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INTRODUCTION TO THE THEORY OF SUPERMANIFOLDS

D. A. Leites

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Introduction

Recent years have seen the rapid development of a new branch of mathematics. Articles on the subject are easily recognized, as nearly all the terms used bear the prefix "super". Some of the ideas embodied in "supermathematics" were discussed individually long ago, though mathematicians lacked the stimulus to study them in detail, and it has only recently become clear that

they are all parts of a single whole.

Interest in supermathematics was originally aroused by its applications to physics. Apparently, a unified theory of strong, weak, electromagnetic, and gravitational interactions can be constructed in the language of supermanifolds. For this subject and for the beautiful properties of supersymmetries, we refer to the surveys of Ogievetskii and Mezinchesku [19] and Freedman and Nieuwenhuisen [26], which contain references to hundreds of physical papers, of which we only mention those of Berezin and Marinov [23] and Ogievetskii and Sokachev [20].

Representations of space-time have changed with the development of science, to the point where physicists now reluctantly accept the following fundamental premise.

We live in a (4, 4)*-dimensional supermanifold whose underlying manifold is the ordinary* 4*-dimensional space-time. The group of transformations of this supermanifold is a Lie supergroup whose points make up the Poincaré group.*[1]

This fundamental premise has a philosophical significance that transcends the confines of pure physics.

The first mathematician to realize that he stood on the threshold of a new subject ("supermathematics") was undoubtedly Berezin.

Being concerned with questions of the second quantization, he noticed the possibility of giving a parallel description of the boson and fermion fields and, as early as the 1960s, arrived at the conclusion that there is a non-trivial analogue in analysis, in which the rôle of functions is taken by elements of the Grassmann algebra.

The incubation period lasted for seven years, and then articles appeared in which various issues in supermathematics were presented. They contained the specific notions of a supermanifold and a Lie supergroup, the construction of an analogue to Lie theory for Lie supergroups and Lie superalgebras, a description of the simplest Lie superalgebras, the construction of a theory of integration in supermanifolds, the beginnings of a representation theory of Lie superalgebras and Lie supergroups, and more besides.

It is high time we acknowledged that the prefix "super" was introduced by spirited physicists, who first designated (by the single word super!) some remarkable groups that reshuffled the particles of different statistics. Then they realized the action of these supergroups on "superspaces", and this terminology later spread through the whole subject. Note that in some contemporary articles and in every article written before 1974 Lie superalgebras are called graded Lie algebras. I prefer the term "Lie superalgebra". Firstly, it

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[1] This model will be discussed in detail in the section describing the representations of Lie supergroups. The first papers in which the Poincaré group is extended to a Lie supergroup and the Minkowski space to a superspace, which is more suited to the construction of supergravity, are due to Gol'fand and Likhtman, Volkov and Akulov, and Wess and Zumino (see [19], [34]).

is more accurate, since the term "graded Lie algebra" is inconsistent (Lie superalgebras are not Lie algebras), and secondly, the term Lie superalgebra is elegant and dynamic.

Having said so much about physics, it is not surprising that the "super" point of view is also useful in mathematics (see [8], [16], [27], [29]).

In the large collection of articles on supermathematics and its applications, there are hardly any that give an account of the elementary concepts on which the theory is founded. The purpose of this article is to remedy this deficiency.

In the text we keep to the following terminology.

A statement will be called: an *assertion* if it is obvious or proved in a reference; a *remark* if it is unimportant at a first reading; a *lemma* if it is of an auxiliary nature; a *proposition* if it is important; and a *theorem* if it is very important.

In work with superobjects, consistency in choice of signs plays a special rôle. To avoid errors, we must keep in mind the following rule (Quillen [32]): *when something of parity p moves past something of parity q, the sign* $(-1)^{pq}$ *appears*.

This article covers the simplest ideas in the theory of supermanifolds and comprises the first part of a general survey by Bernshtein and myself, which also contains the theory of integration in supermanifolds, the elements of representation theory of Lie supergroups and Lie superalgebras, and the elements of differential geometry on supermanifolds (see [6], [7], [9], [25], [33]).

I have tried to imbue the presentation of supermanifolds with geometrical clarity. It differs in this from earlier articles, where, under the momentum of [5], all the ideas are expressed in terms of the algebra of functions on a supermanifold.

I am deeply grateful to F. A. Berezin for attracting my interest in 1971 to the problem, which at that time seemed horrendous enough, of "doing analysis in the Grassmann algebra". I thank him and A. L. Onishchik for much help. I also thank B. M. Zupinik, A. N. Rubakov, and B. L. Feigin for many fruitful discussions, and Yu. I. Manin, V. V. Molotkov, A. B. Sosinskii, D. B. Fuks, and V. N. Shander for some useful comments. I am grateful to I. A. Akchurin for psychological preparation for the solution of Berezin's problem and for his advice (in 1968) to read the lectures of Yu. I. Manin [17].

CHAPTER I

LINEAR ALGEBRA IN SUPERSPACES

All the spaces and algebras considered here are over the field $k = \mathbf{R}$ or $\mathbf{C}$ (though everything that is said remains true for any field k with char $k \neq 2$).

The purpose of this chapter is to transfer the basic concepts of linear algebra (as in an undergraduate course) to the "super" case.

§ 1. Linear superspaces

1.1.1. Let $\mathbf{Z}_2 = \{\bar{0}, \bar{1}\}$ denote the field of residues modulo 2. A linear space M is called a *superspace* if it admits a decomposition

$$M = M_{\bar{0}} \oplus M_{\bar{1}}.$$

The elements of $M_{\bar{0}}$ and $M_{\bar{1}}$ are called *homogeneous even* and *odd* elements of M, respectively. If $v \in M_i$, $i \in \mathbf{Z}_2$, we write $p(v) = i$ and call $p(v)$ the *parity* of v. A *subsuperspace* is a subspace $N \subset M$ such that

$$N = (N \cap M_{\bar{0}}) \oplus (N \cap M_{\bar{1}}).$$

If M and N are superspaces, we make $M \oplus N$, $M \otimes N$, and $\mathrm{Hom}(M, N)$ into superspaces by setting

$$(M \oplus N)_i = M_i \oplus N_i, \quad (M \otimes N)_i = \bigoplus_{m+n=i} M_m \otimes N_n,$$

and $\mathrm{Hom}(M, N)_i = \{F \in \mathrm{Hom}(M, N) \mid FM_m \subset N_{m+i}\}$, respectively, where $m, n, i \in \mathbf{Z}_2$.

Morphisms from M to N are elements of $\mathrm{Hom}(M, N)_{\bar{0}}$.

1.1.2. A superspace A is called a *superalgebra* if A is an associative algebra with identity and if the multiplication $A \times A \to A$ is an even bilinear transformation.

A superalgebra A is called *commutative* if $ab = (-1)^{p(a)p(b)}ba$ for homogeneous $a, b \in A$.

Here and later, $(-1)^{\bar{0}} = 1$, $(-1)^{\bar{1}} = -1$. In what follows, we assume that a relation holding only on the homogeneous elements also holds on arbitrary elements providing that it holds on their homogeneous components.

1.1.3. IMPORTANT EXAMPLE. For $\xi = (\xi_1, \ldots, \xi_n)$ we denote by $\Lambda(n)$, $\Lambda[\xi]$, or $k(\xi)$ the exterior algebra (Grassmann algebra) in n variables. This is the algebra generated by $\xi_1, \ldots, \xi_n$ subject to the relations

$$\xi_i \xi_j = -\xi_j \xi_i \quad (i, j = 1, 2, \ldots, n).$$

In particular, $\xi_i^2 = 0$.

Any $f \in \Lambda(n)$ can be written uniquely as a sum

$$f = \sum f_\nu \xi^\nu, \qquad f_\nu \in k,$$

where ν takes all possible values $\nu = (\nu_1, \ldots, \nu_n)$, where each $\nu_i = 0$ or 1, and $\xi^\nu = \xi^{\nu_1} \ldots \xi_n^{\nu_n}$. This expression can also be written in the form:

$$f = \sum_{0 \leqslant j \leqslant n} \sum_{1 \leqslant i_1 < \ldots < i_j \leqslant n} f_{i_1 \ldots i_j} \xi_{i_1} \cdots \xi_{i_j}, \qquad f_{i_1 \ldots i_j} \in k.$$

We introduce on $\Lambda(n)$ the structure of a superalgebra by setting $p(\xi_i) = \bar{1}$.

1.1.4. The *tensor product* of two superalgebras A and B is the superspace $A \otimes B$, together with the structure of a superalgebra given by

$$(x \otimes y)(z \otimes t) = (-1)^{p(y)p(z)} xz \otimes yt,$$

where $x, z \in A$ and $y, t \in B$.

ASSERTION. 1) *If A and B are associative superalgebras, then so is $A \otimes B$.*

2) *The tensor product of commutative superalgebras is also commutative.*

For example,

$$\Lambda(n) \otimes \Lambda(m) = \Lambda(n + m).$$

1.1.5. For any superalgebra A there is a *canonical projection*

$$\pi\colon A \to A/(A_{\bar{1}}) = A_{\bar{0}}/((A_{\bar{1}})^2).$$

Here the symbol (x) denotes the ideal generated by a set x.

We shall often make use of the following result.

LEMMA. *Let A be a commutative superalgebra. Then an element $a \in A$ is invertible if and only if $\pi(a)$ is invertible* (see Proposition 1.7.2).

§2. Modules over superalgebras

1.2.1. Let A be an associative superalgebra with identity.

A *left module over A* (or *left A-module*) is a superspace M together with a left action of A on M, that is, a mapping $A \times M \to M$ satisfying the following conditions:

a) $a(bm) = (ab)m$, $a, b \in A$, $m \in M$,

b) $(a + b)m = am + bm$, $a(m + m') = am + am'$,

c) $1 \cdot m = m$,

d) if $p(a)$ and $p(m)$ are defined, then $p(am) = p(a) + p(m)$.

Right A-modules are defined in a similar way.

1.2.2. Throughout this chapter, all superalgebras are assumed to be commutative unless the contrary is stated.

If A is a commutative superalgebra, then any left A-module can be turned into a right A-module (and vice versa) by setting

$$(*) \qquad ma = (-1)^{p(m)p(a)}am, \quad a \in A, \quad m \in M.$$

The structures of a left and right module over A are compatible:

$$a(mb) = (am)b, \quad a, b \in A, \quad m \in M,$$

that is, every module over a commutative superalgebra is two-sided.

In what follows, as in the "even" commutative case, a module over a commutative superalgebra is said to be two-sided whenever the transition from left action to right action is effected by the formula (*).

1.2.3. Let M and N be A-modules. An *A-homomorphism* or (*linear*) *operator* from M to N is a linear transformation $F\colon M \to N$ such that

$$F(ma) = (F(m))a \quad \text{for all} \quad a \in A, \ m \in M.$$

The superspace of A-homomorphisms from M to N is denoted by $\mathrm{Hom}_A(M, N)$.

In $\mathrm{Hom}_A(M, N)$ we introduce the structure of an A-module by the formula

(*) $$(aF)(m) = a(F(m)), \quad (Fa)(m) = F(am).$$

From the formulae (*) here and in 1.2.2 it follows that

$$aF = (-1)^{p(a)p(F)} Fa.$$

Given operators $F: M \to N$ and $G: N \to L$, their composite $G \circ F: M \to L$ is also an A-linear operator, and

$$(aG) \circ F = a(G \circ F), \quad (Ga) \circ F = G \circ (aF), \quad G \circ (Fa) = (G \circ F)a.$$

1.2.4. Let A be a commutative superalgebra. A superalgebra C is called an *A-algebra* if C is an A-module such that

$$(ac_1)c_2 = a(c_1c_2), \quad (c_1a)c_2 = c_1(ac_2), \quad c_1(c_2a) = (c_1c_2)a,$$

where $a \in A$ and $c_1, c_2 \in C$.

EXAMPLES. 1) $\mathrm{End}_A(M) = \mathrm{Hom}_A(M, M)$ is an associative A-algebra with identity.

2) If C is an associative superalgebra, the set

$$Z(C) = \{z \in C \mid cz - (-1)^{p(c)p(z)}zc = 0 \text{ for all } c \in C\}$$

is called the *centre* of C. It is clearly a $Z(C)$-algebra. The structure of C as an A-algebra is given by the homomorphism $A \to Z(C)$.

1.2.5. The A-module $M^* = \mathrm{Hom}_A(M, A)$ is called the *adjoint* or *dual* of M. The pairing of M^* and M is denoted by $(\ ,\ )$, that is, (m^*, m) is the image of $m \in M$ under the action of $m^* \in M^*$. It follows from the definition that

$$(m^*a, m) = (m^*, am), \quad (am^*, m) = a(m^*, m),$$
$$(m^*, ma) = (m^*, m)a, \quad a \in A, \quad m \in M, \quad m^* \in M^*.$$

For any A-module M there is a canonical homomorphism: $I_M: M \to M^{**} = (M^*)^*$, given by the formula

$$(I_M(m), m^*) = (-1)^{p(m)p(m^*)}(m^*, m).$$

1.2.6. Corresponding to any $F \in \mathrm{Hom}_A(M, N)$ there is an *adjoint operator* $F^* \in \mathrm{Hom}_A(N^*, M^*)$ defined by

(*) $$(F^*n^*, m) = (-1)^{p(F)p(n^*)}(n^*, Fm).$$

This formula (*) yields the following result.

ASSERTION. 1) $p(F) = p(F^*)$.

2) *If $F: M \to N$ and $G: N \to L$ are homomorphisms of A-modules, then* $(GF)^* = (-1)^{p(F)p(G)}F^*G^*$.

3) $$I_N \circ F = F^{**} \circ I_M.$$

1.2.7. The *tensor product of A-modules* M and N is the superspace $M \otimes_A N$, that is, the factor of $M \otimes N$ (the tensor product over k) by the relations $ma \otimes n - m \otimes an = 0$, where $m \in M$, $n \in N$, $a \in A$.

$M \otimes_A N$ becomes an A-module if we define

$$a(m \otimes_A n) = am \otimes_A n, \quad (m \otimes_A n)a = m \otimes_A na.$$

There is a *canonical isomorphism* T between $M \otimes_A N$ and $N \otimes_A M$

$$T\colon m \otimes_A m \mapsto (-1)^{p(m)p(n)} n \otimes_A m.$$

1.2.8. Let $P(M)$ denote the A-module consisting of elements $P(m)$, $m \in M$, with A-action and parity given by

$$p(P(m)) = \bar{1} - p(m), \qquad P(m)a = P(ma), \qquad aP(m) = (-1)^{p(a)} P(am).$$

(Note that the last equation follows from the other two.)

The mapping $m \mapsto P(m)$ is the canonical odd homomorphism from M to $P(M)$.

For any A-homomorphism $F\colon M \to N$, where M and N are A-modules, we define an A-homomorphism $P(F)\colon P(M) \to P(N)$ by setting

$$P(F)(P(m)) = P(Fm).$$

We identify M and $P(P(M))$ by setting $m = P(P(m))$. It is then clear that $F = P(P(F))$ for any homomorphism F.

§3. Matrix algebra

To study the coordinate expression for vectors in free modules and the matrix notation for operators we need a matrix calculus. Since the supercase is somewhat different from the ordinary one, we devote this section to the necessary definitions. The motivation for them will become clear in §4.

1.3.1. A *matrix structure* is a rectangular array whose cells are indexed by pairs consisting of a row number and a column number.

A *supermatrix structure* is a matrix structure with a parity attached to each row and column. (The parities of the i-row and j-column are denoted by $p_{\mathrm{row}}(i)$ and $p_{\mathrm{col}}(j)$, respectively.)

We usually arrange a supermatrix structure in such a way that all the even rows and columns come first, and the odd ones second, so that it can be conveniently written in block form

$$X = \begin{pmatrix} R & S \\ T & U \end{pmatrix},$$

where R, S, T, U are the matrix structures corresponding to the partition into even and odd rows and columns.

If a supermatrix structure has p even rows and q odd rows, and r even columns and s odd columns, we call it a matrix structure of *size* $(p, q) \times (r, s)$. A $(p, q) \times (p, q)$ structure is said to have *order* (p, q).

1.3.2. Given a supermatrix structure X, the set $\{X_{ij} \mid X_{ij} \in A\}$ corresponding to the cells of X is called a *matrix* over A.

The notion of parity in the space of matrices over A is defined as follows:

$$p(X)=\bar{0} \quad \text{if} \quad p(X_{ij})+p_{\text{row}}(i)+p_{\text{col}}(j)=\bar{0} \quad \text{for all} \quad i, j,$$
$$p(X)=\bar{1} \quad \text{if} \quad p(X_{ij})+p_{\text{row}}(i)+p_{\text{col}}(j)=\bar{1} \quad \text{for all} \quad i, j.$$

When a matrix is written in block form

$$X=\begin{pmatrix} R & S \\ T & U \end{pmatrix}$$

in accordance with the partition by parity of rows and columns, the definition of the parity of X can be restated in the form

$$p(X)=\bar{0} \quad \text{if} \quad p(R_{ij})=p(U_{ij})=\bar{0},\ p(S_{ij})=p(T_{ij})=\bar{1},$$
$$p(X)=\bar{1} \quad \text{if} \quad p(R_{ij})=p(U_{ij})=\bar{1},\ p(S_{ij})=p(T_{ij})=\bar{0}.$$

1.3.3. The product of two matrices X and Y is computed in the usual way:

$$(XY)_{ij}=\sum_k X_{ik}Y_{kj}.$$

But in the supercase we consider such a product to be defined only if the columns of X have the same parity as the corresponding rows of Y. In particular, a $(p, q)\times(m, n)$ matrix can only be multiplied on the right by an $(m, n)\times(r, s)$ matrix. Under these conditions, we have $p(XY)=p(X)+p(Y)$.

1.3.4. EXAMPLE. Let $\text{Mat}_{p,q}(A)$ denote the space of matrices of order (p, q) over a superalgebra A. With respect to the parity and multiplication defined above, this space forms an associative superalgebra.

1.3.5. The superspace of $(p, q)\times(m, n)$ matrices over a commutative superalgebra A is made into an A-module by defining (see 1.4.4):

$$(Xa)_{ij}=(-1)^{p(a)p_{\text{col}}(j)}X_{ij}a, \qquad (aX_{ij})=(-1)^{p(a)p_{\text{row}}(i)}aX_{ij}.$$

An alternative definition is as follows. For each pair (r, s), we define a homomorphism of superalgebras $A\to\text{Mat}_{r,s}(A)$, by assigning to each $a\in A$ the diagonal matrix $\text{scalar}_{r,s}(a)=(a_{ij})$, where $a_{ij}=0$ for $i\neq j$ and $a_{ii}=(-1)^{p_{\text{row}}(i)p(a)}a$. If, as usual, the first r rows and columns are even and the others are odd, then

$$\text{scalar}_{r,s}(a)=\text{diag}(\underbrace{a, \ldots, a}_{r}, \underbrace{(-1)^{p(a)}a, \ldots, (-1)^{p(a)}a}_{s}).$$

If X is a $(p, q)\times(m, n)$ matrix, then

$$aX=\text{scalar}_{p,q}(a)\cdot X, \quad Xa=X\cdot\text{scalar}_{m,n}(a),$$

where $\text{scalar}_{p,q}(a)$ and $\text{scalar}_{m,n}(a)$ are the matrices of order (p, q) and (m, n), respectively, corresponding to $a\in A$.

In particular, the associativity of matrix multiplication implies that

$$(aX)Y=a(XY), \quad (Xa)Y=X(aY), \quad X(Ya)=(XY)a.$$

EXERCISE. Determine the centre of $\text{Mat}_{p,q}(A)$.

1.3.6. A *row-vector* (*column-vector*) is a matrix with just one row (column), and that of parity $\bar{0}$ (see 1.4.4).

1.3.7. We now define transposition in the space of matrices over a commutative superalgebra A. This differs from the ordinary definition by the appearance of certain signs. Therefore, in contrast to ordinary transposition, which is denoted by t, we denote *supertransposition* by the symbol st.

Let $X = (X_{ij})$ be a $(p, q) \times (m, n)$ matrix over A. Its *supertranspose* X^{st} is the $(m, n) \times (p, q)$ matrix whose entries have the following form (see 1.3.8 and 1.4.7):

$$(X^{\mathrm{st}})_{ij} = (-1)^{(p_{\mathrm{row}}(i)+p_{\mathrm{col}}(j))(p(X)+p_{\mathrm{row}}(i))} X_{ji} = (-1)^{(p_{\mathrm{row}}(i)+p_{\mathrm{col}}(j))(p(X_{ij})+p_{\mathrm{col}}(j))} X_{ji}$$

where the parities $p_{\mathrm{row}}(i)$ and $p_{\mathrm{col}}(j)$ are with respect to X^{st}. In block notation

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix};$$

this means that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{\mathrm{st}} = \begin{pmatrix} A^t & C^t \\ -B^t & D^t \end{pmatrix} \qquad \text{if} \quad p(X) = \bar{0},$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{\mathrm{st}} = \begin{pmatrix} A^t & -C^t \\ B^t & D^t \end{pmatrix} \qquad \text{if} \quad p(X) = \bar{1}.$$

Note that the double transpose has the form

$$\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{\mathrm{st}}\right)^{\mathrm{st}} = \begin{pmatrix} A & -B \\ -C & D \end{pmatrix}.$$

It is clear that supertransposition satisfies the identity $(\mathrm{st})^4 = \mathrm{id}$.

EXAMPLE. If X is a column-vector with coordinates

$$(x_1, \ldots, x_{m+1}, \ldots, x_{m+n}),$$

the first m being even and the rest odd, then the row-vector X^{st} has the coordinates

$$(x_1, \ldots, x_m, -(-1)^{p(x_{m+1})} x_{m+1}, \ldots, -(-1)^{p(x_{m+n})} x_{m+n}).$$

If $Y = (y_1, \ldots, y_{m+n})$ is a row vector with the first m columns even and the rest odd, then Y^{st} is the column vector with coordinates

$$(y_1, \ldots, y_m, (-1)^{p(y_{m+1})} y_{m+1}, \ldots, (-1)^{p(y_{m+n})} y_{m+n}).$$

1.3.8. ASSERTION. *The transpose of a product is given by*

$$(XY)^{\mathrm{st}} = (-1)^{p(X)p(Y)} Y^{\mathrm{st}} X^{\mathrm{st}}.$$

In particular,

$$(aX)^{\mathrm{st}} = (-1)^{p(a)p(X)} X^{\mathrm{st}} a = a X^{\mathrm{st}}.$$

§4. Free modules

The structure of modules over superalgebras is complicated enough, even in the commutative case. In this section we select a class of modules that are simpler by virtue of properties shared with linear spaces over a field.

All the modules considered are over a superalgebra A.

1.4.1. Let I be a set that is the disjoint union of subsets $I_{\bar{0}}$ and $I_{\bar{1}}$.

A *basis* of an A-module is a set of homogeneous elements $m_i \in M$, $i \in I$, for which $p(m_i) = \bar{0}$ for $i \in I_{\bar{0}}$ and $p(m_i) = \bar{1}$ for $i \in I_{\bar{1}}$, and such that every $m \in M$ is uniquely expressible as a sum $\Sigma \mu_i m_i$, $\mu_i \in A$, where all but finitely many of the μ_i are equal to 0.

An A-module M is said to be *free* if it has a basis with some indexing set $I = I_{\bar{0}} \cup I_{\bar{1}}$. The *dimension* of a free A-module is the element $p + q\varepsilon$ of the ring $\mathbf{Z}[\varepsilon]/(\varepsilon^2 - 1)$, with $p = |I_{\bar{0}}|$, $q = |I_{\bar{1}}|$. It is customary to write $\dim M = (p, q)$, for the sake of brevity.

Unless otherwise stated, we assume modules to be finite-dimensional, that is, $\dim M = (p, q)$ with $p, q < \infty$. We usually enumerate the elements of a basis in a free module so that the even ones come first and then the odd ones.

LEMMA. *The dimension of a free module is independent of the choice of basis.*

PROOF. We consider a homomorphism from the ring

$$A/(A_{\bar{1}}) = A_{\bar{0}}/((A_{\bar{1}})^2)$$

into any field k. Then $M \otimes_A k$ is a superspace over k of dimension (p, q). The invariance of p and q now follows from that of the dimension of a linear space over a field.

1.4.2. We see that free A-modules with a single generator fall into two types:

a) the generator is even, and the module is canonically isomorphic to A,

b) the generator is odd, and the module coincides with $P(A)$.

It is clear that $P(M) = P(A) \otimes_A M$ for any A-module M (see 1.2.8), and that if $\dim M = (p, q)$, then

$$\dim P(M) = (q, p).$$

1.4.3. Let $\{m_i\}$ be a basis for an A-module M. By definition, each element $m \in M$ is uniquely expressible in the form

$$m = \sum \mu_i^l m_i, \qquad \mu_i^l \in A.$$

We call the $\mu_i^l \in A$ the *left coordinates* of m (with respect to the given basis) and write them as a row-vector $\langle \mu_j^l |$, defining $p_{\mathrm{col}}(j) = p(m_j)$ (see §3). Now m can also be written uniquely in the form

$$m = \sum m_i \mu_i^r, \qquad \mu_i^r \in A.$$

We call the μ_i^r the *right coordinates* of m and write them as a column-vector $|\mu_i^r\rangle$, defining $p_{\mathrm{row}}(i) = p(m_i)$.

It is easy to verify that the row-vector of left coordinates is obtained from the column-vector of right coordinates by supertransposition:

$$\langle \mu_i^l | = (|\mu_i^r\rangle)^{\mathrm{st}} \text{ (see § 3).}$$

1.4.4. Let m_j and n_i be the elements of a basis of A-modules M and N, respectively. To any operator $F \in \mathrm{Hom}_A(M, N)$ we assign the matrix $\overline{F} = (F_{ij})$, where

(*) $$Fm_j = \sum n_i F_{ij}.$$

We put $p_{\text{row}}(i) = p(n_i)$, $p_{\text{col}}(j) = p(m_j)$; $\overline{F}$ is a (dim N) × (dim M) matrix. Conversely, every such matrix corresponds to the operator given by (*).

To every homogeneous operator there corresponds a matrix of the same parity. To the product (composite) of operators there corresponds the product of the corresponding matrices. The correspondence $F \to \overline{F}$ is an A-module homomorphism.

If $M = N$ and $m_i = n_i$, the correspondence $F \to \overline{F}$ defines an isomorphism of superalgebras

$$\text{End}_A M \to \text{Mat}_{p,q}(A), \text{where } (p,\ q) = \dim M.$$

Here an element $a \in A \subset \text{End}_A M$ goes over into the matrix scalar(a).

If $M = A$ and $m_1 = 1$, then $\text{Hom}_A(A, N)$ is naturally identified with N; we assign to any operator $F\colon A \to N$ the column-vector of right coordinates of $F(1) \in N$,

1.4.5. We now consider the effect of a change of basis on coordinates of vectors and operators.

Suppose that $\{m_i\}$ and $\{m_i'\}$ are two bases for an A-module M. Then the matrix $C = (c_{ij})$, defined by

$$m_j' = \sum m_i c_{ij},$$

is called the *matrix of coordinate change*. It is clear that C is both even and invertible.

The right coordinates $|\mu_i^r\rangle$, $|\mu_i'^r\rangle$ of a vector $m \in M$, with respect to $\{m_i\}$, $\{m_i'\}$ respectively, are related by the formula

$$|\mu_i^r\rangle = C\,|\mu_i'^r\rangle, \quad \text{that is,} \quad \mu_i^r = \sum c_{ij}\mu_j'^r.$$

Hence, for the left coordinates,

$$\langle\mu_i^l| = \langle\mu_i'^l|C^{\text{st}} \qquad (\text{because } p(C) = \bar{0}).$$

Now let $\{n_i\}$ and $\{n_i'\}$ be two bases for the A-module N. Let $\overline{F}$ be the matrix of an operator $F\colon M \to N$ with respect to $\{m_i\}$, $\{n_i\}$, and $\overline{F}'$ its matrix with respect to $\{m_i'\}$, $\{n_i'\}$. It is easy to verify that these matrices are related by

$$\overline{F} = D\overline{F}'C^{-1},$$

where C and D are the matrices of coordinate change in M and N, respectively. If $M = N$, $m_i = n_i$, and $m_i' = n_i'$, we obtain the formula describing the effect of changing the basis on the coordinates of an operator $F \in \text{End}_A M$:

$$\overline{F} = C\overline{F}'C^{-1}.$$

1.4.6. Let M be a finite-dimensional free A-module, and $\{m_i\}$ a basis of M. Then it is easy to check that the $m_i^* \in M^*$ given by

$$(m_i^*,\ m_j) = \delta_{ij}$$

form a basis for M^*. The bases $\{m_i\}$ and $\{m_i^*\}$ are said to be *dual*; $\{m_i^*\}$ is the *left dual* of $\{m_i\}$, and $\{m_i\}$ is the *right dual* of $\{m_i^*\}$.

It is clear that $p(m_i^*) = p(m_i)$, in particular, dim $M^* = \dim M$.

Let $\{m_i^{**}\}$ be the basis of M^{**} that is left dual to $\{m_i^*\}$. Clearly, $I_M(m_i) = (-1)^{p(m_i)}m_i^{**}$, so that $I_M : M \to M^{**}$ is an isomorphism. This is used to identify M and M^{**}; in particular, we can consider both (m^*, m) and

$$(m,\ m^*) = (-1)^{p(m)p(m^*)}(m^*,\ m).$$

If $\{m_i^*\}$ is a basis for M^*, and $\{m_i'\}$ and $\{m_i\}$ are its left and right duals, then these bases of M are related by $m_i' = (-1)^{p(m_i)}m_i$.

If $m^* \in M^*$, and $a_i = (m^*, m_i)$, then the row-vector $\langle a_i \mid$ coincides with the left coordinate vector of m^* in the basis $\{m_i^*\}$. In particular, if m is a vector with right coordinates $\mid \mu_i\rangle$, then

$$(m^*,\ m) = \sum a_i\mu_i.$$

The row vector $\langle a_i \mid$ coincides with the matrix of $m^* \in M^*$ in the basis $\{m_i\}$, $\{1\}$.

1.4.7. Let M and N be A-modules with bases $\{m_i\}$ and $\{n_i\}$, and let $\bar{F}$ be the matrix of an operator $F: M \to N$ with respect to these. Then, with respect to the left dual bases $\{m_i^*\}$ and $\{n_j^*\}$ of M^* and N^*, the adjoint operator F^* has the matrix $\bar{F}^* = (\bar{F})^{\mathrm{st}}$. This follows easily from the definitions.

1.4.8. Let $F \in \mathrm{Hom}_A(M, N)$. If we order the bases $\{m_i\}$ and $\{n_j\}$ in such a way that the even ones come first and then the odd ones, and if to F there corresponds in the bases $\{m_i\}$, $\{n_j\}$

$$\bar{F} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

then, in the bases $\{P(m_i)\}$ and $\{P(n_i)\}$ for $P(M)$ and $P(N)$, to $P(F)$ (see 1.2.8) there corresponds the matrix

$$\overline{P(F)} = \begin{pmatrix} D & C \\ B & A \end{pmatrix}.$$

§5. Bilinear forms

1.5.1. Let M and N be modules over a commutative superalgebra A. A mapping $B: M \times N \to A$ is called a *bilinear form* if it is additive in both arguments and $B(ma, n) = B(m, an)$ and $B(m, na) = B(m, n)a$, where $m \in M$, $n \in N$, $a \in A$.

EXAMPLE. If $M = N^*$, then $B(n^*, n) = (n^*, n)$ is a bilinear form.

It is readily checked that every bilinear form B determines a morphism of A-modules $\hat{B}$: $M \otimes_A N \to A$, given by $\hat{B}(m \otimes n) = B(m, n)$. Thus the space $\mathrm{Bil}_A(M, N)$ of bilinear forms can be identified with the space $(M \otimes {}_AN)^*$. The space then acquires a natural A-module structure. It is clear that

$$p(B(m,\ n)) = p(B) + p(m) + p(n),$$
$$(aB)(m,\ n) = a \cdot B(m,\ n) = (-1)^{p(a)p(B)}B(am,\ n).$$

1.5.2. Every bilinear form $B \in \mathrm{Bil}_A(M, N)$ determines a homomorphism $B_{M,N^*}: M \to N^*$, by the rule

$$(B_{M,N^*}(m),\ n) = B(m,\ n).$$

The correspondence $B \mapsto B_{M,N^*}$ defines an isomorphism between the A-modules $\mathrm{Bil}_A(M, N)$ and $\mathrm{Hom}_A(M, N^*)$.

Similarly, we can define an isomorphism from $\mathrm{Bil}_A(M, N)$ to $\mathrm{Hom}_A(N, M^*)$, sending a form B to the operator B_{N,M^*} given by

$$(m,\ B_{N,\ M^*}(n)) = (-1)^{p(B)p(m)} B(m,\ n).$$

A bilinear form $B \in \mathrm{Bil}_A(M, N)$ is called *non-degenerate* if B_{M,N^*} and B_{N,M^*} are isomorphisms.

1.5.3. Let M and N be finite-dimensional A-modules with bases $\{m_i\}$ and $\{n_i\}$, respectively. We assign to a form $B \in \mathrm{Bil}_A(M, N)$ the matrix $\bar{B} = B_{ij}$, where

$$B_{ij} = (-1)^{p(m_i)p(B)} B(m_i,\ n_j).$$

This is called the *matrix of the bilinear form.* It is easy to see that $\bar{B}$ is the matrix corresponding to the operator B_{N,M^*} with respect to the bases $\{n_j\}$ and $\{m_j^*\}$, where $\{m_j^*\}$ is the left dual of $\{m_j\}$.

The correspondence $B \mapsto \bar{B}$ defines an isomorphism from $\mathrm{Bil}_A(M, N)$ onto the A-module of matrices with the corresponding superstructure.

If μ^l is the row-vector of left coordinates of $m \in M$ and ν^r the column-vector of right coordinates of $n \in N$, then it is easy to check that

$$B(m,\ n) = (-1)^{p(B)p(m)} \langle \mu^l \mid \bar{B} \mid \nu^r \rangle.$$

1.5.4. Let M and N be A-modules, C and D the matrices of a coordinate change in M and N. Then the matrix $\bar{B}'$ of a form B in the new basis is related to $\bar{B}$ by the formula

$$\bar{B}' = C^{\mathrm{st}}\bar{B}D.$$

In particular, if $M = N$, $m_i = n_i$, and $m_i' = n_i'$, then $\bar{B}' = C^{\mathrm{st}}\bar{B}C$.

1.5.5. Given a form $B \in \mathrm{Bil}_A(M, N)$, we define a form $B^t \in \mathrm{Bil}_A(N, M)$ by setting

$$B^t(n,\ m) = (-1)^{p(n)p(m)} B(m,\ n).$$

It is easy to verify that B^t is indeed a bilinear form.

The correspondence $B \to B^t$ defines an isomorphism of A-modules: $\mathrm{Bil}_A(M, N) \mapsto \mathrm{Bil}_A(N, M)$. Furthermore, $(B^t)^t = B$ for every $B \in \mathrm{Bil}_A(M, N)$.

If $\bar{B}$ is the matrix of B with respect to bases $\{m_i\}$ and $\{n_j\}$ of M and N, then the matrix $\bar{B}^t$ of B^t has the form

$$(\bar{B}^t)_{ij} = (\bar{B}^{\mathrm{st}})_{ij} (-1)^{p_{\mathrm{row}(i)}}.$$

In block notation,

$$\overline{B}=\begin{pmatrix} R & S \\ T & U \end{pmatrix} \mapsto \overline{B}^t, \quad \overline{B}^t=\begin{cases} \begin{pmatrix} R^t & T^t \\ S^t & -U^t \end{pmatrix}, & p(B)=0, \\ \begin{pmatrix} R^t & -T^t \\ -S^t & -U^t \end{pmatrix}, & p(B)=\overline{1}. \end{cases}$$

1.5.6. We now examine the case $M=N$ and denote the space $\mathrm{Bil}_A(M, M)$ by $\mathrm{Bil}_A(M)$.

A form $B \in \mathrm{Bil}_A(M)$ is called *symmetric* if $B^t = B$, and *skew-symmetric* if $B^t = -B$. It is clear that the spaces of symmetric and skew-symmetric forms are A-submodules of $\mathrm{Bil}_A(M)$. Any bilinear form B can be uniquely written as the sum of a symmetric and a skew-symmetric form: $B = \frac{B+B^t}{2} + \frac{B-B^t}{2}$.

Let $\{m_i\}$ be a basis for M. Then to symmetric forms there correspond matrices that, in block notation, are of the form

$$\overline{B}=\begin{pmatrix} R & S \\ T & U \end{pmatrix}, \quad \text{where} \quad R^t=R, \quad U^t=-U, \quad S^t=(-1)^{p(B)}T.$$

To skew-symmetric forms there correspond matrices

$$\overline{B}=\begin{pmatrix} R & S \\ T & U \end{pmatrix}, \quad \text{where} \quad R^t=-R, \; U^t=U, \; S^t=-(-1)^{p(B)}T.$$

§6. The supertrace

1.6.1. For a commutative superalgebra A, the *supertrace* str is defined on $\mathrm{Mat}_{p,q}(A)$ by the formula

$$\mathrm{str}\, X=\sum (-1)^{(p(X)+1)p_{\mathrm{row}(i)}} X_{ii}=\sum (-1)^{(p(X_{ii})+1)p_{\mathrm{row}(i)}} X_{ii}.$$

In other words

$$\mathrm{str}\begin{pmatrix} A & B \\ C & D \end{pmatrix}=\mathrm{tr}\, A-\mathrm{tr}\, D \quad \text{for even matrices},$$
$$\mathrm{str}\begin{pmatrix} A & B \\ C & D \end{pmatrix}=\mathrm{tr}\, A+\mathrm{tr}\, D \quad \text{for odd matrices}.$$

We list the main properties of the supertrace.

ASSERTION. a) $\mathrm{str}(X+Y)=\mathrm{str}\, X+\mathrm{str}\, Y$.

b) *If Z is a matrix of size $(p, q)\times(r, s)$ and W a matrix of size $(r, s)\times(p, q)$, then*

$$\mathrm{str}(ZW) = (-1)^{p(Z)p(W)}\,\mathrm{str}(WZ).$$

In particular, if $X \in \mathrm{Mat}_{p,q}(A)$ and C is an even invertible matrix, then $\mathrm{str}(CXC^{-1})=\mathrm{str}\, X$, *because* $X=(XC^{-1})C$ *and* $CXC^{-1}=C(XC^{-1})$.

c) $\mathrm{str}(aX)=a\,\mathrm{str}\, X$, $\mathrm{str}(Xa)=(\mathrm{str}\, X)a$.

d) $\mathrm{str}(X^{\mathrm{st}})=\mathrm{str}\, X$.

e) $\mathrm{str}(P(X))=-(-1)^{p(X)}\mathrm{str}\, X$.

1.6.2. Let M be a finite-dimensional free A-module. We define the

supertrace str on $\mathrm{End}_A(M)$ by setting $\mathrm{str}\, X = \mathrm{str}\, \overline{X}$, the matrix corresponding to $X \in \mathrm{End}_A(M)$ in some basis. Under a change of basis $\overline{X}$ goes over into $C\overline{X}C^{-1}$, so that the supertrace is unchanged. The properties a)–e) above clearly remain valid for operators.

§7. The Berezinian (Berezin function)

1.7.1. Let $\mathcal{A}$ be a commutative superalgebra, and $\mathrm{GL}_{p,q}(\mathcal{A})$ the multiplicative group of even invertible elements of $\mathrm{Mat}_{p,q}(\mathcal{A})$. This is an analogue to the general linear group in the usual linear algebra. We want to define a homomorphism: $\mathrm{GL}_{p,q}(\mathcal{A}) \to \mathrm{GL}_{1,0}(\mathcal{A}) = \mathcal{A}_{\bar{0}}^{*}$ (the group of invertible elements of $\mathcal{A}_{\bar{0}}$), which is an analogue to the usual determinant.

1.7.2. Let

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in (\mathrm{Mat}_{p,q}(\mathcal{A}))_{\bar{0}}.$$

LEMMA. *X is invertible if and only if A and D are.*

This is an immediate consequence of the following general result.

PROPOSITION. *Let $\mathcal{A}$ be a commutative superalgebra, $\pi: \mathcal{A} \to \tilde{\mathcal{A}} = \mathcal{A}/(\mathcal{A}_{\bar{1}})$ the natural homomorphism, and*

$$\pi\colon \mathrm{Mat}_n(\mathcal{A}) \to \mathrm{Mat}_n(\tilde{\mathcal{A}})$$

the corresponding homomorphism of matrix algebras (where the superstructure is ignored). Then $X \in \mathrm{Mat}_n(\mathcal{A})$ is invertible if and only if $\pi(X)$ is invertible.

PROOF. It is clear that if X is invertible, then so is $\pi(X)$.

Conversely, assume that $\pi(X)$ is invertible. Since $\pi\colon \mathrm{Mat}_n(\mathcal{A}) \to \mathrm{Mat}_n(\tilde{\mathcal{A}})$ is an epimorphism, there is a matrix $Y \in \mathrm{Mat}_n(\mathcal{A})$ such that $\pi(X)\,\pi(Y) = \pi(Y)\,\pi(X) = 1$, that is, $\pi(XY) = \pi(YX) = 1$. It is sufficient to prove that XY and YX are invertible, and we do this for XY, the other case being similar.

Now XY has the form $1 - Z$, with $\pi(Z) = 0$. We claim that Z is nilpotent, that is, $Z^{r+1} = 0$ for some r; from this it follows that $1 + Z + Z^2 + \ldots + Z^r$ is the inverse of $1 - Z = XY$. All the entries in Z belong to the ideal generated by $\mathcal{A}_{\bar{1}}$; hence, there are elements $\xi_1, \ldots, \xi_r \in \mathcal{A}_{\bar{1}}$ such that every entry in Z can be written as a sum of products of the ξ_i. Then the same is true of the entries in Z^{r+1}, and in every product there are at least $r+1$ factors. Since these are all equal to zero, it follows that $Z^{r+1} = 0$.

1.7.3. For $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_{p,q}(\mathcal{A})$ we define

$$\mathrm{Ber}\, X = \det(A - BD^{-1}C)(\det D)^{-1}.$$

Note that D is invertible by Lemma 1.7.2. The entries in D and $A - BD^{-1}C$ lie in the commutative algebra $\mathcal{A}_{\bar{0}}$, so that the determinants are well-defined and $\mathrm{Ber}\, X \in \mathcal{A}_{\bar{0}}$.

It is easy to check that $\mathrm{Ber}\, X$ is an invertible element of $\mathcal{A}_{\bar{0}}$.

We call the function Ber the *Berezinian* in honour of F. A. Berezin. It is an

analogue to the usual determinant, as the following theorem shows.

THEOREM. *If* $X,\ Y \in \mathrm{GL}_{p,\,q}(\mathcal{A})$, *then*

$$\mathrm{Ber}(XY) = \mathrm{Ber}(X)\cdot\mathrm{Ber}(Y).$$

1.7.4. We carry out the proof of the theorem in a number of steps.

a) We consider the subgroups G_+, G_0, and G_- of $\mathrm{GL}_{p,\,q}(\mathcal{A})$ given by

$$G_+ = \left\{\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}\right\}, \quad G_0 = \left\{\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\right\}, \quad G_- = \left\{\begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix}\right\}.$$

Every matrix $X \in \mathrm{GL}_{p,\,q}(\mathcal{A})$ can be written in the form $X = X_+X_0X_-$, with $X_+ \in G_+$, $X_0 \in G_0$, and $X_- \in G_-$. For if $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, then

$$X = \begin{pmatrix} 1 & BD^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A-BD^{-1}C & 0 \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & BD^{-1} \\ 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} A-BD^{-1}C & 0 \\ 0 & D \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 \\ D^{-1}C & 1 \end{pmatrix}.$$

From

$$\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & B' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & B+B' \\ 0 & 1 \end{pmatrix}$$

it follows that every matrix X_+ can be written as a product of elementary matrices, that is, matrices of the form $\begin{pmatrix} 1 & E \\ 0 & 1 \end{pmatrix}$, where E has only one non-zero entry.

b) We claim that $\mathrm{Ber}(XY) = \mathrm{Ber}\,X\cdot\mathrm{Ber}\,Y$ whenever $X \in G_+$ or $X \in G_0$, and similarly, whenever $Y \in G_0$ or $Y \in G_-$.

Taking, for example, the case

$$X = \begin{pmatrix} 1 & F \\ 0 & 1 \end{pmatrix} \in G_+, \quad Y = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

we see that

$$\mathrm{Ber}\,(XY) = \mathrm{Ber}\begin{pmatrix} A+FC & B+FD \\ C & D \end{pmatrix} = \det\left(A+FC-(B+FD)\,D^{-1}C\right)\det D^{-1} =$$
$$= \det\left(A-BD^{-1}C\right)\det D^{-1} = \mathrm{Ber}\,Y = \mathrm{Ber}\,X\cdot\mathrm{Ber}\,Y.$$

The other cases are treated similarly.

c) We claim that $\mathrm{Ber}(XY) = \mathrm{Ber}\,X\cdot\mathrm{Ber}\,Y$ for any elementary matrix

$$Y = \begin{pmatrix} 1 & E \\ 0 & 1 \end{pmatrix} \in G_+.$$

Writing X in the form $X_+X_0X_-$, we have

$$\mathrm{Ber}\,XY = \mathrm{Ber}\,X_+X_0(X_-Y) = \mathrm{Ber}\,X_0\cdot\mathrm{Ber}\,X_-Y,$$
$$\mathrm{Ber}\,X\cdot\mathrm{Ber}\,Y = \mathrm{Ber}\,X_0\cdot\mathrm{Ber}\,X_-\cdot\mathrm{Ber}\,Y,$$

by b). Therefore, it is sufficient to prove that

$$\mathrm{Ber}\,(X_-Y) = \mathrm{Ber}\,X_-\cdot\mathrm{Ber}\,Y = 1.$$

Let $X_- = \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix}$. Then

$$\operatorname{Ber} X_Y = \operatorname{Ber}\begin{pmatrix} 1 & E \\ C & 1+CE \end{pmatrix} = \det(1 - E(1+CE)^{-1}C)\det^{-1}(1+CE).$$

Note that E has only one non-zero entry, namely, some $\xi \in \mathcal{A}_{\bar{1}}$. It follows that in the matrices E, CE, and $E(1 + CE)^{-1}C$, all the entries are divisible by ξ, so that the product of any two of them is zero. Hence, we can invoke the following lemma.

d) LEMMA. *Let L be a square matrix over a commutative ring R, and suppose that the product of any two of its entries is zero. Then* $(1 + L)^{-1} = 1 - L$ *and* $\det(1 + L) = 1 + \operatorname{tr} L$.

The proof is immediate.

e) We now complete the calculation in c). We have $(1 + CE)^{-1} = 1 - CE$, and $E \cdot CE = 0$. Therefore

$$\operatorname{Ber} X_Y = \det(1 - EC)\det^{-1}(1 + CE) = (1 - \operatorname{tr} EC)(1 + \operatorname{tr} CE)^{-1}.$$

It is easy to check that $\operatorname{tr} CE = -\operatorname{tr} EC$, because the entries in E and C belong to $\mathcal{A}_{\bar{1}}$. Thus,

$$\operatorname{Ber}(X_Y) = (1 + \operatorname{tr} CE)^{-1}(1 + \operatorname{tr} CE) = 1,$$

which proves c).

f) Let G be the set of matrices $Y \in \mathrm{GL}_{p,q}(\mathcal{A})$ such that $\operatorname{Ber}(XY) = \operatorname{Ber} X \cdot \operatorname{Ber} Y$ for all $X \in \mathrm{GL}_{p,q}(\mathcal{A})$. It is clear that G is a group, since for all Y_1, Y_2 in G,

$$\operatorname{Ber}(X \cdot Y_1 Y_2) = \operatorname{Ber}(XY_1)\, Y_2 = \operatorname{Ber}(XY_1)\operatorname{Ber} Y_2 = \\ = \operatorname{Ber} X \operatorname{Ber} Y_1 \operatorname{Ber} Y_2 = \operatorname{Ber} X \cdot \operatorname{Ber} Y_1 Y_2.$$

As shown in b) and c), G contains G_- and G_0, and all the elementary matrices $Y \in G_+$. Since $\mathrm{GL}_{p,q}(\mathcal{A})$ is generated by these matrices (see a)), it follows that $G = \mathrm{GL}_{p,q}(\mathcal{A})$, that is, $\operatorname{Ber}(XY) = \operatorname{Ber} X \cdot \operatorname{Ber} Y$ for all X, $Y \in \mathrm{GL}_{p,q}(\mathcal{A})$.

1.7.5. Let $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_{p,q}(\mathcal{A})$. In the definition of $\operatorname{Ber} X$ the rôles played by A and D are disparate. We can also define the function

$$\widetilde{\operatorname{Ber}}\, X = \det(D - CA^{-1}B)\cdot\det^{-1} A.$$

Since

$$X = \begin{pmatrix} 1 & 0 \\ CA^{-1} & 1 \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix},$$

it follows from Theorem 1.7.3 that

$$\operatorname{Ber} X = \det(D - CA^{-1}B)^{-1}\det A = (\widetilde{\operatorname{Ber}}\, X)^{-1}.$$

1.7.6. Let $X \in \mathrm{GL}_{p,q}(\mathcal{A})$. Then $\operatorname{Ber}(X^{\mathrm{st}}) = \operatorname{Ber} X$.

For if $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, then

$$\operatorname{Ber}(X^{\mathrm{st}}) = \operatorname{Ber}\begin{pmatrix} A^t & C^t \\ -B^t & D^t \end{pmatrix} =$$

$$= \det(A^t + C^t (D^t)^{-1} B^t) \det^{-1} D^t = \det(A - BD^{-1}C) \det^{-1} D.$$

1.7.7. For any $\mathcal{A}$-module M, let $\mathrm{GL}(M) = \mathrm{GL}_{\mathcal{A}}(M)$ denote the group of invertible even elements of $\mathrm{End}_{\mathcal{A}}(M)$. If M is a finite-dimensional free $\mathcal{A}$-module, then we define the function Ber on $\mathrm{GL}(M)$ by setting $\operatorname{Ber} X = \operatorname{Ber} \overline{X}$ for $X \in \mathrm{GL}_{\mathcal{A}}(M)$, where $\overline{X}$ is the matrix of X in some basis. Now $\operatorname{Ber} X$ is well defined because under a change of basis $\overline{X}$ is transformed into $C\overline{X}C^{-1}$, a matrix with the same Berezinian. It is clear that $\operatorname{Ber} XY = \operatorname{Ber} X \cdot \operatorname{Ber} Y$.

It follows from 1.7.5 that $\operatorname{Ber}(P(X)) = (\operatorname{Ber} X)^{-1} = \widetilde{\operatorname{Ber}} X$.

If M^* is the dual module and $X^* \in \mathrm{GL}(M^*)$ the adjoint of $X \in \mathrm{GL}(M)$, then it follows from 1.7.6 and 1.4.7 that

$$\operatorname{Ber} X^* = \operatorname{Ber} X.$$

If M and N are finite-dimensional free $\mathcal{A}$-modules and $F \in \mathrm{GL}(M)$, $H \in \mathrm{GL}(N)$, then

$$\operatorname{Ber}(F \oplus H) = \operatorname{Ber} F \cdot \operatorname{Ber} H.$$

1.7.8. In the "even" case there is a relation between the trace and the determinant given by the formula $\det \exp X = \exp \operatorname{tr} X$. We need an analogue to this equation.

LEMMA. *Let $X \in \mathrm{Mat}_{p,\,q}(\mathcal{A})$ be an even matrix such that the product of any two entries of it is zero. Then*

$$\operatorname{Ber}(1 + X) = 1 + \operatorname{str} X.$$

For if $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, then

$$\operatorname{Ber}(1+X) = \det(1 + A - B(1+D)^{-1} C) \det^{-1}(1+D) =$$

$$= \det(1+A) \cdot \det(1-D) = 1 + \operatorname{tr} A - \operatorname{tr} D,$$

because $B(1+D)^{-1}C = 0$, $\operatorname{tr} A \cdot \operatorname{tr} D = 0$, $(1+D)^{-1} = 1 - D$, $\det(1-D) = 1 - \operatorname{tr} D$, and $\det(1+A) = 1 + \operatorname{tr} A$, by Lemma 1.7.4 d).

§8. Tensor algebras

1.8.1. Let A be a commutative superalgebra and B a commutative A-algebra. Then with any A-module M we can associate the B-module $M_B = B \otimes_A M$; this correspondence is called a *change of rings*. It is easy to check that there are canonical isomorphisms

$$(M \oplus N)_B = M_B \oplus N_B, \quad (P(M))_B = P(M_B),$$

$$(M \otimes_A N)_B = M_B \otimes_B N_B.$$

1.8.2. The tensor product of two A-modules was defined in 1.2.7. The tensor product of A-modules $M_1, \ldots, M_n$ is the superspace

$M_1 \otimes_A \ldots \otimes_A M$, obtained from the tensor product of the M_i over k factored by the relations

$$m_1 \otimes \ldots \otimes m_i a \otimes m_{i+1} \otimes \ldots \otimes m_n = m_1 \otimes \ldots \otimes m_i \otimes a m_{i+1} \otimes \ldots \otimes m_n \ (i = 1, 2, \ldots, n-1).$$

1.8.3. The *tensor algebra* $T(M)$ of an A-module M is defined as the superspace $\bigoplus_{n \geqslant 0} T^n(M)$, where $T^0(M) = A$ and $T^n(M) = M \otimes_A \ldots \otimes_A M$ (n-copies), endowed with the structure of an algebra through the tensor product (over A), namely, if $x \in T^n(M)$ and $y \in T^k(M)$, then

$$xy = x \otimes_A y \in T^{n+k}(M).$$

In other words,

$$(m_1 \otimes m_2 \otimes \ldots \otimes m_n)(m_1' \otimes \ldots \otimes m_k') = m_1 \otimes \ldots \otimes m_n \otimes m_1' \otimes \ldots \otimes m_k'.$$

The superalgebra $T(M)$ admits a natural $\mathbf{Z}$-grading by degree:

$$\deg x = n \Leftrightarrow x \in T^n(M).$$

$T(M)$ can be described as the universal object for mappings into associative A-algebras. Let $i: M \to T(M)$ be the natural homomorphism of A-modules; it is clear that i is an embedding and that $T(M)$ is generated by the elements of M.

LEMMA. a) *The pair* $(T(M), i)$ *has the following universal property:*

If B is an A-algebra and $\alpha: M \to B$ is an even homomorphism of A-modules, then there is a unique homomorphism $T(\alpha): T(M) \to B$ such that $T(\alpha) \circ i = \alpha$.

b) *The universal property determines the pair* $(T(M), i)$ *uniquely up to isomorphism.*

The proof is left as an exercise.

1.8.4. Let $\mathcal{J}_S$ be the two-sided ideal of $T(M)$ generated by the elements of the form $xy - (-1)^{p(x)p(y)}yx$, where x and y are homogeneous elements of $M = T^1(M)$. The A-algebra $T(M)/\mathcal{J}_S$ is called the *symmetric algebra* of M and is denoted by $S(M) = S_A(M)$. We can equip $S(M)$ with the grading given by $S(M) = \bigoplus_{n \geqslant 0} S^n(M)$, where $S^n(M) = T^n(M) (\mathrm{mod}\ \mathcal{J}_S)$. Since

$$\mathcal{J}_S \subset T^2(M) \oplus T^3(M) \oplus \ldots,$$

it follows that

$$S^1(M) = T^1(M) = M;$$

we denote the corresponding homomorphism: $M \to S(M)$ by the same symbol i as before. It is clear that the elements of $S^1(M)$ generate $S(M)$, and since they commute in the supersense, $S(M)$ is a commutative A-algebra.

1.8.5. For any A-module M we define the exterior algebra $E(M) = E_A(M)$ by putting $E(M) = S(P(M))$. It is clear that $E(M)$ is a commutative A-algebra. The morphisms of A-modules $P: M \to P(M)$ and $i: P(M) \to S(P(M))$ combine to give a canonical odd morphism $\iota = i \circ P: M \to E(M)$.

Let $M = L \oplus N$ be a direct sum of A-modules. We define an A-homomorphism $\alpha\colon M \to S(L) \otimes S(N)$, by setting $\alpha((l, n)) = l \otimes 1 + 1 \otimes n$, and we extend this to a homomorphism of superalgebras

$$\hat{\alpha}\colon S(M) \to S(L) \otimes S(N).$$

LEMMA. *The homomorphism α is an isomorphism.*

It is sufficient to verify that the pair $(S(L) \otimes S(N), \alpha)$ satisfies the conditions of Lemma 1.8.3, and this is trivial.

The canonical isomorphism

$$E(M) = E(L) \otimes E(N)$$

is established in a similar way.

Thus,

$$S^n(M) = \bigoplus_{i=0}^{n} S^i(L) \otimes S^{n-i}(N),$$

$$E^n(M) = \bigoplus_{i=0}^{n} E^i(L) \otimes E^{n-i}(N)$$

for any n.

§9. Lie superalgebras and derivations of superalgebras

In this section we assume that char $k \neq 2$ or 3, but not that the superalgebras are associative or have an identity.

1.9.1. A superalgebra $\mathfrak{L}$ over a commutative superalgebra A is called a *Lie superalgebra* if the multiplication in $\mathfrak{L}$ (usually written $[\ ,\ ]$ and called *commutation*) satisfies the following laws:

(1) $$[x, y] = -(-1)^{p(x)p(y)}[y, x],$$

(2) $$(-1)^{p(x)p(y)}[x, [y, z]] + (-1)^{p(y)p(x)}[y, [z, x]] + (-1)^{p(z)p(y)}[z, [x, y]] = 0.$$

We will be chiefly concerned with Lie superalgebras over k.

The law (2) is called the *Jacobi identity*.

1.9.2. If M is a superspace, we make $\mathrm{End}_k(M)$ into a Lie superalgebra by setting for $F, G \in \mathrm{End}_A(M)$

(1) $$[F, G] = FG - (-1)^{p(F)p(G)}GF.$$

We use the notation $(\mathrm{End}_k(M))_L$ to emphasize that we are thinking of the multiplication (1).

In general, any associative superalgebra A gives rise to a Lie superalgebra A_L if we define

(2) $$[a, b] = ab - (-1)^{p(a)p(b)}ba$$

for $a, b \in A$. Conditions (1) and (2) above are easily checked.

1.9.3. Let us look at the laws (1) and (2) of 1.9.1 in more detail.

When $x, y, z \in \mathscr{L}_{\bar{0}}$, (1) and (2) show that $\mathscr{L}_{\bar{0}}$ is an ordinary Lie algebra.

If two of the elements x, y, z are even and the third is odd, then the Jacobi identity shows that the operation $[\ ,\ ]$ defines a representation of $\mathscr{L}_{\bar{0}}$ on $\mathscr{L}_{\bar{1}}$. But if two of them are odd and the third is even, then we see that the bilinear symmetric mapping $[\ ,\ ]: \mathscr{L}_{\bar{1}} \times \mathscr{L}_{\bar{1}} \to \mathscr{L}_{\bar{0}}$ is a homomorphism of $\mathscr{L}_{\bar{0}}$-modules. The proof of this requires one application of (2).

Thus, to specify a Lie superalgebra $\mathscr{L}$ we need only prescribe:

1) a Lie algebra $\mathscr{L}_{\bar{0}}$;

2) an $\mathscr{L}_{\bar{0}}$-module $\mathscr{L}_{\bar{1}}$, such that

$$[x, y] = -[y, x],$$

where $x \in \mathscr{L}_{\bar{1}}, y \in \mathscr{L}_{\bar{0}}$ and $[\ ,\ ]$ denotes the action of $\mathscr{L}_{\bar{0}}$ on $\mathscr{L}_{\bar{1}}$;

3) a symmetric bilinear mapping $\mathscr{L}_{\bar{1}} \times \mathscr{L}_{\bar{1}} \to \mathscr{L}_{\bar{0}}$ also denoted by $[\ ,\ ]$, that is a homomorphism of $\mathscr{L}_{\bar{0}}$-modules; here the mapping $[\ ,\ ]$ defined by 2) and 3) must satisfy for all $x, y, z \in \mathscr{L}_{\bar{1}}$ the identity

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

1.9.4. It is easy to see that a change of rings carries one Lie superalgebra into another (that is, if $\mathscr{L}$ is a Lie superalgebra over a commutative superalgebra A, and B is a commutative A-algebra, then $\mathscr{L}_B = B \otimes_A \mathscr{L}$ is a Lie superalgebra over B).

PROPOSITION. *Let $\mathscr{L}$ be a superalgebra (with operation $[\ ,\ ]$) over a commutative superalgebra A. Then $\mathscr{L}$ is a Lie superalgebra if and only if, under any change of rings, the following conditions are fulfilled:*

a) *if $X \in (\mathscr{L}_B)_{\bar{0}}$, then $[X, X] = 0$,*

b) *if $X \in (\mathscr{L}_B)_{\bar{1}}$, then $[X, [X, X]] = 0$.*

PROOF. We prove the Jacobi identity for $X_1, X_2, X_3 \in \mathscr{L}$. We consider the superalgebra $B = A[\alpha_1, \alpha_2, \alpha_3]$, where $p(\alpha_i) = p(X_i) + 1$. Then

$$X = \sum \alpha_i X_i \in (\mathscr{L}_B)_{\bar{1}},$$

so that $[X, [X, X]]$ can be written in the form

$$\sum \alpha_1^{n_1} \alpha_2^{n_2} \alpha_3^{n_3} X_{n_1 n_2 n_3}, \quad \text{where} \quad X_{n_1 n_2 n_3} \in \mathscr{L}.$$

It is not hard to check that the coefficient X_{111} of $\alpha_1 \alpha_2 \alpha_3$ is precisely the left-hand side of the Jacobi identity, and this is equal to zero by b). The skew-symmetry is proved similarly.

REMARK. This proposition also has something to say about Lie algebras, regarded as Lie superalgebras whose odd part is zero.

1.9.5. Let A be an arbitrary superalgebra (not necessarily associative or commutative). A linear operator $D \in \mathrm{End}_k(A)$ is called a *derivation* of A if

$$D(ab) = D(a) \cdot b + (-1)^{p(D)p(a)} aD(b).$$

The superspace $\mathrm{Der}_k A$ of all derivations of A is a Lie subsuperalgebra of $(\mathrm{End}_k A)_L$.

Note that, because of (1), the Jacobi identity (2) can be rewritten in the form

$$[x, [y, z]] = [[x, y], z] + (-1)^{p(x)p(y)} [y, [x, z]].$$

Thus, the operator adx of left multiplication by x, that is, $adx(y) = [x, y]$ for $x, y \in \mathcal{L}$, is a derivation of $\mathcal{L}$.

1.9.6. A left *module over a Lie superalgebra* $\mathcal{L}$ is a superspace M together with an even mapping $\mathcal{L} \times M \to M$ such that for $g_1, g_2 \in \mathcal{L}$ and $m \in M$:

$$[g_1, g_2]m = g_1(g_2 m) - (-1)^{p(g_1)p(g_2)} g_2(g_1 m).$$

CHAPTER II

ANALYSIS IN SUPERSPACES AND SUPERDOMAINS

The difference between supermanifolds and manifolds consists in the fact that, while the functions on a manifold form a commutative algebra, those on a supermanifold form a commutative superalgebra. In the study of supermanifolds, we cannot get by without sheaves and ringed spaces. We need only the very basic definitions and the simplest facts from the theory of sheaves, and a good introduction is to be found in [17] or [21].

Supermanifolds are formed by glueing together simpler pieces, superdomain in a superspace. In this chapter we study analysis on superdomains.

§ 1. Definition of superspaces and superdomains

Let X be a topological space. Suppose that to each open subset $U \subset X$ there is assigned a set $\mathcal{F}(U)$ and that for any open subsets $U \subset V$ there is a restriction mapping r_U^V: $\mathcal{F}(V) \to \mathcal{F}(U)$ such that $r_U^V \circ r_V^W = r_U^W$ for any open subsets $U \subset V \subset W$; $\mathcal{F}(\varnothing)$ is a singleton and r_U^U is the identity mapping.

We often write $\Gamma(U, \mathcal{F})$ instead of $\mathcal{F}(U)$ and $\xi|_U$ instead of $r_U^V(\xi)$ for $\xi \in \mathcal{F}(V)$.

A family $\mathcal{F}$ of sets $F(U)$ and mappings r_U^V is called a *sheaf* if for any collection $\{U_i\}$ of open sets in X with $U = \cup U_i$ the following conditions (of uniqueness and glueing) are satisfied:

1) if $\xi, \eta \in F(U)$ and $\xi|_{U_i} = \eta|_{U_i}$, then $\xi = \eta$;

2) if $\xi_i \in F(U_i)$ and $\xi_i|_{U_i \cup U_j} = \xi_j|_{U_i \cup U_j}$ for all i, j, then there is a $\xi \in \mathcal{F}(U)$ such that $\xi|_{U_i} = \xi_i$ for all i.

If $\mathcal{F}$ and $\mathcal{G}$ are sheaves over X, then a *morphism* h: $\mathcal{F} \to \mathcal{G}$ is a collection of mappings h_U: $\mathcal{F}(U) \to \mathcal{G}(U)$, one for each open $U \subset X$, such that $r_U^V h_U = h_U r_U^V$ for any $U \subset V$ of X. If all the h_U are inclusions, then $\mathcal{F}$ is called a *subsheaf* of $\mathcal{G}$.

If all the $\mathcal{F}(U)$ are groups or modules or superalgebras, . . ., and the r_U^V are homomorphisms of these structures, then $\mathcal{F}$ is called a *sheaf of groups*, or *modules*, or *superalgebras*, ...

The *restriction* of a sheaf $\mathcal{F}$ over X to a sheaf over an open subset $U \subset X$ is defined in the obvious way and denoted by $\mathcal{F}\,|_U$.

A *ringed space* is a pair $(X, \mathcal{F})$, where X is a topological space and $\mathcal{F}$ is a sheaf of rings over X. If $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ are ringed spaces, a *morphism* φ: $(X, \mathcal{F}) \to (Y, \mathcal{G})$ is a collection $(\tilde{\varphi}, \varphi_U^*)$, where $\tilde{\varphi}$: $X \to Y$ is a continuous mapping and φ_U^*: $\mathcal{G}(U) \to \mathcal{F}(\tilde{\varphi}^{-1}(U))$ a homomorphism of rings, one for each open subset U of Y, compatible with the restriction mappings, that is, $r_{\tilde{\varphi}^{-1}(U)}^{\tilde{\varphi}^{-1}(V)} \circ \varphi_V^* = \varphi_U^* \circ r_U^V$.

EXAMPLE (the structure sheaf on a manifold). Let $k = \mathbf{R}$ and let M be a Hausdorff space with a countable base and locally homeomorphic to an open subset of k^m. Suppose that every point $p \in M$ has an open neighbourhood V and that h_V is a homeomorphism from V onto an open subset of k^m such that for any open set $U \subset V$ the function f: $U \to k$ is smooth (that is, infinitely differentiable) on $h_V(U)$.

Let $U = \cup U_i$, where the U_i are open subsets of M. A mapping f: $U \to k$ is called a *smooth function* if $f_{|U_i}$ is smooth for all i. A ringed space $(M, \mathcal{O}_M)$ is called a *smooth manifold of dimension m* if $\Gamma(U, \mathcal{O}_M)$ is an algebra of smooth functions on M.

2.1.1. Let $(k^p, \mathcal{O}_{k^p})$ be a ringed space. A ringed space

$$\mathcal{K}^{p,q} = (k^p, \mathcal{O}_{\mathcal{K}^{p,q}}), \quad \text{where} \quad \Gamma(U, \mathcal{O}_{\mathcal{K}^{p,q}}) = \Gamma(U, \mathcal{O}_{k^p}) \otimes \Lambda(q),$$

is called a *smooth superspace*. In other words, functions on $\mathcal{K}^{p,q}$ are functions on k^p with values in $\Lambda(q)$. Here k^p is called the *underlying space* of $\mathcal{K}^{p,q}$.

Let U be a domain in k^p. A ringed space

$$\mathcal{U}^{p,q} = (U, \mathcal{O}_{\mathcal{K}^{p,q}|U})$$

is called a *superdomain of dimension* (p, q).

We denote superdomains by script letters $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$, . . ., often omitting reference to the dimension. The underlying domains of superdomains $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$ are denoted by the corresponding Latin capitals, U, V, W. If u is a point in an underlying domain U, then we also refer to u as a point in $\mathcal{U}$ and write $u \in \mathcal{U}$.

In what follows, we restrict our attention to smooth real superdomains, writing $\mathcal{O}_{p,q}$ instead of $\mathcal{O}_{\mathcal{K}^{p,q}}$ and $C^\infty(\mathcal{U})$ instead of $\Gamma(U, \mathcal{O}_{p,q})$. Every function $f \in C^\infty(\mathcal{U})$ can be written uniquely in the form

$$f(u, \xi) = \sum_{\nu=(\nu_1, \ldots, \nu_q)} f_\nu(u)\, \xi_1^{\nu_1} \ldots \xi_q^{\nu_q} = \sum_\nu f_\nu \xi^\nu,$$

where $\nu_i = 0$ or 1 and $f_\nu(u)$ is a smooth function of the coordinates $u = (u_1, \ldots, u_q)$ on U.

The functions $u_1, \ldots, u_p \in C^\infty(\mathcal{U})_{\bar{0}}$ and $\xi_1, \ldots, \xi_q \in C^\infty(\mathcal{U})_{\bar{1}}$ are called *coordinates on* $\mathcal{U}$ (the u_i *even* and the ξ_j *odd* coordinates). The set of these functions is called a *coordinate system* on $\mathcal{U}$. For convenience, we often write a coordinate system in the form $x = (x_1, \ldots, x_{p+q})$, where

$$x_1 = u_1, \ldots, x_p = u_p, x_{p+1} = \xi_1, \ldots, x_{p+q} = \xi_q.$$

2.1.2. A *morphism* φ of a superdomain $\mathcal{U} = (U^p, \mathcal{O}_{p,q}|_U)$ into a superdomain $\mathcal{V} = (V^m, \mathcal{O}_{m,n}|_V)$ is a morphism of ringed spaces $\varphi: \mathcal{U} \to \mathcal{V}$. If $u \in \mathcal{U}$, we often denote the point $\tilde{\varphi}(u) \in \mathcal{V}$ simply by $\varphi(u)$.

A morphism $\varphi: \mathcal{U} \to \mathcal{V}$ is called an *isomorphism* (or a *diffeomorphism*) if there is a morphism $\psi: \mathcal{V} \to \mathcal{U}$ such that $\varphi \circ \psi$ and $\psi \circ \varphi$ are identity morphisms. In this case, $p = m$ and $q = n$. If $x = (u, \xi)$ is a coordinate system on $\mathcal{U}$, and $y = (v, \eta)$ a coordinate system on $\mathcal{V}$, and if $\varphi: \mathcal{U} \to \mathcal{V}$ is an isomorphism, then the set of functions

$$\varphi^*(y) = (\varphi^*(v_1), \ldots, \varphi^*(v_p), \varphi^*(\eta_1), \ldots, \varphi^*(\eta_q))$$

is *also called a coordinate system on* $\mathcal{U}$, and the passage from x to $\varphi^*(y)$ is called a *change of coordinates*. We usually identify the rings $C^\infty(\mathcal{U})$ and $C^\infty(\mathcal{V})$ by means of the isomorphism φ^* and we do not distinguish between the coordinates $\varphi^*(y)$ and y.

2.1.3. EXAMPLES. a) A superdomain $\mathcal{U}$ of dimension $(p, 0)$ is simply a domain U in $\mathbf{R}^p$.

b) The underlying space of a superspace of dimension $(0, q)$ consists of a single point. But the ring of functions on $\mathcal{R}^{0,q}$ is non-trivial (it is isomorphic to $\Lambda(q)$). Therefore, this superspace has many non-trivial automorphisms.

c) Let $E \to U^p$ be a fibration with q-dimensional fibre, and $\mathcal{L}_{\Lambda E}$ the sheaf of sections of the exterior algebra ΛE of this fibration. Then $\mathcal{U}^{p,q} = (U^p, \mathcal{L}_{\Lambda E})$ is a superdomain. Let $u_1, \ldots, u_p$ be coordinates on U^p, and $\xi_1, \ldots, \xi_q$ a basis in the fibre of E. A morphism $\varphi: \mathcal{U}^{p,q} \to \mathcal{U}^{p,q}$ is defined by the formulae

$$(*) \qquad \begin{cases} \varphi^*(u_i) = \varphi_0^i(u) + \underline{\sum_{i_1 < \ldots < i_{2r}} \varphi^i_{i_1 \ldots i_{2r}}(u)\, \xi_{i_1} \ldots \xi_{i_{2r}}}, \\ \varphi^*(\xi_j) = \underline{\sum_{j_1 < \ldots < j_{2r+1}} \varphi^j_{j_1 \ldots j_{2r+1}}(u)\, \xi_{j_1} \ldots \xi_{j_{2r+1}}}. \end{cases}$$

The difference between the fibration ΛE and the superdomain $\mathcal{U}^{p,q}$ is best understood by looking at the underlined terms in (*). *In the category of superdomains there are many more morphisms than in the category of fibrations, namely, those for which the underlined terms are non-zero.*

d) Let $\mathcal{U}^{p,q}$ be a superdomain with underlying domain U. The canonical homomorphism $\pi: C^\infty(\mathcal{U}) \to C^\infty(U)$ (see 2.1.1) induces a morphism of superdomains $\pi: U \to \mathcal{U}$, where U is regarded as a superdomain of dimension $(p, 0)$. This π is called the *canonical embedding of the underlying domain.*

Thus, the underlying domain is always a subsuperdomain. Note that the embedding $\pi\colon U \to \mathcal{U}$ always induces an isomorphism of the underlying domains, but when $q > 0$, it is not an isomorphism of superdomains.

The canonical embedding π is compatible with morphisms of superdomains: every morphism $\varphi\colon \mathcal{U} \to \mathcal{V}$ determines a morphism $\widetilde{\varphi}\colon U \to V$ of the underlying domains regarded as superdomains, and $\pi \circ \varphi = \widetilde{\varphi} \circ \pi$.

Note that we can also define a homomorphism $\rho\colon C^\infty(U) \to C^\infty(\mathcal{U})$, by setting $\rho(f) = f$. This gives us a projection $\rho\colon \mathcal{U} \to U$. However, in contrast to π, ρ is not compatible with coordinate changes.

2.1.4. Let $\mathcal{U} = (U, \mathcal{O}_{\mathcal{G}_l p,q}|_U)$ be a superdomain, and V an open subset of U. Then the restriction to V of the sheaf $\mathcal{O}_{\mathcal{G}_l p,q}$ induces on V the structure of a superdomain $\mathcal{V} = (V, \mathcal{O}_{\mathcal{G}_l p,q}|_V)$. In this case we call $\mathcal{V}$ an *open subsuperdomain of* $\mathcal{U}$.

The canonical embedding $\varphi\colon \mathcal{V} \to \mathcal{U}$ induces a homomorphism $\varphi^*\colon C^\infty(\mathcal{U}) \to C^\infty(\mathcal{V})$. The function $\varphi^*(f)$ is called the *restriction* of $f \in C^\infty(\mathcal{U})$ to $\mathcal{V}$ and is denoted by $f|_{\mathcal{V}}$.

If $\varphi\colon \mathcal{U} \to \mathcal{W}$ is a morphism of superdomains and $\mathcal{W}'$ an open subsuperdomain of $\mathcal{W}$, we denote by $\varphi^{-1}(\mathcal{W}')$ the open subsuperdomain of $\mathcal{U}$ corresponding to $U' = \widetilde{\varphi}^{-1}(W') \subset U$.

If $u \in \mathcal{U}$, a *neighbourhood* of u in $\mathcal{U}$ is an open subsuperdomain $\mathcal{V} \subset \mathcal{U}$ such that $u \in \mathcal{V}$.

As in ordinary analysis, we often say that some property of superdomains holds "in a neighbourhood of u". For example, the statement that a function $f \in C^\infty(\mathcal{U})$ is "invertible in a neighbourhood of u" means that there is a neighbourhood $\mathcal{V}$ of u in $\mathcal{U}$ such that $f|_{\mathcal{V}}$ is invertible. The statement that a morphism $\varphi\colon \mathcal{U} \to \mathcal{W}$ "is a diffeomorphism in a neighbourhood of u" means that there is a neighbourhood $\mathcal{V}$ of u in $\mathcal{U}$ and a neighbourhood $\mathcal{V}'$ of $w = \varphi(u)$ in $\mathcal{W}$ such that $\varphi|_{\mathcal{V}}\colon \mathcal{V} \to \mathcal{V}'$ is a diffeomorphism.

2.1.5. Let $\varphi\colon \mathcal{U} \to \mathcal{V}$ be a morphism of superdomains. We claim that the mapping $\widetilde{\varphi}\colon U \to V$ of underlying domains in the definition of φ is always smooth.

LEMMA. *Let $f \in C^\infty(\mathcal{V})$, $u \in U$. Then $\varphi^*(f)(u) = \widetilde{f}(\widetilde{\varphi}(u))$.*

In particular, if $y = (v_1, \ldots, v_m, \eta_1, \ldots, \eta_n)$ is a coordinate system on $\mathcal{V}$, then $\widetilde{\varphi}\colon U \to V$ is given by

$$\widetilde{\varphi}(u) = v = (h_1(u), \ldots, h_m(u)), \quad \textit{where } h_i = \varphi^*(v_i) \in C^\infty(U),$$

hence, is a smooth mapping.

PROOF. Let $f^* = \varphi^*(f) \in C^\infty(\mathcal{U})$ and $v = \widetilde{\varphi}(u) \in V$. Suppose that $\widetilde{f}^*(u) \neq \widetilde{f}(v)$. By adding a constant to f, we may assume that $\widetilde{f}^*(u) = 0$ and $\widetilde{f}(v) \neq 0$. Let V' be a neighbourhood of v on which $\widetilde{f}$ is different from zero and so is invertible. It follows from 1.1.5 that $\widetilde{f}|_{\mathcal{V}'}$ is also invertible.

Let $\mathcal{U}' = \varphi^{-1}(\mathcal{V}')$. Then $f^*_{\mathcal{U}} |_{\mathcal{U}'} = \varphi^*(f |_{\mathcal{V}'})$ is invertible. Therefore, $\widetilde{f}^*$ is invertible on $\mathcal{U}'$, but this contradicts the fact that $u \in \mathcal{U}'$ and $\widetilde{f}^*(u) = 0$.

2.1.6. A superdomain $\mathcal{U}$ can be recovered from a knowledge of $C^\infty(\mathcal{U})$ as an abstract superalgebra.

With every $u \in \mathcal{U}$ we associate the homomorphism s_u: $C^\infty(\mathcal{U}) \to \mathbf{R}$ given by $s_u(f) = \widetilde{f}(u)$. Then $s_u(f)$ is called the value of f at u. We again emphasize that a function $f \in C^\infty(\mathcal{U})$ is not determined by its values at points.

It is clear that distinct homomorphisms correspond to distinct points.

LEMMA. *Any homomorphism of* $\mathbf{R}$*-algebras* s: $C^\infty(\mathcal{U}) \to \mathbf{R}$ *has the form* $s = s_u$ *for some* $u \in \mathcal{U}$.

This lemma shows that the points in U are in one-to-one correspondence with the homomorphisms $C^\infty(\mathcal{U}) \to \mathbf{R}$, hence U can be recovered from the abstract superalgebra $C^\infty(\mathcal{U})$. The topology on $\mathcal{U}$ and the structure of the ringed space $(U, \mathcal{O}_{\mathcal{R}^{p,q}} |_{\mathcal{U}})$ can also be recovered from this superalgebra; this follows, for example, from Theorem 2.1.7 below.

PROOF. Let $x = (u_1, \ldots, u_p, \xi_1, \ldots, \xi_q)$ be a coordinate system on $\mathcal{U}$. We claim that the point $u = (s(u_1), \ldots, s(u_p))$ belongs to U and that $s = s_u$. To prove this, we consider $f_i = u_i - s(u_i)$.

Suppose that $u \notin U$, and consider the function $h = f_1^2 + \ldots + f_p^2$. Then $\widetilde{h} \in C^\infty(U)$ is different from zero at all points of U and is thus invertible. By 2.1.1, $h \in C^\infty(\mathcal{U})$ is also invertible. But this contradicts the fact that $s(h) = 0$. Hence, $u \in U$.

Now suppose that $s(f) \neq s_u(f)$ for some $f \in C^\infty(\mathcal{U})$, and consider the function

$$h = f_1^2 + \ldots + f_p^2 + (f - s_u(f))^2.$$

As above, it can be shown that h is invertible; but this contradicts the fact that $s(h) = 0$. Thus, $s = s_u$, and this completes the proof.

2.1.7. In classical analysis, an important rôle is played by the "coordinate expression" of mappings. If X is a domain with coordinates $(x_1, \ldots, x_p)$ and Y is a domain with coordinates $(y_1, \ldots, y_m)$, a morphism φ: $X \to Y$ can be specified by $y_i = f_i(x)$ $(i = 1, \ldots, m)$. In other words, φ is uniquely determined by the set of functions

$$f_i = \varphi^*(y_i) \in C^\infty(X).$$

A similar assertion holds in the supercase. It plays an important part in everything that follows.

THEOREM. *Let* $\mathcal{U}$ *and* $\mathcal{V}$ *be superdomains and* $y = (v_1, \ldots, v_m, \eta_1, \ldots, \eta_n)$ *a coordinate system on* $\mathcal{V}$.

a) *Let* ψ^*: $C^\infty(\mathcal{V}) \to C^\infty(\mathcal{U})$ *be any morphism of superalgebras. We consider the set of functions* y_i^* $(i = 1, \ldots, m+n)$, *where* $y_i^* = v_i^* = \psi^*(v_i)$ *for* $i = 1, \ldots, m$, *and* $y^*_{m+j} = \eta_j^* = \psi^*(\eta_j)$ *for* $j = 1, \ldots, n$. *Then the set of functions* (v^*, η^*) *satisfies the following condition*:

(*) $v_i^* \in (C^\infty(\mathcal{U}))_{\bar{0}}$, $\eta_j^* \in (C^\infty(\mathcal{U}))_{\bar{1}}$; *if* $u \in \mathcal{U}$, *then* $v = (\widetilde{v_1^*}, \ldots, \widetilde{v_m^*}(u))$ *belongs to* V.

b) *Let*

$$y^* = (v_1^*, \ldots, v_m^*, \eta_1^*, \ldots, \eta_n^*)$$

be an arbitrary set of functions in $C^\infty(\mathcal{U})$ *satisfying* (*). *Then there is one and only one homomorphism of superalgebras* $\psi^*: C^\infty(\mathcal{V}) \to C^\infty(\mathcal{U})$ *for which* $\Psi^*(v_i) = v_i^*$, $\Psi^*(\eta_j) = \eta_j^*$.

c) *To every homomorphism of superalgebras* $\psi^*: C^\infty(\mathcal{U}) \to C^\infty(\mathcal{V})$ *there corresponds one and only one morphism of superdomains* $\varphi: \mathcal{U} \to \mathcal{V}$ *such that* $\varphi^*: C^\infty(\mathcal{V}) \to C^\infty(\mathcal{U})$ *is the same as* ψ^*.

It is clear from this that the morphisms of $\mathcal{U}$ into $\mathcal{V}$ are in one-to-one correspondence with the homomorphisms of superalgebras $C^\infty(\mathcal{V}) \to C^\infty(\mathcal{U})$. Furthermore, these morphisms can be specified in terms of coordinates by prescribing a set of functions $y^* = (v^*, \eta^*)$ satisfying (*). We have used this expression of morphisms in Example c) of 2.1.3.

Note that when $\mathcal{V}$ is a superspace, the second part of (*) holds automatically.

For the proof of the theorem we need two technical lemmas.

2.1.8. LEMMA (Hadamard's lemma). *Let* $\mathcal{U}$ *be a superdomain with coordinates*

$$(u_1, \ldots, u_p, \xi_1, \ldots, \xi_q), \quad u \in \mathcal{U}.$$

Let I_u *be the ideal in* $C^\infty(\mathcal{U})$ *generated by the functions*

$$u_1 - s_u(u_1), \ldots, u_p - s_u(u_p), \xi_1, \ldots, \xi_q.$$

Then for any function $f \in C^\infty(\mathcal{U})$ *and any integer* $k \geqslant 0$ *there is a polynomial* P_k *in the coordinates*

$$y = (u_1, \ldots, u_p, \xi_1, \ldots, \xi_q)$$

such that

$$f - P_k \in I_u^{k+1}.$$

PROOF. By explicitly writing out the elements of the ring $C^\infty(\mathcal{V})$ as in 2.1.1 it is sufficient to verify the lemma in the case $q = 0$. But then it is just the classical Hadamard lemma.

2.1.9. COROLLARY. *The kernel of the homomorphism* $s_u: C^\infty(\mathcal{U}) \to \mathbf{R}$ (see 2.1.6) *is* I_u.

For it is clear that $I_u \subset \operatorname{Ker} s_u$, and it follows from Hadamard's lemma that $C^\infty(\mathcal{U}) = \mathbf{R} + I_u$.

2.1.10. LEMMA. *Let* $\mathcal{U}^{p,q}$ *be a superdomain,* $f, f' \in C^\infty(\mathcal{U})$, *and* k *an integer greater than* q. *If* $f - f' \in I_u^k$ *for every* $u \in \mathcal{U}$, *then* $f = f'$.

This lemma shows that, although a function $f \in C^\infty(\mathcal{U})$ cannot be recovered from its values at the points of $\mathcal{U}$, that is, from its images in $C^\infty(\mathcal{U})/I_u = \mathbf{R}$, it can be recovered from a knowledge of "its values up to a small exponent k, where $k > q$", that is, a knowledge of its images in $C^\infty(\mathcal{U})/I_u^k$ for all $u \in \mathcal{U}$.

PROOF OF THE LEMMA. We consider the function $f'' = f - f'$. Now $f'' \in I_u^k$ for any $u \in U$. It follows that f'' belongs to the ideal generated by the functions $u_i - s_u(u_i)$ $(i = 1, \ldots, p)$, since the product of any $q + 1$ odd functions is equal to zero. Hence, all the coefficients f_β in the expansion $f'' = \Sigma f_\beta \xi^\beta$ vanish at u. Since this holds for all u, we see that $f'' = 0$ and $f = f'$.

2.1.11. PROOF OF THEOREM 2.1.7. We carry this out in a number of steps.

(1) Let $\psi^*: C^\infty(\mathcal{V}) \to C^\infty(\mathcal{U})$ be a homomorphism of superalgebras, and $u \in \mathcal{U}$. We consider the homomorphism $s_u: C^\infty(\mathcal{U}) \to \mathbf{R}$ (see 2.1.6) and the composite homomorphism $s = s_u \circ \psi: C^\infty(\mathcal{V}) \to \mathbf{R}$. As was shown in 2.1.6, s has the form s_v, where v is some point in $\mathcal{V}$. We consider the ideals $I_u = \operatorname{Ker} s_u \subset C^\infty(\mathcal{U})$ and $I_v = \operatorname{Ker} s_v \subset C^\infty(\mathcal{V})$ (see 2.1.9). It follows from the definition of v that $\psi^*(I_v) \subset I_u$. In particular, if $(v_1^0, \ldots, v_m^0)$ are the coordinates of v, then $v_i - v_i^0 \in I_v$ for all i, so that $\psi^*(v_i - v_i^0) = v_i^* - v_i^0 \in I_u$, hence, $v^0 = \tilde{v}_i^*(u)$. Thus, the point with the coordinates $(\tilde{v}_1^*(u), \ldots, \tilde{v}_m^*(u))$ is v and so belongs to $\mathcal{V}$. This proves a).

(2) Let $y^* = (v^*, \eta^*) \in C^\infty(\mathcal{U})$ be a given set of functions. We claim that the homomorphism $\psi^*: C^\infty(\mathcal{V}) \to C^\infty(\mathcal{U})$, if it exists, is uniquely determined by the condition $\psi^*(y_i) = y_i^*$.

For $f \in C^\infty(\mathcal{V})$, we show that $\psi^*(f)$ is uniquely determined by the set y^*. By Lemma 2.1.10, it is sufficient to verify that for any $u \in \mathcal{U}$ the image of $\psi^*(f)$ in the quotient algebra $C^\infty(\mathcal{U})/I_u^{q+1}$ is uniquely determined.

First of all, it is clear from (1) that y^* determines a unique point $v \in \mathcal{V}$ such that $\psi^*(I_v) \subset I_u$. We choose a polynomial $P(y)$ such that $f - P(y) \in I_v^{q+1}$ (see 2.1.8). Then $\psi^*(f) - P(y^*) \in \psi^*(I_v^{q+1}) \subset I_u^{q+1}$, that is, the image of $\psi^*(f)$ in $C^\infty(\mathcal{U})/I_u^{q+1}$ coincides with that of $P(y^*)$, hence, is determined by y^*. Thus, ψ^* is unique.

(3) Let $y^* \in C^\infty(\mathcal{U})$ be a given set of functions satisfying (*). Then the functions $\tilde{y}_1^*, \ldots, \tilde{y}_m^*$ determine a smooth mapping $\tilde{\varphi}: U \to V$.

We claim that there is a homomorphism $\psi^*: C^\infty(\mathcal{V}) \to C^\infty(\mathcal{U})$ such that $\psi^*(y_i) = y_i^*$ (which is unique by (2)).

Let $C^\infty(V)$ be the subring of $C^\infty(\mathcal{V})$ consisting of those functions that only depend on v. Since every $f \in C^\infty(\mathcal{V})$ can be written uniquely in the form $\Sigma f_\beta \xi^\beta$, with $f_\beta \in C^\infty(V)$ (see 2.1.1), it is sufficient to construct a homomorphism $\psi^*: C^\infty(V) \to C^\infty(\mathcal{U})$. Hence, we may assume that $n = 0$.

We write each y_i^* in the form $y_i' + y_i''$, where $y_i' = \tilde{y}_i^* \in C^\infty(U) \subset C^\infty(\mathcal{U})$ and $y_i'' = y_i^* - y_i'$. It is clear that y_i'' is nilpotent, indeed, $(y_i'')^{q+1} = 0$.

The mapping $\widetilde{\varphi}\colon U \to V$ induces a homomorphism $\widetilde{\varphi}^*\colon C^\infty(V) \to C^\infty(U)$, for which $\widetilde{\varphi}^*(y_i) = y_i'$. We want to modify $\widetilde{\varphi}^*$ to a homomorphism $\psi^*\colon C^\infty(V) \to C^\infty(\mathcal{U})$ such that

$$\psi^*(y_i) = y_i^* = y_i' + y_i''.$$

For any $f \in C^\infty(V)$, let $\hat{f}$ denote its Taylor series. This means that we consider the function

$$f(y_1 + z_1,\ y_2 + z_2,\ \ldots,\ y_m + z_m),$$

where the point $(z_1, \ldots, z_m)$ varies over a small neighbourhood of zero, expressed as a Taylor series in z: $\hat{f} = \Sigma f_\alpha z^\alpha$, where α ranges over n-tuples of non-negative integers and $f_\alpha \in C^\infty(V)$. We regard $\hat{f}$ as a formal power series in z, that is, we are not concerned with its convergence. It is clear that

$$\widehat{(f+h)} = \hat{f} + \hat{h},\quad \widehat{fh} = \hat{f}\,\hat{h},$$

where the right-hand sides are the sum and product of formal power series in z.

Let $\psi^*(f)$ denote the element of $C^\infty(\mathcal{U})$ obtained from $\hat{f}$ by replacing f_α by $\varphi^*(f_\alpha)$ and z_i by y_i'', $i = 1, \ldots, m$; this makes sense because y_i'' is nilpotent. It is clear that

$$\psi^*(f + h) = \psi^*(f) + \psi^*(h),\quad \psi^*(fh) = \psi^*(f)\psi^*(h).$$

Furthermore, $\psi^*(y_i) = y_i^*$, so that ψ^* is the required homomorphism.

(4) Let $\varphi\colon \mathcal{U} \to \mathcal{V}$ be a morphism of superdomains. We claim that it is uniquely determined by the homomorphism of superalgebras $\varphi^*\colon C^\infty(\mathcal{V}) \to C^\infty(\mathcal{U})$.

First of all, note that, by 2.1.5, the mapping $\widetilde{\varphi}\colon U \to V$ can be recovered uniquely from φ^*.

Next, for any open subset $V' \subset V$, the homomorphism

$$\varphi^*_{V'}\colon \Gamma(V',\ \mathcal{O}_{m,n}) \to \Gamma(\widetilde{\varphi}^{-1}(V'),\ \mathcal{O}_{p,q})$$

is uniquely determined, since we know its effect on the v_i and η_i. Hence, the morphism $\varphi\colon \mathcal{U} \to \mathcal{V}$ of ringed spaces is determined by φ^*.

(5) Let $\psi^*\colon C^\infty(\mathcal{V}) \to C^\infty(\mathcal{U})$ be a given homomorphism. Then the set of functions $\widetilde{y}^* = \psi^*(\widetilde{y})$ satisfies (*), by (1). The set of functions $\widetilde{y}_1^*, \ldots, \widetilde{y}_m^* \in C^\infty(U)$ determines a smooth mapping $\widetilde{\varphi}\colon U \to V$. If V' is an open subset of V and $U' = \widetilde{\varphi}^{-1}(V')$, then, by (3), the set of functions $v_i^*|_{U'}$, $\eta_j^*|_{U'}$, determines a homomorphism

$$\varphi^*_{V'} : \Gamma(V',\ \mathcal{O}_{m,n}) \to \Gamma(U',\ \mathcal{O}_{p,q}).$$

By the uniqueness proved in (2), the system of homomorphisms $\varphi^*_{V'}$ is compatible with restriction homomorphisms. Therefore, we obtain a morphism $\varphi\colon \mathcal{U} \to \mathcal{V}$ of ringed spaces for which $\varphi^* = \psi^*$. This completes the proof of Theorem 2.1.7.

2.1.12. Let $\mathcal{U}$ and $\mathcal{V}$ be superdomains with coordinates (u, ξ) and (v, η), respectively. The *product* of $\mathcal{U}$ and $\mathcal{V}$ is the superdomain $\mathcal{U} \times \mathcal{V}$ with underlying domain $U \times V$ and coordinates (u, v, ξ, η). We consider the projection morphisms

$$\mathrm{pr}_{\mathcal{U}}\colon \mathcal{U} \times \mathcal{V} \to \mathcal{U} \quad \text{and} \quad \mathrm{pr}_{\mathcal{V}}\colon \mathcal{U} \times \mathcal{V} \to \mathcal{V},$$

given in terms of coordinates by the formulae

$$\mathrm{pr}^*_{\mathcal{U}}(u_i) = u_i, \quad \mathrm{pr}^*_{\mathcal{U}}(v_i) = 0, \quad \mathrm{pr}^*_{\mathcal{U}}(\xi_i) = \xi_i, \quad \mathrm{pr}^*_{\mathcal{U}}(\eta_i) = 0,$$
$$\mathrm{pr}^*_{\mathcal{V}}(u_i) = 0, \quad \mathrm{pr}^*_{\mathcal{V}}(v_i) = v_i, \quad \mathrm{pr}^*_{\mathcal{V}}(\xi_i) = 0, \quad \mathrm{pr}^*_{\mathcal{V}}(\eta_i) = \eta_i.$$

LEMMA. *If $\mathcal{W}$ is a superdomain, then for any pair of morphisms $\varphi\colon \mathcal{W} \to \mathcal{U}$ and $\psi\colon \mathcal{W} \to \mathcal{V}$ there is one and only one morphism $\varphi \times \psi\colon \mathcal{W} \to \mathcal{U} \times \mathcal{V}$ such that*

$$\mathrm{pr}_{\mathcal{U}} \circ \varphi \times \psi = \varphi \quad \textit{and} \quad \mathrm{pr}_{\mathcal{V}} \circ \varphi \times \psi = \psi.$$

This lemma follows at once from the coordinate expression of morphisms.

§2. Vector fields and Taylor series

2.2.1. Let $\mathcal{U}^{p,q}$ be a superdomain and $x = (u, \xi)$ a coordinate system on $\mathcal{U}$. We define a derivation $\frac{\partial}{\partial x_i}\colon C^\infty(\mathcal{U}) \to C^\infty(\mathcal{U})$ called a *partial derivative*, by setting

$$\frac{\partial}{\partial u_i}(f\xi_1^{\nu_1} \dots \xi_q^{\nu_q}) = \frac{\partial f}{\partial u_i}\xi_1^{\nu_1} \dots \xi_q^{\nu_q},$$
$$\frac{\partial}{\partial \xi_j}(f\xi_1^{\nu_1} \dots \xi_q^{\nu_q}) = \nu_j(-1)^{\nu_1+\dots+\nu_{j-1}} f\xi_1^{\nu_1} \dots \xi_j^{\nu_j - 1} \dots \xi_q^{\nu_q},$$

where $f \in C^\infty(U)$.

LEMMA. *The $\partial/\partial u_i$ are even and the $\partial/\partial \xi_j$ are odd. They satisfy the Leibniz rule*

$$\frac{\partial}{\partial u_i}(fg) = \frac{\partial f}{\partial u_i} g + f \frac{\partial g}{\partial u_i},$$
$$\frac{\partial}{\partial \xi_j}(fg) = \frac{\partial f}{\partial \xi_j} g + (-1)^{p(f)} f \frac{\partial g}{\partial \xi_j}.$$

PROOF. By linearity, we may assume that f and g are monomials in ξ, and for monomials the proof is immediate.

2.2.2. Let Der $C^\infty(\mathcal{U})$ denote the Lie superalgebra of derivations of $C^\infty(\mathcal{U})$ (see 1.9.5). Then Der $C^\infty(\mathcal{U})$ has the structure of a $C^\infty(\mathcal{U})$-module, given by

$$(fD)(g) = f(D(g)), \text{ where } f,\ g \in C^\infty(\mathcal{U}),\ D \in \mathrm{Der}\, C^\infty(\mathcal{U}).$$

It is clear that

$$[D_1, fD_2] = D_1(f)\, D_2 + (-1)^{p(f)p(D_1)} f\,[D_1, D_2].$$

Let $f, g \in C^\infty(\mathcal{U})$ and let $\hat{f}$ denote the operator $g \mapsto fg$. Then $[D, \hat{f}] = \widehat{D(f)}$.

By analogy with classical analysis, we call the derivations on $C^\infty(\mathcal{U})$ a *vector field* on $\mathcal{U}$.

2.2.3. LEMMA. *The Lie superalgebra* Der $C^\infty(\mathcal{U})$ *is a free* $C^\infty(\mathcal{U})$*-module with basis* $\{\partial/\partial x_i\}$.

PROOF. Let $D \in \mathrm{Der}\, C^\infty(\mathcal{U})$. We set $D_i = D(x_i)$ and consider the field $D' = D - \Sigma D_i \partial/\partial x_i$. Since $D'(x_i) = 0$ for all i, we can see that $D'(P) = 0$ for any polynomial $P \in \mathbf{R}[x]$. By Lemma 2.1.10, it is sufficient to prove that $D'(f) \in I_u^{q+1}$ for any $f \in C^\infty(\mathcal{U})$ and any $u \in U$. Let P be a polynomial such that $f - P \in I_u^{q+1}$ (see 2.1.8). Then

$$D'(f) = D'(f) - D'(P) = D'(f - P) \in I_u^{q+1}.$$

This completes the proof.

2.2.4. Let φ: $\mathcal{U}^{p,q} \to \mathcal{V}^{m,n}$ be a morphism of superdomains and $x = (u, \xi)$, and $y = (v, \eta)$ coordinate systems on $\mathcal{U}$ and $\mathcal{V}$ respectively.

The *matrix of derivatives of the coordinates y by the coordinates x is defined as* I_{xy}

$$(I_{xy})_{ik} = \frac{\partial}{\partial x_i}\, \varphi^*(y_k).$$

The set of partial derivatives of any function $f \in C^\infty(\mathcal{V})$ is written in the form of a column-vector $|\, \partial f/\partial y_i\rangle$ (and similarly for $g \in C^\infty(\mathcal{U})$).

THEOREM (rule for differentiating a composite function).

$$(*) \qquad \frac{\partial \varphi^*(f)}{\partial x_i} = \sum_{k=1}^{m+n} \frac{\partial \varphi^*(y_k)}{\partial x_i}\, \varphi^*\left(\frac{\partial f}{\partial y_k}\right) \qquad (i = 1, 2, \ldots, p+q),$$

or, in matrix notation,

$$(**) \qquad \left|\frac{\partial \varphi^*(f)}{\partial x}\right\rangle = I_{xy} \left|\varphi^* \frac{\partial f}{\partial y}\right\rangle.$$

In particular, if ψ: $\mathcal{V} \to \mathcal{W}$ *is a morphism of superdomains and* $z = (w, \zeta)$ *a coordinate system on* $\mathcal{W}$*, then*

$$(***) \qquad I_{xy} \cdot \psi^*(I_{yz}) = I_{xz}.$$

COROLLARY. *If x and y are two coordinate systems on* $\mathcal{U}$*, then*

$$\frac{\partial}{\partial x_i} = \sum \frac{\partial y_k}{\partial x_i} \frac{\partial}{\partial y_k}.$$

In ordinary analysis, the Jacobian matrix is used instead of I_{xy}, which is its transpose. Therefore, we also introduce the *Jacobian matrix* J_φ of a mapping φ by $J_\varphi = (I_{xy})^{\mathrm{st}}$. In other words, J_φ is of size $(m, n) \times (p, q)$ and is given by the formula

$$J_{ki} = (-1)^{(p(y_k)+1)p(x_i)} \frac{\partial}{\partial x_i} \varphi^*(y_k)$$

or, in block form,

$$J_\varphi = \begin{pmatrix} \frac{\partial v}{\partial u} & -\frac{\partial v}{\partial \xi} \\ \frac{\partial \eta}{\partial u} & \frac{\partial \eta}{\partial \xi} \end{pmatrix}.$$

From (***) it follows that $J_{\psi\circ\varphi} = \varphi^*(J_\psi) \cdot J_\varphi$ (see 1.3.8); we recall that $p(J) = \bar{0}$.

2.2.5. PROOF OF THEOREM 2.2.3. Let D denote the difference between the left- and right-hand sides of (*) in 2.2.3. It follows from 2.2.2 that

$$D(fg) = D(f)\varphi^*(g) + (-1)^{p(D)p(f)} \varphi^*(f)D(g),$$

where $f, g \in C^\infty(\mathcal{V})$, and $D(f), D(g) \in C^\infty(\mathcal{U})$. Moreover, it is easy to verify that $D(y_r) = 0$ for $r = 1, \ldots, m+n$. Therefore, it follows from Leibniz' rule that if f is a polynomial in y, then $D(f) = 0$. The proof that $D(f) = 0$ for any function f goes just as in Lemma 2.2.3.

2.2.6. In ordinary analysis, the Taylor series of a function f is a power series that approximates f asymptotically in a neighbourhood of a given point x. The coefficients of this series can be expressed in terms of the derivatives of f at x.

Since in the supercase the value of a function at a point carries little information, we have to use Taylor series that depend on additional parameters.

We begin by formulating Hadamard's lemma for functions depending on a parameter.

Let $\mathcal{U}$ and $\mathcal{V}$ be superdomains with coordinate systems x and y, respectively, and let $\mathcal{U} \times \mathcal{V}$ be their product (see 2.1.12). Suppose that 0 belongs to $\mathcal{V}$, and let $I_{\mathcal{U}}$ be the ideal in $C^\infty(\mathcal{U} \times \mathcal{V})$ generated by the set of functions $y = (v, \eta)$.

GENERAL HADAMARD LEMMA. *Let $f \in C^\infty(\mathcal{U} \times \mathcal{V})$. Then for any integer $r \geqslant 0$ we can select a polynomial $P_r = \sum_{|n|+|\nu|=r} P_{n,\nu} v^n \eta^\nu$ in $y = (v, \eta)$ with coefficients $P_{n,\nu} \in C^\infty(\mathcal{U})$ such that $f - P_r \in I_{\mathcal{U}}^{r+1}$.*

We shall prove this in 2.2.11 but first we derive Taylor's formula from it.

2.2.7. Let $\mathcal{W}$ be a superdomain with coordinates u, v, ξ, η. Let $\mathcal{U}$ be the subsuperdomain of $\mathcal{W}$ defined by $v = 0, \eta = 0$. (This means that in the underlying domain W we select a subdomain U of W by means of the equation $v = 0$ and leave in it of the odd coordinates only ξ.) The formulae

$$\varphi^*(u) = u, \quad \varphi^*(\xi) = \xi, \quad \varphi^*(v) = 0, \quad \varphi^*(\eta) = 0$$

yield a natural embedding $\varphi: \mathcal{U} \to \mathcal{W}$. Let

$$f|_{\mathcal{U}} = f|_{v=0} = \varphi^*(f).$$

It follows from the general Hadamard lemma that

$$C^\infty(\mathscr{W}) = C^\infty(\mathscr{U}) \oplus I_{\mathscr{U}}$$

in a neighbourhood of u, where $C^\infty(\mathscr{U})$ is naturally embedded in $C^\infty(\mathscr{W})$ ($u \mapsto u$, $\xi \mapsto \xi$), and $I_{\mathscr{U}}$ is the ideal in $C^\infty(\mathscr{W})$ generated by the functions $y = (v, \eta)$.

2.2.8. TAYLOR'S FORMULA (first form).

THEOREM. *In the notation of* 2.2.6, *let* $f \in C^\infty(\mathscr{U} \times \mathscr{V})$ *and let* r *be a non-negative integer. Then there is a polynomial*

$$P_r = \sum_{|n|+|\nu|\leqslant r} P_{n,\nu} v^n \eta^\nu$$

of degree at most r *in the variable* $y = (v, \eta)$, *with coefficients* $P_{n,\nu} \in C^\infty(\mathscr{U})$, *such that* $f - P_r \in I_{\mathscr{U}}^{r+1}$. *This polynomial is unique, and its coefficients can be expressed in terms of* f *by the formula*

$$P_{n\nu} = \frac{N(\nu)}{n!}\left[\left(\frac{\partial}{\partial v}\right)^n \left(\frac{\partial}{\partial \eta}\right)^\nu f\right]\Big|_{\mathscr{U}}.$$

Here $\mathscr{U}$ is embedded in $\mathscr{W}$ as in 2.2.7, $n = (n_1, \ldots, n_p)$,

$$\nu = (\nu_1, \ldots, \nu_q),\ n! = n_1! \ldots n_p!,\ N(\nu) = (-1)^{|\nu|(|\nu|-1)/2}.$$

The P_r constructed in the theorem is called the *Taylor polynomial* of degree r of f in the variable $y = (v, \eta)$. The formal power series

$$\hat{f} = \sum P_{n\nu} v^n \eta^\nu$$

is called the *Taylor series* of f in y.

PROOF OF THE THEOREM. The existence of P_r follows from the general Hadamard lemma. Next, applying to $f - P_r$ the operator

$$D^{n\nu} = \left(\frac{\partial}{\partial v}\right)^n \left(\frac{\partial}{\partial \eta}\right)^\nu,$$

where $|n| + |\nu| \leqslant r$, we obtain a function in $I_{\mathscr{U}}$. It follows that

$$D^{n\nu} f\,|_{\mathscr{U}} = D^{n\nu} P_r\,|_{\mathscr{U}}.$$

But it is easy to see that the right-hand side is equal to $N(\nu) n!\, P_{n\nu}$, as required.

2.2.9. TAYLOR'S FORMULA (second form). Let $\mathscr{U}^{p,q}$ be a superdomain with coordinates $x = (u, \xi)$, and let $f \in C^\infty(\mathscr{U})$. We introduce additional coordinates $\Delta x = (\Delta u, \Delta\xi)$ and consider the function $f(x + \Delta x)$. In more detail, on the superdomain $\mathscr{U} \times \mathscr{R}^{p,q}$ with coordinates $(u, \Delta u, \xi, \Delta\xi)$ we define a morphism s: $\mathscr{U} \times \mathscr{R}^{p,q} \to \mathscr{U}$ by $s^*(x_i) = x_i + \Delta x_i$ (in a neighbourhood of $u \times \{0\}$). Then $f^* = s^*(f)$ is a function of x and Δx and is usually written $f(x + \Delta x)$.

Applying Taylor's formula to Δx, we obtain the formula in its second form:

$$f(x + \Delta x) = \sum_{|n|+|\nu|\leqslant r} P_{n,\nu} (\Delta u)^n (\Delta\xi)^\nu + \omega,$$

where the remainder term ω has a zero in Δx of order at least $r+1$, that is, it belongs to $I_{\mathcal{U}}^{r+1}$, where $I_{\mathcal{U}}$ is the ideal generated by the variables Δx.

2.2.10. The second form of Taylor's formula yields the following "geometrical" interpretation of the partial derivatives, which corresponds to the usual definition of classical analysis. Let $\mathcal{U}$ be a superdomain with coordinates $x = (u, \xi)$, and let $f \in C^\infty(\mathcal{U})$. For a fixed i we consider the function

$$h(x, t) = f(x_1, \ldots, x_i + t, \ldots, x_{p+q}) - f(x_1, \ldots, x_{p+q}),$$

where t is a coordinate of the same parity as x_i. By Hadamard's lemma, there is then a function $g(x, t)$ such that $t\,g(x, t) = h(x, t)$ (note that if t is an odd variable, then $g(x, t)$ is not uniquely determined by this equation). From the definition we have

$$\frac{\partial f}{\partial x_i} = g(x, t)\Big|_{t=0}.$$

It follows from 2.2.9 that this expression is the same as that defined in 2.2.1, in particular, is independent of the choice of $g(x, t)$.

2.2.11. PROOF OF THE GENERAL HADAMARD LEMMA (see 2.2.6).

(1) We claim that it is sufficient to prove the general Hadamard lemma for $r = 0$. For assuming that this has been done, we proceed by induction on r. Let P_r be a polynomial such that $\omega = f - P_r \in I_{\mathcal{U}}^{r+1}$. This means that

$$\omega = \sum_{|n|+|\nu|=r+1} f_{n\nu} v^n \eta^\nu, \quad \text{where} \quad f_{n\nu} \in C^\infty(\mathcal{U} \times \mathcal{V}).$$

We write each of the coefficients $f_{n\nu}$ in the form $C_{n\nu} + f'_{n\nu}$, with $C_{n\nu} \in C^\infty(\mathcal{U})$ and $f'_{n\nu} \in I_{\mathcal{U}}$ (this is possible by the case $r = 0$ of the lemma). Setting

$$P_{r+1} = P_r + \sum_{|n|+|\nu|=r+1} C_{n\nu} v^n \eta^\nu,$$

we then find that $f - P_{r+1} \in I_{\mathcal{U}}^{r+2}$. Thus, we have constructed P_{r+1} from P_r, and this completes the proof.

(2) It remains to consider the case $r = 0$. Using the explicit expression for the elements of $C^\infty(\mathcal{U} \times \mathcal{V})$ as polynomials in ξ and η with coefficients depending on u and v, we see that it is sufficient to carry out the proof in the purely even case. Next, an induction on the dimension allows us to reduce everything to the case of a single variable v. Thus, it is sufficient to show that if $f(u_1, \ldots, u_p, v)$ is a smooth function, then

$$g(u_1, \ldots, u_p, v) = \frac{1}{v}[f(u_1, \ldots, u_p, v) - f(u_1, \ldots, u_p, 0)]$$

extends to a smooth function of $(u_1, \ldots, u_p, v)$. But it is easy to check that g can be given explicitly by the formula

$$g(u_1, \ldots, u_p, v) = \int_0^1 \frac{\partial f}{\partial v}(u_1, \ldots, u_p, tv)\, dt,$$

in which the right-hand side is a smooth function of u and v. This completes the proof of the general Hadamard lemma.

§3. The inverse function theorem and the implicit function theorem

2.3.1. THEOREM (the inverse function theorem). *Let $\mathcal{U}$ and $\mathcal{V}$ be superdomains with coordinate systems $x = (u, \xi)$ and $y = (v, \eta)$, respectively; let $\varphi\colon \mathcal{U} \to \mathcal{V}$ be a morphism of superdomains and $u_0 \in \mathcal{U}$. Then the following conditions are equivalent:*

a) *φ is a diffeomorphism in a neighbourhood of u_0,*

b) *the Jacobian matrix J_φ of φ is invertible at u_0.*

Note that $\partial v/\partial \xi$ and $\partial \eta/\partial u$ are odd functions, so that they vanish at u_0. Therefore, b) is equivalent to the invertibility of the matrices $\partial v/\partial u$ and $\partial \eta/\partial \xi$ at u_0.

2.3.2. We break up the proof of Theorem 2.3.1 into several steps.

(1) a) $\Rightarrow$ b). By passing over to neighbourhoods, we may assume that φ is an isomorphism, that is, a change of variables, and this case was analysed in 2.2.5.

(2) We now embark on the proof of the reverse implication. Let I_U denote the ideal in $C^\infty(\mathcal{U})$ that singles out the underlying domain U; this is the ideal generated by the variables ξ.

We show first that we may restrict our attention to the case when $\widetilde{\varphi}\colon U \to V$ is a diffeomorphism and $v_i^* = u_i (\mathrm{mod}\ I_U)$ (that is, $\widetilde{v} = u$; see 2.1.3). For since $\partial v/\partial u$ is an invertible matrix, we can invoke the classical inverse function theorem and replace U and V by subdomains U' and V' such that $\widetilde{\varphi}\colon U' \to V'$ is a diffeomorphism. We can then replace the coordinates u on U' by new coordinates $u' = \widetilde{v}^*$ on U' without disturbing the coordinates ξ. It follows that $v = u\ (\mathrm{mod}\ I_U)$.

(3) We may assume that $\partial \eta/\partial \xi$ is invertible everywhere on U, for otherwise we replace U by a neighbourhood of u_0 where it is invertible. Without disturbing the coordinates u, we make a linear change of the coordinates ξ:

$$\xi'_j = \sum_k f_{jk}\xi_k, \quad \text{where} \quad (f_{jk}) = \widetilde{\left(\frac{\partial \eta_j}{\partial \xi_k}\right)}.$$

Note that this is a change of coordinates, since it has the inverse

$$\xi_k = \sum g_{kj}\xi'_j,$$

where the $g_{kj} \in C^\infty(U)$ can be found from the system of equations

$$\sum g_{kj} f_{ji} = \delta_{ki},$$

which can be solved, because (f_{jk}) is invertible.

In the new coordinates

$$v_i^* = u_i (\mathrm{mod}\ I_U), \quad \eta_i^* = \xi'_i (\mathrm{mod}\ I_U^2),$$

and we can assume that this is already true for the original coordinates $x = (u, \xi)$.

(4) We now construct a morphism $\psi\colon \mathscr{V} \to \mathscr{U}$ inverse to φ.

We define morphisms $\psi_k^*\colon \mathscr{V} \to \mathscr{U}$ by setting

$$\psi_0^*(x_i) = y_i,$$

$$\psi_{k+1}^*(x_i) = \psi_k^*(x_i) + \psi_0^*(x_i - \varphi^*\psi_k^*(x_i)).$$

We claim that ψ_k^* is the required morphism for $k > q$. To see this, we consider the mappings

$$\Delta_k\colon C^\infty(\mathscr{U}) \to C^\infty(\mathscr{U}),\ \Delta_k(f) = \varphi^*\psi_k^*(f) - f$$

(they are not homomorphisms). By construction, $\Delta_{k+1}(x_i) = \Delta_0(\Delta_k(x_i))$. Since the homomorphism $\varphi^*\psi_0^*\colon C^\infty(\mathscr{U}) \to C^\infty(\mathscr{U})$ satisfies the condition

$$\varphi^*\psi_0^*(u_i) = u_i(\mathrm{mod}\ I_U),\ \varphi^*\psi_0^*(\xi_j) = \xi_j(\mathrm{mod}\ I_U^2),$$

it follows that $\Delta_0(I_U^k) \subset I_U^{k+1}$. Hence, $\Delta_k(f) \subset I_U^k$ for all $f \in C^\infty(\mathscr{U})$. In particular, $\Delta_k(f) = 0$ for $k > q$, that is, $\varphi^*\psi_k^*$ is the identity homomorphism.

(5) We have now constructed a morphism $\psi\colon \mathscr{V} \to \mathscr{U}$ such that $\psi\varphi$ is the identity, that is, φ has a left inverse. If $v_0 = \varphi(u_0)$, then the Jacobian matrix J_ψ of ψ is invertible at v_0, because $J_\psi(v_0)J_\varphi(u_0) = 1$. It follows from what has been proved that there is a morphism $\varphi'\colon \mathscr{U} \to \mathscr{V}$ such that $\varphi'\psi$ is the identity. But it is clear that $\varphi' = \varphi'\psi\varphi = \varphi$, that is, ψ is the two-sided inverse of φ.

2.3.3. COROLLARY. *Let $\varphi\colon \mathscr{U} \to \mathscr{V}$ be a morphism of superdomains such that the mapping $\tilde{\varphi}\colon U \to V$ of the underlying domains is one-to-one and the Jacobian matrix J_φ is invertible at every point $u \in U$. Then φ is a diffeomorphism.*

PROOF. We construct a morphism $\psi\colon \mathscr{V} \to \mathscr{U}$ inverse to φ. We put $\tilde{\psi} = \tilde{\varphi}^{-1}$, which is well defined, since $\tilde{\psi}$ is one-to-one. If $v \in V$ and $u = \tilde{\varphi}^{-1}(v)$, then by the inverse function theorem φ maps a neighbourhood $\mathscr{U}'$ of u isomorphically onto a neighbourhood $\mathscr{V}'$ of v. Hence, on $\mathscr{V}'$ there is a morphism $\psi'\colon \mathscr{V}' \to \mathscr{U}'$ inverse to φ. It is clear that $\tilde{\psi}' = \tilde{\psi}$, so that $\tilde{\psi}$ is continuous in a neighbourhood of v; by the same token $\tilde{\psi}\colon V \to U$ is a continuous mapping. We identify the topological spaces U and V by means of the homeomorphisms $\tilde{\varphi}$ and $\tilde{\psi}$.

Let $\mathcal{O}_{\mathscr{U}}$ and $\mathcal{O}_{\mathscr{V}}$ be the structure sheaves on U and V.

Then φ induces a morphism of sheaves $\varphi^*\colon \mathcal{O}_{\mathscr{V}} \to \mathcal{O}_{\mathscr{U}}$, which, by the inverse function theorem, locally in a neighbourhood of each point $u \in \mathscr{U}$, has an inverse $\psi_u^*\colon \mathcal{O}_{\mathscr{U}} \to \mathcal{O}_{\mathscr{V}}$. Clearly this inverse is unique, so that the morphisms ψ_u^* on different neighbourhoods agree on their intersection. Therefore, they induce a global inverse morphism of sheaves $\psi^*\colon \mathcal{O}_{\mathscr{U}} \to \mathcal{O}_{\mathscr{V}}$, that is, a morphism of ringed spaces $\psi\colon \mathscr{V} \to \mathscr{U}$ inverse to φ.

2.3.4. COROLLARY. *Let $\varphi\colon \mathscr{U} \to \mathscr{V}$ be a morphism of superdomains. Suppose that the Jacobian matrix J_φ is invertible at every point $u \in U$ and that φ does not glue points together, that is, $\tilde{\varphi}(u) \neq \tilde{\varphi}(u')$ for distinct points*

$u, u' \in U$. Then there is an open subdomain $\mathscr{V}' \subset \mathscr{V}$, such that $\varphi(\mathscr{U}) \subset \mathscr{V}'$ and $\varphi\colon \mathscr{U} \to \mathscr{V}'$ is an isomorphism.

PROOF. Let $V' = \widetilde{\varphi}(U)$. If $v \in V'$ and $u = \widetilde{\varphi}^{-1}(v)$, then by the inverse function theorem, there are neighbourhoods U'' of u and V'' of v such that $\widetilde{\varphi}\colon U'' \to V''$ is a diffeomorphism. In particular, V' is an open subset of V. Let $\mathscr{V}'$ denote the corresponding subdomain of $\mathscr{V}$. Then applying Corollary 2.3.3 to the morphism $\varphi\colon \mathscr{U} \to \mathscr{V}'$, we see that it is an isomorphism.

2.3.5. COROLLARY. *Let $\mathscr{U}^{p,q}$ be a superdomain, $x = (u, \xi)$ a coordinate system on $\mathscr{U}$, and $y = (v_1, \ldots, v_p, \eta_1, \ldots, \eta_q)$ a system of functions satisfying the following conditions*:

a) $v_i \in (C^\infty(\mathscr{U}))_{\bar{0}}, \eta_j \in C^\infty(\mathscr{U}))_{\bar{1}}$;

b) *the matrices of partial derivatives $A = (\partial v/\partial u)$ and $D = (\partial \eta/\partial \xi)$ are invertible at every point $u \in \mathscr{U}$*;

c) *the $\widetilde{v}_i$ separate points, that is, if $u, u' \in U$ and $u \neq u'$, then $\widetilde{v}_i(u) \neq \widetilde{v}_i(u')$ for some i.*

Then $y = (v, \xi)$ is a coordinate system on $\mathscr{U}$.

To prove this it is sufficient to apply Corollary 2.3.4 to the morphism $\varphi\colon \mathscr{U} \to \mathscr{R}^{p,q}$ defined by the coordinate expression $\varphi^*(v_i') = v_i$, $\varphi^*(\eta_j') = \eta_j$, where $y' = (v', \eta')$ is a coordinate system on $\mathscr{R}^{p,q}$.

2.3.6. THEOREM (the implicit function theorem). *Let $\mathscr{U}$, $\mathscr{V}$, and $\mathscr{W}$ be superdomains with coordinate systems x, y, and z, respectively; let $\varphi\colon \mathscr{U} \times \mathscr{V} \to \mathscr{W}$ be a morphism of superdomains, and let u_0 and v_0 be points of U and V with $w_0 = \widetilde{\varphi}(u_0, v_0)$. Suppose that the Jacobian matrix $J_{\varphi|u} = (\partial z/\partial x)$ is invertible at $(u_0, v_0) \in \mathscr{U} \times \mathscr{V}$.*

Then there is one and only one morphism $\psi\colon \mathscr{W} \times \mathscr{V} \to \mathscr{U}$ defined in a neighbourhood of $w_0 \times v_0$ such that $\widetilde{\psi}(w_0 \times v_0) = u_0$ and the composite

$$\mathscr{W} \times \mathscr{V} \xrightarrow{\psi \times \mathrm{pr}_{\mathscr{V}}} \mathscr{U} \times \mathscr{V} \xrightarrow{\varphi} \mathscr{W}$$

coincides with the projection $\mathrm{pr}_{\mathscr{W}}\colon \mathscr{W} \times \mathscr{V} \to \mathscr{W}$.

In the classical case this assertion can be expressed in the language of points as follows:

$$\varphi(\psi(w, v), v) = w \text{ for all } v \in V, w \in W.$$

PROOF. We consider the morphism $\varphi' = \varphi \times \mathrm{pr}_{\mathscr{V}}\colon \mathscr{U} \times \mathscr{V} \to \mathscr{W} \times \mathscr{V}$. By hypothesis, $J_{\varphi'}$ is invertible at (u_0, v_0). Therefore, by the inverse function theorem, in a neighbourhood of (w_0, v_0) there is an inverse morphism $\psi'\colon \mathscr{W} \times \mathscr{V} \to \mathscr{U} \times \mathscr{V}$. It is clear that the morphism $\psi = \mathrm{pr}_{\mathscr{U}} \circ \psi'$ satisfies the conditions of the theorem. The uniqueness of ψ follows from the fact that $\psi \times \mathrm{pr}_{\mathscr{V}}\colon \mathscr{W} \times \mathscr{V} \to \mathscr{U} \times \mathscr{V}$ is inverse to φ', hence, is uniquely determined.

2.3.7. Let $\mathscr{U}$ be a superdomain and $J = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ an even matrix of functions on $\mathscr{U}$. The *rank* of J at a point $u \in \mathscr{U}$ is defined as the pair of numbers

$$\operatorname{rk} J(u) = (\operatorname{rk}(A(u), \operatorname{rk} D(u)).$$

Let $\varphi\colon \mathcal{U} \to \mathcal{V}$ be a morphism of superdomains,

$$x = (u_1, \ldots, u_p, \xi_1, \ldots, \xi_q) \text{ and } y = (v_1, \ldots, v_m, \eta_1, \ldots, \eta_n)$$

coordinate systems on $\mathcal{U}$ and $\mathcal{V}$, and J_φ the Jacobian matrix of φ.

A morphism φ is called an *immersion* at $u \in \mathcal{U}$ if $\operatorname{rk} J_\varphi(u) = (p, q)$, and a *submersion* at $u \in \mathcal{U}$ if $\operatorname{rk} J_\varphi(u) = (m, n)$.

THEOREM. a) *A morphism φ is an immersion at u_0 if and only if in some neighbourhood of u_0 it is an embedding of subsuperdomains, that is, $\mathcal{V}$ can be written as $\mathcal{U} \times \mathcal{W}$ and φ as*

$$\varphi\colon \mathcal{U} = \mathcal{U} \times \{pt\} \xrightarrow{\mathrm{id} \times \iota} \mathcal{V} = \mathcal{U} \times \mathcal{W},$$

where $\iota\colon \{pt\} \to \mathcal{W}$ is the inclusion of a point.

b) *A morphism φ is a submersion at u_0 if and only if in some neighbourhood of u_0 it coincides with a projection, that is, $\mathcal{U}$ can be written as $\mathcal{V} \times \mathcal{W}$, and φ as*

$$\varphi\colon \mathcal{U} = \mathcal{V} \times \mathcal{W} \xrightarrow{\operatorname{pr}^{\mathcal{V}}} \mathcal{V}.$$

In other words, if φ is an immersion, then in a neighbourhood of $\varphi(u_0) \in \mathcal{V}$ we can choose a coordinate system $y' = (v', \eta')$ such that $\varphi^*(v_i') = u_i$ and $\varphi^*(\eta_j') = \xi_j$ for $1 \leqslant i \leqslant p$, $1 \leqslant j \leqslant q$, and $\varphi^*(v_i') = \varphi^*(\eta_j') = 0$ for $i > p$, $j > q$. But if φ is a submersion, then in a neighbourhood of $u_0 \in \mathcal{U}$ we can choose a coordinate system $x' = (u', \xi')$ such that $\varphi^*(v_i) = u_i'$, $\varphi^*(\eta_j) = \xi_j'$ for $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$.

PROOF. a) By renumbering the coordinates y, we may assume that the matrices

$$\left(\frac{\partial}{\partial u_k} \varphi^*(v_i)\right), \quad 1 \leqslant i, k \leqslant p \quad \text{and} \quad \left(\frac{\partial}{\partial \xi_l} \varphi^*(\eta_j)\right), \quad 1 \leqslant j, l \leqslant q$$

are invertible. We consider the superspace $\mathcal{R}^{m-p, n-q}$ with coordinates $u_{p+1}, \ldots, u_m, \xi_{q+1}, \ldots, \xi_n$, and let $\mathcal{W} = \mathcal{U} \times \mathcal{R}^{m-p, n-q}$. We specify a morphism $\psi\colon \mathcal{W} \to \mathcal{V}$ by the formula

$$\psi^*(y_i) = \varphi^*(y_i) \text{ for } 1 \leqslant i \leqslant p, \; m+1 \leqslant i \leqslant m+q,$$
$$\psi^*(y_i) = \varphi^*(y_i) + z_i \text{ for } p < i \leqslant m, \; m+q < i \leqslant m+n.$$

It is easy to check that J_ψ is invertible at $w_0 = u_0 \times (0) \in \mathcal{W}$, hence, by the inverse function theorem, ψ is a local diffeomorphism in a neighbourhood of this point. Therefore we can replace $\mathcal{V}$ by $\mathcal{W}$ and φ by the natural embedding

$$\mathcal{U} = \mathcal{U} \times \{0\} \to \mathcal{W} = \mathcal{U} \times \mathcal{R}^{m-p, n-q}.$$

The converse, which asserts that the embedding of a manifold is an

immersion, follows easily from the fact that the property of being an immersion at u_0 does not depend on the choice of coordinate system.

Part b) of the theorem is proved similarly.

2.3.8. In ordinary analysis, an important part is played by the theorem on mappings of constant rank, which is a generalization of Theorem 2.3.7 a) and b). It asserts that if the rank of J_φ is constant in a neighbourhood of a point $u \in U$, then locally φ can be written in the form

$$\varphi\colon \mathcal{U} \xrightarrow{\varphi_1} \mathcal{W} \xrightarrow{\varphi_2} \mathcal{V},$$

where φ_1 is a projection (submersion) and φ_2 is an embedding (immersion). We prove an analogue to this assertion in the supercase.

We first define a matrix of functions of constant rank. Let J be an even $(p, q) \times (m, n)$ matrix of functions on a superdomain $\mathcal{U}$. Then J is called a *matrix of constant rank* (k, l) if by means of a transformation

$$(*) \qquad J \mapsto g_1 J g_2, \qquad g_1 \in \mathrm{GL}_{p,q}(C^\infty(\mathcal{U})), \quad g_2 \in \mathrm{GL}_{m,n}(C^\infty(\mathcal{U}))$$

it can be brought to the form

$$J = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \qquad \text{where} \quad A = \begin{pmatrix} 1_k & 0 \\ 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1_l & 0 \\ 0 & 0 \end{pmatrix}.$$

Clearly, if J is a matrix of constant rank (k, l), then $\operatorname{rk} J(u) = (k, l)$ for all $u \in \mathcal{U}$. However, the converse is false; for example, if $\mathcal{U} = \mathcal{R}^{0,1}$ and $J = \begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix}$, then $\operatorname{rk} J(0) = (0, 0)$ at the only point 0, but J is not a matrix of constant rank $(0, 0)$.

LEMMA. *Let J be an even matrix of functions on $\mathcal{U}$, and let u_0 be a point of $\mathcal{U}$ with* $\operatorname{rk} J(u_0) = (k, l)$. *Then*:

a) *in some neighbourhood of u_0, by a transformation (*) J can be brought to the form $J' = 1_{k,l} \oplus X$, where X is a matrix of size $(p-k, q-l) \times (m-k, n-l)$ such that $X(u_0) = 0$;*

b) *for J to be a matrix of constant rank (in a neighbourhood of u_0) it is necessary and sufficient that there is a matrix $\gamma \in \mathrm{GL}_{p,q}(C^\infty(\mathcal{U}))$ such that only $k + l$ rows of the matrix γJ are different from zero. A similar result holds for columns*;

c) *suppose that some rows of J can be deleted to give a matrix I of constant rank (k, l). Suppose further that J has constant rank in some neighbourhood of a point $u \in U$. Then in a neighbourhood of u each row J_i of J can be written in the form*

$$J_i = \sum f_{ik} I_k, \quad \text{where } I_k \text{ is a row of } I \text{ and } \quad f_{ik} \in C^\infty(\mathcal{U}).$$

A similar result holds for columns.

PROOF. a) Multiplying by a matrix of constants, we can bring $J(u_0)$ to the form $J(u_0) = 1_{k,l} \oplus 0$. Hence, J can be decomposed into block form

$J = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$, so that $P(u_0) = 1_{k,l}$, $Q(u_0) = R(u_0) = S(u_0) = 0$. By 1.6.2, P is invertible in a neighbourhood of u_0, therefore, multiplying by an invertible matrix, we may assume that $P \equiv 1_{k,l}$. It is clear that the elementary transformations of J are of type (*). By using them we can achieve that Q and R are invertible at 0. This proves a).

b) If $\gamma_1 J \gamma_2 = I_{k,l} \oplus 0$, then at most $k + l$ rows of $\gamma_1 J$ are different from zero. Conversely, assume that J can be reduced by a transformation of type (*) to a matrix with only $k + l$ non-zero rows, and let I be the matrix formed from these rows. By performing (*)-transformations on I, we can bring it to the form $I = 1_{k,l} \oplus X$ (see a)). But since I has only $k + l$ rows, it follows that $X = 0$,as required in b).

c) Replacing J by $J\gamma$, for some $\gamma \in \mathrm{GL}_{m,n}(C^\infty(\mathcal{U}))$, we may assume that J has only $k + l$ non-zero columns (see b)), and by discarding the others, we can take $m = k$, $n = l$. It follows from b) that I is a matrix of constant rank. Hence, it can be reduced by a (*)-transformation to the form $1_{k,l} \oplus 0$; these transformations can be extended to J. Therefore, we may assume from the outset that I has this form. But since I has altogether $k + l$ columns, c) is obvious in this case.

2.3.9. THEOREM (on morphisms of constant rank). *Let* $\varphi\colon \mathcal{U} \to \mathcal{V}$ *be a morphism of superdomains and* $u_0 \in \mathcal{U}$. *Then* φ *splits in a neighbourhood of* u_0 *into a composite* $\mathcal{U} \xrightarrow{\varphi_1} \mathcal{W} \xrightarrow{\varphi_2} \mathcal{V}$, *where* φ_1 *is a projection (submersion at* u_0*) and* φ_2 *is an embedding (immersion at* $w_0 = \varphi_1(u_0)$*), if and only if* J_φ *is a matrix of constant rank in a neighbourhood of* u_0.

PROOF. The necessity of the condition is obvious. For the sufficiency, let $x = (u, \xi)$ be coordinates on $\mathcal{U}$ and $y = (v, \eta)$ on $\mathcal{V}$ such that $x_i(u_0) = y_i(\varphi(u_0)) = 0$.

We use I_{xy} instead of the Jacobian matrix. Let $\mathrm{rk}\, I_{xy}(u) = (k, l)$. Then we can choose from the coordinates y a subset Ξ of $k + l$ coordinates such that the corresponding submatrix I has rank (k, l) at u_0. We may assume that Ξ consists of the first k even coordinates and the first l odd coordinates.

We consider the superspace $\mathcal{W} = \mathcal{R}^{k,l}$ with coordinates $z = (w_1, \ldots, w_k, \zeta_1, \ldots, \zeta_l)$, that is, the coordinates of $\mathcal{W}$ all belong to Ξ. Let $\varphi_1\colon \mathcal{U} \to \mathcal{W}$ be the morphism defined by

$$\varphi_1^*(w_i) = \varphi^*(v_i), \quad \varphi_1^*(\zeta_j) = \varphi^*(\eta_j), \qquad i,\ j \in \Xi.$$

Here the matrix I_{xz} coincides with I, so that φ_1 is a submersion at u_0. Therefore, by Theorem 2.3.7, we can adjust the coordinate system on $\mathcal{U}$ in such a way that

$$\varphi_1^*(w_i) = u_i = \varphi^*(v_i),\ \varphi_1^*(\zeta_j) = \xi_j = \varphi^*(\eta_j), \text{ where } i,\ j \in \Xi.$$

Let $\psi\colon \mathcal{W} \to \mathcal{U}$ be the embedding morphism defined by $\psi^*(u_i) = w_i$, $\psi^*(\xi_j) = \zeta_j$ for $i, j \in \Xi$ and $\psi^*(u_i) = \psi^*(\xi_j) = 0$ for $i, j \notin \Xi$. Putting

$\varphi_2 = \varphi \circ \psi$: $\mathscr{W} \to \mathscr{T}$, we have $\varphi_2^*(v_i) = w_i$ and $\varphi_2^*(\eta_j) = \zeta_j$ for $i, j \in \Xi$. Hence, φ_2 is an immersion, and we can adjust the coordinates y not in Ξ, without affecting those in Ξ, in such a way that $\varphi_2^*(v_i) = \varphi_2^*(\eta_j) = 0$ for $i, j \notin \Xi$.

We claim that φ coincides with the morphism $\varphi' = \varphi_2 \circ \varphi_1$. To see this, we consider the matrix I_{xy}; it has the form

$$\left(\begin{array}{c|c} 1_{k,l} & S \\ \hline 0 & T \end{array}\right),$$

where the blocks correspond to the partition of y into Ξ and the rest. Since I_{xy}, by hypothesis, is a matrix of constant rank, all its columns are linear combinations of the first $k + l$ (see 2.2.8 c)), so that $T \equiv 0$. This means that for any of the coordinates y,

$$\frac{\partial \varphi^*(y)}{\partial u_i} = \frac{\partial \varphi^*(y)}{\partial \xi_j} = 0 \quad \text{for} \quad i, j \notin \Xi.$$

Clearly, the equation remains valid with $\varphi'^*(y)$ in place of $\varphi^*(y)$. Furthermore,

$$\varphi^*(y)|_{\mathscr{W}} = \varphi'^*(y)|_{\mathscr{W}}$$

by construction. Theorem 2.3.9 is, therefore, a consequence of the following lemma.

2.3.10. LEMMA. *Let f_1 and f_2 be two functions on a superdomain with coordinates (u, v, ξ, η) such that*

$$\frac{\partial f_\nu}{\partial \eta_i} = \frac{\partial f_\nu}{\partial \eta_j} \equiv 0 \quad (\nu = 1, 2)$$

for any i and j, and

$$f_1|_{\mathscr{W}} = f_2|_{\mathscr{W}},$$

where $\mathscr{W}$ is the subsuperdomain defined by the equations $v = 0$, $\eta = 0$ (see 2.2.7). Then $f_1 \equiv f_2$ in some neighbourhood of $\mathscr{W}$.

PROOF. We put $f = f_1 - f_2$ and write

$$f = \sum f_{\alpha\beta} \xi^\alpha \eta^\beta.$$

From the equations it follows that $f_{\alpha\beta} = 0$ when $\beta \neq 0$. Next,

$$\frac{\partial f_{\alpha,0}}{\partial v_j} = 0 \quad \text{for all} \quad j.$$

Since $f_{\alpha 0}|_{\mathscr{W}} = 0$, it follows that $f_{\alpha 0} = 0$ in a neighbourhood of $\mathscr{W}$, that is, $f \equiv 0$.

§4. Integration in superdomains

2.4.1. By analogy with classical analysis, we wish to define a notion of integration in superdomains that is independent of the coordinate system. This problem splits into three steps:

a) We must define a suitable object for integration, namely, a volume form. In classical analysis, this is the expression $f(y)\,dy$, which is transformed into $f(y(x))\mathrm{det}(I_{yx})dx$ under a change of coordinates, where I_{yx} is the matrix of derivatives of the y with respect to the x;

b) we must define the notion of the integral of such an object in a given coordinate system;

c) we must show that the integral of the form is invariant under a change of coordinates. However, this is not true for all changes of coordinates, but only for the oriented ones. Therefore, as a preliminary we need the concept of orientation on a domain.

2.4.2. Let $\mathcal{U}^{m,n}$ be a superdomain and x and y two coordinate systems on $\mathcal{U}$. The *Jacobian* of the change of coordinates from x to y is defined as $\mathrm{Ber}\, I_{xy}$ and is denoted by $D(y)/D(x)$. Note that $D(y)/D(x) = \mathrm{Ber}\, J$, where J is the Jacobian matrix of the change of coordinates from x to y.

A *volume form* on a superdomain $\mathcal{U}$ is an object ρ which in the coordinate system $x = (u, \xi)$ of U can be written as

$$(*) \qquad \rho = f\Delta_x, \quad \text{where} \quad f \in C^\infty(\mathcal{U}),$$

and $f\Delta_x$ and $h\Delta_y$ in different coordinate systems x and y are related by the formula

$$(**) \qquad h\Delta_y = \left(\frac{D(y)}{D(x)}\, h\right)\Delta_x = f\Delta_x.$$

Let $\mathrm{Vol}(\mathcal{U})$ denote the space of volume forms on $\mathcal{U}$. We endow $\mathrm{Vol}(\mathcal{U})$ with the structure of a $C^\infty(\mathcal{U})$-module by setting

$$p(\Delta_x) = m - n(\mathrm{mod}\ 2) \quad \text{and} \quad f(g\Delta_x) = (fg)\Delta_x, \text{ where } f,\ g \in C^\infty(\mathcal{U}).$$

This structure is compatible with coordinate changes, since $D(y)/D(x)$ is always even.

2.4.3. Let $\mathcal{U}$ be a superdomain and $f \in C^\infty(\mathcal{U})$. We consider the set U_f of points u such that $f = 0$ in a neighbourhood of u. Clearly, U_f is open. Its complement $U \setminus U_f$ is called the *support* of f and written $\mathrm{supp}\, f$. (A function f is sometimes called *finite* if $\mathrm{supp}\, f$ is compact.)

2.4.4. Let $\mathcal{U}^{m,n}$ be a superdomain with coordinate system $x = (u, \xi)$ and let $\rho = f\Delta_x$ be a volume form on $\mathcal{U}$ of compact support.

We define the *integral* of ρ with respect to x by

$$(*) \qquad \int\limits_{(\mathcal{U}, x)} \rho = \int\limits_{(\mathcal{U}, x)} f\Delta_x = (-1)^{\frac{n(n-1)}{2}+mn} \int\limits_{\check{U}} f_{1\ldots 1}\, du,$$

where $f_{1\ldots1}$ is the coefficient of the monomial $\xi_1 \ldots \xi_n$ of maximal degree in the decomposition $f = \Sigma f_\beta \xi^\beta$.

REMARK. The sign on the right-hand side ensures that the analogue of Fubini's theorem holds and that

$$\int \xi \Delta_\xi = 1.$$

2.4.5. An *orientation* on a superdomain $\mathcal{U}$ is one on the underlying domain U. A coordinate system $x = (u, \xi)$ on $\mathcal{U}$ determines an orientation on $\mathcal{U}$: it is the orientation on U corresponding to the coordinate system $(u_1, \ldots, u_m)$.

THEOREM. *Let $x = (u, \xi)$ and $y = (v, \eta)$ be two coordinate systems on $\mathcal{U}$, and ρ a volume form on $\mathcal{U}$ of compact support. Then*

$$\int_{(\mathcal{U}, x)} \rho = \pm \int_{(\mathcal{U}, y)} \rho,$$

where the +-sign is taken when x and y determine the same orientation, and the −-sign otherwise.

Note that the assumption that ρ has compact support is essential, as the following example shows.

EXAMPLE. Let $\mathcal{U}^{1,2}$ be a superdomain in $\mathcal{R}^{1,2}$ with coordinates $x = (u, \xi_1, \xi_2)$ whose underlying domain U is the interval $(0, 1)$. Let $y = (v, \eta_1, \eta_2)$ be another coordinate system with $v = u + \xi_1\xi_2$, $\eta_1 = \xi_1$, $\eta_2 = \xi_2$. Let $\rho = v\Delta_y$ be a volume form on $\mathcal{U}$. Since $D(y)/D(x) = 1$, it follows that $\rho = v(x)\Delta_x$. Therefore,

$$\int_{(\mathcal{U}, x)} \rho = \int_{(\mathcal{U}, x)} (u + \xi_1\xi_2)\,\Delta_x = -\int_U du = -1,$$

$$\int_{(\mathcal{U}, y)} \rho = \int_{(\mathcal{U}, y)} v\Delta_y = 0.$$

2.4.6. To prove the theorem we define an action of the Lie superalgebra $\mathrm{Der}\, C^\infty(\mathcal{U})$ on the superspace $\mathrm{Vol}(\mathcal{U})$.

LEMMA. *Let x be a coordinate system on $\mathcal{U}$. Then*

a) *there is a unique action of $\mathrm{Der}\, C^\infty(\mathcal{U})$ on $\mathrm{Vol}(\mathcal{U})$ satisfying the following conditions*:

(i) $D(f\rho) = D(f)\,\rho + (-1)^{p(D)p(f)} f D(\rho)$,

(ii) $(fD)\,\rho = (-1)^{p(f)p(D)} D(f\rho)$,

(iii) $\frac{\partial}{\partial x_i}\Delta_x = 0$ *for any i*:

b) *this action is independent of the choice of coordinate system.*

PROOF. a) Given an action satisfying (i)–(iii), we have

$$(*)\qquad \left(f\frac{\partial}{\partial x_i}\right)(g\Delta_x)=(-1)^{p(x_i)p(f)}\frac{\partial}{\partial x_i}(fg\Delta_x)=$$
$$=\left[(-1)^{p(x_i)p(f)}\left(\frac{\partial f}{\partial x_i}\right)g+f\frac{\partial g}{\partial x_i}\right]\Delta_x,$$

which proves the uniqueness. On the other hand, it is easy to see that (*) defines a bilinear pairing $\mathrm{Der}\, C^\infty(\mathcal{U})\times \mathrm{Vol}(\mathcal{U})\to \mathrm{Vol}(\mathcal{U})$ satisfying (i)–(iii). It remains to check that (*) defines an action of the Lie superalgebra, that is,

$$D_1D_2\rho-(-1)^{p(D_2)p(D_1)}D_2D_1\rho=[D_1D_2]\rho,\qquad D_1,\ D_2\in \mathrm{Der}\, C^\infty(\mathcal{U}),\quad \rho\in \mathrm{Vol}\,(U).$$

By (i) and (ii) it is sufficient to check the trivial case

$$D_1=\frac{\partial}{\partial x_k},\qquad D_2=\frac{\partial}{\partial x_l}.$$

b) We shall show in 2.4.7 that if $A\in \mathrm{GL}_{r,s}(C^\infty(\mathcal{U}))$ and $D\in \mathrm{Der}\ C^\infty(\mathcal{U})$, then

$$(**)\qquad D(\mathrm{Ber}\, A)\cdot \mathrm{Ber}\, A^{-1}=\mathrm{str}\,(D(A)\cdot A^{-1}),$$

where $D(A)_{ij}=(-1)^{p(D)p_{\mathrm{row}(i)}}DA_{ij}$.

Let x and y be coordinate systems on $\mathcal{U}$, and

$$A_{ij}=\frac{\partial y_j}{\partial x_i},\qquad B_{kr}=\frac{\partial x_r}{\partial y_k},\qquad C_{ij}^m=\left[\frac{\partial}{\partial x_m}(A)\right]_{ij}.$$

It is clear that $AB=1$ and

$$\frac{\partial}{\partial x_i}=\sum A_{ij}\frac{\partial}{\partial y_j},\qquad \frac{\partial}{\partial y_r}=\sum B_{rk}\frac{\partial}{\partial x_k}.$$

We claim that the two actions of $\mathrm{Der}\, C^\infty(\mathcal{U})$ on $\mathrm{Vol}\,\mathcal{U}$ corresponding to x and y are the same. By (i) and (ii), it is sufficient to verify that under the two actions the forms $\partial/\partial x_i\Delta y$ coincide for each i. Calculating this form in the coordinate system x, we have

$$\frac{\partial}{\partial x_i}\Delta_y=\frac{\partial}{\partial x_i}(\mathrm{Ber}\, A\cdot\Delta_x)=\left(\frac{\partial}{\partial x_i}\mathrm{Ber}\, A\right)\mathrm{Ber}\, A^{-1}\cdot\Delta_y.$$

while in the coordinate system y,

$$\frac{\partial}{\partial x_i}\Delta_y=\sum_j A_{ij}\frac{\partial}{\partial y_j}\Delta_y=\sum_j(-1)^{p(y_j)p(A_{ij})}\frac{\partial A_{ij}}{\partial y_j}\Delta_y=$$
$$=\sum_j(-1)^{p(y_j)p(A_{ij})}\sum_k B_{jk}\frac{\partial A_{ij}}{\partial x_k}\Delta_y=\sum_{j,\,k}(-1)^{p(y_j)p(A_{ij})}B_{jk}C_{kj}^i\Delta_y=$$
$$=\sum_j(-1)^{p(y_j)(p(y_i)+p(y_j))}(BC^i)_{jj}\Delta_y=\mathrm{str}\,(BC^i)\Delta_y.$$

It follows from (**) that these volume forms coincide.

2.4.7. To prove (**), we write $D=\Sigma D_k\,\partial/\partial x_k$ so that the proof reduces to

the case $D = \partial/\partial x_k$. We argue as in 2.2.10.

Let $\mathscr{R}$ be a (1,0)- or (0, 1)-dimensional superspace with coordinate t of the same parity as x_k. We consider in $\mathscr{U} \times \mathscr{R}$ a neighbourhood $\mathscr{V}$ of $\mathscr{U} \times \{0\}$ and define a morphism φ: $\mathscr{V} \to \mathscr{U}$ by $\varphi^*(x_i) = x_i$ for $i \neq k$ and $\varphi^*(x_k) = x_k + t$. Using 2.2.4 we find that $\varphi^*(A) \equiv [A + t\,\partial/\partial x_k A]$ (mod t^2). Hence,

$$\operatorname{Ber}\varphi^*(A) \equiv \operatorname{Ber} A \cdot \operatorname{Ber}\left(1 + tA^{-1}\frac{\partial}{\partial x_k} A\right) \equiv \operatorname{Ber} A\left(1 + t \operatorname{str} A^{-1}\frac{\partial}{\partial x_k} A\right) (\operatorname{mod} t^2)$$

(since $\operatorname{Ber}(1 + tX) = 1 + t \operatorname{str} X$, see 1.7.8).

Thus,

$$\varphi^*(\operatorname{Ber} A) \equiv \operatorname{Ber}\varphi^* A + t \operatorname{Ber} A \operatorname{str} A^{-1}\frac{\partial}{\partial x_k} A \,(\operatorname{mod} t^2).$$

Since the coefficient of t is $\partial/\partial x_k \operatorname{Ber} A$, the result follows.

2.4.8. LEMMA. *Let ρ be a volume form of compact support on a superdomain $\mathscr{U}$ and D a derivation. Then for any coordinate system x on $\mathscr{U}$,*

$$\int_{(\mathscr{U}, x)} D(\rho) = 0.$$

PROOF. We may assume that $D = h\,\partial/\partial x_i$. Then, by 2.4.6(i),

$$D(\rho) = \pm\frac{\partial}{\partial x_i}(h\rho),$$

hence we may restrict our attention to the case $D = \partial/\partial x_i$.

Let $\rho = f\Delta_x$, where $f = \Sigma f_\beta \xi^\beta$. Then $D(\rho) = \partial f/\partial x_i \Delta_x$. If $x_i = \xi_k$ is an odd coordinate, then the coefficient $(\partial f/\partial \xi_k)_{1\ldots1}$ is zero, so that $\int D(\rho) = 0$. If $x_i = u_i$ is an even coordinate, then

$$\left(\frac{\partial f}{\partial x_i}\right)_{1\ldots1} = \frac{\partial}{\partial u_i} f_{1\ldots1}.$$

Therefore,

$$(*) \qquad \int_{(\mathscr{U}, x)} D(\rho) = \int_U \frac{\partial g}{\partial u_i}\, du,$$

where $g = f_{1\ldots1}$ is of compact support on U.

Consequently, we may replace U by the whole space $\mathbf{R}^n$. Applying Fubini's theorem and using the fact that on a line an integral of the form $\int \left(\frac{\partial}{\partial u} g\right) du$ of a function g of compact support vanishes, we find that the right-hand side of (i) is zero. Hence, $\int D(\rho) = 0$.

2.4.9. Turning to the proof of Theorem 2.4.5, let $\rho = f\Delta_x$. We split f into $f = f_0 + f_1$, where $f_1 = f_{1\ldots1}\xi_1 \ldots \xi_q$ is the monomial of maximal degree in the decomposition $f = \Sigma f_\beta \xi^\beta$ and f_0 is the sum of the remaining terms, and we set $\rho_0 = f_0\Delta_x$, $\rho_1 = f_1\Delta_x$.

It is clear that f_0 has the form $f_0 = \Sigma \partial/\partial\xi_i(f_i')$, where the f_i' are of compact support. Therefore,

$$\rho_0 = \sum \frac{\partial}{\partial \xi_i}(\rho_i') = \sum D_i(\rho_i') \quad \text{where} \quad D_i \in \operatorname{Der} C^\infty(\mathcal{U}),$$

and ρ_i' is a form of compact support on $\mathcal{U}$. But by Lemma 2.4.8, the integral of ρ_0 is zero in any coordinate system y. Hence, we may assume in what follows that $\rho = \rho_1 = f\Delta_x$, where $f = g(u)\xi_1 \ldots \xi_q$ and $g \in C_c^\infty(\mathcal{U})$.

We rewrite ρ in the coordinate system $y = (v, \eta)$. Let $\mathcal{J}_U$ be the ideal that distinguishes U (that is, the ideal generated by the ξ or, equivalently, the η). It is clear that $\rho \in \mathcal{J}_u^q \operatorname{Vol}_c(\mathcal{U})$, so that it has the form $\rho = (h\eta_1 \ldots \eta_q)\Delta_y$ in the coordinate system $y = (v, \eta)$. We must show that

$$\int g\, du_1 \ldots du_p = \pm \int h\, dv_1 \ldots dv_p.$$

By the classical theorem on change of variables on U, it is sufficient to verify that $g = h\, D(v)/D(u)$, where $D(v)/D(u)$ is the Jacobian for the change of coordinates from u to v on U.

We have

$$\rho = g\xi_1 \ldots \xi_q \Delta_x = h\eta_1 \ldots \eta_q \Delta_y = h\frac{D(y)}{D(x)}\eta_1 \ldots \eta_q.$$

Let

$$(I_1)_{ik} = \left(\frac{\partial v_k}{\partial u_i}\right), \qquad (I_2)_{jl} = \left(\frac{\partial \eta_l}{\partial \xi_j}\right).$$

Then

$$\frac{D(y)}{D(x)} = \det I_1 \det{}^{-1}(I_2) \pmod{\mathcal{J}_U}, \qquad \eta_l = \sum \xi_i \frac{\partial \eta_l}{\partial \xi_i} \pmod{\mathcal{J}_U^2},$$

so that

$$\eta_1 \ldots \eta_q = \det I_2(\xi_1, \ldots \xi_q) \pmod{\mathcal{J}_U^{q+1}}.$$

Thus, cancelling $\xi_1 \ldots \xi_q$, we find that

$$g = h \det I_2 \det I_1 \det{}^{-1} I_2 \equiv h \det I_1 \pmod{\mathcal{J}_U}.$$

But by definition, der $I_1 \pmod{I_U}$ is the ordinary Jacobian $D(v)/D(u)$, and this completes the proof.

CHAPTER III

SUPERMANIFOLDS

§1. Definition of a supermanifold

3.1.1. A *supermanifold* $\mathcal{M}$ is a ringed space $(M, \Theta_{\mathcal{M}})$, where $\Theta_{\mathcal{M}}$ is a sheaf of commutative superalgebras on M, such that

a) M is a Hausdorff space with a countable base,

b) every point $m \in M$ has a neighbourhood U such that the ringed space $(U, \Theta_{\mathcal{M}}(U))$ is isomorphic to a superdomain $\mathcal{U}$.

A *morphism* of supermanifolds is a morphism of the corresponding ringed spaces. A morphism $\varphi : \mathcal{N} \to \mathcal{M}$ is called a *diffeomorphism* if it has an inverse morphism $\psi : \mathcal{M} \to \mathcal{N}$ (that is, $\varphi \circ \psi = \mathrm{id}_{\mathcal{M}}$, $\psi \circ \varphi = \mathrm{id}_{\mathcal{N}}$).

3.1.2. There is an equivalent definition of a supermanifold that does not appeal to the notion of a sheaf.

Let M be a Hausdorff space with a countable base. A chart on M is a pair $(\mathcal{U}, c)$, where $\mathcal{U}$ is a superdomain and c: $U \to M$ a homeomorphism from the underlying domain U to an open subset of M, usually identified with U.

Let $(\mathcal{U}_1, c_1)$ and $(\mathcal{U}_2, c_2)$ be two charts with $c_1(U_1)$, $c_2(U_2) \subset M$ and $W = c_1(U_1) \cap c_2(U_2)$. We put $U_1' = c_1^{-1}(W) \subset U_1$, $U_2' = c_2^{-1}(W) \subset U_2$, and let $\gamma'_{U_1 U_2} : U_1' \to U_2'$ denote the composite $c_2^{-1} \circ c_1$, which is clearly a homeomorphism.

A *compatibility* between two charts $(\mathcal{U}_1, c_1)$ and $(\mathcal{U}_2, c_2)$ is an isomorphism of superdomains $\gamma_{\mathcal{U}_1 \mathcal{U}_2}$: $\mathcal{U}_1' \to \mathcal{U}_2'$ such that the underlying mapping $\tilde{\gamma}_{\mathcal{U}_1 \mathcal{U}_2}$ coincides with $\gamma'_{U_1 U_2}$.

An *atlas* is a collection of charts $\{(\mathcal{U}_\alpha, c_\alpha)\}$, where α ranges over some indexing sets, and a set of compatibilities $\gamma_{\alpha,\beta} = \gamma_{\mathcal{U}_\alpha, \mathcal{U}_\beta}$ between $(\mathcal{U}_\alpha, c_\alpha)$ and $(\mathcal{U}_\beta, c_\beta)$ for all index pairs α, β such that

(i) the set of $c_\alpha(U_\alpha)$ covers M;

(ii) for all α, β, γ, the composite $\gamma_{\alpha\beta} \circ \gamma_{\beta\alpha} \circ \gamma_{\gamma\alpha}$ is the identity on the open superdomain in $\mathcal{U}_\alpha$, on which it is defined;

(iii) $\gamma_{\alpha\alpha}$: $\mathcal{U}_\alpha \to \mathcal{U}_\alpha$ is the identity for all α. In particular, $\gamma_{\alpha\beta} = \gamma_{\beta\alpha}^{-1}$.

A *supermanifold* is a space together with a preferred atlas on it. Let $\mathcal{M} = (M, \{(\mathcal{U}_\alpha, c_\alpha)\})$ and $\mathcal{N} = (N, \{(\mathcal{V}_\beta, c'_\beta)\})$ be two supermanifolds. A *morphism* φ: $\mathcal{M} \to \mathcal{N}$ is a continuous mapping $\tilde{\varphi}$: $M \to N$ and a collection of morphisms of superdomains $\varphi_{\alpha\beta}$: $\mathcal{U}_\alpha \to \mathcal{V}_\beta$ such that $\varphi_{\alpha\beta}\gamma_{\gamma\alpha} = \gamma_{\delta\beta}\varphi_{\gamma\delta}$ for any $\alpha, \beta, \gamma, \delta$.

Note the difference between the definition of a supermanifold and that of a manifold. In the case of a manifold there is no need to define the compatibilities $\gamma_{\mathcal{V},\mathcal{U}}$, since $\gamma_{\mathcal{V},\mathcal{U}}$ is completely determined by $\gamma'_{V,U}$. But this is not so in the supercase, because a morphism of superdomains is not determined by a mapping of points. Similarly, in the case of manifolds, (ii)

and (iii) in the definition of an atlas are automatically fulfilled, since they hold for the mappings γ'.

3.1.3. The following lemma is useful in the construction of supermanifolds.

LEMMA. *Let M be a Hausdorff space with a countable base and $\{U_\alpha\}$ a family of open domains in M covering M. Let $\mathcal{U}_\alpha = (U_\alpha, \mathcal{O}_{\mathcal{U}_\alpha})$ be a superdomain and α, β some pair of indices such that $U_\alpha \subset U_\beta$ and $\gamma_{\beta,\alpha}: \mathcal{U}_\alpha \to \mathcal{U}_\beta$ is an embedding compatible with that of U_α in U_β. Suppose that:*

a) *if $\gamma_{\delta\beta}$ and $\gamma_{\beta\alpha}$ are defined, then so is $\gamma_{\delta\alpha}$ and $\gamma_{\delta\alpha} = \gamma_{\delta\beta} \circ \gamma_{\beta\alpha}$. The embedding $\gamma_{\alpha\alpha}$ is always defined and is the identity;*

b) *if a point $m \in M$ belongs to $U_\alpha \cap U_\beta$, then there is a δ such that $m \in U_\delta$ and the embeddings*

$$\gamma_{\alpha\delta}: U_\delta \to \mathcal{U}_\alpha \quad \text{and} \quad \gamma_{\beta\delta}: U_\delta \to \mathcal{U}_\beta$$

are defined.

Then there is a supermanifold $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$ and a collection of morphisms $c_\alpha: \mathcal{U}_\alpha \to \mathcal{M}$ compatible with the embeddings of the U_α in M and such that $c_\beta \circ \gamma_{\beta\alpha} = c_\alpha$ when $\gamma_{\beta\alpha}$ is defined. This supermanifold $\mathcal{M}$ and collection of morphisms c_α is uniquely determined up to isomorphism.

3.1.4. Let $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$ be a supermanifold. We consider on M the sheaf $\mathcal{O}_M$ of commutative algebras, where $\Gamma(U, \mathcal{O}_M) = \pi\Gamma(U, \mathcal{O}_{\mathcal{M}})$ (and $\pi: A \to A/(A_{\bar{1}})$). It follows from 3.1.1 that the ringed space $(M, \mathcal{O}_M)$ is an ordinary smooth manifold. We call it the *underlying manifold* of $\mathcal{M}$ and denote it by M.

If $\varphi: \mathcal{M} \to \mathcal{N}$ is a morphism, then it follows from 2.1.7 that the underlying mapping $\tilde{\varphi}: M \to N$ is smooth.

As in 2.1.3, we can construct a canonical embedding of the underlying manifold $\pi: M \to \mathcal{M}$, where M is regarded as a supermanifold. Here $\tilde{\varphi} \circ \pi = \pi \circ \varphi$ for any morphism $\varphi: \mathcal{M} \to \mathcal{N}$.

A supermanifold $\mathcal{M}$ is called *connected* if M is connected, and *compact* if M is compact.

3.1.5. If $\mathcal{M}$ is a supermanifold, then to any open subset $M' \subset M$ there corresponds a supermanifold $\mathcal{M}' = (M', \mathcal{O}_{\mathcal{M}}|_{M'})$; such a supermanifold is said to be an open subsupermanifold of $\mathcal{M}$. The *union* and *intersection* of open subsupermanifolds are defined in the natural way, and so is the *inverse image* of an open subsupermanifold under a morphism. A *neighbourhood* of a subset $X \subset M$ is an open subsupermanifold $\mathcal{M}'$ of $\mathcal{M}$ such that $M' \supset X$.

A morphism of supermanifolds $\varphi: \mathcal{N} \to \mathcal{M}$ is called an *open embedding* if it induces a diffeomorphism of $\mathcal{N}$ with an open subsupermanifold $\mathcal{M}' \subset \mathcal{M}$; we usually do not distinguish between $\mathcal{N}$ and $\mathcal{M}'$.

An open subsupermanifold $\mathcal{U}$ of $\mathcal{M}$ is called a *superdomain* if it is diffeomorphic with a superdomain.

It is clear that if $\mathcal{M}$ is a connected supermanifold, then all its subsuperdomains have the same dimension (p, q), which we call the *dimension* of $\mathcal{M}$.

A *local coordinate system* $x = (u, \xi)$ on $\mathscr{M}$ is a subsuperdomain $\mathscr{U}$ together with a coordinate system x on U.

If $\mathscr{M}'$ is an open subsupermanifold of $\mathscr{M}$ and $f \in C^\infty(\mathscr{M})$, then the image of f under the natural homomorphism $C^\infty(\mathscr{M}) \to C^\infty(\mathscr{M}')$ is called the *restriction* of f to $\mathscr{M}'$ and is denoted by $f|_{\mathscr{M}'}$.

As in 2.4.3 for any function $f \in C^\infty(\mathscr{M})$ we can define its support $\operatorname{supp} f \subset M$. The space of functions with compact support is denoted by $C_c^\infty(\mathscr{M})$.
If $\mathscr{M}'$ is an open subsupermanifold of $\mathscr{M}$, it is clear that the superalgebra $C_c^\infty(\mathscr{M}')$ is embedded in $C_c^\infty(\mathscr{M})$.

3.1.6. Let $\mathscr{M}$ and $\mathscr{N}$ be two supermanifolds. For subsuperdomains $\mathscr{U} \subset \mathscr{M}$ and $\mathscr{V} \subset \mathscr{N}$ we specify on $U \times V \subset M \times N$ the structure of a superdomain $\mathscr{U} \times \mathscr{V}$. If $\mathscr{U}' \subset \mathscr{U}$ and $\mathscr{V}' \subset \mathscr{V}$, let $\gamma_{\mathscr{U} \times \mathscr{V}, \mathscr{U}' \times \mathscr{V}'}$ denote the natural embedding $\mathscr{U}' \times \mathscr{V}' \to \mathscr{U} \times \mathscr{V}$. It follows from Lemma 3.1.3 that the set of superdomains $\mathscr{U} \times \mathscr{V}$ and of embeddings $\gamma_{\mathscr{U} \times \mathscr{V}, \mathscr{U}' \times \mathscr{V}'}$ uniquely determine the structure of a supermanifold.

The resulting supermanifold $\mathscr{M} \times \mathscr{N}$ is called the *product* of $\mathscr{M}$ and $\mathscr{N}$. Projection morphisms $\mathrm{pr}_{\mathscr{M}} : \mathscr{M} \times \mathscr{N} \to \mathscr{M}$ and $\mathrm{pr}_{\mathscr{N}} : \mathscr{M} \times \mathscr{N} \to \mathscr{N}$ are defined in the natural way.

It is easy to check that Lemma 2.1.12 remains valid with supermanifolds in place of superdomains. If $\alpha : \mathscr{L} \to \mathscr{M}$ and $\beta : \mathscr{L} \to \mathscr{N}$ are morphisms, let $\alpha \times \beta : \mathscr{L} \to \mathscr{M} \times \mathscr{N}$ denote the morphism defined by $\mathrm{pr}_{\mathscr{M}} \circ (\alpha \times \beta) = \alpha$, $\mathrm{pr}_{\mathscr{N}} \circ (\alpha \times \beta) = \beta$.

Suppose that we are given a family $\{\mathscr{U}_\alpha\}$ of open subsupermanifolds of a supermanifold $\mathscr{M}$ (or, what is the same thing, a family of open subsets $\{U_\alpha\}$ in the underlying manifold M). Then $\{\mathscr{U}_\alpha\}$ is called an (*open*) *covering* if the union of the U_α is M. If $\{\mathscr{U}_\alpha\}$ and $\{\mathscr{V}_\beta\}$ are two coverings, then we say that $\{\mathscr{U}_\alpha\}$ is a *refinement* of $\{\mathscr{V}_\beta\}$ if for any α there is a β such that $\mathscr{U}_\alpha \subset \mathscr{V}_\beta$.
A covering $\{\mathscr{U}_\alpha\}$ is called *locally finite* if any compact subset K of M meets only finitely many of the U_α.

3.1.7. LEMMA. *Let $\{\mathscr{V}_\beta\}$ be a covering of a supermanifold $\mathscr{M}$. Then there is a covering $\{\mathscr{U}_\alpha\}$ and a family of functions $\{\varphi_\alpha \in (C_c^\infty(\mathscr{M}))_{\bar{0}}\}$ such that*

a) *$\{\mathscr{U}_\alpha\}$ is a refinement of $\{\mathscr{V}_\beta\}$;*

b) *$\{\mathscr{U}_\alpha\}$ is locally finite;*

c) *$\operatorname{supp} \varphi_\alpha$ is a compact set contained in $\mathscr{U}_\alpha$, and the $\widetilde{\varphi}_\alpha \in C^\infty(M)$ are non-negative;*

d) $\Sigma \varphi_\alpha = 1$.

(It follows from b) and c) that on any compact subset $K \subset M$ this sum has only finitely many non-zero terms, so that the sum makes sense.)

The family of functions $\{\varphi_\alpha\}$ is called a *partition of unity on a refinement of* $\{\mathscr{V}_\beta\}$.

PROOF. It follows from the usual theorem on the partition of unity that there is a covering $\{U_\alpha\}$ and a set of functions $\psi_\alpha \in C^\infty(M)$ satisfying a)–d). Taking a refinement of $\{U_\alpha\}$, if necessary, we may assume that each U_α

is the underlying domain of a superdomain $\mathcal{U}_\alpha$. We can then take a function $\varphi'_\alpha \in (C^\infty(\mathcal{U}_\alpha))_{\bar{0}}$ such that $\widetilde{\varphi'_\alpha} = \psi_\alpha$ and supp φ'_α = supp ψ_α; these functions satisfy a), b), and c).

We set $\varphi = \sum \varphi'_\alpha$. It is clear that $\widetilde{\varphi} = \sum \widetilde{\varphi'_\alpha} = \sum \psi_\alpha = 1$, so that φ is locally invertible. Hence, φ is also globally invertible, and we can take $\varphi_\alpha = \varphi^{-1}\varphi'_\alpha$.

3.1.8. COROLLARY (localization principle). *Let X be a closed subset of M, $\mathcal{U}$ an open subsupermanifold containing X, and $f \in C^\infty(\mathcal{U})$. Then there is an open subsupermanifold $\mathcal{V}$ containing X and contained in $\mathcal{U}$, and a function $h \in C^\infty(\mathcal{M})$ such that $f|_{\mathcal{V}} = h|_{\mathcal{V}}$ and* supp $h \subset$ supp f. *If X is compact, then $h \in C_c^\infty(\mathcal{U}) \subset C_c^\infty(\mathcal{M})$.*

PROOF. Let $\{\varphi_\alpha\}$ be a partition of unity on a refinement of the covering $\{U, M \setminus X\}$. Let I_X denote the set of indices α for which supp φ_α intersects X; it is clear that supp $\varphi_\alpha \subset U$ for $\alpha \in I_X$. Let $h = \sum \varphi_\alpha f$, with $\alpha \in I_X$. All the terms in this sum belong to $C_c^\infty(\mathcal{U}) \subset C_c^\infty(\mathcal{M})$, so that $h \in C^\infty(\mathcal{M})$ and supp $h \subset$ supp f. If X is compact, then I_X is a finite set, hence, $h \in C_c^\infty(\mathcal{U})$.

3.1.9. We give a simple example to show how the localization principle is applied.

LEMMA (localization of a derivation). *Let $\mathcal{M}$ be a supermanifold, $D: C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ a derivation, and $\mathcal{U}$ an open subsupermanifold of $\mathcal{M}$. Then there is one and only one derivation $D': C^\infty(\mathcal{U}) \to C^\infty(\mathcal{U})$ such that $Dg|_{\mathcal{U}} = D'(g|_{\mathcal{U}})$ for any $g \in C^\infty(\mathcal{M})$.*

PROOF. First of all we show that if $\mathcal{V}$ is an open subsupermanifold of $\mathcal{M}$, then $Dh|_{\mathcal{V}}$ for any $h \in C^\infty(\mathcal{M})$ is determined by $h|_{\mathcal{V}}$; in other words, if $h_{\mathcal{V}} = 0$, then $Dh|_{\mathcal{V}} = 0$. For let $v \in \mathcal{V}$. By 3.1.8, there is a function $\varphi \in C_{\bar{0}}^\infty(\mathcal{M})$ such that supp $\varphi \subset V$ and $\varphi = 1$ in some neighbourhood $\mathcal{W}$ of v. Therefore, $\varphi h = 0$, so that $D(\varphi h) = D\varphi \cdot h + \varphi \cdot Dh = 0$. Since $h|_{\mathcal{W}} = 0$ and $\varphi|_{\mathcal{W}} = 1$, it follows that $Dh|_{\mathcal{W}} = 0$. Since this holds for any point v, $Dh|_{\mathcal{V}} = 0$.

Now let $f \in C^\infty(\mathcal{U})$ and $u \in \mathcal{U}$. By 3.1.8, we can select a function $h \in C^\infty(\mathcal{M})$ that agrees with f in some neighbourhood $\mathcal{V}$ of u. From what has been shown above it follows that in this neighbourhood Dh does not depend on h but only on f. Therefore, these functions, defined on neighbourhoods of different points of $\mathcal{U}$, can be glued together to give a single function on $\mathcal{U}$, which we denote by $D'f$. In this way we have constructed an operator D', which is easily seen to be a derivation. The uniqueness of D' is a simple consequence of the method of proof.

§2. Subsupermanifolds

3.2.1. Let $\mathcal{W}$ be a supermanifold with coordinates (u, v, ξ, η) and $\mathcal{U}$ the subsupermanifold given by $v = 0$, $\eta = 0$. In this case the embedding $\varphi: \mathcal{U} \to \mathcal{W}$ is called *standard*.

3.2.2. Let $\varphi: \mathscr{N}' \to \mathscr{M}$ be a morphism of supermanifolds. Then φ is called a *regular closed embedding* if:

a) the set $N' = \widetilde{\varphi}(N) \subset M$ is closed and $\widetilde{\varphi} : N \to N'$ is a homeomorphism, where N' has the induced topology;

b) for any $n \in \mathscr{N}'$ there is a neighbourhood $\mathscr{U}$ of n in $\mathscr{N}'$ and a neighbourhood $\mathscr{V}'$ of $\varphi(n)$ in $\mathscr{M}$ such that $\varphi: \mathscr{U} \to \mathscr{V}'$ is the standard embedding.

A morphism $\varphi: \mathscr{N}' \to \mathscr{M}$ is called a *regular embedding* if it is a regular closed embedding into some open subsupermanifold $\mathscr{M}' \subset \mathscr{M}$. In other words, in a) we require the set $\widetilde{\varphi}(N)$ to be locally closed (the intersection of an open and a closed set).

It follows from 2.3.8 that b) is equivalent to the following condition:

b') the morphism φ is an immersion at each point $n \in \mathscr{N}'$, that is, the rank of the Jacobian matrix of φ at n is equal to $p + q$, where (p, q) is the dimension of $\mathscr{N}'$ in a neighbourhood of n. Here the Jacobian matrix is calculated with respect to local coordinate systems in neighbourhoods of n and $\varphi(n)$; it is clear that its rank at a point is independent of the choice of these systems.

3.2.3. LEMMA. a) *Let* $\varphi: \mathscr{N}' \to \mathscr{M}$ *be a regular embedding and* $\psi_1, \psi_2 : \mathscr{L} \to \mathscr{N}'$ *two morphisms. If* $\varphi \circ \psi_1 = \varphi \circ \psi_2$, *then* $\psi_1 = \psi_2$.

b) *Let* $\varphi': \mathscr{N}' \to \mathscr{L}$ *and* $\varphi'': \mathscr{L} \to \mathscr{M}$ *be morphisms such that* $\varphi = \varphi'' \circ \varphi'$ *is a regular embedding. Then* φ' *is also regular. The same is true for regular closed embeddings.*

PROOF. a) It is clear that $\widetilde{\psi}_1 = \widetilde{\psi}_2$. Therefore, we need only check that ψ_1 and ψ_2 coincide locally, that is, we may assume that $\mathscr{L}$, $\mathscr{M}$ and $\mathscr{N}'$ are superdomains and $\varphi: \mathscr{N}' \to \mathscr{M}$ is a standard embedding of supermanifolds. But in this case the assertion of the lemma follows from the coordinate expression of mappings. b) It is easy to see that φ' satisfies conditions a) and b) of 3.2.2.

3.2.4. Two regular embeddings $\varphi : \mathscr{N}' \to \mathscr{M}$ and $\varphi': \mathscr{N}'' \to \mathscr{M}$ are called *equivalent* if there are morphisms $\psi': \mathscr{N}'' \to \mathscr{N}'$ and $\psi: \mathscr{N}' \to \mathscr{N}''$ such that $\varphi'\psi = \varphi$ and $\varphi\psi' = \varphi'$. It follows from 3.2.3 that ψ and ψ' are uniquely determined and that $\psi\psi'$ and $\psi'\psi$ are identity mappings.

An equivalence class of regular embeddings is called a *subsupermanifold*, and an equivalence class of regular closed embeddings is called a *closed subsupermanifold*.

We say that a subsupermanifold $\psi : \mathscr{N}'' \to \mathscr{M}$ is embedded in a subsupermanifold $\varphi: \mathscr{N}' \to \mathscr{M}$ (and write $\mathscr{N}'' \subset \mathscr{N}'$) if there is a morphism $\psi': \mathscr{N}'' \to \mathscr{N}'$ such that $\varphi \circ \psi' = \psi$. It follows from 3.2.3 that this morphism is uniquely determined and is a regular embedding, that is, it defines $\mathscr{N}''$ as a subsupermanifold of $\mathscr{N}'$. More generally, if $(\mathscr{N}', \varphi)$ is a subsupermanifold of $\mathscr{M}$ and $\psi: \mathscr{W}' \to \mathscr{M}$ a morphism of supermanifolds, then we write $\psi(\mathscr{W}') \subset \mathscr{N}'$ to indicate that there is a morphism $\psi' : \mathscr{W}' \to \mathscr{N}'$ (unique by 3.2.3. a)) such that $\varphi \circ \psi' = \psi$.

3.2.5. Let $\varphi\colon \mathscr{N} \to \mathscr{M}$ be a regular closed embedding. Then the homomorphism $\varphi^*\colon C^\infty(\mathscr{M}) \to C^\infty(\mathscr{N})$ is an epimorphism. For if $f \in C^\infty(\mathscr{N})$, then it follows easily from the definition of a regular closed embedding that any point $m \in \mathscr{M}$ has a neighbourhood $\mathscr{V}_m$ such that $f_m|_{\mathscr{N} \cap \mathscr{V}_m} = f|_{\mathscr{N} \cap \mathscr{V}_m}$ for some $f_m \in C^\infty(\mathscr{V}_m)$. Let φ_α be a partition of unity corresponding to a refinement $\{\mathscr{U}_\alpha\}$ of $\{\mathscr{V}_m\}$, and let f_α denote the restriction of f_m to $\mathscr{U}_\alpha$ (where $\mathscr{U}_\alpha \subset \mathscr{V}_m$). Then the function $h = \sum \varphi_\alpha \cdot f_\alpha$ belongs to $C^\infty(\mathscr{M})$, and $h|_{\mathscr{N}} = f$.

It can be shown that the converse is also true: if $\varphi : \mathscr{N} \to \mathscr{M}$ is a morphism of supermanifolds such that $\varphi^* : C^\infty(\mathscr{M}) \to C^\infty(\mathscr{N})$ is an epimorphism, then φ is a regular closed embedding.

3.2.6. We now show how to specify a subsupermanifold by equations. Let $\varphi\colon \mathscr{N} \to \mathscr{M}$ be a subsupermanifold and $f \in C^\infty(\mathscr{M})$. We set $\mathscr{I}_{\mathscr{N}} = \{f \in C^\infty(\mathscr{M}) \mid \varphi^*(f) = 0\}$.

PROPOSITION. a) *Let $\mathscr{N}$ be a closed subsupermanifold of $\mathscr{M}$. Then the ideal $\mathscr{I}_{\mathscr{N}}$ satisfies the following conditions*:

1) *if $m \in M$ is a point such that $\mathscr{I}_{\mathscr{N}} \subset \mathscr{I}_m$, then in $\mathscr{I}_{\mathscr{N}}$ there are functions $f_1, \ldots, f_r$ that are homogeneous relative to parity and such that they generate $\mathscr{I}_{\mathscr{N}}$ in some neighbourhood of m, and their differentials are linearly independent at m*;

2) *if $f_1, \ldots, f_n, \ldots$ lie in $\mathscr{I}_{\mathscr{N}}$ and any compact subset $K \subset M$ intersects only finitely many of the* supp f_i, *then $f = \Sigma f_i$ also lies in $\mathscr{I}_{\mathscr{N}}$.*

b) *Conversely, if $\mathscr{I}$ is an ideal of $C^\infty(\mathscr{M})$ satisfying* 1) *and* 2), *then there is a unique closed subsupermanifold $\mathscr{N} \subset \mathscr{M}$ such that $\mathscr{I} = \mathscr{I}_{\mathscr{N}}$.*

Note that the condition $\mathscr{I}_{\mathscr{N}} \subset \mathscr{I}_m$ in 1) simply means that all the $f \in \mathscr{I}_{\mathscr{N}}$ vanish at m.

Without 2), b) is no longer valid. For example, if $\mathscr{M} = \mathscr{R}^{1,0}$ and $\mathscr{I} = C_c^\infty(\mathscr{M})$, then $\mathscr{I}$ trivially satisfies 1), but, of course, it does not correspond to any subsupermanifold. Essentially 2) is equivalent to the ideal $\mathscr{I}$ being a global section of some sheaf of ideals of the structure sheaf.

PROOF OF THE PROPOSITION. a). This follows easily from the definitions and the localization principle 3.1.8. Let us prove b). Let $N = \{m \in M \mid f(m) = 0 \text{ for all } f \in \mathscr{I}\}$; it is clear that N is a closed subset of M. By 2.3.8, for any point $n \in N$ there is a local coordinate system x in some neighbourhood $\mathscr{U}_n$ of n in $\mathscr{M}$ such that $f_1, \ldots, f_r$ occur among the coordinates

$$x = (u_1, \ldots, u_p, \xi_1, \ldots, \xi_q);$$

moreover, we may assume that the f_i generate $\mathscr{I}$ in this neighbourhood. Then the remaining coordinates define on $\mathscr{U}_n \cap \mathscr{N}$ the structure of a superdomain, standardly embedded in $\mathscr{U}_n$. As n runs over N, these superdomains combine to form a supermanifold, and there is a regular closed embedding $\varphi\colon \mathscr{N} \to \mathscr{M}$. Clearly, $\mathscr{I} \subset \mathscr{I}_{\mathscr{N}}$.

We claim that $\mathscr{I}_{\mathscr{N}} \subset \mathscr{I}$; let $f \in \mathscr{I}_{\mathscr{N}}$. By construction, for each point of $\mathscr{I}_{\mathscr{N}}$

there is a neighbourhood $\mathcal{U}_m$ such that $f|_{\mathcal{U}_m}$ is generated by the restriction of $\mathcal{J}$. Let φ_β be a partition of unity on a refinement of $\{\mathcal{U}_m\}$. Then it is clear that all the $\varphi_\beta f$ belong to $\mathcal{J}$, and by 2), $f = \Sigma \varphi_\beta f \in \mathcal{J}$.

The uniqueness of $\mathcal{N}$ is an easy consequence of the following lemma.

3.2.7. LEMMA *Let* $\varphi\colon \mathcal{N} \to \mathcal{M}$ *be a closed subsupermanifold of* $\mathcal{M}$, *and* $\mathcal{J}_{\mathcal{N}}$ *the corresponding ideal of* $C^\infty(\mathcal{M})$. *Let* $\psi\colon \mathcal{L} \to \mathcal{M}$ *be a morphism such that* $\psi^*(\mathcal{J}_{\mathcal{N}}) = 0$. *Then there is a unique morphism* $\psi'\colon \mathcal{L} \to \mathcal{N}$ *such that* $\varphi\psi' = \psi$.

PROOF. The uniqueness is proved in 3.2.3. Let us prove the existence.

It is clear that $\tilde{\psi}(L) \subset \tilde{\varphi}(N)$, and since $\tilde{\varphi}\colon N \to \tilde{\varphi}(N)$ is a homeomorphism, there is a continuous mapping $\tilde{\psi}'\colon L \to N$ for which $\tilde{\varphi}\tilde{\psi}' = \tilde{\psi}$.

It is sufficient to construct the morphism ψ' locally, since by uniqueness it is compatible with intersections. Therefore, we may assume that $\mathcal{L}$, $\mathcal{M}$, and $\mathcal{N}$ are superdomains and that $\varphi\colon \mathcal{N} \to \mathcal{M}$ is a standard embedding. In this case the assertion of the lemma follows from the coordinate expression of mappings.

3.2.8. In this sub-section and the next we describe two methods of constructing subsupermanifolds based on Proposition 3.2.6.

Let $\mathcal{M}$ be a supermanifold and $m \in \mathcal{M}$. We define the cotangent space $T^*_m(\mathcal{M})$ as $\mathcal{J}_m/\mathcal{J}_m^2$ and the tangent space $T_m(\mathcal{M})$ as the adjoint of $T^*_m(\mathcal{M})$. It follows from the localization principle 3.1.8 that the tangent space is invariant under passage to a neighbourhood of m.

Every morphism $\varphi\colon \mathcal{N} \to \mathcal{M}$ induces a morphism of tangent spaces $\partial\varphi\colon T_n(\mathcal{N}) \to T_{\varphi(n)}(\mathcal{M})$. If $\mathcal{N}$ is a subsupermanifold, then $\partial\varphi$ is an embedding, and we identify $T_n(\mathcal{N})$ with the subsuperspace $\partial\varphi(T_n(\mathcal{N})) \subset T_{\varphi(n)}(\mathcal{M})$.

Let $\mathcal{N}$ be a subsupermanifold of a supermanifold $\mathcal{M}$ and $\psi\colon \mathcal{L} \to \mathcal{M}$ a morphism of supermanifolds. Then ψ is said to be *transversal* to $\mathcal{N}$ at $l \in \mathcal{L}$ if either $\tilde{\psi}(l) \notin \mathcal{N}$ or the image $\partial\psi(T_l(\mathcal{L}))$ of $T_l(\mathcal{L})$ in $T_m(\mathcal{M})$ at $m = \tilde{\psi}(l)$, together with $T_m(\mathcal{N})$, gives the whole space $T_m(\mathcal{M})$.

LEMMA. *Let* $\psi\colon \mathcal{L} \to \mathcal{M}$ *be a morphism transversal to a subsupermanifold* $\mathcal{N}$ *at all points* $l \in \mathcal{L}$, *and let* $\mathcal{J}$ *be the ideal of* $C^\infty(\mathcal{L})$ *generated by* $\psi^*(\mathcal{J}_{\mathcal{N}})$. *Then* $\mathcal{J}$ *distinguishes a certain subsupermanifold* $\mathcal{L}'$ *of* $\mathcal{L}$.

PROOF. It is easy to check that $\mathcal{J}$ satisfies 3.2.6, 1) and 2) and hence determines a subsupermanifold.

Let $\psi^{-1}(\mathcal{N})$ denote the subsupermanifold of $\mathcal{L}$ just constructed; it is easily verified that its underlying manifold coincides with $\tilde{\psi}^{-1}(N)$.

3.2.9. PROPOSITION. *Let* $\psi\colon \mathcal{L} \to \mathcal{M}$ *be a morphism of supermanifolds and* $m \in \mathcal{M}$. *Suppose that* ψ *for every* $l \in \mathcal{L}$ *with* $\psi(l) = m$ *is a morphism of constant rank in a neighbourhood of* l. *Let* $\mathcal{J}$ *be the ideal of* $C^\infty(\mathcal{L})$ *generated by* $\psi^*(\mathcal{J}_m)$. *Then* $\mathcal{J}$ *distinguishes a subsupermanifold* $\mathcal{L}'$ *of* $\mathcal{L}$.

PROOF. Using the theorem on morphisms of constant rank and the localization principle, we can easily verify that $\mathcal{J}$ satisfies 3.2.6, 1) and 2) and hence determines a subsupermanifold.

Let $\psi^{-1}(m)$ denote this subsupermanifold; its underlying manifold coincides with $\widetilde{\psi}^{-1}(m)$.

§3. Families

3.3.1. In classical analysis, a very important role is played by families of objects of a certain type that depend smoothly on one or several parameters. For example, this could be the family of smooth mappings $\varphi_t\colon N \to M$, the family of vector fields X_t on M, or the family of coordinate systems $\{x_i\}_t$, all depending smoothly on t.

We consider the first example, that of smooth mappings $\varphi_t\colon N \to M$. To define such a family, we specify for each value of t a mapping $\varphi_t\colon N \to M$ and require that the φ_t "depend smoothly on t". This means that if W is the manifold of values of the parameter t, then the resulting mapping $\psi\colon W \times N \to M$ is smooth.

In the supercase, we should like to have families depending on odd parameters t (so that, for example, the "manifold" $\mathcal{W}$ of "values" of t would coincide with $\mathcal{R}^{0,q}$).

? 3.2. Let $\mathcal{W}$ be a supermanifold, which we shall refer to as the supermanifold of parameters. A *smooth family of morphisms* $\varphi_{\mathcal{W}}\colon \mathcal{N} \to \mathcal{M}$ *parametrized by* $\mathcal{W}$ (or simply a *$\mathcal{W}$-family*) is an arbitrary morphism $\varphi\colon \mathcal{W} \times \mathcal{N} \to \mathcal{M}$. If w is a point of $\mathcal{W}$ and $i\colon w \to \mathcal{W}$ the morphism of point inclusion, then the morphism $\varphi\circ(i\times\mathrm{id})$ is the *value* of $\varphi_{\mathcal{W}}$ at w. Since we are in the supercase, a family of morphisms is by no means determined by the set of its values at all points.

We now describe in this example what a substitution of parameters is. Let $\alpha\colon \mathcal{U} \to \mathcal{W}$ be a morphism of supermanifolds and $\varphi\colon \mathcal{W}\times\mathcal{N} \to \mathcal{M}$ a $\mathcal{W}$-family of morphisms of $\mathcal{N}$ into $\mathcal{M}$. We wish to effect a substitution of the parameter and define a $\mathcal{U}$-family of morphisms of $\mathcal{N}$ into $\mathcal{M}$, which, in classical terminology, is obtained from φ by "expressing the coordinates of $\mathcal{W}$ in terms of those of $\mathcal{U}$". This $\mathcal{U}$-family, which we denote by φ^{α}, is constructed as follows:

$$\varphi^{\alpha}\colon \mathcal{U}\times\mathcal{N} \xrightarrow{(\alpha\circ\mathrm{pr}_{\mathcal{U}})\times\mathrm{pr}_{\mathcal{N}}} \mathcal{W}\times\mathcal{N} \xrightarrow{\varphi} \mathcal{M}.$$

We say that φ^{α} is obtained from φ by a change of the parameter α.

3.3.3. We also make use of an equivalent definition of a $\mathcal{W}$-family of morphisms of $\mathcal{N}$ into $\mathcal{M}$, when instead of a morphism $\varphi\colon \mathcal{W} \times \mathcal{N} \to \mathcal{M}$ we take a morphism $\varphi'\colon \mathcal{W} \times \mathcal{N} \to \mathcal{W} \times \mathcal{M}$ that is compatible with projection onto $\mathcal{W}$, that is, $\mathrm{pr}_1(\mathcal{W}\times\mathcal{N}) = \mathrm{pr}_1(\mathcal{W}\times\mathcal{M})\circ\varphi'$. It is clear that for every such φ' there is a unique $\varphi = \mathrm{pr}_1\circ\varphi'$, and vice versa.

Given $\mathcal{W}$-families of morphisms $\varphi_{\mathcal{W}}\colon \mathcal{N}\to\mathcal{M}$ and $\psi_{\mathcal{W}}\colon \mathcal{M}\to\mathcal{L}$, the composite $(\psi\circ\varphi)_{\mathcal{W}}$ is defined by setting

$$\psi\circ\varphi\colon \mathcal{W}\times\mathcal{N}\xrightarrow{\varphi}\mathcal{W}\times\mathcal{M}\xrightarrow{\psi}\mathcal{W}\times\mathcal{L}.$$

A $\mathcal{W}$-family of morphisms $\varphi_{\mathcal{W}}\colon \mathcal{N}\to\mathcal{M}$ is called a *family of diffeomorphisms* if there is a $\mathcal{W}$-family of morphisms $\psi_{\mathcal{W}}\colon \mathcal{M}\to\mathcal{N}$ such that $\varphi\circ\psi=\psi\circ\varphi=\mathrm{id}$. This is equivalent to the requirement that $\varphi:\mathcal{W}\times\mathcal{N}\to\mathcal{W}\times\mathcal{M}$ is a diffeomorphism.

If $\alpha\colon \mathcal{U}\to\mathcal{W}$ is a morphism (change of parameters), then from a given $\mathcal{W}$-family of morphisms $\varphi\colon \mathcal{W}\times\mathcal{N}\to\mathcal{W}\times\mathcal{M}$ we can define a $\mathcal{U}$-family φ^{α} as follows:

$$\varphi^{\alpha}\colon \mathcal{U}\times\mathcal{N}\xrightarrow{\mathrm{pr}_{\mathcal{U}}\times(\alpha\circ\mathrm{pr}_{\mathcal{U}})\times\mathrm{pr}_{\mathcal{N}}}\mathcal{U}\times\mathcal{W}\times\mathcal{N}\xrightarrow{\mathrm{pr}_{\mathcal{W}}\times(\varphi\circ\mathrm{pr}_{\mathcal{W}}\times\mathcal{N})}$$

$$\to\mathcal{U}\times(\mathcal{W}\times\mathcal{M})\xrightarrow{\mathrm{pr}_{\mathcal{U}}\times\mathcal{M}}\mathcal{U}\times\mathcal{M}.$$

Clearly, $(\varphi\circ\psi)^{\alpha}=\varphi^{\alpha}\circ\psi^{\alpha}$.

In particular, a given morphism $\psi\colon \mathcal{N}\to\mathcal{M}$ can be regarded as a pt-family, where "pt" $=\mathcal{R}^{0,0}$ is the manifold consisting of a single point. Then to every supermanifold $\mathcal{W}$ there corresponds a family ψ^{p}, where $p\colon \mathcal{W}\to\mathrm{pt}$ is the unique mapping onto a singleton; in other words, $\psi^{p}_{\mathcal{W}}=\psi\circ\mathrm{pr}_{\mathcal{N}}\colon \mathcal{W}\times\mathcal{N}\to\mathcal{M}$. Such a family of morphisms is said to be *constant*.

3.3.4. A *$\mathcal{W}$-family of points* of a supermanifold $\mathcal{M}$ is a morphism $\varphi_{\mathcal{W}}\colon \mathcal{W}\to\mathcal{N}$. Instead of $\varphi_{\mathcal{W}}$ we often use the corresponding morphism $\varphi\colon \mathcal{W}\xrightarrow{\mathrm{id}\times\varphi_{\mathcal{W}}}\mathcal{W}\times\mathcal{M}$, for which $\mathrm{pr}_{\mathcal{W}}\circ\varphi=\mathrm{id}$. If $\alpha\colon \mathcal{U}\to\mathcal{W}$ is a morphism, then the family of points $\varphi^{\alpha}_{\mathcal{W}}$ obtained by a change of the parameter α has the form

$$\varphi^{\alpha}_{\mathcal{W}}=\varphi_{\mathcal{W}}\circ\alpha\colon \mathcal{U}\xrightarrow{\alpha}\mathcal{W}\xrightarrow{\varphi_{\mathcal{W}}}\mathcal{M}.$$

Given a morphism $\beta\colon \mathcal{M}\to\mathcal{L}$ and a $\mathcal{W}$-family of points of a supermanifold $\mathcal{M}$, we can construct in the natural way a $\mathcal{W}$-family $\psi=\beta\circ\varphi$ of points on $\mathcal{L}$.

It is clear that a $\mathcal{W}$-family of morphisms of pt into $\mathcal{M}$ is the same as a $\mathcal{W}$-family of points of $\mathcal{M}$.

3.3.5. A *$\mathcal{W}$-family of functions* on a supermanifold $\mathcal{M}$ is a function f on $\mathcal{W}\times\mathcal{M}$; the set of these families is denoted by $C^{\infty}(\mathcal{M};\mathcal{W})$.

A *$\mathcal{W}$-family of even functions* is an even function on $\mathcal{W}\times\mathcal{M}$.

If $\alpha:\mathcal{U}\to\mathcal{W}$ is a change of parameter and $f\in C^{\infty}(\mathcal{M};\mathcal{W})$, then $f^{\alpha}=\beta^{*}(f)\in C^{\infty}(\mathcal{M};\mathcal{U})$, where $\beta=(\alpha\circ\mathrm{pr}_{\mathcal{W}})\times\mathrm{pr}_{\mathcal{M}}\colon \mathcal{U}\times\mathcal{M}\to\mathcal{W}\times\mathcal{M}$.

Note that it follows from the theorem on the coordinate expression of mappings that $\mathcal{W}$-families of even functions on $\mathcal{M}$ are in one-to-one correspondence with $\mathcal{W}$-families of morphisms $\mathcal{M}\to\mathcal{R}^{1,0}$.

3.3.6. Given a $\mathscr{W}$-family of points of a supermanifold $\mathscr{M}$, that is, a morphism $\varphi: \mathscr{W} \to \mathscr{W} \times \mathscr{M}$, let us define the families of tangent vectors and cotangent vectors at the points of φ. We begin with the cotangent vectors.

Since $\mathrm{pr}_{\mathscr{W}} \circ \varphi$ is the same as $\mathrm{id}: \mathscr{W} \to \mathscr{W}$, we see that φ is a regular closed embedding.

Let

$$\mathscr{I}_\varphi = \mathrm{Ker}\ \varphi^* = \{f \in C^\infty(\mathscr{W} \times \mathscr{M}) \mid \varphi^*(f) = 0\}.$$

It is clear that

$$C^\infty(\mathscr{W} \times \mathscr{M}) = C^\infty(\mathscr{W}) \oplus \mathscr{I}_\varphi,$$

where $C^\infty(\mathscr{W})$ is naturally embedded in $C^\infty(\mathscr{W} \times \mathscr{M})$ by means of the homomorphism $\mathrm{pr}^*_{\mathscr{W}}(\mathscr{W} \times \mathscr{M})$.

A *$\mathscr{W}$-family of cotangent vectors* (at the points of φ) is an element of $\mathscr{I}_\varphi/\mathscr{I}_\varphi^2$.

These $\mathscr{W}$-families form a $C^\infty(\mathscr{W})$-module, which we denote by $T^*_\varphi(\mathscr{M})$ and call the *module of cotangent vectors to the family* φ. The adjoint $C^\infty(\mathscr{W})$-module $T_\varphi(\mathscr{M}) = \mathrm{Hom}_{C^\infty(\mathscr{W})}(T^*_\varphi(\mathscr{M}), C^\infty(\mathscr{W}))$ is called the *module of tangent vectors to the family* φ. We can identify $T_\varphi(\mathscr{M})$ with the module of $C^\infty(\mathscr{W})$-linear homomorphisms $\partial: C^\infty(\mathscr{W} \times \mathscr{M}) \to C^\infty(\mathscr{W})$, which satisfy Leibniz' rule.

If f is a $\mathscr{W}$-family of functions on $\mathscr{M}$, that is, $f \in C^\infty(\mathscr{W} \times \mathscr{M})$, let δf denote the $\mathscr{W}$-family of cotangent vectors of the form

$$\delta f = (f - \varphi^*(f))(\mathrm{mod}\ \mathscr{I}_\varphi^2).$$

(see 2.4.2).

Let $\psi: \mathscr{N} \to \mathscr{M}$ be a morphism of supermanifolds. A *ψ-derivation* is an arbitrary $\mathbf{R}$-linear mapping $X: C^\infty(\mathscr{M}) \to C^\infty(\mathscr{N})$ satisfying Leibniz' rule.

Let Der_ψ denote the space of all ψ-derivations.

Now suppose that we are given a $\mathscr{W}$-family of points of a supermanifold $\mathscr{M}$, that is, a morphism $\varphi: \mathscr{W} \to \mathscr{W} \times \mathscr{M}$, and a family of tangent vectors $\partial \in T_\varphi(\mathscr{M})$. As we have seen, ∂ determines a φ-derivation $\partial: C^\infty(\mathscr{W} \times \mathscr{M}) \to C^\infty(\mathscr{W})$ (clearly, not every ψ-derivation is of this form, but only those that vanish on $C^\infty(\mathscr{W}) \subset C^\infty(\mathscr{W} \times \mathscr{M})$). By restricting ∂ to the subalgebra $C^\infty(\mathscr{M}) \subset C^\infty(\mathscr{W} \times \mathscr{M})$, we obtain a linear mapping $X_\partial: C^\infty(\mathscr{M}) \to C^\infty(\mathscr{W})$, which is easily seen to be a $\varphi_{\mathscr{W}}$-derivation. Thus, we have constructed a canonical homomorphism

$$r: T_\varphi(\mathscr{M}) \to \mathrm{Der}_{\varphi_{\mathscr{W}}}(\partial \mapsto X_\partial).$$

PROPOSITION. *r is an isomorphism.*

In other words, the proposition asserts that every $\varphi_{\mathscr{W}}$-derivation $X: C^\infty(\mathscr{M}) \to C^\infty(\mathscr{W})$ extends uniquely to a $\bar{\varphi}$-derivation

$$\partial\colon C^\infty(\mathscr{U}\times\mathscr{M})\to C^\infty(\mathscr{U}) \quad \text{such that} \quad \partial(C^\infty(\mathscr{U}))=0.$$

3.3.7. Before proving Proposition 3.3.6, we examine in more detail the structure of Der_ψ for a given morphism $\psi\colon \mathscr{N}\to\mathscr{M}$.

1) *The space* Der_ψ *is a* $C^\infty(\mathscr{N})$*-module under the action* $(fX)(h)=f(X(h))$, *where* $f\in C^\infty(\mathscr{N})$, $h\in C^\infty(\mathscr{M})$, *and* $X\in\mathrm{Der}_\psi$.

2) *If* $X\in\mathrm{Der}_\psi$, *then* $X(1)=0$.

3) *Let* $\mathscr{M}$ *be a superdomain with coordinate system* $\{x_i\}$. *To any set of functions* $f=\{f_i\in C^\infty(\mathscr{N})\}$ *there corresponds a mapping* $X_f\colon C^\infty(\mathscr{M})\to C^\infty(\mathscr{N})$ *given by* $X_f(h)=\Sigma f_i\,\psi^*(\partial h/\partial x_i)$ *that is a* ψ*-derivation. Any* ψ*-derivation* X *is of this form, and the set of functions* f *is uniquely determined by* X.

The uniqueness follows from the fact that f is determined by the ψ-derivation X_f, because $f_i=X_f(x_i)$. Next, by subtracting X_f from X, where $f_i=X(x_i)$, we may assume that $X(x_i)=0$. We claim that in this case $X=0$.

Let $h\in C^\infty(\mathscr{M})$ and $n\in\mathscr{N}$. It is sufficient to prove that $X(h)\in\mathscr{J}_n^k$ for any k. Let $m=\psi(n)$. Since $\psi^*(\mathscr{J}_m)\subset\mathscr{J}_n$ and $X(x_i)=0$, it follows from Leibniz' rule that $X(\mathscr{J}_m^{k+1})\subset\mathscr{J}_n^k$ and $X(P(x_i))=0$ for any polynomial P. But by 2.2.8, we can choose $P(x_i)$ in such a way that $h-P(x_i)\in\mathscr{J}_m^{k+1}$. Therefore, $X(h)=X(h-P(x_i))+X(P(x_i))=X(h-P(x_i))\in\mathscr{J}_n^k$, as required.

4) *Let* $\mathscr{M}'$ *be an open subsupermanifold of* $\mathscr{M}$ *containing* $\psi(\mathscr{N})$. *In other words, there is a factorization* $\psi\colon\mathscr{N}\xrightarrow{\psi'}\mathscr{M}'\xrightarrow{i}\mathscr{M}$, *where* ψ' *is a morphism and* i *an open embedding. Let* $i_*\colon\mathrm{Der}_{\psi'}\to\mathrm{Der}_\psi$ *be the mapping defined by* $i_*X(h)=X(i^*(h))$, *where* $X\in\mathrm{Der}_{\psi'}$ *and* $h\in C^\infty(\mathscr{M})$. *Then* i^* *is an isomorphism, in other words, the space* Der_ψ *depends only on a neighbourhood of* $\psi(\mathscr{N})$.

We show first of all that if h_1, $h_2\in C^\infty(\mathscr{M})$ agree on a subsupermanifold $\mathscr{U}\subset\mathscr{M}$, then $X(h_1)$ and $X(h_2)$ agree on $\psi^{-1}(\mathscr{U})$. For let $h=h_1-h_2$ and let $n\in\mathscr{N}$ be a point for which $\psi(n)\in\mathscr{U}$. To show that $X(h)$ vanishes on a neighbourhood of n, we invoke the localization principle 3.1.8 to construct a function $\varphi\in C_0^\infty(\mathscr{M})$ such that $\varphi=1$ on a neighbourhood of $\psi(n)$ and $\mathrm{supp}_\varphi\subset\mathscr{U}$. Then $\varphi h=0$, so that $X(\varphi h)=X(\varphi)\cdot\psi^*(h)+\psi^*(\varphi)\cdot X(h)=0$. Since $\psi^*(h)=0$ and $\varphi^*(\varphi)=1$ on a neighbourhood of n, we see that $X(h)=0$ on a neighbourhood of n.

Next, we claim that if $X\in\mathrm{Der}_{\psi'}$ and $i_*(X)=0$, then $X=0$. Let $h'\in C^\infty(\mathscr{M}')$ and $n\in\mathscr{N}$. We must check that $X(h')=0$ on a neighbourhood of n. Using the localization principle we choose a function $h\in C^\infty(\mathscr{M})$ that agrees with h' on a neighbourhood of $\psi(n)$. Then $X(h')=X(i^*(h))=i_*(X)(h)=0$ on a neighbourhood of n.

It remains for us to show that every ψ-derivation $Y\in\mathrm{Der}_\psi$ has the form $i^*(X)$, with $X\in\mathrm{Der}_{\psi'}$. Let $h\in C^\infty(\mathscr{M}')$. We choose a covering $\{\mathscr{V}_\alpha\}$ of $\mathscr{N}$ and a set of functions $h_\alpha\in C^\infty(\mathscr{M})$ such that h_α agrees with h on a neighbourhood of $\psi(\mathscr{V}_\alpha)$; this is possible by the localization principle. Now we consider the function $f_\alpha=Y(h_\alpha)|_{\mathscr{V}_\alpha}$ on $\mathscr{V}_\alpha$. We have already shown that f_α does not

depend on the choice of h_α. Therefore, f_α and f_β agree on $\mathscr{V}_\alpha \cap \mathscr{V}_\beta$. Hence, all the f_α are restrictions of some $f \in C^\infty(\mathscr{N})$. We put $X(h') = f$. Then it is easy to check that X is a ψ'-derivation and $i^*(X) = Y$.

5) *Let $\mathscr{N}'$ be an open subsupermanifold in $\mathscr{N}$ and $\psi|_{\mathscr{N}'}: \mathscr{N}' \to \mathscr{M}$ the corresponding morphism. We define a mapping $\mathrm{Der}_\psi \to \mathrm{Der}_{\psi|\mathscr{N}'}$ by $X \mapsto X|_{\mathscr{N}'}$, where $X|_{\mathscr{N}'}(h) = X(h)|_{\mathscr{N}'}$. Then $\mathrm{Der}_{\psi|\mathscr{N}'}$ forms a sheaf on $\mathscr{N}'$.*

The proof is the same as that in 3.3.5.

3.3.8. PROOF OF PROPOSITION 3.3.6. Let $\mathscr{V}$ be an open subsupermanifold in $\mathscr{W}$, and let $\varphi_{\mathscr{V}^\circ} = \mathrm{id} \times \varphi_{\mathscr{W}}|_{\mathscr{V}^\circ}: \mathscr{V} \to \mathscr{V} \times \mathscr{M}$, $\psi_{\mathscr{V}^\circ} = \varphi|_{\mathscr{V}^\circ}: \mathscr{V} \to \mathscr{W} \times \mathscr{M}$. Then $\mathrm{Der}_{\varphi_{\mathscr{V}^\circ}} = \mathrm{Der}_{\psi_{\mathscr{V}^\circ}}$, by 3.3.7, 4). Hence from 3.3.7, 5) it follows that the correspondences $\mathscr{V} \mapsto T_{\varphi_{\mathscr{V}^\circ}}(\mathscr{M})$ and $\mathscr{V} \to \mathrm{Der}_{\varphi_{\mathscr{W}}|_{\mathscr{V}^\circ}}$ are sheaves. Now it is sufficient to verify that every point $w \in \mathscr{W}$ has a neighbourhood $\mathscr{V}$ such that $r_{\mathscr{V}^\circ}: T_{\varphi_{\mathscr{V}^\circ}} \to \mathrm{Der}_{\varphi_{\mathscr{W}}|_{\mathscr{V}^\circ}}$ is an isomorphism. We may assume that $\mathscr{V}$ is a superdomain and that $\varphi(\mathscr{V}) \subset \mathscr{M}'$, where $\mathscr{M}'$ is an open superdomain in $\mathscr{M}$. Since changing from $\mathscr{M}$ to $\mathscr{M}'$ leaves $T_{\varphi_{\mathscr{V}^\circ}}$ and $\mathrm{Der}_{\varphi_{\mathscr{W}}|_{\mathscr{V}^\circ}}$ unaffected (see 3.3.7, 4)), we see that it is sufficient to prove Proposition 3.3.6 in the case when $\mathscr{W}$ and $\mathscr{M}$ are superdomains, and this follows easily from 3.3.7, 3).

3.3.9. We list the basic properties of families of tangent vectors and cotangent vectors at the points of a family $\varphi: \mathscr{W} \times \mathscr{U} \to \mathscr{M}$.

a) Let $\alpha: \mathscr{U} \to \mathscr{W}$ be a change of parameters. If $\omega = \delta f \in T^*_\varphi(\mathscr{M})$, with $f \in C^\infty(\mathscr{M}, \mathscr{W})$, then we put $\omega^\alpha = \delta f^\alpha$, where f^α is the $\mathscr{U}$-family of functions obtained from f by the change of parameters α. If $\partial \in T_\varphi(\mathscr{M})$, then we define $\partial^\alpha \in T_{\varphi^\alpha}(\mathscr{M})$ by $X_{\partial^\alpha}(h) = \alpha(X_\partial(h))$ for all $h \in C^\infty(\mathscr{M})$ (here $X_\partial \in \mathrm{Der}_{\varphi_{\mathscr{W}}}$ and $X_{\partial^\alpha} \in \mathrm{Der}_{\varphi^\alpha_{\mathscr{W}}}$). It is clear that

$$\langle \partial^\alpha, f^\alpha \rangle = \alpha^* \langle \partial, f \rangle.$$

b) Let $\beta: \mathscr{W} \times \mathscr{M} \to \mathscr{W} \times \mathscr{L}$ be a $\mathscr{W}$-family of morphisms of $\mathscr{M}$ and $\mathscr{L}$, φ a $\mathscr{W}$-family of points of $\mathscr{M}$, and $\psi = \beta \circ \varphi$ the corresponding family of points of $\mathscr{L}$. Then there are homomorphisms

$$\beta_*: T_\varphi(\mathscr{M}) \to T_\psi(\mathscr{L}),$$
$$\beta^*: T^*_\psi(\mathscr{L}) \to T^*_\varphi(\mathscr{M}),$$

where

$$\beta_*(\partial): C^\infty(\mathscr{W} \times \mathscr{L}) \to C^\infty(\mathscr{W}),$$
$$\beta_*(\partial)(h) = \partial(\beta^*(h)), \qquad \partial \in T_\varphi(\mathscr{M}).$$

Clearly, β_* and β^* are adjoint to one another, that is,

$$\langle \partial, \beta^*(\omega) \rangle = \langle \beta_*(\partial), \omega \rangle, \qquad \partial \in T_\varphi(\mathscr{M}), \quad \omega \in T^*_\psi(\mathscr{L}).$$

c) If $\beta: \mathscr{M} \to \mathscr{L}$ is an embedding of an open subsupermanifold, then it

follows from 3.2.6 and 3.3.7, 4) that β_* and β^* are isomorphisms.

d) Let $\mathscr{M}$ be a superdomain with coordinate system x. Then it follows easily from 3.3.7, 3) that the elements δx_i form a basis for $T^*_{\varphi}(\mathscr{M})$ as a $C^\infty(\mathscr{W})$ -module. The dual basis of $T_\varphi(\mathscr{M})$ consists of the elements $\partial_i = \partial_{x_i}$, given by

$$X_{\partial_i}(h) = \varphi_0^*\left(\frac{\partial h}{\partial x_i}\right), \qquad h \in C^\infty(\mathscr{M}), \qquad X_{\partial_i} \in \operatorname{Der} \varphi_0.$$

3.3.10. We turn to the important case when $\mathscr{W} = \mathscr{M}$ and $\varphi\colon \mathscr{W} \to \mathscr{M}$ is the identity mapping. In this case a $\mathscr{W}$-family of tangent (or cotangent) vectors is called a *vector* (or *covector*) *field on* $\mathscr{M}$. We denote the $C^\infty(\mathscr{M})$ -module of vector fields by $D_1(\mathscr{M})$ and that of covector fields by $D^1(\mathscr{M})$.

It follows from Proposition 3.2.6 that $D_1(\mathscr{M})$ is canonically isomorphic to $\operatorname{Der} C^\infty(\mathscr{M})$, and we often identify these modules.

If $\mathscr{U}$ is an open subsupermanifold in $\mathscr{M}$, then using 3.3.7, 4), it is easy to construct mappings $D_1(\mathscr{M}) \to D_1(\mathscr{U})$ and $D^1(\mathscr{M}) \to D^1(\mathscr{U})$ that are compatible with the restriction of functions and with the pairing $D_1(\mathscr{M}) \times D^1(\mathscr{M}) \to C^\infty(\mathscr{M})$.

By means of these mappings, D^1 and D_1 can be regarded as sheaves of $\mathcal{O}_{\mathscr{M}}$-modules, and

a) if $f \in C^\infty(\mathscr{M})$, then $\delta f \in D^1(\mathscr{M})$;

b) if $\partial \in D_1(\mathscr{M})$ and $f \in C^\infty(\mathscr{M})$, then $\partial(f) \in C^\infty(\mathscr{M})$;

c) if $\partial \in D_1(\mathscr{M})$ and $\omega \in D^1(\mathscr{M})$, then $\langle \partial, \omega \rangle \in C^\infty(\mathscr{M})$, with $\langle \partial, \delta f \rangle = \partial(f)$;

d) for each morphism $\varphi\colon \mathscr{M} \to \mathscr{L}$ there is a homomorphism $\varphi^*\colon D^1(\mathscr{L}) \to D^1(\mathscr{M})$, defined as follows. If $\omega \in D^1(\mathscr{L}) = T^*_{\mathrm{id}}(\mathscr{L})$, we can use φ as a change of parameters to obtain an $\mathscr{M}$-family ω^φ of tangent vectors at the points of the $\mathscr{M}$-family φ. By applying the operator φ^* we obtain an $\mathscr{M}$-family $\varphi^*(\omega^\varphi) \in T_{\mathrm{id}}(\mathscr{M})$, which we denote by $\varphi^*(\omega)$.

It is clear that for any function $f \in C^\infty(\mathscr{M})$,

$$\varphi^*(\delta f) = \delta(\varphi^*(f)).$$

Note that, in general, it is not possible to define the homomorphism $\varphi_*\colon D_1(\mathscr{M}) \to D_1(\mathscr{L})$ nor $\varphi^*\colon D_1(\mathscr{L}) \to D_1(\mathscr{M})$. But when $\varphi\colon \mathscr{M} \to \mathscr{L}$ is an open embedding, then $\varphi^*\colon D_1(\mathscr{L}) \to D_1(\mathscr{M})$ is defined: it is the *restriction of a vector field to a subsupermanifold*. If φ is a diffeomorphism, then φ^* is an isomorphism, and we define $\varphi_*\colon D_1(\mathscr{M}) \to D_1(\mathscr{L})$ by putting $\varphi_* = (\varphi^*)^{-1}$.

3.3.11. Let us find out what a family of families is. If we consider $\mathscr{U}$-families of objects of type X, then it is natural to define a $\mathscr{W}$-family of such $\mathscr{U}$-families simply as a $\mathscr{W} \times \mathscr{U}$-family of objects of type X.

In terms of this, it is a simple matter to define the concept of a $\mathscr{W}$-family of vector fields on $\mathscr{M}$. For this purpose we consider the morphism $\varphi = \mathrm{pr}_{\mathscr{M}}\colon \mathscr{W} \times \mathscr{M} \to \mathscr{M}$ and define a *$\mathscr{W}$-family of vector fields* on $\mathscr{M}$ as a

$\mathcal{W} \times \mathcal{M}$-family of tangent vectors at the points of the $\mathcal{W} \times \mathcal{M}$-family φ. If α: $\mathcal{U} \to \mathcal{W}$ is a change of parameter, then from every $\mathcal{W}$-family of vector fields ∂ on $\mathcal{M}$, we can construct in the natural way the $\mathcal{U}$-family ∂^α of vector fields on $\mathcal{M}$. Families of covector fields are defined similarly. The $\mathcal{W}$-families of vector and covector fields on $\mathcal{M}$ are denoted by $D_1(\mathcal{M}; \mathcal{W})$ and $D^1(\mathcal{M}; \mathcal{W})$, respectively.

The following properties are easy to verify by using Proposition 3.3.6.

a) Let ∂ be a $\mathcal{W}$-family of vector fields on $\mathcal{M}$. Then to each $\mathcal{W}$-family $f \in C^\infty(\mathcal{M}; \mathcal{W})$ there corresponds a new $\mathcal{W}$-family $\partial(f) \in C^\infty(\mathcal{M}; \mathcal{W})$. The mapping $f \mapsto \partial f$ is a derivation of the algebra $C^\infty(\mathcal{W} \times \mathcal{M}) = C^\infty(\mathcal{M}; \mathcal{W})$. Distinct families ∂ correspond to distinct derivations.

b) Let f be a family of functions that are constant on $\mathcal{M}$, that is, $f \in C^\infty(\mathcal{W}) \subset C^\infty(\mathcal{W} \times \mathcal{M})$; then $\partial f = 0$.

c) Every derivation of $C^\infty(\mathcal{W} \times \mathcal{M})$ satisfying b) corresponds to a unique $\mathcal{W}$-family of vector fields on $\mathcal{M}$.

Thus, a $\mathcal{W}$-family of vector fields on $\mathcal{M}$ is simply a vector field on $\mathcal{W} \times \mathcal{M}$ (speaking informally, this means that it is directed along the fibres of the projection $\mathcal{W} \times \mathcal{M} \to \mathcal{W}$).

3.3.12. Let φ: $\mathcal{W} \to \mathcal{W} \times \mathcal{M}$ be a $\mathcal{W}$-family of points of a supermanifold $\mathcal{M}$. It is useful to have a geometrical interpretation of $\mathcal{W}$-families of tangent vectors at the points of φ.

Let $\mathcal{R}$ be a one-dimensional superspace with coordinate t, and let $\{0\} \in \mathcal{R}$ be the point given by $t = 0$. A *$\mathcal{W}$-family of $p(t)$-curves* (beginning at the points of φ) is a morphism C: $\mathcal{V} \to \mathcal{W} \times \mathcal{M}$, where $\mathcal{V}$ is an open subsupermanifold in $\mathcal{W} \times \mathcal{R}$ containing $\mathcal{W} = \mathcal{W} \times \{0\}$ such that it is compatible with projection onto $\mathcal{W}$ (that is, $\mathrm{pr}_1(\mathcal{W} \times \mathcal{M}) \circ C|_{\mathcal{W}} = \varphi$). Two such morphisms C_1 and C_2 are called *equivalent* if $C_1^*(f) \equiv C_2^*(f) \pmod{\mathcal{J}_{\mathcal{W}}^2}$ for any $f \in C^\infty(\mathcal{W} \times \mathcal{M})$, where $\mathcal{J}_{\mathcal{W}}$ is the ideal generated by t.

To each $\mathcal{W}$-family of curves C there corresponds a family of tangent vectors $\partial_C \in T_\varphi(\mathcal{M})$, given by $\partial_C = \delta C(\partial_t)$. In other words, if $f \in C^\infty(\mathcal{W} \times \mathcal{M})$, then

$$\partial_C(f) = \left(\frac{\partial}{\partial t} C^*(f)\right)\Big|_{\mathcal{W}}.$$

It is clear that equivalent curves correspond to one and the same family of tangent vectors, and inequivalent ones to distinct families. It can be shown that any family $\partial \in T_\varphi(\mathcal{M})$ is of the form $\partial = \partial_C$ for some family of curves C. When $\mathcal{M}$ is a superdomain, this follows at once from the coordinate expression for mappings.

Let us consider the case when $\mathcal{W} = \mathcal{M}$ and $\varphi = id_{\mathcal{M}}$ is the identity morphism. Then a family of curves C can be thought of as a morphism C: $\mathcal{V} \to \mathcal{M}$, where $\mathcal{V} \subset \mathcal{R} \times \mathcal{M}$. In other words, C can be interpreted as a family of "local" diffeomorphisms of $\mathcal{M}$ parametrized by t, where the diffeomorphism at $t = 0$ is the identity. This corresponds to the usual inter-

pretation of a vector field ∂_C as the directrix to a one-parameter family of diffeomorphisms.

3.3.13. Let $\mathcal{M}$ be a supermanifold. Then a *$\mathcal{W}$-family of (local) coordinate systems* on $\mathcal{M}$ is an open subsupermanifold $\mathcal{U} \subset \mathcal{W} \times \mathcal{M}$ and an open embedding φ: $\mathcal{U} \to \mathcal{W} \times \mathcal{R}^{p,q}$ compatible with projections onto $\mathcal{W}$. If x_i are coordinates on $\mathcal{R}^{p,q}$ then we use the same letters to denote the functions $\varphi^*(x_i) \in C^\infty(\mathcal{U})$. It is clear that these functions uniquely determine the family of coordinate systems φ. We denote φ by $\{x_i; \mathcal{W}\}$.

Given a $\mathcal{W}$-family of coordinate systems on $\mathcal{M}$, the derivations $\partial/\partial x_i$: $C^\infty(\mathcal{U}) \to C^\infty(\mathcal{U})$ are uniquely determined by the conditions

$$\frac{\partial}{\partial x_i}(x_k) = \delta_{ik}, \quad \frac{\partial}{\partial x_i} f = 0,$$

for $f \in C^\infty(\mathcal{W})$ (see 3.3.7).

Given two $\mathcal{W}$-families of coordinate systems $\{x_i; \mathcal{W}\}$ and $\{y_k; \mathcal{W}\}$ on one and the same superdomain $\mathcal{U}$, we can define the matrix of a change of coordinates $I_{xy} = (I_{ik})$ and the Jacobian matrix $J = (I_{xy})^{st}$.

Then

$$\frac{\partial f}{\partial x_i} = \sum_k \frac{\partial}{\partial x_i}(y_k) \frac{\partial f}{\partial y_k},$$

that is

$$\left| \frac{\partial f}{\partial x_i} \right\rangle = I_{xy} \left| \frac{\partial f}{\partial y_k} \right\rangle$$

(see Ch. II, §2).

NOTES

Commutative superalgebras first appeared in the work of Grassmann, which contains the definition of an exterior (Grassmann) algebra. These algebras came into common use by mathematicians when algebras of differential forms and products in cohomology began to be widely studied. Lie superalgebras were apparently discovered by Whitehead when he defined his product in homotopy groups.

Subsequently, commutative superalgebras and Lie superalgebras made frequent appearances in various branches of topology (cohomology algebras, Hopf algebras, and Steenrod algebras). Milner and Moore [31], while engaged in the study of Hopf algebras, discovered a connection between these and Lie superalgebras. In his work on rational homotopy theory, Quillen [32] strengthened this connection by defining for Lie superalgebras an analogue of the exponential mapping (the Hurewicz homomorphism).

Berezin and Kats [5] constructed formal Lie superalgebras and extended the Lie theory to them. Most significant in their work was the precise statement that they were concerned with a generalization of classical analysis, where odd variables appear on equal terms alongside the usual even variables. In this

article they raised the problem of defining analogues of Lie groups in the large (as opposed to formal Lie groups) having Lie superalgebras instead of Lie algebras. They gave several examples to show what they had in mind.

Using Berezin's idea that in the definition of these analogues Grothendieck's language of schemes could be useful, Leites first defined algebraic [12] and then smooth supermanifolds and Lie supergroups [13], [14]. A more detailed study was published by Berezin in Leites in [4].

Even before supermanifolds had been defined, Berezin formulated the notion of an integral in the supercase and wrote down (in a letter to Kats in 1971) a formula for the Jacobian of a change of variables. He proved this formula in [3] in the purely odd case. It was proved in general by his student Pakhomov [18]. The multiplicativity of the Berezinian was first proved by Leites [15]. The supertrace was defined in 1973 by Leites and Feigin, and used by Kats to classify the simple Lie superalgebras [10]. Both the supertrace and the Berezinian were defined independently by Arnowitt, Coleman and Nath (see [9]).

Differential forms on supermanifolds or, more precisely, on superspaces were first defined by Kats and Koronkevich [11] (see also the article by Segal [33], in which are defined spaces ringed by non-commutative superalgebras, the "quantized supermanifolds").

In the proofs of the theorems, all the simplifications (over the original ones) are due to Bernshtein, and so is the account of the point functor in the language of families, the correct formulation and proof of the inverse function theorem and the implicit function theorem, Definition 1.9.4, and the sections on Taylor series and subsupermanifolds.

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