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Internal Coalgebras in Varieties

Laurent Poinsot · Hans-E. Porst

To Bob Lowen for his 70^{th} birthday

Abstract Various approaches to internal coalgebras in a cocomplete category $\mathcal{C}$ and in a variety $\mathcal{V}$ in particular have been discussed for example by Kan, Freyd and Bergman. We describe the respective categories in terms of Lawvere theories $\mathcal{T}$ which in particular leads to the observation that internal $\mathcal{T}$-coalgebras in a cocomplete category $\mathcal{C}$ are nothing but the restrictions of left adjoints $\mathbf{Alg}\mathcal{T} \to \mathcal{C}$ to the dual of $\mathcal{T}$. This result can be seen as a generalization of the famous Eilenberg-Watts Theorem. We prove that the categories $\mathbf{Coalg}(\mathcal{T}, \mathcal{C})$ of internal $\mathcal{T}$-coalgebras in a locally presentable category $\mathcal{C}$ are locally presentable and have a left adjoint forgetful functor into $\mathcal{C}$. This generalizes and strengthens Bergman's recent completeness result for categories of internal coalgebras in varieties. Moreover, we show that Freyd's canonical constructions of internal coalgebras in a variety are left adjoints. Special instances of the respective right adjoints are the group-like elements and the primitive elements functors known from Hopf algebra theory and, in the case where $\mathcal{V}$ is a commutative variety, coreflectors from the category $\mathbf{Coalg}(\mathcal{T}, \mathcal{V})$ into $\mathcal{V}$.

Keywords Internal coalgebra, locally presentable category, covariety, (commutative) variety.

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Laurent Poinsot
LIPN, UMR CNRS 7030, University Paris 13, Sorbonne Paris Cité, and CReA, École de l'Air, France.
E-mail: laurent.poinsot@lipn.univ-paris13.fr

Hans-E. Porst
Department of Mathematical Sciences, University of Stellenbosch, Stellenbosch, South Africa.
E-mail: porst@math.uni-bremen.de
Permanent address: Department of Mathematics, University of Bremen, 28359 Bremen, Germany

Introduction

There are several concepts of what a category of coalgebras in a category $\mathcal{C}$ may be as, for example,

1. a particular subcategory of the category $\mathbf{Coalg}F$ of *functor coalgebras* for some functor $\mathcal{C} \xrightarrow{F} \mathcal{C}$, for example,
   (a) a comonadic category over $\mathcal{C}$,
   (b) a *co-Birkhoff subcategory* of $\mathbf{Coalg}F$, that is, a subcategory with appropriate closure properties see (e.g. [3]),
   (c) a *coequational category* in the sense of [3], that is, subcategory of $\mathbf{Coalg}F$ defined by co-equational arrows (that is, a generalization of being projective with respect to a family of regular monomorphisms) or even a *covariety*, a coequational category with cofree objects,
2. a *coequational category* in the sense of [20],
3. the category of *coalgebras* in a cocomplete category $\mathcal{C}$ for an algebraic theory $\boldsymbol{T}$ in the sense of [15].

Hopf algebraists may first of all associate this notion with the category of comonoids in the monoidal category $\mathbf{Mod}_R$ of R-modules over a commutative ring R; but this is *not* the concept this note is dealing with.

The common feature of all the concepts mentioned above is, that they describe algebras of a certain type in $\mathcal{C}^{\mathsf{op}}$, the dual of $\mathcal{C}$. For various relations between the concepts mentioned in 1. and 2. above we refer to [2], [3] and [20].

The topic of this paper is the concept mentioned in 3. above with an emphasis on the case where $\mathcal{C}$ is a variety. To make this clear we call these coalgebras *internal coalgebras in* $\mathcal{C}$, because they are internal algebras in $\mathcal{C}^{\mathsf{op}}$ in the usual sense. For some relations between this concept and those mentioned in 1. and 2. above see Section 2.5.

The first appearance of the concept of an internal coalgebra in a variety in this sense, more precisely of a comonoid in the variety of groups, is due to Kan [19]. His approach was innovative in 1958: he first formulated the group axioms in the form of commutative diagrams in a category $\mathcal{C}$ with products, and then simply dualized this concept of a group in a category.

Nearly a decade later Freyd in his seminal paper [15], apparently unaware of Kan's paper, introduced equationally defined algebras in a category $\mathcal{C}$ with products in the language of universal algebra (using the fundamental idea of Lawvere's concept of an algebraic theory) and dualized this in order to obtain the notion of a coalgebra in a category with coproducts. He apologized for his explicit avoidance of algebraic theories with the words *"I reject his formulation for expository reasons peculiar to our times — the language of categories is still too new."* Specifically he observed that every internal algebra in $\mathcal{C}$ can equivalently be described as a contravariant representable functor $\mathcal{C} \to \mathcal{V}$ from a category $\mathcal{C}$ with finite products into a variety $\mathcal{V}$ such that, dually, an internal coalgebra in $\mathcal{C}$ "is" a covariant representable functor $\mathcal{C} \to \mathcal{V}$, defined on category $\mathcal{C}$ with finite coproducts.

The obvious description of coalgebras by means of the explicit use of Lawvere theories, though in our opinion the most natural one, seems not to have appeared in print. Even the recent textbook [9], despite its frequent use of categorical concepts and Freyd's apology no longer applying, does not use this option. Each of these descriptions has its advantages: For example, using Lawvere theories, the so called *canonical* coalgebra structures on free algebras appear "automatically" since the dual of the Lawvere theory of a variety $\mathcal{V}$ is a full subcategory of $\mathcal{V}$ closed under coproducts, or, using $\mathcal{T}$-representable functors one trivially observes that every algebra in a commutative variety $\mathcal{V}$ carries the structure of an internal coalgebra in a canonical way. The idea to consider coalgebras in $\mathcal{C}$ as models of Lawvere theories in $\mathcal{C}^{\mathsf{op}}$ becomes particularly convincing, when $\mathcal{C}$ not only has finite coproducts but even is cocomplete: The fact that the category $\mathbf{Alg}\mathcal{T}$ of set-valued models of a Lawvere theory $\mathcal{T}$ is the free completion of $\mathcal{T}^{\mathsf{op}}$ under sifted colimits specializes immediately to the observation that $\mathcal{T}$-coalgebras in a cocomplete category $\mathcal{C}$ are nothing but the restrictions of left adjoints $\mathbf{Alg}\mathcal{T} \to \mathcal{C}$ to $\mathcal{T}^{\mathsf{op}}$. This equivalence of the concepts of coalgebras and adjunctions can be perceived as a generalization of the Eilenberg-Watts Theorem, as will be shown below, and provides, for example, a very simple conceptual understanding of the familiar adjunctions between the *Restriction of scalars*-functor ${}_S\mathbf{Mod} \to {}_R\mathbf{Mod}$, the *Extension of scalars*-functor ${}_R\mathbf{Mod} \to {}_S\mathbf{Mod}$, and the *Coextension of scalars*-functor ${}_R\mathbf{Mod} \to {}_S\mathbf{Mod}$ with respect to a homomorphism $\phi\colon R \to S$ for not necessarily commutative rings R and S. We therefore start this note by presenting all of these descriptions in some detail and proving their equivalence.

Using standard categorical arguments, we provide the following improvements of the main results of the more recent paper [8] for the case of $\mathcal{C}$ being a variety: For every Lawvere theory $\mathcal{T}$ the category $\mathbf{Coalg}(\mathcal{T}, \mathcal{C})$ of $\mathcal{T}$-coalgebras in a locally presentable category $\mathcal{C}$ is a locally presentable category, hence complete in particular (see [8, Thm. 4.7]) and its underlying functor into $\mathcal{C}$ not only has a right adjoint (see [8, Thm. 4.9]) but even is comonadic which, in particular relates 3. and 1. (a) above for locally presentable categories. We show moreover that in the particular case, where in $\mathcal{C}$ for every monomorphism m the coproduct $m + m$ is a monomorphism, $\mathbf{Coalg}(\mathcal{T}, \mathcal{C})$ even is a covariety in the sense of [3]. The converse does not hold.

The canonical constructions of coalgebras in a variety mentioned above are functorial. In the case of a commutative variety $\mathcal{V}$ with Lawvere theory $\mathcal{T}_\mathcal{V}$ the canonical construction provides a full embedding of $\mathcal{V}$ into $\mathbf{Coalg}(\mathcal{T}_\mathcal{V}, \mathcal{V})$. Moreover, all canonical constructions are left adjoints. The respective right adjoints can be viewed as a generalization of both, the *group-like elements* and the *primitive elements* functors, known from Hopf algebra theory. We show by examples that these canonical constructions may or may not provide all coalgebras in a variety.

We add an Appendix containing arguments of a more algebraic flavor for some crucial results as well as couple of examples.

1 Preliminaries

A *signature* is a pair $\Sigma = (\bar{\Sigma}, \bar{\Sigma} \xrightarrow{ar} \mathbb{N})$ consisting of a set $\bar{\Sigma}$ and an *arity map* $\bar{\Sigma} \xrightarrow{ar} \mathbb{N}$. Equivalently, Σ is a family $(\Sigma_n)_n$ of sets. Correspondingly, we call the category $\mathbf{Set}^{\mathbb{N}}$ the *category of signatures*. By $\mathbf{Alg}\Sigma$ we denote the category of Σ-algebras in the sense of universal algebra. Denoting by $T(\Sigma)_n$ the set of n-ary Σ-terms in the sense of universal algebra (see e.g. [24]) one gets a signature $T(\Sigma)$. The interpretation of a term $t \in T(\Sigma)_n$ on a Σ-algebra A will be denoted by $A^n \xrightarrow{t^{\mathsf{A}}} A$. Since the familiar definitions of t^{A} are either given element-wise or use freeness of the algebra of Σ-terms and none of these methods works for arbitrary internal algebras in a category $\mathcal{C}$ with finite products, Freyd [15] suggested an inductive definition using products, which however has some shortcomings as already observed in the review [7].

We therefore use the following inductive definition of the interpretation $A^n \xrightarrow{t^{\mathsf{A}}} A$ of a Σ-term $t \in T(\Sigma)_n$ in a Σ-algebra A in $\mathcal{C}$:

1. $x_j^{\mathsf{A}} := A^n \xrightarrow{\pi_j} A$ is the j^{th} product projection, for the variables $x_1, \dots x_n$.
2. $t^{\mathsf{A}} := A^n \xrightarrow{\langle (t_i^{\mathsf{A}})_i \rangle} A^{ar(\tau)} \xrightarrow{\tau^{\mathsf{A}}} A$, if $t = \tau(t_1, \cdots, t_{ar(\tau)})$ with $t_i \in T(\Sigma)_n$, where $\langle (t_i^{\mathsf{A}})_i \rangle$ denotes the map with coordinates t_i^{A}. In particular
   (a) $\sigma_n^{\mathsf{A}} := A^n \xrightarrow{!} 1 \xrightarrow{\sigma^{\mathsf{A}}} A$ for each $t = \sigma \in \Sigma_0$, where ! denotes the unique morphism,
   (b) $\sigma(x_1, \dots, x_n)^{\mathsf{A}} = A^n \xrightarrow{\sigma^{\mathsf{A}}} A$ for each $\sigma \in \Sigma_n$.

Since every homomorphism $\mathsf{A} \xrightarrow{f} \mathsf{A}'$ satisfies $t^{\mathsf{A}'} \circ f^n = f \circ t^{\mathsf{A}}$ for every $t \in T(\Sigma)_n$, every such t determines a natural transformation $\tilde{t} \colon |-|^n \Rightarrow |-|$ with components $\tilde{t}_{\mathsf{A}} = t^{\mathsf{A}}$, where $\mathbf{Alg}\Sigma \xrightarrow{|-|} \mathbf{Set}$ denotes the forgetful functor. A *set of Σ-equations* is a family $\mathfrak{E} = (\mathfrak{E}_n)_n$ with $\mathfrak{E}_n \subset T(\Sigma)_n \times T(\Sigma)_n$. As usual we denote pairs $(s,t) \in \mathfrak{E}$ by $s = t$. A Σ-algebra A is said to *satisfy the equation* $s = t$ iff $t^{\mathsf{A}} = s^{\mathsf{A}}$. $\mathbf{Alg}(\Sigma, \mathfrak{E})$ denotes the full subcategory of $\mathbf{Alg}\Sigma$ of algebras satisfying all equations in $\mathfrak{E}$ and is called a *variety*.

Writing $t = \tau(t_1, \dots t_m)$ with $\tau \in \Sigma_m$ and $t_i \in T(\Sigma)_k$ and $s = \sigma(s_1, \dots s_n)$ with $\sigma \in \Sigma_n$ and $s_j \in T(\Sigma)_k$ a Σ-algebra A satisfies $s = t$ if and only if the following diagram commutes.

$$\begin{array}{ccc} A^k & \xrightarrow{\langle (t_i^{\mathsf{A}})_i \rangle} & A^m \\ {\scriptstyle \langle (s_j^{\mathsf{A}})_j \rangle}\downarrow & & \downarrow{\scriptstyle \tau^{\mathsf{A}}} \\ A^n & \xrightarrow[\sigma^{\mathsf{A}}]{} & A \end{array} \tag{1}$$

Satisfaction of an equational axiom $s = t$, thus, is the requirement for Diagram (1) to commute. Since the paths of this diagram are the components of the natural transformations $\tilde{t}$ and $\tilde{s}$, the variety $\mathbf{Alg}(\Sigma, \mathfrak{E})$ is – in the terminology of [4] – the *joint equifier* $\mathbf{Equ}(\tilde{s}, \tilde{t})_{s=t \in \mathfrak{E}}$.

The categorification of these notions is due to Lawvere [22], as is well known: Pairs $(\Sigma, \mathfrak{E})$ are substituted by *Lawvere theories* $\mathcal{T}$ and varieties by the full subcategories $\mathbf{Alg}\mathcal{T}$ of $\mathbf{Set}^{\mathcal{T}}$ consisting of all finite product preserving functors. There is the canonical forgetful functor $\mathbf{Alg}\mathcal{T} \xrightarrow{|-|} \mathbf{Set}$ given by evaluation at $1 \in \mathsf{ob}\mathcal{T}$. This functor creates limits.

We recall the following basic facts of this categorification as follows, introducing some notations used in this note at the same time.

I. Lawvere theories form, together with finite product preserving functors as morphisms, the category **Lawv**. The dual of a skeleton of the category of finite sets is a Lawvere theory $\mathcal{N}$. This is the initial object of **Lawv**. For any Lawvere theory $\mathcal{T}$ we denote by T the unique **Lawv**-morphism $\mathcal{N} \to \mathcal{T}$. There is a forgetful functor $\mathbf{Lawv} \to \mathbf{Set}^{\mathbb{N}}$ acting as $\mathcal{T} \to \mathcal{T}(n,1) =: \Sigma_{\mathcal{T}}$. This functor has a left adjoint. The *free theory* $\mathcal{T}_\Sigma$ *over a signature* Σ has as its morphism sets $\mathcal{T}_\Sigma(n,1)$ the sets $T(\Sigma)_n$ (see [6]); in other words $\Sigma_{\mathcal{T}_\Sigma} = T(\Sigma)$. There is a concrete isomorphism

$$\mathbf{Alg}\mathcal{T}_\Sigma \simeq \mathbf{Alg}\Sigma. \tag{2}$$

II. There exists a dual biequivalence between Lawvere theories and varieties. In more detail (see e.g. [6]):
   (a) For every variety $\mathcal{V} = \mathbf{Alg}(\Sigma, \mathfrak{E})$ there exists an essentially unique Lawvere theory $\mathcal{T}_\mathcal{V}$ such that $\mathcal{V}$ is concretely equivalent to $\mathbf{Alg}\mathcal{T}_\mathcal{V}$. $\mathcal{T}_\mathcal{V}$ is the dual of any full subcategory $\langle F_\mathcal{V} n \mid n \in \mathbb{N}\rangle$ of $\mathcal{V}$ formed by choosing, for each $n \in \mathbb{N}$, a free $\mathcal{V}$-algebra $F_\mathcal{V} n$ in such a way that $F_\mathcal{V} n \neq F_\mathcal{V} m$ for $n \neq m$. Equivalently, $\mathcal{T}_\mathcal{V}(n,1)$ is the set of natural transformations $|-|^n \Rightarrow |-|$. In particular, $\mathbf{Alg}\mathcal{N} \simeq \mathbf{Set}$.
   (b) For every Lawvere theory $\mathcal{T}$ the category $\mathbf{Alg}\mathcal{T}$ is concretely equivalent to some variety $\mathcal{V} = \mathbf{Alg}(\Sigma, \mathfrak{E})$, equivalently, $\mathcal{T}$ is isomorphic to $\mathcal{T}_\mathcal{V}$ for some $\mathcal{V}$.
   (c) For a morphism $\mathcal{S} \xrightarrow{\Phi} \mathcal{T}$ of Lawvere theories $\Phi^* \colon \mathbf{Alg}\mathcal{T} \to \mathbf{Alg}\mathcal{S}$ denotes the functor given by $(\mathcal{T} \xrightarrow{\mathsf{A}} \mathbf{Set}) \mapsto (\mathcal{S} \xrightarrow{\Phi} \mathcal{T} \xrightarrow{\mathsf{A}} \mathbf{Set})$, called the *algebraic functor induced by* Φ. This is a concrete functor admitting a left adjoint Φ_*, and every concrete functor $\mathcal{V} \xrightarrow{U} \mathcal{W}$ between varieties determines a morphism in **Lawv** by restricting its left adjoint to a functor $\langle F_\mathcal{W} n \mid n \in \mathbb{N}\rangle \to \langle F_\mathcal{V} n \mid n \in \mathbb{N}\rangle$.

   In particular, for any Lawvere theory $\mathcal{T}$, $\mathbf{Alg}\mathcal{T} \xrightarrow{T^*} \mathbf{Alg}\mathcal{N} \simeq \mathbf{Set}$ is the forgetful functor, and $F_\mathcal{T} := T_* = \mathbf{Set} \to \mathbf{Alg}\mathcal{T}$ the *free algebra functor*.

III. Every Lawvere theory $\mathcal{T}$ is a regular quotient of a free one. In more detail: If $\mathbf{Alg}\mathcal{T}$ is concretely isomorphic to the variety $\mathbf{Alg}(\Sigma, \mathfrak{E})$, then there exists a quotient morphism $\mathcal{T}_\Sigma \xrightarrow{[-]} \mathcal{T}$ in **Lawv**. This functor identifies terms s and t, whenever (s,t) is deducible from $\mathfrak{E}$, that is, if every algebra in $\mathbf{Alg}(\Sigma, \mathfrak{E})$ satisfies the equation (s,t) (see e.g. [6], [28]).

   Then $\mathbf{Alg}\mathcal{T} \xrightarrow{[-]^*} \mathbf{Alg}\mathcal{T}_\Sigma$ is a full embedding and $\mathbf{Alg}\mathcal{T}$ is a *Birkhoff subcategory* of $\mathbf{Alg}\mathcal{T}_\Sigma$, that is, $\mathbf{Alg}\mathcal{T}$ is closed in $\mathbf{Alg}\mathcal{T}_\Sigma$ under products, sub-

algebras, and (retract carried) quotients. Moreover, each such subcategory $\mathbf{Alg}\mathcal{T}_\Sigma$ is equivalent to some $\mathbf{Alg}\mathcal{T}'$ for a regular quotient $\mathcal{T}'$ of $\mathcal{T}_\Sigma$.

IV. For every algebraic theory $\mathcal{T}$, the category $\mathbf{Alg}\mathcal{T}$ is free completion of $\mathcal{T}^{\mathsf{op}}$ under sifted colimits. In more detail this is (see [6, Theorem 4.13, Remark 4.14]): If $\mathcal{C}$ is a category with sifted colimits then for every functor $\mathsf{A}\colon \mathcal{T}^{\mathsf{op}} \to \mathcal{C}$ there exists an essentially unique sifted colimit preserving functor $K_\mathsf{A}\colon \mathbf{Alg}\mathcal{T} \to \mathcal{C}$ which extends A. The assignment $\mathsf{A} \mapsto K_\mathsf{A}$ provides an equivalence between the functor category $[\mathcal{T}^{\mathsf{op}}, \mathcal{C}]$ and the category of sifted colimit preserving functors from $\mathbf{Alg}\mathcal{T}$ to $\mathcal{C}$.
Moreover, if $\mathcal{C}$ is cocomplete and A preserves finite coproducts, then K_A is a left adjoint of the functor $H_\mathsf{A}\colon \mathcal{C} \to \mathbf{Alg}\mathcal{T}$ with $H_\mathsf{A}C = \mathcal{C}(\mathsf{A}-, C)$.

Given an arbitrary category $\mathcal{C}$ with finite products one defines categories $\mathbf{Alg}(\mathcal{T}, \mathcal{C})$, $\mathbf{Alg}(\Sigma, \mathcal{C})$ and $\mathbf{Alg}((\Sigma, \mathfrak{E}), \mathcal{C})$ of *internal algebras in* $\mathcal{C}$, equipped with underlying functors $|-|_\mathcal{C}$, by replacing in the definitions of $\mathbf{Alg}\Sigma$, $\mathbf{Alg}(\Sigma, \mathfrak{E})$ and $\mathbf{Alg}\mathcal{T}$, respectively, "set" by "$\mathcal{C}$-object" and "map" by "$\mathcal{C}$-morphism". Most of the above then can immediately be generalized to internal algebras in an arbitrary category $\mathcal{C}$ with finite products as, in particular, the following fact.

Fact 1 If $\mathcal{T}_\mathcal{V}$ is the Lawvere theory of a variety $\mathcal{V} = \mathbf{Alg}(\Sigma, \mathfrak{E})$, then $\mathbf{Alg}(\mathcal{T}_\mathcal{V}, \mathcal{C})$ is concretely equivalent over $\mathcal{C}$ to the full subcategory $\mathbf{Equ}(\tilde{s}, \tilde{t})_{s=t\in\mathfrak{E}}$ of the category $\mathbf{Alg}(\Sigma, \mathcal{C})$.

However, unlike to the case of $\mathcal{C} = \mathbf{Set}$, none of the following is true in general.

i. The forgetful functor $\mathbf{Alg}(\mathcal{T}, \mathcal{C}) \xrightarrow{|-|_\mathcal{C}} \mathcal{C}$ has a left adjoint.
ii. The functors $\Phi^*_\mathcal{C}$ (see II (c) above) have left adjoints.
iii. Every natural transformation $|-|^n_\mathcal{C} \Rightarrow |-|_\mathcal{C}$ is determined by some $t \in \mathcal{T}(n, 1)$.
iv. The correspondence $\mathcal{T} \mapsto \mathbf{Alg}(\mathcal{T}, \mathcal{C})$ is essentially one-to-one.
v. Every Birkhoff subcategory of $\mathbf{Alg}(\mathcal{T}_\Sigma, \mathcal{C})$ is equivalent to some $\mathbf{Alg}(\mathcal{T}', \mathcal{C})$ for a regular quotient $\mathcal{T}'$ of $\mathcal{T}_\Sigma$.

Statements i. and ii. hold, provided that $\mathcal{C}$ is locally finitely presentable; in this case $\mathbf{Alg}(\mathcal{T}, \mathcal{C})$ is locally finitely presentable (see [25], [18], [21]). In particular, there is a free algebra functor $\mathcal{C} \to \mathbf{Alg}(\mathcal{T}, \mathcal{C})$.

2 Internal coalgebras

2.1 Coalgebras for a variety

The following approach has been taken by Kan [19] as early as 1958 in the special situation, where $\mathcal{C} = \mathbf{Mon}$, the category of monoids and $\mathcal{V} = \mathbf{Grp}$, the variety of groups. It is the first appearance of this concept known to us.

Definition 2 Let $\mathcal{C}$ be a category with finite coproducts. For every signature Σ and every set of Σ-equations $\mathfrak{E}$ the category of *internal* $(\Sigma, \mathfrak{E})$*-coalgebras in* $\mathcal{C}$ is the category $\mathbf{Coalg}((\Sigma, \mathfrak{E}), \mathcal{C}) := \mathbf{Alg}((\Sigma, \mathfrak{E}), \mathcal{C}^{\mathsf{op}})^{\mathsf{op}}$. With $\mathfrak{E} = \emptyset$, one gets $\mathbf{Coalg}(\Sigma, \mathcal{C}) = \mathbf{Alg}(\Sigma, \mathcal{C}^{\mathsf{op}})^{\mathsf{op}}$. If $\mathcal{V} = \mathbf{Alg}(\Sigma, \mathfrak{E})$ is the variety presented by $(\Sigma, \mathfrak{E})$ we also write $\mathbf{Coalg}(\mathcal{V}, \mathcal{C})$ instead of $\mathbf{Coalg}((\Sigma, \mathfrak{E}), \mathcal{C})$.

Every $t \in \mathcal{T}_\Sigma(n, 1)$ determines the natural transformation $\hat{t} := \tilde{t}^{\mathsf{op}} \colon ||-|| \Rightarrow n \cdot ||-||$ from the forgetful functor $|-|^{\mathsf{op}} = \mathbf{Coalg}(\Sigma, \mathcal{C}) \xrightarrow{||-||} \mathcal{C}$ into its n-fold copower. By dualization of Fact 1 one obtains

Fact 3 $\mathbf{Coalg}((\Sigma, \mathfrak{E}), \mathcal{C})$ is the joint equifier $\mathbf{Equ}(\hat{s}, \hat{t})_{s=t \in \mathfrak{E}}$ in $\mathbf{Coalg}(\Sigma, \mathcal{C})$.

If $\mathcal{C}$ also has products this leads to a description of $\mathbf{Coalg}((\Sigma, \mathfrak{E}), \mathcal{C})$ as a category of functor coalgebras. Considering the functor $Q_\Sigma \colon \mathcal{C} \to \mathcal{C}$ with $X \mapsto \prod_{n \in \mathbb{N}} (n \cdot X)^{|\Sigma_n|}$, the category $\mathbf{Coalg}Q_\Sigma$ of functor coalgebras for this functor is concretely isomorphic to $\mathbf{Coalg}(\Sigma, \mathcal{C})$, and we may consider the natural transformations $\hat{s}$ and $\hat{t}$ as natural transformations $|| - || \Rightarrow n \cdot || - ||$, where now $\mathbf{Coalg}Q_\Sigma \xrightarrow{||-||} \mathcal{C}$ is the forgetful functor. Then $\mathbf{Coalg}((\Sigma, \mathfrak{E}), \mathcal{C})$ is the joint equifier $\mathbf{Equ}(\hat{s}, \hat{t})_{s=t \in \mathfrak{E}}$ in $\mathbf{Coalg}Q_\Sigma$.

Adopting the language of universal algebra, a Σ-coalgebra in $\mathcal{C}$ is a pair $\mathsf{A} = (A, (A \xrightarrow{\sigma_\mathsf{A}} n \cdot A)_{\sigma \in \Sigma_n})$, where A is a $\mathcal{C}$-object and $A \xrightarrow{\sigma_\mathsf{A}} n \cdot A$ is a $\mathcal{C}$-morphism, called the *n-ary co-operation determined by* $\sigma \in \Sigma_n$. A $(\Sigma, \mathfrak{E})$-coalgebra in $\mathcal{C}$ then is a Σ-coalgebra in $\mathcal{C}$ making all Diagrams (1) in $\mathcal{C}^{\mathsf{op}}$ commute. This is often expressed by saying that it *satisfies duals of the equational axioms* defining $\mathcal{V}$. Denoting by $[((t_i)_\mathsf{A})_i)]$ denotes the morphism with components $(t_i)_\mathsf{A} := (\hat{t}_i)_\mathsf{A}$ this means explicitly that each of the following diagrams commutes in $\mathcal{C}$.

$$\begin{array}{ccc} A & \xrightarrow{\sigma_\mathsf{A}} & n \cdot A \\ {\scriptstyle \tau_\mathsf{A}}\downarrow & & \downarrow{\scriptstyle [((s_j)_\mathsf{A})_j]} \\ m \cdot A & \xrightarrow[[((t_i)_\mathsf{A})_i]]{} & k \cdot A \end{array}$$

2.2 Coalgebras for a Lawvere theory

A more elegant way to define coalgebras in a category $\mathcal{C}$ with finite coproducts is the following.

Definition 4 Let $\mathcal{C}$ be a category with finite coproducts. For every Lawvere theory $\mathcal{T}$ the category of *internal $\mathcal{T}$-coalgebras in $\mathcal{C}$* is the category $\mathbf{Coalg}(\mathcal{T}, \mathcal{C}) = \mathbf{Alg}(\mathcal{T}, \mathcal{C}^{\mathsf{op}})^{\mathsf{op}}$.

A *$\mathcal{T}$-coalgebra in $\mathcal{C}$* thus is a finite coproduct preserving functor $\mathcal{T}^{\mathsf{op}} \xrightarrow{\mathsf{A}} \mathcal{C}$ and a morphism $\mathsf{A} \to \mathsf{B}$ of $\mathcal{T}$-coalgebras is a natural transformation $\mu \colon \mathsf{A} \Rightarrow \mathsf{B}$.

$||-|| := \mathbf{Coalg}(\mathcal{T},\mathcal{C}) \xrightarrow{|-|_{\mathcal{C}^{\mathsf{op}}}} \mathcal{C}$ denotes the *forgetful functor*. By duality this functor is faithful, reflects isomorphisms and creates colimits. By item 1 of Section 1 and Equation (2) one has $\mathbf{Coalg}(\Sigma,\mathcal{C}) \simeq \mathbf{Coalg}(\mathcal{T}_\Sigma,\mathcal{C})$. By Fact 1 of Section 1 one obtains

Lemma 5 *Let $\mathcal{T}$ be the Lawvere theory of the variety $\mathcal{V} = \mathbf{Alg}(\Sigma,\mathfrak{E})$ and $\mathcal{C}$ a category with finite coproducts. There is a concrete equivalence*

$$\mathbf{Coalg}(\mathcal{T},\mathcal{C}) \simeq \mathbf{Coalg}((\Sigma,\mathfrak{E}),\mathcal{C})$$

2.3 Coalgebras as representable functors

Definition 6 Given a Lawvere theory $\mathcal{T}$ and $\mathcal{C}$ a category with finite coproducts. A functor $\mathcal{C} \xrightarrow{H} \mathbf{Alg}\mathcal{T}$ is called *$\mathcal{T}$-representable (on $\mathcal{C}$) by $C \in \mathsf{ob}\mathcal{C}$* or an *$\mathbf{Alg}\mathcal{T}$-lift of $\mathcal{C}(C,-)$*, if $\mathcal{C} \xrightarrow{H} \mathbf{Alg}\mathcal{T} \xrightarrow{|-|} \mathbf{Set} = \mathcal{C} \xrightarrow{\mathcal{C}(C,-)} \mathbf{Set}$, that is, if this composition is a representable functor in the usual sense.

$\mathbf{Rep}(\mathcal{T},\mathcal{C})$ denotes the category of $\mathcal{T}$-representable functors on $\mathcal{C}$ with all natural transformations as its morphisms.

$\mathbf{Rep}(\mathcal{T},\mathcal{C})^{\mathsf{op}}$ is concrete over $\mathcal{C}$ by means of the faithful functor $\{-\}_{\mathcal{T}}$ given by $(H \xrightarrow{\mu} H') \mapsto (C' \xrightarrow{f} C)$, if $|\mu| = \mathcal{C}(f,-)$.

Occasionally we will in an obvious way use the notion of $\mathcal{V}$-lifts and $\mathcal{V}$-representable functors, where $\mathcal{V}$ is a variety.

It is essentially well known (see e.g. [23, Chapter III.6] or [9, Chapter 10]), that the $\mathbf{Alg}\mathcal{T}$-lifts of $\mathcal{C}(C,-)$ correspond essentially one-to-one to the $\mathcal{T}$-coalgebra structures on C:

If $\mathcal{T}^{\mathsf{op}} \xrightarrow{\mathsf{A}} \mathcal{C}$ is a $\mathcal{T}$-coalgebra in $\mathcal{C}$ then, for each C in $\mathcal{C}$, the functor

$$\mathcal{C}(\mathsf{A}-,C) := \mathcal{T} \xrightarrow{\mathsf{A}^{\mathsf{op}}} \mathcal{C}^{\mathsf{op}} \xrightarrow{\mathcal{C}^{\mathsf{op}}(C,-)=\mathcal{C}(-,C)} \mathbf{Set} \tag{3}$$

preserves finite products and so belongs to $\mathbf{Alg}\mathcal{T}$. Moreover, this construction is functorial and the resulting functor $\mathcal{C} \xrightarrow{H_{\mathsf{A}}} \mathbf{Alg}\mathcal{T}$ with $H_{\mathsf{A}}C = \mathcal{C}(\mathsf{A}-,C)$ satisfies $|H_{\mathsf{A}}C| = H_{\mathsf{A}}C(1) = \mathcal{C}(\mathsf{A}1,C)$, that is, H_{A} is $\mathcal{T}$-representable with representing object $\mathsf{A}1$. Conversely, if $\mathcal{C} \xrightarrow{H} \mathbf{Alg}\mathcal{T}$ is $\mathcal{T}$-representable with representing object A, then $A = \mathsf{A}1$ for the $\mathcal{T}$-coalgebra $\mathcal{T}^{\mathsf{op}} \xrightarrow{\mathsf{A}} \mathcal{C}$ which maps any $\tau \in \mathcal{T}(n,1)$, that is, any natural transformation $\tau\colon |-|^n \Rightarrow |-|$ to the $\mathcal{C}$-morphism $A \xrightarrow{\bar{\tau}} n \cdot A$ corresponding by the Yoneda lemma to the natural transformation $\tau_H\colon \mathcal{C}(n \cdot A,-) = |H-|^n \Rightarrow |H-| = \mathcal{C}(A,-)$. We so have proved the following lemma.

Lemma 7 *For every Lawvere theory $\mathcal{T}$ and every category $\mathcal{C}$ with finite coproducts the assignment $\mathsf{A} \mapsto H_{\mathsf{A}}$ defines a concrete equivalence of the categories $\mathbf{Coalg}(\mathcal{T},\mathcal{C})$ and $\mathbf{Rep}(\mathcal{T},\mathcal{C})^{\mathsf{op}}$, where the underlying $\mathcal{C}$-object of a coalgebra A corresponds to the representing object of H_{A}.*

Remark 8 A representable functor $\mathcal{C}(C,-)$ may have non-isomorphic $\mathbf{Alg}\mathcal{T}$-lifts $H, H' \colon \mathcal{C} \to \mathbf{Alg}\mathcal{T}$, equivalently, a $\mathcal{C}$-object may carry more than one structure of a $\mathcal{T}$-coalgebra.
Denoting by **Ring** and **Mon** the varieties of unital rings and monoids, respectively, there are for example the functors $\Phi, \Psi \colon \mathbf{Ring} \to \mathbf{Mon}$ mapping a unital ring R to its additive monoid $(R,+,0)$ and its multiplicative monoid $(R,\times,1)$, respectively. These are **Mon**-lifts of the representable functor $\mathbf{Ring}(\mathbb{Z}[x],-)$, which is up to equivalence the forgetful functor of **Ring**. The functors $\Phi', \Psi' \colon \mathbf{Ring} \to \mathbf{Ring}$, where Φ' is the identity and Ψ' maps a ring to its opposite are **Ring**-lifts of $\mathbf{Ring}(\mathbb{Z}[x],-)$.

2.4 Coalgebras as adjunctions

Every functor $\mathcal{C} \xrightarrow{H} \mathbf{Alg}\mathcal{T}$ admitting a left adjoint K is $\mathcal{T}$-representable with representing object $KF1$, where $F1$ denotes the free $\mathcal{T}$-algebra over a singleton. The converse has been claimed to hold for every cocomplete category $\mathcal{C}$ in [15], but proved only for the case of $\mathcal{C}$ being a variety. This is easily proved by applying Dubuc's adjoint triangle theorem [14] or [12, Thm. 4.5.6], which was not yet available to Freyd: If $\mathcal{C}$ has copowers, the functor $\mathcal{C}(\mathsf{A}1,-)$ has a left adjoint; since $|-|$ is monadic the hypotheses of this theorem are satisfied, provided that $\mathcal{C}$ also has reflexive coequalizers. Hence, $|-| \circ H_{\mathsf{A}} = \mathcal{C}(\mathsf{A}1,-)$ implies the existence of a left adjoint K_{A} of H_{A}.

An algebraic proof of this fact has been given in [9, Thm. 10.4.3], where for every algebra A in a variety $\mathcal{V}$ a $\mathcal{C}$-object $K\mathsf{A}$ is constructed such that there is a bijection $\mathcal{C}(K\mathsf{A},C) \simeq \mathcal{V}(\mathsf{A},HC)$ for each C in $\mathcal{C}$. For the sake of the reader not too familiar with the categorical methods used here we provide a simplification and correction of that proof in the Appendix.

Most elegantly one obtains this and more by means of the fact that $\mathbf{Alg}\mathcal{T}$ is the free completion of $\mathcal{T}^{\mathsf{op}}$ under sifted colimits, which implies (see item IV of Section 1) that, for any cocomplete category $\mathcal{C}$ and any Lawvere theory $\mathcal{T}$, the $\mathcal{T}$-coalgebras in $\mathcal{C}$ are nothing but the restrictions of left adjoints $\mathbf{Alg}\mathcal{T} \xrightarrow{K} \mathcal{C}$ to $\mathcal{T}^{\mathsf{op}}$, considered as a subcategory of $\mathbf{Alg}\mathcal{T}$, and the $\mathcal{T}$-representable functors on $\mathcal{C}$ are the right adjoints of those functors K. In other words

Theorem 9 *For every Lawvere theory $\mathcal{T}$ and every cocomplete category $\mathcal{C}$ there are concrete equivalences*

$$\mathbf{Coalg}(\mathcal{T},\mathcal{C}) \simeq \mathbf{Rep}(\mathcal{T},\mathcal{C})^{\mathsf{op}} \simeq \mathbf{Adj}(\mathbf{Alg}\mathcal{T},\mathcal{C})$$

where $\mathbf{Adj}(\mathbf{Alg}\mathcal{T},\mathcal{C})$ is the category of adjunctions from $\mathbf{Alg}\mathcal{T}$ to $\mathcal{C}$.

Note that the category of adjunctions $\mathbf{Adj}(\mathbf{Alg}\mathcal{T},\mathcal{C})$ in the sense of [23] is equivalent to the category of left adjoints from $\mathbf{Alg}\mathcal{T}$ to $\mathcal{C}$ and is a concrete category over $\mathcal{C}$ by means of the functor mapping a natural transformation $K \xrightarrow{\mu} K'$ between left adjoints to the $\mathcal{C}$-morphism $K(F1) \xrightarrow{\mu_{F1}} K'(F1)$ with $F1$ the free $\mathcal{T}$-algebra on one generator.

2.5 Properties of the categories of coalgebras

Let $\mathcal{C}$ be a category with finite coproducts.

1. If $\mathcal{T}$ is a regular quotient of $\mathcal{T}_\Sigma$, then $\mathbf{Coalg}(\mathcal{T},\mathcal{C})$ is a *co-Birkhoff subcategory* of $\mathbf{Coalg}(\mathcal{T}_\Sigma,\mathcal{C})$, that is, $\mathbf{Coalg}(\mathcal{T},\mathcal{C})$ is closed in $\mathbf{Coalg}(\mathcal{T}_\Sigma,\mathcal{C})$ under
   (a) coproducts,
   (b) epimorphic quotients, and
   (c) coretract carried subalgebras.
2. Dually to the definition of Φ^*, for every **Lawv**-morphism $\mathcal{S} \xrightarrow{\Phi} \mathcal{T}$ and for every category $\mathcal{C}$ with finite coproducts, there is a faithful functor
$$\widetilde{\Phi^*_{\mathcal{C}}} := (\Phi^*_{\mathcal{C}^{\mathsf{op}}})^{\mathsf{op}} \colon \mathbf{Coalg}(\mathcal{T},\mathcal{C}) \to \mathbf{Coalg}(\mathcal{S},\mathcal{C})$$
   This functor commutes with the underlying functors into $\mathcal{C}$. It preserves colimits since the underlying functors create those.
   Considering coalgebras as representable functors this functor acts as $H \mapsto \Phi^* \circ H$; in the language of Section 2.1 it acts as
$$\widetilde{\Phi^*_{\mathcal{C}}}(\mathsf{A}) = (A, (\Phi(\tau)_{\mathsf{A}})_{\tau\in\mathcal{S}(n,1)})$$
3. Let $\mathcal{T}$ be a Lawvere theory. Then every coproduct preserving functor $\mathcal{C} \xrightarrow{S} \mathcal{D}$ between cocomplete categories maps coalgebras to coalgebras, that is, S can be lifted to a functor $\mathbf{Coalg}(\mathcal{T},\mathcal{C}) \xrightarrow{\hat{S}} \mathbf{Coalg}(\mathcal{T},\mathcal{D})$ by mapping a $\mathcal{T}$-coalgebra A to $S \circ \mathsf{A}$. In the language of Section 2.1, $\hat{S}$ acts as
$$\hat{S}(\mathsf{A}) = \big(S(A), (S(A) \xrightarrow{S(\sigma_{\mathsf{A}})} S(n\cdot A) \simeq n\cdot S(A))_{\sigma\in\mathcal{T}(n,1)}\big)$$
   $\hat{S}$ preserves all colimits which are preserved by S.

Theorem 10 *Let $\mathcal{T}$ be a Lawvere theory and $\mathcal{C}$ a locally presentable category. Then the following hold:*

1. $\mathbf{Coalg}(\mathcal{T},\mathcal{C})$ *is locally presentable.*
2. *Every hom-functor* $\mathbf{Coalg}(\mathcal{T},\mathcal{C}) \to \mathbf{Set}$ *has a left adjoint.*
3. *For every* $\mathcal{S} \xrightarrow{\Phi} \mathcal{T}$ *the functor* $\widetilde{\Phi^*_{\mathcal{C}}} \colon \mathbf{Coalg}(\mathcal{T},\mathcal{C}) \to \mathbf{Coalg}(\mathcal{S},\mathcal{C})$ *has a right adjoint* $\underline{\Phi_{\mathcal{C}}}$.
4. $\mathbf{Coalg}(\mathcal{T},\mathcal{C}) \xrightarrow{||-||} \mathcal{C}$ *has a right adjoint* $C_{\mathcal{T},\mathcal{C}}$ *and, moreover, is comonadic.*
5. *The functor* $\mathbf{Coalg}(\mathcal{T},\mathcal{C}) \xrightarrow{\hat{S}} \mathbf{Coalg}(\mathcal{T},\mathcal{D})$ *of item 3 above has a right adjoint* $\underline{S}$, *provided that* S *preserves colimits.*

Proof: The first statement follows from the equivalence $\mathbf{Coalg}(\mathcal{T},\mathcal{C}) \simeq \mathbf{Adj}(\mathbf{Alg}\mathcal{T},\mathcal{C})$, since for locally presentable categories $\mathcal{C}$ and $\mathcal{D}$ the category $\mathbf{Adj}(\mathcal{C},\mathcal{D})$ is locally presentable (see [10]). For a more algebraic argument see the Appendix.

Item 2 is clear since locally presentable categories have coproducts.

The claimed adjunctions of the other statements follow from item 1.: Since the forgetful functors create colimits, the functors supposed to have right adjoints preserve those and, hence, have a right adjoint by (the dual of) the Special Adjoint Functor Theorem because their domains are locally presentable. Comonadicity follows by the Beck-Paré-Theorem: Assuming that $\mathcal{T}$ is a regular quotient of $\mathcal{T}_\Sigma$ the category $\mathbf{Coalg}(\mathcal{T},\mathcal{C})$, considered as a subcategory of $\mathbf{Coalg}Q_\Sigma$, is closed under absolute equalizers. □

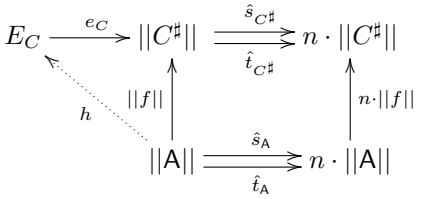
The next result relates the categories of internal coalgebras to covarieties in the sense of [3]. Certainly, without any further argument, $\mathbf{Coalg}(\mathcal{T},\mathcal{C})$ is a covariety of S-coalgebras for the functor $S = C_{\mathcal{T},\mathcal{C}} \circ ||-||$ by (the proof of) [3, 6.21].

Proposition 11 *Let the Lawvere theory $\mathcal{T}$ be a regular quotient of $\mathcal{T}_\Sigma$, for some signature Σ, and let $\mathcal{C}$ be a locally presentable category in which, for each monomorphism m, the morphism $m + m$ is a monomorphism as well. Then the category $\mathbf{Coalg}(\mathcal{T},\mathcal{C})$ is a covariety of Q_Σ-coalgebras.*

Proof: $\mathbf{Coalg}(\mathcal{T},\mathcal{C})$ is equivalent to some category $\mathbf{Coalg}((\Sigma,\mathfrak{E}),\mathcal{C})$ which is, in the notation of Section 2.1, the joint equifier $\mathbf{Equ}(\hat{s},\hat{t})_{s=t\in\mathfrak{E}}$, considered as a full subcategory of $\mathbf{Coalg}Q_\Sigma$. Thus, by item 4 of Theorem 10 $\mathbf{Coalg}(\mathcal{T},\mathcal{C})$ is a covariety of Q_Σ-coalgebras, if and only if it is a coequational category of Q_Σ-coalgebras in the sense of [3].

Define a class of coequation arrows as follows: Denote, for each C in $\mathcal{C}$, by $C^\sharp$ the cofree Q_Σ-coalgebra over C and form, for each $(s,t) \in \mathfrak{E}$, the equalizer $E_C \xrightarrow{e_C} ||C^\sharp||$ of the pair of $\mathcal{C}$-morphisms $\hat{s}_{C^\sharp}, \hat{t}_{C^\sharp} : ||C^\sharp|| \to n \cdot ||C^\sharp||$. Since Q_Σ is a constructive covarietor by [5, Thm. 2.6] each e_C is a coequation arrow. It remains to show that a Q_Σ-coalgebra A belongs to $\mathbf{Equ}(\hat{s},\hat{t})_{s=t\in\mathfrak{E}}$ if and only if for each C in $\mathcal{C}$ and for each morphism $\mathsf{A} \xrightarrow{f} C^\sharp$ in $\mathbf{Coalg}Q_\Sigma$ its underlying $\mathcal{C}$-morphism $||f||$ factors through e_C.

For each such f the squares of the following diagram commute by naturality of $\hat{s}$ and $\hat{t}$.



Consequently, by the equalizer property of e_C the dotted h exists making the triangle commute for each $\mathsf{A} \in \mathbf{Equ}(\hat{s},\hat{t})_{s=t\in\mathfrak{E}}$.

Conversely, if A allows for such an h, for each $\mathsf{A} \xrightarrow{f} C^\sharp$, it does so in particular for $\mathsf{A} \xrightarrow{\eta_\mathsf{A}} ||\mathsf{A}||^\sharp$, the unit of the adjunction $||-|| \dashv (-)^\sharp$. One obtains the equation $n \cdot ||\eta_A|| \circ \hat{s}_\mathsf{A} = \hat{s}_{||\mathsf{A}||^\sharp} \circ e_{||\mathsf{A}||} \circ h = \hat{t}_{||\mathsf{A}||^\sharp} \circ e_{||\mathsf{A}||} \circ h = n \cdot ||\eta_A|| \circ \hat{t}_\mathsf{A}$. Since $||\eta_\mathsf{A}||$ is a split monomorphism by definition, so is $n \cdot ||\eta_\mathsf{A}||$, such that $\hat{s}_\mathsf{A} = \hat{t}_\mathsf{A}$ follows. □

Examples 12 The additional assumption of the proposition above is satisfied

1. in every category with finite biproducts, hence in all (left or right) module categories $\mathbf{Mod}_R$, where R is a (not necessarily commutative) unital ring,
2. in **Grp**, the category of groups, and in **Mon**, the category of monoids by the construction of coproducts in these categories,
3. in ${}_c\mathbf{Alg}_R$, the category of commutative R-algebras for a commutative ring R if and only if R is a von Neumann regular ring. This follows from the facts that (a) the coproduct $A + B$ in ${}_c\mathbf{Alg}_R$ is given by $A \otimes B$ and (b) the implication m injective $\implies m \otimes m$ injective holds in ${}_c\mathbf{Alg}_R$ iff R is a von Neumann regular ring (see [27, p. 703]).

It follows from item 3 that this condition is not satisfied in the category of commutative unital rings ${}_c\mathbf{Ring} = {}_c\mathbf{Alg}_{\mathbb{Z}}$, since the ring of integers is not von Neumann regular.

Remark 13 We now can answer the following question in the negative: *Is every covariety over $\mathcal{C}$ in the sense of [3] a category of internal coalgebras?* Recall from [3, Thm. 6.21] that, for any complete category $\mathcal{C}$, this question is equivalent to the question whether every comonadic category over $\mathcal{C}$ is a category of internal coalgebras in $\mathcal{C}$. Consider the full subcategory **Tor** of **Ab** spanned by all commutative torsion groups. Its embedding into **Ab** has a right adjoint and is comonadic; hence **Tor** is a covariety over **Ab**.

Assume that $\mathbf{Tor} = \mathbf{Coalg}(\mathcal{T}, \mathbf{Ab})$ for some Lawvere theory $\mathcal{T}$, represented as a regular quotient of some $\mathcal{T}_\Sigma$, such that $\mathbf{Coalg}(\mathcal{T}, \mathbf{Ab})$ is equivalent to an equifier in the functor category $\mathbf{Coalg}Q_\Sigma$ according to Fact 3. Since **Ab** has biproducts the functor $\mathbf{Ab} \xrightarrow{Q_\Sigma} \mathbf{Ab}$ preserves products since $Q_\Sigma X = \prod_{n\in\mathbb{N}}(n \cdot X)^{|\Sigma_n|} = \prod_{n\in\mathbb{N}} X^{|n\Sigma_n|}$, such that the forgetful functor $\mathbf{Coalg}Q_\Sigma \to \mathbf{Ab}$ creates products and the equifier corresponding to $\mathbf{Coalg}(\mathcal{T}, \mathbf{Ab})$ is closed under products. Since **Tor** fails to be closed under products in **Ab** it cannot be of the form $\mathbf{Coalg}(\mathcal{T}, \mathbf{Ab})$.

2.6 Examples of coalgebras

1. The probably first example of cogroups in a category with finite coproducts $\mathcal{C}$, that is of **Grp**-coalgebras in $\mathcal{C}$, stems from homotopy theory (see [19],[15]). Let $\mathcal{C} = \mathbf{HTop}$ be the category of pointed topological space as objects and homotopy classes of continuous maps as morphisms. The coproduct of two pointed space (X, x_0) and (Y, y_0) in **HTop** is their wedge sum $X \vee Y$, the topological sum of X and Y with x_0 and y_0 identified and taken as its base point.
   By definition the n^{th} homotopy group $\pi_n(X, x_0)$ of a pointed space (X, x_0) has as its underlying set the hom-set $\mathbf{HTop}(S^n, X)$. Thus, the functor $\mathbf{HTop} \xrightarrow{\pi_n} \mathbf{Grp}$ is a **Grp**-representable functor with representing object S^n.

It is well known that, when collapsing the equator S^{n-1} of an n-sphere S^n to a point s_0, the quotient is homeomorphic to $S^n \vee S^n$. The cogroup structure (μ, s_0) on S^n corresponding to the functor π_n is given by the quotient map $S^n \xrightarrow{\mu} S^n \vee S^n$ together with the constant map with value s_0. Note that in **Top** the cogroup axioms will hold only up to homotopy.

2. Let $\mathcal{V}$ be the variety $\mathbf{Mod}_R$ of R-modules over a commutative ring R and M an R-module. The inner hom-functor $[M, -]\colon \mathbf{Mod}_R \to \mathbf{Mod}_R$ is $\mathcal{V}$-representable with representing object M and has a left adjoint $M \otimes -$. The corresponding coalgebra has as its underlying object the module M and the co-operation $M \xrightarrow{\sigma_M} n \cdot M$ corresponding to some $\sigma \in \mathcal{T}_{\mathcal{V}}(n, 1) = \mathcal{V}(F_{\mathcal{V}}1, F_{\mathcal{V}}n)$ is $M \simeq M \otimes F_{\mathcal{V}}1 \xrightarrow{M\otimes\sigma} M \otimes F_{\mathcal{V}}n \simeq n \cdot M$.
This obviously generalizes to an arbitrary commutative variety $\mathcal{V}$, that is, a variety whose theory $\mathcal{T}_{\mathcal{V}}$ is commutative, since commutative varieties are symmetric monoidal closed. For more of this see Section 3.2. For a generalization of a different kind, namely to adjunctions between module categories over not necessarily commutative rings see the discussion of bimodules in the Appendix.
3. Given a variety $\mathcal{V}$, the inclusion $\mathcal{T}_{\mathcal{V}}^{\mathsf{op}} = \langle F_{\mathcal{V}}n \mid n \in \mathbb{N}\rangle \hookrightarrow \mathcal{V}$ obviously is a $\mathcal{T}_{\mathcal{V}}$-coalgebra $\mathcal{V}[1]$ in $\mathcal{V}$ with $||\mathcal{V}[1]|| = F_{\mathcal{V}}1$. This already has been observed by Kan [19] and Freyd [15]. For more on this construction see Section 3.1 below.

3 Coalgebras in varieties

3.1 Canonical coalgebras on a free algebra

We start this section with the following observation, which follows from Lemma 7 since the $\mathcal{T}_{\mathcal{W}}$-representable functors H_{A} with with $||\mathsf{A}|| = F_{\mathcal{V}}1$, the free $\mathcal{V}$-algebra on one generator are precisely the algebraic functors $\mathcal{V} \to \mathcal{W}$.

Proposition 14 ([15]) *Let $\mathcal{V}$ and $\mathcal{W}$ be varieties. Then $\mathcal{T}_{\mathcal{W}}$-coalgebras on $F_{\mathcal{V}}1$ correspond one-to-one (up to isomorphism) to morphisms of Lawvere theories $\mathcal{T}_{\mathcal{W}} \to \mathcal{T}_{\mathcal{V}}$.*

Kan already had constructed comonoids in groups on each free group [19] before. We extend these observations as follows.

Defining $\mathcal{W}$-coalgebras in a variety $\mathcal{V}$ is equivalent to defining a right adjoint $\mathcal{V} \to \mathcal{W}$. The best known examples of such right adjoints are algebraic functors Φ^* induced by a morphism of Lawvere theories $\mathcal{T}_{\mathcal{W}} \xrightarrow{\Phi} \mathcal{T}_{\mathcal{V}}$. Since Φ^* commutes with the underlying functors, it is $\mathcal{W}$-representable with representing object $F_{\mathcal{V}}1$, the free $\mathcal{V}$-algebra on one generator. We denote this $\mathcal{W}$-coalgebra in $\mathcal{V}$ by $\mathcal{V}_\Phi[1]$. With notations as in item 3 of Section 2.6 and item 2 of Section 2.5 one has $\mathcal{V}_\Phi[1] = \widetilde{\Phi^*_{\mathcal{V}}}(\mathcal{V}[1])$.

Since $\mathbf{Coalg}(\mathcal{W}, \mathcal{V})$ has coproducts we can form the coalgebra $\mathcal{V}_\Phi[X] = X \cdot \mathcal{V}_\Phi[1]$ for each set X. Considered as an adjunction this is $X \cdot \Phi_* \dashv \Phi^{*X}$

with the $\mathcal{W}$-representable functor Φ^{*X} represented by $F_{\mathcal{V}}X$. The respective co-operations on $F_{\mathcal{V}}X$ are the $\mathcal{V}$-homomorphisms $F_{\mathcal{V}}X \xrightarrow{X\cdot\Phi(\sigma)} X\cdot F_{\mathcal{V}}n \simeq n\cdot F_{\mathcal{V}}X$ for all $\sigma \in \mathcal{T}(n,1) = \mathcal{W}(F_{\mathcal{W}}1, F_{\mathcal{W}}n)$ and $n \in \mathbb{N}$.

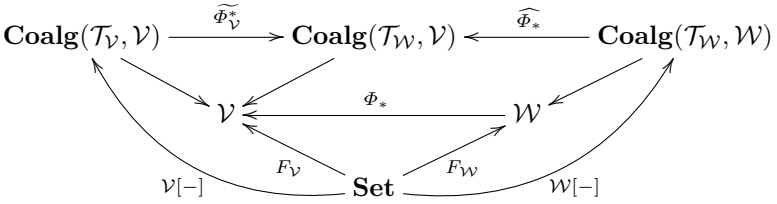
We note for further use that this co-operation acts on the free generators $x \in X$ of $F_{\mathcal{V}}X$ as $X \xrightarrow{\eta_X} F_{\mathcal{V}}X \xrightarrow{\langle\nu_k\rangle} (F_{\mathcal{V}}(n\cdot X))^n \xrightarrow{\Phi(\sigma)^{F_{\mathcal{V}}(n\cdot X)}} F_{\mathcal{V}}(n\cdot X) \simeq n\cdot F_{\mathcal{V}}X$, where $\langle\nu_k\rangle$ is the homomorphism whose coordinates are the coproduct injections $F_{\mathcal{V}}X \xrightarrow{\nu_k} n\cdot F_{\mathcal{V}}X$. In other words,

$$\sigma_{\mathcal{V}_\Phi[X]} \circ \eta_X = \Phi(\sigma)^{F_{\mathcal{V}}(n\cdot X)} \circ \langle\nu_k\rangle \circ \eta_X \tag{4}$$

We so have defined a functor $\mathcal{V}_\Phi[-]\colon \mathbf{Set} \to \mathbf{Coalg}(\mathcal{W},\mathcal{V})$. We will simply write $\mathcal{V}[-]$ instead of $\mathcal{V}[-]_{id}$. The following equivalences then are obvious.

$$\widetilde{\Phi^*_{\mathcal{V}}} \circ \mathcal{V}[-] \simeq \mathcal{V}_\Phi[-] \simeq \widehat{\Phi_*} \circ \mathcal{W}[-] \tag{5}$$

This situation is illustrated by the following diagram, where the unlabeled arrows are the forgetful functors. Here all cells obviously commute, while commutativity of the outer frame represents Equation (5).



(6)

Since the functor $\mathcal{V}_\Phi[-]$ is nothing but the functor mapping a set X to the X-fold copower $X\cdot\mathcal{V}_\Phi[1]$ one gets

Theorem 15 *For every morphism of Lawvere theories $\mathcal{T} \xrightarrow{\Phi} \mathcal{T}_{\mathcal{V}}$ the functor $\mathcal{V}_\Phi[-]$ is a left adjoint of* $\hom(\mathcal{V}_\Phi[1],-)\colon \mathbf{Coalg}(\mathcal{T},\mathcal{V}) \to \mathbf{Set}$.

Identifying $\mathbf{Coalg}(\mathcal{T},\mathcal{V})$ and $\mathbf{Adj}(\mathbf{Alg}\mathcal{T},\mathcal{C})$ the hom-set $\hom(\mathcal{V}_\Phi[1],\mathsf{A})$ is $\mathrm{nat}(\Phi_*,\mathsf{A})$, the set of natural transformations $\Phi_* \Rightarrow \mathsf{A}$, for each $\mathcal{T}$-coalgebra A. Since every natural transformation $(\mu_n)_n\colon \Phi_* \Rightarrow \mathsf{A}$ is uniquely determined by the value of its component μ_1 at the free generator 1 of $F1$, since $\mu_n = n\cdot\mu_1$, the free $\mathcal{T}$-algebra on a singleton, the set $\hom(\mathcal{V}_\Phi[1],\mathsf{A})$ can be identified with a subset of $A = \mathsf{A}(1)$. We provide a description of this set in terms of elements of coalgebra A as follows.

Let A be an internal $\mathcal{T}$-coalgebra in $\mathcal{V}$ with $||\mathsf{A}|| = A$. Denote, for $\tau \in \mathcal{T}(n,1)$, by τ_{A} the underlying map of the $\mathcal{V}$-homomorphism $\mathsf{A}\tau$, by $\Phi(\tau)^{n\cdot A}$ the interpretation of $\Phi\tau$ in the $\mathcal{V}$-algebra $n\cdot A$, and by $A \xrightarrow{\langle\nu_k\rangle} (n\cdot A)^n$ the homomorphism with coordinates the coproduct injections $\nu_k,\dots\nu_n\colon A \to n\cdot A$ in $\mathcal{V}$. Denote by $\mathcal{G}_\Phi\mathsf{A}$ the intersection of equalizers in $\mathbf{Set}$

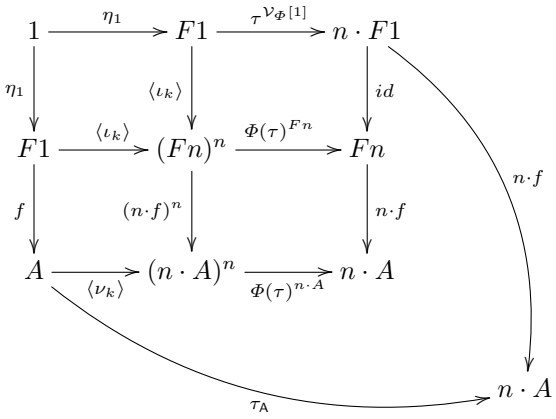
$$\mathcal{G}_\Phi\mathsf{A} \;=\; \bigcap_{\tau\in\mathcal{T}(n,1),n\in\mathbb{N}} \mathbf{Equ}(\tau_{\mathsf{A}}, (\Phi\tau)^{n\cdot A}\circ\langle\nu_k\rangle) \tag{7}$$

Proposition 16 *Let $\Phi\colon \mathcal{T} \to \mathcal{T}_\mathcal{V}$ be a morphism of Lawvere theories and $\mathsf{A}\colon \mathcal{T}^{\mathsf{op}} \to \mathcal{V}$ a $\mathcal{T}$-coalgebra in $\mathcal{V}$. Then there exists a bijection between morphisms $\mu\colon \Phi_* \Rightarrow \mathsf{A}$ in $\mathbf{Coalg}(\mathcal{T},\mathcal{V})$ and the set $\mathcal{G}_\Phi\mathsf{A}$. $\mathcal{G}_\Phi$ becomes a subfunctor of the forgetful functor $\mathbf{Coalg}(\mathcal{T},\mathcal{V}) \xrightarrow{||-||} \mathcal{V} \xrightarrow{|-|} \mathbf{Set}$ by restricting a coalgebra morphism $\mathsf{A} \xrightarrow{f} \mathsf{B}$ to a map $\mathcal{G}_\Phi\mathsf{A} \to \mathcal{G}_\Phi\mathsf{B}$.*

Proof: Consider $\mathcal{T}^{\mathsf{op}}$ as a full subcategory of $\mathbf{Alg}\mathcal{T}$ and denote by Fn the free $\mathcal{T}$-algebra on $\{1,\dots,n\}$. The claim is equivalent to the following one:
For $\mathcal{T}$-coalgebras A in $\mathcal{V}$ and $\mathcal{V}$-homomorphisms $F1 \xrightarrow{f} A = \mathsf{A}(1)$ the following holds.

> The family $(n \cdot f)_n$ is a natural transformation $\Phi_* \Rightarrow \mathsf{A}$ if and only if $f(1) \in \mathcal{G}_\Phi\mathsf{A}$.

Using Equation (4), this becomes evident by the following diagram

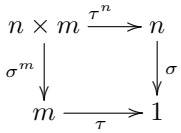


(8)

Functoriality of $\mathcal{G}_\Phi$ follows immediately from the definition of free completions (see IV of Section 1). □

3.2 Coalgebras in commutative varieties

Recall that a Lawvere theory $\mathcal{T}$ is called *commutative*, if for all $\sigma \in \mathcal{T}(n,1), \tau \in \mathcal{T}(m,1)$ the following diagram commutes.



Varieties $\mathcal{V}$ whose Lawvere theory $\mathcal{T}_\mathcal{V}$ is commutative are known under various names as, e.g., *entropic varieties*, *distribute varieties* or *commutative varieties* (see [13] for the respective references). Though the first option seems to be the

more popular choice amongst universal algebraists, we will use the second one because of its match with the notion of a commutative theory.

Since limits in functor categories are computed pointwise one immediately gets from this definition, that for any commutative Lawvere theory $\mathcal{T}$ and for every internal $\mathcal{T}$-algebra A in a category $\mathcal{C}$ with finite products the A interpretation $\mathsf{A}\tau$ of any m-ary operation $\tau \in \mathcal{T}(m,1)$ is a morphism $\mathsf{A}^m \to \mathsf{A}$ in $\mathbf{Alg}(\mathcal{T},\mathcal{C})$, that is, for every $\sigma \in \mathcal{T}(n,1)$ and every $n \in \mathbb{N}$ the following diagram commutes in $\mathcal{V}$, where $A := \mathsf{A}1$.

$$\begin{array}{ccc} A^{n\times m} & \xrightarrow{(\tau^{\mathsf{A}})^n} & A^n \\ {\scriptstyle \sigma^{(\mathsf{A}^m)}=(\sigma^{\mathsf{A}})^m}\downarrow & & \downarrow{\scriptstyle \sigma^{\mathsf{A}}} \\ A^m & \xrightarrow[\tau^{\mathsf{A}}]{} & A \end{array}$$

By dualization we obtain

Proposition 17 *Let $\mathcal{T}$ be a commutative Lawvere theory. Then for every variety $\mathcal{V}$ the category* $\mathbf{Coalg}(\mathcal{T},\mathcal{V})$ *is* cocommutative*, that is, for every $\mathcal{T}$-algebra $\mathcal{T} \xrightarrow{\mathsf{A}} \mathcal{V}^{\mathsf{op}}$ and every $\tau \in \mathcal{T}(m,1)$ the $\mathcal{V}$-homomorphism $\mathsf{A}\tau = A \xrightarrow{\tau_{\mathsf{A}}} m \cdot A$ is a morphism in* $\mathbf{Coalg}(\mathcal{T},\mathcal{V})$.

In a commutative variety $\mathcal{V}$ the hom-sets $\mathcal{V}(A,B)$ form subalgebras $[A,B]$ of the products B^A. $\mathcal{V}$ then is a monoidal closed category with internal hom-functor $[-,-]$ so defined (see [13], [12]). In particular, for each $\mathcal{V}$-algebra A one has an adjunction $A \otimes - \dashv [A,-]$ on $\mathcal{V} = \mathbf{Alg}\mathcal{T}_{\mathcal{V}}$ and, thus, a *canonical* $\mathcal{T}_{\mathcal{V}}$-coalgebra NA with underlying $\mathcal{V}$-algebra A. Equivalently, NA is the $\mathcal{T}_{\mathcal{V}}$-coalgebra in $\mathcal{V}$ with corresponding $\mathcal{T}_{\mathcal{V}}$- representable functor $H_{NA} = \mathcal{V} \xrightarrow{[A,-]} \mathcal{V}$.

The co-operation $A \xrightarrow{\sigma_{NA}} n \cdot A$ of NA corresponding to $\sigma \in \mathcal{T}_{\mathcal{V}}(n,1)$ is (use Equation (4) and the fact that operations in $\mathcal{V}$ are homomorphisms, and Example 2, respectively in Section 2.6) equivalently given by

$$A \xrightarrow{\sigma_{NA}} n \cdot A = A \xrightarrow{\langle \nu_1,\ldots,\nu_n\rangle} (n \cdot A)^n \xrightarrow{\sigma^{n\cdot A}} n \cdot A$$
$$A \xrightarrow{\sigma_{NA}} n \cdot A = A \simeq A \otimes F_{\mathcal{V}}1 \xrightarrow{A\otimes\sigma} A \otimes F_{\mathcal{V}}n \simeq n \cdot A$$

This defines a full embedding

$$N\colon \mathcal{V} \to \mathbf{Coalg}(\mathcal{T}_{\mathcal{V}},\mathcal{V}) \text{ with } \; || - || \circ N = id_{\mathcal{V}} \tag{9}$$

Though N occasionally is an equivalence, that is, up to isomorphism $\mathcal{V}$ only admits canonical $\mathcal{T}_{\mathcal{V}}$-coalgebras, this is not the case general (see Examples 21 below).

Theorem 18 *Let $\Phi\colon \mathcal{T} \to \mathcal{T}_{\mathcal{V}}$ be a theory morphism into a commutative theory $\mathcal{T}_{\mathcal{V}}$. Then the following hold, with notation as above:*

1. *The functor* $N_\Phi = \mathcal{V} \xrightarrow{N} \mathbf{Coalg}(\mathcal{T}_\mathcal{V}, \mathcal{V}) \xrightarrow{\widetilde{\Phi^*_\mathcal{V}}} \mathbf{Coalg}(\mathcal{T}, \mathcal{V})$ *is full and faithful.*
2. $N_\Phi(F_\mathcal{V}X) = \mathcal{V}_\Phi[X]$ *and* $||-|| \circ N_\Phi = id_\mathcal{V}$.
3. *The* $\mathcal{T}$*-representable functor corresponding to* $N_\Phi A$ *is* $H_{N_\Phi A} = \mathcal{V} \xrightarrow{[A,-]} \mathcal{V} \xrightarrow{\Phi^*} \mathbf{Alg}\mathcal{T}$*, for any* $\mathcal{V}$*-algebra* A*. The co-operation* $\sigma_{N_\Phi A}$ *corresponding to* $\sigma \in \mathcal{T}(n,1)$ *is explicitly given by* $\sigma_{N_\Phi A} = \Phi(\sigma)^{n\cdot A} \circ \langle \nu_1, \cdots, \nu_n \rangle$.
4. *The hom-functor* $\hom(\mathcal{V}_\Phi[1], -)$ *factors as* $\mathbf{Coalg}(\mathcal{T}, \mathcal{V}) \xrightarrow{\mathcal{G}_\Phi} \mathcal{V} \xrightarrow{|-|} \mathbf{Set}$*. The functor* $\mathcal{G}_\Phi$ *is faithful and satisfies the equation* $\mathcal{G}_\Phi N_\Phi = id_\mathcal{V}$.
5. N_Φ *is left adjoint to* $\mathcal{G}_\Phi$*; hence* $\mathcal{V}$ *is (equivalent to) a full coreflective subcategory of* $\mathbf{Coalg}(\mathcal{T}, \mathcal{V})$.

Proof: N_Φ is faithful trivially. Considering coalgebra as representable functors N_Φ acts as $A \mapsto \Phi^* \circ [A, -]$; thus, $N_\Phi f \colon \Phi^*[B,-] \Rightarrow \Phi^*[A,-]$ is the natural transformation with components $\Phi^*[B,C] \xrightarrow{\Phi^*[f,C]} \Phi^*[A,C]$, for any $\mathcal{V}$-morphism $A \xrightarrow{f} B$. Now the components $\Phi^*[B,C] \xrightarrow{\mu_C} \Phi^*[A,C]$ of any natural transformation have underlying maps $\mathcal{V}(B,C) \xrightarrow{|\mu_C|} \mathcal{C}(A,C)$ and these correspond one-to-one to $\mathcal{V}$-morphisms f. This proves 1.

The first identity of item 2 follows from Equation (4) since $\mathcal{V}$ is commutative, hence all operations are homomorphisms; the second one is obvious, as is item 3.

Concerning item 4 observe first that $\hom(\mathcal{V}_\Phi[1], -)$ factors over $\mathcal{V}$ by Proposition 16 since the maps τ_A are homomorphisms in the commutative variety $\mathcal{V}$ (we here don't distinguish between the set valued functor $\mathcal{G}_\phi$ and its lift). The lifted functor $\mathbf{Coalg}(\mathcal{T}, \mathcal{V}) \xrightarrow{\mathcal{G}_\Phi} \mathcal{V}$ is obviously faithful and satisfies the equation $\mathcal{G}_\Phi N_\Phi = id_\mathcal{V}$.

Finally, item 4 implies by Dubuc's Adjoint Triangle Theorem that the functor $\mathcal{G}_\Phi$ has a left adjoint L, which then satisfies the equation $L(F_\mathcal{V}X) = \mathcal{V}_\Phi[X]$ trivially. Now $N_\Phi(F_\mathcal{V}1) = L(F_\mathcal{V}1)$ follows by item 2, and this implies $L = N_\Phi$ by Lemma 19 below. □

Lemma 19 *Let* $\mathcal{V}$ *be a commutative variety. A functor* $S \colon \mathcal{V} \to \mathbf{Coalg}(\mathcal{T}, \mathcal{V})$ *with* $||-|| \circ S = id_\mathcal{V}$ *is, up to isomorphism, determined by* $SF_\mathcal{V}1$.

Proof: By assumption for each $\mathcal{V}$-homomorphism $F_\mathcal{V}1 \xrightarrow{f} A$ the $\mathcal{T}$-coalgebra morphism Sf has f as its underlying morphism in $\mathcal{V}$ and this satisfies, for each each $\sigma \in \mathcal{T}(n,1)$,

$$F_\mathcal{V}1 \xrightarrow{f} A \xrightarrow{\sigma_{SA}} n \cdot A = F_\mathcal{V}1 \xrightarrow{\sigma_{SF_\mathcal{V}1}} A \xrightarrow{n\cdot f} n \cdot A$$

This proves the claim since the family $(F_\mathcal{V}1 \xrightarrow{f} A)_{f \in \mathcal{V}(F_\mathcal{V}1, A)}$ is jointly epimorphic. □

Proposition 20 *Let $\mathcal{V}$ be a commutative variety. The assignment $\Phi \mapsto N_\Phi$ defines an essentially bijective correspondence between morphisms of Lawvere theories $\Phi\colon \mathcal{T} \to \mathcal{T}_\mathcal{V}$ and functors $S\colon \mathcal{V} \to \mathbf{Coalg}(\mathcal{T}, \mathcal{V})$ with $||-|| \circ S = id_\mathcal{V}$.*

Proof: For every functor $S\colon \mathcal{V} \to \mathbf{Coalg}(\mathcal{T}, \mathcal{V})$ with $||-|| \circ S = id_\mathcal{V}$ there exists by Proposition 14 the algebraic functor $H_{SF_\mathcal{V}1}\colon \mathcal{V} \to \mathbf{Alg}\mathcal{T}$. Let $\Phi_S\colon \mathcal{T} \to \mathcal{T}_\mathcal{V}$ be the morphism of Lawvere theories with ${\Phi_S}^* = H_{SF_\mathcal{V}1}$. The assignment $S \mapsto \Phi_S$ is essentially injective by Lemma 19.

Items 2. and 3. of Theorem 18 imply $H_{N_\Phi F_\mathcal{V}1} \simeq \Phi^*$ for each Φ. In other words, the assignment $S \mapsto \Phi_S$ is essentially bijective. □

The following examples show that the functors N and N_Φ respectively, may or may not be equivalences.

Examples 21 1. If $\mathcal{V}$ is a commutative variety all of whose algebras are free, then N is an equivalence as is easily seen. By [16] there are essentially four such varieties: **Set**, the variety of pointed sets $\mathbf{Set}_*$, $\mathbf{Vect}_k$, and $\mathbf{Aff}_k$, the variety of affine spaces over a field k.
In other words, if $\mathcal{V}$ is any of these varieties, then every $\mathcal{T}_\mathcal{V}$-coalgebra in $\mathcal{V}$ is of the form $\mathcal{V}[X]$ as it is the case for the variety of groups (see [19]).

2. Let $\mathcal{V}$ be a commutative semi-additive variety, in other words a commutative variety enriched over the category ${}_c\mathbf{Mon}$ of commutative monoids. Since each such variety has finite biproducts (see e.g. [11]) one obtains, using the Eckmann-Hilton Argument and the fact that the only comonoids in a cartesian category are trivial, the following equivalences
   (a) $\mathbf{Coalg}(\mathcal{T}_{{}_c\mathbf{Mon}}, {}_c\mathbf{Mon}) \simeq {}_c\mathbf{Mon} \simeq \mathbf{Coalg}(\mathcal{T}_\mathbf{Mon}, {}_c\mathbf{Mon})$,
   (b) $\mathbf{Coalg}(\mathcal{T}_{{}_c\mathbf{Mon}}, \mathbf{Ab}) \simeq \mathbf{Ab} \simeq \mathbf{Coalg}(\mathcal{T}_\mathbf{Mon}, \mathbf{Ab})$,
   (c) $\mathbf{Coalg}(\mathcal{T}_\mathbf{Grp}, \mathbf{Ab}) \simeq \mathbf{Ab} \simeq \mathbf{Coalg}(\mathcal{T}_\mathbf{Ab}, \mathbf{Ab})$,
   (d) $\mathbf{Coalg}(\mathcal{T}_\mathbf{Ab}, \mathbf{Mod}_R) \simeq \mathbf{Mod}_R$.
   Moreover, the forgetful functors of the above categories of coalgebras are equivalences, concrete over ${}_c\mathbf{Mon}$ and $\mathbf{Ab}$, respectively, having the respective functors N and N_Φ as inverses.

3. Let $\mathcal{V} = {}_c\mathbf{Sem}$ be the variety of commutative semigroups, and let $\Phi^*\colon \mathbf{Ab} \to {}_c\mathbf{Sem}$ be the forgetful functor. Note that ${}_c\mathbf{Sem}$ is commutative but fails to be semi-additive.
   The coproduct $\mathsf{S} + \mathsf{T}$ in $\mathcal{V}$ of two commutative semigroups S, T is given as $(\mathsf{S}^0 \times \mathsf{T}^0) \setminus \{(0,0)\}$ with the component-wise addition, where S^0 (respectively T^0) denotes the unitarization of S (resp. T), and with coprojections $\nu_1(a) := (a,0)$, $a \in S$, and $\nu_2(b) := (0,b)$, $b \in T$ (see [17]). Then $N(\mathsf{S})$ is an internal $\mathcal{T}_{{}_c\mathbf{Sem}}$-coalgebra in ${}_c\mathbf{Sem}$ with comultiplication $\mu_{N\mathsf{S}} = \mu^{2\cdot\mathsf{S}} \circ \langle \nu_1, \nu_2 \rangle$, i.e., $\mu_{N\mathsf{S}}(a) = (a,0) + (0,a) = (a,a)$, $a \in S$.
   (a) Now, let S be a commutative semigroup that contains an idempotent element e (e.g., S is the underlying semigroup of a commutative monoid, and e is the corresponding identity). Let $E_e\colon \mathsf{S} \to \mathsf{S}+\mathsf{S}$ be the semigroup homomorphism given by $E_e(a) := (e,e)$, $a \in S$. This endows S with a structure of an internal $\mathcal{T}_{{}_c\mathbf{Sem}}$-coalgebra in ${}_c\mathbf{Sem}$ not isomorphic to that

obtained by applying N when S is not reduced to e. This shows that N cannot be an equivalence between ${}_c\mathbf{Sem}$ and $\mathbf{Coalg}(\mathcal{T}_{{}_c\mathbf{Sem}}, {}_c\mathbf{Sem})$.

(b) The comultiplication in $N_\Phi(\mathsf{A})$, for an abelian group A, thus is given by $a \mapsto (a, a)$. But A carries also a trivial internal $\mathcal{T}_{\mathbf{cSem}}$-coalgebra structure, namely, $Z_{\mathsf{A}}(a) := (0, 0)$. This clearly implies that N_Φ is not an equivalence of the categories $\mathbf{Ab}$ and $\mathbf{Coalg}(\mathcal{T}_{{}_c\mathbf{Sem}}, \mathbf{Ab})$.

4 Appendix

An algebraic proof of [9, Thm. 10.4.3]

As mentioned in Section 2.4 the existence of a left adjoint for a $\mathcal{V}$-representable functor on a cocomplete category is shown by algebraic methods in [9]. The argument given there is based on choosing (arbitrary) presentations of algebras in $\mathcal{V}$; it is incomplete, however, in that it is lacking the proof of naturality of the bijections of the respective hom-sets. Since we doubt this can be fixed without resorting to canonical presentations we here provide a modification of that approach.

Proposition 22 *Let $\mathcal{V}$ be a variety and $\mathcal{C}$ a cocomplete category. Then every $\mathcal{V}$-representable functor $H\colon \mathcal{C} \to \mathcal{V}$ has a left adjoint K.*

Proof: Recall first that, for every $\mathbf{Alg}\mathcal{T}$-algebra A, there is the *canonical presentation by generators and relations* given by the coequalizer diagram

$$F|K_A| \underset{s_A}{\overset{r_A}{\rightrightarrows}} F|A| \xrightarrow{\epsilon_A} A \tag{10}$$

where ϵ_A is the counit of the adjunction $F \dashv |-|$, $K_A \underset{q}{\overset{p}{\rightrightarrows}} F|A| \xrightarrow{\epsilon_A} A$ its kernel pair and (r_A, s_A) is the pair of homomorphic extensions of the projections p and q. Denoting by R the representing object of H we have, for every set X, a natural isomorphism

$$\mathcal{V}(FX, H-) \simeq \mathcal{V}(F1, H-)^X = \mathcal{C}(R, -)^X \simeq \mathcal{C}(X \cdot R, -) \tag{11}$$

and for every $\mathcal{V}$-morphism $f\colon FX \to FY$ the natural transformation

$$\mathcal{V}(f, H-)\colon \mathcal{C}(Y \cdot R, -) \simeq \mathcal{V}(FY, H-) \Rightarrow \mathcal{V}(FX, H) \simeq \mathcal{C}(X \cdot R, -) \tag{12}$$

which, by the Yoneda lemma, determines a $\mathcal{C}$-morphism $\phi\colon X \cdot R \to Y \cdot R$ with $\mathcal{C}(\phi, -) \simeq \mathcal{V}(f, H-)$.

The canonical presentation (10) of any $\mathcal{V}$-algebra A then yields natural transformations

$$\begin{array}{ccc} \mathcal{V}(A, H-) & \overset{\mathcal{V}(\epsilon_A, H-)}{\Longrightarrow} & \mathcal{V}(F|A|, H-) \underset{\mathcal{V}(s_A, H-)}{\overset{\mathcal{V}(r_A, H-)}{\rightrightarrows}} \mathcal{V}(F|K_A|, H-) \\ \| & & \| \\ \mathcal{C}(|A| \cdot R, -) & & \underset{\mathcal{C}(\sigma_A, -)}{\overset{\mathcal{C}(\rho_A, -)}{\rightrightarrows}} \mathcal{C}(|K_A| \cdot R, -) \end{array}$$

Since for each $\mathcal{V}$-algebra B the functor $\mathcal{V}(-, B)$ maps coequalizers to equalizers one obtains from (10) for each C in $\mathcal{C}$ the equalizer diagram

$$\mathcal{V}(A, HC) \xrightarrow{\mathcal{V}(\epsilon_A, HC)} \mathcal{C}(|A| \cdot R, C) \underset{\mathcal{C}(\sigma_A, C)}{\overset{\mathcal{C}(\rho_A, C)}{\rightrightarrows}} \mathcal{C}(|K_A| \cdot R, C)$$

Considering the coequalizer $|K_A| \cdot R \underset{\sigma_A}{\overset{\rho_A}{\rightrightarrows}} |A| \cdot R \xrightarrow{\upsilon_A} KA$ in $\mathcal{C}$ we obtain the equalizer diagram

$$\mathcal{C}(KA, C) \xrightarrow{\mathcal{C}(\upsilon_A, C)} \mathcal{C}(|A| \cdot R, C) \underset{\mathcal{C}(\sigma_A, C)}{\overset{\mathcal{C}(\rho_A, C)}{\rightrightarrows}} \mathcal{C}(|K_A| \cdot R, C)$$

in **Set**, for each C in $\mathcal{C}$. Comparison with the equalizer diagram above shows that for each C in $\mathcal{C}$ and each A in $\mathcal{V}$ there is a bijection

$$\mathcal{C}(KA, C) \xrightarrow{\alpha_{A,C}} \mathcal{V}(A, HC)$$

making the following diagram commute.

$$\begin{array}{ccc} \mathcal{C}(KA, C) & \xrightarrow{\mathcal{C}(\upsilon_A, C)} & \mathcal{C}(|F|A|| \cdot R, C) \\ {\scriptstyle \alpha_{A,C}} \downarrow & \nearrow_{\mathcal{V}(\epsilon_A, HC)} & \\ \mathcal{V}(A, HC) & & \end{array} \tag{13}$$

The claim now follows by [23, Chapter IV.1, Corollary 2], provided that these bijections are natural in C. But this is obvious: Since $\mathcal{V}(\epsilon_A, H-)$ and $\mathcal{C}(\upsilon_A, -)$ are natural and $\mathcal{V}(\epsilon_A, HC)$ is monic, for each C in $\mathcal{C}$, commutativity of Diagram (13) implies that the family $(\alpha_{A,C})_C$ is natural. □

A more algebraic argument for local presentability of $\mathbf{Coalg}(\mathcal{T},\mathcal{C})$

An argument with a slightly more algebraic flavor for local presentability of $\mathbf{Coalg}(\mathcal{T},\mathcal{C})$ for any locally presentable category $\mathcal{C}$ (see Theorem 10) is the following:

Assume $\mathbf{Alg}\mathcal{T} \simeq \mathbf{Alg}(\Sigma,\mathfrak{E})$. By Fact 3 $\mathbf{Coalg}(\mathcal{T},\mathcal{C}) \simeq \mathbf{Equ}(\hat{s},\hat{t})_{s=t\in\mathfrak{E}}$, where the equifier is a subcategory of the category $\mathbf{Coalg}Q_\Sigma$, since locally presentable categories have products.

If $\mathcal{C}$ is locally λ-presentable, one can choose a regular cardinal μ larger than λ and the cardinality of Σ, such that the functor Q_Σ is μ-accessible; hence, $\mathbf{Coalg}Q_\Sigma$ is locally μ-presentable (see e.g. [4, 2.j]) and all functors $||-|| \Rightarrow n\cdot||-||$ are μ-accessible. But then the equifier $\mathbf{Equ}(\hat{s},\hat{t})_{s=t\in\mathfrak{E}}$ is accessible by [4, 2.76]. Since the category $\mathbf{Coalg}(\mathcal{T},\mathcal{C})$ is cocomplete, it is locally presentable.

Examples of coalgebras

Bimodules. This generalizes Example 2.6.2. For any not necessarily commutative rings R and S the category of left S-left, R-right bimodules ${}_S\mathbf{Mod}_R$ coincides with $\mathbf{Coalg}({}_R\mathbf{Mod},{}_S\mathbf{Mod})$ as follows immediately from the definitions. One so obtains immediately

The Eilenberg-Watts theorem. For every bimodule ${}_SU_R$ the ${}_R\mathbf{Mod}$-representable functor corresponding to ${}_SU_R$ is ${}_S\mathbf{Mod} \xrightarrow{{}_S\hom({}_SU,-)} {}_R\mathbf{Mod}$. The corresponding adjunction is $(U_R\otimes_R -) \dashv {}_S\hom({}_SU,-)$. These are the only adjunctions from ${}_R\mathbf{Mod}$ to ${}_S\mathbf{Mod}$.

Change of rings. Consider a ring homomorphism $R \xrightarrow{\phi} S$ (equivalently, a morphism of Lawvere theories $\Phi\colon \mathcal{T}_{{}_R\mathbf{Mod}} \to \mathcal{T}_{{}_S\mathbf{Mod}}$).

1. The algebraic functor $\Phi^*\colon \mathcal{V} = {}_S\mathbf{Mod} \to {}_R\mathbf{Mod}$ is the ${}_R\mathbf{Mod}$-representable functor ${}_S\hom(S,-)$, known as the *restriction of scalars*-functor. The corresponding S-left, R-right bimodule structure on S is $\mathcal{V}_\Phi[1]$. By the above its left adjoint Φ_* is the functor $\mathcal{V}_\Phi[1]_R\otimes_R -$, known as the *extension of scalars*-functor.
2. Since the forgetful functors from module categories into $\mathbf{Ab}$ create colimits, the functor Φ^*, being a concrete functor over $\mathbf{Ab}$, preserves them. Consequently, the restriction of Φ^* to the dual of the theory of ${}_S\mathbf{Mod}$ is an ${}_S\mathbf{Mod}$-coalgebra in ${}_R\mathbf{Mod}$ and Φ^* is its unique (sifted) colimit preserving extension, hence a left adjoint with right adjoint G being the ${}_S\mathbf{Mod}$-lift of ${}_R\hom(\Phi^*S,-)$. In other words, G is the *coextension of scalars*-functor.

We so have obtained, without any calculations, the familiar adjunctions *extension of scalars* $\dashv$ *restriction of scalars* $\dashv$ *coextension of scalars*.

Special linear groups. Assigning to a commutative unital ring R the (special linear) group $SL(n,R)$ of $n\times n$-matrices with entries in R defines a functor $SL_n\colon {}_1\mathbf{Ring} \to \mathbf{Grp}$ from the variety of commutative unital rings into that of groups. This functor is $\mathbf{Grp}$-representable.

By [23, III. Propostion 2] we only need to find a commutative ring A_n such that there exists a U-universal map $\{\star\} \xrightarrow{\eta} |SL(n, A_n)|$. Since $\mathbb{Z}$ is the free group over a singleton $\{\star\}$ it suffices to find a SL_n-universal group homomorphism $\mathbb{Z} \xrightarrow{u} SL(n, A_n)$. In other words, we need to find a ring A_n in ${}_1\mathbf{Ring}$ together with a particular $n \times n$-matrix $H = (h_{ij})$ with $\det H = 1$ over A_n such that for every $n \times n$-matrix $M = (m_{ij})$ with $\det M = 1$ over any R in ${}_1\mathbf{Ring}$ there exists a unique ring homomorphism $A_n \xrightarrow{\phi} R$ such that $\phi(h_{ij}) = m_{ij}$, for each pair (i, j) with $1 \leq i, j \leq n$
Noting that det and the constant e can be considered as $n \times n$-ary (derived) operations in the theory of commutative rings (alternatively, as elements of $F(n \times n)$, the free commutative ring over $n \times n$), we may form the ring A_n in ${}_1\mathbf{Ring}$, presented by the set $n \times n$ of generators and the relation $\det \simeq 1$, that is, the coequalizer $F(n \times n) \xrightarrow{q} A_n$ of the pair of ring homomorphisms $\det^\sharp, 1^\sharp \colon F\{\star\} \to F(n \times n)$ in ${}_1\mathbf{Ring}$ given by $\star \mapsto \det$ and $\star \mapsto 1$, respectively.
Now choose $H = (h_{ij})$ to be the $n \times n$ identity matrix (δ_{ij}) over A_n. For every M in $SL(n, R)$ for some commutative ring R we have the unique ring homomorphism $\psi \colon F(n \times n) \to R$ with $\delta_{ij} \mapsto m_{ij}$ and this satisfies the equation $\psi \circ \det^\sharp = \psi \circ 1^\sharp$. Consequently, there exists a unique ring homomorphism $A_n \xrightarrow{\phi} R$ with $\phi \circ q = \psi$ and this satisfies $\phi(h_{ij}) = m_{ij}$.
Since every ring homomorphism $A_n \xrightarrow{\xi} R$ satisfying the condition $\xi(h_{ij}) = m_{ij}$ for every such M maps $\det H$ to $\det M$ and 1 to 1 trivially, it satisfies the equation $\xi \circ q = \psi$ and, thus, coincides with ϕ by the coequalizer property. This proves the claim.

Cocommutative Hopf algebras. Since in the variety ${}_c\mathbf{Alg}_R$ of R-algebras over a commutative ring R the monoidal structure coincides with the cocartesian structure, that is, $\otimes = +$ in ${}_c\mathbf{Alg}_R$, the cogroups in ${}_c\mathbf{Alg}_R$ are the commutative Hopf algebras over R.

More on $\mathcal{G}_\Phi\mathsf{A}$

Remarks 23 The following facts are easy to verify.

1. $\mathcal{G}_\Phi\mathsf{A}$ contains all $\mathcal{V}$-constants in A.
2. If $\mathcal{T}_\Sigma \xrightarrow{[-]} \mathcal{T}$ is a regular quotient, then
$$\mathcal{G}_\Phi\mathsf{A} = \bigcap_{\sigma \in \Sigma_n, n \in \mathbb{N}} \mathbf{Equ}(\sigma_\mathsf{A}, (\Phi\sigma)^{n \cdot A} \circ \langle \nu_k \rangle).$$
3. By Equation (4) for every set X the following holds: $X \subset \mathcal{G}_\Phi\mathcal{V}_\Phi[X] \subset |F_\mathcal{V}X|$.
4. If $\mathcal{V}$ is commutative, then $\mathcal{G}_\Phi\mathcal{V}_\Phi[X] = F_\mathcal{V}X$ and $\mathbf{Coalg}(\mathcal{T}, \mathcal{V})(\mathcal{V}_\Phi[X], \mathsf{A})$ is a $\mathcal{V}$-subalgebra of $[F_\mathcal{V}X, ||\mathsf{A}||]$.

The following are well known special instances of the functor $\mathcal{G}_\Phi$.

Group-like elements. Let $\mathcal{V} = {}_c\mathbf{Alg}_R$ be the variety of commutative unital algebras over a commutative unital ring R, $\mathcal{W} = \mathbf{Mon}$ that of monoids, and Φ be the morphism $\mathcal{T}_{\mathbf{Mon}} \to \mathcal{T}_{{}_c\mathbf{Alg}_R}$ such that $\Phi^* \colon {}_c\mathbf{Alg}_R \to \mathbf{Mon}$ is the algebraic functor which sends an algebra to its underlying multiplicative monoid. Then $\mathbf{Coalg}(\mathcal{T}_{\mathbf{Mon}}, {}_c\mathbf{Alg}_R)$ is equivalent to the category ${}_c\mathbf{Bialg}_R$ of all commutative unital and counital R-bialgebras.
For any such bialgebra $\mathsf{A} = (A, m, e, \Delta, \epsilon)$ one has, by a simple computation,

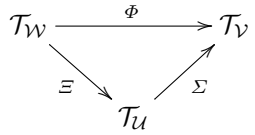
$$\mathcal{G}_\Phi(\mathsf{A}) = \{\, a \in A \colon \Delta(a) = a \otimes a,\ \epsilon(a) = e(1) \,\}.$$

In other words, $\mathcal{G}_\Phi(\mathsf{A})$ is the usual set of group-like elements of the bialgebra A.

The primitive Hopf structure on polynomial algebras. It is well known that, for every commutative ring R, the polynomial algebras in ${}_c\mathbf{Alg}_R$, that is, the free algebras in ${}_c\mathbf{Alg}_R$, carry the structure of an R-Hopf algebra in which the variables are primitive elements (see e.g. [1, p. 92]). Since in ${}_c\mathbf{Alg}_R$ the monoidal structure coincides with the cocartesian structure, that is, $\otimes = +$, these Hopf algebras are cogroups in ${}_c\mathbf{Alg}_R$. Since the forgetful functor $\mathcal{V} := {}_c\mathbf{Alg}_R \xrightarrow{\Phi^*} \mathbf{Ab} =: \mathcal{W}$ mapping an R-algebra to its underlying additive abelian group is an algebraic functor, the functor $\mathcal{V}_\Phi[-]$ is a functor from $\mathbf{Set}$ into the category ${}_{bi}\mathbf{Hopf}_R$ of bicommutative, that is, commutative and cocommutative Hopf algebras over R. Equation (4) implies that every $x \in X$ is a primitive element in $\mathcal{V}_\Phi[X]$.
As is easily seen, for every bicommutative Hopf algebra H the set $\mathcal{G}_\Phi H$ is the set of primitive elements of H. Thus, item 4 of Remarks 23 explains the result mentioned at the beginning of this example.

The proof of the following consequences of Theorem 18 can safely be left to the reader. Concerning item 3 one should note that $\mathcal{G}_\Xi(\underline{\Sigma_*}(C_{\mathcal{T}_\mathcal{W},\mathcal{V}}(\|\mathsf{A}\|)))$ is a $\mathcal{U}$-subalgebra of $||\underline{\Sigma_*}(C_{\mathcal{T}_\mathcal{W},\mathcal{V}}(\|\mathsf{A}\|))||$ by definition, since $\mathcal{U}$ is commutative.

Corollary 24 *Let the following diagram in* **Lawv** *commute, where $\mathcal{T}_\mathcal{U}$ is commutative.*



Then the following hold, where $\underline{\Sigma_*}$ denotes a right adjoint of $\widehat{\Sigma_*}$ and A a $\mathcal{T}_\mathcal{W}$-coalgebra in $\mathcal{V}$:

1. *$\mathcal{G}_\Phi$ factors (up to a natural isomorphism) as*

$$\mathbf{Coalg}(\mathcal{T}_\mathcal{W}, \mathcal{V}) \xrightarrow{\Sigma_*} \mathbf{Coalg}(\mathcal{T}_\mathcal{W}, \mathcal{U}) \xrightarrow{\mathcal{G}_\Xi} \mathcal{U} \xrightarrow{|-|} \mathbf{Set}.$$

2. *$C_{\mathcal{T}_\mathcal{W},\mathcal{U}} \circ \Sigma^* \simeq \underline{\Sigma_*} \circ C_{\mathcal{T}_\mathcal{W},\mathcal{V}}$.*
3. *The $\mathcal{U}$-algebra $\mathcal{G}_\Xi \underline{\Sigma_*} \mathsf{A}$ is a subalgebra of the $\mathcal{U}$-algebra $\mathcal{G}_\Xi(\underline{\Sigma_*}(C_{\mathcal{T}_\mathcal{W},\mathcal{V}}(\|\mathsf{A}\|)))$ $= \mathcal{G}_\Xi(C_{\mathcal{T}_\mathcal{W},\mathcal{U}}(\Sigma^*(\overline{\|\mathsf{A}\|})))$.*

The following applications are in the spirit of Kan's example. Here R always denotes a commutative unital ring and ${}_{bi}\mathbf{Hopf}_R$ the category of bicommutative Hopf algebras over R. Consider the following forgetful (algebraic) functors $\mathcal{V} := {}_c\mathbf{Alg}_R \xrightarrow{\Sigma^*} \mathbf{Mod}_R =: \mathcal{U}$, ${}_c\mathbf{Alg}_R \xrightarrow{\Phi^*} \mathbf{Ab} =: \mathcal{W}$, ${}_c\mathbf{Alg}_R \xrightarrow{\Psi^*} \mathbf{Mon} =: \mathcal{M}$ and $\mathcal{U} \xrightarrow{\Xi^*} \mathcal{W}$. Note that $\mathcal{W}$ and $\mathcal{U}$ are commutative varieties.

More on primitive elements. Every symmetric R-algebra, that is, every free algebra $A = A_M$ over some R-module M with respect to the forgetful functor Σ^*, is known to carry the structure of a bicommutative Hopf algebra A_H in which each $x \in M$ is a primitive element. One easily sees that A_H is obtained by evaluating the functor $P := \mathbf{Mod}_R \xrightarrow{N_\Xi} \mathbf{Coalg}(\mathcal{T}_\mathcal{W}, \mathcal{U}) \xrightarrow{\widehat{\Sigma_*}} \mathbf{Coalg}(\mathcal{T}_\mathcal{W}, \mathcal{V}) \simeq {}_{bi}\mathbf{Hopf}_R$ at the module M.

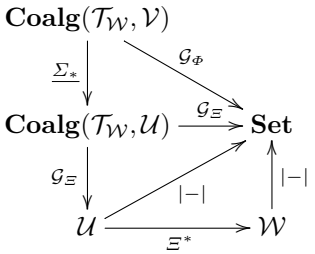
By Corollary 24 the functor ${}_{bi}\mathbf{Hopf}_R \simeq \mathbf{Coalg}(\mathcal{T}_\mathcal{W}, \mathcal{V}) \xrightarrow{\mathcal{G}_\Phi} \mathbf{Set}$, which coincides with the primitive element functor (see above), lifts to a functor ${}_{bi}\mathbf{Hopf}_R \to \mathbf{Mod}_R$. This is a right adjoint of P.

More on group-like elements. Every group algebra A, that is, every free algebra $A = A_G$ over some group G with respect to the forgetful functor $\mathbf{Alg}_R \xrightarrow{\Psi^*} \mathbf{Mon}$, is known to carry the structure of a Hopf algebra A_H.
Denoting by G the functor $\mathbf{Ab} \to {}_c\mathbf{Alg}_R$ which maps an abelian group to its (commutative) group algebra we get $\mathbf{Coalg}(\mathcal{T}_\mathcal{W}, \mathcal{W}) \xrightarrow{\widehat{G}} \mathbf{Coalg}(\mathcal{T}_\mathcal{W}, \mathcal{V})$, since G is a left adjoint and, hence, preserves coproducts. Then A_H is obtained by evaluating the functor

$$\mathbf{Ab} = \mathcal{W} \xrightarrow{N} \mathbf{Coalg}(\mathcal{T}_\mathcal{W}, \mathcal{W}) \xrightarrow{\widehat{G}} \mathbf{Coalg}(\mathcal{T}_\mathcal{W}, \mathcal{V}) \simeq {}_{bi}\mathbf{Hopf}_R$$

at the abelian group G.
One observes that the restriction to ${}_{bi}\mathbf{Hopf}_R$ of the group-like element functor $\mathcal{G}_\Psi$ (see Example above), when lifted to $\mathbf{Ab}$ according to Corollary 24, is right adjoint to $\widehat{G}N$.

Remark 25 In the situation of Theorem 18 and Corollary 24, respectively, the following diagram commutes. (Observe that the square below commutes because $\mathcal{U}$ is commutative and Ξ^* is algebraic.)



In other words, the functor $\mathcal{G}_\Phi$ factors over $\mathcal{W}$, as in the case of group-like elements. If one assumes that $\mathcal{U} \simeq \mathbf{Coalg}(\mathcal{T}_\mathcal{W}, \mathcal{U})$ under the isomorphism

$N_\Xi = C_{\mathcal{T}_\mathcal{W},\mathcal{U}}$ with inverse $\mathcal{G}_\Xi = \| - \|$, the above result shows that $\mathcal{G}_\Phi(\mathsf{A})$ is the carrier set of the subalgebra $\Sigma^*(\|\mathsf{A}\|)$. In conjunction with Example 21.2. this explains why

1. the set of group-like elements of a bicommutative bialgebra is a sub-monoid of the multiplicative monoid of the underlying algebra.
2. the set of primitive elements of a bicommutative Hopf algebra is a sub-module of the underlying module of the carrier algebra.

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