D}$ as objects. The inclusion $\mathbf{B}^{\to}(\mathscr{D}) \subsetneq \mathbf{B}^{\to}$ is then a full comprehension category. We mention two more points.

(i) For the special case where $\mathscr{D}$ consists of all monics, we write $Sub(\mathbf{B})$ for the category $\mathbf{B}^{\to}(\mathscr{D})$.

(ii) Every comprehension category $\mathscr{P}: \mathbf{E} \to \mathbf{B}^{\to}$ determines a collection of display maps given by its projections $\{\mathscr{P}E \mid E \in \mathbf{E}\}$, see Lemma 4.4.

Example 4.6 (*Full internal subcategories*). Let $\mathbf{B}$ be an LCCC and τ an arbitrary morphism in $\mathbf{B}$. By a standard construction (see e.g. [28] or [19, 2.38]), τ gives rise to an internal category in $\mathbf{B}$ which is called a "full internal subcategory of $\mathbf{B}$". The latter means that the fibration $\Sigma(\tau) \to \mathbf{B}$ obtained by externalization comes equipped with a full and faithful cartesian functor $\Sigma(\tau) \to \mathbf{B}^{\to}$. This functor is then of course a full comprehension category.

Example 4.7 (*Topos comprehension*). Let $\mathbf{B}$ be a topos with subobject classifier $\top: t \to \Omega$. One obtains a comprehension category $\mathbf{B}/\Omega \to \mathbf{B}^{\to}$ by assigning to each "formula" $\varphi: A \to \Omega$ its "extension" $\{\varphi\}$ obtained in Fig. 12. The resulting functor $\mathbf{B}/\Omega \to \mathbf{B}^{\to}$ is full and faithful only on the subcategory $Cart(\mathbf{B}) \subsetneq \mathbf{B}^{\to}$ with pullbacks as morphisms. Hence, one does not obtain a *full* comprehension category.

Next it will be shown how every comprehension category can be turned into a full one.

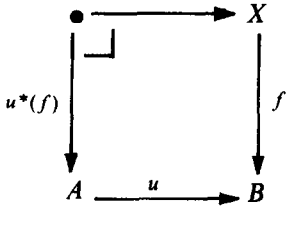


Fig. 11.

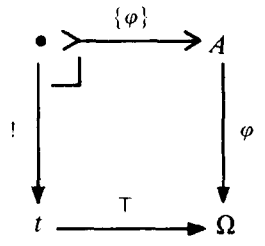


Fig. 12.

Definition 4.8. A category $Comp(\mathbf{B})$ of comprehension categories with basis $\mathbf{B}$ is defined by taking as morphisms $H:(\mathscr{P}:\mathbf{E}\to\mathbf{B}^{\to})\to(\mathscr{Q}:\mathbf{D}\to\mathbf{B}^{\to})$ cartesian functors $H:\mathbf{E}\to\mathbf{D}$ commuting with the projections, i.e. $\mathscr{Q}\circ H=\mathscr{P}$. We write $Comp_{\mathrm{full}}(\mathbf{B})$ for the full subcategory of full comprehension categories with basis $\mathbf{B}$.

More generally, a category $Comp$ is laid down by $(H,K):(\mathscr{P}:\mathbf{E}\to\mathbf{B}^{\to})\to(\mathscr{Q}:\mathbf{D}\to\mathbf{A}^{\to})$ iff $H:\mathbf{E}\to\mathbf{D}$ is a cartesian functor which together with $K:\mathbf{B}\to\mathbf{A}$ commutes with the projections, i.e. $\mathscr{Q}\circ H=K^{\to}\circ\mathscr{P}$.

Lemma 4.9. *The inclusion functor* $Comp_{\mathrm{full}}(\mathbf{B})\subsetneq Comp(\mathbf{B})$ *has a left adjoint. In Ehrhard's* [9] *notation, it will be denoted by* $(-)^{\heartsuit}$.

Proof. Given a comprehension category $\mathscr{P}:\mathbf{E}\to\mathbf{B}^{\to}$, one forms a *full* comprehension category $\mathscr{P}^{\heartsuit}:\mathbf{E}^{\heartsuit}\to\mathbf{B}^{\to}$, called by Ehrhard the *heart of* $\mathscr{P}$, as follows. The category $\mathbf{E}^{\heartsuit}$ has objects $E\in\mathbf{E}$; morphisms $(u,v):E\to E'$ in $\mathbf{E}^{\heartsuit}$ are given by maps $u:pE\to pE'$ and $v:\mathscr{P}_0E\to\mathscr{P}_0E'$ in $\mathbf{B}$ such that $u\circ\mathscr{P}E=v\circ\mathscr{P}E'$. The functor $\mathscr{P}^{\heartsuit}:\mathbf{E}^{\heartsuit}\to\mathbf{B}$ is then described by $E\mapsto\mathscr{P}E$ and $(u,v)\mapsto(u,v)$.

The unit morphism $\mathscr{P}\to\mathscr{P}^{\heartsuit}$ is given by a functor $\eta_{\mathscr{P}}:\mathbf{E}\to\mathbf{E}^{\heartsuit}$ with $E\mapsto E$ and $f\mapsto\langle pf,\mathscr{P}_0f\rangle$. □

Example 4.10 (*Categories with attributes*). (see [5, 7, 24]) We loosely follow the exposition in [24], but use fibrations instead in indexed categories. A category with attributes consists of a *discrete* fibration $p:\mathbf{E}\to\mathbf{B}$ (i.e. a fibration whose fibres are discrete categories) together with a functor $\mathscr{P}_0:\mathbf{E}\to\mathbf{B}$ and a natural transformation $\mathscr{P}:\mathscr{P}_0\dot{\to}p$ such that for every $E\in\mathbf{E}$ above $B\in\mathbf{B}$ and $u:A\to B$ there is a pullback of the form shown in Fig. 13, where $\bar{u}(E)$ denotes the (unique) cartesian lifting of u. Moggi writes $B\cdot E$ for $\mathscr{P}_0E$ and $u\cdot E$ for $\mathscr{P}_0\bar{u}(E)$. In the above formulation it is clear that one obtains a comprehension category $\mathscr{P}:\mathbf{E}\to\mathbf{B}^{\to}$ with discrete fibres. As pointed out in [3], these categories with attributes correspond to full split comprehension categories: if $\mathscr{P}$ is a category with attributes, then its full completion $\mathscr{P}^{\heartsuit}$ (see Lemma 4.9) is a full split comprehension category. In the reverse direction, one simply forgets the arrows in the fibres. These transitions are inverse to each other.

Indeed in a *full* comprehension category $\mathscr{P}:\mathbf{E}\to\mathbf{B}^{\to}$ morphisms in the total category $\mathbf{E}$ correspond bijectively to morphisms in $\mathbf{B}^{\to}$ between projections. Hence, the

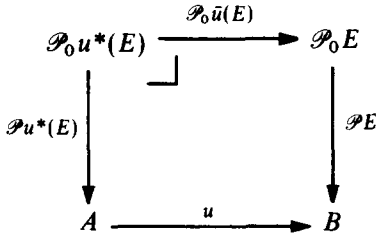


Fig. 13.

presence of these morphisms in **E** does not seem to be necessary. We think it does make sense not to throw them away (as is done for categories with attributes) for the following two reasons.

(i) In "natural" presentations of concrete examples the fibre categories are not discrete.

(ii) As shown in Section 3, in fibred category theory one defines "fibred structure" in the total category via fibred adjunctions. We follow this approach in defining products and sums for comprehension categories in Section 5. For categories with attributes, one has to define products and sums in terms of maps in the basis, since the fibres are discrete. The result is a rather "ad hoc" formulation (see e.g. [24, Definition 6.6]). In our approach this description comes out as a result of the fibred adjunctions, see Proposition 5.15. Especially the Beck–Chevalley condition (which ensures that the relevant structure is obtained uniformly) comes out in a natural way, since it is encoded in the notion of a fibred adjunction (see Proposition 3.4).

Example 4.11 (*Constant comprehension categories*). Let **B** be a category with cartesian products. A category $\bar{\mathbf{B}}$ is defined with pairs of objects (A, X) from **B** as objects. Morphisms $(A, X) \to (B, Y)$ in $\bar{\mathbf{B}}$ are given by two maps $u: A \to B$ and $f: A \times X \to Y$ in **B**. Composition is described by $(v, g) \circ (u, f) = (v \circ u, g \circ \langle u \circ \pi, f \rangle)$ and identities by (id, π'). The first projection $\bar{\mathbf{B}} \to \mathbf{B}$ is then a split fibration.

One obtains a full comprehension category $Cons_{\mathbf{B}}: \bar{\mathbf{B}} \to \mathbf{B}^{\to}$ by $(A, X) \mapsto$[the projection $\pi: A \times X \to A$]. This comprehension category will be called *constant* because there are no dependencies involved.

The "comprehension categories with unit" which will be introduced next are Ehrhard's *D-categories* from [9, 8]. We decided to change the name of these structures in order to provide more clarity and uniformity. The name "unit" indicates the presence of unit types, i.e. of types inhabited by exactly one term.

Definition 4.12. A *comprehension category with unit* is given by a fibration $p: \mathbf{E} \to \mathbf{B}$ provided with a terminal object functor $1: \mathbf{B} \to \mathbf{E}$, which has a right adjoint $\mathscr{P}_0: \mathbf{E} \to \mathbf{B}$.

The ensuing functor $\mathscr{P}: \mathbf{E} \to \mathbf{B}^{\to}$ given by $E \mapsto p(\varepsilon_E)$, where $\varepsilon: 1\mathscr{P}_0 \dot{\to} Id$ is counit, then forms a comprehension category (see below for the proof).

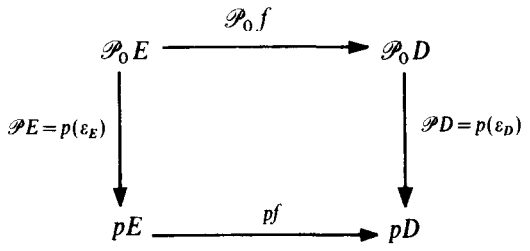


Fig. 14.

Using the adjunction $1 \dashv \mathscr{P}_0$ and the fact that p is a fibration one can verify that the functor $\mathscr{P}: \mathbf{E} \to \mathbf{B}^{\to}$ described in Definition 4.12 is a comprehension category. It suffices to check that for cartesian $f: E \to D$ in $\mathbf{E}$, the morphism $\mathscr{P}(f) = (pf, \mathscr{P}_0 f): \mathscr{P}E \to \mathscr{P}D$ is a pullback in $\mathbf{B}$ (see Fig. 14). Therefore, assume we are presented with morphisms $u: A \to \mathscr{P}_0 D$, $v: A \to pE$ satisfying $\mathscr{P}D \circ u = pf \circ v$. The transpose $\hat{u} = \varepsilon_D \circ 1u: 1A \to D$ in $\mathbf{E}$ satisfies $p(\hat{u}) = \mathscr{P}D \circ u = pf \circ v$. Since f is (strong) cartesian there is a unique $\varphi: 1A \to E$ above v with $f \circ \varphi = \hat{u}$. Then one can take as mediating arrow the transpose $\check{\varphi}: A \to \mathscr{P}_0 E$ since

$$\mathscr{P}_0 f \circ \check{\varphi} = \mathscr{P}_0 f \circ \mathscr{P}_0 \varphi \circ \eta_A = \mathscr{P}_0 \hat{u} \circ \eta_A = \check{\hat{u}} = u,$$

$$\mathscr{P}E \circ \check{\varphi} = p(\varepsilon_E \circ 1\mathscr{P}_0 \varphi \circ 1\eta_A) = p(\varphi \circ \varepsilon_{1A} \circ 1\eta_A) = p(\varphi) = u.$$

Using the uniqueness of φ one easily obtains that $\check{\varphi}$ is also unique.

Of the examples mentioned before, the comprehension category $Id_{\mathbf{B}}$ from Example 4.5 has a unit. The adjunctions involved are $cod \dashv id_{(-)} \dashv dom$, see also Definition 3.5. In case the calculus considered in Example 4.3 has a type-theoretic unit (i.e. a singleton type), the comprehension category constructed there has a unit. In Example 4.11, the constant comprehension category $Cons_{\mathbf{B}}$ has a unit if the category $\mathbf{B}$ has a terminal object.

In the sequel we shall loosely speak about "a comprehension category with unit $\mathscr{P}: \mathbf{E} \to \mathbf{B}^{\to}$", thereby meaning that there is a terminal object functor $1: \mathbf{B} \to \mathbf{E}$ which is a left adjoint to $\mathscr{P}_0$ in such a way that the counit is above $\mathscr{P}$.

After the following technical lemma, some more examples are described.

Lemma 4.13. *Let $\mathscr{P}: \mathbf{E} \to \mathbf{B}^{\to}$ be a comprehension category with unit, say via $1: \mathbf{B} \to \mathbf{E}$. Then*

(i) *for $E \in \mathbf{E}$ above A one has $|E| \cong \mathbf{E}_A(1A, E)$;*

(ii) *for $E \in \mathbf{E}$ and $u: B \to pE$ one has $\mathbf{B}/pE(u, \mathscr{P}E) \cong \mathbf{E}_B(1B, u^*(E))$;*

(iii) *$\mathscr{P}1: \mathscr{P}_0 1 \xrightarrow{\cdot} Id$ is an isomorphism; hence, $\mathscr{P}$ preserves the fibred terminal.*

Proof. (i): By the adjunction $1 \dashv \mathscr{P}_0$.

(ii): By (i) and the observations following Lemma 4.4 one obtains

$$\mathbf{E}_B(1B, u^*(E)) \cong |u^*(E)| \cong \mathbf{B}/pE(u, \mathscr{P}E).$$

(iii): The unit $\eta: Id \dot{\to} 1\mathscr{P}_0$ is an iso since 1 is full and faithful (see Definition 3.5). But $\mathscr{P}1 \circ \eta = p\varepsilon 1 \circ p1\eta = p(\varepsilon 1 \circ 1\eta) = id$. □

Example 4.14 (*Family models over sets*). For an arbitrary category **C**, one defines the category $Fam(\mathbf{C})$ containing set-indexed families of objects and arrows of **C**, in the following way. Objects are given by a set I and a map $X: I \to Obj\,\mathbf{C}$, written as (I, X) or $\{X_i\}_{i\in I}$ depending on what is most convenient. Morphisms $(u,f):(I, X)\to(J, Y)$ are maps $u: I\to J$ in **Sets** and $f: I\to Mor\,\mathbf{C}$ such that $f_i: X_i\to Y_{u(i)}$ in **C**; notation $\{f_i: X_i\to Y_{u(i)}\}_I$. Given another morphism $\{g_j: Y_j\to Z_{v(j)}\}_J$, composition in $Fam(\mathbf{C})$ is layed down by $(v, g)\circ(u, f)=(v\circ u, \lambda i\in I.g_{u(i)}\circ f_i)$. The first projection $p: Fam(\mathbf{C})\to$**Sets** is a split fibration: for $u: I\to J$ one can take $u^*(J, Y)=(I, Y\circ u)$ and $\bar{u}(J, Y)=(u, \lambda i\in I.id_{Y_{u(i)}})$.

So far the construction is quite familiar, see e.g. [2]; it can be described as the Grothendieck construction applied to the functor $I\mapsto \mathbf{C}^I$. One easily verifies that the fibration $Fam(\mathbf{C})\to$**Sets** has finite limits and exponents iff **C** has them. In case the category **C** has a terminal object t such that all collections $\mathbf{C}(t, A)$ are small, then the fibration $Fam(\mathbf{C})\to$**Sets** can be extended to a comprehension category with unit. As terminal object functor $1:$**Sets**$\to Fam(\mathbf{C})$ one takes

$$I\mapsto \{t\}_I,$$

$$u: I\to J\mapsto (u, \lambda i\in I.id_t).$$

The required functor $\mathscr{P}_0: Fam(\mathbf{C})\to$**Sets** is defined by disjoint union,

$$\{X_i\}_I\mapsto \dot{\bigcup_{i\in I}}.\,\mathbf{C}(t, X_i)=\{\langle i, x\rangle \mid i\in I \text{ and } x: t\to X_i\},$$

$$\{f_i: X_i\to Y_{u(i)}\}_I\mapsto \lambda\langle i, x\rangle\in\mathscr{P}_0(I, X)\cdot\langle u(i), f_i\circ x\rangle.$$

It is easy to verify that the resulting comprehension category with unit $\mathscr{P}: Fam(\mathbf{C})\to$ **Sets**$^{\to}$ has projections $\mathscr{P}(I, X)=\lambda\langle i, x\rangle.i: \mathscr{P}_0(I, X)\to I$. This comprehension category will be called the "family model of **C**". We investigate some of its properties.

(i) The comprehension category $Fam(\mathbf{C})\to$**Sets**$^{\to}$ is full if and only if the global sections functor $\mathbf{C}(t, -): \mathbf{C}\to$**Sets** is full and faithful.

The proof is easily established using that a comprehension category is full if and only if it is fibrewise a full and faithful functor.

(ii) The fibration $Fam(\mathbf{C})\to$**Sets** has products (sums), see Definition 3.10, if and only if C has infinite products (coproducts).

The if-part is proved by defining for a function $u: I\to J$ product and sum functors $Fam(\mathbf{C})_J\to Fam(\mathbf{C})_I$ by $\{Y_j\}_J\mapsto\{\Pi_{j\in u^{-1}(i)}\cdot Y_j\}_I$ and $\{Y_j\}_J\mapsto\{\coprod_{j\in u^{-1}(i)}\cdot Y_j\}_I$, respectively. The only-if-part is obtained by looking at the fibred above the terminal object, which is isomorphic to **C**.

(iii) The category $Fam(\mathbf{C})$ can be understood as the completion of **C** with respect to set-indexed colimits. First, note that $Fam(\mathbf{C})$ has infinite coproducts itself: given a map $Y: J\to Fam(\mathbf{C})$, say for $j\in J$ with $Y_j=(I_j, X_j)=\{X_{j,i}\}_{i\in I_j}$, then one can take $\coprod_J Y_j=\{X_{\pi m, \pi' m}\}_{m\in\dot{\cup}_j\cdot I_j}$.

Furthermore, there is a functor $\eta_{\mathbf{C}}: \mathbf{C} \to Fam(\mathbf{C})$ given by indexing over a terminal set, see [2, 3.5]. Note that $\{X_i\}_I \cong \coprod_I \eta_{\mathbf{C}}(X_i)$ in $Fam(\mathbf{C})$. The completion mentioned above means that for any category $\mathbf{D}$ with infinite coproducts and any functor $H: \mathbf{C} \to \mathbf{D}$ there is a unique (up-to-isomorphism) coproduct-preserving functor $H': Fam(\mathbf{C}) \to \mathbf{D}$ with $H'\eta_{\mathbf{C}} \cong H$ (see Fig. 15). Obviously, one has (up-to-isomorphism) $H'\{X_i\}_I = \coprod_I HX_i$.

Applying this result to the global sections functor $\mathbf{C}(t, -): \mathbf{C} \to \mathbf{Sets}$ and to the Yoneda functor $Y: \mathbf{C} \to \mathbf{Sets}^{\mathbf{C}^{op}}$, one obtains Fig. 16, relating some important functors. Here, the comprehension functor $\mathscr{P}_0: Fam(\mathbf{C}) \to \mathbf{Sets}$ reappears.

Example 4.15 (*Internal family model*). Hyland [15] describes the family model in internal category theory using an internal global sections functor. We briefly review the construction. Let $\mathbf{B}$ be a category with finite limits and $\mathbf{C} = \langle C_0, C_1, \ldots \rangle$ be an internal category in $\mathbf{B}$ with internal terminal object $\underline{t}: t \to C_0$. The internal category $\mathbf{C}$ gives rise to a split fibration $\Sigma(\mathbf{C}) \to \mathbf{B}$ by externalization. Similar to the previous example, the (internal) terminal object allows us to obtain a comprehension category with unit. Therefore, a terminal object functor $\mathbf{B} \to \Sigma(\mathbf{C})$ is defined by $A \mapsto [\underline{t} \circ !_A : A \to C_0]$. The comprehension functor $\mathscr{P}_0: \Sigma(\mathbf{C}) \to \mathbf{B}$ (see Fig. 17) is obtained

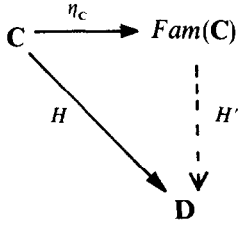


Fig. 15.

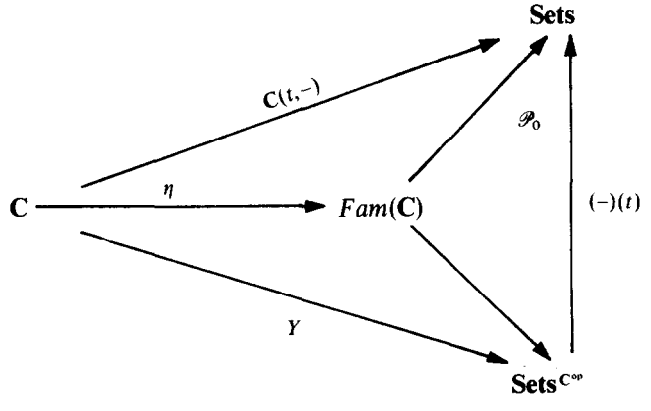


Fig. 16.

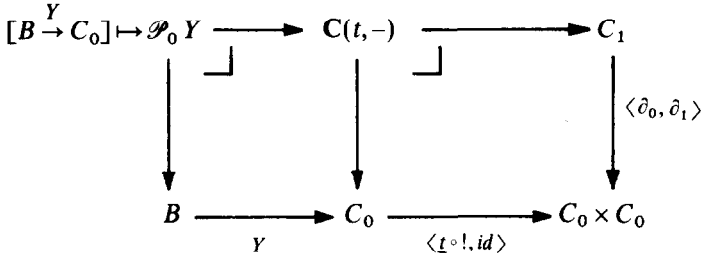


Fig. 17.

by following the construction in [15, 0.1]. In informal notation, one constructs $\mathscr{P}_0 Y$ as $\{\langle b, x\rangle \mid b \in B$ and $x : t \to Y_b\}$; it is like in Example 4.14. The resulting projection is the above arrow $\mathscr{P}_0 Y \to B$. Thus, one obtains a comprehension category with unit. In [15] it is defined that $\mathbf{C}$ is a "full subcategory of $\mathbf{B}$" iff all the (fibrewise) functors $\Sigma(\mathbf{C})_B \to \mathbf{B}/B$ are full and faithful, i.e. iff the above comprehension category $\Sigma(\mathbf{C}) \to \mathbf{B}^{\to}$ is full.

Example 4.16 (*Fibrations as projections*). There are two ways to obtain a comprehension category $Fib \to \mathbf{Cat}^{\to}$. First one has the "inclusion" $Fib \subsetneq \mathbf{Cat}^{\to}$ using Proposition 2.6. This comprehension category does not have a unit.

Alternatively one can define a comprehension functor $Fib \to \mathbf{Cat}$ by $[p : \mathbf{E} \to \mathbf{B}] \mapsto Cart(\mathbf{E})$, using the notation introduced before Definition 3.9. One obtains a (full) comprehension category with unit, since $Fib(Id_{\mathbf{A}}, p) \cong \mathbf{Cat}(\mathbf{A}, Cart(\mathbf{E}))$. The functor $Fib \to Fib$ given by $[p : \mathbf{E} \to \mathbf{B}] \mapsto [|p| : Cart(\mathbf{E}) \to \mathbf{B}]$ forms a morphism from the second to the first comprehension category described here.

The following notion of "reflexive comprehension category" is adapted from [10]. Apart from the fact that it describes two fundamental structures (see after the definition), it will turn out to be useful in analyzing Lawvere's comprehension.

Definition 4.17. A comprehension category $\mathscr{P} : \mathbf{E} \to \mathbf{B}^{\to}$ on a base category $\mathbf{B}$ with pullbacks will be called *reflexive* if $\mathscr{P}$ has a fibred left adjoint in such a way that the counit is an isomorphism.

By a standard result about adjunctions, $\mathscr{P}$ is then a full and faithful functor and, thus, a full comprehension category.

We briefly mention the two examples from [10]. Let $\mathbf{B}$ be a topos; the comprehension category of monics $Sub(\mathbf{B}) \subsetneq \mathbf{B}^{\to}$ described in Example 4.5(i) is reflexive, since every morphism in a topos has a unique epi-mono factorization. The example $Fam_{\mathrm{eff}}(\mathbf{M}) \to \omega\text{-}\mathbf{Set}^{\to}$ of modest ω-sets described in Example 5.14(v) also forms an example.

Example 4.18 (*Lawvere's comprehension*). Historically Lawvere [21] first described comprehension in categorical terms, using his "hyperdoctrines satisfying the comprehension scheme". Both Pavlović and Curien pointed out the relevance for the present

work. We start with a translation of Lawvere's notion into fibred category theory. Subsequent refinements are our additions.

Let $p: \mathbf{E} \to \mathbf{B}$ be a bifibration with a terminal object described by a functor $1: \mathbf{B} \to \mathbf{E}$. One defines a functor $\mathscr{S}: \mathbf{B}^{\to} \to \mathbf{E}$ on objects by $u: A \to B \mapsto \Sigma_u(1A)$, see Proposition 2.3(i) and on morphisms by Fig. 18. This bifibration p will be called a *Lawvere category* if the functor $\mathscr{S}$ has an (ordinary) right adjoint $\mathscr{P}: \mathbf{E} \to \mathbf{B}^{\to}$ with $cod \circ \mathscr{P} = p$ and vertical unit (or equivalently, vertical counit). A standard result, which is not hard to verify, yields that $\mathscr{P}$ is then a cartesian functor (see e.g. [33, Lemma 4.5]).

This notion gives rise to ramifications. One can call p a *full Lawvere category* if the counit of the adjunction is an isomorphism. Further, suppose that the category $\mathbf{B}$ has pullbacks and that the fibration p admits sums, i.e. the sum functors $\Sigma_u \dashv u^*$ additionally satisfy the Beck–Chevalley condition. It is left to the reader to verify that $\mathscr{S}$ becomes a cartesian functor $cod \to p$. We then call p a *cartesian Lawvere category* if $\mathscr{S}$ has a fibred right adjoint; it will be called a *full cartesian Lawvere category* if the counit of this adjunction is an isomorphism.

In the next two results, Lawvere's approach will be related to ours. The right adjoint required in Lawvere's definition forms a comprehension category in our sense. Actually, we introduced a comprehension category simply as such a functor, without any of the above prerequirements about the fibration involved.

Result (i). *The notions "Lawvere category" and "bifibration + comprehension category with unit" coincide.*

Proof. We show how one can go back-and-forth between these notions. Let $p: \mathbf{E} \to \mathbf{B}$ be a bifibration with terminal object via $1: \mathbf{B} \to \mathbf{E}$. Assume first that p is a Lawvere category, say with $\mathscr{P}: \mathbf{E} \to \mathbf{B}^{\to}$ as right adjoint to the above functor $\mathscr{S}$. The functor $I = \mathscr{S} \circ id_{(-)}: \mathbf{B} \to \mathbf{E}$ is isomorphic to the terminal object functor $1: \mathbf{B} \to \mathbf{E}$, since $IA = \mathscr{S}(id_A) = \Sigma_{id_A}(1A) = (id_A)_*(1A) \cong 1A$. Moreover, I has a right adjoint $\mathscr{P}_0 = dom \circ \mathscr{P}$ by composition of adjoints: $I = \mathscr{S} \circ id_{(-)} \dashv dom \circ \mathscr{P} = \mathscr{P}_0$. Hence, one also has $1 \dashv \mathscr{P}_0$. Using that the counit of the adjunction $\mathscr{S} \dashv \mathscr{P}$ is vertical one obtains that the counit of the adjunction $1 \dashv \mathscr{P}_0$ is above $\mathscr{P}$. Hence, $\mathscr{P}$ is a comprehension category with unit.

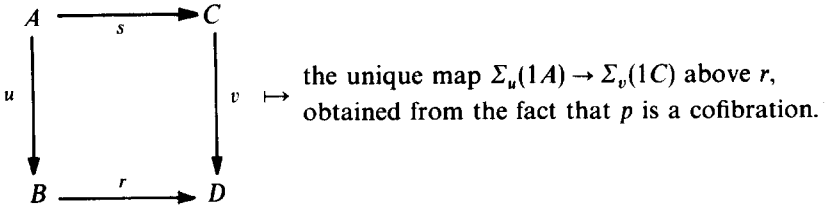


Fig. 18.

The other way round, assume that there is an adjointness $1 \dashv \mathscr{P}_0$; the resulting comprehension category $\mathscr{P}: \mathbf{E} \to \mathbf{B}^{\to}$ (see the construction in Definition 4.12) is then a right adjoint of the above functor $\mathscr{S}$, since for $u: A \to B$ in $\mathbf{B}$ one has

$$\begin{aligned}
\mathbf{E}(\mathscr{S}u, E) &\cong \dot{\bigcup_{v: B \to pE}} \mathbf{E}_B(\Sigma_u(1A), v^*(E)) \quad \text{(see Definition 2.2)} \\
&\cong \dot{\bigcup_{v: B \to pE}} \mathbf{E}_A(1A, u^*v^*(E)) \\
&\cong \dot{\bigcup_{v: B \to pE}} \mathbf{E}_A(1A, (v \circ u)^*(E)) \\
&\cong \dot{\bigcup_{v: B \to pE}} \mathbf{B}/pE(v \circ u, \mathscr{P}E) \quad \text{[by using Lemma 4.13(ii)]} \\
&\cong \mathbf{B}^{\to}(u, \mathscr{P}E).
\end{aligned}$$

Turning a Lawvere category first into a comprehension category with unit and then into a Lawvere category again, one ends up with the original adjointness. The other way round, starting from a comprehension category with unit one obtains a similar result. □

Result (ii). *The following notions describe the same structures*:

(a) *full cartesian Lawvere category*;
(b) *reflexive comprehension category*;
(c) *fibration with sums + full comprehension category with unit.*

Proof. The implication (a) ⇒ (b) and (a) ⇔ (c) are obvious, either by definition, or (i) above. We shall do (b) ⇒ (c). Therefore, let $\mathscr{P}: \mathbf{E} \to \mathbf{B}^{\to}$ be a reflexive comprehension category with $\mathscr{R}: \mathbf{B}^{\to} \to \mathbf{E}$ as fibred left adjoint to $\mathscr{P}$. Sums are obtained by the following definition, essentially due to [9]. For $u: A \to B$ in $\mathbf{B}$ one introduces a functor $\sum_u: \mathbf{E}_A \to \mathbf{E}_B$ by

$$E \mapsto \mathscr{R}(u \circ \mathscr{P}E) \qquad f \mapsto \mathscr{R}(\langle id_B, \mathscr{P}_0 f \rangle).$$

Then $\sum_u$ is a right adjoint to u^* using that the pullback functor $u^{\#}$ in $\mathbf{B}$ has a left adjoint by composition, see the example at the end of Section 3.

$$\begin{aligned}
\mathbf{E}_B(\textstyle\sum_u(E), D) &= \mathbf{E}_B(\mathscr{R}(u \circ \mathscr{P}E), D) \\
&\cong \mathbf{B}/B(u \circ \mathscr{P}E, \mathscr{P}D) \\
&\cong \mathbf{B}/A(\mathscr{P}E, u^{\#}(\mathscr{P}D)) \\
&\cong \mathbf{B}/A(\mathscr{P}E, \mathscr{P}u^*(D)) \quad \text{(see Lemma 4.4)} \\
&\cong \mathbf{E}_A(E, u^*(D)) \quad \text{(by fullness)}.
\end{aligned}$$

Beck–Chevalley holds, since $\mathscr{R}$ is a cartesian functor. Hence, p admits sums. The unit-part of the statement follows as in the proof above, using this time that

$I = \mathscr{R} \circ id_{(-)} : \mathbf{B} \to \mathbf{E}$ is a terminal object functor, due to the reflection $\mathscr{R} \dashv \mathscr{P}$, see e.g. [10, Lemma 3]. One obtains a *full* comprehension category because the counit of the adjunction is an iso and, thus, is $\mathscr{P}$ a full and faithful functor. □

The family models $Fam(\mathbf{C}) \to \mathbf{Sets}^{\to}$ yield cartesian Lawvere categories in case $\mathbf{C}$ has infinite coproducts, see Example 4.14(ii). One obtains *full* cartesian Lawvere categories (i.e. reflexive comprehension categories) for $\mathbf{C} = \cdot \to \cdot$ and $\mathbf{C} = \mathbf{Sets}$. In fact, these examples have been mentioned already, because of the equivalences $Fam(\cdot \to \cdot) \simeq Sub(\mathbf{Sets})$, see the topos example after Definition 4.17, and $Fam(\mathbf{Sets}) \simeq \mathbf{Sets}^{\to}$, see Example 4.5.

Example 4.19 (*Family models over categories*). The construction from Example 4.14 yielding the family models $Fam(\mathbf{C}) \to \mathbf{Sets}^{\to}$ can also be performed with **Cat** as base category. Essentially this construction occurs in [21], although not described in such a way. For a fixed category $\mathbf{C}$, the Grothendieck construction applied to the (contravariant) functor which send a (small) category $\mathbf{A}$ to the functor category $\mathbf{C}^{\mathbf{A}}$, yields a split fibration $Fam(\mathbf{C}) \to \mathbf{Cat}$. The objects in the total category $Fam(\mathbf{C})$ are pairs $(\mathbf{A}, X)$ with X a functor $\mathbf{A} \to \mathbf{C}$; morphisms $(\mathbf{A}, X) \to (\mathbf{B}, Y)$ are pairs (U, α) with U a functor $\mathbf{A} \to \mathbf{B}$ and α a natural transformation $X \xrightarrow{\cdot} YU$. In case the category $\mathbf{C}$ has a terminal object t such that all collections $\mathbf{C}(t, A)$ are small, one obtains a comprehension category with unit $Fam(\mathbf{C}) \to \mathbf{Cat}^{\to}$. Obviously, one defines the terminal object functor $1 : \mathbf{Cat} \to Fam(\mathbf{C})$ by $\mathbf{A} \mapsto$[the functor $\mathbf{A} \to \mathbf{C}$ which is constant t]; the comprehension functor $\mathscr{P}_0 : Fam(\mathbf{C}) \to \mathbf{Cat}$ is laid down by $(\mathbf{A}, X) \mapsto$[the comma category $(t \downarrow X)$]. The resulting functor $Fam(\mathbf{C}) \to \mathbf{Cat}^{\to}$ assigns to $X : \mathbf{A} \to \mathbf{C}$ the projection $(t \downarrow X) \to \mathbf{A}$.

Results comparable to those in Example 4.14 about families over **Sets** can be obtained.

Result (i). *The comprehension category* $Fam(\mathbf{C}) \to \mathbf{Cat}^{\to}$ *is full if and only if the functor* $\mathbf{C}(t, -) : \mathbf{C} \to \mathbf{Sets}$ *is full and faithful.*

The next (familiar) result resembles Example 4.5(ii), except that with **Cat** as base category, "Beck–Chevalley" becomes problematic. It yields examples of Lawvere categories, see Result (i) of Example 4.18.

Result (ii). $Fam(\mathbf{C}) \to \mathbf{Cat}$ *is a bifibration if and only if the category* $\mathbf{C}$ *has all (small) colimits.*

Proof. The implication ($\Rightarrow$) follows as in Example 4.5(ii) by looking at the fibre above the terminal. The reverse implication involves left Kan extensions cf. [21, p. 5]. Assume that functors $U : \mathbf{A} \to \mathbf{B}$ and $X : \mathbf{A} \to \mathbf{C}$ are given; we have to define $\Sigma_U(X) : \mathbf{B} \to \mathbf{C}$. For $B \in \mathbf{B}$, one considers the comma category $(U \downarrow B)$ and the projection functor $(U \downarrow B) \to \mathbf{A}$; composition with X yields a functor $X_B : (U \downarrow B) \to \mathbf{C}$. One takes $\Sigma_U(X) = Colim(X_B)$. □

Definition 4.20 gives a "nondegeneracy" condition for comprehension categories. It will be used in Result (ii) of Example 4.21 and in Lemma 5.4.

Definition 4.20. A comprehension category $\mathscr{P}: \mathbf{E} \to \mathbf{B}^{\to}$ is called *nonempty* if in every context there is at least one inhabited type, i.e. if for every $A \in \mathbf{B}$ there is at least one $E \in \mathbf{E}$ above A with $|E| \neq \emptyset$.

Equivalently, if $\mathbf{B}$ contains a terminal object t, if there is an object $E \in \mathbf{E}$ above t which has a global section $t \to \mathscr{P}_0 E$ in $\mathbf{B}$; that is, if there is at least one inhabited closed type.

Trivially, comprehension categories with unit are nonempty.

Example 4.21 (*Pavlović's comprehension*). In a companion work [27] to ours, Pavlović introduces a notion of "comprehensiveness' for fibrations. The description one finds below is due to Streicher; it is slightly different from the one in [27].

In Definition 4.12 we introduced a *comprehension category with unit* as what Ehrhard [8] called a *D-category*. In fact we came to the notion of a comprehension category by looking for a "D-category without unit". In a similar way one can think about Pavlović's comprehension. The essential point about comprehension categories with unit (i.e. D-categories) is that the fibrewise global sections functors are representable, see Lemma 4.13(ii). In an arbitrary category $\mathbf{C}$ with terminal object t, one has for each $X \in \mathbf{C}$ a bijective correspondence,

$$\frac{\sigma: Id_{\mathbf{C}} \dot{\to} X}{f: t \to X},$$

between cocones $Id_{\mathbf{C}} \dot{\to} X$ and global sections of X. Hence, if one does not want to assume a terminal object t, then it makes sense to consider cocones $Id_{\mathbf{C}} \dot{\to} X$.

For a fibration $p: \mathbf{E} \to \mathbf{B}$ the situation is slightly more subtle. For an object $E \in \mathbf{E}$ above $A \in \mathbf{B}$, one has a fibration $dom_A: \mathbf{B}/A \to \mathbf{B}$; by change-of-base one obtains a projection,

$$\Pi_A: \mathbf{E} \underset{p,\, dom_A}{\times} \mathbf{B}/A \to \mathbf{E}.$$

For an arrow $u: B \to A$ we write "$\sigma: \Pi_B \ddot{\to} E$ over u" if $\sigma: \Pi_B \dot{\to} E$ is a cocone which satisfies: for each $E' \in \mathbf{E}$ and $w: pE' \to B$ one has that $\sigma_{E', w}$ is above $u \circ w$. Such σ's are the appropriate cocones for fibred categories: in case p has a fibred terminal object, say via $1: \mathbf{B} \to \mathbf{E}$, one obtains for $u: B \to A$ a bijective correspondence,

$$\frac{\sigma: \Pi_B \ddot{\to} E \text{ over } u}{f: 1B \to E \text{ above } u}.$$

For $\sigma: \Pi_B \ddot{\to} E$ over u, one takes $f = \sigma_{1B, id_B}: 1B \to E$. The other way, for $E' \in \mathbf{E}$ and $w: pE' \to B$ one takes $\sigma_{E', w} = f \circ \breve{w}: E' \to E$ above $u \circ w$, where $\breve{w}$ is the transpose across the adjunction $p \dashv 1$, see Definition 3.5.

Back to the general situation without terminal, one can form for each $E \in \mathbf{E}$ above A a functor from $(\mathbf{B}/A)^{\mathrm{op}}$ to **Sets** (or to a suitably larger universe) in the following way.

$$B \xrightarrow{u} A \mapsto \{\sigma : \Pi_B \stackrel{\cdot\cdot}{\to} E \text{ over } u\},$$

$$(B \xrightarrow{u} A) \xrightarrow{v} (B' \xrightarrow{u'} A) \mapsto \text{the function which sends } \sigma' : \Pi_{B'} \stackrel{\cdot\cdot}{\to} E \text{ over } u' \text{ to } \sigma : \Pi_B \stackrel{\cdot\cdot}{\to} E \text{ over } u, \text{ given by } \sigma_{E',w} = \sigma'_{E',v \circ w}.$$

One says that the fibration $p : \mathbf{E} \to \mathbf{B}$ is *comprehensive* if all these functors are representable. In more elementary formulation, p is comprehensive if for each $E \in \mathbf{E}$ there is a "representing" arrow $\mathscr{P}E : \mathscr{P}_0 E \to pE$ in $\mathbf{B}$ and a cocone $\tilde{E} : \Pi_{\mathscr{P}_0 E} \stackrel{\cdot\cdot}{\to} E$ over $\mathscr{P}E$ such that for every $u : B \to pE$ and $\sigma : \Pi_B \stackrel{\cdot\cdot}{\to} E$ over u, there is a unique $v : B \to \mathscr{P}_0 E$ which satisfies

$$\mathscr{P}E \circ v = u,$$

$$\sigma_{E',w} = (\tilde{E})_{E',v \circ w} \quad \text{for each } w : pE' \to B.$$

The next three results relate comprehensive fibrations and comprehension categories; the first and third ones occur in [27].

Result (i). *Let $p : \mathbf{E} \to \mathbf{B}$ be a comprehensive fibration. In that case, there is a comprehension category $\mathscr{P} : \mathbf{E} \to \mathbf{B}^{\to}$ with $p = cod \circ \mathscr{P}$.*

Proof. The object part of the functor $\mathscr{P} : \mathbf{E} \to \mathbf{B}^{\to}$ is obtained by choosing representing arrows. For a morphism $f : E_1 \to E_2$ in $\mathbf{E}$, one obtains a cocone $f \circ \tilde{E}_1 : \Pi_{\mathscr{P}_0 E_1} \stackrel{\cdot\cdot}{\to} E_2$ over $pf \circ \mathscr{P}E_1$. Hence, there is a unique arrow, say $\mathscr{P}_0 f$, from $\mathscr{P}_0 E_1$ to $\mathscr{P}_0 E_2$ with

$$\mathscr{P}E_2 \circ \mathscr{P}_0 f = pf \circ \mathscr{P}E_1,$$

$$f \circ (\tilde{E}_1)_{E,w} = (\tilde{E}_2)_{E',\mathscr{P}_0 f \circ w} \quad \text{for each } w : pE' \to \mathscr{P}_0 E_1.$$

By uniqueness one obtains a functor. In case f is cartesian, the resulting diagram $\mathscr{P}f$ in $\mathbf{B}$ is a pullback: suppose $u_1 : A \to pE_1$ and $u_2 : A \to \mathscr{P}_0 E_2$ with $pf \circ u_1 = \mathscr{P}E_2 \circ u_2$ are given. There is a cocone $\sigma : \Pi_A \stackrel{\cdot\cdot}{\to} E_2$ over u_2 with components $\sigma_{E',w} = (\tilde{E}_2)_{E',u_2 \circ w} : E' \to E_2$, which is above $\mathscr{P}E_2 \circ u_2 \circ w = pf \circ u_1 \circ w$. Since f is (strong) cartesian one obtains arrows $\tau_{E',w} : E' \to E_1$ above $u_1 \circ w$ with $f \circ \tau_{E',w} = \sigma_{E',w}$. These form a cocone $\tau : \Pi_A \stackrel{\cdot\cdot}{\to} E_1$ over u_1. Thus, one obtains the mediating arrow $A \to \mathscr{P}_0 E_1$. □

Result (ii). *Let $\mathscr{P} : \mathbf{E} \to \mathbf{B}^{\to}$ be a full nonempty comprehension category. The fibration $p = cod \circ \mathscr{P}$ is then comprehensive.*

Proof. For $E \in \mathbf{E}$ we construct an appropriate cocone $\tilde{E} : \Pi_{\mathscr{P}_0 E} \stackrel{\cdot\cdot}{\to} E$ over $\mathscr{P}E$ as follows. For $w : pE' \to \mathscr{P}_0 E$ one has a morphism $(w \circ \mathscr{P}E', \mathscr{P}E \circ w) : \mathscr{P}E' \to \mathscr{P}E$ in $\mathbf{B}^{\to}$. By fullness one obtains a unique arrow $(\tilde{E})_{E',w}$ above $\mathscr{P}E \circ w$ with $\mathscr{P}_0((\tilde{E})_{E',w}) = w \circ \mathscr{P}E'$.

Let $u: B \to pE$ together with $\sigma: \Pi_B \dashrightarrow E$ over u be given. Since $\mathscr{P}$ is nonempty we may assume an $F \in \mathbf{E}$ above B and a section $v \in |F|$ of $\mathscr{P}F$. Put $\sigma = \sigma_{F, id_F}: F \to E$, which is above u. We claim that $\mathscr{P}_0\sigma \circ v: B \to \mathscr{P}_0 E$ is the appropriate mediating arrow. Indeed,

$$\mathscr{P}E \circ \mathscr{P}_0\sigma \circ v = p\sigma \circ \mathscr{P}F \circ v = u \circ id = u.$$

Further, for $w: pE' \to B$ one has $(v \circ w \circ \mathscr{P}E', w): \mathscr{P}E' \to \mathscr{P}F$ in $\mathbf{B}^{\to}$; hence, there is a unique $g: E' \to F$ above w with $\mathscr{P}_0 g = v \circ w \circ \mathscr{P}E'$. But then $\sigma_{E', w} = \sigma \circ g$ (since σ is a cocone) satisfies the conditions which uniquely determine $(\tilde{E})_{E', \mathscr{P}_0\sigma \circ v \circ w}$. □

Result (iii). *Let $p: \mathbf{E} \to \mathbf{B}$ be a fibration with terminal object functor $1: \mathbf{B} \to \mathbf{E}$ then*

$$p \text{ is comprehensive} \Leftrightarrow \text{there is a comprehension category with unit } \mathscr{P}: \mathbf{E} \to \mathbf{B}^{\to} \text{ satisfying } p = cod \circ \mathscr{P}.$$

Proof. $\Rightarrow$: As in the proof of (i), the object part of $\mathscr{P}$ is obtained by choosing representing arrows. Then for $E \in \mathbf{E}$ above A one has

$$\begin{aligned}
\mathbf{E}(1B, E) &\cong \dot{\bigcup_{u: B \to A}} \mathbf{E}_u(1B, E) && \text{(see Definition 2.2)} \\
&\cong \dot{\bigcup_{u: B \to A}} \{\sigma: \Pi_B \dashrightarrow E \text{ over } u\} && \text{[by (1)]} \\
&\cong \dot{\bigcup_{u: B \to A}} \mathbf{B}/A(u, \mathscr{P}E) \\
&\cong \mathbf{B}(B, dom(\mathscr{P}E)) \\
&= \mathbf{B}(B, \mathscr{P}_0 E).
\end{aligned}$$

$\Leftarrow$: Similarly, for $E \in \mathbf{E}$ above A and $u: B \to A$ one has

$$\begin{aligned}
\{\sigma: \Pi_B \dashrightarrow E \text{ over } u\} &\cong \mathbf{E}_u(1B, E) && \text{[see (1)]} \\
&\cong \mathbf{E}_B(1B, u^*(E)) \\
&\cong \mathbf{B}/A(u, \mathscr{P}E) && \text{[by Lemma 4.13(ii)].}
\end{aligned}$$

Hence, the functor $(\mathbf{B}/A)^{op} \to \mathbf{Sets}$ described in the beginning is representable. □

Final remark 4.22. As is shown above, the main forms of comprehension can be ordered linearly; in increasing strength:

(1) Comprehension categories.
(2) Pavlović's comprehension.
(3) Ehrhard's comprehension (described by comprehension categories with unit).
(4) Lawvere's comprehension.

5. Quantification

Products and sums (both "weak" and "strong") for comprehension categories are described in this section. A comprehension category is then called *closed* if it has a unit, products and strong sums (see Definition 5.13). Such a structure gives a syntax-free description of a calculus with a unit and dependent products and sums. There are many examples of such closed comprehension categories. A few are described here; more of them, and more related results, may be found in [17, 18]. A comparison with other "closed" structures for type dependency occurs in [3].

Products and sums for comprehension categories are described by right and left adjoints to weakening functors. This can be done with fibred adjunctions (following the approach in [8, 9]) or, equivalently (using Proposition 3.4), with fibrewise adjunctions plus a Beck–Chevalley condition. Both are given.

A comprehension category $\mathscr{P}: \mathbf{E} \to \mathbf{B}^{\to}$ determines a category $Cart(\mathbf{E}) \subsetneq \mathbf{E}$ with cartesian arrows only. By restriction one obtains two functors $|p|$ and $|\mathscr{P}_0|$ from $Cart(\mathbf{E})$ to $\mathbf{B}$. Change-of-base of $p: \mathbf{E} \to \mathbf{B}$ along these functors yields two fibrations,

$$|p|^*: Cart(\mathbf{E}) \underset{|p|, p}{\times} \mathbf{E} \to Cart(\mathbf{E}) \qquad |\mathscr{P}_0|^*(p): Cart(\mathbf{E}) \underset{|\mathscr{P}_0|, p}{\times} \mathbf{E} \to Cart(\mathbf{E}),$$

see Proposition 2.6. The natural transformation $\mathscr{P}: \mathscr{P}_0 \dot{\to} p$ can be lifted to a cartesian functor $\langle \mathscr{P} \rangle : |p|^*(p) \to |\mathscr{P}_0|^*(p)$ by

$$(E, D) \mapsto (E, \mathscr{P}E^*(D)),$$

$$(E, D) \xrightarrow{(f,g)} (E', D') \mapsto (f, h), \text{ where } h: \mathscr{P}E^*(D) \to \mathscr{P}E'^*(D') \text{ is the unique map above } \mathscr{P}_0 f \text{ making an obvious square in } \mathbf{E} \text{ commute.}$$

In [8, Proposition 3] one can find the proof that $\langle \mathscr{P} \rangle$ is a cartesian functor.

Finally, one says that the comprehension category $\mathscr{P}$ has *products* (*sums*) if the above functor $\langle \mathscr{P} \rangle$ has a fibred right (left) adjoint.

An equivalent fibrewise description of products and sums can be obtained from Proposition 3.4. One then requires that both

- for every $E \in \mathbf{E}$, every weakening functor $\mathscr{P}E^*: \mathbf{E}_{pE} \to \mathbf{E}_{\mathscr{P}_0 E}$ has a right adjoint Π_E (left adjoint Σ_E).
- the "Beck–Chevalley" condition holds, i.e. for every cartesian morphism $f: E \to E'$ in $\mathbf{E}$ one has that the canonical natural transformation

$$(pf)^* \Pi_{E'} \dot{\to} \Pi_E (\mathscr{P}_0 f)^* \qquad (\Sigma_E (\mathscr{P}_0 f)^* \dot{\to} (pf)^* \Sigma_{E'})$$

  is an isomorphism.

This fibrewise formulation will be used in the rest of this paper. Note that if $\mathscr{P}$ is a *full* comprehension category one can equivalently formulate the second "Beck–Chevalley" condition as: for every pullback in $\mathbf{B}$ of the form shown in Fig. 19, one has

$$r^* \Pi_{E'} \cong \Pi_E s^*$$

canonically. See Lemma 2.5(ii); similarly for sums.

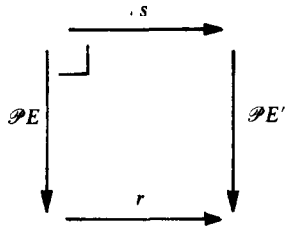


Fig. 19.

In the next few paragraphs, these products and sums are investigated more closely from a type-theoretic perspective. Since one only needs *full* comprehension categories for the description of type dependency, we often restrict ourselves to those structures.

Products 5.1. Let $\mathscr{P}: \mathbf{E} \to \mathbf{B}^{\to}$ be a comprehension category with products. Objects $E \in \mathbf{E}$ are called types and elements of $|E| = \{u : pE \to \mathscr{P}_0 E \mid \mathscr{P}E \circ u = id\}$ are called terms of type E. For types $E, D \in \mathbf{E}$ with $pD = \mathscr{P}_0 E$ one can think of D as containing a variable of type E. The product type $\Pi_E.D$ above pE can then be formed. There is a "canonical" map

$$|\Pi_E.D| \to |D|$$

$$u \mapsto u \cdot var^E,$$

which can be understood as application of the term $u : \Pi_E.D$ to a variable of type E. One obtains $u \cdot var^E$ as $\mathscr{P}_0(\varepsilon_D) \circ u'$, where $\varepsilon_D : \mathscr{P}E^* \Pi_E.D \to D$ is counit and u' is obtained as the unique mediating arrow in Fig. 20, using Lemma 4.4.

From a type-theoretic perspective one expects this map $|\Pi_E.D| \to |D|$ to be an isomorphism, with the inverse given by λ-abstraction, see [30]. However, this is not automatic. We first show that one obtains an isomorphism here iff the functor $\mathscr{P}$ preserves products, i.e. if it produces products on its own projections in the slices of the base category.

Lemma 5.2. *Let $\mathscr{P}: \mathbf{E} \to \mathbf{B}^{\to}$ be a comprehension category with products. The following two statements are equivalent:*

(i) $|\Pi_E.D| \cong |D|$ *for all appropriate* E, D;

(ii) $\mathscr{P}$ *preserves products, i.e. for all appropriate* E, D *and* $u : A \to pE$ *one has*

$$\mathbf{B}/pE(u, \mathscr{P}(\Pi_E.D)) \cong \mathbf{B}/\mathscr{P}_0 E(\mathscr{P}E^{\#}(u), \mathscr{P}D),$$

where the pullback functor $\mathscr{P}E^{\#}$ is described in Lemma 4.4.

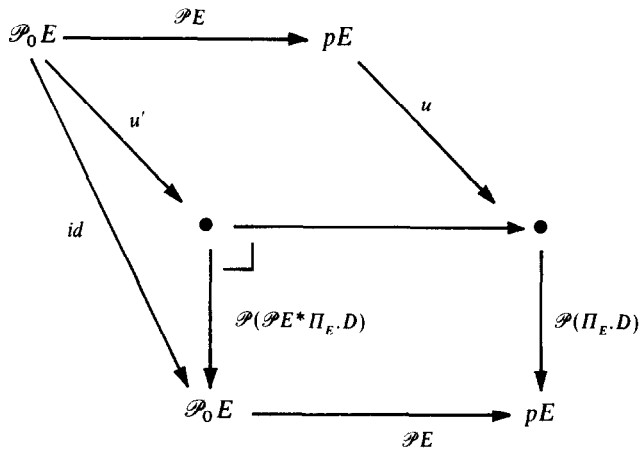


Fig. 20.

Proof. (i) $\Rightarrow$ (ii):

$$\begin{aligned}
\mathbf{B}/pE(u, \mathscr{P}(\Pi_E.D)) &\cong \mathbf{B}/A(id, u^{\#}\mathscr{P}(\Pi_E.D)) \\
&\cong \mathbf{B}/A(id, \mathscr{P}u^*(\Pi_E.D)) \qquad \text{(see Lemma 4.4)} \\
&= |u^*(\Pi_E.D)| \\
&\cong |\Pi_{u^*(E)}.\mathscr{P}E^{\#}(u)^*(D)| \qquad \text{(by Beck–Chevalley)} \\
&\cong |\mathscr{P}E^{\#}(u)^*(D)| \qquad \text{(by assumption)} \\
&= \mathbf{B}/\mathscr{P}_0u^*(E)(id, \mathscr{P}(\mathscr{P}E^{\#}(u)^*(D))) \\
&\cong \mathbf{B}/\mathscr{P}_0E(\mathscr{P}E^{\#}(u), \mathscr{P}D).
\end{aligned}$$

(ii) $\Rightarrow$ (i):

$$\begin{aligned}
|\Pi_E.D| &= \mathbf{B}/pE(id, \mathscr{P}(\Pi_E.D)) \\
&\cong \mathbf{B}/\mathscr{P}_0E(\mathscr{P}E^{\#}(id), \mathscr{P}D) \qquad \text{(by assumption)} \\
&\cong \mathbf{B}/\mathscr{P}_0E(id, \mathscr{P}D)) \\
&= |D|. \qquad \square
\end{aligned}$$

Lemma 5.3. *A comprehension category with unit preserves products.*

Proof. By Lemma 5.2, since

$$\begin{aligned}
|\Pi_E.D| &\cong \mathbf{E}_A(1A, \Pi_E.D) \qquad \text{[by Lemma 4.13(i), where } A = pE\text{]} \\
&\cong \mathbf{E}_{\mathscr{P}_0E}(\mathscr{P}E^*(1A), D) \\
&\cong \mathbf{E}_{\mathscr{P}_0E}(1\mathscr{P}_0E, D) \\
&\cong |D|. \qquad \square
\end{aligned}$$

Whether products are also preserved without units is a subtle matter. It might be instructive to take a look at Example 4.3 (term model). For ease of exposition we omit the braces [–] denoting equivalence classes. The fibre above a context Γ contains types $\Gamma \vdash \sigma : \mathit{Type}$ as objects; morphisms $\Gamma \vdash \sigma : \mathit{Type} \to \Gamma \vdash \tau : \mathit{Type}$ are terms $M(x)$ with $\Gamma, x{:}\sigma \vdash M(x) : \tau$. Hence, even when Γ is the empty context, such a term in a fibre is supposed to depend on *at least one* variable, viz. $x{:}\sigma$. This makes it a bit difficult to describe closed terms in the fibres. In case one can take σ to be a unit (i.e. a singleton type), this does not cause any problems.

Below, we describe products as preserved by assuming nonemptiness of the relevant comprehension category, see Definition 4.20. Then we may assume that at least one inhabited type $\Gamma \vdash N_0 : \rho$. For a term $\Gamma, x{:}\sigma \vdash M : \tau$ one can form $\Gamma \vdash \lambda x{:}\sigma.M : \Pi x{:}\sigma.\tau$ as follows. First, a dummy dependence $\Gamma, x{:}\sigma, z{:}\rho \vdash M : \tau$ is introduced; this yields M as morphism in the fibre above the context $\Gamma, x{:}\sigma$. Hence, one can form, $\Gamma, z{:}\rho \vdash \lambda x{:}\sigma.M : \Pi x{:}\sigma.\tau$ by transposition across the adjunction. By substituting N_0 for the dummy variable $z{:}\rho$ one obtains $\Gamma \vdash \lambda x{:}\sigma.M : \Pi x{:}\sigma.\tau$ as required.

Lemma 5.4. *A nonempty full comprehension category preserves products.*

Proof. Again we rely on Lemma 5.2. Let $\mathscr{P} : \mathbf{E} \to \mathbf{B}^{\to}$ be a nonempty full comprehension category. Suppose types $E, D \in \mathbf{E}$ with $pD = \mathscr{P}_0 E$ and a term $w \in |D|$ are given; let us write $A = pE$. Since $\mathscr{P}$ is nonempty there is a type F above A and a term $v \in |F|$ of type F. By "weakening" we obtain a type $F' = \mathscr{P}E^*(F)$ and a term $v' \in |F'|$; the latter is obtained in the same way as u' in 5.1. One has $w \circ \mathscr{P}F' : \mathscr{P}F' \to \mathscr{P}D$ in $\mathbf{B}/\mathscr{P}_0 E$. Using fullness, there is a unique $f : \mathscr{P}E^*(F) \to D$ above $\mathscr{P}_0 E$ such that $\mathscr{P}_0 f = w \circ \mathscr{P}F'$. The transpose $\hat{f} : F \to \Pi_E.D$ above A enables us to define

$$\lambda_E.w = \mathscr{P}_0(\hat{f}) \circ v \in |\Pi_E.D|.$$

Note that the term v is used to remove the "dummy" dependency on F.

We have to show that $\lambda_E.-$ is the inverse of the canonical map $|\Pi_E.D| \to |D|$ described in 5.1.

$$\begin{aligned}
(\lambda_E.w) \cdot var^E &= \mathscr{P}_0(\varepsilon_D) \circ (\lambda_E.w)' && \text{(see 5.1)} \\
&= \mathscr{P}_0(\varepsilon_D) \circ \mathscr{P}_0(\mathscr{P}E^*(\hat{f})) \circ v' && \text{(by a diagram chase)} \\
&= \mathscr{P}_0(f) \circ v' \\
&= w \circ \mathscr{P}F' \circ v' \\
&= w.
\end{aligned}$$

The other way round, for $u \in |\Pi_E.D|$, one has $u \circ \mathscr{P}F : \mathscr{P}F \to \mathscr{P}(\Pi_E.D)$ in $\mathbf{B}/A$ and,

hence, a unique $g: F \to \Pi_E.D$ above A with $\mathscr{P}_0 g = u \circ \mathscr{P}F$. The transpose $\check{g}: \mathscr{P}E^*(F) \to D$ satisfies

$$\begin{aligned}\mathscr{P}_0(\check{g}) &= \mathscr{P}_0(\varepsilon_D) \circ \mathscr{P}_0(\mathscr{P}E^*(g)) \\ &= \mathscr{P}_0(\varepsilon_D) \circ u' \circ \mathscr{P}F' \qquad \text{(by a diagram chase)} \\ &= u \cdot var^E \circ \mathscr{P}F'.\end{aligned}$$

Hence, $\lambda_E.(u \cdot var^E) = \mathscr{P}_0(\hat{\check{g}}) \circ v = \mathscr{P}_0(g) \circ v = u \circ \mathscr{P}F \circ v = u$. □

Sums 5.5. Type theoretically one distinguishes the so-called weak and strong sums. Categorically the weak ones are obtained by left adjoints to weakening functors, as described at the beginning of this section. For strong sums one has a simple additional requirement, see Definition 5.8. We start with a syntactical description in order to clarify the difference between weak and strong. The formation and introduction rules are the same in both cases.

$$\frac{\Gamma \vdash \sigma : Type \quad \Gamma, x{:}\sigma \vdash \tau : Type}{\Gamma \vdash \Sigma x{:}\sigma.\tau : Type}, \qquad \frac{\Gamma \vdash M : \sigma \quad \Gamma \vdash N : \tau[x := M]}{\Gamma \vdash \langle M, N \rangle : \Sigma x{:}\sigma.\tau}.$$

The *weak* elimination rule is given by

$$\frac{\Gamma \vdash P : \Sigma x{:}\sigma.\tau \quad \Gamma \vdash \rho : Type \quad \Gamma, x{:}\sigma, y{:}\tau \vdash Q : \rho}{\Gamma \vdash Q \textbf{ where } \langle x, y \rangle := P : \rho}.$$

In the *strong* elimination rule, the type ρ may contain an extra variable $w{:}\Sigma x{:}\sigma.\tau$:

$$\frac{\Gamma \vdash P : \Sigma x{:}\sigma.\tau \quad \Gamma, w{:}\Sigma x{:}\sigma.\tau \vdash \rho : Type \quad \Gamma, x{:}\sigma, y{:}\tau \vdash Q : \rho[w := \langle x, y \rangle]}{\Gamma \vdash Q \textbf{ where } \langle x, y \rangle := P : \rho[w := P]},$$

both with conversions

$$Q \textbf{ where } \langle x, y \rangle := \langle M, N \rangle \;=\; Q[x := M][y := N]$$

$$Q[w := \langle x, y \rangle] \textbf{ where } \langle x, y \rangle := P \;=\; Q[w := P].$$

Alternatively, one can formulate the strong elimination rule with first and second projections. For a term $P: \Sigma x{:}\sigma.\tau$ one requires terms $\pi P: \sigma$ and $\pi' P: \tau[x := \pi P]$ with conversions

$$\pi\langle M, N \rangle = M, \qquad \pi'\langle M, N \rangle = N, \qquad \langle \pi P, \pi' P \rangle = P.$$

This formulation allows "packing and unpacking". One can define these projections with the strong elimination rule as follows.

$$\pi P = x \textbf{ where } \langle x, y \rangle := P, \qquad \pi' P = y \textbf{ where } \langle x, y \rangle := P.$$

In order to obtain the second projection one takes $\rho(w)=\tau[x:=\pi w]$ in the above strong elimination rule. Then obviously $\pi\langle M, N\rangle = M$ and $\pi'\langle M, N\rangle = N$, but also

$$\begin{aligned}\langle \pi P, \pi' P\rangle &= \langle \pi\langle x, y\rangle, \pi'\langle x, y\rangle\rangle \textbf{ where } \langle x, y\rangle := P\\ &= \langle x, y\rangle \textbf{ where } \langle x, y\rangle := P\\ &= P.\end{aligned}$$

The other way round, one easily obtains the above strong elimination rules if first and second projection are available. One simply takes

$$Q \textbf{ where } \langle x, y\rangle := P \;=\; Q[x:=\pi P][y:=\pi' P].$$

For types $\Gamma\vdash\sigma: \mathit{Type}$ and $\Gamma\vdash\tau: \mathit{Type}$ one can define a *cartesian product* type $\sigma\times\tau$ as $\Sigma x{:}\sigma.\tau$, where $x\notin FV(\tau)$ is a fresh variable. With the strong elimination rule one has projections (using the above ones) and surjectivity of pairing. These projections for cartesian product types are also definable with the weak elimination rule. One takes the definitions of πP and $\pi' P$ used above; the second one may now be formed because $x\notin FV(\tau)$.

Next we describe weak sums in a full comprehension category $\mathscr{P}: \mathbf{E}\to\mathbf{B}^{\to}$ with sums (as described at the beginning of this section).

A useful observation about full comprehension categories is that for objects $E, D\in\mathbf{E}$ in the same fibre, say above $A\in\mathbf{B}$, one has

$$\mathbf{E}_A(E, D)\cong\mathbf{B}/A(\mathscr{P}E, \mathscr{P}D)\cong|\mathscr{P}E^*(D)|. \tag{2}$$

The latter isomorphism is described just after Lemma 4.4.

As a consequence, for types $E, D, F\in\mathbf{E}$ with $pD=\mathscr{P}_0 E$ and $pE=pF$ one obtains

$$\begin{aligned}|\mathscr{P}(\Sigma_E.D)^*(F)| &\cong \mathbf{E}_{pE}(\Sigma_E.D, F) && \text{[by (2)]}\\ &\cong \mathbf{E}_{\mathscr{P}_0 E}(D, \mathscr{P}E^*(F)) && \text{(by the adjunction)}\\ &\cong |\mathscr{P}D^*\mathscr{P}E^*(F)| && \text{[by (2) again].}\end{aligned}$$

We recall that functors of the form $\mathscr{P}(-)^*$ are weakening functors. Thus, these isomorphisms yield the conversion rules for weak sums: using variables $x{:}E, y{:}D, z{:}\Sigma_E.D$ in a suggestive way one obtains

- for $u\in|\mathscr{P}(\Sigma_E.D)^*(F)|$ a new term $u[z:=\langle x, y\rangle]\in|\mathscr{P}D^*\mathscr{P}E^*(F)|$,
- and for $v\in|\mathscr{P}D^*\mathscr{P}E^*(F)|$ one obtains v **where** $\langle x, y\rangle := z\in|\mathscr{P}(\Sigma_E.D)^*(F)|$ such that

$$v \textbf{ where } \langle x, y\rangle := \langle x, y\rangle = v$$

$$u[z:=\langle x, y\rangle] \textbf{ where } \langle x, y\rangle := z = u.$$

The conversion rules for sums mentioned in the beginning of 5.5 are substitution instances of these.

Lemma 5.6. *Let $\mathscr{P}: \mathbf{E} \to \mathbf{B}^{\to}$ be a full comprehension category with unit, products and sums. The fibration $p = cod \circ \mathscr{P}$ is then a fibred CCC.*

Proof. For objects $E, D \in \mathbf{E}$ in the same fibre one takes $E \times D = \Sigma_E.\mathscr{P}E^*(D)$ and $E \Rightarrow D = \Pi_E.\mathscr{P}E^*(D)$. It is elementary but laborious to verify that everything works as it should. □

In Result (ii) of Example 4.19 we saw that for family models over **Cat**, "Beck–Chevalley" is problematic if one requires left adjoints to all substitution functors. Lemma 5.7 states that there are no problems if one requires adjoints to weakening functors only.

Lemma 5.7. *The comprehension category $Fam(\mathbf{C}) \to \mathbf{Cat}^{\to}$ has sums in case $\mathbf{C}$ has infinite coproducts. The same holds for products.*

Proof. For $X: \mathbf{A} \to \mathbf{C}$ and $Y: (X \downarrow t) \to \mathbf{C}$ one defines $\Sigma_X.Y: \mathbf{A} \to \mathbf{C}$ by $A \mapsto \coprod_{x \in \mathbf{C}(t, XA)} Y(A, x)$. □

Definition 5.8. A comprehension category $\mathscr{P}: \mathbf{E} \to \mathbf{B}^{\to}$ has *strong* sums if it has sums in such a way that for objects $E, D \in \mathbf{E}$ with $pD = \mathscr{P}_0 E$ one has that the "canonical" map $\mathscr{P}_0 D \to \mathscr{P}_0(\Sigma_E.D)$ is an isomorphism.

Translating this definition into a more type-theoretic formulation, one has strong sums if the contexts $\Gamma, x{:}\sigma, y{:}\tau$ and $\Gamma, z{:}\Sigma x{:}\sigma.\tau$ are isomorphic. The canonical map takes $x{:}\sigma$ and $y{:}\tau$ to the pair $\langle x, y \rangle{:}\Sigma x{:}\sigma.\tau$. In categorical formulation, it is $\mathscr{P}_0(in_{E,D}): \mathscr{P}_0 D \to \mathscr{P}_0(\Sigma_E.D)$, where $in_{E,D}: D \to \Sigma_E.D$ is the composition $\overline{\mathscr{P}E}(\Sigma_E.D) \circ \eta_D$.

It is not hard to prove that for strong sums one has an isomorphism between the set $|\Sigma_E.D|$ of terms of type $\Sigma_E.D$ and the set $\{(u, v) \mid u \in |E| \text{ and } v \in |u^*(D)|\}$.

Next we turn to a formulation of the type-theoretic notion of strong sums in ordinary category theory. "Packing and unpacking" is the main aspect. The following notion is also relevant in topos theory, see e.g. [1, 7.1].

Definition 5.9. Let $\mathbf{C}$ be a category with terminal object t and let $X: I \to \mathbf{C}$ describe a set-indexed collection of objects of $\mathbf{C}$ with colimiting cone $\{in_i: X_i \to \coprod_I X\}$. This sum is called strong if the functor between comma categories

$$(t \downarrow X) \to (t \downarrow \coprod_I X),$$

$$(i \in I, f: t \to X_i) \mapsto in_i \circ f,$$

is an isomorphism.

Equivalently, one can require that

$$\dot{\bigcup_{i \in I}}.\mathbf{C}(t, X_i) \cong \mathbf{C}(t, \coprod_I X)$$

canonically, which implies that the global sections functor $\mathbf{C}(t, -)$ preserves $\coprod_I X$. Note that in case $\{in_i: X_i \to \coprod_I X\}$ is a *disjoint sum* (i.e. the pullback of in_i, in_j yields an initial object if $i \neq j$ and all in_i's are monics) one has that the above map $\dot{\bigcup}_{i \in I} \mathbf{C}(t, X_i) \to \mathbf{C}(t, \coprod_I X)$ is injective.

In many categories sums are not strong ("weak"), but in **Sets** for example, they are.

Lemma 5.10. *Let* $\mathbf{C}$ *be a category with strong sums and small collections* $\mathbf{C}(t, A)$. *The functor* $\mathbf{C}(t, -): \mathbf{C} \to \mathbf{Sets}$ *then has a full and faithful left adjoint.*

Proof. For a set I, put $\hat{I} = \coprod_I t \in \mathbf{C}$; this yields a functor $\hat{-}: \mathbf{Sets} \to \mathbf{C}$. Then

$$\mathbf{C}(\hat{I}, A) = \mathbf{C}(\coprod_I t, A) \cong \Pi_I \mathbf{C}(t, A) = \mathbf{C}(t, A)^I = \mathbf{Sets}(I, \mathbf{C}(t, A));$$

so, $\hat{-}$ is a left adjoint of $\mathbf{C}(t, -)$. The unit of the adjunction is an isomorphism, since

$$\mathbf{C}(t, \hat{I}) \cong \dot{\bigcup_{i \in I}} \mathbf{C}(t, t) \cong I. \qquad \square$$

Lemma 5.11. *Let* $\mathbf{C}$ *be a category with small collections* $\mathbf{C}(t, A)$. *Then*

$\mathbf{C}$ *has strong sums* $\Leftrightarrow$ *the comprehension category* $Fam(\mathbf{C}) \to \mathbf{Sets}^{\to}$ *has strong sums.*

Similarly for $Fam(\mathbf{C}) \to \mathbf{Cat}^{\to}$.

Proof. We perform the calculation for the implication ($\Rightarrow$). For objects $\{X_i\}_I \in Fam(\mathbf{C})$ and $Y: \mathscr{P}_0\{X_i\} \to Obj\,\mathbf{C}$ one has $\Sigma_{\{X_i\}}.\{Y_{i,x}\}: I \to Obj\,\mathbf{C}$ defined by $i \mapsto \coprod_{x \in \mathbf{C}(t, X_i)}. Y_{i,x}$, see 4.14(ii). Then

$$\mathscr{P}_0(\Sigma_{\{X_i\}}.\{Y_{i,x}\}) = \left\{ \langle i, z \rangle \;\middle|\; i \in I \text{ and } z: t \to \coprod_{x \in \mathbf{C}(t, X_i)}. Y_{i,x} \right\}$$

$$\cong \{\langle i, \langle x, y \rangle \rangle \mid i \in I,\ x: t \to X_i \text{ and } y: t \to Y_{i,x}\} \qquad \text{(by strongness)}$$

$$\cong \{\langle \langle i, x \rangle, y \rangle \mid i \in I,\ x: t \to X_i \text{ and } y: t \to Y_{i,x}\}$$

$$= \mathscr{P}_0\{Y_{i,x}\}. \qquad \square$$

Having seen the above description of strong sums, Carboni pointed out that in certain categories strongness of sums is equivalent to indecomposability of the terminal object. This is shown next.

Proposition 5.12. *Let* $\mathbf{C}$ *be a distributive category, i.e. a category with finite limits, a strict initial object and universal disjoint sums. Then*

$\mathbf{C}$ *has strong sums* $\Leftrightarrow$ *the terminal object* t *is indecomposable.*

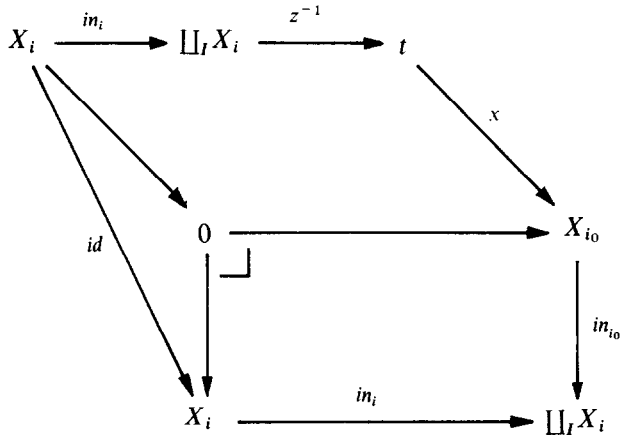


Fig. 21.

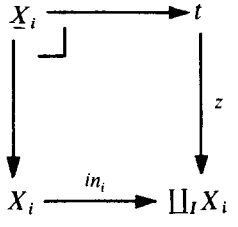


Fig. 22.

Proof. $\Rightarrow$: Suppose $t \cong \coprod_I X_i$, say by $z: t \to \coprod_I X_i$. Since the sum $\coprod_I X_i$ is strong, there is a unique $i_0 \in I$ and an $x: t \to X_{i_0}$ with $in_{i_0} \circ x = z$. Then $t \cong X_{i_0}$ and for $i \neq i_0$ one has $X_i \cong 0$ since there is an arrow $X_i \to 0$ as in Fig. 21. Hence, t is indecomposable.

$\Leftarrow$: Suppose an arrow $z: t \to \coprod_I X_i$ is given. For each $i \in I$, one forms the pullback shown in Fig. 22. Since sums are universal, $\{\underline{X}_i \to t\}$ is again a colimiting cone. Hence, $t \cong \coprod_I \underline{X}_i$ and, so, there is a unique $i_0 \in I$ with $t \cong \underline{X}_{i_0}$. Thus, one obtains a unique arrow $t \to \underline{X}_i \to X_{i_0}$, say x, satisfying $in_{i_0} \circ x = z$. □

The next notion combines many aspects which we have investigated separately. The rest of this section will be devoted to examples and one final lemma.

Definition 5.13. A *closed* comprehension category (CCompC) is a full comprehension category with unit which has products and strong sums.

Examples 5.14 (*Examples of closed comprehension categories*). (i) Let $\mathbf{B}$ be a category with finite limits. The identity functor on $\mathbf{B}^{\to}$ then forms a full comprehension category with unit and strong sums. Moreover,

$$Id_{\mathbf{B}^{\to}} \text{ is a CCompC} \Leftrightarrow \mathbf{B} \text{ is an LCCC.}$$

(ii) Let **B** be a category with finite products. The functor $Cons_{\mathbf{B}}: \bar{\mathbf{B}} \to \mathbf{B}^{\to}$ from Example 4.11 forms a full comprehension category with unit and strong sums. Moreover,

$$Cons_{\mathbf{B}} \text{ is a CCompC} \Leftrightarrow \mathbf{B} \text{ is a CCC.}$$

These two examples show that finite products and exponents are related like finite limits and local exponentials.

(iii) The instantiation $Fam(\mathbf{Sets}) \to \mathbf{Cat}^{\to}$ from Example 4.19 is a CCompC: fullness is obtained from Result (i) of Example 4.19 and products and sums from Lemma 5.7. The sums are strong by Lemma 5.11.

(iv) The term model construction described in Example 4.3 applied to a calculus with unit-type, products and strong sums yields a CCompC.

(v) Finally, we briefly describe the well-known realizability models (see e.g. [15, 8, 25]) as closed comprehension categories. The category ω-**Set** has objects $A = (|A|, \vdash_A)$, where $|A|$ is a set and $\vdash_A \subseteq \mathbb{N} \times |A|$ is a relation satisfying $\forall a \in |A|.\ \exists n \in \mathbb{N}.\ n \vdash_A a$. Morphisms $f: A \to B$ in ω-**Set** are given by functions $f: |A| \to |B|$ for which there is a *realizer* $n \in \mathbb{N}$ such that $\forall a \in |A|.\ \forall m \in \mathbb{N}.\ m \vdash_A a \Rightarrow n \cdot m \vdash_B f(a)$, where $n \cdot m$ denotes the result of the nth partial recursive function applied to m. It is not hard to verify that ω-**Set** is an LCCC.

The full subcategory **M** of the so-called "modest" ω-sets has objects $A = (|A|, \vdash_A)$ satisfying $\forall a, a' \in |A|.\ \forall n \in \mathbb{N},\ n \vdash_A a$ and $n \vdash_A a' \Rightarrow a = a'$. As shown in [10], the inclusion functor $\mathbf{M} \subsetneq \omega$-**Set** has a left adjoint F, which constitutes a reflection. For $A = (|A|, \vdash_A) \in \omega$-**Set**, one first defines a relation $\smile$ on $|A|$ by $a \smile a' \Leftrightarrow \exists n \in \mathbb{N}.\ n \vdash_A a$ and $n \vdash_A a'$. Then one takes $\sim$ to be the transitive closure of $\smile$. Finally, one can put $FA = (|A|/\sim, \vdash_{\theta A})$, with $n \vdash_{\theta A} [a] \Leftrightarrow \exists a' \in [a].\ n \vdash_A a'$.

Let **C** be ω-**Set** or **M**. The category $Fam_{\mathrm{eff}}(\mathbf{C})$ has pairs (A, X) with $A \in \omega$-**Set** and $X: |A| \to Obj\,\mathbf{C}$ as objects. A morphism $(f, \alpha): (A, X) \to (B, Y)$ consists of a map $f: A \to B$ in ω-**Set** and an *effective* family $\alpha = \{\alpha_a\}_{a \in |A|}$ of functions $\alpha_a: |X_a| \to |Y_{f(a)}|$; effectivity here means that the family itself has a realizer, i.e. $\exists n \in \mathbb{N}.\ \forall a \in |A|.\ \forall m \in \mathbb{N}.\ m \vdash_A a \Rightarrow n \cdot m$ realizes α_a. The first projection $Fam_{\mathrm{eff}}(\mathbf{C}) \to \omega$-**Set** is then a split fibration. Both instantiations $\mathbf{C} = \omega$-**Set** and $\mathbf{C} = \mathbf{M}$ yield a closed comprehension category $Fam_{\mathrm{eff}}(\mathbf{C}) \to \omega\text{-}\mathbf{Set}^{\to}$.

The first one is easy since there is a fibred equivalence (Fig. 23). Therefore, we first define a functor $\mathcal{Q}_0: Fam_{\mathrm{eff}}(\omega\text{-}\mathbf{Set}) \to \omega\text{-}\mathbf{Set}$ by $(A, X) \mapsto (\bigcup_{a \in |A|} \cdot |X_a|, \vdash)$, with

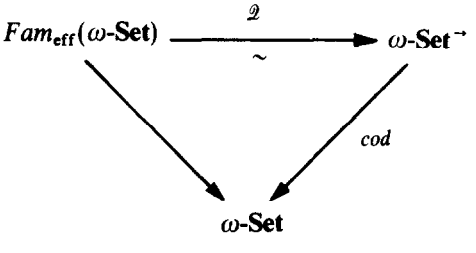


Fig. 23.

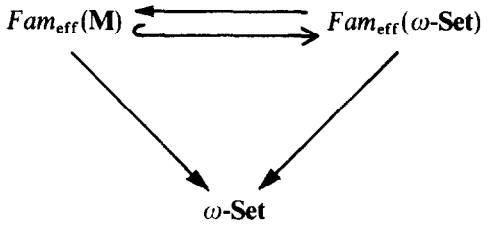


Fig. 24.

$n \vdash (a,x) \Leftrightarrow fst(n) \vdash_A a$ and $snd(n) \vdash_{X_a} x$. On morphisms $\mathcal{Q}_0$ is described by $(f,\alpha) \mapsto \lambda(a,x).(f(a),\alpha_a(x))$; the latter has a realizer because α is an effective family. Finally, $\mathcal{Q}(A,X)$ becomes the projection $\mathcal{Q}_0(A,X) \to A$ in ω-$\mathbf{Set}^{\to}$ and $\mathcal{Q}(f,\alpha)$ becomes $(f, \mathcal{Q}_0(f,\alpha))$.

The reflection $\mathbf{M} \leftrightarrows \omega$-**Set** lifts to a fibred reflection (Fig. 24). As a result one obtains a reflexive comprehension category $Fam_{\mathrm{eff}}(\mathbf{M}) \to \omega$-$\mathbf{Set}^{\to}$. Also this comprehension category is closed. This completes the examples.

The unit, products and sums of a closed comprehension category have been defined in terms of fibred adjunctions in the total category. Lemma 5.15 states that one obtains the corresponding structure on the projection maps in the slices of the base category as a result, cf. the discussion at the end of Example 4.10.

Lemma 5.15. *Let $\mathcal{P}: \mathbf{E} \to \mathbf{B}^{\to}$ be a CCompC. Then considered as a functor, $\mathcal{P}$ preserves units, sums and products.*

Proof. Units are preserved by Lemma 4.13(iii) and products by Lemma 5.3. Sums are preserved because they are strong: $\mathcal{P}(\Sigma_E.E') \cong \mathcal{P}E \circ \mathcal{P}E' = \Sigma_{\mathcal{P}E'}.\mathcal{P}E'$ in $\mathbf{B}/pE$. □

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