JFP, 49 pages, 2023. © Cambridge University Press 2023
doi:10.1017/xxxxx

The graphical theory of monads

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Abstract

The formal theory of monads shows that much of the theory of monads can be developed in the abstract at the level of 2-categories. This means that results about monads can established once and for all, and simply instantiated in settings such as enriched category theory.

Unfortunately, these results can be hard to reason about as they involve more abstract machinery. In this paper, we present the formal theory of monads in terms of string diagrams — a graphical language for 2-categorical calculations. Using this perspective, we show that many aspects of the theory of monads, such as the Eilenberg–Moore and Kleisli resolutions of monads, liftings, and distributive laws, can be understood in terms of systematic graphical calculational reasoning.

This paper will serve as an introduction both to the formal theory of monads and to the use of string diagrams, in particular, their application to calculations in monad theory.

1 Introduction

Street's formal theory of monads (Street, 1972) shows that a large part of the theory of monads is independent of the specifics of categories, functors, and natural transformations, and can be developed in the abstract. This insight both clarifies the nature of the original theory, and allows the transfer of results to other settings, such as enriched category theory. Unfortunately, much of the original work is phrased in terms of various 2-categorical constructions, in particular, 2-adjunctions, and the use of this machinery can make the material inaccessible to many.

In this paper, we show that the formal theory of monads can be developed in much more elementary terms, by systematic calculation using equational reasoning, not with ordinary mathematical symbols, but with *diagrams*. Our aims are twofold. Firstly, we aim to illustrate the power of the graphical language of string diagrams, by explicitly proving non-trivial results in formal monad theory as our running example. Secondly, we hope to present formal monad theory in a more elementary light, opening the power of Street's vision to a broader audience.

We assume some basic knowledge of category theory, but otherwise provide a self-contained account of the required material. We begin with background on string diagrams and 2-categories, and the mathematical structures of interest in Sections 2 and 3. This

material is adapted from that in Hinze and Marsden (2023), although the account we present here is specialized to the needs of later calculations. The subsequent sections present entirely new material that has not appeared in print before. In Section 4 we introduce the key abstraction of an Eilenberg–Moore object. Section 5 develops what is probably the fundamental result of formal monad theory, that Eilenberg–Moore objects imply that every monad arises from an adjunction, and Section 6 then shows that this adjunction is a canonical choice. Sections 7 and 9 present results about lifting 1-cells and monads, and require significantly more involved calculations which provide serious illustrations of string-diagrammatic techniques. The intervening Section 8 shows how duality can be exploited to recover results about Kleisli constructions and comonads for free, and relates these dualities to the symmetries of our diagrams.

1.1 Contributions

The contributions of the paper are as follows:

- We provide the first string-diagrammatic account of axiomatic monad theory in a 2-categorical setting. Our work presents an elementary, graphical formulation of Streets formal theory of monads, obviating the need for complex machinery such as 2-adjunctions or constructions of auxiliary 2-categories. We cover several fundamental aspects, beginning with an explicit diagrammatic definition of Eilenberg–Moore objects and their universal property. This is followed by explicit proofs that:
  - Every monad arises from an adjunction.
  - The adjunction is terminal among such resolutions.
  - There is a one-to-one correspondence between so called Eilenberg–Moore laws and liftings of 1-cells.
  - There is a one-to-one correspondence between Beck distributive laws and monad liftings to Eilenberg–Moore objects.
- We emphasize geometric intuition for calculational moves such as “dragging”, “bending” and “splitting” wires, and “sliding beads” along wires. Our approach highlights the need for good notational choices, particularly in this graphical setting where there is great freedom to express ideas. We include illustrations of good versus bad diagrammatic choices to develop the intuition of readers new to these techniques — Section 2.5 gives a foretaste.
- We introduce some diagrammatic notational innovations:
  - A diagrammatic technique to focus on regions of interest in diagrams where proof steps will occur.
  - A simple systematic approach to Eilenberg–Moore objects, emphasizing the connection to the base category.
  - The elementary, but crucial, use of explicit identity morphisms to both consistently maintain type information, and to isolate the role of equations between 1-cells in our proofs.

1.2 Related work

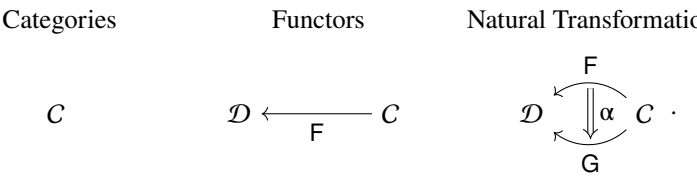
Book length expositions of string diagrams, in the style of the present work can be found in Hinze and Marsden (2023), and its forthcoming sequel (Hinze and Marsden, 2024). Hinze and Marsden (2016) develop the theory of distributive laws using string diagrams for categories, functors, and natural transformations — the calculations transfer directly to the more abstract setting of the current paper. Other topics dealt with using similar techniques can be found in Piróg and Wu (2016) and Hinze and Marsden (2016). Earlier accounts of string diagrams for categorical calculations, with some stylist differences, can be found in Curien (2008) and Marsden (2014).

String diagrams can be used in a variety of settings, with probably the most common being various types of monoidal categories. These methods have been applied in many areas, including quantum theory (Coecke and Kissinger, 2017; Coecke and Gogioso, 2022; Heunen and Vicary, 2019), natural language semantics (Coecke et al., 2010), signal flow graphs (Bonchi et al., 2015), control theory (Baez and Erbele, 2015), economic game theory (Ghani et al., 2018) and (Ghani et al., 2018), Markov processes (Baez et al., 2016), analogue (Baez and Fong, 2015) and digital (Ghica and Jung, 2016) electronics and hardware architecture (Brown and Hutton, 1994), machine learning (Fong et al., 2019), linear algebra (Sobocinski, 2019), and logic (Clingman et al., 2021). General background on these different graphical calculi can be found in Selinger (2011). Although each of these diagrammatic languages has its own distinctive "feel", they have much in common, and the present work could serve as an introduction to the general methods.

2 Background: string diagrams

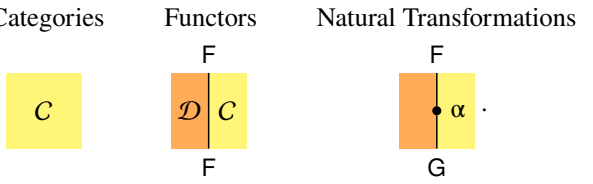
We begin our introduction to string diagrams in terms of conventional category theory. That is, we consider categories, functors, and natural transformations. Once we have introduced the required notation, we move to the more abstract setting of 2-categories, of which **Cat**, the 2-category of categories, functors, and natural transformations, is the paradigmatic example. A systematic account of this style of graphical reasoning for elementary category theory can be found in Hinze and Marsden (2023).

We first recall the traditional notation most commonly used in category theory.



Categories are drawn as points or vertices, zero-dimensional objects, typically depicted as letters C, D, and so on. A functor is drawn as a one-dimensional object, in the form of an arrow between two categories. Finally, natural transformations appear as two-dimensional objects, as double arrows intuitively filling the region between a parallel pair of functor arrows. Given its appearance, the notion for a natural transformation is sometimes referred to as an **eye diagram**.

String diagrams invert these notational conventions, considering their **Poincaré dual**.



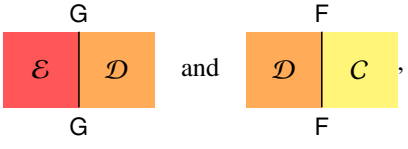
Now categories are two-dimensional objects, depicted as coloured regions, functors remain one-dimensional, with the slight tweak that they are now drawn vertically, as lines, or **wires**, separating the coloured regions of their domain to the right and codomain to the left. Functors can only exit the diagram at the top or bottom boundary, where their wire is labelled with their name. Natural transformations are now the focus of attention, drawn as zero-dimensional **vertices**, appearing on the wires denoting their domain above and codomain below. This change of emphasis better reflects the significance of the concepts. As described in Mac Lane (1998):

> *"Category" has been defined in order to be able to define "functor" and "functor" has been defined in order to be able to define "natural transformation."*

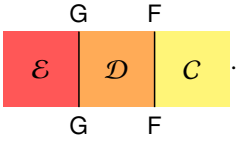
With the string diagram notation, natural transformations are the focus of attention, with categories and functors auxiliary concepts that provide vital type information about how they can be combined. In what follows, the names $\mathcal{C}$, $\mathcal{D}$, ... will range over categories, F, G, ... over functors, and α, β, ... over natural transformations.

2.1 Composition

Of course, these concepts do not live in isolation, we build more complicated functors and natural transformations by composition, and this is reflected in our diagrams. Given functors $\mathsf{G} : \mathcal{D} \to \mathcal{E}$ and $\mathsf{F} : \mathcal{C} \to \mathcal{D}$, in pictures,



their composite $\mathsf{G} \circ \mathsf{F} : \mathcal{C} \to \mathcal{E}$ is drawn as follows:



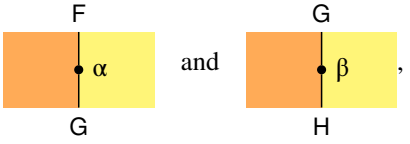
As a special case, we draw the identity functor on $\mathcal{C}$ as an empty region:



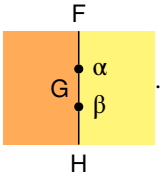
As will be discussed later, it will occasionally be useful to break this convention, and explicitly draw dashed identity wires for clarity.

Using these conventions, both the unitality equations $\mathsf{Id}_{\mathcal{D}} \circ \mathsf{F} = \mathsf{F} = \mathsf{F} \circ \mathsf{Id}_{C}$ and the associativity equation $(\mathsf{H} \circ \mathsf{G}) \circ \mathsf{F} = \mathsf{H} \circ (\mathsf{G} \circ \mathsf{F})$ are built into the notation.

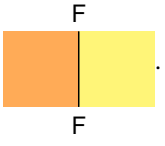
Given natural transformations,



we depict their **vertical composite** $\beta \cdot \alpha : \mathsf{F} \dot{\to} \mathsf{H}$ as the following diagram:



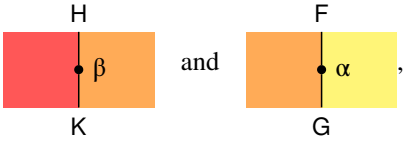
As a special case, an identity natural transformation is represented by the edge for the corresponding functor:



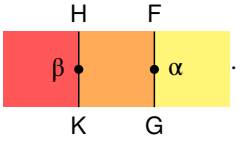
As with identity functors, it will be useful to break the convention for identity natural transformations on occasion, and explicitly draw them as white circles. This will be discussed in detail later.

Again, using these conventions the unitality equations $id_{\mathsf{G}} \cdot \alpha = \alpha = \alpha \cdot id_{\mathsf{F}}$ and the associativity equation $(\gamma \cdot \beta) \cdot \alpha = \gamma \cdot (\beta \cdot \alpha)$ are built into the notation.

There is a second notion of composition. For natural transformations



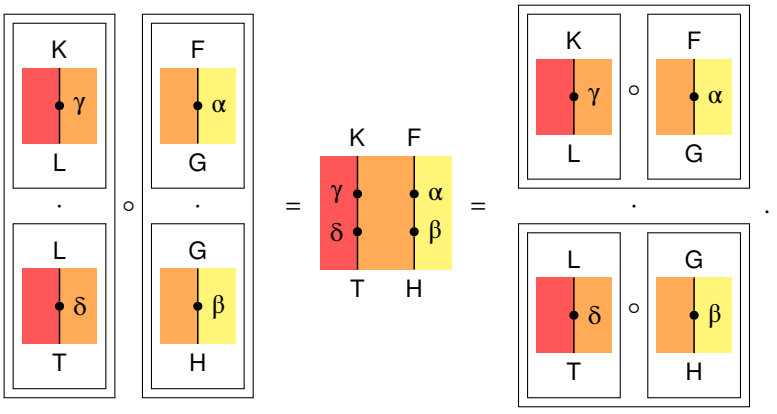
we denote their **horizontal composite** $\beta \circ \alpha : \mathsf{H} \circ \mathsf{F} \dot{\to} \mathsf{K} \circ \mathsf{G}$ via horizontal diagrammatic composition:



These two forms of composition satisfy the **interchange law**:

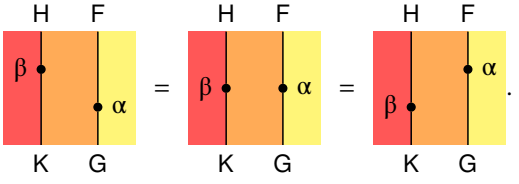
$$(\delta \cdot \gamma) \circ (\beta \cdot \alpha) = (\delta \circ \beta) \cdot (\gamma \circ \alpha). \tag{2.1}$$

which again is built into the notation:



Although unitality and associativity can be built into conventional linear notation by leaving out identities and brackets around composition, *the interchange law is an intrinsically two-dimensional notion*.

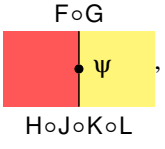
As a corollary of the notational convention for identities, we obtain what Dubuc suggestively calls the **elevator equations** (Dubuc and Szyld, 2013). Intuitively, we can slide vertices up and down wires, which is a very useful manoeuvre to be able to perform during proofs.



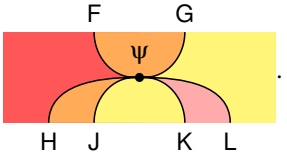
(2.2)

2.2 Natural transformations between composite functors

So far, all our natural transformations have been "one-in one-out". To perform useful calculations, we need to look in more detail at natural transformations between composite functors. For example, we could draw the natural transformation $\psi : F \circ G \nrightarrow H \circ J \circ K \circ L$ as:



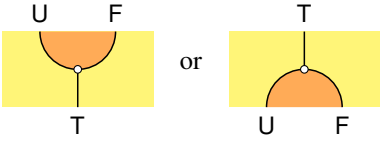
but this is probably not the best choice as it mixes symbolic and graphical notation. Instead, we draw separate wires for each element of the composite:



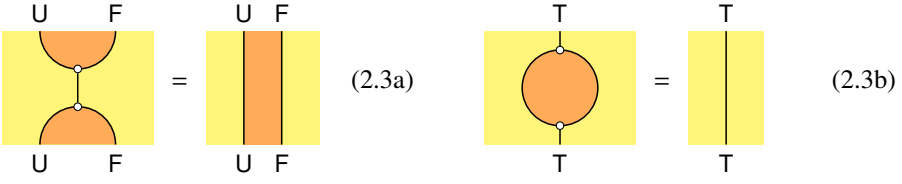
This explicitly exposes the composite wires as multiple threads, which makes it possible to access them directly in our calculations.

2.3 Identity natural transformations

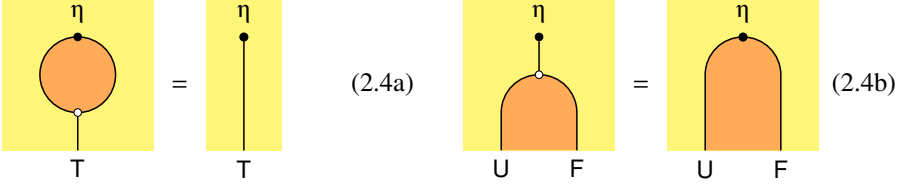
A related topic is the handling of equations *between functors*. For example, if we have an equation such as $T = U \circ F$, we can exploit this in our diagrams using explicit **identity vertices**, depicted using special white circles, in our diagrams:



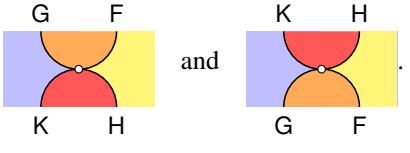
These explicit identity vertices allow us to "expand" and "collapse" composite wires, as is convenient during our proofs. Identity vertices satisfy obvious cancellation identities:



We can also fuse identities with other vertices, for example for $\eta : \mathsf{Id} \dot{\to} T$, with $T = U \circ F$ as before:



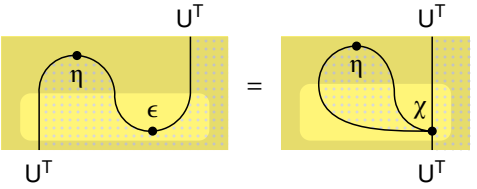
This use of explicit identities extends in the obvious way to more complex equations, such as $G \circ F = K \circ H$ which has composite functors on both sides of the equation, and will be witnessed by vertices of the form:



The notion of an Eilenberg–Moore object introduced in Section 4 introduces equations such as these, and so it is important to have a diagrammatic technique to handle them smoothly.

2.4 Focusing

During calculations, it can sometimes be hard even for experts to identify exactly how a diagram has been changed during a proof step. To address this, we introduce a visual cue to focus on a particularly part of a diagram, by highlighting the region of interest. For example, in the following two diagrams

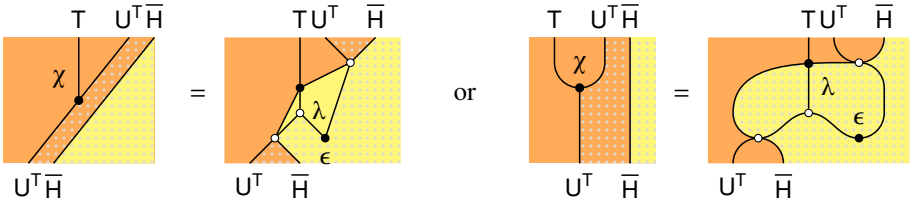


an equality has been applied to the rectangular regions that appear to be in the spotlight, whilst the regions in shadow remain passive. This graphical trick is a purely cosmetic visual aid, and is not a formal part of the string diagram notation. We indicate the introduction or moving of these highlights in proofs with the hint "focus", and their removal to reveal a full diagram with "unfocus".

2.5 Diagram Evolution

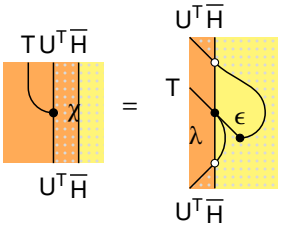
On one level of abstraction a string diagram is simply a planar graph: vertices are connected by lines, partitioning the plane into regions. (Lines are implicitly oriented from top to bottom, so they must not have a horizontal tangent.) Two-dimensional notation provides considerable artistic freedom. It is quite rare that one gets a drawing "right" the first time — a diagram typically goes through a series of evolutionary steps. To illustrate, we have recorded the genesis of one of the central calculational rules in the paper, Equation (7.2). For the following discussion, it is not important to understand the significance of this equation.

The ugly. Drawing string diagrams is to some extent a matter of personal taste: some authors simply connect the vertices by straight lines, others prefer curvy diagrams. Our initial attempt at Equation (7.2) is shown on the right below.



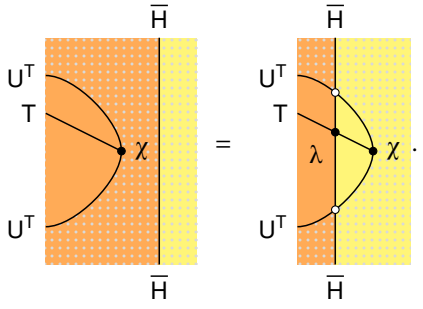
Even though they look quite different, the two equations are semantically equivalent. (In general, string diagrams that are equivalent up to planar isotopy denote the same natural transformation. A planar isotopy is a continuous deformation of a plane diagram that preserves cusps, crossings, and the property of having no horizontal tangents.) However, neither style provides any intuition about the calculational rule or its applicability. Especially the larger diagrams appear to be cryptic muddles of wires. A geometric intuition is, however, important to be able to identify situations in which the rewrite rule can be profitably applied.

The bad. We made some progress when we drew the path that connects the two occurrences of $\overline{H}$ as a continuous, *straight vertical* line.



The vertical line can be seen as a border. The rendering of the equation suggests a calculational manoeuvre: stuff on the left of the border transmogrifies into stuff on the right, or vice versa. However, the diagram on the right-hand side is still a little unwieldy.

The good. For the final version of the equation, we unfolded the definition of the vertex ϵ, expressing it in terms of χ.



This tweak was enlightening as the resulting equation provides a clear geometric intuition: when applied from left to right, the vertex χ and the three wires attached to it are dragged across the border. When a wire crosses another wire, it "creates" a vertex, which is why the diagram on the right has three additional vertices. Sections 7 and 9 detail the significance of these vertices.

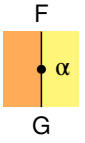
2.6 2-Categories

So far, we have talked of composing categories, functors, and natural transformations. A **2-category** is an abstraction of the composition of these entities, in the same way we can think of categories as an abstraction of how functions between sets compose. Instead of categories, functors, and natural transformations, we now talk of **0-cells**, **1-cells**, and **2-cells** respectively. There are two sorts of composition, horizontal and vertical, and these satisfy all the equations previously introduced. The string diagrammatic notation transfers seamlessly to this abstract setting.

There are several reasons to move to this level of abstraction. Results can be proved at greater generality, exploiting the fact that many other categories "look similar" to **Cat**, the 2-category of categories, functors, and natural transformations. It is also clarifying to understand which results depend on the specifics of ordinary category theory, and which

only depend on higher-level structure. Mathematicians study group theory by abstracting away from concrete groups for the same reason.

The relationship between the abstract 2-categorical terminology, the concrete example of **Cat**, and the string diagrammatic notation is summarized below:

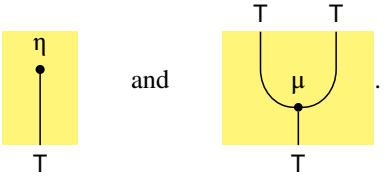
Abstractly	0-cells	1-cells	2-cells
Concretely	Categories	Functors	Natural Transformations
Graphically			 .

3 Background: key mathematical structures

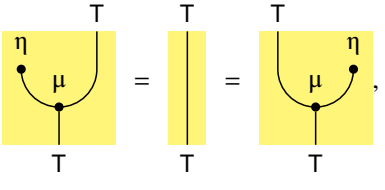
Now that we have fixed our setting and notational convention, it is time to introduce the key mathematical structures of interest. This will also serve as an introduction to applying the string diagrammatic notation introduced in Section 2.

3.1 Monads

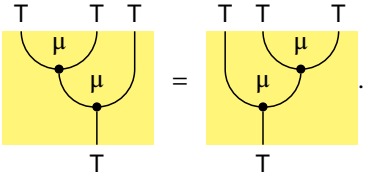
Our fundamental object of study is that of a monad. A **monad** on 0-cell C consists of a 1-cell $\mathsf{T} : C \to C$, and **unit** and **multiplication** 2-cells



The unit $\eta : \mathsf{Id} \dot{\to} \mathsf{T}$ looks a tad like a lollipop, the multiplication $\mu : \mathsf{T} \circ \mathsf{T} \dot{\to} \mathsf{T}$ resembles a tuning fork. Unit and multiplication are required to satisfy **unitality** and **associativity** equations:

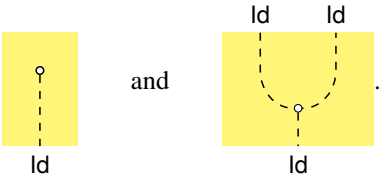


(3.1a)



(3.1b)

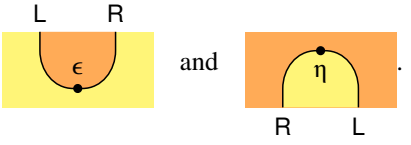
As a trivial but useful example of a monad, we note that the identity $\mathsf{Id} : C \to C$ carries the structure of a monad:



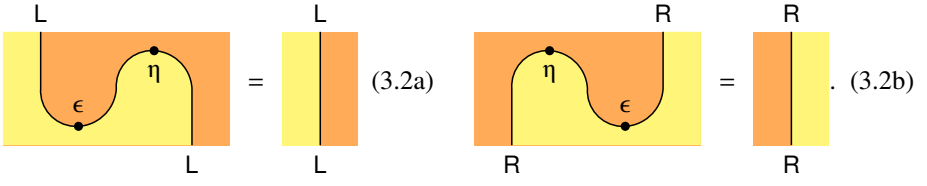
Verifying all the axioms boils down to confirming equations of the form:

3.2 Adjunctions

An **adjunction** between a pair of 1-cells $\mathsf{L} : \mathcal{D} \to C$ and $\mathsf{R} : C \to \mathcal{D}$ consists of a pair of **counit** and **unit** 2-cells:



The counit $\epsilon : \mathsf{L} \circ \mathsf{R} \dot{\to} \mathsf{Id}$ resembles a curved cup, the unit $\eta : \mathsf{Id} \dot{\to} \mathsf{R} \circ \mathsf{L}$ a cap. They are required to satisfy the following **snake equations**, which intuitively allow us to straighten out bends in our string diagrams, by pulling a wire straight.



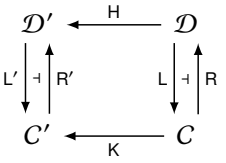
In the case of categories, functors, and natural transformations, this definition is equivalent to the many other formulations of adjunctions (Fokkinga and Meertens, 1994), but is particularly convenient for graphical reasoning.

An adjoint situation with **left adjoint** $\mathsf{L} : \mathcal{D} \to C$ and **right adjoint** $\mathsf{R} : C \to \mathcal{D}$ is often denoted $\mathsf{L} \dashv \mathsf{R} : C \rightharpoonup \mathcal{D}$, with the understanding that the units are given implicitly.

3.3 Maps of adjunctions

As we have introduced adjunctions, as usual in category theory, we are beholden to consider morphisms between these structures.

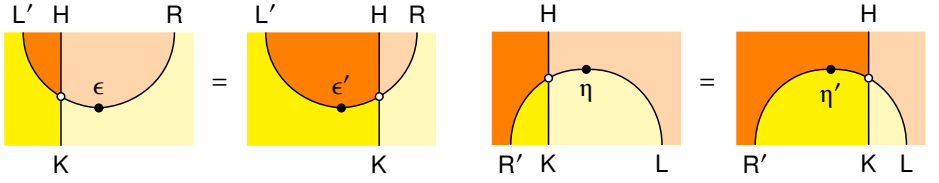
For adjunctions $\mathsf{L} \dashv \mathsf{R} : \mathcal{C} \rightharpoonup \mathcal{D}$ and $\mathsf{L}' \dashv \mathsf{R}' : \mathcal{C}' \rightharpoonup \mathcal{D}'$, and $\mathsf{H} : \mathcal{D} \to \mathcal{D}'$ and $\mathsf{K} : \mathcal{C} \to \mathcal{C}'$ as in the diagram below:



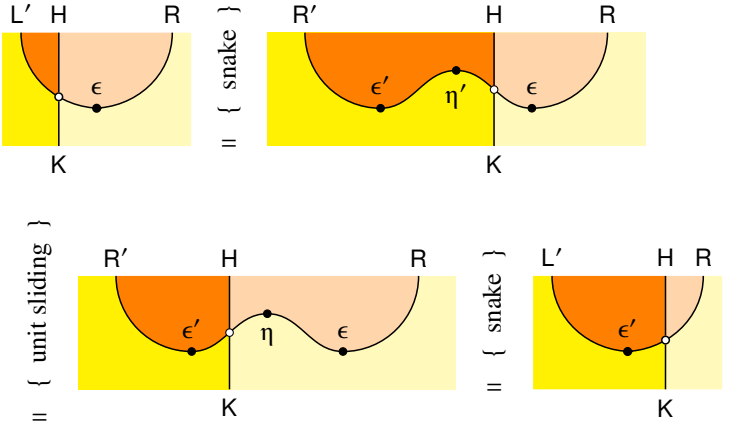
we say that the pair H, K is a **maps of adjunctions** if the following conditions hold:

$$\mathsf{K} \circ \mathsf{L} = \mathsf{L}' \circ \mathsf{H} \qquad \mathsf{H} \circ \mathsf{R} = \mathsf{R}' \circ \mathsf{K} \qquad \mathsf{K} \circ \epsilon = \epsilon' \circ \mathsf{K} \qquad \mathsf{H} \circ \eta = \eta' \circ \mathsf{H} \tag{3.3}$$

The conditions relating the units and counits have a nice visual representation exploiting identity 2-cells:



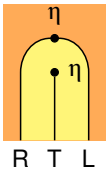
which we shall refer to as the **sliding equations** for obvious reasons. In fact, each of the sliding equations implies the other. To show unit sliding implies counit sliding, we calculate:



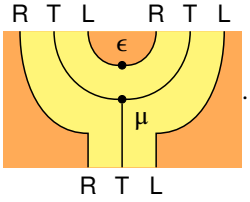
The other direction is symmetrical.

3.4 Huber's construction

Given a monad T and an adjunction $\mathsf{L} \dashv \mathsf{R}$, we can build a new monad with underlying 1-cell $\mathsf{R} \circ \mathsf{T} \circ \mathsf{L}$ using **Huber's construction** (Huber, 1961). The resulting has a unit and multiplication, graphically given by "wrapping up" the monad T using the adjunction:

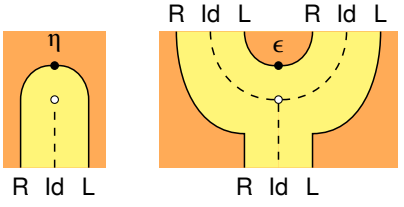


(3.4a)



(3.4b)

The special case we are particularly interested in is when the monad T is the identity monad. The resulting composites:

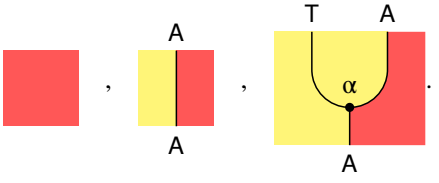


then show that every adjunction induces a monad. The proofs of the axioms, (3.1a) and (3.1b), are left as instructive exercises to the reader; the solutions can be found in Hinze and Marsden (2023).

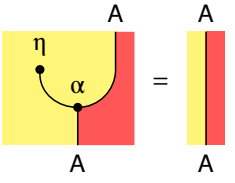
3.5 Monad actions and action morphisms

Adjunctions and monads are central concepts of category theory. The third key structure we require is the less well-known notion of a monad action. Intuitively, an action can be thought of as a generalizations of the possibly more familiar idea of an Eilenberg–Moore algebra.

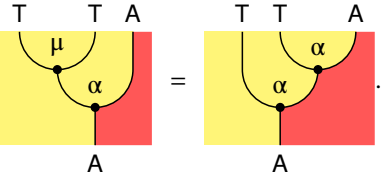
Given a monad $(\mathsf{T} : C \to C, \eta, \mu)$, a **left action** of T, or **left T-action**, consists of a 0-cell $\mathcal{E}$, a 1-cell $\mathsf{A} : \mathcal{E} \to C$ and a 2-cell $\alpha : \mathsf{T} \circ \mathsf{A} \nrightarrow \mathsf{A}$, graphically:



We occasionally say α is a **left T-action on** A for emphasis, and that $\mathcal{E}$ is its **underlying 0-cell**. The 2-cell α must respect the unit and multiplication of the monad T, in that the following equations hold.



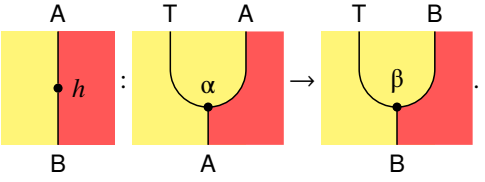
(3.5a)



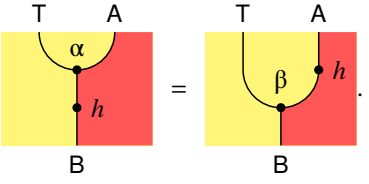
(3.5b)

For conciseness, we often refer to an action using the diagram for its underlying 2-cell, as this implicitly defines all the other data.

Given a pair of actions of the *same* monad with the *same* underlying 0-cell, an **action morphism**, written $h : (\mathsf{A}, \alpha) \to (\mathsf{B}, \beta)$, graphically:

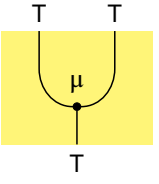


is a 2-cell $h : \mathsf{A} \dot{\to} \mathsf{B}$ such that the **right-turn axiom** holds:



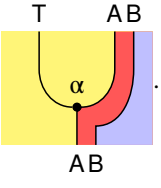
(3.6)

As a first example of an action, we note that Equations (3.1a) and (3.1b) imply that the monad multiplication



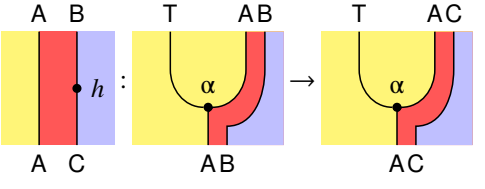
is a left T-action.

Furthermore, given any left action $\alpha : \mathsf{T} \circ \mathsf{A} \dot{\to} \mathsf{A}$ and 1-cell $\mathsf{B} : \mathcal{F} \to \mathcal{E}$, we can form a new action by "outlining" on the right-hand side, as follows:

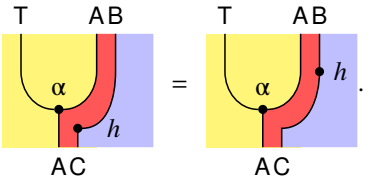


(3.7)

In fact, for a fixed left T-action $\alpha : \mathsf{T} \circ \mathsf{A} \dot{\to} \mathsf{A}$ the outlining operation is functorial. If we have an arbitrary 2-cell $h : \mathsf{B} \dot{\to} \mathsf{C}$, then the composite

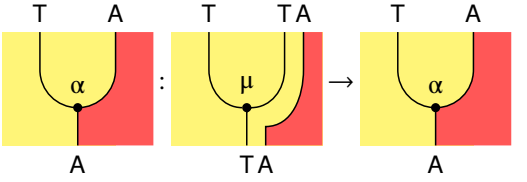


is a T-action morphism, as by the elevator equations (2.2):



Associativity and preservation of identities is trivial to see diagrammatically. Outlining will be a key feature of the universal property presented in Section 4.

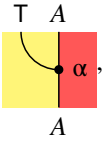
With these observations in place, we note that Equation (3.5b) is equivalent to saying a left T-action $\alpha : \mathsf{T} \circ \mathsf{A} \nrightarrow \mathsf{A}$ is simultaneously an action morphism of type:



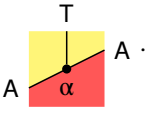
(3.8)

We make use of this idea in Section 5.

As with all string diagrams, it is worth considering different ways of depicting left actions. These can allow us to highlight the intuitions for manipulations performed in proofs, or simply be more convenient in certain situations. For example, we will often use the compact, less symmetrical rendering



which emphasizes the idea of T acting on the left of A. It is also sometimes convenient to adjust the orientation, so that the A wire lies in a more passive horizontal direction:



When reasoning with string diagrams, for example the proofs in this paper, we would encourage readers to play with alternative diagrammatic renderings to help clarify their understanding of the proof moves involved.

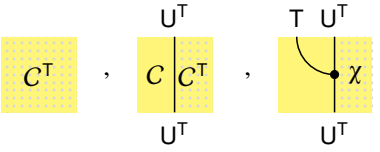
4 From Eilenberg–Moore categories to Eilenberg–Moore objects

Using the definition of monad, a reasonable amount of monad theory can be developed within a 2-category. For example, the graphical arguments about composing monads using distributive laws and the Yang-Baxter equation presented in Hinze and Marsden (2016), building upon the original work of Beck (1969) and Cheng (2011) in more conventional notation, transfer smoothly to the 2-categorical setting.

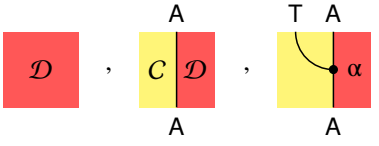
However, for many of the more interesting results, we need a bit more. In ordinary category theory, there are two categories associated with a monad T, the Eilenberg–Moore category (Eilenberg and Moore, 1965), commonly denoted C^{T}, and the Kleisli category (Kleisli, 1965), often denoted C_{T}. These constructions cannot be directly transferred to the 2-categorical setting, as they involve the explicit construction of new categories in terms of objects and arrows of the base category, and the structures of the monads involved. This presents a challenge for developing more serious aspects of monad theory at this level of abstraction.

Street (1972) resolved this problem by identifying the correct abstraction of the Eilenberg–Moore construction in the 2-categorical setting. In this section we introduce this machinery in diagrammatic terms, laying the foundations for the proofs in subsequent sections. Further background in more traditional notation can be found in Kelly and Street (2006) and Lack (2009); Lack and Street (2002).

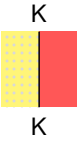
Given a monad $(\mathsf{T} : C \to C, \eta, \mu)$, a left T-action



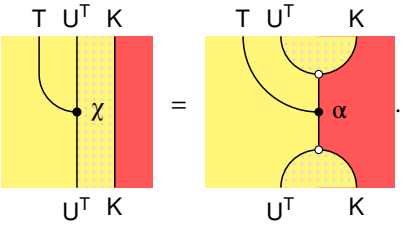
is **universal** if it satisfies the following two properties. *First*, for every left T-action



there exists a *unique* **comparison** 1-cell

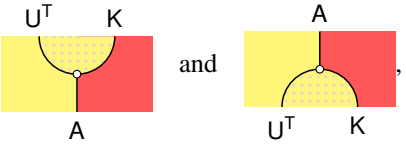


such that $\mathsf{U}^{\mathsf{T}} \circ \mathsf{K} = \mathsf{A}$ and

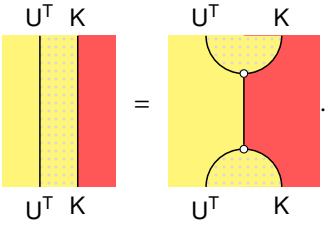


(4.1)

In Equation (4.1), we introduce identity cells:

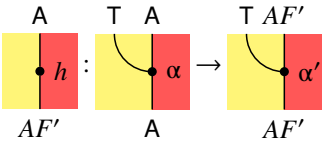


to ensure both diagrams have the same types at their boundary. These identities enjoy the following **fusion equation** between two parallel wires and an hourglass-shaped composite — shapes that make many appearances in what follows.

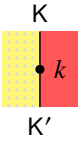


(4.2)

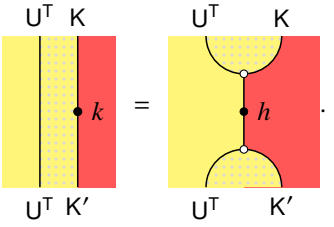
Second, given two left actions, α and α', with induced comparison 1-cells, K and K', and a T-action morphism



between them, there exists a *unique* **comparison** 2-cell



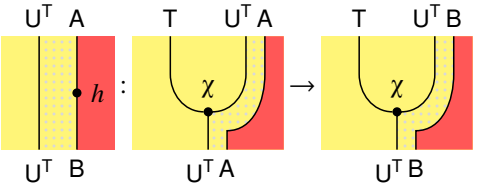
such that the following equality holds:



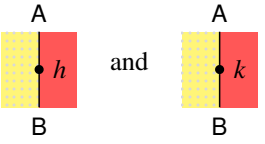
(4.3)

A 0-cell carrying the structure of a universal left action is referred to as an **Eilenberg–Moore object** for T, denoted C^{T}. In the graphical representation, an Eilenberg–Moore object C^{T} is represented by a "dotted" region, reusing the colour of C. Note we loosen some of our other conventions, and choose not to write χ^{T} making the monad explicit, as the monad should be clear from the context. (This is primarily for space reasons to avoid big labels inside our diagrams.) As one might expect, in the case of categories, functors, and natural transformations, the Eilenberg–Moore object of a monad is the same thing as its Eilenberg–Moore category (Eilenberg and Moore, 1965).

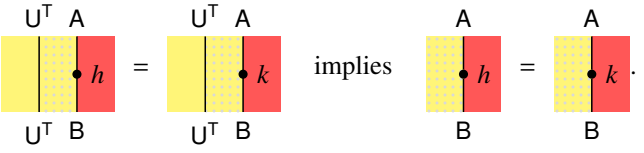
We make an observation which will prove important for calculations. Given a 2-cell $h : \mathsf{A} \nrightarrow \mathsf{B}$ with $\mathsf{A}, \mathsf{B} : \mathcal{D} \to C^{\mathsf{T}}$, the composite



is an action morphism due to the functoriality of "outlining". Consequently, for 2-cells



the uniqueness part of the universal property immediately yields that



We will exploit this property by saying U^{T} is **left-cancellative**.

How is the universal property used in calculations? The typical pattern in later sections is that by carefully choosing suitable left T-actions and T-action morphisms, the universal property yields new 1-cells and 2-cells, and each case these satisfy equations that we can exploit in our calculations. We shall see many examples of this technique in what follows.

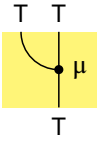
5 Every monad with an Eilenberg–Moore object is induced by an adjunction

We begin our exploration of how the universal property of Eilenberg–Moore objects can be applied to establish results in monad theory. Our aim is to show that every monad $(\mathsf{T} : C \to C, \eta, \mu)$ with an Eilenberg–Moore object is induced by an adjunction. One could describe this as the **fundamental result of formal monad theory**, as every subsequent result hinges on the existence of this adjunction. To establish the result we proceed in two steps: first, we construct an adjunction; second, we show that this adjunction gives the original monad, via Huber's construction.

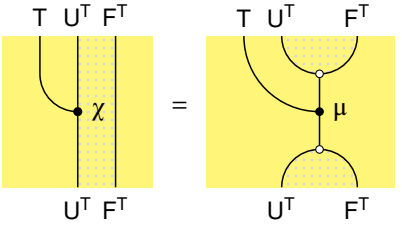
5.1 Every Eilenberg–Moore object induces an adjunction

The Eilenberg–Moore object immediately gives us a 1-cell $\mathsf{U}^{\mathsf{T}} : C^{\mathsf{T}} \to C$, which serves as the right adjoint. As a first step towards establishing an adjunction, we would like to find a

candidate left adjoint of type $C \to C^{\mathsf{T}}$. To this end, we recall that the monad multiplication



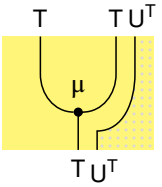
is simultaneously a left monad action. Therefore, by the universal property there exists a unique comparison 1-cell $\mathsf{F}^{\mathsf{T}} : C \to C^{\mathsf{T}}$ such that $\mathsf{U}^{\mathsf{T}} \circ \mathsf{F}^{\mathsf{T}} = \mathsf{T}$ and



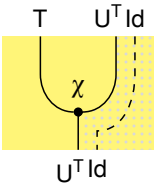
(5.1)

Since $\mathsf{U}^{\mathsf{T}} \circ \mathsf{F}^{\mathsf{T}} = \mathsf{T}$, an obvious choice for the unit of our adjunction is the unit of the monad T. Finally, the counit $\epsilon : \mathsf{F}^{\mathsf{T}} \circ \mathsf{U}^{\mathsf{T}} \dot{\to} \mathsf{Id}$ is constructed using a suitable comparison 2-cell. To this end, we need to find an action morphism between actions whose comparison 1-cells are $\mathsf{F}^{\mathsf{T}} \circ \mathsf{U}^{\mathsf{T}}$ and Id, respectively. We do not have to look far:

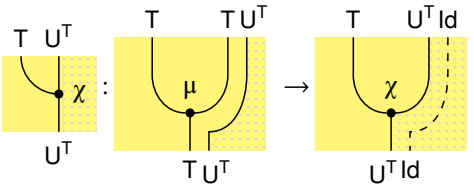
- the comparison 1-cell of the source action has to be $\mathsf{F}^{\mathsf{T}} \circ \mathsf{U}^{\mathsf{T}}$; as $\mathsf{U}^{\mathsf{T}} \circ (\mathsf{F}^{\mathsf{T}} \circ \mathsf{U}^{\mathsf{T}}) = \mathsf{T} \circ \mathsf{U}^{\mathsf{T}}$, the "outlined" multiplication, see also (3.7), is of the right type:



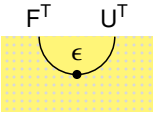
- the comparison 1-cell of the target action has to be Id; the universal action χ itself is of the required type:



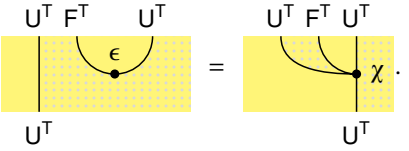
Now recall from Section 3.5 that an action is simultaneously an action morphism. In our case, the universal action χ is a morphism of the desired type:



Consequently, the universal property gives a comparison 2-cell:

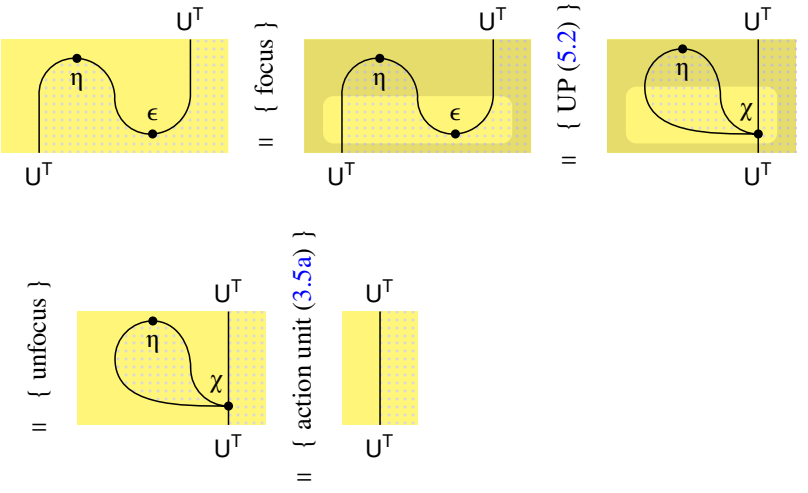


satisfying the following instance of (4.3):



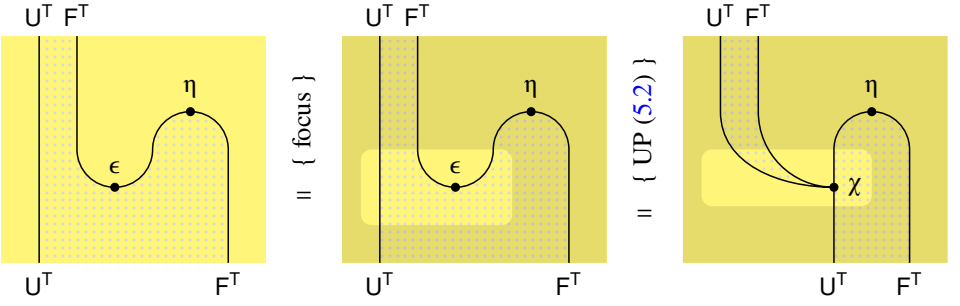
(5.2)

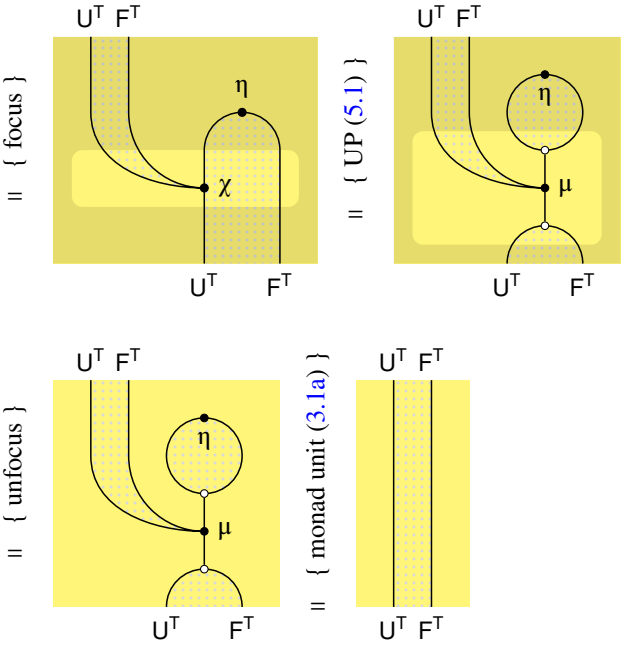
We now have 1-cells $\mathsf{F}^\mathsf{T} : \mathcal{C} \to \mathcal{C}^\mathsf{T}$ and $\mathsf{U}^\mathsf{T} : \mathcal{C}^\mathsf{T} \to \mathcal{C}$, and 2-cells $\eta : \mathsf{Id} \dot{\to} \mathsf{U}^\mathsf{T} \circ \mathsf{F}^\mathsf{T}$ and $\epsilon : \mathsf{F}^\mathsf{T} \circ \mathsf{U}^\mathsf{T} \dot{\to} \mathsf{Id}$. To show that these are such that $\mathsf{F}^\mathsf{T} \dashv \mathsf{U}^\mathsf{T}$, it remains to confirm that η and μ satisfy the snake equations. To establish equation (3.2a), we reason as follows:



The second snake equation (3.2b) is slightly more subtle to prove. Trying to establish the equation directly, we quickly discover there is no obvious calculation step we can apply. The plan therefore is to exploit the fact that U^T is left-cancellative, and first prove an equality for a more complex diagram, which allows us to get the proof off the ground.

We calculate:

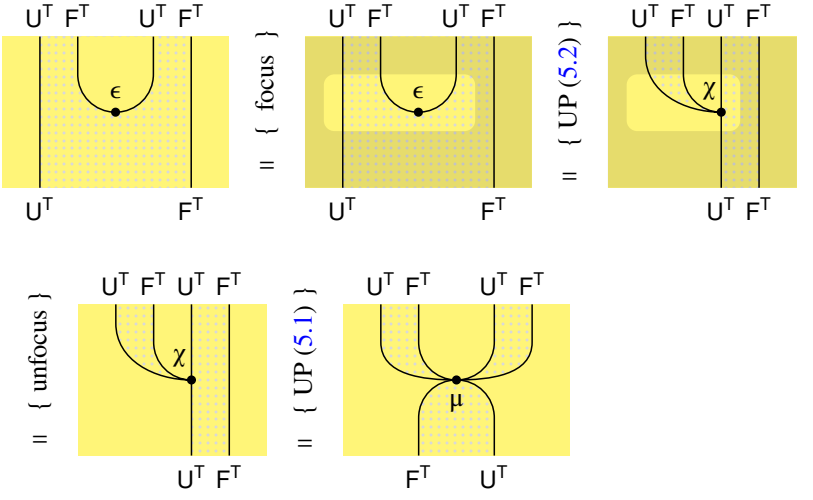




Applying that U^T is left-cancellative completes the proof.

5.2 Huber's construction yields the original monad

We now wish to confirm that this adjunction induces the original monad, via Huber's construction. We have already seen that $U^T \circ F^T = T$, and the unit of the adjunction is the unit of the monad by construction. It remains to confirm that the induced monad has the same multiplication. This follows straightforwardly from equations we have already established:

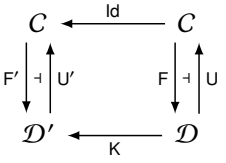


6 Eilenberg–Moore comparison and the terminal resolution

Given a monad $(\mathsf{T} : C \to C, \eta, \mu)$, an adjunction $\mathsf{F} \dashv \mathsf{U} : \mathcal{D} \multimap C$ is called a **resolution** of T, if the adjunction induces the monad via Huber's construction. We wish to compare different resolutions of the same monad. To do so, for resolutions

$$\mathsf{F} \dashv \mathsf{U} : \mathcal{D} \multimap C \quad \text{and} \quad \mathsf{F}' \dashv \mathsf{U}' : \mathcal{D}' \multimap C$$

it is natural to consider maps of adjunctions between them, as introduced in Section 3.3. As the monad, and its base 0-cell are fixed, we restrict to monad morphisms with one component the identity, as in the following diagram:



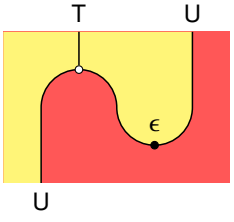
For Huber's construction to yield the same monad, we must have $\eta = \eta'$. This implies the unit sliding equation holds trivially, and so as the sliding equations imply each other, the counit sliding equation must hold. Therefore, a morphisms of adjunctions between resolutions of the same monad is equivalent to requiring a $\mathsf{K} : \mathcal{D} \to \mathcal{D}'$ such that

$$\mathsf{K} \circ \mathsf{F} = \mathsf{F}' \tag{6.1a}$$

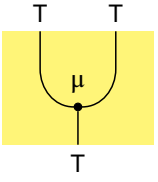
$$\mathsf{U}' \circ \mathsf{K} = \mathsf{U}. \tag{6.1b}$$

In Section 5 we saw that if a monad T has an Eilenberg–Moore object, then this induces an adjunction $\mathsf{F}^{\mathsf{T}} \dashv \mathsf{U}^{\mathsf{T}}$, which is a resolution of T. Our aim is to show this adjunction is a canonical choice.

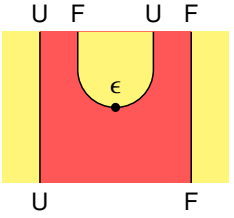
Assume we have a resolution $\mathsf{F} \dashv \mathsf{U} : \mathcal{E} \multimap C$ of monad $\mathsf{T} : C \to C$. It is an easy exercise in applying the snake equation to verify this yields a left action as the composite:



and therefore, by the universal property of Eilenberg–Moore objects induces a 1-cell $\mathsf{K} : \mathcal{E} \to C^{\mathsf{T}}$ such that $\mathsf{U}^{\mathsf{T}} \circ \mathsf{K} = \mathsf{U}$. Recall that the monad multiplication

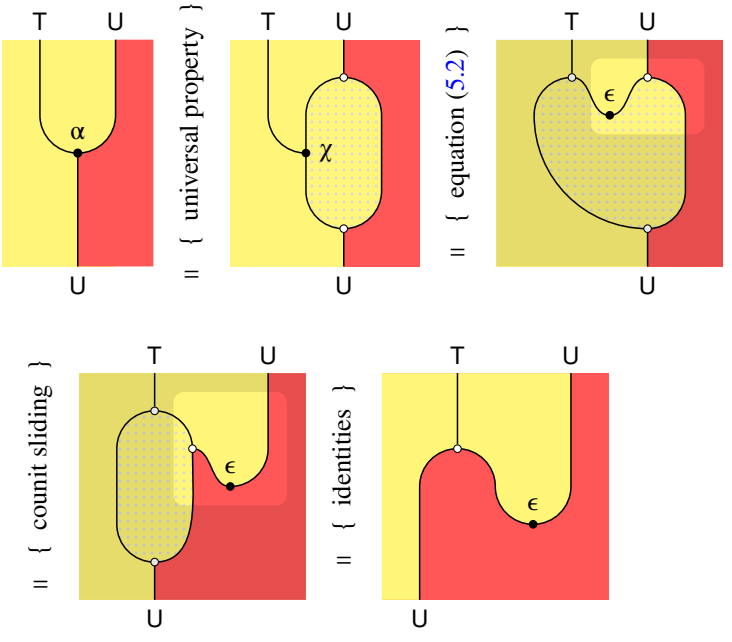


induced the comparison 1-cell F^T, and Huber's "outlined" counit



induces the 1-cell $K \circ F$. As both adjunctions induce the same monad via Huber's construction, and in particular the same multiplication, by the uniqueness part of the universal property $K \circ F = F^T$.

We have shown that there exists a map of adjunctions $F \dashv U \to F^T \dashv U^T$, we would like to show this map is unique. To do so, assume we have a left T-action inducing a map of adjunctions $K : \mathcal{E} \to \mathcal{C}^T$. By the universal property, this action must be of the form $(U, \alpha : T \circ U \dot{\to} U)$. We would like to show α is uniquely determined, and hence so is the comparison 1-cell. We calculate



In the proof we highlight the crucial role played by the use of a somewhat distorted form the counit sliding equation.

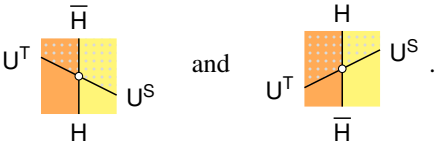
We have shown there is a unique map of adjunctions from every resolution to the Eilenberg–Moore resolution. This is referred to as the Eilenberg–Moore resolution being the **terminal resolution**.

7 Eilenberg–Moore laws classify liftings

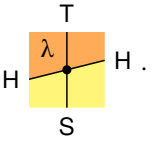
If we have $\mathsf{H} : C \to \mathcal{D}$ and monads $\mathsf{S} : C \to C$ and $\mathsf{T} : \mathcal{D} \to \mathcal{D}$, it is natural to ask when this induces a 1-cell between the corresponding Eilenberg–Moore objects $\overline{\mathsf{H}} : C^{\mathsf{S}} \to \mathcal{D}^{\mathsf{T}}$. It turns out that there is a one-to-one correspondence between certain well-behaved $\overline{\mathsf{H}}$, so-called liftings, and 2-cells known as Eilenberg–Moore laws. Establishing this relationship is the aim of this section.

7.1 Liftings and Eilenberg–Moore laws

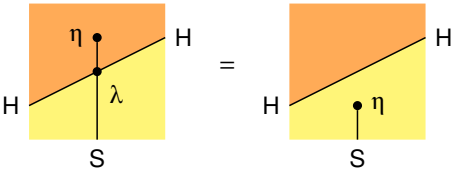
For monads $\mathsf{S} : C \to C$ and $\mathsf{T} : \mathcal{D} \to \mathcal{D}$ and 1-cell $\mathsf{H} : C \to \mathcal{D}$, we say that a 1-cell $\overline{\mathsf{H}} : C^{\mathsf{S}} \to \mathcal{D}^{\mathsf{T}}$ is a **lifting** of H, if $\mathsf{U}^{\mathsf{T}} \circ \overline{\mathsf{H}} = \mathsf{H} \circ \mathsf{U}^{\mathsf{S}}$, that is, we have identity vertices:



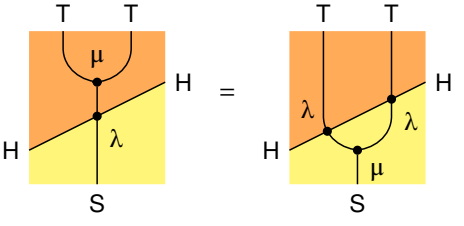
Furthermore, we say that



is an **Eilenberg–Moore law** if it satisfies the following two coherence conditions with respect to the monad structure:



(7.1a)



(7.1b)

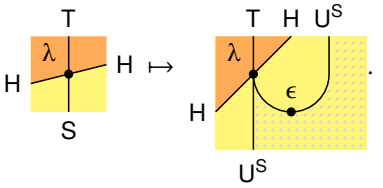
We draw these diagrams in this way to emphasize the intuition that the Eilenberg–Moore law allows us to "drag" the monad structure across the H wire.

7.2 A one-to-one correspondence between Eilenberg–Moore laws and left actions

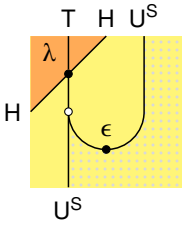
Let $\mathsf{S} : C \to C$ and $\mathsf{T} : \mathcal{D} \to \mathcal{D}$ be monads with Eilenberg–Moore objects. Our first goal is to establish a one-to-one correspondence between

1. Eilenberg–Moore laws of type $\mathsf{T} \circ \mathsf{H} \dot{\to} \mathsf{H} \circ \mathsf{S}$ and
2. left T-actions on $\mathsf{H} \circ \mathsf{U}^{\mathsf{S}}$.

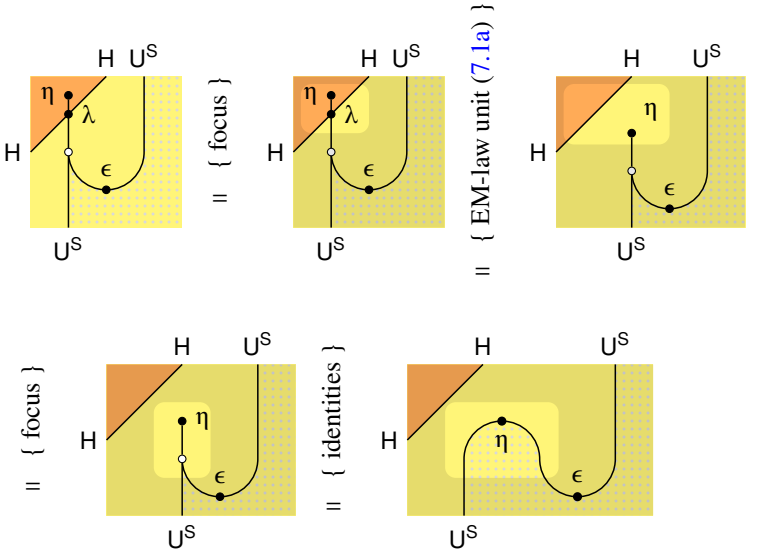
Using the results of Section 5, there is an adjunction $\mathsf{F}^{\mathsf{S}} \dashv \mathsf{U}^{\mathsf{S}} : C^{\mathsf{S}} \multimap C$ inducing S via Huber's construction. Using this adjunction, we show that every Eilenberg–Moore law yields a left T-action via the mapping:

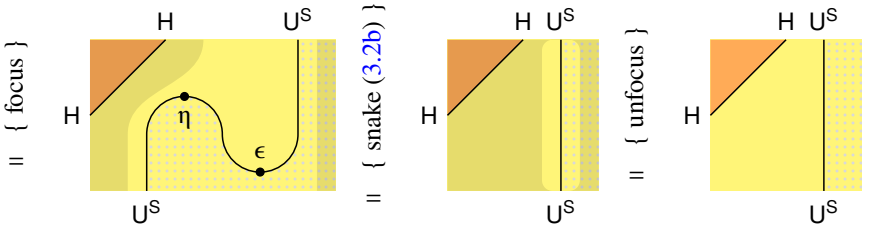


Before establishing the required equations, we pause to consider an alternative rendering of the diagram on the right, in which we add an explicit identity vertex which "expands" the S wire.

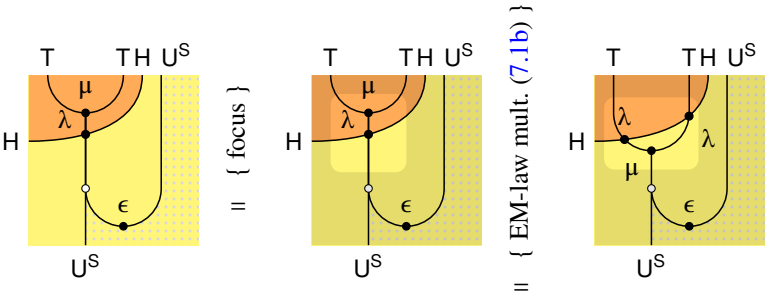


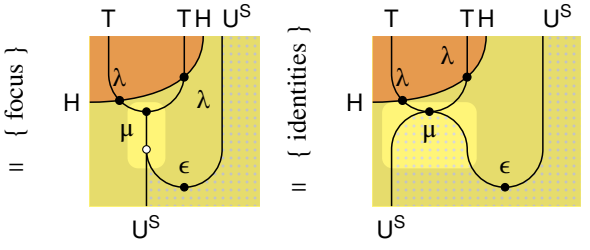
This formulation will prove more convenient in the subsequent calculations. Using this rendition, we verify the unit axiom (3.5a) as follows:

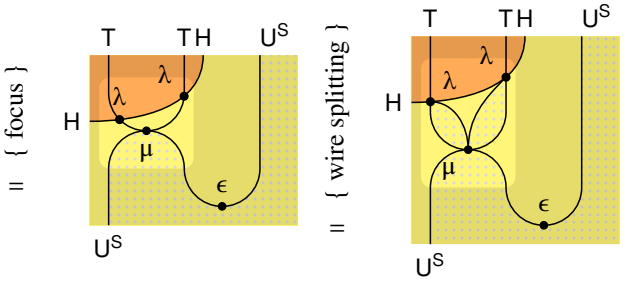


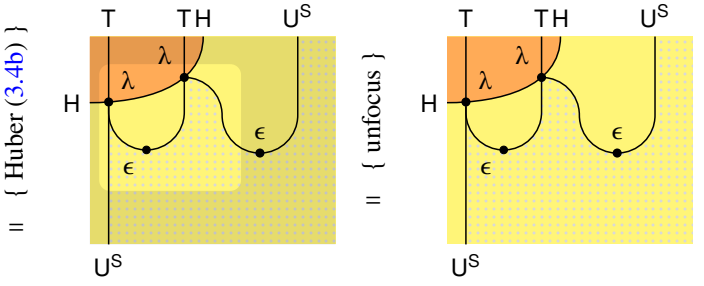


To establish the left action multiplication axiom (3.5b), we reason as follows:

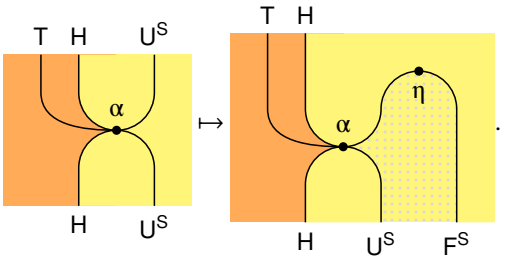




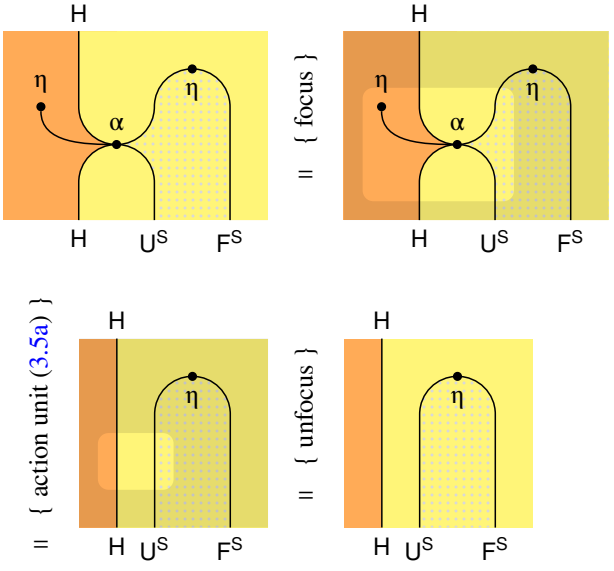




In the other direction, given a left T-action we can form an Eilenberg–Moore law by "bending a wire down":

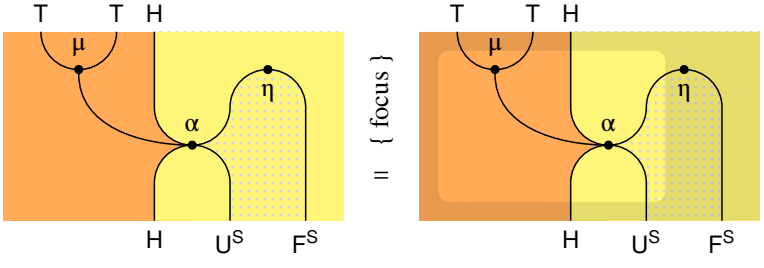


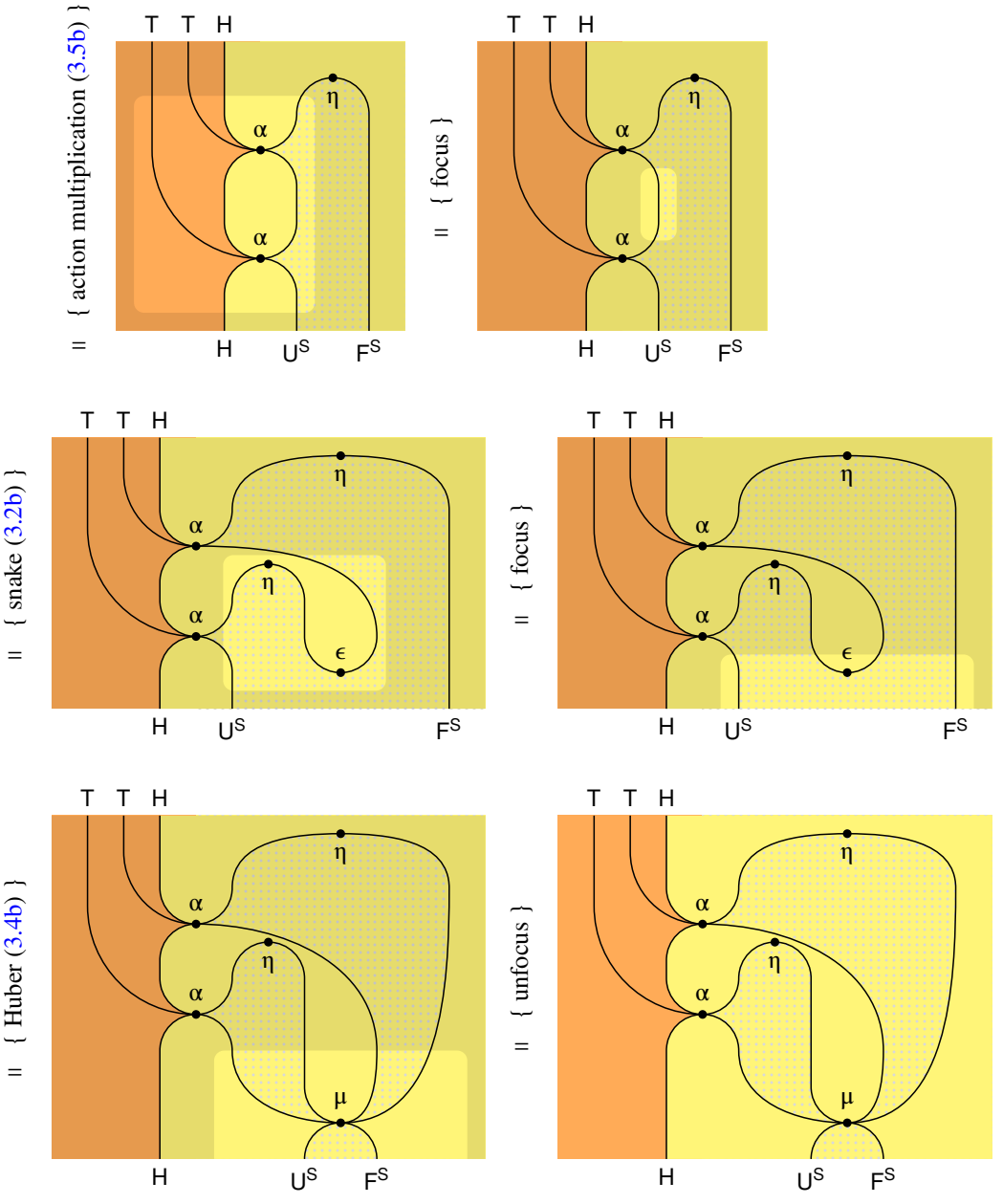
At the bottom of the diagram on the right, we exploit that $S = U^S \circ F^S$. Of course, we must verify this composite satisfies the two axioms of an Eilenberg–Moore law. For the unit axiom (7.1a), we calculate:



We see that, apart from the focusing steps, this is actually a one-step proof simply applying the left S-action unit axiom.

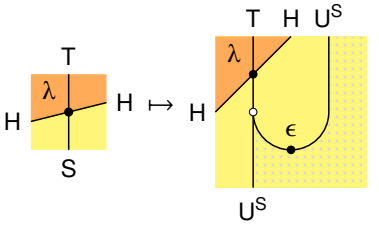
The multiplication proof (7.1b) is slightly more complicated.



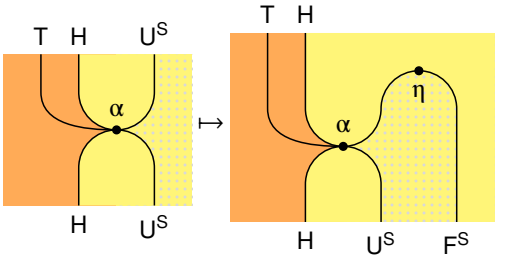


Apart from the focusing steps, we required three non-trivial equations to complete this proof. Possibly the most interesting step is to deliberately complicate the diagram by the insertion of an extra "kink" in a wire via the snake equation, which enables a subsequent proof step.

Finally, it is not hard to see that the mappings



and



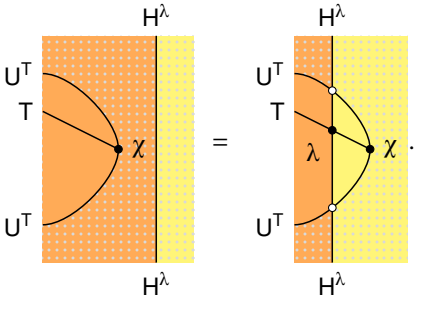
establish a one-to-one correspondence between Eilenberg–Moore laws of type $\mathsf{T}\circ\mathsf{H} \dot{\to} \mathsf{H}\circ\mathsf{S}$ and left T-actions on $\mathsf{H}\circ\mathsf{U}^{\mathsf{S}}$ — the snake equations do the trick.

7.3 A one-to-one correspondence between left actions and liftings

Driving the proof home, the universal property of Eilenberg–Moore objects immediately gives a one-to-one correspondence between

1. Left T-actions on $\mathsf{H}\circ\mathsf{U}^{\mathsf{S}}$ and
2. liftings $\overline{\mathsf{H}} : \mathcal{C}^{\mathsf{S}} \to \mathcal{D}^{\mathsf{T}}$ such that $\mathsf{U}^{\mathsf{T}}\circ\overline{\mathsf{H}} = \mathsf{H}\circ\mathsf{U}^{\mathsf{S}}$.

We use the notation $\mathsf{H}^{\lambda} : \mathcal{C}^{\mathsf{S}} \to \mathcal{D}^{\mathsf{T}}$ for the lifted 1-cell induced by an Eilenberg–Moore law $\lambda : \mathsf{T}\circ\mathsf{H} \dot{\to} \mathsf{H}\circ\mathsf{S}$. From the discussion above, by the universal property and the characterization of the counit (5.2), this will satisfy:



(7.2)

This equation is an important rewrite rule, so let us try to provide some geometric intuition, see also Section 2.5. Observe that the 1-cells H and H^{λ} are drawn as straight, vertical lines — think of them as a border. Applied from left to right, the equation allows us to drag the universal action across this border. As a result, the three wires emerging from χ also

cross the border, "creating" three vertices: an instance of the Eilenberg–Moore law and two identity 2-cells. In calculations, this move is indicated by the hint "drag".

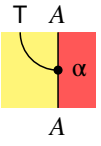
We consider liftings in more detail in Section 9.

8 Kleisli objects and duality

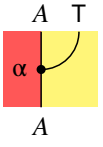
When doing mathematics, or anything else for that matter, it is always nice to get things for free. In the case of the monad theory we have been developing, it pays to consider if we can exploit some of the symmetries of our diagrams. For example, what happens if we mirror our diagrams about a vertical or horizontal axis? By pursuing this idea, we recover a lots of additional results in the theory of both monads and comonads.

We begin by considering reflection about a vertical axis. If we examine the key notion of a monad, reflecting all the diagrams in this way results in exactly the same structure.

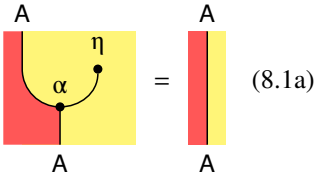
Fortunately, things get more interesting when we consider monad actions. Given a monad $(\mathsf{M} : C \to C, \eta, \mu)$, a **left action** of T on $\mathsf{A} : \mathcal{E} \to C$ was defined to be a



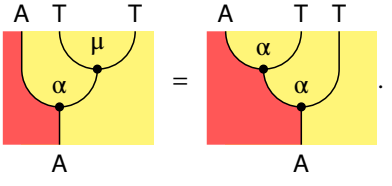
such that Equations (3.5a) and (3.5b) hold. By mirroring about the vertical axis, we discover the notion of a right T-action as an



satisfying the equations



(8.1a)

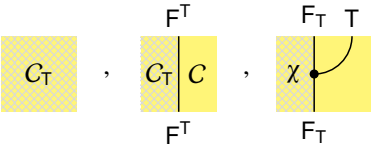


(8.1b)

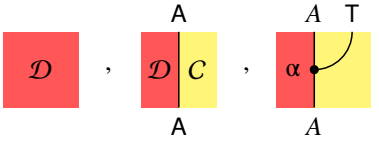
Similarly, a morphism of right actions is required to satisfy the vertical mirror image of Equation (3.6).

Taking this mirroring idea further, we can dualize the notion of Eilenberg–Moore object. We will do this in full detail, so that interested readers can contrast with Section 4 to see the impact of dualizing the definitions given there.

For a monad T, we say that a right action



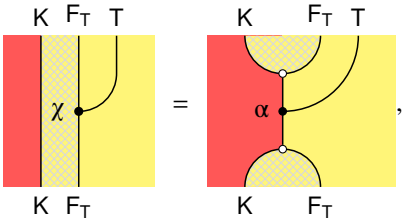
is **universal** if it satisfies the following two properties. *First*, for every right T-action



there exists a *unique* **comparison** 1-cell



such that $\mathsf{K} \circ \mathsf{F_T} = \mathsf{A}$ and

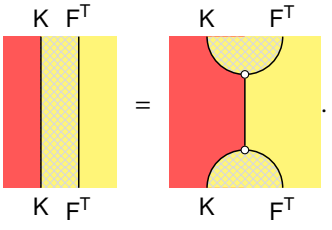


(8.2)

where we have introduced identity cells

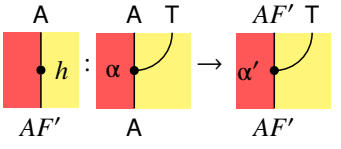


These enjoy the following **fusion equation**



(8.3)

Second, given two right actions α and α', with induced comparison 1-cells K and K', and a right T-action morphism

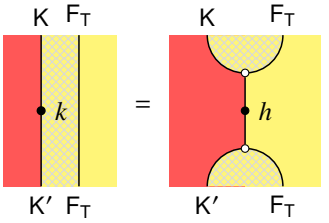


(8.4)

there exists a *unique* **comparison** 2-cell



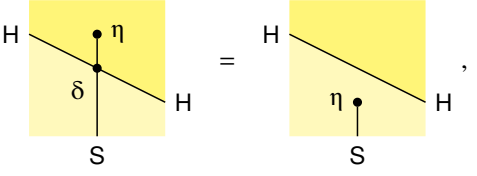
such that the following equality holds:



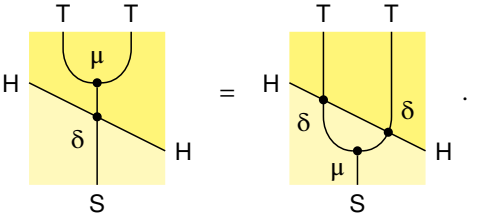
The 0-cell carrying the structure of a universal right action is referred to as a **Kleisli object** for T, denoted $\mathcal{C}_\mathsf{T}$. Using a similar convention to that for Eilenberg–Moore objects, graphically, a Kleisli object $\mathcal{C}_\mathsf{T}$ is represented by a "hashed" region, reusing the colour of $\mathcal{C}$. Again, the terminology relates to the fact that the Kleisli object for a monad in **Cat** is its Kleisli category (Kleisli, 1965). Observant readers will notice that as well as taking the vertical reflection of our diagrams, we have adjusted some of the names. This is a cosmetic change to ensure we follow standard naming conventions, and we shall continue to do so in what follows.

By mirroring the arguments in Section 5 and 6, we get new results for free. Every monad T with a Kleisli object arises from an adjunction $\mathsf{F}_\mathsf{T} \dashv \mathsf{U}_\mathsf{T} : \mathcal{C}_\mathsf{T} \rightharpoonup \mathcal{C}$ via Huber's construction, and furthermore this adjunction is the initial resolution.

Similarly, if we reflect Equations (7.1a) and (7.1b) about their vertical axis, we get the following equations, for H of appropriate type:



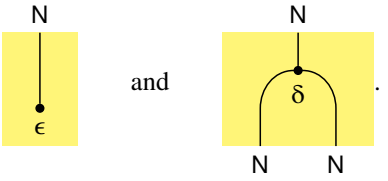
(8.5a)



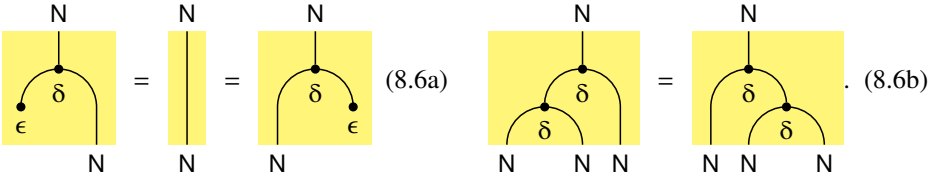
(8.5b)

These are the axioms of what is known as a **Kleisli law**. Again, via an entirely formal process of mirroring the proofs of Section 7, we can derive new results. Specifically, there is a one-to-one correspondence between Kleisli laws for $\mathsf{H} : \mathcal{D} \to \mathcal{C}$ and liftings $\overline{\mathsf{H}} : \mathcal{C}_\mathsf{S} \to \mathcal{D}_\mathsf{T}$ such that $\overline{\mathsf{H}} \circ \mathsf{F}^\mathsf{T} = \mathsf{F}^\mathsf{T} \circ \mathsf{H}$.

Given the additional results we have found simply by reflecting diagrams about the vertical axis, it is natural to ask what happens if we reflect about the horizontal axis instead. If we reflect the diagrams for the key notion of monad on C, we are lead to a pair



satisfying the equations



Such a triple $(\mathsf{N} : C \to C, \epsilon : \mathsf{N} \dot{\to} \mathsf{Id}, \delta : \mathsf{N} \dot{\to} \mathsf{N} \circ \mathsf{N})$, satisfying Equations (8.6a) and (8.6b), in the setting of categories, functors, and natural transformations is precisely the usual definition of a **comonad**. The general case is the abstraction of comonads to the 2-categorical setting. Therefore, by reflecting all the diagrams in the previous sections about the horizontal axis, we derive further results showing every comonad arises via a canonical adjunction, and lifting results for comonads.

In total, by combining vertical and horizontal reflections, every definition or proof we introduce yields three further mirror images. This is a powerful principle, as without doing further work each time we get four concepts and sets of results for the price of one. In more mathematical language, we are applying dualities of 2-categories:

1. Each 2-category has a dual given by reversing all of the 1-cells. Taking the vertical reflection of our diagrams precisely corresponds to considering definitions and proofs in this dual 2-category.
2. Each 2-category has another dual, given by reversing all of the 2-cells. Taking the horizontal reflection of our diagrams then corresponds to considering structures in this second dual 2-category.
3. The two dualities can be combined, and diagrammatically this related to mirroring about both axes.

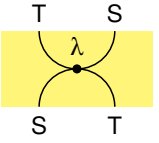
The string diagrammatic notation plays particularly well with these 2-categorical dualities, as it is easy to visualize their impact on definitions, axioms, and equational proofs.

9 Beck's distributive laws classify monad liftings

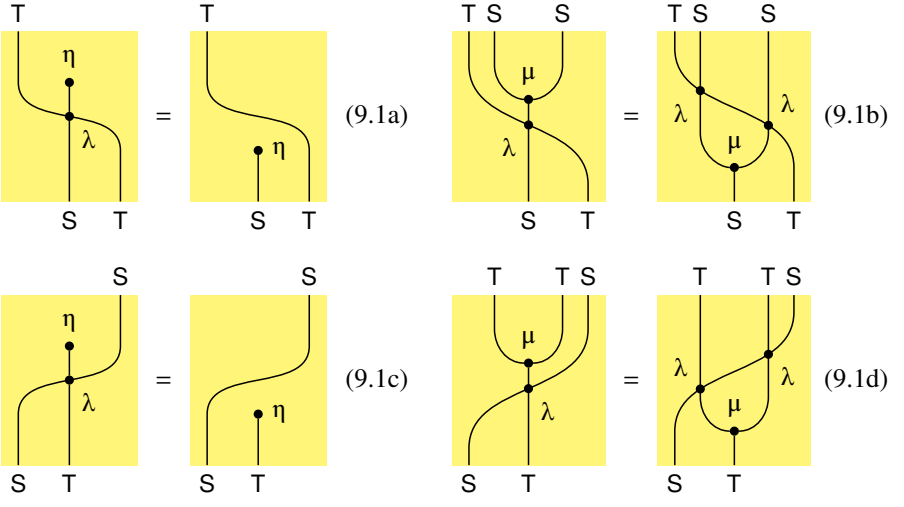
In Section 7 we established tight conditions under which we could lift a 1-cell to the level of Eilenberg–Moore objects. In this section, we continue this line of thought, considering when we can lift a *monad* to the Eilenberg–Moore object of another.

9.1 Beck distributive laws

A **Beck distributive law** or simply **distributive law** of type $\mathsf{T}\circ\mathsf{S} \dot{\to} \mathsf{S}\circ\mathsf{T}$ is an Eilenberg–Moore law which is simultaneously a Kleisli law. That is,

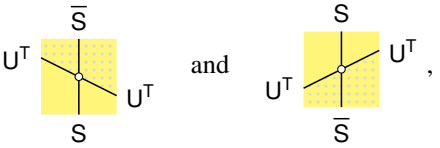


satisfies the following equations:

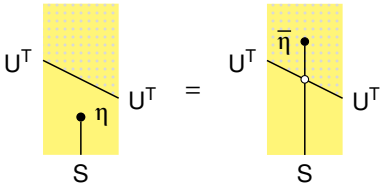


9.2 Monad Liftings

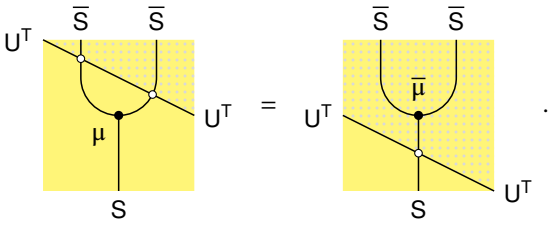
We say that a monad $(\overline{\mathsf{S}} : C^{\mathsf{T}} \to C^{\mathsf{T}}, \overline{\eta}, \overline{\mu})$ is a **lifting** of the monad $(\mathsf{S} : C \to C, \eta, \mu)$ if $\mathsf{U}^{\mathsf{T}} \circ \overline{\mathsf{S}} = \mathsf{S} \circ \mathsf{U}^{\mathsf{T}}$, that is, we have identity vertices:



and exploiting these identity vertices, the units and multiplications satisfy the equations:

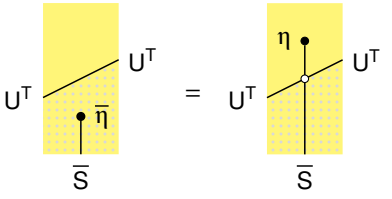


(9.2a)

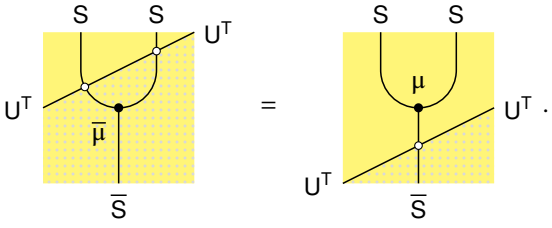


(9.2b)

This is equivalent to requiring that the following dual equations hold:



(9.3a)

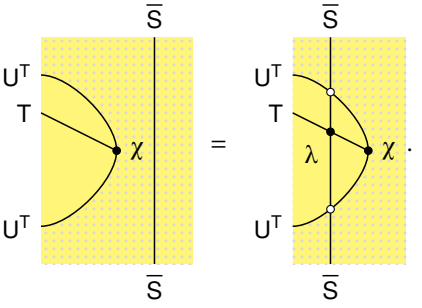


(9.3b)

That we have two equivalent pairs of equations for liftings, Equations (9.2a) and (9.2b), and Equations (9.3a) and (9.3b), can be seen as witnessing a certain bias in their diagrammatic rendering. Both capture the single pair of symbolic equations $U^T \circ \overline{\eta} = \eta \circ U^T$ and $U^T \circ \overline{\mu} = \mu \circ U^T$. That the symbolic notation is unbiased in this respect comes at the cost of omitting type information that is explicit at the boundaries of our diagrams.

9.3 Lifting the monad 1-cell

Given a distributive law $\lambda : T \circ S \nrightarrow S \circ T$, we can lift the 1-cell S to $\overline{S} := S^{\lambda} : C^T \to C^T$, satisfying the following instance of Equation (7.2):

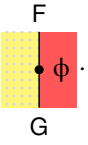


(9.4)

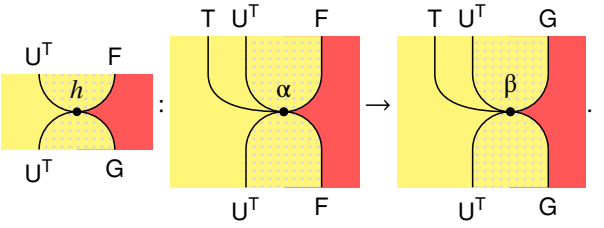
Our goal is now to show that the additional axioms of a Beck distributive law mean that unit and multiplication also lift to give a monad on C^{T}.

9.4 Lifting the unit

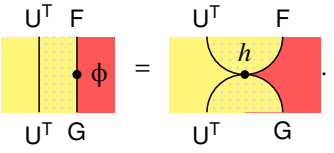
We begin by aiming to find a candidate unit for the lifted monad. To this end let us briefly pause to explain a general recipe for constructing 2-cells. Say, we need a 2-cell of type



The idea is, of course, to appeal to the universal property, which gives a suitable 2-cell, *provided* there is an action morphism of type:

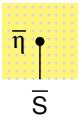


Then ϕ is given as the unique comparison 2-cell, satisfying

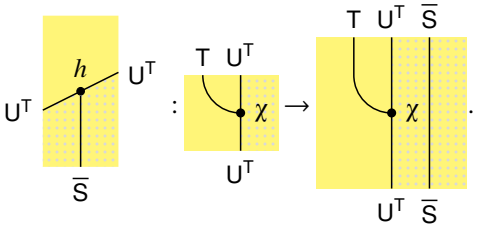


There are two proof obligations: we need to show (1) that α and β are actions; and (2) that h is an action morphism between them. Fortunately, these obligations are often easy to discharge if we use general constructions such as "outlining". To illustrate, let us apply the recipe to the problem at hand.

Since we aim to construct a lifted unit,



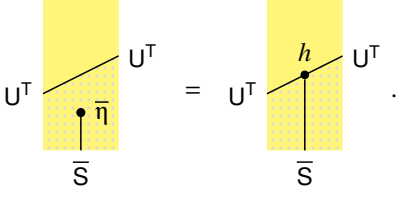
we need to find an action morphism of type



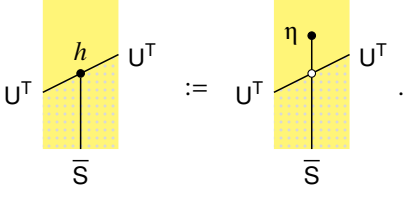
(9.5)

The actions are constructed by outlining the universal action (the only action around): once with the source of $\overline{\eta}$, the identity 1-cell $\overline{\mathsf{Id}}$, and a second time with $\overline{S}$, the target of $\overline{\eta}$.

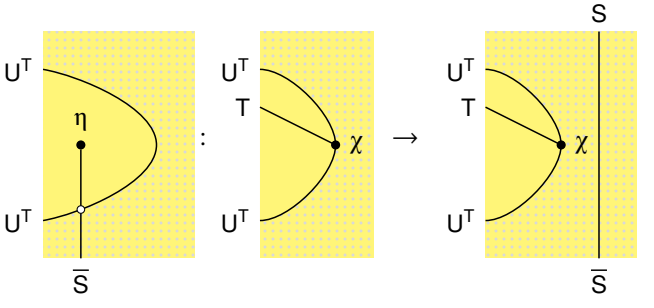
The universal property then induces a *unique* $\overline{\eta}$ such that



Equation (9.3a), the condition required of the unit of a lifted monad, suggests defining

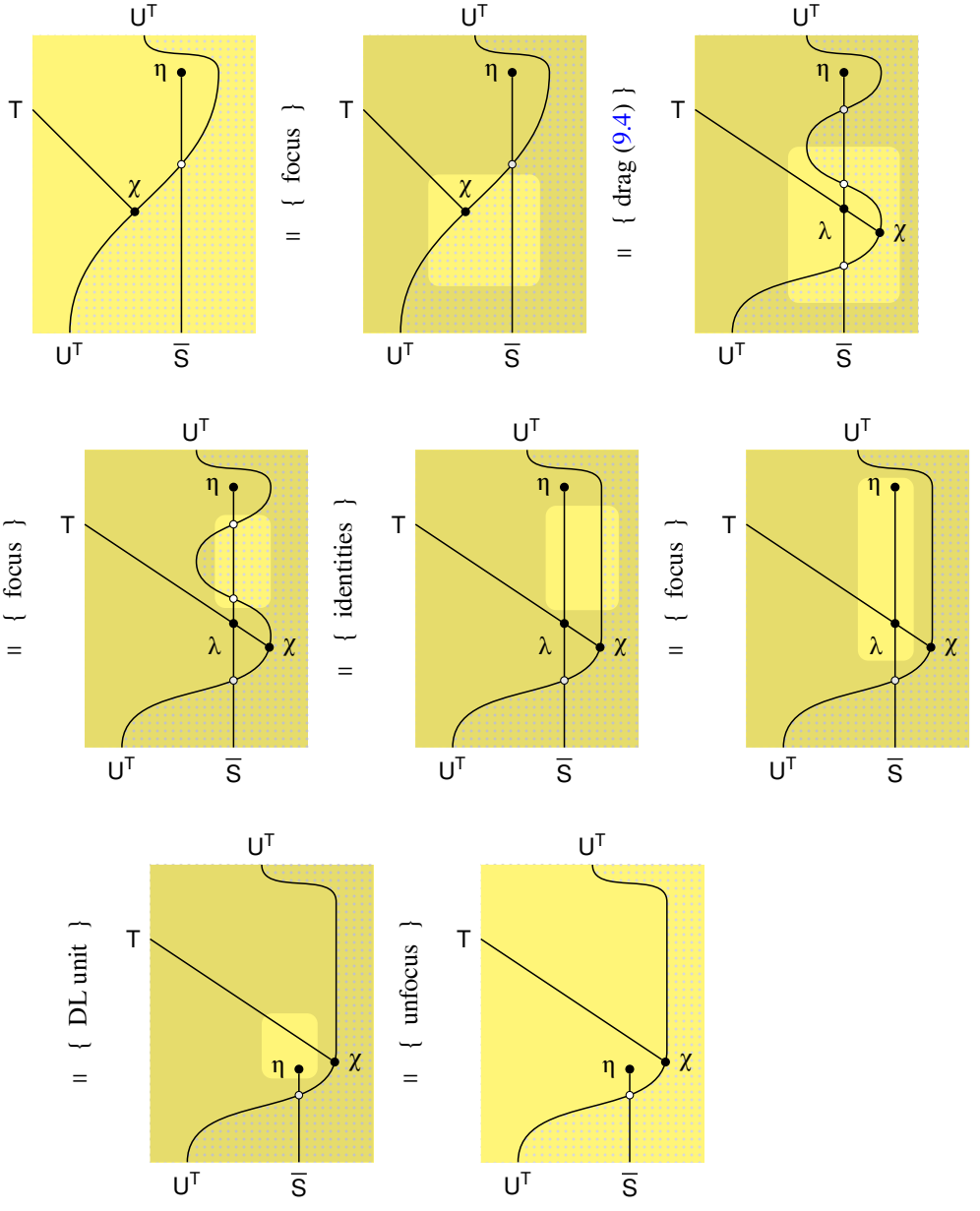


so that the requirement holds by definition. All that remains to be done is to discharge the proof obligations: (1) $\chi \circ \overline{\mathsf{Id}}$ and $\chi \circ \overline{S}$ are actions via "outlining"; (2) to show that $\eta \circ U^T$ is an action morphism between them, we first consider an alternative rendering of (9.5) that is more suitable for our graphical calculations:



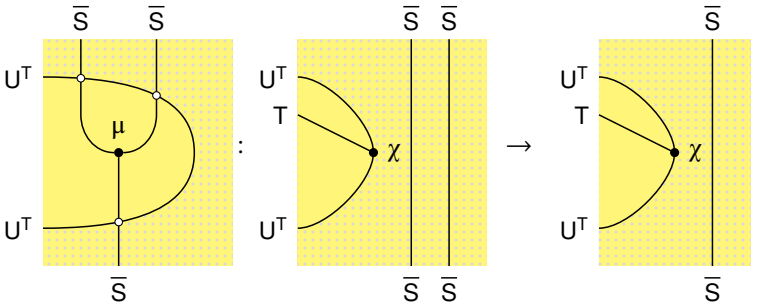
(9.6)

The 1-cell U^{T} is consistently drawn as a cone, in line with the style of (9.4), which is one of the two non-trivial steps in the proof below. For the right-turn axiom (3.6), we need to vertically paste the diagrams: for the left-hand side, we place the morphism below the source action, and for the right-hand side above. Stacking the diagrams vertically turns the 1-cell U^{T} into a zig-zag wire. To establish the axiom, we start with its right-hand side:

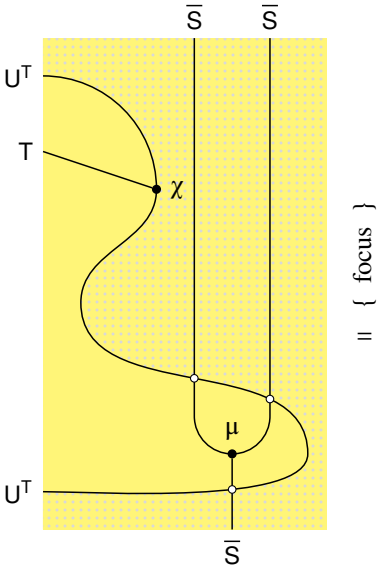


9.5 Lifting the multiplication

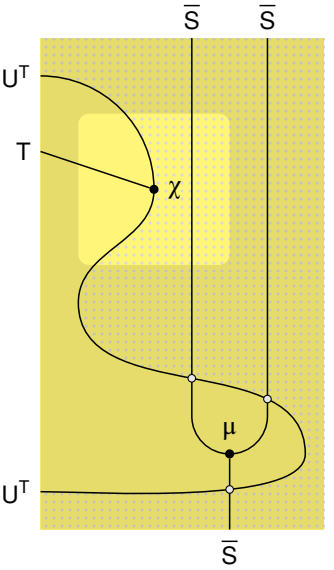
To find a candidate multiplication for the lifted monad, we follow the same steps as in the previous section. In particular, we use the condition $U^T \circ \overline{\mu} = \mu \circ U^T$ required of the multiplication of a lifted monad, (9.2b) and (9.3b), to fix the action morphism. We obtain the following counterpart of (9.6):



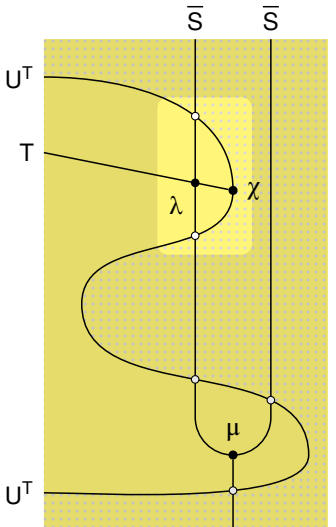
We verify the right-turn axiom (3.6) as follows:



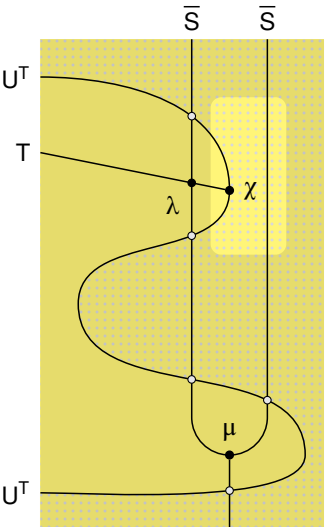
= { focus }

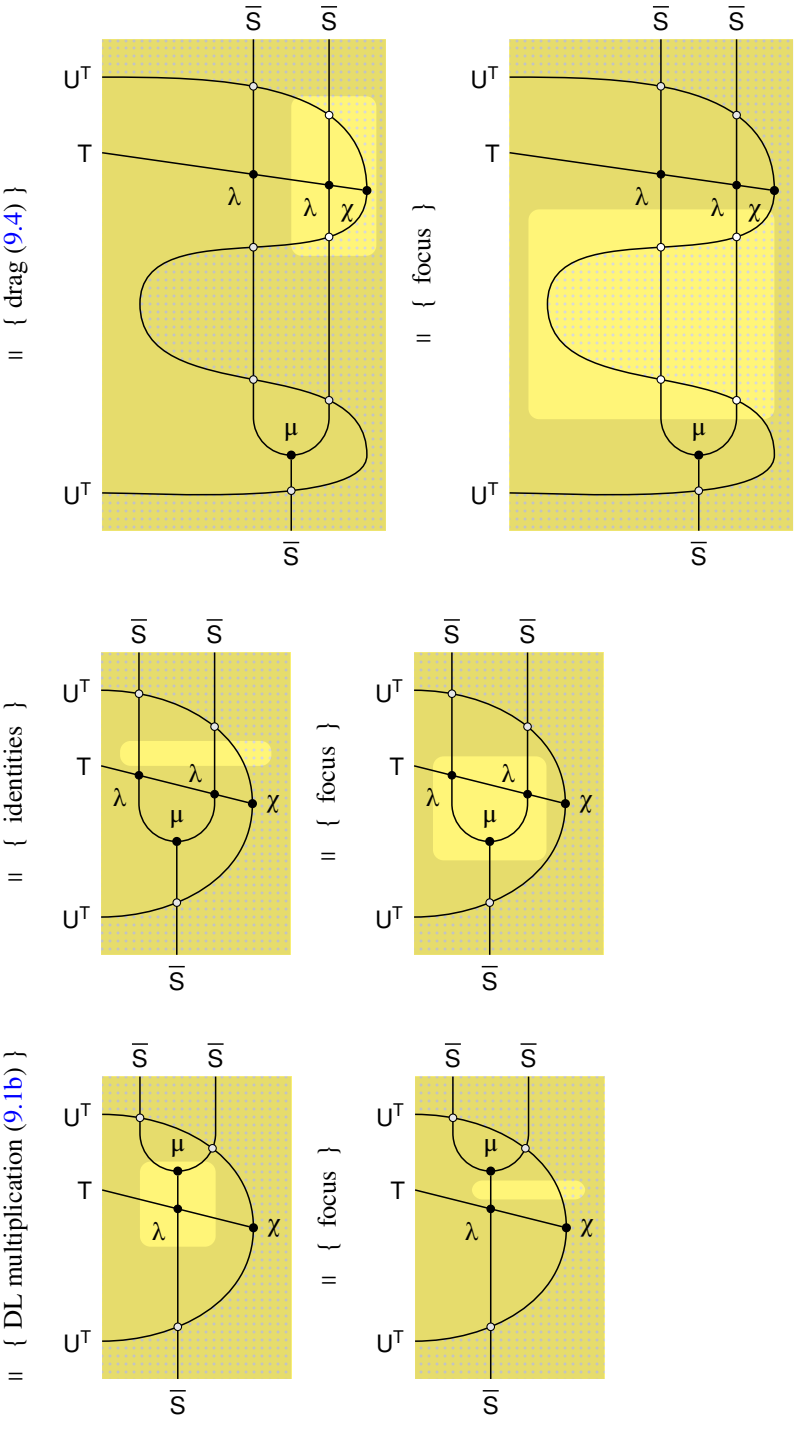


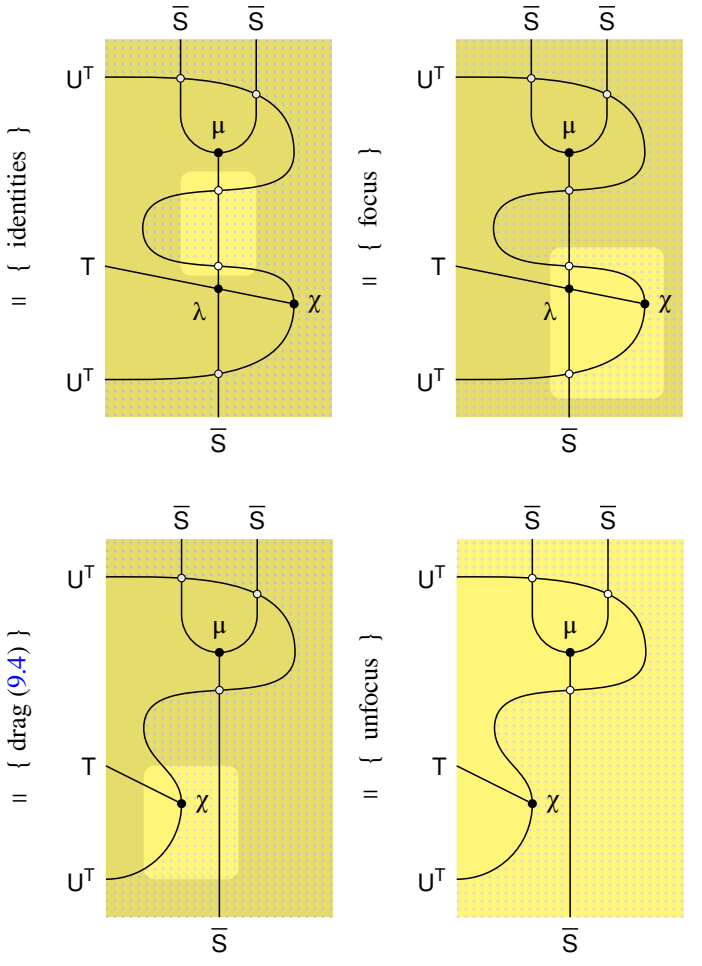
= { drag (9.4) }



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= { drag (9.4) }
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= { identities }
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= { DL multiplication (9.1b) }
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U^T
T
λ
χ
μ
S̄

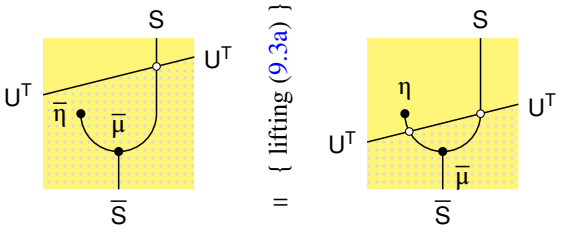


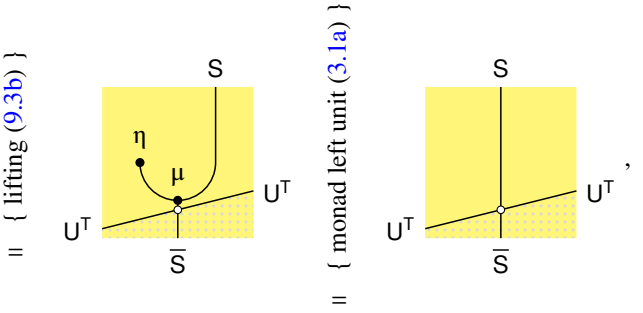
The proof is quite long, but it contains only four non-trivial steps: we "drag" the universal action thrice (as there are three occurrences of $\overline{S}$) and we "drag" the multiplication" once.

9.6 Lifting the monad

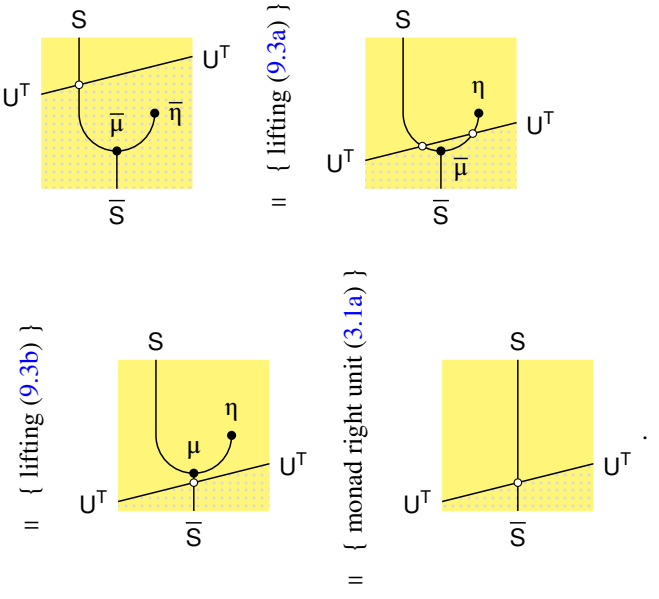
It remains to establish that $\overline{\eta}$ and $\overline{\mu}$ satisfy the three monad axioms. The pattern in each case is the same, we exploiting the lifting Equations (9.3a) and (9.3b) to "slide away the veil", revealing a construction in terms of the original monad. We can then apply the axioms of that monad to further our proof.

For the left unit axiom (3.1a):

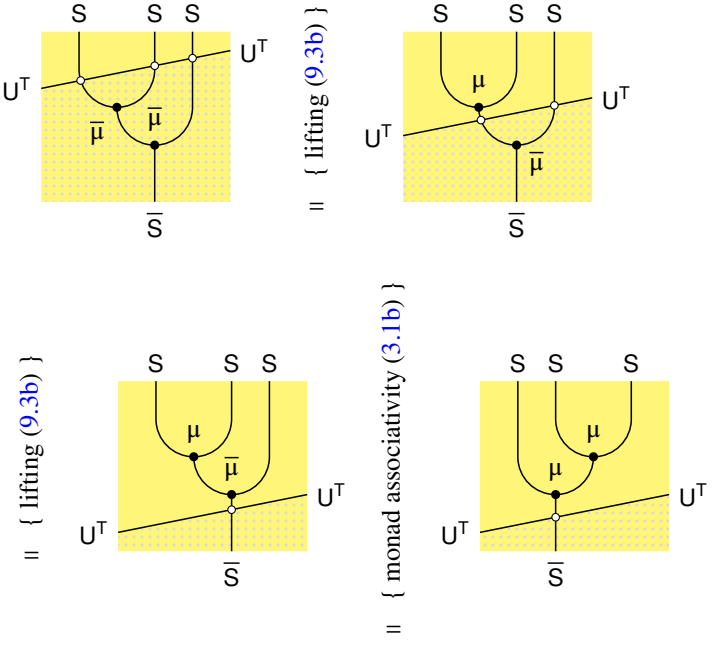


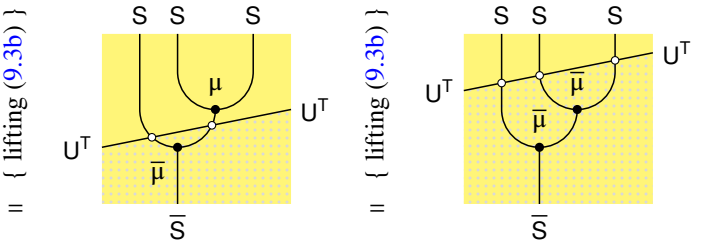


and for the right unit axiom (3.1a):



Finally, for the associativity axiom (3.1b):

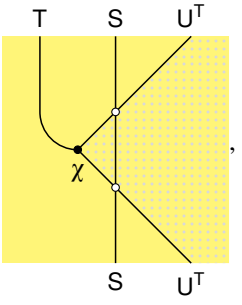




Therefore, as U^{T} is left-cancellative, we have established that the required equations hold.

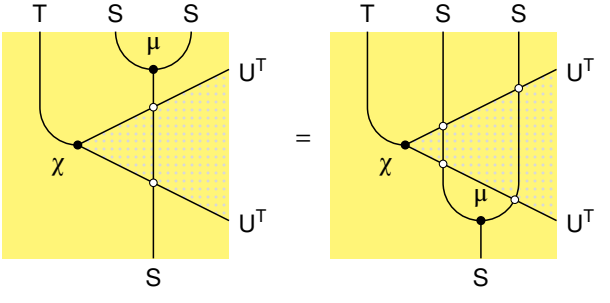
9.7 A lifting induces a distributive law

We would now like to show that every lifting of $(\mathsf{S} : C \to C, \eta, \mu)$ arises in this way. To do so, we assume a lifting $(\overline{\mathsf{S}} : C^{\mathsf{T}} \to C^{\mathsf{T}}, \overline{\eta}, \overline{\mu})$. As $\mathsf{U}^{\mathsf{T}} \circ \overline{\mathsf{S}} = \mathsf{S} \circ \mathsf{U}^{\mathsf{T}}$, the lifted 1-cell $\overline{\mathsf{S}}$ is induced by the left T-action:



,

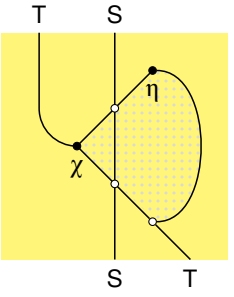
where we introduce some identity vertices to conveniently arrange the boundary wires. Observe that χ is drawn creatively: the U^{T} wires now point to the right, in order to make the subsequent calculations more visually appealing. For example, we can establish the following intuitive equation, a simple corollary of Equations (9.2b) and (9.3b):



(9.7)

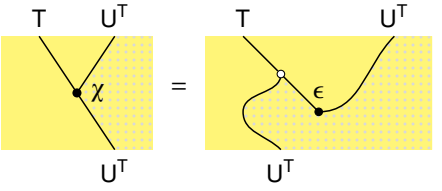
An analogous law holds for the unit.

As we have seen earlier, a left action induces an Eilenberg–Moore law by "bending a wire down":



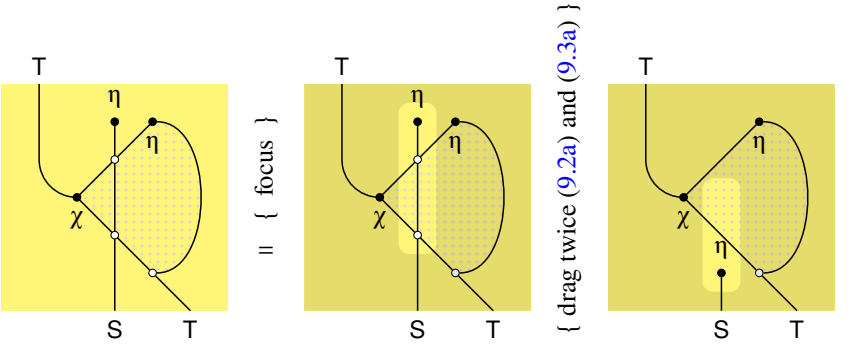
To show that this composite is a Beck distributive law, we must confirm that it also satisfies the axioms of a Kleisli law.

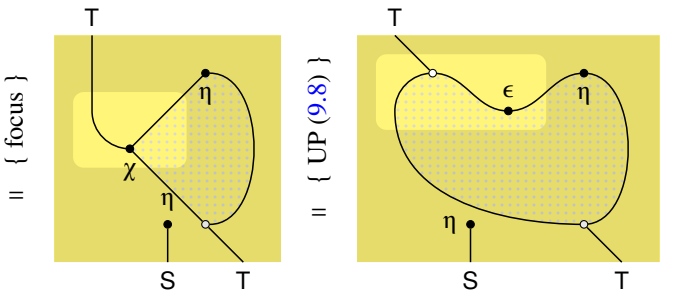
For the unit axiom, we first redraw Equation (5.2), introducing an explicit identity vertex.

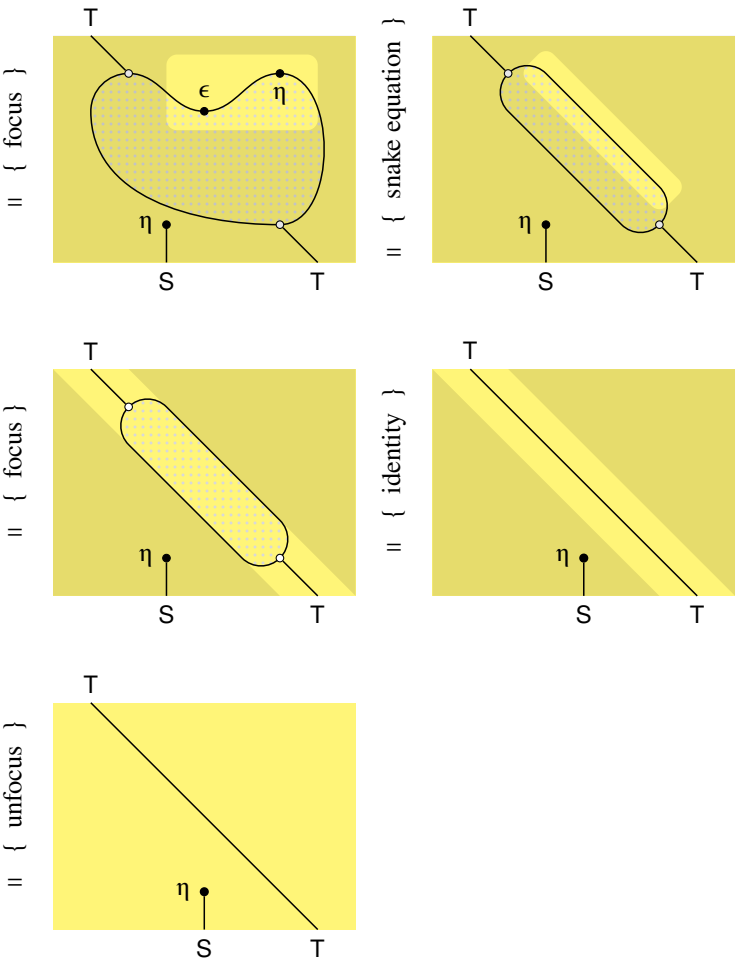


(9.8)

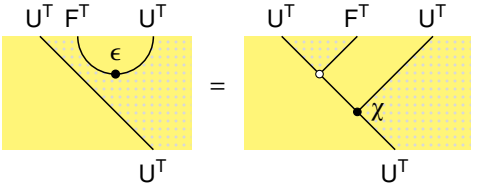
This variant allows us to express the universal action in terms of the counit. With this handy identity in place, we can proceed to establish the Kleisli unit axiom (9.1a):



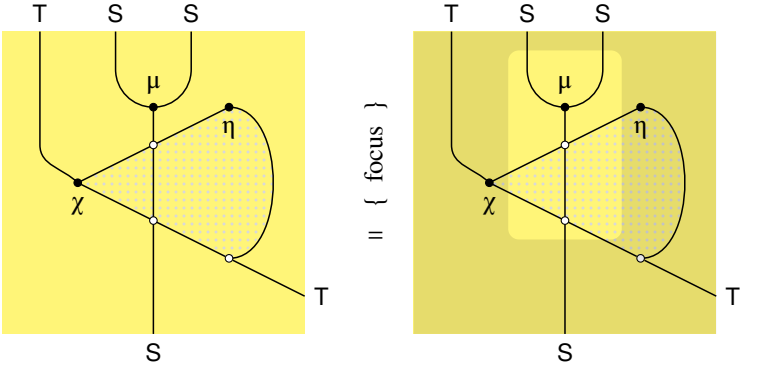




We proceed in a similar way to establish the multiplication axiom. First we redraw Equation (5.2) a second time, this time introducing an explicit identity vertex on the left arm of the universal action.



For the proof of the Kleisli multiplication axiom (9.1b), we argue:



= { drag twice (9.7) }
= { focus }
= { snake equation }
= { focus }
= { elevator }
= { focus }
= { universal property }
= { unfocus }
T
S
η
χ
μ
ϵ

It is best to read the proof backwards. The goal is clear: we need to "merge" the two copies of the candidate law into one. To this end, we first replace the lower universal action by the counit, creating a snake between the two vertical paths. Alas, we cannot immediately pull the wire straight, as this would transmogrify the extremities of the identity vertices attached to the snake, turning an arm into a leg and vice versa. To enable the snake equation, we first need to raise the right identity vertex above the level of the left one. Here, we make *essential* use of the elevator equations (2.2). The rest is routine: we pull the string straight and then drag the multiplication upwards, across the U^T wires.

All that remains to be done is to show that the distributive law induced by a lifted monad induces the original monad. But this is straightforward, as unit and multiplication of a lifted monad are *uniquely* defined. Assume that we have two lifted units, then $\mathsf{U}^\mathsf{T} \circ \overline{\eta} = \eta \circ \mathsf{U}^\mathsf{T} = \mathsf{U}^\mathsf{T} \circ \overline{\eta'}$ and consequently $\overline{\eta} = \overline{\eta'}$ as U^T is left-cancellative. An analogous argument shows that lifted multiplications are unique.

10 Conclusion

A great deal more monad theory can be developed graphically in the style of this paper. For example, the convenient notation for Kan extensions (Kan, 1958) presented in Hinze (2012) transfers to the 2-categorical setting. The techniques of that paper enable a diagrammatic account of the theory of codensity monads (Kock, 1966). A graphical formulation of Kan extensions and codensity monads in a graphical style more consistent with the present work will appear in Hinze and Marsden (2024).

If we allow ourselves to move beyond the 2-categorical setting, diagrammatic reasoning can be pushed even further. One shortcoming of working in a 2-category is that there is no convenient abstraction of hom sets, and this can place some categorical ideas out of reach. Moving to the setting of double categories (Ehresmann, 1963), or even more abstractly to virtual equipments (Cruttwell and Shulman, 2009) can address this problem by providing a connection with profunctors. Myers (2016) presents a graphical language suitable for these settings, very similar to that used in the present work, and this notation was exploited by Arkor and McDermott (2023) to give graphical arguments about relative monads (Altenkirch et al., 2015), which require this additional flexibility. We leave the exploration of these more advanced techniques to further work.

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