

General Functorial Semantics, I
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GENERAL FUNCTORIAL SEMANTICS, I.

By JOHN R. ISBELL.*

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Introduction. The taproot of the general (or free-form) functorial semantics of this paper is the observation that an adjoint connection between two categories is normally induced by an "object living in both categories" Indeed, half of the Introduction will go to clarifying the statement. Consider an example to begin with, Pontrjagin duality. I mean the duality between compact and discrete abelian groups; the semantics of the generally locally compact version is too complex to begin with. In the compact-discrete setting, one need only consider the circle S to appear as a compact group S_1 and also as a discrete group S_2. The character group of a compact group X_1 or of a discrete group X_2 has the underlying set $\mathrm{Hom}(X_i, S_i)$; its structure (as discrete group or compact group respectively) is found by regarding that set as a subset of a power of the other aspect S_{3-i} of S. As a subset, it determines a subobject, the character of X_i.

Note that this description of the duality is not particularly relevant to showing that it is a duality. That fact is a theorem of topological groups. That character groups yield an adjoint connection is a theorem of category theory.

Of course there are a number of further familiar examples, such as Stone duality, represented by the compact Boolean algebra $\{0, 1\}$. By an (adjoint) connection I mean just a pair of contravariant functors $F: \mathcal{C}^{op} \to \mathcal{D}$,

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$H: \mathcal{D}^{op} \to \mathcal{C}$, adjoint on the right. To explain an object living in two categories takes longer. A very large central class of illustrations is given by the notion of an algebra in a category. From Lawvere's thesis [17] we have the meaning of a model A of an algebraic theory $\boldsymbol{T}$ in a (finitely) left complete category $\mathcal{C}$. Moreover, my "taproot" is in this class of cases a reformulation of an old theorem of Freyd [6], giving a bijection from the isomorphism classes of connections between the variety $\mathcal{S}_{\boldsymbol{T}}$ of $\boldsymbol{T}$-algebras and $\mathcal{C}$ to the isomorphism classes of $\boldsymbol{T}$-algebras in $\mathcal{C}$. The theorem is readily extended to an equivalence between the category of connections $\mathcal{S}_{\boldsymbol{T}}(x)\mathcal{C}$ and the category $\mathcal{C}_{\boldsymbol{T}}$ of $\boldsymbol{T}$-algebras in $\mathcal{C}$. One cannot yet reduce it to characters; A has underlying object A_2 of $\mathcal{C}$ but not yet an underlying algebra A_1 in $\mathcal{S}_{\boldsymbol{T}}$.

This seems an appropriate place for more history. The free-form theories used here are essentially as long known as Freyd's theorem, coming from a manuscript of Gabriel's (fuller in some ways than the version being published by Gabriel and Ulmer [7]). Pultr called them "relational theories" in giving a direct generalization [24] of Freyd's theorem to them. (Pultr's announcement at a conference in Waterloo and mine at a conference in Kanpur [15] were independent and nearly simultaneous.) It is important also that Lawvere used to stress adjoint connections as a generalization of Galois connections of order theory; that is the only antecedent of this paper (that I know of) in which two categories enter symmetrically, and it impressed me.

Now the gist of the form to be given to the "taproot" is as follows. A category over $\mathcal{S}$, $U: \mathcal{C} \to \mathcal{S}$, if complete and rather nice (e.g. any normal completion of a small category), and grounded by a representable functor U, is the category of models of a theory $\operatorname{Str} U$ after Gabriel. (Completeness is essential. Inverse limits of models of these theories are models.) The category $\mathcal{C}(x)\mathcal{D}$ of connections between two of these is equivalent to the category of models of $\operatorname{Str} U$ in $\mathcal{D}$, and also to the category of models of the theory of $\mathcal{D}$ in $\mathcal{C}$. In general, a connection $A = (F, H)$ is *over* a character theory; it has underlying objects A_1 in $\mathcal{C}$, A_2 in $\mathcal{D}$, and the effect of F and H is partially described by the Pontrjagin construction for A_1 and A_2, but not determined by A_1 and A_2. For $\mathcal{C} = \mathcal{D} =$ topological spaces, A_1 and A_2 do not determine A. However, if $\mathcal{C}$ and $\mathcal{D}$ are rather nice and both groundings are faithful and one of them reflects isomorphisms, all connections are character theories of the "pointwise" sort described in the first paragraph.

The preceding paragraph summarizes Theorems 1.2, 2.6, 2.8, 3.4, 3.8. For the topological spaces, see 3.10. General Pontrjagin "duality" is another example of a non-pointwise connection, but not on topological abelian groups (not a connection); one must use k-groups.

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It is important for this paper that universes are used, and theories may be bigger than the universe. It is important because big theories do not arise only from a love of generality. *The complete theory of groups is big.* Its objects are all of the (left) continuous covariant functors from groups to sets. Actually that statement is imprecise; still, for any class of different simple groups, their coproduct functor is an object of the theory. The hom sets are not small. Roughly, working with theories of variable form requires this. That assertion is made partially precise in 1.5. Note that Gabriel-Ulmer [7] and Pultr [24] do not attempt to take the theory of a "fact." Indeed, results not involving that construction Str (the adjoint of the semantic functor), such as 1.2, 2.6, 2.8, 3.4, and 3.8, mostly remain true when restricted to small theories and can be relativized to reasonable theories with small hom sets. That is more or less true of the five illustrations.

There is a poorer, still very general, semantics which does not require big theories. The idea is to force the groundings $U: \mathscr{E} \to \mathscr{S}$ to be faithful. In that *reduced* semantics, the completion of every small theory has small hom sets.

The term "general functorial semantics" refers primarily to the main system; the theories are called *ruled* theories, and one may speak more precisely of "ruled semantics." A broader notion is introduced for one result (1.11) to the effect that any semantics stronger than ruled semantics works. Apropos of "works": adjoint functors called (with reason) "semantics" and "structure" are relatively trivial to produce. However, a system can scarcely be said to work unless the structure of the semantics of a theory $\boldsymbol{T}$ is semantically equivalent to $\boldsymbol{T}$, i.e. Sem Str Sem $\boldsymbol{T}$ is equivalent (in some strong sense) to Sem $\boldsymbol{T}$. That is what 1.11 establishes.

Apropos of "equivalent": a reader may regret to learn that we actually have *pretheories*, *midtheories*, and *theories*. But the motivation is strong. A pretheory is precisely after Gabriel (except for cardinality), so that its category of models lacks a distinguished grounding. To illustrate the significance, tensor products (like those known [6] for algebraic theories) make the category of theories monoidal, but pretheories are even closed monoidal. A midtheory is the obvious enrichment. Theories form a quotient category of the midtheories, by a rather small equivalence relation which makes things come out "on the nose." Illustrations: for theories, Sem Str Sem $\boldsymbol{T}$ is isomorphic with Sem $\boldsymbol{T}$. Fixed-form theories—for a precise illustration, the category of algebraic theories—can be isomorphically embedded in complete theories, preserving the models.

Facts (and prefacts and midfacts) unfortunately are not merely categories

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over $\mathcal{S}$ (as in algebraic functorial semantics). I do not know if there is a reasonably general free-form semantics in which they are. However, there is reason to suspect it. A *fact* as defined here is a category over $\mathcal{S}$ with an additional element of structure called a *coregulus*, resembling the *regulus* (after Gabriel) which is an essential constituent of a theory. The coregulus may be eliminable, because for a reasonable theory $\boldsymbol{T}$, the coregulus of Sem $\boldsymbol{T}$ is canonically determined by the category (as the *fine* coregulus; 3.5).

Three points about the structure of the category *Theo* of theories are, I believe, of major importance, though this paper does not develop them. (1) Besides morphisms (= *interpretations*) there are natural transformations. These are never considered in algebraic semantics, because they are semantically unnatural. Here, the equivalence relation which takes us from midtheories to theories is defined by means of certain restricted natural transformations, which seem to be the right ones to study further. For algebraic theories, the restricted natural transformations are invertible and unique. In reasonable reduced theories they are still unique (2.6); so they merely preorder the interpretations. The preordering is semantically natural. (2) *Theo* and the subcategories of reduced theories and of complete theories are not finitely complete. (2.14) (3) But the category of reduced theories complete relative to reduced theories is complete. (2.14.1)

The (varietal) theory of $\mathcal{S}^{op}$ has, but the theory of abelian groups (even of vector spaces over any field) lacks, the property that its tensor product with any varietal theory is (up to semantic equivalence) varietal.

An appendix refutes a result I professed to prove in [12], that every iterated limit of representable functors on a small category is a twofold iterate, a limit of limits. In fact there is no small bound on the number of iterations required. My applications of this claimed result in [12] and [13] amounted only to truncating transfinite inductions, and all are easily corrected.

I am indebted to J. W. Duskin for a number of conversations on this material. I am indebted to the referee for a number (thirty-six) of suggestions. The Introduction owes a good bit to K. H. Hofmann and K. Keimel.

Chapter I. Semantic connections.

Since some important individuals in the category of theories to be constructed here (such as the complete theory of groups) are bigger than the universe in which the theories will be modeled, we must begin with some conventions, at least, about foundations. The conventions are these. In Zermelo-Fraenkel set theory (or any reasonable variant) we assume there is at least one uncountable inaccessible cardinal and we distinguish one of them,

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called ∞. We distinguish also a larger cardinal q such that the cardinals less than q are closed under (binary) exponentiation. Sets, and categories, smaller than ∞ are called *small*; set of power ∞ are *large*. A set of power less than q is *at most illegitimate*, or *ami*. However, a category will be called *legitimate* if (only) its hom sets are small.

We need the "universes" $|\mathcal{S}|$ of size ∞ and $|\mathcal{S}!|$ of size q. $|\mathcal{S}|$ is defined as the set of all sets of small rank, $|\mathcal{S}!|$ as the set of all those sets whose rank α is such that every set of rank α is ami. (To recall the standard notion of rank: the rank of a set is the least ordinal exceeding the ranks of its elements.) The categories $\mathcal{S}$, $\mathcal{S}!$ consist of these sets (respectively) and all functions between them. Subject to a reservation, every category in this paper is assumed, or can be shown, to have both its set of objects and its set of morphisms elements of $|\mathcal{S}!|$. Reservation: each of the several exceptions will be called a *metacategory* at least when it is first introduced, save for $\mathcal{S}!$, which we now remark to be a metacategory. Without exception, a *diagram* is a diagram ($=$ functor) whose domain is not a metacategory. Finally, we use the symbol $\mathcal{S}ets$ for any ami full category of $\mathcal{S}!$ that contains $\mathcal{S}$ and is extensive enough for the purpose of the moment; in other words, a $\mathcal{S}ets$-valued functor is a bounded ami set-valued functor.

Remark. The number of grades by size (small, large, ami, meta) seems irreducible. The inaccessibility of ∞ is intuitively necessary, though if one wishes only to justify formal theorems, one can doubtless evade the assumption. There may be one real possibility for reducing q. The illegitimate categories we need seem to have skeletons no bigger than 2^∞. Probably there is a sufficiently rich semantics with q the cardinal successor of 2^∞; but it will need some more refined measure than rank to delimit it.

Now recall that a *morphism* from a diagram $D: \mathcal{A} \to \mathcal{C}$ to a diagram $E: \mathcal{B} \to \mathcal{C}$ consists of a functor $H^f: \mathcal{A} \to \mathcal{B}$ and a natural transformation $H^t: D \to EH^f$. An inductive cone over D is a special case, precisely the case that $\mathcal{B}$ is the category 1 with exactly one morphism. (1 is a right zero of *Cat*, so H^f is uniquely determined.) Projective cones are not morphisms, but comorphisms in the following sense. It will be convenient to regard opposite categories as having the same objects and morphisms (with opposite multiplication), and to use same notation for $D: \mathcal{A} \to \mathcal{C}$ and for $D: \mathcal{A}^{op} \to \mathcal{C}^{op}$. Thus a morphism $(H^f, H^t) = H: D \to E$ as above has an opposite composed of $H^f: \mathcal{A}^{op} \to \mathcal{B}^{op}$ and $H^t: EH^f \to D$; such an ordered pair is a *comorphism* from (note) $E: \mathcal{B}^{op} \to \mathcal{C}^{op}$ to $D: \mathcal{A}^{op} \to \mathcal{C}^{op}$. One might use more superscripts, but this notation will suffice for this paper.

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A morphism $H: D \to E$ induces a natural transformation $H^\sharp: L^E \to L^D$ of the colimit functors [12] of the diagrams (taking inductive cones over E to cones with base EH^t and then with base D). H will be called a *colens* if $H^\sharp$ is a natural equivalence. Thus H is a (represented) colimit if and only if it is a colens and a cone. *Lenses* are defined dually (comorphisms H such that $H^\flat$ is a natural equivalence of limit functors).

A *pretheory* $\boldsymbol{T}$ consists of a category $\mathcal{J}$ and a set Γ (not necessarily ami) of morphisms of diagrams in $\mathcal{J}$. $\mathcal{J}$ is the category *of* or *underlying* $\boldsymbol{T}$; Γ is the *regulus*, and its elements the *rules*. (I am indebted to A. J. Lohwater for a canvass of the meanings of "regulus." Note, a typical rule is not something like a commutative law for an operation *, but a specification that * is a binary operation.)

A *model* of a pretheory $(\mathcal{J}, \Gamma)$ in a category $\mathcal{C}$ is a functor $M: \mathcal{J}^{op} \to \mathcal{C}$ taking each rule in Γ to a lens in $\mathcal{C}$. ("Taking": in the notation established above, we get diagrams $MD: \mathcal{A}^{op} \to \mathcal{C}$ and $ME: \mathcal{B}^{op} \to \mathcal{C}$, and M *takes* (H^t, H^t) to the ordered pair $M \circ H$ made of $H^t: \mathcal{A}^{op} \to \mathcal{B}^{op}$ and the longitudinal (Godement) composition $M \circ H^t$.)

1.0.0. *A morphism of diagrams H in $\mathcal{C}$ is a colens if and only if each representable contravariant functor on $\mathcal{C}$ takes H to a lens.*

Proof. Testing whether $h_X \circ H$ is a lens reduces to evaluating the natural transformation $(h_X \circ H)^\flat$ at a singleton P in $\mathcal{S}ets$ since everything is a copower of P and limit functors preserve colimits. The domain $\mathrm{Hom}(K_P, h_X E)$ becomes identified with $\mathrm{Hom}(E, K_X)$ if we identify functions $P \to h_X E(B)$ with their values $E(B) \to X$, and similarly for the codomain. For (every) $h_X \circ H$ to be a lens means that H connects these sets bijectively, which is exactly the condition for H to be a colens.

We define a pretheory to be *regular* if its regulus consists of colenses. From 1.0.0:

1.0.1. *A pretheory is regular if and only if all representable contravariant functors on its category are models.*

Dualizing 1.0.0:

1.0.2. *A functor $M: \mathcal{J}^{op} \to \mathcal{C}$ is a model if and only if every $h^X M$ is a model.*

A pretheory not known or assumed to be regular may be called *singular*. $\boldsymbol{T}$ is *conical* if its regulus consists entirely of inductive cones; *light* if its

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category is small and its rules are morphisms of small diagrams; *small* if it is light and its regulus is small set. $\boldsymbol{T}$ is *legitimate* if its category is legitimate.

It may be possible to do semantics using only conical theories. At least, (but not published) that for every small pretheory $(\mathcal{J}, \Gamma)$ there exist a small conical theories are more convenient and very general. Gabriel has shown conical pretheory $(\mathcal{U}, \Delta)$ and a full faithful functor $\mathcal{J} \to \mathcal{U}$ inducing an equivalence from the category of models of $(\mathcal{U}, \Delta)$ in $\mathcal{S}$ to the category of models of $(\mathcal{J}, \Gamma)$.

It is possible to do semantics using only regular theories (2.9, 2.10), but singular theories are often convenient, e.g. for tensoring. It should be mentioned that, while a proof of stability for conical semantics (the analogue of 1.4) would be impressive and instructive, it would not be conclusive. For instance, Theorem 2.7 below is fairly important and the obvious conical analogue is false. (The same remarks apply to Theorem 1.2, but I do not think stability can be proved without finding a suitable analogue of this.)

1.1. *Every limit of models is a model.*

Proof sketch. If $(\mathcal{J}, \Gamma)$ is being modeled in $\mathcal{E}$, one must consider a diagram of models $N: \mathcal{N} \to Cat(\mathcal{J}^{op}, \mathcal{E})$ with limit M. Adjointed to it is $N^-: \mathcal{N} \times \mathcal{J}^{op} \to \mathcal{E}$ with pointwise limit M along $\mathcal{N}$. For $H: D \to E$ in Γ, where $D: \mathcal{A} \to \mathcal{J}$, $N^-(1 \times D): \mathcal{N} \times \mathcal{A}^{op} \to \mathcal{E}$ has pointwise limit MD. We have bijections between the projective cones over MD, $N^-(1 \times D)$, $N^-(1 \times E)$, and ME, with composition $(M \circ H)^\sharp$.

Let $\mathrm{Mod}(\boldsymbol{T}, \mathcal{E})$ denote the category of models of $\boldsymbol{T}$ in $\mathcal{E}$ (full in $Cat(\mathcal{J}^{op}, \mathcal{E})$). 1.0.2 and 1.1 nearly characterize the operators $\mathrm{Mod}(\boldsymbol{T},)$. They seemingly do not do it on the present foundations, perhaps on no foundations. What matters for semantics is the following. First, 1.0.2 reduces the description of $\mathrm{Mod}(\boldsymbol{T},)$ to $\mathrm{Mod}(\boldsymbol{T}, \mathcal{S}ets)$, and models in legitimate categories to models in $\mathcal{S}$. Then:

1.2. THEOREM. *A full subcategory $\mathcal{M}$ of $Cat(\mathcal{J}^{op}, \mathcal{S})$ is $\mathrm{Mod}(\boldsymbol{T}, \mathcal{S})$ for some singular pretheory $\boldsymbol{T}$ on $\mathcal{J}$ if and only if every functor $\mathcal{J}^{op} \to \mathcal{S}$ which is an iterated limit of objects of $\mathcal{M}$ in some category $Cat(\mathcal{J}^{op}, \mathcal{S}ets)$ belongs to $\mathcal{M}$.*

Proof. The condition is necessary, by 1.1 and the preservation and reflection of lenses by $\mathcal{S} \subset \mathcal{S}ets$. Suppose $\mathcal{M}$ satisfies it, and let Γ be the set of all morphisms of diagrams in $\mathcal{J}$ which each functor in $\mathcal{M}$ takes to a lens. Then $\mathrm{Mod}((\mathcal{J}, \Gamma), \mathcal{S})$ contains $\mathcal{M}$. For any $F: \mathcal{J}^{op} \to \mathcal{S}$, construct an $\mathcal{M}$-reflection F' as follows. In a suitable category $Cat(\mathcal{J}^{op}, \mathcal{S}ets)$, form a

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product Π indexed by the union of sets $\mathrm{Hom}(F, M)$, M in $|\mathcal{M}|$, with α-th factor the codomain of α. Map F to Π by the obvious map π whose α-th coordinate is α. Though $Cat(\mathcal{J}^{op}, \mathcal{S}ets)$ need not be complete, π has a right image F' with respect to morphisms to $\mathcal{M}$, which is an iterated limit of objects of $\mathcal{M}$, by 2.6 of [9]. The initial factor $p: F \to F'$ is $\mathcal{M}$-epic, and every $\alpha: F \to M$ factors across F'. If p is invertible, F is in $\mathcal{M}$. It remains to verify that if p is not invertible it yields a morphism of diagrams H which is in Γ but not taken by F to a lens.

Construct the universal diagrams D, D', which exhibit F, F' as colimits of representable functors. For instance, $D: \mathcal{A} \to \mathcal{J}$ has objects $a \in |\mathcal{A}|$ indexed by the union of all values $F(X)$, with $D(a) = X$ for a in $F(X)$. The morphism $a \to b$ in $\mathcal{A}$ are indexed by the morphisms $t: D(a) \to D(b)$ such that $F(t)(b) = a$; and $D(t) = t$. To define $H: D \to D'$, put $H^t(a) = p_X(a)$ for a in $F(X)$, $H^t(t) = t$. (t is in $\mathcal{A}'$ since p is natural.) Then $D'H^t$ is D, and we may put $H^t = 1$. Observe that, applying a Yoneda functor $Y: \mathcal{J} \to Cat(\mathcal{J}^{op}, \mathcal{S}ets)$, YD and YD' have colimit objects F and F' (2.1.a of [9]), and H induces p.

That $M \circ H$ is a lens need only be checked at $P \in |\mathcal{S}|$, just as for 1.0.0. It means that H takes inductive cones over YD' with vertex M bijectively to inductive cones over YD. Since morphisms $F \to M$ are uniquely divisible by p, this is true. Similarly, $F \circ H$ being a lens would mean, at P, that H induces an isomorphism of the colimits F and F'. The proof is complete.

1.2.1. Corollary. *Every pretheory $\boldsymbol{T}$ on a small category $\mathcal{J}$ has the same models in $\mathcal{S}$, and hence in every legitimate category, as some light pretheory $\boldsymbol{T}'$ on $\mathcal{J}$.*

This is a corollary of the proof. The universal diagrams D, D' are small; so one need only define the regulus of $\boldsymbol{T}'$ as the set of all morphisms of small diagrams taken to lenses by models of $\boldsymbol{T}$.

One naturally asks whether, for small $\mathcal{J}$, 1.2 can be simplified to refer only to iterated limits in $Cat(\mathcal{J}^{op}, \mathcal{S})$. It is not known; but one would need a quite different proof, for closure under iterated limits does not imply reflectiveness (MR 41 #5444).

A *morphism* of pretheories is a functor between their categories taking every rule to a rule. Evidently the pretheories and their morphisms constitute a metacategory *PTh*.

We define a *prefact* $\boldsymbol{F}$ to consist of a category $\mathcal{J}$ and a set Λ of comorphisms of diagrams in $\mathcal{J}$ (the *coregulus* of $\boldsymbol{F}$). A *morphism* is a functor

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taking corules to corules. This gives us a (meta-) category PFa and an isomorphism $\delta: PTh \to PFa$ induced by the involution of Cat which takes every category to its opposite. It will be convenient to refer to this isomorphism (like its underlying automorphism) as "duality."

An *arity* of a prefact in a category $\mathscr{C}$ is a (covariant) functor from its category to $\mathscr{C}$ taking corules to lenses. The prefact is *regular* if all representable functors are arities; equivalently, in view of $1.0.1^{op}$, corules are lenses. Other terms such as "conical" carry over in the same way ("legitimate" is the only one of these really used below).

For each pretheory $\boldsymbol{T}$ we define $\mathrm{PSem}\,\boldsymbol{T}$ as the (regular) fact on $\mathrm{Mod}(\boldsymbol{T}, \mathscr{S})$ whose corules are those morphisms of diagrams that are lenses pointwise on $\mathscr{T}$, i.e. the lenses that are preserved by $\mathrm{Mod}(\boldsymbol{T}, \mathscr{S}) \subset Cat(\mathscr{T}^{op}, \mathscr{S})$. (If $\boldsymbol{T}$ is regular and legitimate then $\mathrm{Mod}(\boldsymbol{T}, \mathscr{S})$ contains the representable functors and all its lenses become corules.) For a morphism of pretheories $F: \boldsymbol{T} \to \boldsymbol{T}'$, $Cat(F, \mathscr{S})$ evidently takes models to models and lenses of $Cat(\mathscr{T}'^{op}, \mathscr{S})$ to lenses of $Cat(\mathscr{T}^{op}, \mathscr{S})$; so it induces a morphism of prefacts $\mathrm{PSem}\,F$. Since $Cat(\ , \mathscr{S})$ is functorial, so is $\mathrm{PSem}: PTh^{op} \to PFa$.

We define $\mathrm{PStr}: PFa^{op} \to PTh$ as $\delta^{-1}\mathrm{PSem}\,\delta^{-1}$. Note that the underlying category of $\mathrm{PStr}\,\boldsymbol{F}$ is $\mathrm{Ari}(\boldsymbol{F}, \mathscr{S})^{op}$.

1.3. THEOREM. PSem *and* PStr *are adjoint on the right.*

Proof. This is a specialization of the familiar adjunction between $Cat(\ ^{op}, \mathscr{S})$ and $Cat(\ , \mathscr{S})^{op}$. For any $\mathscr{T}$ and $\mathscr{T}$, functors $G: \mathscr{T} \to Cat(\mathscr{T}, \mathscr{S})^{op}$ may be identified with $G': \mathscr{T}^{op} \to Cat(\mathscr{T}, \mathscr{S})$ which correspond adjointly to $G^-: \mathscr{T}^{op} \times \mathscr{T} \to \mathscr{S}$, which correspond obviously with $G^*: \mathscr{T} \times \mathscr{T}^{op} \to \mathscr{S}$, which correspond adjointly to $G^0: \mathscr{T} \to Cat(\mathscr{T}^{op}, \mathscr{S})$ (natural throughout). For G to take $\mathscr{T}$ into $\mathrm{Ari}(\boldsymbol{F}, \mathscr{S})^{op}$ means that G^0 takes corules of $\boldsymbol{F}$ to lenses; similarly, for G to take the rules of $\boldsymbol{T}$ to colenses means that G^0 is into $\mathrm{Mod}(\boldsymbol{T}, \mathscr{S})$.

We define an *equivalence* of prefacts $\boldsymbol{F}$, $\boldsymbol{F}'$ as a member of a pair of morphisms $\boldsymbol{F} \to \boldsymbol{F}'$, $\boldsymbol{F}' \to \boldsymbol{F}$, both of whose compositions are naturally equivalent to identities. A morphism of pretheories F is a *presentation* if $\mathrm{PSem}\,F$ is an equivalence.

1.4. THEOREM. *For every pretheory* $\boldsymbol{T}$, *the adjunction map* $\eta_{\boldsymbol{T}}: \boldsymbol{T} \to \mathrm{PStr}\,\mathrm{PSem}\,\boldsymbol{T}$ *is a presentation.*

For later generalization of 1.4 (to stronger languages) we want two or three refinements, all involving much the same situation and proof. First:

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1.4.1. *Let* $I: \boldsymbol{F} \to \mathrm{PSem}\,\boldsymbol{T}$ *embed a full subcategory* $\mathcal{J}$ *of* $\mathrm{Mod}(\boldsymbol{T}, \mathcal{S})$ *such that every model is an iterated limit of objects of* $\mathcal{J}$ *in some* $Cat(\mathcal{J}^{op}, \mathcal{S}ets)$. *Let* $\boldsymbol{F} = (\mathcal{J}, \Lambda)$, *where* Λ *is the set of lenses preserved by* $\mathcal{J} \subset Cat(\mathcal{J}^{op}, \mathcal{S})$. *Then* $\mathrm{PSem}\,\mathrm{PStr}\,I$ *is full and faithful.*

1.4.2. *In* 1.4.1, *if every model is a subobject of a small product of objects of* $\mathcal{J}$, *or every object of* $\mathrm{PStr}\,\boldsymbol{F}$ *is a quotient of a small coproduct of objects* $\mathrm{PStr}\,I(A)$, *then* $\mathrm{PSem}_{\eta_{\boldsymbol{T}}}\mathrm{PSem}\,\mathrm{PStr}\,I$ *is an equivalence.*

What is $\mathrm{PSem}_{\eta_{\boldsymbol{T}}}$ doing here? Reducing complications. We shall prove 1.4.2. Then 1.4 follows (putting $\boldsymbol{F} = \mathrm{PSem}\,\boldsymbol{T}$). 1.4.1 will come out of the same proof. Later we shall also need

1.4.3. *In* 1.4.2, $\mathrm{PStr}\,I$ *is an equivalence.*

But this is an easy corollary of 1.4.2 with 1.4^{op} and the observation that PStr takes equivalences (specifically $\mathrm{PSem}\,\mathrm{PStr}\,I$) to equivalences. In the direction of completeness:

1.4.4. *In* 1.4.1 *without more assumption,* $\mathrm{PSem}\,\mathrm{PStr}\,I$ *need not be an equivalence.*

Proof. Example first (for 1.4.4), seems a good plan. A *speckled set* is a set X with a base point 0 and an indexed partition of the non-zero points into ∞ subsets X_α, indexed by the small ordinals. Morphisms of speckled sets $X \to Y$ are those functions that take 0 to 0 and every X_α into $Y_\alpha \cup \{0\}$. Let $\boldsymbol{T}$ be the category of all speckled sets in $\mathcal{S}$ with the fine regulus, the set of all colenses. Take $\mathcal{J} \subset \mathrm{Mod}(\boldsymbol{T}, \mathcal{S})$ on the representable functors. Now, whatever else is in $\mathrm{Mod}(\boldsymbol{T}, \mathcal{S})$, it includes a product L of ∞ representable functors represented by two-point speckled sets X^α, $X_\alpha{}^\alpha$ consisting of one point and $X_\beta{}^\alpha$ empty for $\alpha \neq \beta$. (For L is a product of models, and it is small-valued on $\mathcal{J}$, since for X in $\mathcal{J}$, almost all X_α are empty and $\mathrm{Hom}(X, X^\alpha)$ singletons.) Similarly $\mathrm{PStr}\,\boldsymbol{F}$ includes an isomorphic copy of $\boldsymbol{T}$ (image of $\eta_{\boldsymbol{T}}$) and a coproduct A of all X^α. Let us omit "P." Consider the adjunction map $ev_{\mathrm{Sem}\,\boldsymbol{T}} = e: \mathrm{Sem}\,\boldsymbol{T} \to \mathrm{Sem}\,\mathrm{Str}\,\mathrm{Sem}\,\boldsymbol{T}$. There cannot be an object L' of $\mathrm{Sem}\,\mathrm{Str}\,\boldsymbol{F}$ such that $\mathrm{Sem}\,\mathrm{Str}\,I(L')$ is isomorphic with $e(L)$. For, for each α, $e(L)\,(\eta_{\boldsymbol{T}}(X^\alpha))$ reduces to $L(X^\alpha))$, a two-element set. So $L'(\mathrm{Str}\,I(\eta_{\boldsymbol{T}}(X^\alpha)))$ would have two elements, and $L'(A)$ would not be small.

That is the only obstacle, the smallness question. We shall find an equivalence between $\mathrm{Sem}\,\boldsymbol{T}$ and a prefact of $\mathcal{S}ets$-valued models of $\mathrm{Str}\,\boldsymbol{F}$. Let us write v for $(\mathrm{Str}\,I)\eta_{\boldsymbol{T}}$. Note that v simply takes T in $\mathcal{J}$ (and f in $\mathcal{J}$) to the corresponding evaluation on $\mathcal{J} \subset Cat(\mathcal{J}^{op}, \mathcal{S})$, except that the codomain

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of v is $\mathrm{Ari}(\boldsymbol{F}, \mathcal{S})^{op}$ (not $\mathrm{Ari}(\boldsymbol{F}, \mathcal{S})$). Here 1.2^{op} applies; the coregulus of $\boldsymbol{F}$ consists of all lenses preserved by all of the functors in $v(\mathcal{T})$, so every object of $\boldsymbol{U} = \mathrm{Str}\,\boldsymbol{F}$ is an iterated colimit in some $Cat(\mathcal{T}, \mathcal{S}ets)^{op}$ of objects $v(T)$.

Consider for a moment the metacategory $Cat(\mathcal{T}^{op}, \mathcal{S}!)$, in order to define the sub- (meta-) category $\mathrm{Hom}!\boldsymbol{T}$, as the union of all $Cat(\mathcal{T}^{op}, \mathcal{S}ets)$. (It is possible, but distracting, to find one category $\mathcal{S}ets$ sufficient for this proof.) Let $\mathrm{Sem}!\boldsymbol{T}$ be the smallest full subcategory of $\mathrm{Hom}!\boldsymbol{T}$ which contains $\mathcal{T}$ and whose intersection with each $Cat(\mathcal{T}^{op}, \mathcal{S}ets)$ is closed under forming limits. By hypothesis this contains $\mathrm{Sem}\,\boldsymbol{T}$; by 1.2, $\mathrm{Sem}\,\boldsymbol{T}$ is (on) the set of all $\mathcal{S}$-valued functors in $\mathrm{Sem}!\boldsymbol{T}$. To be precise, $\mathrm{Sem}!\boldsymbol{T}$ was defined as a category and contains only the underlying category of $\mathrm{Sem}\,\boldsymbol{T}$; but we shall regard $\mathrm{Sem}!\boldsymbol{T}$ as a (meta-) prefact with the coregulus of all lenses preserved in $\mathrm{Hom}!\boldsymbol{T}$. (It happens that all lenses are so preserved, but we do not need that.)

In $\mathrm{Hom}!\boldsymbol{U}$, let $\mathcal{M}!$ be the smallest full subcategory containing the image of $\mathcal{T}$ under the adjunction map $ev(\boldsymbol{F}$ to $\mathrm{Sem}\,\mathrm{Str}\,\boldsymbol{F} = \mathrm{Sem}\,\boldsymbol{U})$ and closed under bounded limits (limits in $Cat(\mathcal{U}^{op}, \mathcal{S}ets)$). By 1.2 again, $\mathrm{Sem}\,\boldsymbol{U}$ is on the full subcategory $\mathcal{M}$ of $\mathcal{M}!$ on the $\mathcal{S}$-valued functors.

Similarly, in $Cat(\mathcal{T}, \mathcal{S}!)^{op}$, let $\mathrm{Moh}!\boldsymbol{F}$ be the union of all $Cat(\mathcal{T}, \mathcal{S}ets)^{op}$ and let $\mathrm{Str}!\boldsymbol{F}$ be the smallest full subcategory closed under bounded colimits and containing $v(T)$. Now there is a contravariant (evaluation) functor τ from $\mathcal{T}$ to $Cat(\mathrm{Sem}!\boldsymbol{T}, \mathcal{S}!)$, $\tau(T)(M) = M(T)$. Each $\tau(T)$ is an arity. Then since everything in $\mathrm{Sem}!\boldsymbol{T}$ is an iterated limit in $\mathrm{Hom}!\boldsymbol{T}$ of objects of $\mathcal{T}$, we have a recursive formula for $\tau(T)(L)$; given any representation of L as a limit of objects L_i on which $\tau(T)$ is known, $\tau(T)(L) = \lim \tau(T)(L_i)$. Consequently for any morphism $m: v(T) \to v(T')$ ($v(X)$ being the opposite of the restriction of $\tau(X)$ to $\mathcal{T}$), we can define $\tau_1(m): \tau(T') \to \tau(T)$ recursively; $\tau_1(m)_M = m_M$ for M in $\mathcal{T}$, and for any representation of L as a limit of objects K for which $\tau_1(m)_K$ is known, $\tau_1(m)_L = \lim \tau_1(m)_K$. Defining $\tau_1(v(T))$ as $\tau(T)$, we have a contravariant functor τ_1 from the full image of v to $\mathrm{Ari}(\mathrm{Sem}!\boldsymbol{T}, \mathcal{S}!)$. We are in business. Recursively in $\mathrm{Str}!\boldsymbol{F}$, if τ_α has been defined over some iterated colimits, extending τ_1 and with each $\tau_\alpha(A)$ an extension of $A: \mathcal{T} \to \mathcal{S} \subset \mathcal{S}!$, we can extend over a new colimit B by $\tau_\beta(B) = \lim \tau_\alpha(A_i)$ for any representation of B as colimit of A_i's. $\tau_\beta(B)$ is an arity since it is a limit of arities; it extends B since the $\tau_\alpha(A_i)$ extends A_i. We have the formula $\tau_\beta(B)(L) = \lim \tau_\beta(B)(L_i)$ for any limit L of L_i's in $\mathrm{Sem}!\boldsymbol{T}$, and use it to define τ_β on morphisms. We conclude with τ_1 extended to $\sigma: (\mathrm{Str}!\boldsymbol{F})^{op} \to \mathrm{Ari}(\mathrm{Sem}!\boldsymbol{T}, \mathcal{S}!)$.

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All we need of σ is its restriction to $\boldsymbol{U}$. We have ev^-: Sem$!\boldsymbol{T} \to \mathcal{M}!$, $ev^-(L)(A) = \sigma(A)(L)$. Offhand this is into Hom$!\boldsymbol{U}$, but since σ is arity-valued, ev^- preserves pointwise lenses. By construction, $ev^- \mid \mathcal{F}$ is naturally equivalent to ev. Every object X of $\mathcal{M}!$ is isomorphic with an $ev^-(L)$ since X is obtainable from $ev(\mathcal{F})$ by iterated formation of limits which one can copy in Sem$!\boldsymbol{T}$. If X is in $\mathcal{M}$, $ev^-(L)$ is small-valued on $\mathcal{U}$ and L is certainly small-valued on $\mathcal{I}$; so L is isomorphic with an object of Sem$\,\boldsymbol{T}$.

There is a functor Sem$!v$: Sem$!\boldsymbol{U} \to$ Sem$!\boldsymbol{T}$, composition with v. It extends Sem$\,v$ and preserves pointwise lenses. From the definitions, (Sem$\,v)ev = I$. So (Sem$!\,v)ev^-$ is a limit-preserving endofunctor of Sem$!\boldsymbol{T}$ whose restriction to $\mathcal{F}$ is naturally equivalent to that restriction of the identity. Such a natural equivalence λ has a unique extension (by iterated limits) to a natural equivalence from (Sem$!\,v)ev^-$ to 1.

Sem$!\,v$ is also faithful, as follows. Every object of $\mathcal{U}$ is in fact, in Moh$!\boldsymbol{F}$, an iterated colimit of objects $v(T_i)$. Then in $Cat(\mathcal{F}, \mathcal{S})^{op}$, any two different morphisms $A \to B$ have different compositions with some $v(T_i) \to A$. This is a rule; if it were only one $f\colon v(T_i) \to A$, it would be the pushout

$$\begin{array}{ccc} v(T_i) & \xrightarrow{\ f\ } & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle 1} \\ A & \xrightarrow{\ 1\ } & A, \end{array}$$

and for many f's, join them all to three A's in the same way. Any model of $\boldsymbol{U}$ preserves the rule. (Things in $\mathcal{M}!$ are models by 1.1.) Therefore any two natural transformations of models agreeing on $v(\mathcal{I})$ are identical.

This does it. When the final factor of an equivalence is faithful, both factors are full, both faithful, both equivalences. Under the first condition of 1.4.2, ev^- takes every object L of Sem$\,\boldsymbol{T}$ to a subobject of a small product of objects of $\mathcal{M}$ in $\mathcal{M}!$, which is small-valued and thus isomorphic with an object of $\mathcal{M}$. (Being a subobject of a small product is expressible by a corule; compare the displayed diagram.) Under the second condition of 1.4.2, note that objects of $\mathcal{M}!$ small-valued on the image of Str$\,I$ are small-valued. Either way, $ev^- \mid$ Sem$\,\boldsymbol{T}$ is naturally equivalent to an $\mathcal{M}$-valued functor ϵ, and ϵ Sem$\,v$ and (Sem$\,v)\epsilon$ are still naturally equivalent to identities. Hence 1.4.2. As noted before, this makes Sem$\,\eta_{\boldsymbol{T}}$ an equivalence. Then since Sem$\,v$ is full and faithful, so is Sem Str$\,I$.

Now we define the (meta-) category *MTh* of *midtheories* to consist of arrows $\boldsymbol{1} \xrightarrow{P} \boldsymbol{T}$ in *PTh*, where $\boldsymbol{1}$ is the right zero category with the empty

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coregulus. (That is not the right zero pretheory.) Thus a midtheory $\boldsymbol{T}$ consists of a category $\mathcal{T}$, a *point* $P \in |\mathcal{T}|$, and a regulus; morphisms are point-preserving pretheory morphisms. Similarly *MFa* consists of the objects-over-$\boldsymbol{S}$, $\boldsymbol{F} \to \boldsymbol{S}$ in *PFa*, where $\boldsymbol{S}$ is $\mathcal{S}$ with the *fine* coregulus consisting of all lenses. A *midfact* $\boldsymbol{F}$ consists of a prefact $\boldsymbol{F}^-$ and an arity $U: \boldsymbol{F}^- \to \boldsymbol{S}$, the *grounding*.

This makes the requirement $U'G = U$ for a morphism of midfacts $G: \boldsymbol{F} \to \boldsymbol{F}'$ seem very natural. For light facts, it is not. We have no actual need to deal with light facts, but Theorem 3.8 and some constructions apply to them. It seems that one should pass from $(\mathcal{T}, U, \Lambda)$ to the category of representable functors in $Cat(\mathcal{T}, \mathcal{S})^{op}$, with the grounding represented by U. It is not so clear what to do with Λ; in 3.8, Λ is the fine regulus, and it is clear. (3.8 also has special features making contravariant representable functors usable and preferable.)

Duality does not lift to midtheories and midfacts, but (1.3.a) adjoint PSem and PStr do lift. Of course, MSem $\boldsymbol{T}$ is defined as PSem $\boldsymbol{T}^-$ grounded by evaluation at the point; the morphism PSem F is a morphism of midfacts if F is a morphism of midtheories. MStr $\boldsymbol{F}$ is defined as PStr $\boldsymbol{F}^-$ pointed by grounding, and this also takes morphisms (by PStr) to morphisms. To lift the proof of adjointness, 1.3, note that $G(P) = U$ if and only if G^0 followed by evaluation at P is U.

Theorem 1.4, 1.4.1, 1.4.2, lift as well. One must reexamine the proof, but there is no difficulty in doing it over $\mathcal{S}$!. While reexamining, note that the natural equivalences λ of composites with identities can be so chosen that $U \circ \lambda$ is an identity. Similarly $1.4.2^{op}$ lifts and the needed natural equivalences β can be so chosen that β_P is an identity.

1.5. If $\boldsymbol{T}^0$ is a (mid-) theory of groups, Sem $\boldsymbol{T}^0$ the usual midfact of groups, then in the first place, Str Sem $\boldsymbol{T}^0$ is illegitimate. For the coproduct functor of any family of pairwise non-isomorphic simple groups is small-valued. By 1.1^{op}, these are arities. But a large coproduct has 2^∞ idempotent endomorphisms. In the second place, this is not an accident caused by our having too many theories. Consider the midtheory $\boldsymbol{T}^1$ on the arrow category $P \to X$ with empty regulus. A morphism of it into Str Sem $\boldsymbol{T}^0$ is a diagram $Z \to A$. If A is "represented" by a large group, there are ∞ morphisms $\boldsymbol{T}^1 \to$ Str Sem $\boldsymbol{T}^0$ agreeing on objects, i.e. agreeing on the subtheory $\boldsymbol{T}^2$ carried by the objects of $\boldsymbol{T}^1$. Adjointness translates this into morphisms of the legitimate midfacts Sem $\boldsymbol{T}^i$; so this situation must exist in any semantics having these theories and their models. Finally, morphisms $\boldsymbol{T}^1 \to \boldsymbol{T}$ agreeing

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on $\boldsymbol{T}^2$ are precisely morphisms taking X to a fixed object A (and P to the point Z of $\boldsymbol{T}$); they can differ only in their values on the arrow.

Next, the (meta-) category *Theo* of *theories* is defined as the quotient of *MTh* by the equivalence relation which identifies pairs of morphisms $\boldsymbol{T} \to \boldsymbol{T}'$ related by a natural equivalence α such that $\alpha_P = 1_{P'}$ (P, P' the points). We shall call such a natural equivalence a *syntactic equivalence*. So a theory is a midthoery; a morphism of theories, or *intepretation*, is a syntactic equivalence class of morphisms (of midtheories). Similarly a *syntactic equivalence* of morphisms of midfacts $\boldsymbol{F} \to \boldsymbol{F}'$ is a natural equivalence α such that $U' \circ \alpha$ is an identity; *Fact* is the quotient of *MFa* by syntactic equivalence.

1.3.b. MSem and MStr factor across adjoint (on the right) functors Sem: $Theo^{op} \to Fact$, Str: $Fact^{op} \to Theo$. For by inspection of the proof of 1.3.a, G_1 and G_2 are syntactically equivalent if and only if $G_1{}^0$ and $G_2{}^0$ are so.

To process 1.4 and 1.4^{op}, let us introduce some more terminology. A *connection* from $\mathscr{C}$ to $\mathscr{D}$ is a (contravariant) functor $F: \mathscr{C}^{op} \to \mathscr{D}$ having an adjoint on the right. That adjoint H is then determined up to natural equivalence. F is a *Galois* connection if (for one, hence all, adjoints H) the adjunction transformations $\eta: 1 \to HF$, $\overline{\eta}: 1 \to FH$, yield natural equivalences $\overline{\eta} \circ F$ and $\eta \circ H$. Equivalently, the left inverses $F \circ \eta$, $H \circ \overline{\eta}$ are natural equivalences. A connection F is *universal* if FH and HF are naturally equivalent to the identities (which implies, of course, that η and $\overline{\eta}$ are invertible).

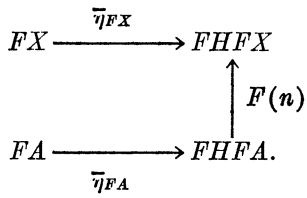
I do not have an example where F and FHF as well as H and HFH are naturallly equivalent, but the connection is not Galois. Nevertheless the Galois property seems considerably stronger; it means that the restrictions of η and $\overline{\eta}$ to the full images (of H resp. F) are invertible.

1.6. *Every Galois connection $F: \mathscr{C}^{op} \to \mathscr{D}$ is naturally equivalent to a connection IF_0R, where R is a reflector upon a full subcategory, F_0 is a universal connection, where I embeds a full reflective subcategory.*

Proof. Let $\mathscr{D}_0$, $\mathscr{C}_0$ be the full images of F and its adjoint H. Then F and H induce adjoint functors $F_0: \mathscr{C}_0{}^{op} \to \mathscr{D}_0$ and $H_0: \mathscr{D}_0{}^{op} \to \mathscr{C}_0$. As we noted above, the restricted adjunction transformations are invertible; F_0 is universal. Moreover, $HF: \mathscr{C} \to \mathscr{C}_0 \subset \mathscr{C}$ is a reflector with reflection maps η_X, as follows. It suffices to test the factoring across η_X of morphisms $X \to HFA$, since every object HY of $\mathscr{C}_0$ is isomorphic with $HFHY$. By adjointness, every $m: X \to HFA$ is $H(g)\eta_X$ for exactly one $g: FA \to FX$ (adjointed to m). H takes $\mathscr{D}(FA, FX)$ injectively to $\mathscr{C}(HFX, HFA)$ a

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fortiori; if $g \neq h$, even $H(g)\eta_X$ and $H(h)\eta_X$ differ. Finally, for any $n: HFX \to HFA$, form



Since these $\overline{\eta}$'s are invertible, one can fill in $g: FA \to FX$; adjunction yields $n = H(g)$. So we have a reflector R, $\mathcal{D}_0$ must be likewise reflective, and the composite IF_0R is FHF.

On the other hand, 1.7 and 1.8 give some sufficient conditions for a connection to be Galois.

1.7. *A connection F is Galois if $\overline{\eta} \circ F$ is invertible.*

Proof. We need to invert $\eta \circ H$ too, i.e. (locally) each $\eta_{HD}: HD \to HFHD$. This is possible at least for D of the form FC, i.e. $\eta \circ HF$ is invertible, as follows. Since $H \circ \overline{\eta}$ is a one-sided inverse of $\eta \circ H$, so is $H \circ \overline{\eta} \circ F$ of $\eta \circ HF$. Since $\overline{\eta} \circ F$ is invertible, so is $H \circ \overline{\eta} \circ F$, and its right inverse $\eta \circ HF$. Thus F induces a Galois connection between its domain $\mathcal{C}$ and its full image $\mathcal{D}_0 \subset \mathcal{D}$. By the structure theorem 1.6, the objects in $H(\mathcal{D}_0)$ are the objects of a full reflective subcategory of $\mathcal{C}$. HD is a retract of $HFHD$ and therefore isomorphic with HFC for some C; so η_{HD} is invertible.

1.7.1. *If $FHFH$ is naturally equivalent to FH by one of the distinguished transformations $\overline{\eta} \circ FH$, $F \circ \eta \circ H$, $FH \circ \overline{\eta}$, then the connection is Galois.*

Proof. Of course $F \circ \eta \circ H$ is a one-sided (left) inverse of the other named transformations, which are on this hypothesis the same, and all are invertible. 1.7 applies to the induced connection between the domain of H and its full image. So $\eta \circ H$ is invertible, and 1.7 applies to H.

One can formulate several more corollaries. First, there is an evident prolongation of 1.7.1 by finite induction. (Formal statement omitted because of the notation problem.) Second, there is an evident translation to adjoint covariant functors. Using these, consider a category $\mathcal{A}$ with a reflector R_0 upon a full subcategory $\mathcal{D}$ and a coreflector C_0 upon a full subcategory $\mathcal{C}$. Let $R_0 \dashv I$, $J \dashv C_0$, $R = IR_0$, $C = JC_0$. Crudely, if RC is idempotent then CR is idempotent.

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Correctly: The adjunction transformations $C \to 1 \to R$ yield unique transformations $\alpha: CRC \to RC$, $\beta: CRC \to RCRC$; also $a: CR \to RCR$, $b: CRCR \to RCR$. We have:

1.7.2. *Given a reflector and coreflector as just described, if α and β are related by an isomorphism of their codomains then a and b are related by an isomorphism of their domains.*

Proof. If $\theta\beta = \alpha$, θ invertible, apply $J \dashv C_0$ to the diagram and consider the adjoint functors $F = R_0J$, $G = C_0I$. $C_0\theta J$ turns out to invert one of the distinguished transformations (as in 1.7.1) $GFG = GFGFG$. So all of these distinguished transformations are invertible except $1 \to GF$ and $FG \to 1$; and CR, $CRCR$ have a suitable isomorphism.

In the situation preliminary to 1.7.2 (nothing idempotent), one has a Galois connection between $\mathcal{A}$ and $\mathcal{D}^{op}$. 1.7.2 describes the case that the induced connection between $\mathcal{C}$ and $\mathcal{D}^{op}$ is again Galois. Of course this is a somewhat exceptional case, for which one would like to have criteria. Here is one. I have no ready application for it, but it did come up in some work which may find a place (later) around Chapter VI.

Call a Galois connection $F: \mathcal{A}^{op} \to \mathcal{B}$ *essential* if for every monomorphism $m: X \to HY$ in $\mathcal{A}$ (H adjoint to F, Y in $\mathcal{B}$), Fm is an epimorphism relatively in the full image of F. In view of 1.6, this is the same as to say that the reflection of m into full $H(\mathcal{B})$ is a monomorphism (absolutely). Note, X admits a monomorphism into a *complete* object HY if and only if its reflection morphism (completion) is monic. Accordingly F is essential if and only if when X is embeddable in a complete object, the completion HFX is the smallest complete object containing X.

1.8. *An essential Galois connection $F: \mathcal{C}^{op} \to \mathcal{D}$, restricted to a monocoreflective full subcategory of $\mathcal{C}$, is again Galois.*

Proof. Let H be the adjoint connection, J the coreflector, I the adjoint insertion. Consider a coreflection (mono-) morphism $m: IJHFIX \to HFIX$; we have epic $Fm: FHFIX \to FIJHFIX$. Since F is Galois, adjunction $\bar{\eta}_{FIX}: FIX \to FHFIX$ is isomorphic. and $Fm\bar{\eta}_{FIX}$ is epimorphic. But $Fm\bar{\eta}_{FIX}$ is the adjunction morphism $\bar{\epsilon}_{FIX}$ for the connection FI, since the composite adjunction takes it (via m) to $Jm = 1_{JHFIX}$. An $\bar{\epsilon}_{FIX}$ (the form being $\bar{\eta}_{FZ}$) has a left inverse, so when epic it is invertible. By 1.7, FI is Galois.

1.9. THEOREM. Sem *is a Galois connection.*

Proof. This means merely that the transformations we have, *ev*: Sem

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$\rightarrow$ Sem Str Sem and the "dual," are invertible. Invertibility of a natural transformation is a local property, and we have already verified it in lifting 1.4 and 1.4^{op} to midtheories.

As was indicated in the Introduction, the connection just finished is the main one for this paper, but we shall also regard it as an instance of a more general scheme. The same (pre-, mid-) facts are used throughout, but the theories (pre-, midtheories) so far considered are *ruled*. In the broader context the symbols *PTh*, *MTh*, *Theo*, Sem, Str (PSem, etc.) should be preceded by "$\mathcal{R}$" to avoid ambiguity. It matters less for Sem (PSem, MSem), since the different instances are just different restrictions of one semantic functor. The notion of a presentation of theories therefore will not vary. (Of course, a morphism of midtheories is called a presentation only if MSem takes it to a *syntactic* equivalence.)

A ruled theory is *complete* if it is isomorphic with some theory $\mathcal{R}\,\mathrm{Str}\,\boldsymbol{F}$. *Ruled-complete* facts are defined symmetrically. From the proof of 1.9 (and omitting "meta-"):

1.9.1. Corollary. *The full subcategories $\mathcal{R}CTh$ of $\mathcal{R}Theo$, $\mathcal{R}CFa$ of Fact, consisting of $\mathcal{R}$-complete theories and facts, are reflective, and* $\mathcal{R}$Sem *induces a universal connection between them.*

To adapt the argument of 1.5 to $\mathcal{R}Theo$, one need only use an arity A "represented" by a group G whose elements fall into ∞ orbits under the automorphism group. This is easily arranged in a coproduct of simple groups G_α. For instance, the centralizer of an element of one summand G_α is contained in G_α, and one can distinguish ∞ elements by the sizes of their centralizers.

Turning to general languages, we need first a language-holder $\mathcal{E}^+$. This is a set-valued functor on the metacategory of all ami categories, defined as follows. Let I be a set of q indices. For each category $\mathcal{T}$, for each object T of $\mathcal{T}$, let $V(T)$ be a copy of I; its elements are called *variables*. For each morphism $f\colon T \rightarrow T'$ we introduce a symbol f^* called a *shift*. Suppressing an induction, a *term* of *type* T is a variable in $V(T)$ or a finite string of shifts followed by a variable in $V(T')$, the shifts being given by morphisms $T \rightarrow T_1 \rightarrow \cdots \rightarrow T'$. Beginning a transfinite induction, an *atomic sentence* is a string $\tau = \tau'$, where τ and τ' are terms of the same type. It has either one or two *free variables*, the variables which end τ and τ'. Sentences are generated from atomic sentences σ_i by negation, $\sim\sigma_1$; *conjunction*, $\Lambda(\sigma_i)$, of ami sets of sentences; and *quantification*, $(\exists X)\sigma_1$, for any ami set X of variables. We cannot (yet) confine quantification to the free variables of σ_1,

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as will be seen, but we must keep track of free variables. The free variables of $\sim\sigma_1$ are the same as in σ_1; of $\Lambda(\sigma_i)$, the union; of $(\exists X)\sigma_1$, the variables free in σ_1 and not belonging to X. Then $\mathcal{E}^+(\mathcal{T})$ is the set of all these sentences.

For a functor $F\colon \mathcal{T} \to \mathcal{T}'$, $\mathcal{E}^+(F)\colon \mathcal{E}^+(\mathcal{T}) \to \mathcal{E}^+(\mathcal{T}')$ is defined as follows. Map each $V(T)$ bijectively via I to $V(FT)$; take each shift f^* to $(Ff)^*$; and map terms to terms, sentences to sentences, accordingly. Evidently $\mathcal{E}^+$ is a functor.

$\mathcal{E}^+$ will not be called a language. The elements are good sentences, but the functions $\mathcal{E}^+(F)$ can take (e.g.) $(\exists x)(\exists y)\sim f^*x = g^*y$ to $(\exists z)(\exists z)\sim h^*z = h^*z$. The largest usable subfunctor of $\mathcal{E}^+$ is the *safe sentence* functor, where $\sigma \in \mathcal{E}^+(\mathcal{T})$ is safe if no two free variables of σ are identified by $\mathcal{T} \to 1$. I call that $\mathcal{E}$.

(Within the safe sentences, one can exclude $(\exists x)(\exists x)\sigma$. But there is no end of economies; $\sim\sim\sigma$ is not needed, and one could also build safety into the formation rules. Similar matters recur below in 1.10.)

A *language* is an arbitrary subfunctor of $\mathcal{E}$. An $\mathcal{L}$-*pretheory*, for any language $\mathcal{L}$, consists of a category $\mathcal{T}$ and a subset $\mathfrak{S}$ of $\mathcal{L}(\mathcal{T})$. A *morphism* of $\mathcal{L}$-pretheories, $(\mathcal{T}, \mathfrak{S}) \to (\mathcal{T}', \mathfrak{S}')$, is a functor $F\colon \mathcal{T} \to \mathcal{T}'$ such that $\mathcal{L}(F)(\mathfrak{S}) \subset \mathfrak{S}'$. Then $\mathcal{L}$-pretheories form a metacategory $\mathcal{L}PTh$.

Next we define "place," "satisfy," "validate," "model." The definitions differ from the usual ones because we must allow for models $M\colon \mathcal{T}^{op} \to \mathcal{S}$ some of whose values are empty. (Mostowski [22] and others have written notes on this, and Hailperin a longer paper [9]. They use different conventions; we follow Mostowski. As he points out, the predicate calculus of possibly empty models is definitely different (no modus ponens, no prenex form, $\cdots$) from the usual predicate calculus. Open formulas are all valid in the empty set; so a theory expressed by open formulas may have an unintended meaning. Fortunately, closed formulas mean what they seem to. The sketchy instructions in 1.10 below refer to closed formulas.)

For any function $M\colon \mathcal{T}^{op} \to \mathcal{S}ets$ and any subset J of the set of all variables over $\mathcal{T}$, a (partial) J-*place* of M is a function p on J such that for $x \in J \cap V(T)$, $p(x) \in M(T)$. The inductive definition of a J-place p of M *satisfying* a sentence σ begins immediately after we note that satisfaction is a ternary relation among p, M, and σ. (For convenience:) J is exactly the set $D(\sigma)$ of variables occurring free in σ. For the rest of the definition, M is fixed and $J = D(\sigma)$. Then for atomic σ, of the form $f_r{}^*\cdots f_1{}^*x = g_s{}^*\cdots g_1{}^*y$ $(r, s \geqq 0)$, p satisfies σ provided $M(f_1\cdots f_r)(p(x)) = M(g_1\cdots g_s)(p(y))$. Inductively, p satisfies $\sim\sigma$ provided p fails to satisfy σ. It satisfies $\Lambda(\sigma_i)$ provided each σ_i is satisfied by the restriction of p to $D(\sigma_i)$. It satisfies

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$(\exists X)\sigma_1$ provided it can be extended to a $D(\sigma_1)$-place which satisfies σ_1. This defines satisfaction. *M validates* σ if every $D(\sigma)$-place of M satifies σ. M is a *model* of $(\mathcal{T}, \mathfrak{S})$ if it validates every sentence in $\mathfrak{S}$.

For any $\mathcal{L}$-pretheory $\boldsymbol{T}$, $\mathrm{Mod}(\boldsymbol{T}, \mathcal{S})$ and $\mathrm{PSem}\,\boldsymbol{T}$ (or $\mathcal{L}\mathrm{PSem}\,\boldsymbol{T}$) are defined exactly as before: category of all models, pointwise coregulus. If $F: \boldsymbol{T} \to \boldsymbol{T}'$ is a morphism and M is a model of $\boldsymbol{T}'$, MF is a model of $\boldsymbol{T}$ (because all sentences are safe); and we have a functor PSem (contravariant). For any prefact $\boldsymbol{F}$, $\mathcal{L}PStr\,\boldsymbol{F}$ is $\mathrm{Ari}(\boldsymbol{F}, \mathcal{S})^{op} = \mathcal{C}$ with the set of all sentences in $\mathcal{L}(\mathcal{C})$ validated by all of the evaluations at objects of $\boldsymbol{F}$. Adjointness (Theorem 1.3) carries over at once to every language $\mathcal{L}$.

1.10. Ruled theories have been called $\mathcal{R}$-theories, and $\mathcal{R}$ has been called a language. This is justified by the existence of languages in $\mathcal{E}$ which are capable of expressing all rules and only rules. (For instance, paraphrase a morphism of diagrams $D \to E$, "For each element at the colimit of D, there is an element at the colimit of E from which it comes, and any two of those are equal." The next step is to replace elements "at the colimit" by their coordinates, and routine further steps yield the general form of sentences of a suitable language.) There is not a sublanguage $\mathcal{L}$ of $\mathcal{E}$ such that $\mathcal{L}PTh$ is isomorphic with $PTh(\mathcal{R}\,PTh)$ by an isomorphism preserving underlying categories and sets of models. (I doubt if the side conditions are necessary.) If this should be wanted, one could get it by narrowing both notions of theory. The idea would be to require theories in languages $\mathcal{L} \subset \mathcal{E}$ to be closed under permissible changes of variables, and to require ruled theories to be invariant under the like changes, replacing diagrams $D: \mathcal{A} \to \mathcal{T}$ by $DJ: \mathcal{A}' \to \mathcal{A} \to \mathcal{T}$, where J is an isomorphism. This is not different in kind from the passages from midtheories to theories, and some such step might become important. For present purposes, it is not important. Note that nothing in the proofs so far would be affected if we did restrict reguli in that way, nor if we paraphrased them in a suitable sublanguage of $\mathcal{E}$. Thus, with the obvious restatements of the needed definitions (presentation, midtheory, syntactic equivalence, theory): *all relevant results,* 1.1 *through* 1.9.1, *remain true for* $\mathcal{L}$*-theories for any language* $\mathcal{L}$ *such that each* $\mathcal{L}(\mathcal{T})$ *can express any rule in* $\mathcal{T}$ *and nothing else.*

For the next result, let $\mathcal{R}$ stand for any of these languages.

1.11. *If* $\mathcal{R}$ *is a sublanguage of* $\mathcal{L}$ *then the adjunction maps* $\eta_{\boldsymbol{T}}: \boldsymbol{T} \to \mathcal{L}\mathrm{PStr}\,\mathcal{L}\mathrm{PSem}\,\boldsymbol{T}$ *are presentations. So is* $ev_{\boldsymbol{F}}: F \to \mathcal{L}\mathrm{PSem}\,\mathcal{L}\mathrm{PStr}\,\boldsymbol{F}$, *in the sense that* $\mathcal{L}\mathrm{PStr}\,ev_{\boldsymbol{F}}$ *is an equivalence.* $\mathcal{L}\mathrm{Sem}$ *is a Galois connection.*

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Proof. Let $\boldsymbol{T} = (\mathcal{T}, \mathfrak{S})$ and let $\boldsymbol{F}_1 = \mathrm{PSem}\,\boldsymbol{T}$. Let $\mathfrak{S}_1$ be the set of all sentences in $\mathcal{L}(\mathcal{T})$ validated by all functors in $\boldsymbol{F}_1$. Then $\mathfrak{S}_1 \supset \mathfrak{S}$ and $\boldsymbol{T}$ may be replaced by $\boldsymbol{T}_1 = (\mathcal{T}, \mathfrak{S}_1)$; $\mathrm{PSem}\,\boldsymbol{T}_1$ is still $\boldsymbol{F}_1$. Let $\boldsymbol{T}_0$ be $\mathcal{T}$ with $\mathfrak{S}_1 \cap \mathcal{R}(\mathcal{T})$. The identity functor is a morphism $i: \boldsymbol{T}_0 \to \boldsymbol{T}_1$. $\mathrm{PSem}\,i$: $\boldsymbol{F} \to \mathrm{PSem}\,\boldsymbol{T}_0$ is full and faithful (since i is on an identity functor), and everything in $\mathrm{PSem}\,\boldsymbol{T}_0$ must be an iterated limit in some $Cat(\mathcal{T}^{op}, \mathcal{S}ets)$ of objects of $\boldsymbol{F}_1$, since (by 1.2) the set of such limits is the set of models for some ruled theory on $\mathcal{T}$, whose axioms are validated by functors in $\boldsymbol{F}_1$.

Omitting "P," fill out a commutative rectangle on i, $v: \boldsymbol{T}_0 \to \mathcal{R}\,\mathrm{Str}\,\boldsymbol{F}_1$, $j: \mathcal{R}\,\mathrm{Str}\,\boldsymbol{F}_1 \to \mathcal{L}\mathrm{Str}\,\boldsymbol{F}_1$ (on the identity functor), and $\eta: \boldsymbol{T}_1 \to \mathcal{L}\,\mathrm{Str}\,\boldsymbol{F}_1$. Sem takes it to a commutative rectangle. $\mathrm{Sem}\,j$ is faithful since j is on an identity, and $\mathrm{Sem}\,v$ is faithful by 1.4 and 1.4.1. Then $\mathrm{Sem}\,\eta$ is an initial factor of a faithful functor and therefore faithful. It is also a final factor of an identity, $(\mathrm{Sem}\,\eta)ev = 1$. So $\mathrm{Sem}\,\eta$ is an equivalence, and it is identical with $\mathrm{Sem}\,\eta_{\boldsymbol{T}}$.

The second assertion follows in the same manner, using 1.4.3. The last assertion follows by the same remarks that led from 1.4 and 1.4^{op} to 1.9.1.

Chapter II. Some structure of theories.

Let us define algebraic (pre-, mid-) theories as follows, imitating Lawvere [17] as closely as possible. Fix throughout the discussion a sequence of index categories $J_0, J_1, \cdots$, each J_n discrete on n objects. Relative to these, a pretheory $\boldsymbol{T} = (\mathcal{T}, \Gamma)$ is *standard algebraic* if $\mathcal{T}$ is small, with exactly $\aleph_0$ objects $T_0, T_1, \cdots$, and Γ has exactly $\aleph_0$ members H_n, which are (actually) universal inductive cones whose base is the constant T_1-valued functor on J_n and whose vertex is T_n. Then plainly the full subcategory $\mathcal{A}PTh$ of PTh on the standard algebraic pretheories is isomorphic with Lawvere's category of algebraic theories. The object T_1 is distinguishable in $\boldsymbol{T}$ (though not necessarily in $\mathcal{T}$), and there is a functor $\mathcal{A}PTh \to MTh$ which makes T_1 the point; it is full and faithful, thus isomorphic from $\mathcal{A}PTh$ to its image, the standard algebraic midtheories. So is its composite with the quotient functor $MTh \to Theo$. (A syntactic equivalence must be 1 at T_1, thus also at its copowers.) Finally:

2.1. $\mathrm{Str}\,\mathrm{Sem}$ *is full and faithful on the standard algebraic theories.*

Proof. Since the completion $\mathrm{Str}\,\mathrm{Sem}\,\boldsymbol{T}$ is a complete reflection of $\boldsymbol{T}$ (1.9.1), it will suffice to show that $\boldsymbol{T}$ is a standard algebraic coreflection of $\mathrm{Str}\,\mathrm{Sem}\,\boldsymbol{T}$ with coreflection map $\eta_{\boldsymbol{T}}$. Indeed $\mathcal{A}Theo$ is not coreflective; but its coreflectivizer evidently contains all those fine theories $\boldsymbol{U}$ which include

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n-th copowers of the point for all n and have all $\mathrm{Hom}(P, nP)$ small. The coreflection of such a $\boldsymbol{U}$ is determined up to isomorphism in *Theo* by putting $T_n = nP$. Now for every legitimate regular midtheory $\boldsymbol{T}$, $\eta_{\boldsymbol{T}}$ is full and faithful, by two applications of the Yoneda Lemma. Hence for T in $\mathcal{A}Theo$, $\eta_{\boldsymbol{T}}$ is a coreflection map coreflecting $\mathrm{Str}\,\mathrm{Sem}\,\boldsymbol{T}$ into $\mathcal{A}Theo$.

Consider the virtues of $\mathcal{A}Theo$ and $\mathcal{A}MTh$ ($\mathcal{A}PTh$ seems too artificial) with a view to generalizing. This category of theories (being Lawvere's) can be fully and rather naturally described in classical algebraic language. It can be done in predicate calculus, with some adjustments to take account of empty models and of morphisms. To take account of morphisms will require developing Freyd's notion of an elementary theory [5, p. 91] or some variant. Still the resources of the predicate calculus will help.

Note that Lawvere's definition of the *set of n-ary operations* of an algebraic theory as $\mathrm{Hom}(P, nP)$ is appropriate for a very broad class of theories: certainly for those which have standard algebraic coreflections and for all complete theories. On all these, the set of n-ary operations is functorial, and is a representable functor represented by (the standard counterpart of) Lawvere's free theory on an n-ary operation.

Algebra is not in such good order in *MTh*, but we have a functorial partial cross-section of the *semantic equivalence classes* (where two midtheories are semantically equivalent if their completions, or equivalently their midfacts of models, are syntactically equivalent).

Third, though $\mathcal{A}Theo$ is not coreflective, it has a big coreflectivizer. Lawvere's semantics has to deal with restricted facts (excluding e. g. the fact of complete lattices) essentially because his theories are not quite a coreflective system. $\mathcal{A}MTh$ is "less coreflective"; for instance, one cannot coreflect the completions.

I shall use the phrase "reduced semantics" a little vaguely. A "maximum system" will be defined, for which we can quickly establish the *sine qua non*—a Galois semantic connection—and also coreflectiveness of the midtheories. So one can treat arbitrary facts, and one can work on the more intuitive level of midtheories and midfacts. Then I shall describe more vaguely a narrower system with the same relatively complete theories and facts which recovers the crucial first virtue, intelligibility in terms of predicate calculus. The vagueness could be avoided, but I do not at all want to avoid it. The purpose in connecting the system with the predicate calculus is to connect with *theorems* of predicate calculus; I have not yet done that, and most likely some modification will be desirable.

We call an object X of a midtheory $\boldsymbol{T}$ *reduced* in $\boldsymbol{T}$ if there is a small

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set of morphisms $f_i: P \to X$ such that for every model M of $\boldsymbol{T}$ in $\mathcal{S}ets$, the functions $M(f_i)$ separate the points of $M(X)$.

2.2. *An arity in* MStr $\boldsymbol{F}$ *is a reduced object if and only if it is a subfunctor of a small power of the grounding.*

Proof. Since evaluations are models, the morphisms $P \to X$ showing that X is reduced embed it in a small power of the grounding. Conversely, if $f: X \to U^m$ is a monic natural transformation, then in $\mathrm{Ari}(\boldsymbol{F}, \mathcal{S})^{op}$ f appears as an epic $\phi: mP \to X$, preserved by $\mathrm{Ari}(\boldsymbol{F}, \mathcal{S})^{op} \subset Cat(\mathfrak{F}, \mathcal{S})^{op}$. In any category, $\phi: W \to X$ is epic if and only if

$$\begin{array}{ccc} W & \xrightarrow{\phi} & X \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle 1} \\ X & \xrightarrow{1} & X \end{array}$$

is a pushout. So this cone is a rule of MStr $\boldsymbol{F}$, and all models preserve it.

A midtheory is *reduced* if all its objects are reduced. The reduced midtheories form a full subcategory MTh^R of MTh, which maps upon a full subcategory $Theo^R$ of $Theo$.

2.3. MTh^R *is coreflective in* MTh. *The coreflector preserves syntactic equivalence and therefore induces a functor from* $Theo$ *to* $Theo^R$, *which is also a coreflector.*

The proof is quite straightforward. Note, the coreflection of a midtheory $\boldsymbol{T}$ need not be the subtheory $\boldsymbol{T}^1$ on the reduced objects, for they need not be reduced in $\boldsymbol{T}^1$. But a morphism must preserve reduced objects, so a reduced midtheory mapping into $\boldsymbol{T}$ goes into $\boldsymbol{T}^1$, hence into $\boldsymbol{T}^2 = (\boldsymbol{T}^1)^1$, and so on.

Defining MStr^R (respectively Str^R) as MStr (Str) followed by the coreflector, we have at once:

2.4. MSem | MTh^R *and* MStr^R *are adjoint on the right.* Sem | $Theo^R$ *and* Str^R *are adjoint on the right.*

Although the reduced semantic functors are mere restrictions, it is sometimes convenient to use the shorter symbols MSem^R, Sem^R.

2.5. THEOREM. Sem^R *is a Galois connection.*

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The proof amounts to observing that the proof of 1.4 did not require all the conditions. We are now using $\boldsymbol{U}' = \mathrm{MStr}^R \boldsymbol{F} = \mathrm{Str}^R \boldsymbol{F}$ (for short) instead of $\boldsymbol{U} = \mathrm{Str}\,\boldsymbol{F}$. But no properties of $\boldsymbol{U}$ were invoked beyond $\boldsymbol{T} \xrightarrow{v} \boldsymbol{U} \to \mathrm{Str}!\boldsymbol{F}$ and the description of the rules of $\boldsymbol{U}$ as the pointwise rules, and (for conventional reasons, but conventions we are using) the arrow into $\mathrm{Str}!\boldsymbol{F}$ going into small-valued functors. For $\boldsymbol{U}'$, or any other fact so situated, the morphism $ev\colon \mathrm{Sem}\,\boldsymbol{T} \to \mathrm{Sem}\,\boldsymbol{U}'$ is syntactically inverted by $\mathrm{Sem}\,v$.

This is "maximum" reduced semantics. It is not the absolute maximum on the present foundations; the essential semantic consideration is 2.2, which would hold if we defined reduced objects by reference to small-valued models only. But that would be ridiculous. A reduced theory should be reduced by virtue of its form, and all models should respect the form.—The problem, what forms of rules suffice to reduce an object, is probably hard and does not seem pressing; but it does seem reasonable.

The obvious reducing rule is diagrammed in 2.2, and we note that complete reduced theories do have these rules. For the following unexploited discussion, let us consider a "special" reduced theory to have at least enough copowers nP to express such a pushout rule reducing each object X.

For fine theories having an object nP, it is appropriate to define the *set of n-ary relations* as the set of quotients of nP. This is a functor, the restriction of a functor on *Theo* represented by a theory on three objects P, nP, X, some obvious morphisms, and two rules (that nP is a copower and $nP \to X$ is epic). In special reduced semantics one will use such a free theory on an n-ary relation to define the set of n-ary relations in general. (Observe, "quotient" is a mere categorical concept; one wants assurance that the putative relations are modeled by actual relations.)

At any rate, in special reduced theories each object is an n-ary relation in at least one way. Pick one. Recall that in Lawvere's theories the most general morphism is from mP to nP (m and n finite) and is determined by its coordinates, which are m n-ary operations. Here a morphism runs from an m-ary relation A to an n-ary relation B (m and n small) and analyzes into m *secondary n-ary operations*, or more specifically, *B-ary operations* (defined only on B-related n-tuples). The translation incorporates, for each $\phi\colon A \to B$ analyzed into $\phi_i\colon P \to B$, an axiom

$$A(\phi_1(x_1, \cdots, x_n), \cdots, \phi_m(x_1, \cdots, x_n)).$$

Multiplication of morphisms is described essentially the same way as for algebraic theories; this extends without difficulty to the fictitious morphisms

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introduced next. To express a conical rule, introduce the colimit as a defined relation D. The cone connects the vertex B to D by a family of B-ary operations. (The theorem that D "holds" for these operations amounts to a conjunction of axioms.) The rule requires this arrow $D \to B$ to be invertible, and is expressed by introducing the inverse as a family of D-ary operations and recording that the compositions are identities. Finally, a general rule is expressed in the same way, but both of the relations are defined relations.

As far as I know, the second notable virtue of algebraic semantics, the functorial cross-section of semantic equivalence classes of theories, is not purchasable for reduced semantics. For reduced-complete facts, it is, as follows.

2.5.1. Call a midfact $(\mathfrak{F}, U, \Lambda)$ *deflated* if every isomorphism f in $\mathfrak{F}$ such that $U(f)$ is an identity is (already) an identity. Trivially, a syntactic equivalence of morphisms $\boldsymbol{F}' \to \boldsymbol{F}$ is an identity if $\boldsymbol{F}$ is deflated; so $MFa \to Fact$ is isomorphic from deflated midfacts to the corresponding deflated facts. Trivially, the full subcategory of MFa on deflated midfacts is reflective. The reflection map $\boldsymbol{F}' \to \boldsymbol{F}$ to a deflated reflection is an equivalence of categories over $\mathfrak{S}$ if every automorphism in $\boldsymbol{F}'$ that lies over a identity is an identity, and a syntactic equivalence if also every isomorph of a corule of $\boldsymbol{F}'$ is a corule. If $\boldsymbol{F}'$ is MStr $\boldsymbol{T}$ for a reduced theory $\boldsymbol{T}$, these conditions hold. So for reduced-complete midfacts, the deflated reflections are syntactically equivalent, hence still reduced-complete, and they cross-section the syntactic equivalence classes. By 2.5, these are the semantic equivalence classes.

Some remarks are called for on Linton's *varietal* and *equational* theories [18]. To convert the definition of a standard algebraic pretheory to "standard varietal," replace "small" by "legitimate" and $\aleph_0$ by ∞; one needs discrete index categories of every small power. 2.1 and the remarks on the virtues require no substantial change. The category of varietal theories is not (right) complete. For a standard equational theory, omit "legitimate." Their semantics applies to arbitrary facts; every complete theory has a standard equational coreflection. However, Str Sem is neither full nor faithful on standard equational theories.

(That it is not full is perhaps not quite obvious. Form an equational theory $\boldsymbol{T}$ all of whose models in $\mathfrak{S}$ are on sets A equivalent to A^m for all small m, by introducing operations and axioms effecting such bijections. If $\boldsymbol{U}$ is the equational coreflection of the complete reflection of $\boldsymbol{T}$, Str Sem is not full on $Theo(\boldsymbol{U}, \boldsymbol{T})$.)

There is no difficulty in showing that equational semantics is a Galois

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connection. Most of the work in 1.4 went into showing that there are no more objects (up to isomorphism) in Sem Str Sem $\boldsymbol{T}$ than in Sem $\boldsymbol{T}$ (in Str Sem Str $\boldsymbol{F}$ than in Str $\boldsymbol{F}$); when theories have a fixed form such as the equational form, this is trivial. Incidentally, this applies as well to "ruled equational" theories consisting of a standard equational theory and any additional rules. Torsion-free abelian groups are the models of one such (ruled algebraic) theory.

We now have all the semantic connections that are going to be mentioned. But we have them only on objects and morphisms—not on natural transformations. It is left for the concerned reader to verify (readily) that the connections extend to natural transformations of morphisms of theories and of facts. I want to indicate the role of the natural transformations, and why they have no role in equational semantics.

A *syntactic transformation* between two morphisms of midthtories $\boldsymbol{T} \to \boldsymbol{T}'$ is a natural transformation α such that $\alpha_P = 1_{P'}$. Preliminary formulation: in standard or complete equational midtheories, every syntactic transformation is a syntactic equivalence. (In standard ones, an identity. For complete ones, apply an analogue of 2.1 amplified for natural transformations.)

One might want to admit some other transformations of morphisms. I cannot show that no others are of any use. But if naturality is required, there is nothing left except certain endomorphisms of P'. For the case of natural transformations from the identity morphism of an equational theory to itself, it is the monoid of unary operations which are *central*, i.e. homomorphic operations. The general equational case is similar.

Next observation, in final form:

2.6.2. *If F, G are morphisms from a reduced midtheory to a regular midtheory, there is no more than one syntactic transformation from F to G.*

Proof. For each object X of the domain we have a set of morphisms $f_i\colon P \to X$ taken to a separating family of functions by every model. Since $h_{GX}F$ is a model, there are not two elements of $h_{GX}F(X)$ mapped by every $h_{GX}F(f_i)$ to $G(f_i)$.

For reduced *theories*, accordingly, syntactic transformations partially order the interpretations into a regular theory. Note, *forward is downward*; if there is a syntactic transformation from F to G, G interprets the domain relations by smaller relations.

2.6.3. A complete midtheory $\boldsymbol{U}$ is a right small-complete category

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(1.1^{op}); so a small diagram D of morphisms K_i and natural transformations, from any midtheory $\boldsymbol{T}$ to $\boldsymbol{U}$, has a pointwise colimit L, which is a morphism of pretheories (1.1). If $\boldsymbol{T}$ is reduced, one inquires whether the K_i have a greatest lower bound. *They do if they have a common upper bound.* For then D can be set up as a pushout diagram of syntactic transformations, and L will preserve the point.

In applying 2.6.3, an obvious candidate for common upper bound would interpret all n-ary relations by the largest n-ary relation nP. Of course that need not be a morphism.

I said 2.6.2 is in final form. It can be generalized (e.g. by considering reflection $Theo \to CTh$), but not radically. Consider midfacts, for which *syntactic transformations* of morphisms $\boldsymbol{F} \to \boldsymbol{F}'$ are defined as natural transformations α such that $U' \circ \alpha = 1$. The natural statements are:

2.6.1.^{op} *A syntactic transformation of morphisms into a midfact whose grounding reflects isomorphisms is a syntactic equivalence.*

2.6.2.^{op} *If F, G are morphisms into a midfact whose grounding is faithful, there is no more than one syntactic transformation from F to G.*

The proofs are one-line; if $U'(\alpha_X)$ must be 1, the first assumption makes α_X invertible and the second makes it unique.

Now the theoretical counterpart of the First Isomorphism Theorem assumed in 2.6.1^{op} is fairly definite, though not entirely so for illegitimate or incomplete theories (which covers a great deal of territory). *P should generate $\mathcal{T}$* (in the original sense [8], not the sense of Freyd). If $\mathcal{T}$ is legitimate and right extremal-complete, this is equivalent [12] to every object's being an iterated colimit of copies of P. For a precise result 2.6.1 comparable with 2.6.1^{op} and 2.6.2, let us say that a theory $\boldsymbol{T}$ is *colimited* by a set S of its objects if $\mathcal{T}$ has no proper full subcategory $\mathcal{P}$ containing S and all vertices of conical rules whose base is small and in $\mathcal{P}$.

2.6.1. *A syntactic transformation of morphisms of midtheories must be a syntactic equivalence if the domain is colimited by its point and the codomain is regular.*

The proof is a straightforward transfinite induction over vertices of conical rules. Both morphisms take those rules to colimit cones; the syntactic transformation, being invertible on the base, is invertible at the vertex.

For the next topic it is necessary to say something about my incorrect description of *stable functors* [12, p. 647] as limits of limits of representable

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functors. The definition (a page earlier) tells us they are the iterated limits of representable functors, with the iteration in $Cat(\mathcal{C}, \mathcal{S}ets)$ as in 1.2. A brief appendix to this paper will show that two steps (limits of limits) do not suffice. Enough is saved from the erroneous proof to justify my applications in [12] and in [13].

2.7. THEOREM. *The midtheories of the form* $\mathrm{MStr}\,\boldsymbol{F}$ *for legitimate regular facts* $\boldsymbol{F}$ *are characterized up to syntactic equivalence as those* $(\mathcal{T}, P, \Gamma)$ *such that* Γ *is fine, the full subcategory* $\mathcal{K}$ *of* $\mathcal{T}$ *on the objects* T *such that* h_T *is small-valued is right adequate, and every stable covariant* $\mathcal{S}ets$*-valued functor on* $\mathcal{T}$ *that is small-valued on* $\mathcal{K}$ *is representable.*

Proof. For legitimate regular $\boldsymbol{F} = (\mathcal{T}, U, \Lambda)$, $\mathcal{T}$ is fully embedded in $\mathcal{T} = \mathrm{Ari}(\boldsymbol{F}, \mathcal{S})^{op}$ by a Yoneda functor Y, and therefore Γ consists of all colenses. $Y(\mathcal{T})$ is right adequate in $\mathcal{T}$ and contained in $\mathcal{K}$, so $\mathcal{K}$ is right adequate. Let $G: \mathcal{T} \to \mathcal{S}ets$ be a stable functor small-valued on $\mathcal{K}$. Then G is an iterated limit of representables, and GY is a small-valued iterated limit of arities, hence an arity A. Thus $G \mid Y(\mathcal{T})$ is naturally equivalent to $h^A \mid Y(\mathcal{T})$. Since G and h^A preserve all limits in $\mathcal{T}$, and every object is a canonical limit of objects in $Y(\mathcal{T})$, G and h^A are naturally equivalent.

Conversely, suppose $(\mathcal{T}, P, \Gamma)$ satisfies the given conditions. The subcategory $\mathcal{K}$ is legitimate and $h^P \mid \mathcal{K} = U$ is small-valued. Let Λ be the set of all lenses in $\mathcal{K}$ that remain lenses in $\mathcal{T}$, and put $\boldsymbol{F} = (\mathcal{K}, U, \Lambda)$. The right subregular representation $F: \mathcal{T} \to Cat(\mathcal{K}, \mathcal{S})^{op}$ is full and faithful, since $\mathcal{K}$ is right adequate. F takes $\mathcal{T}$ into $\mathrm{Ari}(\boldsymbol{F}, \mathcal{S})^{op}$ by definition of Λ, and P to the point of $\mathrm{MStr}\,\boldsymbol{F}$ by definition of U. A colens in $\mathcal{T}$ is taken to a lens by any representable h_X, whether X is in $\mathcal{K}$ or not; hence F takes it to a colens, and F is a morphism $\boldsymbol{T} \to \mathrm{Str}\,\boldsymbol{F}$.

Now consider (in the light of 1.2) the set $\mathcal{A}$ of functors in $Cat(\mathcal{K}, \mathcal{S})$ of the form $h^T \mid \mathcal{K}$, and their isomorphs. For any iterated limit L_0 of them in $Cat(\mathcal{K}, \mathcal{S}ets)$ which is small-valued, form the like iterated limit L of representable functors in $Cat(\mathcal{T}, \mathcal{S}ets)$. L is stable, and $L \mid \mathcal{K} = L_0$ is small-valued; so L is representable, and $L_0 \in \mathcal{A}$. Then $\mathcal{A}$ is $\mathrm{Ari}(\boldsymbol{K}, \mathcal{S})$ for the fact $\boldsymbol{K}$ on $\mathcal{K}$ constructed in the proof of 1.2^{op}; and inspection shows that $\boldsymbol{K}$ is $\boldsymbol{F}$. Thus F is an equivalence of categories. Since both $\mathbf{T}$ and $\mathrm{MStr}\,\boldsymbol{F}$ have all colenses, F is a syntactically invertible morphism of midtheories.

The conditions of 2.7 are a bit awkward; the generality is also great. (For instance, we did not find P in $\mathcal{K}$, and clearly it is not there in general.) Let me describe the tighter connections between $\boldsymbol{T}$, $\boldsymbol{F}$, and $\mathcal{K}$ which hold "normally" (for instance, if $\boldsymbol{T}$ is also the completion of a legitimate regular

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theory). In 2.7 we have the fact $\boldsymbol{K}$ determining $\boldsymbol{T}$. Normally, the category $\mathcal{K}$ determines $\boldsymbol{T}$. $\mathcal{K}$ has also the closure property that every stable small-valued covariant functor which has a small-valued conjugate is representable. $\mathcal{T}$ consists of all the stable small-valued functors on $\mathcal{K}$; and (normally, recall) the induced theory $\boldsymbol{T}_0$ on $\mathcal{K} \subset \mathcal{T}$ is a presentation of $\boldsymbol{T}$. One can define the *legitimate part* $\boldsymbol{T}_0$ of $\mathrm{MStr}\,\boldsymbol{F}$ for every legitimate fact whose grounding arity U is in $\mathcal{K} \subset \mathrm{MStr}\,\boldsymbol{F}$: $\boldsymbol{T}_0$ is $(\mathcal{K}, U, \Gamma_0)$, where Γ_0 consists of all colenses preserved in $\mathrm{Ari}(\boldsymbol{F}, \mathcal{S})^{op}$. This is attractive, but not functorial.

2.7.R.^{op} *The midfacts of the form* $\mathrm{MSem}\,\boldsymbol{T}$ *for legitimate regular reduced theories* $\boldsymbol{T}$ *are characterized up to syntactic equivalence as those* $(\mathcal{T}, U, \Lambda)$ *such that* Λ *is fine,* U *is representable, the full subcategory* $\mathcal{L}$ *of* $\mathcal{T}$ *on the objects* X *such that* U *is injective on every* $\mathcal{T}(X, Y)$ *is left adequate, and every stable contravariant* $\mathcal{S}$*ets-valued functor on* $\mathcal{T}$ *that is small-valued on* $\mathcal{L}$ *is representable.*

There is no difficulty in modifying the proof of 2.7 to prove this. The statements and proofs of 2.7^{op} and 2.7.R are also apparent. A more important variant, easily deduced from 2.7^{op}, is as follows. Call a fact *modeled* if every stable contravariant $\mathcal{S}$-valued functor is representable.

2.8. *A legitimate fact is* (*up to isomorphism in Fact*) $\mathrm{Sem}\,\boldsymbol{T}$ *for some legitimate regular theory* $\boldsymbol{T}$ *if and only if it is fine and modeled and its grounding is representable.*

The important characterization of facts $\mathrm{Sem}\,\boldsymbol{T}$ for small theories $\boldsymbol{T}$ is Gabriel's [7]. It will not be explicitly used in this paper, but all parts of it and the next result in Gabriel's manuscript will be used. The categories are right extremal-complete (legitimate), with a small left adequate subcategory $\mathcal{L}$ such that, for some small cardinal m, the covariant functors represented by objects in $\mathcal{L}$ commute with m-directed colimits. (A diagram is m-directed if every m objects in it have a common successor and every m morphisms $A \to B$ in it have the same composition with some $B \to C$ in the diagram.) Of course U and Λ (not present in Gabriel's formulation) are respectively representable and fine. The necessity of these conditions (all that will be used here) is not hard to prove. Some of them have been extended to $\mathrm{Mod}(\boldsymbol{T}, \mathcal{S})$ for light conical $\boldsymbol{T}$ by Freyd and Kelly (to be published); these categories are right small-complete, with a small left adequate subcategory, and co-well-powered. Gabriel proved co-well-poweredness for $\boldsymbol{T}$ small.

In 2.7 and 2.8, "regular" (combined with "legitimate") is not redun-

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dant. Of course regular theories suffice for semantics, since complete theories are regular. One thinks of regularizing a singular $\boldsymbol{T}$, or passing to the "image" of $\eta_{\boldsymbol{T}}$. It is not the same, and I doubt if every (pre-, mid-) theory has a regular reflection. (Many do. Even if they do, the image of $\eta_{\boldsymbol{T}}$ is different and more important.)

The image we want is not, I think, a categorical image in MTh. Here we are trying to get closer to the models, not to respect the fine structure of theories; and we care only about legitimate-valued models, which reduce by 1.0.2 to $\mathcal{S}$-valued models.

The construct wanted here is the *full image* of $\eta_{\boldsymbol{T}}$ or (equivalently) the *fully regular* reflection of $\boldsymbol{T}$. The basic definitions: a *subtheory* is a categorical subobject in MTh, i.e. a subobject represented by a monomorphism, which (clearly) is the same as a one-to-one morphism. A morphism of MTh is *full* if it is a full functor, and a subtheory is *full* if its category is a full subcategory. The *full image* of $F\colon \boldsymbol{T}_1 \to \boldsymbol{T}_2$ is the factorization across the smallest possible full subtheory of $\boldsymbol{T}_2$.

The full image may be seen to be functorial from MTh^2 to MTh^3 (from morphisms in MTh to diagrams $A \to B \to C$ in MTh). Hence the full image object $\boldsymbol{T}_r$ of $\eta_{\boldsymbol{T}}$, the middle object of the full image diagram, is a functor of $\boldsymbol{T}$. Call it $\mathrm{Frr}\,\boldsymbol{T}$. The whole diagram is a factorization of $\eta\colon 1 \to \mathrm{Frr} \to \mathrm{Str}\,\mathrm{Sem}$. Call the initial factor $\rho\colon 1 \to \mathrm{Frr}$ *fully regulated reflection*. It remains to call $\boldsymbol{T}$ *fully regular* if $\rho_{\boldsymbol{T}}$ is an isomorphism, and to check:

2.9. *Fully regular reflection is a reflection upon the full subcategory of fully regular midtheories.*

Proof sketch. Observe that a model M of $\boldsymbol{T}$ in $\mathcal{E}$ is the same thing as an interpretation of $\boldsymbol{T}$ in $(\mathcal{E}^{op}, M(\mathrm{P}), \Gamma)$, where Γ is the set of all colenses. A theory is fully regular if and only if it is embeddable as a full subtheory of some $\mathrm{MStr}\,\boldsymbol{F}$, which is if and only if it is fully embeddable in a product of regular midtheories on $\mathcal{S}^{op}$. Then $\boldsymbol{T}_r$ is fully regular. Any morphism $\boldsymbol{T} \to \boldsymbol{U}$ is pushed by ρ to $\boldsymbol{T}_r \to \boldsymbol{U}_r$; if $\boldsymbol{U}$ is fully regular, $\boldsymbol{T} \to \boldsymbol{U}$ factors $\boldsymbol{T} \to \boldsymbol{T}_r \to \boldsymbol{U}_r = \boldsymbol{U}$. Uniqueness of the factorization (into a product of theories carried by $\mathcal{S}^{op}$) reduces to equality of two models of $\boldsymbol{T}_r$ in $\mathcal{S}$ when they are equal on $\boldsymbol{T}$, which routine analysis proves.

The device used here, of converting models to morphisms, is important. I doubt if the artificial theories on $\mathcal{S}^{op}$ will bear any semantic significance. (Their categories of models are equivalent to $\mathcal{S}^{op}$, of course.) The point is that "models are morphisms."

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2.10. COROLLARY. *The reflection map $\rho_{\boldsymbol{T}}$ is a presentation, for every midtheory $\boldsymbol{T}$.*

Proof sketch. MSem $\rho_{\boldsymbol{T}}$ is not merely an equivalence, it is an isomorphism. For the models M of $\boldsymbol{T}$ in $\mathcal{S}$ are the morphisms M_1 from $\boldsymbol{T}$ to certain fully regular midtheories $\boldsymbol{U}$ on $\mathcal{S}^{op}$, which factor uniquely as $M_2\rho_{\boldsymbol{T}}$, $M_2\colon \boldsymbol{T}_r \to \boldsymbol{U}$; and M_2 is the same as a model M_3 of $\boldsymbol{T}_r$ in $\mathcal{S}$. One readily checks that the bijection $M \mapsto M_3$ extends to morphisms of models and preserves the groundings and coreguli.

To justify the terminology one should remark: a fully regular theory is regular, a legitimate regular theory is fully regular.

It is not trivial that the fully regular reflection of a small theory $\boldsymbol{T}$ is legitimate. It is trivial, given Gabriel's unpublished result that $\mathrm{Mod}(\boldsymbol{T}, \mathcal{S})$ is reflective in $Cat(\mathcal{J}^{op}, \mathcal{S})$; for the category of $\boldsymbol{T}_r$ is Gabriel's reflection of the left regular (Yoneda) representation of $\mathcal{J}$.

The fully regular reflection of a legitimate theory need not be legitimate. Consider a category of ∞ objects A_i, each with endomorphism monoid Z_2, with two morphisms $A_i \to A_0$ for each i (on which the endomorphism groups act freely), and a rule that A_0 ought to be the coproduct of the others. A model in $\mathcal{S}$ then consists of a family of sets S_i on which Z_2 acts, S_0 being the product of the others. If no S_i is empty, almost all of them will be singletons. But in the fully regular reflection, the A_i are undamaged and A_0 becomes their coproduct, with more than ∞ endomorphisms.

One can attach "R" and "^{op}" to 2.9 and 2.10 freely, with trivial variations of the proofs. One of these variants calls for comment: the reduced-fully regular facts admit a simple characterization. Call a midfact *faithful* if its grounding is faithful.

2.11. *The adjunction map $\boldsymbol{F} \to \mathrm{MSem}\,\mathrm{MStr}^R \boldsymbol{F}$ is full and faithful if and only if $\boldsymbol{F}$ is regular and faithful.*

Proof. The necessity is obvious. On the other hand, if F is regular and faithful, we get a somewhat stronger conclusion which for later reference will be labeled:

2.11.1. *Every representable arity is small-valued and reduced.*

For a faithful fact is legitimate. The natural transformations from h^X to U are determined (Yoneda lemma) by the elements of $U(X)$, which are small in number. And the universal map from h^X to a power of U with factors indexed by $U(X)$ is monic.

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In the Introduction I suggested that going from ruled to reduced semantics amounted to adding rules to make the facts faithful. In a quite insufficient but precise sense, 2.11 justifies that. A real justification would be a proof that every fact that is faithful and ruled-complete is reduced-complete (Sem of a reduced theory). I do not know whether it is true.

Part of the point of reduced semantics is to diminish the association of illegitimate theories to legitimate facts. One might a priori want to stop it; but then one could not treat fairly reasonable facts such as the complete Boolean algebras. Most of the hypothesis of the next result is in this ad hoc definition of a *solid* fact $(\mathcal{J}, U, \Lambda)$. $\mathcal{J}$ is legitimate, right wide-complete, left extremal-complete, and satisfies the ascending chain condition on quotients. (The chain condition is, of course, that there is no well-ordered chain of order type ∞. This is not the first time it has been found useful [13].) U is representable. Λ consists of lenses and contains all small limits.

From the results of Gabriel and of Freyd and Kelly recalled above (after 2.8), *the fact of models of a light conical theory is solid.* The required properties of $\mathcal{J}$ are immediate from those results, except for the left extremal-completeness, which follows from right (wide-) completeness in the presence of a small left adequate subcategory [12; 4.7]. The required representation of U is by the reflection of h^P *in* $\mathrm{Mod}(\boldsymbol{T}, \mathcal{S})$, which is reflective (Gabriel, Freyd-Kelly). Since it is reflective, Λ is the fine regulus.

2.12. *A reduced arity on a solid fact is representable.*

Proof. $(\mathcal{J}, U, \Lambda)$ being solid, there is a representation of U as h^P for some P. A reduced arity A is a subfunctor of a small power of U, which is h^Q for the corresponding copower Q of P. If R is the intersection of all quotients R_i of Q such that h^{R_i} contains A, then h^R contains A. For every proper quotient S_α of R, there is $f_\alpha\colon R \to T_\alpha$ in $A(T_\alpha)$ but not factoring across S_α. Start an induction with the product S_0 of the empty family; since A is an arity, $R \to S_0$ is in $A(S_0)$. Having $R \to S_\alpha$ in $A(S_\alpha)$, S_α a proper quotient, take T_α as above and $V_\alpha = S_\alpha \times T_\alpha$. The maps that we have from R to S_α and T_α yield $R \to V_\alpha$, which is in $A(V_\alpha)$ from its coordinates. Now the right image of $R \to V_\alpha$ is a multistrict subobject $S_{\alpha+1}$ of V_α; since A is an arity, $R \to S_{\alpha+1}$ is in $A(S_{\alpha+1})$ by a secondary induction. We have $R \to S_{\alpha+1} \to S_\alpha$; since $R \to T_\alpha$ does not factor across S_α, $S_{\alpha+1}$ is a larger quotient than S_α. Similarly, at a limit ordinal β, the preceding $R \to S_\alpha$ (each in $A(S_\alpha)$) yield a map into a product and upon a right image S_β (in $A(S_\beta)$). The ascending chain of quotients must end at a small ordinal, and it can end only when S_α is not a proper quotient. So 1_R is in $A(R)$, and $A = h^R$.

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2.12.1. COROLLARY. *The reduced completion* $\mathrm{MStr}^R\,\mathrm{MSem}\,\boldsymbol{T}$ *of a light conical reduced theory* $\boldsymbol{T}$ *is legitimate, and the adjunction map* $\eta_{\boldsymbol{T}}$ *is left adequate.*

For, since all reduced arities are representable, the category (of) $\mathrm{MStr}^R\,\mathrm{MSem}\,\boldsymbol{T}$ and functor $\eta_{\boldsymbol{T}}$ are equivalent to $\mathrm{MSem}\,\boldsymbol{T}$ and a reflected Yoneda representation. Another by-product of 2.12 seems to deserve a more distinctive label; it extends Lawvere's result that algebraic functors have adjoints. (Precisely, it extends only the case of degree 1, but the general algebraic case is immediate from that subcase.)

2.13. *The functor underlying a morphism from a solid fact to a regular faithful fact has a left adjoint.*

Proof. For the underlying functor of $G\colon \boldsymbol{F}_1 \to \boldsymbol{F}_2$ to have an adjoint, the functors $h^X G = \mathrm{MStr}\,G(h^X)$ must be representable. By 2.11.1, h^X is reduced; so $\mathrm{MStr}\,G(h^X)$ is reduced; so by 2.12, it is representable.

In the next result, *CTh* is $\mathfrak{R}\,CTh$ of 1.9.1. Recall that *CFa* is equivalent to CTh^{op}.

2.14. *PTh, MTh, and MFa have limits and colimits of diagrams such that the family of underlying categories of the objects has an ami coproduct in Cat. Theo, Fact, and CTh have products and coproducts of such families of objects, but none of the three has equalizers or coequalizers.*

Proof sketch. The first sentence can be proved by a straightforward string of explicit constructions of products, equalizers, and the duals. I mean explicit relative to the corresponding constructions in *Cat*; those also are explicitly known, but the construction of a coequalizer is mildly complex and one does not need the details (for the present purpose). One does need to note that in *MTh*, product theories have product underlying categories, and the category of a coproduct is a disjoint union with points identified; and similarly in *MFa*, product facts are on pullback categories and coproduct facts on coproduct categories. Then next, the quotient maps $MTh \to Theo$ and $MFa \to Fact$ preserve products and coproducts. Specifically, (1) each $MTh(\boldsymbol{T}, \times \boldsymbol{T}_i) \cong \times MTh(\boldsymbol{T}, \boldsymbol{T}_i)$ reduces to its set of syntactic equivalence classes, which is evidently the same as the (product) set of coordinatewise syntactic equivalence classes. (2) Equally evidently, syntactic equivalence in $MTh(+\boldsymbol{T}_j, \boldsymbol{T})$ is coordinatewise. (3-4) The like identifications in *MFa* are also evident; homming into a product, this depends on the requirement $U \circ \alpha = 1$ for a syntactic equivalence.

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Since *CTh* is reflective in *Theo*, it is closed under products; since *CTh* is full, those products are *CTh*-products. The reflection takes coproducts to *CTh*-coproducts.

The six negative assertions will follow from the absence of equalizers and coequalizers in *CTh* (hence in its dual *CFa*. If *Fact* had equalizers *CFa* would be closed under them, coequalizers would reflect, and similarly for *Theo*). Consider the following pointed categories as theories with empty regulus. $\boldsymbol{U}^3$ consists of P, X, and five non-identity morphisms, namely an involution s of X and $t, u, ts, us: X \to P$. $\boldsymbol{U}^2$ is the subcategory omitting u and us; $\boldsymbol{U}^1$ is the subcategory omitting also t and ts. Consider the following morphisms of midtheories. $I: \boldsymbol{U}^1 \subset \boldsymbol{U}^2$; $J: \boldsymbol{U}^2 \subset \boldsymbol{U}^3$; $K: \boldsymbol{U}^2 \to \boldsymbol{U}^3$ extending JI but taking t to u, ts to us; $L: \boldsymbol{U}^3 \to \boldsymbol{U}^3$ exchanging t with ts. In *MTh*,

$$\begin{array}{ccc} \boldsymbol{U}^1 & \xrightarrow{I} & \boldsymbol{U}^2 \\ {\scriptstyle I}\downarrow & & \downarrow{\scriptstyle J} \\ \boldsymbol{U}^2 & \xrightarrow{K} & \boldsymbol{U}^3 \end{array}$$

is a pushout. But not in *Theo*, for $\boldsymbol{U}^3$ has the automorphism L ($\neq 1$ in *Theo*) such that $LJ = J$ in *Theo*, while $LK = K$ already in *MTh*. Also, while K equalizes L and 1 in *MTh*, an equalizer in *Theo* would have to let J factor through it. Even after applying Str Sem, the indicated pushout $\boldsymbol{V}$ and equalizer $\boldsymbol{W}$ will not exist. That is, after applying MStr MSem $= {}^\circ$, the "syntactic pushout" $\boldsymbol{V}$ and "syntactic equalizer" $\boldsymbol{W}$ do not exist. For that, note that each $\boldsymbol{U}^i$ goes fully and one-to-one into $\boldsymbol{U}^{i\circ}$. Suppose $\boldsymbol{W}$ exists. It has a unique point P and an object X unique up to isomorphism, the image of X from $\boldsymbol{U}^{1\circ}$. Since X has only two automorphisms, s and 1, determining it up to isomorphism suffices to tell the morphisms t and ts from u and us though not from each other. Then up to isomorphism, the image of $\boldsymbol{W}$ in $\boldsymbol{U}^{3\circ}$ contains the images of J and of K, hence all of $\boldsymbol{U}^3$. (More precisely, select a copy of X in $\boldsymbol{W}$ and let $\mathcal{W}$ be the full subcategory on P and X; note that $\boldsymbol{W} \to \boldsymbol{U}^{3\circ}$ takes $\mathcal{W}$ into a copy of $\boldsymbol{U}^3$, unique up to a syntactic equivalence from $1: \boldsymbol{U}^{3\circ} \to \boldsymbol{U}^{3\circ}$ to some other endofunctor. Factoring J° through $\boldsymbol{W}$ in *Theo* gives $J_0: \boldsymbol{U}^2 \subset \boldsymbol{U}^{2\circ} \to \boldsymbol{W}$, which, followed by $\boldsymbol{W} \to \boldsymbol{U}^{3\circ}$, is syntactically equivalent to $J: \boldsymbol{U}^2 \to \boldsymbol{U}^3 \subset \boldsymbol{U}^{3\circ}$; so the image of $\mathcal{W}$ contains the equivalent of the image of J. Similarly it contains the equivalent of the image of K. This makes the equivalent of $\boldsymbol{U}^3$.) So $\boldsymbol{W} \to \boldsymbol{U}^{3\circ}$ does not equalize L and 1 up to syntactic equivalence; contradiction.

Finally, the argument since "Suppose $\boldsymbol{W}$ exists" is also justified by the

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alternative assumption that $\boldsymbol{W}$ is the desired pushout; so that doesn't exist either.

2.14.1. The category of deflated facts is as complete as *MFa*, because it is isomorphic with a full reflective subcategory of *MFa*. (To amplify "as complete": if they have a limit it is deflated; if they have a colimit they have a deflated colimit.) The category of reduced-complete facts is equivalent (2.5.1) to that of reduced-complete deflated facts, which is reflective in the deflated facts; to reflect (deflated) $\boldsymbol{F}$, pass to $\mathrm{Sem}\,\mathrm{Str}^R \boldsymbol{F}$ in *Fact* and then to its isomorphic deflated reflection. We conclude: *the categories of reduced-complete facts and of complete reduced theories have limits and colimits of bounded diagrams.*

Chapter III. Tensoring and connections.

The metacategory of pretheories *PTh* is a monoidal closed category $(\mathcal{V}_0, \otimes, I, r, l, a; p; V, \hom \mathcal{V}, i, j, L)$ as defined in the first 55 pages of [3]. Fortunately Eilenberg and Kelly have provided a test much simpler than the definition, in II.5.7 of [3]. First, a monoidal category is merely a category $\mathcal{V}_0$ with a binary operation $\otimes$ and a constant I making it a monoid up to coherent isomorphic corrections r, l of I to a right and left identity and a of $\otimes$ to associativity. (Details are given e.g. in [3], pages 471-472.) A *normalization* of it is a set-valued functor V naturally equivalent to h^I (with a distinguished natural equivalence). Assume the weak transportability condition that every bijection $V(\boldsymbol{T}) \to X$ is $V(f)$ for some isomorphism $f: \boldsymbol{T} \to \boldsymbol{T}'$. Then the normalized monoidal structure underlies a (unique) monoidal closed structure if and only if each of the functors $\mathrm{Hom}(_ \otimes \boldsymbol{U}, \boldsymbol{V})$ is representable.

The tensor product $\boldsymbol{T}_1 \otimes \boldsymbol{T}_2$ of two objects $\boldsymbol{T}_j = (\mathcal{T}_j, \Gamma_j)$ of *PTh* is $\mathcal{T}_1 \times \mathcal{T}_2$ with the following "wedge" regulus Γ_w. For each rule H in Γ_1 and object Y of $\mathcal{T}_2$, let us say $H = (H^f, H^t): D \to E$, $D: \mathcal{A} \to \mathcal{T}_1$, $E: \mathcal{B} \to \mathcal{T}_1$; we form D^Y (and E^Y) into $\mathcal{T}_1 \times \mathcal{T}_2$ with coordinates D (resp. E) and the constant 1_Y-valued functor, and $H^Y = (H^f, H^t \times 1): D^Y \to E^Y$. For each object X of $\mathcal{T}_1$ and rule H in Γ_2, we form ${}^X H$ similarly. Γ_w consists of all ${}^X H$ and H^Y.

Morphisms $\boldsymbol{T}_j \to \boldsymbol{T}_j'$ $(j = 1, 2)$ have a product functor $\mathcal{T}_1 \times \mathcal{T}_2 \to \mathcal{T}_1' \times \mathcal{T}_2'$, which is readily verified to take the wedge regulus into the wedge regulus; so we have a bifunctor $\otimes$. The identity object I is the category 1 with the empty regulus. Then r, l, a, and the verification of coherence are precisely the same as for the monoidal category on *Cat*, $\times$, 1; one need only observe that the morphisms lift into *PTh*.

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For transportability, we define $V(\boldsymbol{T})$ as the set of objects of $\mathcal{T}$ (with the obvious definition of V on morphisms). This is evidently naturally equivalent to h^I. It remains to construct the hom theories $(\Omega(\boldsymbol{U},\boldsymbol{V}),\Gamma_p)$ which represent $\mathrm{Hom}(_\otimes\boldsymbol{U},\boldsymbol{V})$. $\Omega(\boldsymbol{U},\boldsymbol{V})$ is the category of morphisms from $\boldsymbol{U}$ to $\boldsymbol{V}$ and natural transformations. The "pointwise" regulus Γ_p consists of those morphisms of diagrams $H: D\to E$ in $\Omega(\boldsymbol{U},\boldsymbol{V})$ which are taken to rules of $\boldsymbol{V}$ by all the evaluations $F\mapsto F(X)$ (X an object of $\boldsymbol{U}$).

Using Eilenberg-Kelly's notation $(\boldsymbol{UV})$ for hom theories, we have to see the bijections between $\mathrm{Hom}(\boldsymbol{T},(\boldsymbol{UV}))$ and $\mathrm{Hom}(\boldsymbol{T}\otimes\boldsymbol{U},\boldsymbol{V})$, natural in T. Again the construction is familiar for *Cat* and one need only check that it lifts. Both homsets consist of functors identifiable (naturally) with functors $\mathcal{T}\times\mathcal{U}\to\mathcal{V}$. Such a functor gives a morphism from $\boldsymbol{T}$ to $(\boldsymbol{UV})$ provided it takes objects of $\boldsymbol{T}$ into $\Omega(\boldsymbol{U},\boldsymbol{V})$ and rules of $\boldsymbol{T}$ into Γ_p; this is the same as taking the wedge regulus into the regulus of $\boldsymbol{V}$. In view of [3], we have proved:

3.1. THEOREM. *PTh is monoidal closed with the tensor product* $(\mathcal{T}_1\times\mathcal{T}_2,\Gamma_w)$ *and the grounding* $|\mathcal{T}|$.

The grounding is a natural one, the tensor product is a simple one. It is also semantically sound; for the adjointness of $_\otimes\boldsymbol{U}$ and $(\boldsymbol{U}_)$, and the correspondence between models in $\mathcal{C}$ and morphisms into suitable theories on $\mathcal{C}^{op}$ (seen in the proof of 2.9), yield the usual semantic description of $\otimes$. (Incidentally, $\otimes$ is coherently commutative.)

Since *PTh* is monoidal closed, so is the isomorphic category *PFa*. Accordingly it has a multiplication which for the moment we may call $[x]$ and a "hom" which for the moment will be $[\ ]$.

For theories with points, we supply $\boldsymbol{T}_1^-\otimes\boldsymbol{T}_2^-$ $(\boldsymbol{T}_j^-=(\mathcal{T}_j,\Gamma_j))$ with the point (P_1,P_2). Evidently this yields a bifunctor $\otimes$ on *MTh*, still coherently associative. We shall see shortly that it is still semantically sound.

Tensor products of syntactically equivalent morphisms are easily seen to be syntactically equivalent, so $\otimes$ gives rise also to a coherently associative bifunctor on *Theo*. For that matter, syntactic transformations also multiply. *For the rest of this paper, that structure will not be mentioned.* We work in midtheories and midfacts and systematically omit "mid-" and "*M*."

We need to define the *fact of models* $\mathrm{Sem}(\boldsymbol{T};\boldsymbol{F})$ of a theory $\boldsymbol{T}=(\mathcal{T},P,\Gamma)$ in a fact $\boldsymbol{F}=(\mathcal{F},U,\Lambda)$, The underlying prefact is the hom fact $[\boldsymbol{T}^0\boldsymbol{F}^-]$, where $\boldsymbol{T}^0=(\mathcal{T}^{op},\Gamma^{op})$. To amplify, the underlying category $\mathrm{Mod}(\boldsymbol{T};\boldsymbol{F})=\Omega(\boldsymbol{T}^-,\boldsymbol{F}^0)^{op}$ consists of the functors $M:\mathcal{T}^{op}\to\mathcal{F}$ which take rules in Γ to corules in Λ, and their natural transformations. The coregulus of $\mathrm{Sem}(\boldsymbol{T};\boldsymbol{F})$

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consists of those comorphisms of diagrams in $\mathrm{Mod}(\boldsymbol{T};\boldsymbol{F})$ which are taken to corules of $\boldsymbol{F}$ by all the evaluations $ev^X(M \mapsto M(X))$, X an object of $\boldsymbol{T}$. Finally, the grounding of $\mathrm{Sem}(\boldsymbol{T};\boldsymbol{F})$ is ev^P followed by U, $H \mapsto UH(P)$; this composite preserves the coregulus since its factors do. Having the fact $\mathrm{Sem}(\boldsymbol{T};\boldsymbol{F})$, note that ev^P is now a morphism, the *forgetful functor*.

3.2. *For any two theories and a fact,* $\mathrm{Sem}(\boldsymbol{T}_1 \otimes \boldsymbol{T}_2;\boldsymbol{F})$ *is isomorphic with* $\mathrm{Sem}(\boldsymbol{T}_1;\mathrm{Sem}(\boldsymbol{T}_2;\boldsymbol{F}))$.

The routine proof (note. *isomorphic*) is omitted. To conclude that $\otimes$ is semantically sound, one should specialize the just-invented notion of models in a fact, and note that if we are modeling in the category $\mathcal{S}$, and $\mathrm{Sem}\,\boldsymbol{T}_2$ is fine, then 3.2 means what it appears to mean. Indeed we have a deviation if $\mathrm{Sem}\,\boldsymbol{T}_2$ is not fine. Note that 2.7^{op} gives a very sweeping class of facts $\mathrm{Sem}\,\boldsymbol{T}$ that are fine. We must return to fineness below; 3.5 gives weaker conditions for it. It should be noted now that $\mathrm{Sem}\,\boldsymbol{T}$ is fine if $\boldsymbol{T}$ is small. (This follows from 2.7^{op} and the legitimacy of the fully regular reflection of $\boldsymbol{T}$, which, as was already mentioned, follows from Gabriel's results.)

Note, too:

3.3. *The tensor product of two presentations is a presentation.*

For this, since $\otimes$ is commutative (up to isomorphism), one need only observe that a presentation $\boldsymbol{T}_2 \to \boldsymbol{T}_2'$, making $\mathrm{Sem}(\boldsymbol{T}_2';\mathcal{S}) \to \mathrm{Sem}(\boldsymbol{T}_2;\mathcal{S})$ an equivalence, will do the same for

$$\mathrm{Sem}(\boldsymbol{T}_1;\mathrm{Sem}(\boldsymbol{T}_2';\mathcal{S})) \to \mathrm{Sem}(\boldsymbol{T}_1;\mathrm{Sem}(\boldsymbol{T}_2;\mathcal{S})).$$

Rephrased, $\otimes$ preserves semantic equivalence. Semantic equivalence does not mean having the same (up to equivalence) models in every category; not even the algebraic [17] and varietal [18] theories of groups do that. (Consider their models in the category of finite sets.) But 3.3 gives us the important proposition

3.3.a. *Semantically equivalent theories have equivalent models in every complete fact.*

(A reader who would prefer a more formal statement can readily formulate one.)

Having lifted tensor and hom(Sem(;)), we are ready to go forward; but a side road should be mentioned (and can be skipped). There are fourteen conceivable ways to lift tensor (considered as commutative) or hom by supplying points or groundings. All of them exist in some generality,

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modulo a bookkeeping problem that I have not examined. This will be clear if we examine the one that seems third in interest. If $\boldsymbol{F}_1$, $\boldsymbol{F}_2$ are regular and well behaved (for instance, small) and their groundings are representable by P_1 and P_2, usually (P_1, P_2) will not represent an arity on $\boldsymbol{F}_1^-[x]\boldsymbol{F}_2^-$, but it will as soon as we pass to the fully regular reflection. For well behaved $\boldsymbol{F}_j$, that arity will be small-valued. (Bookkeeping: it is determined only, like the P_j, up to isomorphism. I suppose it can be made functorial, like product and coproduct.) This "optensoring" of facts is not like tensoring the theories of the same facts. For instance, $\boldsymbol{S}[x]\boldsymbol{S}$ reduces to the category 1 $(=(\boldsymbol{S}^{op} \otimes \boldsymbol{S}^{op})_r)$. So does $\boldsymbol{S}[x]\boldsymbol{C}$ for a fine fact $\boldsymbol{C}$ on a category having a two-sided zero object. The multiplication is monoidal on well behaved complete facts (up to equivalence, and applying Sem Str instead of merely $(\ )_r$); the identity is the fact of complete atomic Boolean algebras with every object grounded to a singleton. Optensoring the usual fact of complete atomic Boolean algebras with a suitable fact $(\mathcal{F}, U, \Lambda)$ yields $(\mathcal{F}, V, \Lambda)$, where V is represented by the square of an object representing U. One can compute a little more for other known facts with known duals.

Returning to $\otimes$, we observe that the central fact $\mathrm{Sem}(\boldsymbol{T}_1 \otimes \boldsymbol{T}_2)$ is determined by $\mathrm{Sem}\,\boldsymbol{T}_1$ and $\mathrm{Sem}\,\boldsymbol{T}_2$, as $\mathrm{Sem}(\mathrm{Str}\,\mathrm{Sem}\,\boldsymbol{T}_1 \otimes \mathrm{Str}\,\mathrm{Sem}\,\boldsymbol{T}_2)$, by 1.9 and 3.3. In simple cases (groups and topological spaces were noticed in the Introduction) it results from a simpler construction than $\mathrm{Sem}(\mathrm{Str}(\ ) \otimes \mathrm{Str}(\ ))$, or rather, that operation of "cotensoring" has a simpler description. Note, the formula defines cotensoring at once[1] on $Fact \times Fact$ to $Fact$. The shortcut runs naturally from $Cat \times Cat$ to Cat, though we can elevate the source so as to reach $Fact$.

Recall that a connection from a category $\mathscr{C}$ to a category $\mathcal{D}$ is a contravariant functor from $\mathscr{C}$ to $\mathcal{D}$ having an adjoint on the right. The connections from $\mathscr{C}$ to $\mathcal{D}$, with their natural transformations, form a category $\mathscr{C}(x)\mathcal{D}$, the *connection product*. Evidently $\mathscr{C}(x)\mathcal{D}$ is equivalent to $\mathcal{D}(x)\mathscr{C}$. It is equivalent also to the symmetrically defined category whose objects (S, T, a) are pairs of contravariant functors in both directions with an adjunction on the right, with the obvious morphisms. That construct is probably more convenient for some purposes, but we seem to need a little asymmetry.

The connection product $\mathfrak{C}(x)\boldsymbol{F}$ of a pointed category $\mathfrak{C} = (\mathscr{C}, P)$ and a fact $\boldsymbol{F} = (\mathcal{F}, U, \Lambda)$ is a fact defined as follows. Its category is $\mathscr{C}(x)\mathcal{F}$; its grounding is Uev^P; and it has the pointwise coregulus (as in $[(\mathscr{C}, \emptyset)\boldsymbol{F}^-]$). Pointwise corules go by ev^P to corules and then by U to lenses; so $\mathfrak{C}(x)\mathcal{F}$ is indeed a fact.

[1] Too fast. Probably this construction can be profitably analyzed as the completion of some more delicate construction.

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The theorem we want is to the effect that $\mathrm{Sem}(\boldsymbol{T}_1 \otimes \boldsymbol{T}_2)$ is constructible from $\mathrm{Sem}\,\boldsymbol{T}_1$ and $\mathrm{Sem}\,\boldsymbol{T}_2$ as a connection product. This theorem seems as sure as the Special Adjoint Functor Theorem to develop in several versions with no useful sharpest version. Like the Special Adjoint Functor Theorem, it has a convenient version that is liksly to suffice for most work concerned with particular facts rather than the structure of *Fact*. To wit: *it suffices that both* $\mathrm{Sem}\,\mathbf{T}_1$ *and* $\mathrm{Sem}\,\boldsymbol{T}_2$ *satisfy* 2.8.

This is not the place to exhibit a basic library of variants of the Connection Product Theorem, but two are needed. For one thing, (a) below contains most of Pultr's version [24] and (b) contains most of Freyd's [6].

3.4. THEOREM. *If both* $\mathrm{Sem}\,\boldsymbol{T}_1$ *and* $\mathrm{Sem}\,\boldsymbol{T}_2$ *are legitimate, and the grounding of* $\mathrm{Sem}\,\boldsymbol{T}_1$ *is represented by* R, *then* $\mathrm{Sem}(\boldsymbol{T}_1 \otimes \boldsymbol{T}_2)$ *is syntactically equivalent to* $\mathfrak{C}(x)\ \mathrm{Sem}\,\boldsymbol{T}_2$ *where* $\mathfrak{C} = (\mathrm{Mod}(\boldsymbol{T}_1, \mathcal{S}), R)$, *if either*

(a) $\mathrm{Sem}\,\boldsymbol{T}_1$ *is fine and modeled and* $\mathrm{Sem}\,\boldsymbol{T}_2$ *is fine, or*

(b) $\mathrm{Sem}\,\boldsymbol{T}_1$ *is fine, and every contravariant functor from it to* $\mathcal{S}$ *that preserves small colimits is representable.*

Proof. Consider the metacategory $\mathrm{Hom}!\boldsymbol{T}_j$ (union of all $Cat(\mathcal{T}_j{}^{op}, \mathcal{S}ets)$, from the proof of 1.4), which contains both $\mathrm{Mod}(\boldsymbol{T}_j, \mathcal{S})$ and a Yoneda representation of $\mathcal{T}_j$ which we shall identify with $\mathcal{T}_j$. For any theories, a model M' of $\boldsymbol{T}_1$ in $\mathrm{Sem}\,\boldsymbol{T}_2$ can be extended to a connection $K(M)$ from $\mathrm{Hom}!\boldsymbol{T}_1$ to $\mathrm{Hom}\,!\boldsymbol{T}_2$. This is a standard Kan extension construction, yielding an adjoint pair because we have an adjointness relation from 3.2; so a sketch will suffice. First, 3.2 and commutativity of $\otimes$ identify the models M' (isomorphically) with models M of $\boldsymbol{T}_1 \times \boldsymbol{T}_2$ (in $\mathcal{S}$) and with models M'' of $\boldsymbol{T}_2$ in $\mathrm{Sem}\,\boldsymbol{T}_1$. For objects A of $\boldsymbol{T}_1$, B of $\boldsymbol{T}_2$, we have (literal equality) $M'(A)(B) = M''(B)(A)$. Having embedded the four categories involved into two, $\mathrm{Hom}!\boldsymbol{T}_j$, we get identifications $\mathrm{Hom}(B, M'(A)) \sim \mathrm{Hom}(A, M''(B))$. Now $\mathcal{T}_j$ is left adequate in $\mathrm{Hom}\,!\boldsymbol{T}_j$, so each object X is the colimit of its cospectrum [14] in $\mathcal{T}_j$. That is a bounded diagram. Hence, writing $X = \mathrm{colim}\,A_i$ (X in $\mathrm{Hom}!\boldsymbol{T}_1$), we can define $K(M)(X) = \lim M'(A_i)$ in $\mathrm{Hom}!\boldsymbol{T}_2$. All the morphisms $A \to X$ are coordinates in the representation of X as colimit of its cospectrum; $K(M)$ takes them to corresponding coordinate morphisms. Morphisms $X \to X'$ are determined by their coordinates $A_i \to X'$, and $K(M)$ treats them accordingly. Defining $K'(M) : (\mathrm{Hom}!\boldsymbol{T}_2)^{op} \to \mathrm{Hom}!\boldsymbol{T}_1$ in the same way, the relation of adjointness on the right for $X = \mathrm{colim}\,A_i$, $Y = \mathrm{colim}\,B_k$ reduces to $\lim\lim \mathrm{Hom}(B_k, H'(A_i)) \sim \lim\lim \mathrm{Hom}(A_i, M''(B_k))$. These are not subtle iterated limits but morphisms from a colimit to a limit, and adjointness holds.

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$K(M)(X)(B) = \mathrm{Hom}(B, K(M)(X)) = \mathrm{Hom}(X, M''(B))$ will be small if X is (with $M''(B)$) in $\mathrm{Sem}\,\boldsymbol{T}_1$ and $\mathrm{Sem}\,\boldsymbol{T}_1$ is legitimate. In that case $K(M)(X)$ is a small set-valued model of $\boldsymbol{T}_2$, and $K(M)$ (determined up to natural equivalence) may be chosen to take $\mathrm{Sem}\,\boldsymbol{T}_1$ into $\mathrm{Sem}\,\boldsymbol{T}_2$. If $\mathrm{Sem}\,\boldsymbol{T}_2$ is also legitimate, $K'(M)$ may be chosen to take $\mathrm{Sem}\,\boldsymbol{T}_2$ into $\mathrm{Sem}\,\boldsymbol{T}_1$. In that case we have a connection $I(M)$ from $\mathfrak{C}$ to $\mathrm{Sem}\,\boldsymbol{T}_2$, obtained by extension from the model M. We have, moreover, a full faithful functor I from $\mathrm{Sem}(\boldsymbol{T}_1 \otimes \boldsymbol{T}_2)$ into $\mathfrak{C}(x)\,\mathrm{Sem}\,\boldsymbol{T}_2$, as follows. Functor, because a natural transformation $M \to M_1$ corresponds to a unique natural transformation $M' \to M_1'$, which has a unique extension $K(M) \to K(M_1)$; the morphisms $K(M)(X) \to K(M_1)(X)$ must be defined by limits, and (one verifies) this definition secures naturality. By uniqueness, K (taking M to $K(M)$) is functorial, so its restriction I is too. As for fidelity, two natural transformations $M' \to M_1'$ differing at some object A of $\mathcal{T}_1$ yield different natural transformations $h^Y M' \to h^Y M'_1$, where $Y = M'(A)$. By extension and adjunction we get different natural transformations $h^A I'(M) \to h^A I'(M_1)$. Therefore the natural transformations $I'(M) \to I'(M_1)$ are different; and so must be the adjointed ones $I(M) \to I(M_1)$. Fullness is found on the same path in the opposite direction. Given a natural transformation $\alpha: I(M) \to I(M_1)$, form a longitudinal composition with $M'': \mathcal{T}_2{}^{op} \to \mathrm{Mod}(\boldsymbol{T}_1, \mathcal{S})$, and step up to $\mathrm{Hom}(B, I(M)M''B') \to \mathrm{Hom}(B, I(M_1)M''B')$. I in this position can be replaced by K; and adjunction yields (for B, B' in $\mathcal{T}_2$)

$$\phi: \mathrm{Hom}(M''B', K'(M)B) \to \mathrm{Hom}(M''B', K'(M_1)B).$$

Of course $K'(M)B$ is just $M''B$. Define $\beta: M' \to M_1'$ by $\beta_B = \phi_{BB}(1_B)$. Naturality of β apropos of $f: B \to B'$ follows readily from juxtaposing two rectangles expressing naturality of ϕ apropos of $(1, f)$ and $(f, 1)$. One then verifies $\alpha = I(\beta)$.

Given condition (a), we may assume by 2.8 and 3.3 that $\mathcal{T}_1$ is contained (fully) in $\mathrm{Sem}\,\boldsymbol{T}_1$. The embedding takes rules of $\boldsymbol{T}_1$ to colenses of $\mathrm{Sem}\,\boldsymbol{T}_1$ by 1.0.0. A connection J to $\mathrm{Sem}\,\boldsymbol{T}_2$ takes colenses to lenses since it has an adjoint. Since $\mathrm{Sem}\,\boldsymbol{T}_2$ is fine, $J \mid \mathcal{T}_1$ is a model M'. Since J takes all colimits to limits, which are corules, $K(M)$ is an extension of an isomorph $I(M)$ of J; every connection is isomorphic with a connection $I(M)$. Given condition (b), though $\mathrm{Sem}\,\boldsymbol{T}_2$ may not be fine, still the connections J take colenses to corules. For $\mathrm{Mod}(\boldsymbol{T}_2, \mathcal{S})$ is closed under forming small limits in $Cat(\mathcal{T}_2{}^{op}, \mathcal{S})$; hence all small limits of $\mathrm{Mod}(\boldsymbol{T}_2, \mathcal{S})$ are pointwise limits. So J, followed by an evaluation, is a representable functor on $\mathrm{Mod}(\boldsymbol{T}_1, \mathcal{S})$ and preserves all colenses. That is, J takes every colens to a pointwise lens,

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which is a corule. $J \mid \mathcal{T}_1$ exists as before and is a model; and every connection is isomorphic with a connection $I(M)$.

In case (a) or (b) we have an equivalence of categories, which is also an equivalence of prefacts since both $\mathfrak{C}(x)\operatorname{Sem}\boldsymbol{T}_2$ and $\operatorname{Sem}(\boldsymbol{T}_1; \operatorname{Sem}\boldsymbol{T}_2)$ have pointwise coreguli (and since the Kan extension over new points X is obtained by taking limits of values at old points, so that 1.1 applies). To secure that it is a syntactic equivalence we need only specify that the Yoneda embedding takes the point of $\boldsymbol{T}_1$ to the representing model R. The proof is complete.

For $\operatorname{Sem}\boldsymbol{T}$ to be fine, the evaluation arities on it must be either representable or suitably related to representable functors. This is a precise proposition, by 1.2^{op} (and routine checking), which gives us the meaning of "suitably related." But it is convenient to have a less precise test in terms of functors on $\mathcal{T}$ rather than functors on $\operatorname{Sem}\boldsymbol{T}$. We call $F: \mathcal{T}^{op} \to \mathcal{S}ets$ *importable* on the theory $\boldsymbol{T}$ on $\mathcal{T}$ if, in $Cat(\mathcal{T}^{op}, \mathcal{S}ets)$, it has a reflection in $\operatorname{Mod}(\boldsymbol{T}, \mathcal{S})$, that is, if the set-valued reflection F' constructed in 1.2 is small-valued. An object T of $\mathcal{T}$ is *importable* in $\boldsymbol{T}$ if h_T is importable. The whole theory $\boldsymbol{T}$ is *importable* if it is colimited by the set of importable objects. (Colimiting was defined before 2.6.1.)

Clearly a regular legitimate theory and (from Gabriel's reflection) a small theory are importable. So is the equational theory of complete Boolean algebras [18]; it has only countably many importable objects, but the others are (by rule) colimits of these.

3.5. $\operatorname{Sem}\boldsymbol{T}$ *is fine and modeled if* $\boldsymbol{T}$ *is importable.*

Proof. A lens in $\operatorname{Sem}\boldsymbol{T}$ is a lens pointwise at least at the importable objects of $\boldsymbol{T}$. If $\boldsymbol{T}$ is importable, all the evaluation arities are iterated colimits in $\operatorname{Str}\operatorname{Sem}\boldsymbol{T}$ (limits, as functors) of importable ones; so every lens in $\operatorname{Sem}\boldsymbol{T}$ is a corule.

Considering $\mathcal{M} = \operatorname{Mod}(\boldsymbol{T}, \mathcal{S})$ as a fine pretheory, an $\mathcal{S}$-valued model N on it is stable as an immediate consequence of 1.2. Now every stable functor on $\mathcal{M}$ is the restriction of a representable functor h_{N^-} on some $\operatorname{Mod}(\boldsymbol{T}, \mathcal{S}ets)$, for the set of $\mathcal{S}ets$-valued functors on $\mathcal{M}$ for which that is true contains the representable ones and is closed under forming limits. We have a model $N^-: \mathcal{T}^{op} \to \mathcal{S}ets$ representing N. For an importable object T, h_T having a reflection R in $\mathcal{M}$, $ev^T(N^-) = N^-(T) = \operatorname{Hom}(h_T, N^-) = \operatorname{Hom}(R, N^-) = N(R)$ by (respectively) definition, Yoneda, the following argument, and the fact that N^- represents N. For the third equality (induced by a reflection map R), the set of $\mathcal{S}ets$-valued functors G for which $\operatorname{Hom}(h_T, G) = \operatorname{Hom}(R, G)$

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contains $|\mathcal{M}|$ and is closed under forming limits. The end of the chain of equalities, $N(R)$, is small, so $ev^T(N^-)$ is small. Finally, the set of $U \in |\mathcal{T}|$ for which $ev^U(N^-)$ is small contains the importable ones and is closed under passage from base to vertex of small cones in Γ.

Remark. Using the same hypothesis for the two conclusions that $\mathrm{Sem}\,\boldsymbol{T}$ is fine and modeled sacrifices a little of the natural generality of the proofs. It is clear, at least after the next definition, how one can weaken the hypothesis for each half of 3.5. But the weakening for fineness is slight; and no use has appeared for modeledness of $\mathrm{Sem}\,\boldsymbol{T}$ unless it is also fine.

We call a theory $\boldsymbol{T}$ *coseparated* if $\mathcal{T}$ contains a small set C of objects such that for each object X of $\mathcal{T}$ and model M of $\boldsymbol{T}$ in $\mathcal{S}$, the functions $M(f)$ for $f: W \to X$, $W \in C$, separate the points of $M(X)$. In effect, we need only remark:

3.6. $\mathrm{Sem}\,\boldsymbol{T}$ *is legitimate if* $\boldsymbol{T}$ *is coseparated.*

Proof. If small C coseparates $\boldsymbol{T}$, then models M of $\boldsymbol{T}$ in $\mathcal{S}$ are faithfully represented by their restrictions to the full subcategory $\mathcal{C}$ on C. It does not matter whether $\mathcal{C}$ is small; $M(\mathcal{C})$ is, and $\mathrm{Sem}\,\boldsymbol{T}$ is legitimate.

For the semantic condition that $\boldsymbol{T}$ is coseparated one wants syntactic criteria. Three convenient sufficient conditions are that $\boldsymbol{T}$ is small, or equational, or fine on a coseparated category. Verifying the last, one must note that the actual coseparation by a small set C of objects is describable by colenses, so that the fine theory is a coseparated theory. In fact it is describable by colimits, of substantially the same form as in the proof of 2.2.

Summarizing 3.5 and 3.6 and surrounding remarks, $\mathrm{Sem}\,\boldsymbol{T}$ satisfies 2.8 if $\boldsymbol{T}$ is small or varietal (or equational with $\mathrm{Hom}(P,P)$ small) or fine legitimate coseparated. The small case extends, by the results of Freyd and Kelly, to the light conical case. Moreover, in that case 3.4(b) applies; for $\mathrm{Mod}(\boldsymbol{T}, \mathcal{S})$, being reflective in the functor category on small $\mathcal{T}$, is right complete with a small left adequate subcategory, and the critical condition in 3.4(b) is the conclusion of Lambek's Theorem 7.1 [16].

The next natural question is one on which I know almost nothing except that (by 3.11 below) it is hard. Unfortunately there are several parts to the question. But it is so important that something must be said about it. The whole question is, for what $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ is $\mathrm{Sem}(\boldsymbol{T}_1 \otimes \boldsymbol{T}_2)$ well behaved? The question is sharper in the semantics of varietal theories, where "well behaved" means "a varietal category." In that setting, the counterexamples below (3.11, 3.11.1, 3.11.2) show that it is hard, but the affirmative results

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([21] and below) make a beginning. Here, it would seem reasonable to ask, for what $\boldsymbol{T}_1$ is $\operatorname{Sem}(\boldsymbol{T}_1 \otimes \boldsymbol{T}_2)$ right complete for every small $\boldsymbol{T}_2$? However, I do not know one such theory $\boldsymbol{T}_1$ that is not semantically equivalent to a light conical theory.

$\operatorname{Sem}(\boldsymbol{T}_1 \otimes \boldsymbol{T}_2)$ and the product of the facts $\operatorname{Sem}\boldsymbol{T}_j$ and the product of their underlying categories $\operatorname{Mod}\boldsymbol{T}_j$ (short for $\operatorname{Mod}(\boldsymbol{T}_j, \mathcal{S})$) are connected, ordinarily (e.g. for small $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$), by five distinguished functors. Something non-trivial will be said about

$$\operatorname{Mod}\boldsymbol{T}_1 \times \operatorname{Mod}\boldsymbol{T}_2 \xrightarrow{\otimes} \operatorname{Sem}(\boldsymbol{T}_1 \otimes \boldsymbol{T}_2) \xrightarrow{\operatorname{Rep}} \operatorname{Sem}\boldsymbol{T}_1 \times \operatorname{Sem}\boldsymbol{T}_2.$$

First note the inclusion $I\colon \operatorname{Sem}\boldsymbol{T}_1 \times \operatorname{Sem}\boldsymbol{T}_2 \to \operatorname{Mod}\boldsymbol{T}_1 \times \operatorname{Mod}\boldsymbol{T}_2$; the product of facts is a pullback category, contained in the product category. Under mild restrictions, I preserves limits (a pullback of limit-preserving functors satisfying suitable transportability conditions); so under stronger restrictions, I has an adjoint. I have not looked into these restrictions, for in familiar examples the adjoint can be described and I can see no use to make of it or of I.

Rep (like I) always exists and is strictly unique, being defined by its coordinates $\operatorname{Sem}(\boldsymbol{T}_1 \otimes \boldsymbol{T}_2) \to \operatorname{Sem}\boldsymbol{T}_j$; these are composites of isomorphisms after 3.2 and forgetful functors $\operatorname{Sem}(\boldsymbol{T}; \boldsymbol{F}) \to \boldsymbol{F}$. It is a morphism of complete facts (the product fact $\operatorname{Sem}\boldsymbol{T}_1 \times \operatorname{Sem}\boldsymbol{T}_2$ is the fact of models of the coproduct theory) and hence, under reasonable restrictions, has an adjoint. (It follows from the work of Freyd and Kelly that this holds if both theories are light and conical.) Before continuing with Rep let us try to treat $\otimes$. N. B. The treatment is not satisfactory, but motivates the symbol "$\otimes$" and sharpens the question "When is $\operatorname{Sem}(-\otimes-)$ well behaved?"

The model-tensoring bifunctor $\otimes$ can be described as a composite of arrows that we have already examined, but the description I mean is not even always valid for small theories. Accordingly, for a crude theorem:

3.7. Suppose (a) both $\operatorname{Sem}\boldsymbol{T}_j$ legitimate. Then $\operatorname{Mod}\boldsymbol{T}_j$ maps into $\operatorname{Str}\operatorname{Sem}\boldsymbol{T}_j$ by a Yoneda functor Y as in 2.1. We should like $Y(\operatorname{Mod}\boldsymbol{T}_j)$ to underlie a (regular) theory semantically equivalent to $\boldsymbol{T}_j$. For this it is (necessary and) sufficient that (a′) $\operatorname{Sem}\boldsymbol{T}_j$ satisfy 2.8, by the (statement and) proof of 2.8. (The necessity is not worth much, since this awkward construction can hardly be necessary.) So without assuming more, we have theories $\boldsymbol{T}_j'$ on $\operatorname{Mod}\boldsymbol{T}_j$ and $\boldsymbol{T}_1' \otimes \boldsymbol{T}_2'$ on $\operatorname{Mod}\boldsymbol{T}_1 \times \operatorname{Mod}\boldsymbol{T}_2$, semantically equivalent to the respective unprimed theories. If moreover (b) $(\boldsymbol{T}_1' \otimes \boldsymbol{T}_2')_r$ is legitimate (i.e. every object of $\boldsymbol{T}_1' \otimes \boldsymbol{T}_2'$ is improbable), we can return by a Yoneda functor Y', yielding

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$$\otimes\colon \boldsymbol{T}_1' \otimes \boldsymbol{T}_2' \to (\boldsymbol{T}_1' \otimes \boldsymbol{T}_2')_r \xrightarrow{Y'} \operatorname{Sem}(\boldsymbol{T}_1 \otimes \boldsymbol{T}_2).$$

When (a′) *and* (b) *hold,* $\otimes\colon \operatorname{Mod} \boldsymbol{T}_1 \times \operatorname{Mod} \boldsymbol{T}_2 \to \operatorname{Sem}(\boldsymbol{T}_1 \otimes \boldsymbol{T}_2)$ *exists.*

We may note, without the (routine) proof, that when this description applies (which makes 3.4 apply also), each functor $Q \otimes (\ )$:

$$\operatorname{Mod} \boldsymbol{T}_2 \to \operatorname{Mod} \boldsymbol{T}_1 \otimes \boldsymbol{T}_2,$$

Q in $\operatorname{Mod} \boldsymbol{T}_1$, is (left) adjoint to

$$\operatorname{Mod} \boldsymbol{T}_1 \otimes \boldsymbol{T}_2 \to \operatorname{Mod} \boldsymbol{T}_1(x) \operatorname{Mod} \boldsymbol{T}_2 \to \operatorname{Mod} \boldsymbol{T}_2,$$

where the first arrow is the equivalence of 3.4 and the second is evaluation at Q. Further, the composite just displayed can be defined more generally whenever (every) Q is "exportable" (i.e. Sem $\boldsymbol{T}_1$ is legitimate), by

$$\operatorname{Mod} \boldsymbol{T}_1 \otimes \boldsymbol{T}_2 \to \operatorname{Mod}(\operatorname{Str} \operatorname{Sem} \boldsymbol{T}_1, \operatorname{Mod} \boldsymbol{T}_2) \to \operatorname{Mod} \boldsymbol{T}_2.$$

It is easy to write out a characterization of $\otimes$ based on that. The problem is when $\otimes$ exists. For varietal theories, this is equivalent to the problem when $\operatorname{Sem}(\boldsymbol{T}_1 \otimes \boldsymbol{T}_2)$ is a varietal category. (The verification is immediate from (1) the existence of a tensor multiplication in equational semantics, so that $\operatorname{Sem}(\boldsymbol{T}_1; \operatorname{Sem} \boldsymbol{T}_2)$ as we have defined it is an equational category at least when $\operatorname{Sem} \boldsymbol{T}_2$ is fine; (2) the fact that an equational category is varietal provided it has free algebras; and (3) the equivalence of the notions of a free algebra on $m_1 m_2$ generators in $\operatorname{Sem}(\boldsymbol{T}_1 \otimes \boldsymbol{T}_2)$ and a tensor product of the free algebras on m_i generators in $\operatorname{Sem} \boldsymbol{T}_i$. The easiest way to acquire belief in these observations is to check them, but they occur, more or less, in [6], [18], [6] respectively. In view of (3) we may supplement 3.7: for varietal theories, it is necessary and sufficient for the existence of $\otimes$ that $(b\text{-})(\boldsymbol{T}_1 \otimes \boldsymbol{T}_2)_r$ be legitimate.)

Tensoring should be adjoint to homming. It is, in the following sense. The modified evaluation last displayed, call it e^Q, lifts h^Q

$$\begin{array}{ccc} \operatorname{Mod} \boldsymbol{T}_1 \otimes \boldsymbol{T}_2 & \xrightarrow{e^Q} & \operatorname{Mod} \boldsymbol{T}_2 \\ \downarrow & & \downarrow \\ \operatorname{Mod} \boldsymbol{T}_1 & \xrightarrow{h^Q} & S \end{array}$$

up the forgetful functors.

Suppose we consider

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$$\operatorname{Mod} \boldsymbol{T}_1 \times \operatorname{Mod} \boldsymbol{T}_2 \xrightarrow{\otimes} \operatorname{Mod} \boldsymbol{T}_1(x)\ \operatorname{Mod} \boldsymbol{T}_2 \xrightarrow{\operatorname{Rep}} \operatorname{Mod} \boldsymbol{T}_1 \underset{\mathcal{S}}{\times} \operatorname{Mod} \boldsymbol{T}_2.$$

When 3.4 applies, this differs only notationally from the previous diagram. (This notation will be more convenient for 3.8 and 3.9). There is a marginal reservation; Rep: $\operatorname{Sem}(\boldsymbol{T}_1 \otimes \boldsymbol{T}_2) \to \operatorname{Sem} \boldsymbol{T}_1 \times \operatorname{Sem} \boldsymbol{T}_2$ is a morphism of facts, and the fact construction $\mathfrak{C}(x)\boldsymbol{F}$ uses one coregulus (usually fine) which is abandoned in the present formulation. Finally, the main point about the present diagram is that it generalizes; the $\operatorname{Mod} \boldsymbol{T}_j$ can be replaced by any categories over $\mathcal{S}$. As far as $\otimes$ is concerned, this may be nearly pointless, since existence of $\otimes$ will depend on completeness conditions and complete categories over $\mathcal{S}$ are "ordinarily" categoes $\operatorname{Sem} \boldsymbol{T}$ anyway. We now cease to consider $\otimes$ and pass to

$$\mathfrak{C}(x)\mathfrak{F} \xrightarrow{\operatorname{Rep}} \mathscr{C} \underset{\mathcal{S}}{\times} \mathscr{F}.$$

Here $\mathfrak{C}$ and $\mathfrak{F}$ are categories with distinguished objects, respectively $(\mathscr{C}, P)$, $(\mathscr{F}, Q)$. We must assume that the hom functors h^P, h^Q are $\mathcal{S}$-valued. Let us use the simplest definition of $\mathfrak{C}(x)\mathfrak{F}$ at this moment, which is $\mathfrak{C}(x)\boldsymbol{F}$ as defined previously, where $\boldsymbol{F}$ is the fine fact on $(\mathscr{F}, h^Q)$. Then $\mathfrak{C}(x)\mathfrak{F}$ is a fact by definition. Note that it is obviously regular (and often fine, but that will not be pursued here.) We regard $\mathscr{C} \underset{\mathcal{S}}{\times} \mathscr{F}$ as the product object-over-$\mathcal{S}$, made into a fact by the fine coregulus.

For general $\mathfrak{C}$, $\mathfrak{F}$, there is no guarantee that $\mathscr{C} \underset{\mathcal{S}}{\times} \mathscr{F}$ is even non-empty. From the central case $\mathscr{C} = \operatorname{Mod} \boldsymbol{T}$ we want to retain one bookkeeping assumption: h^P is transportable. (In the sense of [3]: every bijection $\operatorname{Hom}(P, X) \to A$ is $\operatorname{Hom}(P, f)$ for some isomorphism $X \to X'$.) This can always be brought about (for legitimate $\mathscr{C}$) by replacing $\mathscr{C}$ with the equivalent category $\mathscr{C}^-$ of all representable contravariant functors on $\mathscr{C}$ to $\mathcal{S}$. $\mathscr{C}$ and $\mathscr{C}^-$ are equivalent *over* $\mathcal{S}$ if and only if h^P is transportable.

Now, $\mathscr{C}$ being legitimate, $\mathscr{F}$ arbitrary as long as h^Q is $\mathcal{S}$-valued,

$$\operatorname{Rep}^-: \mathfrak{C}(x)\mathfrak{F} \to \mathscr{C}^- \underset{\mathcal{S}}{\times} \mathscr{F}$$

may be defined by its coordinates. The first coordinate is postmultiplication by h^Q. Each connection F has an adjoint G on the right, and h^Q is represented by $G(Q)$; so $h^Q F$ is indeed in $\mathscr{C}^-$. The second coordinate is evaluation at P. Both of these functors, hence also Rep, preserve the (pointwise) lenses in the coregulus of $\mathfrak{C}(x)\mathfrak{F}$, so Rep^- is a morphism of facts. By routine arguments, the equivalences (of categories over $\mathcal{S}$)

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$$\operatorname{Sem}(\boldsymbol{T}_1 \otimes \boldsymbol{T}_2) \to \operatorname{Sem}\boldsymbol{T}_1(x)\ \operatorname{Sem}\boldsymbol{T}_2$$

and

$$\operatorname{Sem}\boldsymbol{T}_1 \times \operatorname{Sem}\boldsymbol{T}_2 \to (\operatorname{Mod}\boldsymbol{T}_1)^- \underset{\mathcal{S}}{\times} \operatorname{Mod}\boldsymbol{T}_2$$

identify Rep^-, up to an isomorphism of functors over $\mathcal{S}$, with the previous Rep.

We are ready for the question "How well does Rep F represent F?" To begin, replace "Rep" by "Rep^-."

3.8. THEOREM. *For $\mathfrak{C} = (\mathcal{C}, P)$ legitimate and $\mathfrak{F} = (\mathcal{F}, Q)$ with h^Q $\mathcal{S}$-valued, Rep^- is full and faithful if every object $\mathcal{C}$ is an iterated colimit of copies of P and h^Q is faithful.*

Proof. First, if merely h^Q is faithful then the first coordinate of Rep^-, $F \mapsto h^Q$, is (already) faithful. (Likewise the second coordinate is faithful if merely h^Q is faithful.)

Using the full hypothesis, consider any morphism $\alpha\colon \mathrm{Rep}^- \to \mathrm{Rep}^- G$, i.e. any $\alpha^1\colon h^Q F \to h^Q G$ and $\alpha^2\colon F(P) \to G(P)$ such that $\alpha^1{}_P\colon h^Q F(P) \to h^Q G(P)$ is postmultiplication by α^2. We need $a\colon F \to G$ such that $h^Q a = \alpha^1$ and $a_P = \alpha^2$. Thus we have a partially defined, $a^0\colon F \mid \mathcal{C}_0 \to G \mid \mathcal{C}_0$ being given by α^2, where $\mathcal{C}_0$ is the full subcategory on P alone. Having a suitable extension a^λ of a^0 over a full subcategory $\mathcal{C}_\lambda$, we want to extend over one more object X which is a colimit of a diagram D in $\mathcal{C}_\lambda$. F takes a universal inductive cone $D \to K_X$ to a (universal) projective cone $K_{FX} \to FD$; and $a^\lambda \circ D\colon FD \to GD$ yields a projective cone $\kappa\colon K_{FX} \to GD$. Since G takes colimits to limits (too), κ is induced by a unique morphism $a_X\colon FX \to GX$. Moreover, $h^Q(a_X)$ must be $\alpha^1{}_X$ because h^Q preserves limits. Then $a^{\lambda+1}$ formed from a^λ and a_X is natural because $h^Q a^{\lambda+1}$ agrees with the natural transformation α^1 and h^Q is faithful. So the recursion runs.

The assumptions on the distinguished objects in 3.8, and also in 3.9, fit in the following scheme. (a) Every object is an iterated colimit of copies of P. (b) h^P reflects isomorphisms. (c) If $h^P(f)$ is a bijection (isomorphism) and f is a monomorphism, then f is an isomorphism. (d) For any monomorphism $f\colon A \to Y$ there is a universal morphism $\hat{f}\colon \hat{A} \to Y$ such that every morphism $fp\colon P \to \hat{A} \to Y$ factors through f. (e) h^P is faithful. Here (c) is the defining condition for P to be a *generator* (in the sense of Grothendieck [8]. When (e) holds, P is called a "generator in the sense of Freyd" or (here, following Semadeni [26]) a *coseparator*. Note that (a) implies all the others; but in a right extremal-complete category (a), (b) and (c) are equivalent. ((c) $\Rightarrow$ (a) from [12], and (b) $\Rightarrow$ (c) trivially.) For 3.9 we want P satisfying (d) and (e), and call such an object an *inducer*.

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Of course (d) may be paraphrased: on the subset underlying a subobject there is a (coarsest) induced structure.

To illustrate, a one-point topological space or partially ordered set, or a discrete topological group of integers, is an inducer. The category of complete topological (abelian) groups has no inducer. I think this is a typical illustration.

3.9. *Let* $\mathfrak{C}_i = (\mathcal{C}_i, P_i)$ *for* $i = 1, 2$, *be legitimate categories having small powers, with inducers* P_i. *Then* Rep$^-$ *on* $\mathfrak{C}_1(x)\mathfrak{C}_2$ *is isomorphic with a composition* RZ, *where* Z *is a reflector upon a full category and* R *is full and faithful.*

Proof. Z will reflect $\mathcal{C}_1(x)\mathcal{C}_2$ upon the subcategory of "coarse" connections, as follows. We write U_i for h^{P_i}. For each connection $F: \mathcal{C}_1{}^{op} \to \mathcal{C}_2$, let $X_2 = F(P_1)$. Define $DF: \mathcal{C}_1{}^{op} \to \mathcal{C}_2$ by $DF(S) = X_2{}^{U_1 S}$, $DF(f) = X_2{}^{U_1 f}$. (This is a generalized corepresentable functor of Ulmer [29]. The parentheses in "$DF(S)$" are for clarity, in "$F(\ )$" for balance.) Define $\delta: F \to DF$ by coordinates, $(\delta_S)_g = F(g): F(S) \to F(P_1)$ for $g \in U_1 S$. One readily verifies that δ is natural. Also, each δ_S is monomorphic, because P_1 coseparates (so that F takes a pushout diagram like that in 2.2 to a pullback). Induce ZF, $i_S: ZF(S) \to DF(S)$ being universal such that $U_2 i_S$ factors through $U_2 \delta_S$. Since $U_2 \delta_S$ factors through itself, we get $\delta_S = i_S r_S$. Evidently ZF is a functor and r and i are natural. F is *coarse* if r is a natural equivalence.

F being a connection with adjoint (on the right) H, ZF is a connection with adjoint ZH. The adjunction $\alpha_{ST}: \mathrm{Hom}(T, ZF(S)) \to \mathrm{Hom}(S, ZH(T))$ is constructed as follows. Morphisms $m^2: T \to ZF(S)$ correspond bijectively to their underlying functions $U_2 m^2: U_2 T \to U_2 ZF(S) = U_2 F(S) \subset (U_2 X_2)^{U_1 S}$. Now $U_2 F$ is represented by $H(P_2) = X_1$. We have identifications of $U_1 X_1 = X_0$ with $U_2 X_2$, and of $U_2 F(S)$ with $\mathrm{Hom}(S, X_1)$; to reduce notation we write these as equalities. Then the most general $U_2 m^2$ is precisely any function $U_2 T \to X_0{}^{U_1 S}$ which takes each element t into $U_1(\mathrm{Hom}(S, X_1))$ and whose s-th coordinate $U_2 T \to X_0$ (for each s in $U_1 S$) is in $U_2(\mathrm{Hom}(T, X_2))$. The adjunction in $\mathcal{S}$ of $(\ ) \times U_1 S$ to $\mathrm{Hom}(U_1 S, \ )$ identifies these with the functions $m_2: U_2 T \times U_1 S \to X_0$ which take verticals $\{t\} \times U_1 S$ morphically into X_2. The natural bijection $U_2 T \times U_1 S \times U_2 T$ takes these m_2 to similarly described m_1, which correspond to $m^1: S \to ZH(T)$ the same way m_2 corresponds to m^2. Putting $\alpha_{ST}(m^2) = m^1$, we have α, which is easily seen to be natural.

Every natural transformation π from a connection F' to a coarse connection F factors across $r': F' \to ZF'$ and even across δ', as follows. For each S,

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$\delta_S\pi_S: F'(S) \to DF(S) = F(P_1)^{U_1S}$ has for s-th coordinate $(s: P_1 \to S)$ $\pi_{P_1}F'(s)$, since π is natural; that is, $\delta_S\pi_S = (\pi_P)^{U_1S}\delta'_S$. Now each r_S' is epic, since U_2 takes it to a surjection. Thus π factors uniquely across r'; ZF' is a coarse reflection of F' and r' is a reflection mapping. It follows that the object function Z can be extended to a reflector upon the full subcategory $\mathscr{C}o$ of coarse connections.

Finally, $R = \mathrm{Rep}^- \mid \mathscr{C}o$ is full and faithful. As noted in 3.8, the present hypotheses suffice to make all of Rep^- faithful. Further, a morphism $a: \mathrm{Rep}^- F \to \mathrm{Rep}^- F'$ (in effect, $(X_1, X_2) \to (X_1', X_2')$) induces a morphism $a_2{}^{U_1}: DF \to DF'$ which takes the subfunctor ZF into ZF' (because a_1 is a morphism); and for this $\alpha: ZF \to ZF'$, it is routine to verify $\mathrm{Rep}^-\alpha = a$. ("$=$," modulo the obvious natural equivalence $\mathrm{Rep}^- ZF \to \mathrm{Rep}^- F$.)

The reflector Z ordinarily has an adjoint besides its coadjoint $\mathscr{C}o \subset \mathscr{C}_1(x)\mathscr{C}_2$; for instance, this certainly holds if Rep^- has an adjoint V (since RZ is naturally equivalent to Rep^- and R is fulll and faithful, $VR \dashv Z$), thus if each $\mathfrak{C}_i$ is $\mathrm{Sem}\,\boldsymbol{T}_i$ where $\boldsymbol{T}_i$ is small (and 3.9 applies). If 3.8 applies too, Z is an equivalence. In the remaining scope of 3.9, I know very little about the adjoint (even) of Z. Clearly whenever it exists it is faithful. If, for instance, $\mathscr{C}_1 = \mathscr{C}_2 =$ completely regular spaces, with the usual point $P_1 = P_2$, does Z have an adjoint? A full adjoint? Is Z an equivalence?

3.10. *Let $\mathfrak{C}_i = (\mathscr{C}_i, P_i)$ for $i = 1, 2$, satisfy* 3.9 *and moreover, for each object Y of $\mathscr{C}_i$ and subset S of $h^{P_i}(Y)$, let the subfunctor of h_Y on the sets of morphisms f into Y such that $h^{P_i}(f)$ factors through S be representable. Then R of* 3.9 *is an equivalence; up to isomorphism,* Rep^- *has a coadjoint right inverse. If $\mathscr{C}_1 = \mathscr{C}_2$, the category of partially ordered sets, of topological spaces, or of T_0, T_1, or T_2 spaces (and the P_i are the usual one-point objects), then* Rep^- *also has an inequivalent adjoint right inverse.*

Proof. For the first part, one simplifies slightly the proof of 3.9. Now that every subset underlies a subobject with induced structure, a coarse connection can be constructed over any (X_1, X_2) in $\mathscr{C}_1 \underset{\mathscr{S}}{\times} \mathscr{C}_2$.

The second part, in the instances asserted here, involves *fine* connections and a functor Fin. For partially ordered sets, $X = (X_1, X_2)$ in $\mathscr{C} \underset{\mathscr{S}}{\times} \mathscr{C}$, A in $\mathscr{C}$, $\mathrm{Fin}\,X(A)$ is the set $\mathrm{Hom}(A, X_1)$ with the following partial order: $f \leqq g$ provided for each a in A, $f(a) \leqq g(a)$ in the order of X_2, and on each component of A, either $f = g$ or both are constant. Clearly this is a partial order. For a morphism $f: A \to A'$, $\mathrm{Fin}\,X(f)$ is the morphism given by the function $\mathrm{Hom}(f, X_1)$—clearly it is a morphism, and $\mathrm{Fin}\,X$ is a functor.

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For topological spaces, similarly, $\mathrm{Fin}\,X(A)$ is on the set $\mathrm{Hom}(A, X_1)$; the limits of a filter Λ of morphisms $A \to X_1$ are those X_2-pointwise limits g such that A has an open covering by sets U_i on each of which either (1) all elements of some $L_i \in \Lambda$, and g, are constant, or (2) all elements of some $L_i \in \Lambda$ agree with g. This is a topology, as is clear when one writes down (and checks) the conditions for a set to be open. It is finer than the X_2-pointwise topology and thus is T_0, T_1, or T_2 if X_2 is so. The functions $\mathrm{Hom}(f, X_1)$ preserve limits of filters, so they are morphisms $\mathrm{Fin}\,X(f)$.

$\mathrm{Fin}\,X$ is a connection whose adjoint on the right is $\mathrm{Fin}^* X = \mathrm{Fin}(X_2, X_1)$. To verify it in the partially ordered case, a morphism from A to $\mathrm{Fin}\,X(B)$ amounts to a function $m: A \times B \to X_0$ (X_0 the common underlying set of X_1 and X_2) taking each $\{a\} \times B$ morphically into X_1 and $A \times \{b\}$ morphically into X_2, such that if $a < a'$, and b and b' are in the same component, then (*) if $m(a, b) \neq m(a', b)$ or $m(a, b') \neq m(a', b')$, then $m(a, b) = m(a, b')$ and $m(a', b) = m(a', b')$. But if (*) holds when $b < b'$, then it holds whenever b and b' are in the same component; for if (e.g.) $m(a, b) \neq m(a', b)$, we get $m(a, c) = m(a, b) \neq m(a', b) = m(a', c)$ for each c comparable with b, and so on back to b'. Observing that (*) is symmetric in a's and b's, we have adjointness. Similarly in the topological case, $A \to \mathrm{Fin}\,X(B)$ amounts to $m: A \times B \to X_0$ taking vertical continuously to X_1 and horizontals to X_2, such that for each a in A, each b in B has a neighborhood U_b such that $a_j \to a$ implies (1) $m(a, \ )$ and, finally, $m(a_j, \ )$ are constant on U_b or (2) finally, $m(a_j, \ ) = m(a, \ )$ on U_b. (If b had no such neighborhood there would be a filter of sets of points $a_{j,N} \to a$ whose images in $\mathrm{Fin}\,X(B)$ fail to converge to the image of a.) Applying this to the filter of neighborhoods of a, we see that m is either constant on horizontals or constant on verticals throughout a neighborhood of (a, b), and $\mathrm{Fin}\,X$ is a connection.

$\mathrm{Fin}\,X$ represents $h^X \mathrm{Rep}$. A morphism $p = (p_1, p_2)$ from X to $\mathrm{Rep}\,F = (F_1, F_2)$ corresponds to $\pi: \mathrm{Fin}\ X \to F$ defined by $h^{P_2}\pi_A: h^{P_2}\mathrm{Fin}\,X(A) = \mathrm{Hom}(A, X_1) \to \mathrm{Hom}(A, F_1) \to h^{P_2}F(A)$ composed of postmultiplication by p_1 and the representation defining F_1. In the topological case π_A is continuous, π is natural, and $p \mapsto \pi$ is bijective as follows. $\mathrm{Hom}(A, X_1)$ goes into $\mathrm{Hom}(A, F_1)$ (for all A, especially $A = X_1$) precisely because p_1 is continuous. It goes continuously precisely because p_2 is continuous and every convergent filter in $\mathrm{Fin}\,X(A)$ derives by a quotient mapping from a coproduct $\sum U_i \to A$; the respect that F must pay to coproducts and quotients secures convergence of the corresponding filters in $F(\sum U_i)$ and in $F(A)$. So π_A is continuous and $p \hookrightarrow \pi$ bijective. π is natural because $h^{P_2}\pi$ is natural and h^{P_2} is faithful. Therefore we have the representations, which make a functor Fin adjoint to

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Rep. Writing down the evident complete description of Fin, one sees that Rep Fin is isomorphic with the identity. Rep$^-$ differs by an equivalence.

Three questions on tensoring varietal theories:

P1. *For what varietal theories* $\boldsymbol{T}$ *is* $\boldsymbol{T} \otimes \boldsymbol{U}$ *semantically equivalent to a varietal theory for every varietal* $\boldsymbol{U}$*?*

P2. *For what varietal* $\boldsymbol{T}$ *is* $\boldsymbol{T} \otimes \boldsymbol{U}$ *semantically varietal for all small varietal* $\boldsymbol{U}$*, i.e.* $\boldsymbol{U}$ *having a rank?*

P3. *What is a reasonable class of varietal theories semantically closed under* $\otimes$*?*

The little imprecisions in the statements of P2 and P3 willl do no harm in this paper since there are no results on those questions.

One might ask that the tensor product theories be syntactically varietal. Not in the present syntax; but there is an *equational* tensor product $\otimes_e$, which we now recall [6] and justify. The justification is incomplete, depending on an application of 3.2 which depends on Sem $\boldsymbol{U}$ being fine, which we do not know unless $\boldsymbol{U}$ is importable. That is much weaker than requiring $\boldsymbol{U}$ to be varietal. (Actually Sem $\boldsymbol{T}$ is fine for all equational $\boldsymbol{T}$, but the only proof I know is longer than it is worth. Some proof will probably be wanted; for a tensor product of varietal theories need not even be semantically importable, see 3.11.2.)

Now observe that a $\boldsymbol{T} \otimes \boldsymbol{U}$-algebra is the same, via 3.2, as a $\boldsymbol{T}$-algebra in the $\boldsymbol{U}$-algebras, which is the same (up to equivalence) as a set which is a $\boldsymbol{T}$-algebra and a $\boldsymbol{U}$-algebra in such a way that every $\boldsymbol{T}$-operation is $\boldsymbol{U}$-homomorphic. But that is the result of a construction of classical universal algebra: unite the operations and laws of $\boldsymbol{T}$ and $\boldsymbol{U}$, and impose additional laws saying that the $\boldsymbol{T}$-operations are $\boldsymbol{U}$-homomorphic. (This is the same, by inspection, as $\boldsymbol{U}$-operations being $\boldsymbol{T}$-homomorphic.) The resulting equational theory is called $\boldsymbol{T} \otimes_e \boldsymbol{U}$; we see that it is semantically equivalent to $\boldsymbol{T} \otimes \boldsymbol{U}$ (Sem $\boldsymbol{U}$ being fine).

The condition that $\boldsymbol{T} \otimes \boldsymbol{U}$ is semantically equivalent to some varietal theory appears to be weaker than the condition that $\boldsymbol{T} \otimes_e \boldsymbol{U}$ is varietal. I do not know if it is weaker. (From Linton's results, $\boldsymbol{T} \otimes \boldsymbol{U}$ is semantically varietal if and only if Sem$(\boldsymbol{T} \otimes \boldsymbol{U})$ is tractable [18]. This is easily seen to happen if and only if Str_e Sem$(\boldsymbol{T} \otimes \boldsymbol{U})$ is varietal, where Str_e is the equational structure functor, taking the subcategory of Ari $(\ , \aleph)$ on the powers of the grounding. Also, if there is a varietal semantic equivalent this is the only one; semantically equivalent varietal theories are syntactically equivalent.)

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At any rate, such results as I have are negative semantic (i.e. strongly negative) and affirmative syntactic (strongly affirmative).

3.11. *There is a varietal theory* $\boldsymbol{T}$ *such that the category of vector spaces (over any field) in* $\boldsymbol{T}$*-algebras is not varietal.*

Proof. The field will be assumed countable for convenience. All vector spaces mentioned will be infinite-dimensional, so that power = dimension.

Let the operations of $\boldsymbol{T}$ be 0-ary 0, unary σ, and m-ary w_m for each small cardinal $m \geqq \aleph_0$ (and composites of these). The laws say that w_m vanishes on each m-tuple whose terms lie in a subalgebra generated by fewer than m elements (by listing all such m-tuples formed from $n < m$ variables. Evidently this makes $\boldsymbol{T}$ varietal.)

We shall construct arbitrarily great $\boldsymbol{T}$-vector spaces generated by a sequence $\{x_n\}$. Let A_0 be a vector space with basis $\{x_n\}$. Let σ vanish on A_0, and let the zero of $\boldsymbol{T}$ be the zero vector. Inductively, A_α will be a vector space whose power m_α exceeds all previous m_β, closed under σ and all previous w_{m_β}. For any skipped aleph m ($m \neq m_\alpha$ for all α), w_m vanishes. We must define w_{m_α} on ${A_\alpha}^{m_\alpha}$, linear and vanishing on the degenerate set D_α of m_α-tuples $\boldsymbol{T}$-generated by fewer than m_α elements. Thus it must vanish on the subspace D_α^- spanned by D_α. We put $w_{m_\alpha}({A_\alpha}^{m_\alpha}) = {A_\alpha}^{m_\alpha}/D_\alpha^-$, using the canonical projection to this quotient space. Call it $E_{\alpha 0}$. Adjoin copies of it, $E_{\alpha n} = \sigma^n(E_{\alpha 0})$. Let $A_{\alpha+1}$ be a direct sum (vector coproduct) of A_α and the direct sum E_α of all $E_{\alpha n}$. Extend σ over $A_{\alpha+1}$ linearly. For $m \leqq m_\alpha$, we have w_m defined on ${A_\alpha}^m$; put $w_m = 0$ on ${E_\alpha}^m$ and extend linearly over ${A_{\alpha+1}}^m = {A_\alpha}^m \oplus {E_\alpha}^m$. We must show that the power $m_{\alpha+1}$ of $A_{\alpha+1}$ exceeds m_α.

At $\alpha = 0$, $D_0 \subset {A_0}^\omega$ is contained in $\cup {B_n}^\omega$ over the subspaces B_n spanned by $\{x_1, \cdots, x_n\}$; by inspection, E_{00} has uncountable dimension, and $m_1 > m_0$. For $\alpha = \beta + 1$, the set of values of w_{m_β} is $E_{\beta 0}$, and the set of values of every other w_m ($m < m_\alpha$) is contained in A_β, a set of power only m_β. Hence if the m_α-tuple (x_γ) is in D_α, then all but fewer than m_α of its distinct terms lie in $\bigcup_n E_{\beta n}$. If it is in D_α^-, then for some integer k, all but fewer than m_α of its (distinct) terms are sums of elements of at most k summands $A_\beta, E_{\beta 0}, E_{\beta 1}, \cdots$. Now $E_{\beta 0}$ must have m_α linearly independent elements y_i (otherwise A_α would not). There are more than m_α subsets S of the set I of indices, each of power m_α, but any two of which have fewer than m_α common elements [4]. For such S's, let ${x_0}^S$ be a list of $\{y_i : i \in S\}$; let

$$ {x_n}^S = {x_0}^S + \sigma({x_0}^S) + \cdots + \sigma^n({x_0}^S). $$

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Let x^S be an m_α-tuple whose terms are all the terms of all $x_n{}^S$, taken once each. Then for any non-zero linear combination c of the x^S, almost all terms of c are formed from the terms of the x^S without cancellation, and c is not in $D_\alpha{}^-$. Consequently the recursion runs up to a limit ordinal λ.

The union A_λ of the expanding sequence $\{A_\alpha : \alpha < \lambda\}$ has larger power m_λ and has the required closure properties. Moreover, $m_\lambda = \sum m_\alpha$ is the supremum of the preceding (increasing) m_α. If an m_λ-tuple (x_γ) is in D_λ, its terms are $\boldsymbol{T}$-generated by a set G of fewer than m_λ elements. Then G has fewer than m_α elements for some $\alpha < \lambda$. So all x_γ, excepting fewer than m_λ of them, lie in $\cup E_{\beta n}$ $(\beta < \alpha,\ n = 0, 1, \cdots)$. Accordingly for (x_γ) in $D_\lambda{}^-$, for some integer k, all different x_γ with fewer than m_λ exceptions are sums of elements of at most k summands A_0, $E_{\beta n}$. Uniting the sets $\{y_i\}$ used at previous isolated ordinals, the same argument applies. The construction runs until we choose to stop it at A_α, putting $w_{m_\alpha} = 0$.

I do not know how badly non-varietal the tensor product theory in 3.11 is. *If the field is finite, the category is not right complete*; for the 1-dimensional subspaces of A_0 spanned by the x_n are subalgebras having no coproduct.

3.11.1. *There is a commutative varietal theory $\boldsymbol{T}$ such that the category of commutative semigroups in $\boldsymbol{T}$-algebras is not finitely complete.*

3.11.2. *The tensor product theory in* 3.11.1 *is not importable, for there are arbitrarily great cyclic algebras.*

Proof. Let the operations of $\boldsymbol{T}$ be w_m for each small $m \geqq 0$, and composites of these. The laws, besides commutativity (w_m is w_n-homomorphic), say that w_m vanishes on each m-tuple whose terms lie in a subalgebra generated by fewer than m elements.

Let C be a two-element zero semigroup and zero $\boldsymbol{T}$-algebra $\{0, c\}$; let $D = \{0, d\}$ be isomorphic. We shall show that C and D have no coproduct, by constructing arbitrarily great algebras E_α generated by $C \cup D$. Each value of an operation will be 0 unless c is one of the arguments, and c will never be a value of an operation. This secures that all operations are homomorphic, and multiplication is associative. (Commutativity is implicitly assumed.) The laws of $\boldsymbol{T}$ will (all) hold because w_m will take non-zero values only on some m-tuples all but one or two of whose terms are distinct non-zero products, and the nonzero products and the exceptional terms c, d, will never be values of a w_n. Let the values of operations not mentioned below (including all values of w_1 and w_2) be 0.

Besides $C \cup D$ take a product $cd = a_3$. Form the sequence $s^3 = (c, d, a_3)$.

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Having the k-term sequence s^k consisting of c, d, and some non-zero products, adjoin an element $b_k = w_k(s^k)$. Adjoin $a_{k+1} = cb_k$, and let s^{k+1} be s^k followed by a_{k+1}.

Let A_ω be $\{a_3, a_4, \cdots\}$. For infinite α, A_α being a set of non-zero products of power m_α greater than that of any preceding A_β, adjoin the set B_α of subsets of A_α of power m_α. They are in no previous B_β, being too big. There are more than m_α of them, for B_α includes at least one of each complementary pair of subsets of A_α. Attach B_α by defining each of its elements s to be the value of w_{m_α} on some m_α-tuple whose terms are c and a list of s. Adjoin a set $A_{\alpha+1}$ equipotent with B_α, and call its elements the products cs, $s \in B_\alpha$. At a limit ordinal λ, let A_λ be the union of the preceding A_α. Since their powers are increasing, it is greater. The construction does not stop until we choose to shop it, by not attaching B_α. (All remaining values of operations are 0.) 3.11.1 is proved.

For 3.11.2, modify the previous construction only by making $w_1(c) = d$ instead of 0. The only change in the reasoning is that the reason for w_2's vanishing on cyclic $\boldsymbol{T}$-algebras is now: w_2 vanishes. This shows immediately that the algebraic tensor product theory is not importable. It is easy (and left to the reader) to deduce the same for the general tensor product.

At this point my original manuscript included a simple artificial example of a varietal category which is not co-well-powered. But a natural example has turned up. Its objects are complete Brouwerian lattices, but the description, and the morphisms, do not involve Brouwer's pseudocomplement operation. The category is described in [1] and called the category of *local lattices*.

3.12. *The local lattices form a non-co-well-powered varietal category.*

The proof will be sketched; much fuller information is given in a manuscript to appear in *Math. Scand.*, entitled "Atomless parts of spaces." First, Benabou in effect showed that the category is varietal [1], lacking only the term "varietal." Second, the complete Boolean algebras are fully embedded in the local lattices, closed under limits and equalizers; and the embedding preserves the grounding. Therefore a free countably generated local lattice has all countably generated complete Boolean algebras as quotients.

Returning to Problem P1: if all operations of $\boldsymbol{T}$ are constant or unary, then in any $\boldsymbol{T} \otimes \boldsymbol{U}$-algebra, the $\boldsymbol{U}$-closure of the $\boldsymbol{T}$-closure of a subset is $\boldsymbol{T}$-closed. Hence all operations of $\boldsymbol{T} \otimes_e \boldsymbol{U}$ are $\boldsymbol{U}$-operations of $\boldsymbol{T}$-operations, and $\boldsymbol{T} \otimes_e \boldsymbol{U}$ is varietal if $\boldsymbol{U}$ is. This little result (from "hence") generalizes to triples in rather general categories [21; 4.6]. So does the sandwich construction 3.13 below, though I have not verified that Manes' conditions (the ground category is "regular" [21]) suffice.

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Compact topological algebras work the other way around; the closure of a subalgebra is a subalgebra. The reason (rephrased from [21] to a form more convenient over $\mathcal{S}$) is that in the theory $\boldsymbol{T}$ of compact (Hausdorff) spaces, every small set $\{w_\alpha\}$ of n_α-ary operations has a *uniformizer*, which is a Πn_α-ary operation w whose value on a special Πn_α-tuple $f\colon \Pi N_\alpha \to A$ whose values are determined by the α-th coordinate of the index—$f(x) = f_\alpha(x_\alpha)$—is $w_\alpha(f_\alpha)$, for all α and f_α. I cannot generalize this very far, but it reaches to the complete, completely distributive Boolean algebras, as follows.

3.13. *If the operations of $\boldsymbol{T}$ are generated by a subtheory $\boldsymbol{T}_0$ in which every small set of operations has a uniformizer and a set T_1 of constant and unary operations such that in every $\boldsymbol{T}$-algebra, the $\boldsymbol{T}_0$-closure of the T_1-closure of any subset is a subalgebra, then $\boldsymbol{T} \otimes_e \boldsymbol{U}$ is varietal for all varietal $\boldsymbol{U}$.*

The proof consists of a routine verification that the $\boldsymbol{T}_0$-closure of the $\boldsymbol{U}$-closure of the T_1-closure of a subset of a $\boldsymbol{T} \otimes \boldsymbol{U}$-algebra is a subalgebra.

3.13 enables us to answer a question raised by R. Davis [2], as follows:

3.13.2. *For every varietal theory $\boldsymbol{T}$, the category of $\boldsymbol{T}$-coalgebras in $\mathcal{S}$ is cotripleable over $\mathcal{S}$.*

For this we wish to apply 3.13 to $\mathcal{S}^{op}$. Why does it apply? There are two reasons, which is one more than we need, but they are the only two methods I know for applying 3.13. (1) Ad hoc. The laws (of union and intersection) are so strong that every operation $w(x_1, x_2, \cdots)$ (of the theory of complete, completely distributive lattices, $\boldsymbol{T}_0$) can be put in the form $\bigvee_i \bigwedge_j x_{ij}$; and so can a small set of operations. (Then put $T_1 = \{0, 1, '\}$.) (2) By convergence. The details are tedious, but everyone can see that in (compact) topological algebras the closure of a subalgebra is a subalgebra. A quick (sufficient) proof is in [21; 5.6]. Analyzing it, one is led to define a *convergence operation* as an m-ary operation ($m \geqq \aleph_0$) w, all of whose binary specializations of the form $w(x, x, \cdots, x, y, y, \cdots)$ (one value x initially, one value y from there on) are equal to y. An operation is *idempotent* if it uniformizes itself and the unary identity operation.

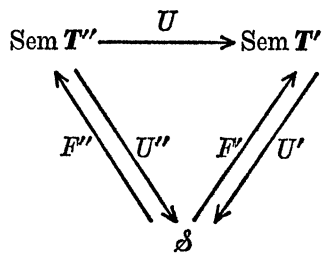
3.13.1. *Any n idempotent operations in a varietal theory which has m-ary convergence operations for all infinite $m \leqq n$ have a uniformizer.*

The proof consists mainly of indices; Πn_α-tuples for $1 \leqq \alpha < \alpha_1$ get shortened to Πn_α-tuples for $\beta \leqq \alpha < \alpha_1$, the β-th operation being used to step from β to $\beta + 1$ and convergence operations at limit ordinals. In complete lattices (not a varietal theory) or m-complete lattices, the operations *lim inf*

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are convergence operations.—Unfortunately, m-complete lattices have too many operations. There are m^m specializations of $\bigwedge$ uniformized by $\bigwedge$, but there are also ordinal strings having no uniformizer. Ont can create applications of 3.13, but I know of none to familiar theories except the basic two.

Now 3.13.2 concerns the dual of the category of models of $\boldsymbol{T}$ in $\mathscr{S}^{op} = \operatorname{Sem}\boldsymbol{T}'$, and asserts that $\operatorname{Sem}(\boldsymbol{T}; \operatorname{Sem}\boldsymbol{T}') = \operatorname{Sem}(\boldsymbol{T} \otimes \boldsymbol{T}')$ is tripleable over $\operatorname{Sem}\boldsymbol{T}'$. By 3.13, it is a varietal category over $\mathscr{S}$, thus tripleable over $\mathscr{S}$. We have the situation



with adjoint pairs of functors, the U's morphisms of facts, and $U'U = U''$. It is more or less known that in this situation (hypotheses indicated by the labeling) $\operatorname{Sem}\boldsymbol{T}''$ is tripleable over $\operatorname{Sem}\boldsymbol{T}'$. Part of the exercise can be modified; $\operatorname{Sem}\boldsymbol{T}'$ being $\mathscr{S}^{op}$, $\operatorname{Sem}\boldsymbol{T}''$ can be any complete fact whose grounding U'' has an adjoint F''. For the preadjoint F of U is defined at least on the objects $F'A$, i.e. $\operatorname{Hom}(F'A, U(\ )) = \operatorname{Hom}(A, U'U(\ )) = \operatorname{Hom}(F''A, \ )$ is representable. But the (largest) domain of a preadjoint F is closed under adjoining colimits of diagrams D such that FD has a colimit [12; 3.9]. Since every complete atomic Boolean algebra is a retract of a free one, and idempotents split in any complete fact, it follows that F is defined everywhere, i.e. it is an adjoint. To finish, we need tripleability of each $\operatorname{Sem}\boldsymbol{T}^{(J)}$ over $U^{(J)}F^{(J)}$; then $\operatorname{Sem}\boldsymbol{T}''$ is tripleable over UF, either by Beck's criterion or directly by using U' to take UF-algebras to $U''F''$-algebras.

Davis has classified the categories of $\boldsymbol{T}$-coalgebras (assuming cotripleableness); each is equivalent to the category of coalgebras over a theory having a rank, and he gives one such theory for each equivalence class [2]. He also shows that there are other categories cotripleable over $\mathscr{S}$, by an example in which U is the forgetful functor from complete atomic Boolean algebras with a completely additive closure operator satisfying $x^{-\prime-} = x^{-\prime}$. The question is open whether all categories cotripleable over $\mathscr{S}$ are duals of varieties. Precisely, if $\mathscr{C}$ is tripleable over $\mathscr{S}^{op}$ with forgetful functor U, is $\mathscr{C}$ tripleable over $\mathscr{S}$ with forgetful functor $U'U$?

The entire gap between 3.13 and P1 can be divided into four or five sub-

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problems, one of which is easily solved. This division seems relevant because the generation of $\boldsymbol{T} \otimes_e \boldsymbol{U}$ has a form like the form of the proof of 3.13: alternately applying operations of $\boldsymbol{T}$ and of $\boldsymbol{U}$. The easy step is this. The theories $\boldsymbol{T}$ such that in $\boldsymbol{T} \otimes \boldsymbol{U}$-algebras, the $\boldsymbol{U}$-closure of the $\boldsymbol{T}$-closure of a subset is always a subalgebra, are exactly the theories generated by constant and unary operations. For if $\boldsymbol{T}$ has a really m-ary operation, one can give its free algebra on m generators $\{x_\alpha\}$ $m-1$ idempotent endomorphisms which move x_1 to x_α and fix the other generators. The $\boldsymbol{U}$-closure of the $\boldsymbol{T}$-closure of $\{x_1\}$ is the union of the cyclic subalgebras on the x_α. Next question: (P1a) If the $\boldsymbol{T}$-closure of the $\boldsymbol{U}$-closure is always a subalgebra, must every small set of operations of $\boldsymbol{T}$ have a uniformizer? (P1b) For $\boldsymbol{T} \otimes_e \boldsymbol{U}$ to be varietal for all $\boldsymbol{U}$, must the alternating construction of it terminate after a small number of steps? Independent of $\boldsymbol{U}$? Three? Finally, there is the gap between $\boldsymbol{T} \otimes_e \boldsymbol{U}$ and its possibly simpler semantic equivalents.

Note: the obvious extension of 3.13 by tensoring two such $\boldsymbol{T}$'s is vacuous.

I said "no results" on P2, but perhaps this remark is worth noting. The obvious answer "theories having a rank" can be extended to theories having suitable cardinal gaps $(m, n]$ such that every word in n variables is a word in a subset of m of the variables. The details depend on the set theory (cardinal exponentiation). I omit details.

I have one result on general Galois connections. Note that limits of connections are pointwise limits (if both of the categories are left complete); in particular, F^2 $(= F \times F)$ and G^2 are adjoint if F and G are. An adjunction map $X \to G^2F^2X$ has coordinates $X \to GF^2X$ factoring across the adjunction $2X \to GF(2X)$ (if $2X$ exists); then $X \to (GF^2X)^2$ factors through the diagonal. When that is an isomorphism, the squaring is vacuous, and $X^2 = X$. If the connection F^2 is Galois, we conclude $G^2 = G$, whence $F^2 = F$.

Of course $F^2 = F$ means that the connection is *simple*, reducing to a universal connection between full reflective schlicht subcategories (schlicht: hom sets have at least one element), and therefore Galois. Summarizing (and removing the needless assumption that products exist):

3.14. *The following properties of a connection between two categories, one of which has finite coproducts, are equivalent: it is* (a) *simple, or* (b) *square and Galois, or* (c) *idempotent.*

Non-simple Galois connections need not be prime (at least on product categories). I do not know if they must be squarefree.

In lectures and preprints I have tried to formulate a problem, When does a universal connection cogenerate $\mathscr{C}(x)\mathscr{C}^{op}$? (Look at any familiar one, such as Pontrjagin duality, to see that it does and "should.") Since $\mathscr{C}(x)\mathscr{C}^{op}$

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is monoidal, equivalent to the category of endofunctors of $\mathscr{C}$ having a left adjoint, one can describe an obstruction monoid to the universal connection's cogenerating (or doing almost anything). This is analogous to the Picard group, of autoequivalences modulo isomorphism of functors. As far as I can see, it is merely analogous. One example where the Picard group vanishes and this obstruction does not is interesting enough for other reasons to be presented below (3.16.1). More superficially, the examples such as Pontrjagin duality and Stone duality and the universal connection between the variety of compact spaces and its dual seem to depend on injective cogenerators (which both of the dual categories of 3.16.1 lack). But in the category of pairs of pointed sets, $1\backslash\mathscr{S} \times 1\backslash\mathscr{S}$, every object is injective and projective, each object (X, Y) neither of whose coordinates is a single point generates and cogenerates, yet the Picard group, Z_2, measures the universal connection's failure to cogenerate.

(The fact that Stone duality is a cogenerating connection is an unpublished result of Banaschewski; I have extended it from Boolean spaces to compact Hausdorff spaces. For Pontrjagin duality it is plain, since one has $\mathscr{C}(x)\mathscr{C}^{op}$ equivalent to $\mathscr{C}^{op}$.)

3.15. THEOREM. *If $\mathscr{C}$ and $\mathscr{C}^{op}$ are each equivalent to the category of all models of a small theory, then they are schlicht.*

Proof. We have already recalled Gabriel's work from which every object of a small theory is importable. Call the resulting models *imported*. We need another observation of Gabriel. A classical (schlicht directed) direct mapping system is called *m-directed*, m a cardinal, if every m objects in it have a common successor. Then if $\boldsymbol{T}$ is small, there exists a small cardinal m such that $\mathrm{Mod}\,\boldsymbol{T}$ is closed under m-directed colimits in $Cat(\mathscr{T}^{op}, \mathscr{S})$ [7].

Our hypotheses imply that $\mathscr{C}$ has two small sets of objects $\{I_i\},\{C_\alpha\}$, corresponding to imported models in $\mathscr{C}$ and $\mathscr{C}^{op}$ respectively. In particular the I_i coseparate and the C_α separate. For some small m, each h^{I_i} preserves m-directed colimits, and every morphism from a power $C_\alpha{}^n$ to C_β $(n > m)$ factors across a partial power $C_\alpha{}^m$. (For in $\mathscr{C}^{op}$, an n-fold copower is an m-directed colimit of the indicated partial copowers.)

Choose regular $n > m$, and for each α (form a reduced power Q_α:) consider a product P_α of n factors F_s $(s \in S)$ each isomorphic with C_α. Form the m-directed system of products $P^k = \Pi[F_s : s \in S_k]$, where S_k runs over all subsets of S omitting fewer than n elements, ordered by reverse inclusion and bonded by projections. Take the colimit Q_α. The diagonal $C_\alpha \to P_\alpha$, followed by $P_\alpha \to Q_\alpha$, remains monomorphic, since two distinct morphisms $T_i \to C_\alpha \to P_\alpha$ remain distinct into all P^k. Yet every morphism

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$Q_\alpha \to C_\beta$ yields a map $P_\alpha \to Q_\alpha \to C_\beta$ which must factor both across a tiny partial power $\Pi[F_s\colon s \notin S_k]$ and across $\Pi[F_s\colon s \in S_k]$. But now, were there two distinct morphisms $f, g\colon I_i \to C_\alpha$, they would remain distinct after $C_\alpha \to P_\alpha \to Q_\alpha \to C_\beta$. The morphisms $f^*, g^*\colon I_i \to C_\alpha \to P_\alpha$ are distinguished by $P_\alpha \to C_\beta$ which factors two ways as above, so distinguished by their s-th coordinates for $s \in S_k \cap (S - S_k)$, which is absurd. Since the I's coseparate and the C's separate, C is schlicht.

The proof of 3.15 evidently did not use the full force of the assumption on $\mathscr{C}$ or on $\mathscr{C}^{op}$. One by-product is

3.15.1. *If A is a separator in a non-schlicht category $\mathscr{C}$ of all models of a small theory, the varietal theory* Str_e *of $h_A\colon \mathscr{C}^{op} \to \mathscr{S}$ has no rank.*

For the (equational) theory of such a hom functor is the dual of the category of powers of A (always varietal), and all we used on the C_α's is that they separate and morphisms $C_\alpha{}^n \to C_\beta$ factor across very small partial powers.

3.15.2. *The conclusion of* 3.15.1 *holds for a separator A in a non-schlicht (elementary) category of models in the sense of Freyd* [5, p. 91].

Proof. Form the set A^n as above and an ultrapower A' which is a coarser quotient of A^n than Q. A' is elementarily equivalent to A and thus in $\mathscr{C}$. The quotient function $p\colon A^n \to A'$ induces a natural transformation $\pi\colon (h_A)^n \to h_{A'}$. The diagonal transformation $\Delta\colon h_A \to (h_A)^n$ yields monic $\pi\Delta$. Now if A is a separator and $\mathscr{C}$ not schlicht, there exist two different morphisms $X \to A$, at which $(\pi\Delta)_X$ must take different values. Since A is a separator, they are still distinguished by some morphism $f\colon A' \to A$ and by the natural transformation ϕ which it induces. However, on the assumption that $\mathrm{Str}_e h_A$ has rank m, $(\phi\pi)_X\colon \mathrm{Hom}(X, A)^n \to \mathrm{Hom}(X, A)$ factors across a partial product $\mathrm{Hom}(X, A)^m$. Since the value of π_X on an n-tuple is independent of any m of its terms, $(\phi\pi)_X$ is constant: contradiction.

It would seem to be a fairly delicate syntactic problem to unite these parallel results. The problem is not mooted by any broad common generalization; for the quasivariety $\mathscr{C}_1$ of rings having sufficiently many homomorphisms into the real field, R, has a separator (namely R) whose algebraic theory has rank 1, if there are no small measurable cardinals. Of course $\mathscr{C}_1$ is not a quasivariety in the elementary sense [27], only in the structural sense [20]: a full subcategory of a variety closed under forming products and subalgebras.

Considering 3.15 as a theorem about universal connections, it does not extend to Galois connections. Indeed, if there are no small measurable cardinals, the variety $\mathscr{S}^{op}$ of complete atomic Boolean algebras is embeddable in the variety with rank $\aleph$ of Boolean σ-algebras, reflectively (the embedding

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being a morphism of facts, induced by the natural interpretation of σ-operations in complete operations) and fully (by the measure condition). Thus there is a non-simple Galois connection between $\mathcal{S}$ and $\mathfrak{Z}$. Once one has such an example, the rank of the bigger variety can be lowered to 1 by using the variety of M-sets, where M is the monoid of endomorphisms of a suitable free algebra (here, a free countably generated σ-algebra). On the other hand, the rankless variety $\mathcal{C}^{op}$ cannot be a quasivariety (in the weaker structural sense) in a variety with rank; for if $\mathcal{Q}$ is a quasivariety in a variety of rank m, then $\mathrm{Str}_e\mathcal{Q}$ also has rank at most m (since free algebras are projective, reflectors preserve copowers, and reflection maps here are coequalizers). Just past this, there is the further question: can $\mathcal{C}$ and $\mathcal{C}^{op}$ be non-schlicht quasivarieties (in varieties) having rank? Note, then neither is a quasivariety in the elementary sense.

Next, just a couple of interesting Galois connections. M. G. Stanley has described [28] a category $\mathcal{R}^1$ and a connection M from it to the category $\mathcal{D}$ of unitary linear algebras over the real field. We shall see that M is Galois (if there are no small measurable cardinals) and dually embeds a subcategory of $\mathcal{R}^1$ that may be called the *real analytic curves* into $\mathcal{D}$. The proof, most of which is in [25] and [28], applies equally well to *real algebraic curves*. The usual meanings of the italicized phrases must bend, but only enough to give complete categories.

An object of $\mathcal{R}^1$ is a Hausdorff space which consists of an open and closed discrete subspace and a real analytic 1-manifold with boundary. A morphism is an analytic mapping such that the image of every non-empty open set has non-empty interior. For each X in $\mathcal{R}^1$, MX is the algebra of extended-real meromorphic functions on X. For $f: X \to X'$, Mf is the homomorphism defined by composition with f. Note: the excluded analytic mappings, constant on some open set, do not in general induce homomorphisms of meromorphic functions. The subalgebra of analytic functions gives another connection, which is not Galois. Whether meromorphic functions on Riemann surfaces give a Galois connection depends on the unsolved problem whether embeddings of fields of meromorphic functions on connected Riemann surfaces are induced by analytic maps [25]. (The problems are equivalent modulo measurable cardinals; one implication will be clear from what follows and the inverse is not hard to deduce from [25].)

There are seven connected real analytic 1-manifolds with boundary: an open, a closed, and a half-open interval, a circle, and three non-separable "long lines." The separable ones have sufficiently many meromorphic functions, but a meromorphic function on a long line is constant. Accordingly the full subcategory $\mathcal{C}_0$ of $\mathcal{R}^1$ on objects with separable components is reflec-

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tive, and M factors across the reflector. It remains to show that $GM \mid \mathscr{C}_0$ is naturally equivalent to 1, where G is the adjoint of M on the right.

G may be described [28] as follows. In $\mathscr{R}^1$, an open interval P is left adequate; so $\mathscr{R}^1$ is naturally fully embedded in the variety of sets on which the monoid $\mathscr{R}^1(P,P)$ acts. (It is right complete and includes P, and is therefore [12; 4.7] reflective.) Then G takes each linear algebra Y to the obvious $\mathscr{R}^1(P,P)$-set on $\mathfrak{D}(Y,MP)$, which is [28] in $\mathscr{R}^1$ (hence of course in $\mathscr{C}_0$). So GMX is the set of unitary linear homomorphisms $h: MX \to MP$, and we must show that for X in $\mathscr{C}_0$ these are all induced by morphisms $P \to X$. (This is true outside $\mathscr{C}_0$; we must note also that different morphisms $P \to X$ induce different homomorphisms, when X is in $\mathscr{C}_0$ and thus has sufficiently many meromorphic functions.)

The needed result may be in print, at least in the connected case. At any rate, Royden's proof [25] of the like result for algebras of complex analytic functions on connected open Riemann surfaces applies with trivial changes to real analytic functions on connected objects of $\mathscr{C}_0$. But a homomorphism $MX \to MP$ must preserve the subalgebras of analytic functions; for $f \in MX$ omits the value ∞ if and only if $1 + f^2$ has a 3^k-th root in MX for all natural numbers k. Since every meromorphic function is a quotient of analytic functions (for future use, this is proved by an algebra formula: on the extended reals $x = x(1+x^2)^{-1}/(1+x^2)^{-1}$), the homomorphism is induced by an analytic function $f: P \to X$. As remarked earlier, f must be a morphism.

Finally, consider $h: MX \to MP$ for disconnected X. The idempotents of MX go to 0 or 1, and those which go to 1 correspond to an ultrafilter F in the set of components of X. We are assuming small cardinals are non-measurable, so either F is principal (and we are done) or F includes a union of disjoint sets of components S_j each of which is not in F. Note that the characterization of functions omitting the value ∞ shifts to any value; so the image of $h(f)$ is contained in the image of f. Then to the sets S_j we may associate a function f constantly equal to j on components in S_j. The image of $h(f)$ must be contained in the set of natural numbers greater than j, for every j, which is absurd. We have:

3.16. *If there are no small measurable cardinals, M is a Galois connection, and $M \mid \mathscr{C}_0$ is a dual embedding.*

If real algebraic 1-manifolds with boundary are defined in the same manner, there is at least one anomaly; the line modulo 1 is a perfectly good manifold having no non-constant meromorphic algebraic functions. If we take Nash's definition [23] (without boundary) and include also a closed

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and a half-open interval, then it takes a little time to verify the essential arguments from [25], [28], and above, but nothing in it requires further comment here. This category of real algebraic curves is dual to its category of algebras of extended-real algebraic functions, if there are no small measurable cardinals.

3.16.1. In $\mathscr{C}_0(x)\mathscr{C}_0{}^{op}$, the universal connection does not cogenerate, for there are "discrete" connections A whose underlying object in $\mathscr{C}_0$ is 0-dimensional. Pic $\mathscr{C}_0$ vanishes. The same holds for algebraic curves.

Appendix. Stable functors.

In [12], a construction of the stable reflection of any object F of $Cat(\mathcal{A}^{op}, \mathcal{S})$ for small $\mathcal{A}$ is mishandled, leading to the incorrect conclusion that if F is already stable it is a limit of limits of representable functors. Later in [12], and in [13], a stable reflection is sometimes introduced, with a reference to that construction. The construction is almost irrelevant, as long as it is established that the full subcategory of stable functors is reflective. However, the simplest way to correct the proofs in question is to edit the original construction as follows. In line -7, p. 646 [12], replace "dominion" by "right image" and "sesquistrict" by "multistrict." Four lines later, change "Since D is the dominion" to "Since q is epic."

As for the actual lengths of representations of stable functors as iterated limits, there is no small bound for general small categories. I do not know if there is even a small bound for each small category. To define length, let a 0-*limit* of a full subcategory $\mathcal{A}$ (or of its objects) mean an object of $\mathcal{A}$; an $\alpha+1$-*limit*, a limit of a diagram in the α limits; and a λ-*limit* for a limit ordinal λ means a μ-limit for some smaller μ.

For every small ordinal α there exists a small category $\mathcal{A}$ and a stable contravariant functor on $\mathcal{A}$ that is not an α-limit of representable functors.

Proof. The category $\mathcal{A}$ will have only three objects (but many morphisms); however, we define a category $\mathcal{B} \supset \mathcal{A}$ which has additional objects $A_1, \cdots, A_{\alpha+2}$. The objects of $\mathcal{A}$ are F, A_0, and D. The full subcategory on A_0 and D has just four morphisms, the identities and two morphisms $A_0 \to D$. Further, every $\mathcal{B}(A_\beta, A_\gamma)$ has one element if $\gamma \leqq \beta$, no element if $\gamma > \beta$; $\mathcal{B}(D, A_\beta)$ is empty, and $\mathcal{B}(A_\beta, D)$ has two elements. For $\beta > 0$ we call the morphisms $A_\beta \to D$ d_β and i_β; every composite $A_\beta \to A_\gamma \to D$ $(\gamma < \beta)$ is d_β.

Call the category so far defined (on D and $A_0, \cdots, A_{\alpha+2}$) $\mathscr{C}$. Convert

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the Eilenberg-MacLane representation of $\mathscr{C}$ into a full embedding in a variety $\mathscr{V}$ of rank 1 as in [11; 4.1], and note that it preserves equalizers and intersections. Let $\mathscr{B}$ be full in $\mathscr{V}$ on $\mathscr{C}$ and a free algebra on one generator, F. $\mathscr{A}$ is the full subcategory on F, A_0, and D.

Since $\mathscr{A}$ includes F, it is left adequate in $\mathscr{V}$, and its left closure is its left regular completion (the category of stable functors) [12]. Each $A_{\beta+1}$ is the equalizer of $A_\beta \rightrightarrows D$, and limit A_λ is the intersection of its predecessors; so they represent stable functors. Now in showing that A_β is not a γ-limit of $\mathscr{A}$ for $\gamma < \beta \leqq \alpha + 1$, we can ignore F, since A_β has no morphism into F (since $F \to A_{\beta+1}$). It will suffice to show that A_γ maps to every γ-limit of $\mathscr{A} \cap \mathscr{C}$. This is evident for $\gamma = 0$. Having it for γ, to map $A_{\gamma+1}$ to the limit of a diagram H in the γ-limits of $\mathscr{A} \cap \mathscr{C}$, map A_γ to each object of H in any way and compose with $A_{\gamma+1} \to A_\gamma$. This gives a projective cone, i.e. $A_{\gamma+1} \to A_\gamma$ equalizes everything, for it equalizes all pairs of morphisms to objects of $\mathscr{A}$ and $\mathscr{A}$ separates its completion. So $A_{\gamma+1}$ maps to the limit of H. Hence A_λ maps to every γ-limit if $\lambda > \gamma$, and at a limit ordinal λ this means A_λ maps to every λ-limit.

Apropos of the problem of constructing similar examples on a fixed small category $\mathscr{A}$, there is an applicable technique, developed and refined through many of the papers of Z. Hedrlin and A. Pultr (notably [10]). Naturally it cannot (as far as is known) digest an arbitrary $\mathscr{C}$ as we did above.

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