SMALL ADEQUATE SUBCATEGORIES

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A full subcategory $\mathscr{A}$ of a category $\mathscr{C}$ is *left adequate* if the left regular representation of $\mathscr{C}$, cut down to $\mathscr{A}$, is still a full embedding of $\mathscr{C}$ in the category of all contravariant functors from $\mathscr{A}$ to sets. A typical example is the subcategory on one free algebra on n generators in a category of algebras with at most n-ary operations ($n>0$) [2]. Roughly, a category with a small left adequate subcategory has a small theory. If a small subcategory $\mathscr{A}$ of $\mathscr{C}$ is both left and right adequate, then $\mathscr{C}$ can be fully embedded in every completion of $\mathscr{A}$, uniquely up to an equivalence.

The only known examples of complete categories with small (two-sided) adequate subcategories have been the sets, the left vector spaces over a division ring, and some obvious combinations of these.† It is still unknown whether the abelian groups have a small right adequate subcategory. The main result of this paper is that† a primitive category of algebras all of whose operations are at most unary has a small adequate subcategory. It is known [1, 4] that every full category of algebras can be fully embedded in one of these.

How small? If there are no measurable cardinals, we use all c-element algebras. In a number of special cases ([2], [4], and below), countable algebras suffice. I do not know if finite algebras can be right adequate in a complete (non-trivial) category of algebras, but there is a large category of algebras in which one algebra with seven elements and seven endomorphisms is adequate. Another example: there is a left complete full category of algebras that is not right complete.† Since I constructed that example, using results of Hedrlín, Pultr, and Vopěnka, the three men named have developed methods for embedding other hand-made horrible examples fully in algebras. There seems to be no category now known to be embeddable in sets and functions but not fully embeddable in a category with a finite adequate subcategory.

Unary algebras

The theorem to be proved involves the following assumption.

(A) *For some small cardinal n, for every small cardinal m, every n-additive two-valued measure is m-additive.*

"Small"; we use for convenience the Grothendieck theory of universes, as described in [5] or many other places, and the small cardinals are the cardinals of members of the universe. The precise statement (A) is wanted for a proof of equivalence, but let us recall that (A) is simply a fact, with $n=\aleph_0$, unless the universe is *both* so large as to stagger the imagination (the set of small inaccessible cardinals, of small inaccessible cardinals k with k inaccessible predecessors, and more rarefied sets, are not small; Hanf–Tarski [7]) *and* oddly shaped ($V\neq L$; Scott [6]). The consistency of not—(A) is gravely in doubt.

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† These results depend on the set-theoretic assumption (A) below.

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THEOREM. *The condition* (A) *is necessary and sufficient for the category $\mathcal{U}$ of small sets and functions to have a small adequate subcategory, and implies that every primitive category of algebras with at most unary operations has a small adequate subcategory.*

The first clause of the theorem is substantially known; the specialization $n=\aleph_0$ is proved in [2], and the equivalence with (A), for left vector spaces instead of sets, is proved in [4]. The methods of proof, for those and other similar results ([2], [3]), have been somewhat awkward. Now we shall use limits, threads, and uniformities.

Given an object X of a category $\mathcal{C}$, the *spectral cone from X into* a full subcategory $\mathcal{A}$ is the diagram on copies A_f of each object A of $\mathcal{A}$, indexed by $\mathcal{C}(X, A)$, with all those maps (morphisms) $k: A_f \to B_g$ such that $kf=g$, and (even if X is in $\mathcal{A}$) one (extra) copy of X and all the maps $f: X \to A_f$.

The subdiagram omitting that copy of X is the *base* of the cone, or the *spectrum* of X in $\mathcal{A}$. The spectral cone is indeed a left conical extension [5] of its base, *i.e.* it has the form and commutativity required for a limit in $\mathcal{C}$ of the spectrum of X in $\mathcal{A}$. The remaining requirement of universality is equivalent to adequacy in the localized version [3]: $\mathcal{A}$ is right adequate for X in $\mathcal{C}$ provided for every W in $\mathcal{C}$ the subregular representation ([5]; $X \to \mathcal{A}(X,)$, $W \to \mathcal{A}(W,)$) takes $\mathcal{C}(W, X)$ bijectively to $\mathrm{Hom}(\mathcal{A}(X,), \mathcal{A}(W,))$. Moreover, one need only marshal the definitions and look to see the equivalence.

It may not be amiss to mention that right adequacy of $\mathcal{A}$ is a much stronger requirement than that everything is a limit of objects of $\mathcal{A}$. An instance to hand: every set is a limit of finite sets, but finite sets are right adequate for no infinite set.

Proof of the theorem. For left adequacy, a free algebra F on one generator suffices [2]. Later we shall notice its cardinal number, p.

In $\mathcal{U}$, the spectrum of a set X in the full subcategory of sets of at most n elements (n infinite) is substantially an inverse mapping system; this is precisely true of the subdiagram on the onto maps, which plainly has the same limit. It is plain, further, that the threads of the inverse limit correspond naturally to filters in X which are Cauchy in the uniformity induced by all partitions of X into at most n subsets; that is, the ultrafilters supporting n-additive two-valued measures.

With the trivial exception of the smaller subsets of an n-element set (retracts of it, taken care of by [2]), the partitions of X into exactly n elements are cofinal, so that if (A) holds a single object is right adequate in $\mathcal{U}$. If (A) fails, the argument shows that every small full subcategory is contained in a full subcategory that is not right adequate, so is not right adequate itself.

Assume (A) and consider a primitive category $\mathcal{C}$ of (at most) unary algebras. Let $\mathcal{A}$ be the full subcategory on algebras of at most n^p elements. (Strictly, we must take just representatives of the isomorphism types to have $\mathcal{A}$ small.) Again we may replace the spectrum of X in $\mathcal{A}$ by the subdiagram on the onto maps. This is an inverse mapping system and its threads correspond naturally to filters in X Cauchy in the uniformity induced by certain partitions of X into at most n^p subsets. In fact every such partition occurs in the uniformity, i.e. it can be refined to a congruence partition of the indicated size. For, given any function ϕ from X to a set S, define $\Phi: X \to S^F$ by $\Phi(x)(w)=\phi(w(x))$, where w is the general element of F written as a word in the generator. The relation $\Phi(x)=\Phi(y)$ is a congruence, for

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$\Phi(v(x))(w)=\phi\left(w(v(x))\right)$, which is $\Phi(v(y))(w)$ if $\Phi(x)$ and $\Phi(y)$ are the same function. Thus the threads of the spectrum are just those corresponding to elements of X, and the theorem is proved.

Of course there is no such result for general algebras. For example, because there are so many simple groups, the category of groups does not even have a small separating subcategory. The full embeddability of every full category of algebras in unary algebras is an easy result I proved in [4]. It is not easy, but proved [1], that one can make do with two unary operations; then $p\leqslant\aleph_0$, and if $n=\aleph_0$, c-element algebras are right adequate.

REMARK. *n^p in the proof of the theorem can be replaced by* $\max(n,p)$ *if there is no more than one unary operation.* This can be proved by a two-stage analysis, first considering the finest partition into u-subalgebras, where u is the unary operation (the u-subalgebras that include values of 0-ary operations need a little special care), then partitioning these as before but taking advantage of the fact that for some x_0 and every x there is a relation $u^i(x)=u^j(x_0)$.

A complete category $\mathscr{C}$ having a small adequate subcategory $\mathscr{A}$ is the unique completion of $\mathscr{A}$ (up to equivalence). This requires (as far as I know) the rather stringent definition of "complete" introduced in [5], which involves limits of small diagrams, intersections of certain families of subobjects that can be large, and the duals. There is a theorem [5; 3.19.a] that a completion in a weaker sense (just the small diagrams) of a small category $\mathscr{A}$ which fails to be complete in this sense not only has the large families of subobjects but also wanders sideways from $\mathscr{A}$. On the other hand, a genuine completion of a small category is well-powered and generated by $\mathscr{A}$ [5; 3.20].

The following proof amounts to showing that the work has been done in [4] and [5].

PROPOSITION. *A category $\mathscr{C}$ having a small adequate subcategory $\mathscr{A}$ can be fully embedded in any completion of $\mathscr{A}$, uniquely up to equivalence.*

Proof. First, any completion $\mathscr{E}$ of $\mathscr{A}$ is a normal completion [5; 3.20], thus uniquely (up to equivalence) representable by grounding couples on $\mathscr{A}$ [5; 3.7.b]. This means that objects X of $\mathscr{E}$, with the maps between them, are determined up to isomorphism by the pair X^π of functors $\mathscr{A}(\ ,X)$, $\mathscr{A}(X,\ )$, coupled by multiplication in $\mathscr{E}$. Now the couples Z^π for Z in $\mathscr{C}$ are mutually conjugate functors coupled by evaluation [2]. Theorem 3.2 of [4] says (under weaker hypotheses) that there exist objects Z' of $\mathscr{E}$ such that $\mathscr{A}(\ ,Z')$ and $\mathscr{A}(Z',\ )$ are naturally equivalent to the corresponding terms of Z^π; and the natural equivalences there constructed are easily seen to preserve the coupling. This takes care of all the maps of $\mathscr{C}$ that have domain or range in $\mathscr{A}$, which suffices for all maps since the objects Z' are obtainable from $\mathscr{A}$ by iterated formation of limits (Remark (2) after 3.2 of [4]).

Examples

From the theorem above and the Hedrlín–Pultr embedding theorem [1], any full category of algebras can be fully embedded in the unique completion of a category with 2^c objects and morphisms (if there are no measurable cardinals). In several substantial special cases, countable algebras suffice. I do not know how far finite algebras may suffice, but at any rate infinity is no absolute barrier.

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EXAMPLE 1. *A large category with a finite adequate subcategory.*

It is the category of sets with base point 0 and *exclusive* mappings, *i.e.* 0-preserving mappings that take no two points to the same non-zero image. Alternatively, the maps are one–one correspondences between subsets of the non-zero points; obviously they can be read backward, and the category is dually isomorphic with itself. Thus to verify that the set T with two non-zero points is left and right adequate, it suffices to check one side. For left adequacy, a natural transformation $\Phi\colon \mathscr{T}(\ ,X)\to\mathscr{T}(\ ,Y)$ ($\mathscr{T}$ full on T) evidently is induced by at most one exclusive map $X\to Y$. The members of the domain of Φ_T may be described as ordered pairs (p,q) in $X\times X$, with $p\neq q$ except for $(0,0)$. Naturality means $\Phi_T(q,p)$, $\Phi_T(p,0)$, $\Phi_T(0,q)$ are (b,a), $(a,0)$, $(0,b)$ when $\Phi_T(p,q)$ is (a,b). In particular, $\Phi_T(p,0)$ has the form $(a,0)$, $a=f(p)$ determines the first co-ordinate of $\Phi_T(p,q)$ for all q, and the second co-ordinate is $f(q)$. That is, f induces Φ. f is exclusive since $\big(f(p),f(q)\big)$ is always a map from T to Y.

Example 1 has equalizers (and duality) but lacks products. I do not know if one can do more with the autoduality trick.

The simplest of a number of artificial counterexamples in category theory is the category $\mathscr{W}$ of (small) ordinal numbers and inclusion functions; it is right complete and well-powered, but has neither dual property. There has been some doubt whether one will ever find such examples except just after constructing them. As mentioned in the introduction, Hedrlín, Pultr, and Vopěnka have represented several of them by algebras by now; so I shall abbreviate the presentation of

EXAMPLE 2. *Assuming* (A), $\mathscr{W}$ *can be fully embedded in a primitive category of algebras.*

Categories for which this is true have been called "full categories of algebras". This is not good, the phrase should refer to a distinguished representation. In view of the theorem of this paper, it seems reasonable to call them *boundable* categories. It is still unknown whether every category embeddable in $\mathscr{U}$ is boundable.

For $\mathscr{W}$, I shall describe operations on the set Ω of all (small) ordinals so that the closed initial segments $I_\alpha=[0,\alpha]$ are subalgebras and a cofinal set of them are connected by no homomorphisms except the embeddings as they sit in Ω. We need the result that the dual of a boundable category is boundable [**4**], the sufficiency of unary operations, and the theorem of Hedrlín–Pultr–Vopěnka that there is an algebra of any power, of fixed type, with no endomorphisms except the identity [**1**, **8**].

Two 0-ary operations with values 0 and 1, a binary operation min, and a binary operation δ (of Kronecker) will secure that no homomorphism whose domain is an initial segment can lower any $\alpha\in\Omega$. In particular, we already have algebras A_α such that there is no homomorphism $A_\alpha\to A_\beta$ if $\beta<\alpha$. Dualizing, we have unary algebras B_α with no map $B_\alpha\to B_\beta$ if $\alpha<\beta$. Since there are only a small number of B's of any fixed power, we can find enough B_α of different powers m_α with m_α monotonic in α. Corresponding to their unary operations u_λ, define binary b_λ so that on $J_\alpha=[0,m_\alpha]$, $b_\lambda(m_\alpha,x)=u_\lambda(x)$. Other values of b_λ can be 0. Now the m_α are flagged, a homomorphism from J_α into Ω must take m_α to some m_β, J_α to J_β (because of min); hence $\alpha\geqslant\beta$, $\alpha=\beta$. Then, treating the Hedrlín–Pultr–Vopěnka rigidifying operations like the operations of the B_α, we have the example.

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