Hypersets as truth values

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Abstract

The hypersets are shown to play the truth values of stable properties of nondeterministc dynamical systems. In fact, the universe of hereditory finite hypersets, with the truth value as an atom added, is shown to be the subobject classifier of the category of simulations of nondeterministic dynamical systems.

1 Introduction

1.1 The transition systems are playing the role of a universal model of behaviors of concurrent systems. They give background to the semantics of various frameworks of concurrent systems, such as process algebras, Petri nets, etc.. This framework can also be regarded as a first approximation to describe the concurrent distributed aspects of living systems.

1.2 In the categorical study of transition systems, there are two possible choices of arrows: either graph maps or simulations.

Usually graph maps are adopted as the arrows[4, 16]. This category is the functor category from the finite category

$$\ell \xleftarrow{\lambda} e \xrightarrow{d_0, d_1} v,$$

where e, v, ℓ stand for the sets of edges, nodes and labels respectively to the category of sets. In particular it is a topos.

The concept of subobjects in this category, however, does not match our intuition of processes. In fact, let P be a process, i.e. a transition system, and S be a subset of states of P. When we say that S with a set τ of transitions forms a subprocess of P, we intend to mean that not only the transitions in τ come from the total process P, but also every transition of the total process P starting from a state of S results in a state of S. But this is not the case in the category with graph maps even when S with τ is a subgraph of P. Another

reason is as follows: Suppose a process P is a dynamical system, i.e., every state have a unique transition from it. A subobject of P in the category with graphs maps may not be a dynamical system, whereas in the category of simulations, it is.

This leads us to consider the category of transition systems whose arrows are simulations maps. This category of simulations is introduced from a point of view different from ours in [9]. It is studied as a mathematical framework of concurrent constrained programming.

1.3 For the sake of simplicity, we take the singleton set as the labeling set and call such transition systems *nondeterministic dynamical systems*. Moreover we consider only systems all of whose nodes have positive out-degrees.

It turns out that subobjects corresponds to stable properties of states: Each subsystem A of a system D defines a unique simulation map χ_A from D to a universe Ω of hypersets , regarded as a nondeterministic dynamical system in a natural way. The χ_A-value of a state $S \in D$ gives the description of the future behavior of the system with the current state S relative to A considered as a stable property. For example, $\chi_A(S) = \bot$ means that the system will never reach the subsystem A from the state S.

Conversely every simulation map χ from D to the hyperset universe Ω determines a unique subsystem A whose characteristic map is precisely χ. In this way, we have a natural bijection between $\mathcal{ND}yn(D, \Omega)$ and the set of subsystems of D. This shows essentially that Ω is a subobject classifier of $\mathcal{ND}yn$.

In fact the category $\mathcal{ND}yn$ is in fact complete, cocomplete, symmetric monoidal closed category with a subobject classifier[10, 13, 14, 15]. This might enable us to regard this category as a universe of a typed linear set theory, similar to the untyped version developed by Shirahata in [11].

1.4 We note that the cocompleteness and the monoidal closedness and the existence of a subobject classifier were proved using elementary category theory[13, 14]. The categorical mechanism behind these proofs became much clearer when $\mathcal{ND}yn$ is considered as the category of coalgebras of the covariant endofunctor $\mathbf{pow}_o$ of $\mathcal{S}\mathbf{et}$, where $\mathbf{pow}_o(X)$ is the set of all the nonempty finite subsets of X and $\mathbf{pow}_o(g)(A) = \left\{ g(a) \;\middle|\; a \in A \right\}$ for $g : X \to Y$ and $A \subseteq X$ [10].

1.5 In §2, we give basic definitions about nondeterministic dynamical systems and recall in §3 a description of a subobject classifier of the topos of the deterministic dynamical systems, which forms a full subcategory of $\mathcal{ND}yn$. We review basic facts about hypersets in §4 and proves in §5 that the universe of hereditory finite hypersets with an atom added can be given a natural structure of nondeterministic dynamical system which is in fact a subobject classifier of $\mathcal{ND}yn$. We give a few examples illustrating the roles of hypersets as

truth values in §6.

2 Definitions

2.1 Nondeterministic dynamical system

A *nondeterministic dynamical system* D is a pair $(|D|, \tau_D)$ of a set $|D|$ and a binary relation $\tau_D \subseteq |D| \times |D|$. Elements of $|D|$ are called *states* and τ_D the *transition relation.*

When $(s_1, s_2) \in \tau_D$, we write $s_1 \rightarrow_D s_2$, which we interpret as the possibility that the system goes from state s_1 to state s_2. We write $s_1 \rightarrow_D^* s_2$ if there is a path from s_1 to s_2 when D is regarded as a directed graph.

A nondeterministic dynamical system D is called *deterministic*, or simply *a dynamical system* if the transition relation is the graph of a selfmap, which is called *the transition map* and is also denoted by τ_D. Hence $s \rightarrow_D t \iff t = \tau_D(s)$.

2.2 Simulations

Let D, D' be nondeterministic dynamical systems.

A map $\varphi : |D| \rightarrow |D'|$ is a *simulation* if

$$\varphi_* \mathrm{Child}_D(s) = \mathrm{Child}_{D'}(\varphi(s))$$

for all $s \in |D|$. Obviously the compositions of simulations are simulations.

2.3 The categories $\mathcal{NDyn}$ and $\mathcal{Dyn}$

Let $\mathcal{NDyn}$ be the category whose objects are nondeterministic dynamical systems $(|D|, \tau)$ satisfying the conditions

- $\forall s \in |D| \exists t \in |D| \left[s \rightarrow t \right]$,
- D is locally finite, namely, the set $\mathrm{Child}_D(s)$ is finite for all $s \in |D|$.

and whose morphisms are simulations. The category of deterministic dynamical systems $\mathcal{Dyn}$, whose arrows are maps between state spaces which commutes with the transition map, is a full subcategory of $\mathcal{NDyn}$ since a simulation between deterministic dynamical systems considered as nondeterministic ones is a $\mathcal{Dyn}$ arrow.

2.4 Subsystem

Let $D = (|D|, \tau) \in \mathcal{NDyn}$. The concept of subobjects coincides with that of *subsystems.* A *subsystem* A of D is a nondeterministic dynamical system whose state space $|A|$ is a subset of $|D|$ and the inclusion map $|A| \rightarrow |D|$ is a simulation, namely,

$$\mathrm{Child}_D(x) = \mathrm{Child}_A(x)$$

for all $x \in |A|$. The set of all the subsystems of D will be denoted by $Sub(D)$.

For $x \in |D|$, we define a subsystem $D.x \in \mathcal{NDyn}$ by

$$|D.x| := \left\{ y \in |D| \;\middle|\; x \rightarrow_D^* y \right\},$$

and

$$u \rightarrow_{D.x} v \overset{def}{\iff} u \rightarrow_D v,$$

which is called *the orbit of D passing through x*. The inclusion map $|D.x| \rightarrow |D|$ is obviously a simulation.

3 The topos of deterministic dynamical system $\mathcal{D}yn$

The full subcategory $\mathcal{D}yn$ of $\mathcal{N}\mathcal{D}yn$ is a topos since it is the functor category $[\mathcal{T}, \mathcal{S}\mathbf{et}]$ where $\mathcal{T}$ is the monoid of the non-negative integers with the addition, considered as a category with one object.

By a well-known construction for the presheaf category, a subobject classifier of $\mathcal{D}yn$ is given by

$$\Omega_{\mathcal{D}yn} = (\{ \top = 0, 1, 2, \cdots, \bot = \infty \}, \tau)$$

whose transition has two fixed points $\bot$, $\top$ and maps n to $n-1$ when $n \neq \top, \bot$. The characteristic map

$$\chi_A : D \rightarrow \Omega_{\mathcal{D}yn}$$

for an invariant subsystem $A \subseteq D$ is described as follows:

- If the orbit from $x \in D$ is disjoint from A then $\chi_A(x) = \bot$,
- if the orbit from $x \in D$ reaches exactly at the n-th transition, then $\chi_A(x) = n$.

Hence, obviously $\chi_A(x) = \top$ if and only if $x \in A$, which implies that if a $\mathcal{D}yn$ arrow $f : D \rightarrow \Omega_{\mathcal{D}yn}$ has the property $f^{-1}(\top) = A$ then $f = \chi_A$. So we can identify a subsystem with its characteristic map just as in the category of sets.

We will see that the sets of hereditory finite hypersets plays for $\mathcal{N}\mathcal{D}yn$ exactly the same role of $\Omega_{\mathcal{D}yn}$ for $\mathcal{D}yn$.

4 Review of hyperset theory

4.1 Anti-foundation axiom

We shall work in the hyperset theory ZFC$^-$+AFA with urelements as is given in [3]. In this framework of set theory, each object is either a set or an urelement. Only sets have elements. The axiom system ZFC$^-$[3, p28] is the Zermero-Fraenkel axiom systems with the axiom of choice but without the foundation axiom, modified in a natural way to incorporate urelements, with two axioms which state that urelements do not have elements and that there are plenty of urelements in a sense.

AFA is the Anti-Foundation Axiom of Aczel[1]: *Every graph has a unique decoration.* Here *a decoration* of a graph $\langle V, E \rangle$ $(E \subseteq V \times V)$ is a family of sets $\left\{ d(x) \;\middle|\; x \in V \right\}$ satisfying

$$d(x) = \left\{ d(y) \;\middle|\; \langle x, y \rangle \in E \right\}$$

for all $x \in V$.

For example the decoration of the graph $\langle \{ e \}, \{ \langle e, e \rangle \} \rangle$ gives us a unique set U with $U = \{ U \}$. Note that the decoration of a graph with no leaves is give by $d(v) = U$ for all node v.

Another example is given by the graph in Figure 1). If we identify the ordered

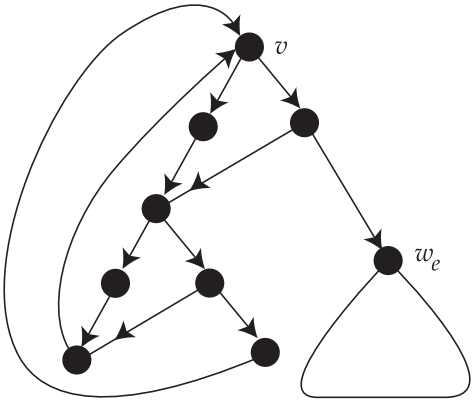


Figure 1: An example of hyperset

pair $\langle x, y\rangle$ with the set $\{\{x\}, \{x, y\}\}$ and $\langle x, y, z\rangle$ with $\langle\langle x, y\rangle, z\rangle$, the decoration of this graph gives us at the node v a set S_e which satisfies

$$S_e = \langle \{S_e\}, \{S_e\}, e\rangle,$$

where e is a usual well-founded set and the graph below the node w_e is the *picture* of the set e whose nodes are the elements of the transitive closure of the set e and u is a child of u' if and only if $u \in u'$.

One of the main results of [1] is that ZFC$^-$+AFA is consistent whenever ZF is consistent. The consistency is proved by constructing a class model, which is the "quotient" of the class of accessible pointed graphs by the maximal bisimulation equivalence.

We denote the class of all the hypersets constructed by a class of urelements $\mathcal{U}$ by $V[\mathcal{U}]$.

4.2 Fixed point theorem

An important result in [1] is the existence of the largest fixed points of set continuous operators.

A map class $\Phi : V[\mathcal{U}] \supseteq \mathcal{X} \mapsto \Phi\mathcal{X}$ is called *set continuous* if

$$\Phi\mathcal{X} = \bigcup \left\{ \Phi x \;\middle|\; x \in \mathbf{pow}\mathcal{X} \right\}.$$

Theorem 4.1 *([1, Theorem 6.5]). If Φ is a set continuous operator, then the class*

$$J_\Phi := \bigcup \left\{ x \in V \;\middle|\; x \subseteq \Phi x \right\}$$

satisfies

1. $J_\Phi \subseteq \Phi J_\Phi$,
2. $\mathcal{X} \subseteq \Phi\mathcal{X}$ *implies* $\mathcal{X} \subseteq J_\Phi$,
3. J_Φ *is the largest fixed point of* Φ.

We remark that the AFA often guarantees nontriviality of these fixed points, whereas fixed points usually reduce to the empty set or coincide with the minimal fixed points in usual ZF set theory.

Let $\mathbf{pow}_o$ be the operator which maps a class $\mathcal{X}$ to the class of finite nonempty subsets of $\mathcal{X}$, which is obviously set-continuous. Applying the above lemma, we obtain its maximal fixed point $V_f[\mathcal{U}]$ in $V[\mathcal{U}]$. In fact

Theorem 4.2 *([3, Theorem 18.3]) If $\mathcal{U}$ is a set, then the class $V_f[\mathcal{U}]$ is a set.*

4.3 Unique solvability of set equations

In $V[\mathcal{U}]$, we can uniquely solve any system of set equations.

Let X be a set. *An X-set e* is a pointed graph $\langle V, E, r\rangle$ with $X \subset V$ satisfying the conditions that the point r does not belong to X and that the nodes in X have the outdegrees 0.

Given a family of sets $\left\{ s_x \;\middle|\; x \in X \right\}$, we can *substitute s_x into x* obtaining a set $e[\{ s_x/x \}]$. More precisely, let g_x be *the pointed graph of the set s_x*, whose node set $V(g_x)$ is the transitive extensional closure of the set $\{ s_x \}$ with s_x being the point and with the edge set $E(g_x)$ consisting of those pairs (u, v) with $v \in u$. Let $e[\{ g_x/x \}]$ be the pointed graph we obtain by pasting to e the g_x's at x. Let d be the decoration of this pasted pointed graph and we define finally

$$e[\{ s_x/x \}] := d(r).$$

Theorem 4.3 *([3, Theorem 8.4]) Let X be a set and $\left\{ e_x \;\middle|\; x \in X \right\}$ a family of X-sets indexed by X itself. Then the system of set equations:*

$$x = e_x, \quad x \in X. \tag{1}$$

has a unique solution, namely, a unique family of sets $\{ b_x \mid x \in X \}$ satisfying

$$b_y = e_y[\{ b_x/x \}] \quad (\forall y \in X).$$

An X-set e is called Φ-*local*, where Φ is a set continuous operator, if for any class B,

$$e[\{ s_x/x \}] \in B$$

whenever $s_x \in B$ for all $x \in X$. Then

Theorem 4.4 *([1, Theorem 6.10],[3, Theorem 16.7]) Let Φ a set continuous operator. If the right hand sides are Φ-local, then its solutions belong to the maximal fixed point of Φ.*

5 Subobject classifier

Let $\Omega = V_f\{ \top \}$ be the set of hereditary finite hypersets constructed from the urelement $\{ \top \}$[3] with this urelement added. Let $\bot$ denote the unique set satisfying

$$\{ \bot \} = \bot.$$

We define the structure of nondeterministic dynamical system in this set by

- $\top \to \top$
- $a \to b \overset{def}{\iff} b \in a$ if a is a set.

Theorem 5.1 *The correspondence $f \mapsto f^{-1}(\top)$ gives a bijection from $\mathcal{NDyn}(D, \Omega)$ to $Sub(D)$ for all $D \in \mathcal{NDyn}$.*

Proof. Let $A \subseteq D$ be a sub $\mathcal{NDyn}$. Consider the following set equation in the unknowns $\{ x_d \mid d \in D \setminus A \}$

$$x_d = \left\{ \begin{array}{l} \bot \\ \{ x_e \mid d \to e \notin A \} \bigcup \{ \top \mid d \to e \in A \}, \end{array} \right.$$

where the former is for the case when the orbit passing through d is disjoint from A. Note that the right hand sides are obviously pow_o-local. By Theorems 4.3 and 4.4, we have a

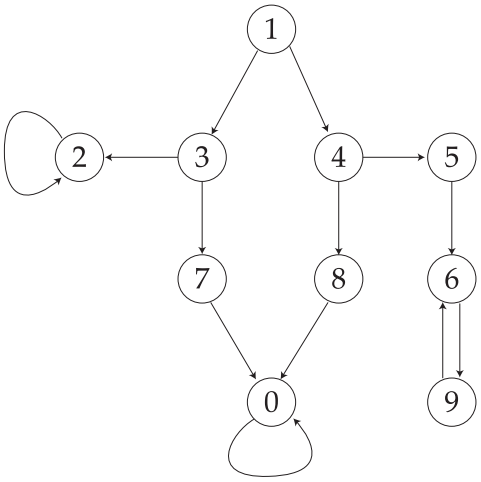
unique solution $x_d = a_d \in \Omega$. The map $\chi_A : d \mapsto a_d$ is obviously a simulation and satisfies $\chi_A^{-1}(\top) = A$. If there is another simulation $g : A \to \Omega$ satisfying the same condition, then $x_d = g(d)$ is also the solution of the above equation, whence g must coincide with χ_A. ∎

We note that in general the difficulty of defining "characteristic map" of a subobject of an object is its uniqueness, which in our case is guaranteed by the solution lemma.

6 Examples

6.1 Example 1

Let D be the following nondeterministic dynamical system:



Let $D_i (i = 1, 2, 3, 4)$ be the subobjects of D with the following nodes:

	nodes
D_1	0
D_2	8, 0
D_3	5, 6, 9
D_4	2, 6, 9, 0

We illustrate the images of χ_{D_i} and tabulate the truth value of each state(Figures 2–5 and Tables 1–4).

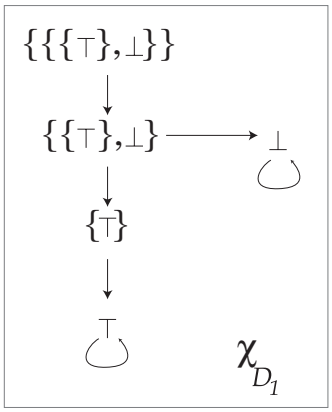


Figure 2: The image of χ_{D_1}

node	D_1
1	{ { { ⊤ }, ⊥ } }
2	⊥
3	{ { ⊤ }, ⊥ }
4	{ { ⊤ }, ⊥ }
5	⊥
6	⊥
7	{ ⊤ }
8	{ ⊤ }
9	⊥
0	⊤

Table 1: The truth values of χ_{D_1}

From this, we see that the states 3 and 4 have the same truth value with respect to the

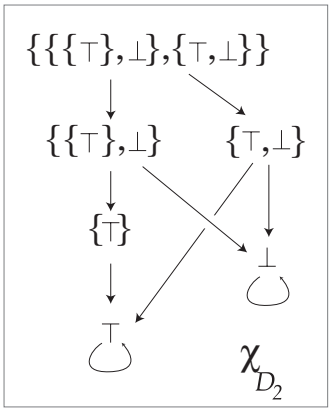


Figure 3: The image of χ_{D_2}

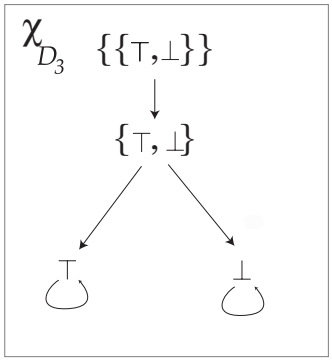


Figure 4: The image of χ_{D_3}

node	D_2
1	$\{\{\{\top\},\bot\},\{\top,\bot\}\}$
2	$\bot$
3	$\{\{\top\},\bot\}$
4	$\{\top,\bot\}$
5	$\bot$
6	$\bot$
7	$\{\top\}$
8	$\top$
9	$\bot$
0	$\top$

Table 2: The truth values of χ_{D_2}

node	D_3
1	$\{\{\top,\bot\}\}$
2	$\bot$
3	$\bot$
4	$\{\top,\bot\}$
5	$\top$
6	$\top$
7	$\bot$
8	$\bot$
9	$\top$
0	$\bot$

Table 3: The truth values of χ_{D_3}

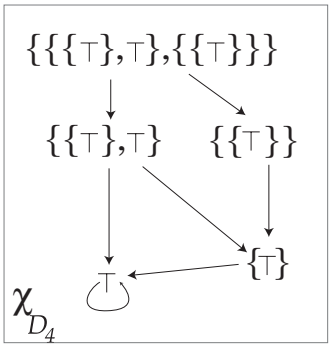


Figure 5: The image of χ_{D_4}

node	D_4
1	$\{\{\{\top\},\top\},\{\{\top\}\}\}$
2	$\top$
3	$\{\{\top\},\top\}$
4	$\{\{\top\}\}$
5	$\{\{\top\}\}$
6	$\{\top\}$
7	$\{\top\}$
8	$\{\top\}$
9	$\top$
0	$\top$

Table 4: The truth values of χ_{D_4}

stable properties D_1 but it is not the case with the stable properties D_2, D_3, D_4.

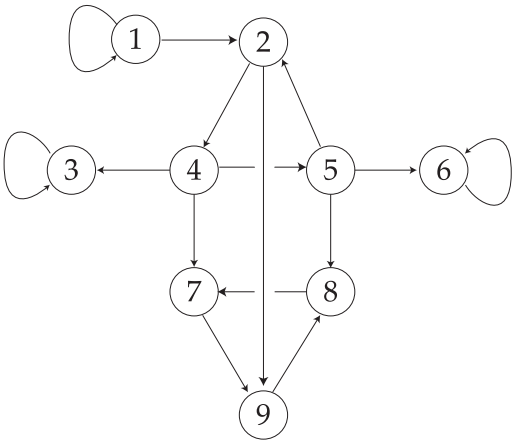
On the other hand, the state 3 has the same truth value $\{\{\top\},\bot\}$ with respect to the stable properties D_1, D_2, D_3 but has different value $\bot$ with respect to D_3.

Note that if the truth value of a state is $\bot$, then it means that the stable property will never hold from that state.

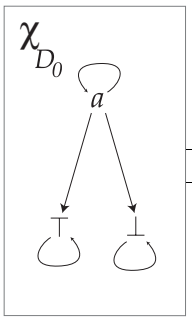
6.2 Example 2

We give now a nondeterministic dynamical system and a stable property which give truth values with non-well founded sets.

Let D be the following nondeterministic dynamical system:



Let A be the subobjects whose nodes are $\{7, 8, 9\}$. Then the image of its classifying map and the truth value of each state are as follows:



node	truth value
1	$\bot$
2	a
3	$\bot$
4	a
5	a
6	$\bot$
7	$\top$
8	$\top$
9	$\top$

Here a is the hyperset defined uniquely as the solution of the set equation $a = \{ a, \top, \bot \}$.

7 Concluding Remarks and Discussions

1. The category $\mathcal{NDyn}$ behaves much like a topos except the product is replaced by a general tensor product. However the logical operations on the subobject classifier seems quite different from that of topos.

2. We note that all the nondeterministic dynamical systems are *bisimilar* to each other. However when we study a process with respect to some stable properties on its states, then even bisimilar processes show utterly different behavior. Indeed only by choosing simulations as arrows, can we obtain subobject classifier which gives natural *truth values* which stable properties take at each state.

3. Most of our results extend to general labeling sets [14]. However, in this case, the existence of terminal objects is nontrivial and its construction is based on the final coalgebra theorems for endofunctors of the set category [1, 2] and the nontriviality of its structure complicates the theory considerably.

As is mentioned in the introduction, the structure of the category $\mathcal{NDyn}$ follows from a general results for certain class of coalgebras[10]. Since it relies however heavily on the advanced category theory, it is rather unclear what is going on in such simple category as $\mathcal{NDyn}$.

References

[1] P. Aczel, Non-well-founded Sets, CSLI Lecture Notes No. 14, Center for the Study of Language and Information (Chicago Univ Press, 1988).

[2] M. Barr, Terminal coalgebras in well-founded set theory, Theoretical Computer Science 114 (1993) 299-315.

[3] J. Barwise and L. Moss, Vicious Circles, (CSLI publications, Stanford, 1996).

[4] A. Joyal, M. Nielsen and G. Winskel, Bisimulation and open maps, Information and Computation 127(1996) 164–185.

[5] M. Mori and Y. Kawahara, A final coalgebra theorem for concurrent computation, in: E. Shibayama ed., The theory of parallel computation and its applications, Surikaisekikenkyusho-Kokyuroku, Vol. 902 (RIMS, Kyoto University 1995), 1–9.

[6] G.M.Kelly, Basic Concepts of Enriched Category Theory, Cambridge University Press 1982.

[7] F. W. Lawvere and S. H. Schanuel, Conceptual Mathematics, (Cambridge University Press, 1997).

[8] S. MacLane, Categories for the Working Mathematician (Springer, Berlin, 1971).

[9] N.P. Mendler, P. Panangaden, P.J.Scott and R.A.G. Seely, A logical view of concurrent constraint programming, Nordic Journal of Computing 2(1995) 182–221.

[10] J.Power, T. Tsujishita and H. Watanabe, An axiomatics for categories of transition systems and simulations, preprint.

[11] M. Shirahata, Linear Set Theory, dissertation, Department of Philosophy, Stanford University 1994.

[12] M. Shirahata, A coherence space semantics for linear set theory, in: N. Suzuki ed., New Aspects in Non-Classical Logics and Their Kripke Semantics Surikaisekikenkyusho-Kokyuroku, Vol. 1010 (RIMS, Kyoto University 1997).

[13] T. Tsujishita and H. Watanabe, Monoidal closedness of the category of simulations, Hokkaido University Preprint Series in Mathematics 392 (1997).

[14] H. Watanabe, The subobject classifier of the category of simulations (draft).

[15] H. Watanabe, The category of simulations (in preparation).

[16] Winskel, Glynn and Nielsen, M., Models for concurrency, in:S. Abramsky, D.B. Gabbay, and T.S.E. Maibaum eds., The Handbook of Logic in Computer Science, Vol 1 (Clarendon, Oxford, 1992),1–139.