Let us start with cohesions. First of all, Lawvere has no concept of cohesion, at most he has a family notion containing loosely all mathematical conceceptualizing of 'spaces' with let us say a bias for 'tame topology' (this comes through in the 2007 como lectures or his writing on functional analysis). Then he has a more less stable system for axiomatic cohesion since 1985, this for him hinges on the topos as generalized space vs. categories of general spaces dichotomy. In my view, it is not clear that this is a natural and useful concept at all: for one the axioms are itself a bunch of concepts like local connectedness, connectedness of Omega, functorial passage to the homotopy category, then the axioms are not really the same between 1985 to 2007 and last Urs who he is the only person whoever has done anything with these ideas in geometry either drops and adds axioms from the list, hence I come to the conclusion that axiomatic cohesion is rather a bunch or bundle of concepts and when I want to convey them to a non-mathematical audience I'd rather focus on the individual concepts than on any particular package of them. Then I ask what are the basic and most important ingredients in the bundle: It don't think it to be the continuity principle or the preservation of products by the components functor nor the Nullstellensatz which are rather technical and only there (when they are there at all! cf. 1985 or 1990 axioms) to oil a machinery whence I think, well the most basic concept involved is that of an adjoint modality. If I assume then that the audience are philosophers the approach over modalities a la Reyes et al. seems extremely advantageous because it relates to Kripke semantics and modal logic with which these people are already familiar with. Then if you look in Reyes' article on modal logic and reference (Notre dame J. 90/91), Reyes-Zolfaghari in the Como 90 proceedings and the more linguistic developments in the various papers coauthored with McNamara throughout the 1990s you'll find a careful motivation of the adjunction analysis of the variable vs. constant set contrast (cohesive vs. discrete), as well as motivations for local connectedness, in the 1999 paper on mass and count nouns the extensive/intensive quantity concepts (even here Lawvere has several such concepts!) are illustrated etc. all this happens around the notion of a 'kind' with a lot of illustrative example, the usual presheaf suspects. Then if your audience understands that they might be prepared for the 2007 axioms but note that even Menni writes in his researchs papers that he does not really understand some of the concepts introduced there!

Concerning the **spec construction**, to some extent, Lawvere derives axiomatic cohesion from it at least since the 1975 Eilenberg-festschrift paper, and the Gaeta topos[1] construction G(C) is also the starting point of the discussion in 1991 Como papers and in the development there is duality between extensive vs. intensive basically space vs. algebra of functions on it: the equation spec(B1xB2)=spec(B1)+spec(B2) in the 1975 paper just states that the category of spaces=specs is an extensive category whereas there dual algebras are just the quantites measured in the ground field k, the toposes arising there like the big Zariski topos, or likely also G(C) which consists of connected specs, satisfy certainly most of the axioms for big toposes aka. cohesion. So if you want

[1] Check out the nLab entry on the Gaeta topos for some further pointers!

to you could start with spec use the Zariski topos and then go on from there to motivate the axioms for cohesions - only as I suggested last time it might be more fun to work out this synthetic algebraic geometries for some unusual example as described/suggested in the 2007 paper on rig geometry, it would certainly involve less hardcore algebraic geometry than the using Zariski topos itself. This is certainly the way Lawvere came to his axioms when in 1975 he realized that the toposes resulting from this algebraic geometry are intuitively categories of spaces not a single generalized space, afterwards trying to get a grip on the difference by using the results of the Grothendieck school on the big Zariski as guiding principle ultimately ending up with the 4-Yoga of adjoint functors (the chocolate medal exercise in SGA4).
Concerning the **adequacy comonad**: First of all, nobody besides Lawvere uses 'adequacy' anymore and by not using the usual terms namely dense and codense you just throw your audience in confusion. The most familiar form of these ideas goes under the name of a codensity monad and is a central concept in categorical shape theory which is a general theory of approximation (There is a nice monograph by Tim Porter with Cordier on this - check out the nLab on codensity monad and shape theory, there is also a very nice TAC article by Tom Leinster on the concept. By the way, there is also an article by Renato Betti on shape theory in 2-categorical context that uses a concept of discreteness which might be linkable to some adjunction a la adjoint modality). What you and Lawvere call 'adequacy comonad' is very likely a special case of the dual concept studied under the name 'model induced cotriple' by Applegate and Tierney in LNM 1969 volume on triples (cf. TAC reprints), the piecing together of spaces by simple spaces is the leading idea and the topos of simplicial sets a guiding example which of course is also a classical guiding example of a big topos (unless Lawvere introduced a continuity axiom in 2007 to rule out these 'combinatorial' examples!), so you could start with SSet as an example of a adequacy comonad and then show that it satisfies also all the cohesive axioms with continuity a late-comer introduced to especially rule it out!