Higher category theory and homotopical algebra

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CHAPTER 1

Presheaves

Presheaves will come back in these lectures many times, and with many disguises. This is the way we express ourselves, at least whenever we use the language of category theory, because of the ubiquitous use of the Yoneda Lemma (which will be recalled below). However, the more we will go into homotopical algebra, the more we will see that this apparently innocent and rather formal looking result, the more we will see how the Yoneda Lemma ramifies into many refinements. We will recall here the basic results needed about presheaves (of sets). These will be used as tools right away, but they also will be revisited with the lenses of homotopical algebra, over and over again. The historical references for this part are D. M. Kan's paper [**Kan58**] (in which the notion of adjoint functor is introduced for the first time), as well as Grothendieck's [**SGA72**, Exposé I] (the presentation we give here is rather close to the latter).

Definition 1.1. Let A be a category. A *presheaf* over A is a functor of the form

$$X : A^{op} \longrightarrow Set$$

For an object a of A, we will denote by

$$X_a = X(a)$$

the evaluation of X at a. The set X_a will sometimes be called the *fibre* of the presheaf X at a, and the elements of X_a thus deserve the name of *sections* of X over a. For a morphism $u : a \longrightarrow b$ in A, the induced map from X_b to X_a often will be written

$$u^* = X(u) : X_b \longrightarrow X_a\,.$$

If X and Y are two presheaves over A, a *morphism* of presheaves $f : X \longrightarrow Y$ simply is a natural transformation from X to Y. In other words, such a morphism f is determined by a collection of maps $f_a : X_a \longrightarrow Y_a$, such that, for any morphism $u : a \longrightarrow b$ in A, the following square commutes.

$$\begin{array}{ccc} X_a & \xrightarrow{f_a} & Y_a \\ {\scriptstyle u^*}\uparrow & & \uparrow{\scriptstyle u^*} \\ X_b & \xrightarrow{f_b} & Y_b \end{array} \qquad f_a\, u^* = u^*\, f_b\,.$$

Presheaves naturally form a category. This category will be written $\widehat{A}$.

Remark 1.2. One checks that a morphism of presheaves $f : X \longrightarrow Y$ is an isomorphism (a monomorphism, an epimorphism) if and only if, for any object a of

A, the induced map $f_a : X_a \longrightarrow Y_a$ is bijective (injective, surjective, respectively). Moreover, the evaluation functors $X \longmapsto X_a$ preserve both limits and colimits (exercise: deduce this latter property by exhibiting a left adjoint and a right adjoint). As a consequence, if $F : I \longrightarrow \widehat{A}$ is a diagram of presheaves and if X is a presheaf, the property that a cone from X to F (a cocone from F to X) exhibits X as a limit (colimit) of F is local in the sense that it can be tested fibrewise. In other words, X is a limit (a colimit) of F if and only if, for any object a of A, the set X_a is a limit (a colimit) of the induced diagram $F_a : I \longrightarrow Set$, respectively.

Definition 1.3. The *Yoneda embedding* is the functor

$$h : A \longrightarrow \widehat{A} \tag{1.3.1}$$

whose value at an object a of A is the presheaf

$$h_a = Hom_A(-, a)\,. \tag{1.3.2}$$

In other words, the evaluation of the presheaf h_a at an object c of A is the set of maps from c to a.

Theorem 1.4 (Yoneda Lemma). *For any presheaf X over A, there is a natural bijection of the form*

$$\begin{aligned} Hom_{\widehat{A}}(h_a, X) &\xrightarrow{\sim} X_a \\ h_a \xrightarrow{u} X &\longmapsto u_a(1_a) \end{aligned}$$

Proof. We only define the map in the other direction: given a section s of X over a, we define a collection of morphisms

$$f_c : Hom_A(c, a) \longrightarrow X_c$$

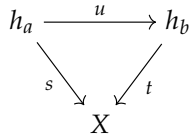
(indexed by obects of C) as follows: for each morphism $u : c \longrightarrow a$, the section $f_c(u)$ is the element $f_c(u) = u^*(s)$. One then checks that this collection defines a morphism $f : h_a \longrightarrow X$, and that the assignment $s \longmapsto f$ is a two-sided inverse of the Yoneda embedding. □

Corollary 1.5. *The Yoneda embedding $h : A \longrightarrow \widehat{A}$ is a fully faithful functor.*

Notation 1.6. The author of these notes prefers to write the isomorphism of the Yoneda embedding as an equality; we will often make an abuse of notations by writing again $f : a \longrightarrow X$ for the morphism of presheaves associated to a section $f \in X_a$ (via the Yoneda Lemma).

Definition 1.7. Let X be presheaf on a category A. The *category of elements* of X (we also call it the *Grothendieck construction* of X) is the category whose objects are couples (a, s), where a is an object of A, while s is a section of X over a, and whose morphisms $u : (a, s) \longrightarrow (b, t)$ are morphisms $u : a \longrightarrow b$ in A, such that $u^*(t) = s$. If we adopt the abuse of notations of paragraph 1.6, this latter condition corresponds, through the Yoneda Lemma, to the commutativity of the

triangle below.



The category of elements of X is denoted by A/X. It comes equiped with a faithful functor

$$\varphi_X : A/X \longrightarrow \widehat{A} \tag{1.7.1}$$

defined on objects by $\varphi_X(a,s) = h_a$, and on morphisms, by $\varphi_X(u) = u$. There is an obvious cocone from φ_X to X defined by the following collection of maps:

$$s : h_a \longrightarrow X\,, \quad (a,s) \in Ob(A/X)\,. \tag{1.7.2}$$

A variation on the Yoneda Lemma is the next statement.

Proposition 1.8. *The collection of maps* (1.7.2) *exhibits the presheaf X as the colimit of the functor* (1.7.1)*.*

Proof. Let Y be an other presheaf on the category A. We have to show that the operation of composing maps from X to Y with the maps (1.7.2) defines a (natural) bijection between morphisms from X to Y and cocones from the functor φ_X to Y in the category of presheaves over A. By virtue of the Yoneda Lemma, a cocone from φ_X to Y can be seen as a collection of sections

$$f_s \in Y_a\,, \quad (a,s) \in Ob(A/X)\,.$$

such that, for any morphism $u : (a,s) \longrightarrow (b,t)$ in A/X, we have the relation $u^*(f_t) = f_s$. This precisely means that the collection of maps

$$\begin{aligned} X_a &\longrightarrow Y_a\,, \quad a \in Ob(A)\,. \\ s &\longmapsto f_s \end{aligned}$$

is a morphism of presheaves. One then checks that this operation is a two sided inverse of the operation of composition with the family (1.7.2). □

Remark 1.9. Until this very moment, we did not mention size (smallness) problems. Well, this is because there was not many. We will come back to size issues little by little. But, whenever we start to be careful with smallness, it is hard to stop. First, when we defined the Yoneda embedding (1.3.1) a first problem arised: for this construction to make sense, we need to work with *locally small categories*[1]. We might say: well, maybe that we did not formulate things properly, since, for instance, even if their formulations seem to need the property that the category A is locally small, the proofs of the Yoneda Lemma (1.4) and of its avatar (1.8) obviously are valid for possibly large categories. Or we could say: well, let us restrict ourselves to locally small categories, since, after

[1]A category is locally small if, for any ordered pair of its objects a and b, morphisms from a to b do form a small set (depending on the set-theoretic foundations the reader will prefer, a small set must either be a set, as opposed to a proper class, or a set which is (in bijection with) an element of a fixed Grothendieck universe).

all, most authors actually require the property of local smallness in the very definition of a category. But Definition 1.1 actually provides examples of categories which are not locally small: for a general locally small category A, the category of presheaves over A may not be locally small (exercise: find many examples). And there are other (less trivial but at least as fundamental) categorical constructions which do not preserve the property of being locally small (e.g. localization). All this means that it might be wiser not to require that categories all are locally small, but, instead, to understand how and why, under appropriate assumptions, certain categorical constructions preserve properties of smallness, or of being locally small. For instance, we can see that, if ever the category A is small[2], the category of presheaves $\widehat{A}$ is locally small. Moreover, the preceding theorem has the following consequence.

THEOREM 1.10 (Kan). *Let A be a small category, together with a locally small category $\mathcal{C}$. For any functor $u : A \longrightarrow \mathcal{C}$, the functor of evaluation at u*

$$u_* : \mathcal{C} \longrightarrow \widehat{A}\,, \quad Y \longmapsto u_*(Y) = \big(a \longmapsto Hom_{\mathcal{C}}(u(a), Y)\big) \tag{1.10.1}$$

has a left adjoint

$$u_! : \widehat{A} \longrightarrow \mathcal{C}\,. \tag{1.10.2}$$

Moreover, there is a unique natural isomorphism

$$u(a) \simeq u_!(h_a)\,, \quad a \in Ob(A)\,, \tag{1.10.3}$$

such that, for any object Y of $\mathcal{C}$, the induced bijection

$$Hom_{\mathcal{C}}(u_!(h_a), Y) \simeq Hom_{\mathcal{C}}(u(a), Y)$$

is the inverse of the composition of the Yoneda bijection

$$Hom_{\mathcal{C}}(u(a), Y) = u_*(Y)_a \simeq Hom_{\widehat{A}}(h_a, u_*(Y))$$

with the adjunction formula

$$Hom_{\widehat{A}}(h_a, u_*(Y)) \simeq Hom_{\mathcal{C}}(u_!(h_a), Y)\,.$$

PROOF. We shall prove that the functor u_* has a left adjoint (the second part of the statement is a direct consequence of the Yoneda Lemma). For each presheaf X over A, we choose a colimit of the functor

$$A/X \longrightarrow \mathcal{C}\,, \quad (a, s) \longmapsto u(a)\,,$$

which we denote by $u_!(X)$. When $X = h_a$ for some object a of A, we have a canonical isomorphism $u(a) \simeq u_!(h_a)$ since $(a, 1_a)$ is a final object of A/h_a. Therefore, for any presheaf X over A, and any object Y of $\mathcal{C}$, we have the following

[2]We remind the reader that this means that it is locally small and that its objects also form a small set.

identifications.

$$\begin{aligned} Hom_{\mathcal{C}}(u_!(X), Y) &\simeq Hom_{\mathcal{C}}(\varinjlim_{(a,s)} u(a), Y) \\ &\simeq \varprojlim_{(a,s)} Hom_{\mathcal{C}}(u(a), Y) \\ &\simeq \varprojlim_{(a,s)} Hom_{\mathcal{C}}(h_a, u_*(Y)) \quad \text{by the Yoneda Lemma} \\ &\simeq Hom_{\mathcal{C}}(\varinjlim_{(a,s)} h_a, u_*(Y)) \\ &\simeq Hom_{\mathcal{C}}(X, u_*(Y)) \quad \text{by Proposition 1.8} \end{aligned}$$

In other words, the object $u_!(X)$ (co)represents the functor $Hom_{\widehat{A}}(X, u_*(-))$. □

REMARK 1.11. Any colimit preserving functor $F : \widehat{A} \longrightarrow \mathcal{C}$ is isomorphic to a functor of the form $u_!$ as above. More precisely, for any such a colimit preserving functor F, if we put $u(a) = F(h_a)$, there is a unique natural isomorphism $u_!(X) = F(X)$ which is the identity whenever the presheaf X is representable (exercise). For instance, for $\mathcal{C} = \widehat{A}$, the identity of $\widehat{A}$ is (canonically isomorphic to) $h_!$, for h the Yoneda embedding.

COROLLARY 1.12. *Any colimit preserving functor $\widehat{A} \longrightarrow \mathcal{C}$ has a right adjoint.*

PROOF. It is sufficient to consider functors of the form $u_!$, for a suitable functor $u : A \longrightarrow \mathcal{C}$ (see the preceding remark). Therefore, by virtue of Theorem 1.10, it has a right adjoint, namely u_*. □

NOTATION 1.13. Let A be a small category. Then the category of presheaves over A is cartesian closed: for any presheaves X and Y, there is an internal Hom, that is a presheaf $\underline{Hom}(X, Y)$ together with natural bijections

$$Hom_{\widehat{A}}(T, \underline{Hom}(X, Y)) \simeq Hom_{\widehat{A}}(T \times X, Y)\,.$$

As can be seen from Theorem 1.10 and Remark 1.11, this object is defined by the formula

$$\underline{Hom}(X, Y)_a = Hom_{\widehat{A}}(h_a \times X, Y)\,.$$

REMARK 1.14. Given a presheaf X, it is equivalent to study maps of codomain X or to study presheaves on the category A/X. To be more precise, one checks that the extension by colimit of the composed functor $A/X \longrightarrow A \xrightarrow{h} \widehat{A}$ sends the final object of $\widehat{A/X}$ to the presheaf X, and the induced functor

$$\widehat{A/X} \xrightarrow{\sim} \widehat{A}/X \tag{1.14.1}$$

is an equivalence of categories. For this reason, even though we will mainly focus on presheaves on a particular category (simplicial sets), it will be convenient to axiomatize our constructions in order to apply to various categories of presheaves.

CHAPTER 2

The category of simplicial sets

We shall write $\mathbf{\Delta}$ for the category whose objects are the finite sets

$$[n] = \{i \in \mathbf{Z} \,|\, 0 \leq i \leq n\} = \{0, \ldots, n\}\,, \quad n \geq 0\,,$$

endowed with their natural order, and whose maps are the (non strictly) order-preserving maps.

DEFINITION 2.1. A *simplicial set* is a presheaf over the category $\mathbf{\Delta}$. We shall write $sSet = \widehat{\mathbf{\Delta}}$ for the category of simplicial sets.

NOTATION 2.2. For $n \geq 0$, we denote by $\Delta^n = h_{[n]}$ the *standard n simplex* (i.e. the presheaf on $\mathbf{\Delta}$ represented by $[n]$).

For a simplicial set X and an integer $n \geq 0$, we write

$$X_n = X([n]) \simeq Hom_{sSet}(\Delta^n, X) \tag{2.2.1}$$

for the set of *n-simplices of X*. A *simplex* of X is an element of X_n for some non-negative integer n. In agreement with the abuse of notations introduced at the number 1.6, an n-simplex x of X can also be seen as a morphism of simplicial sets $x : \Delta^n \longrightarrow X$.

For integers $n \geq 1$ and $0 \leq i \leq n$, we let

$$\partial_i^n : \Delta^{n-1} \longrightarrow \Delta^n \tag{2.2.2}$$

be the map corresponding to the unique strictly order-preserving map from $[n-1]$ to $[n]$ which does not take the value i.

For integers $n \geq 0$ and $0 \leq i \leq n$, the map

$$\sigma_i^n : \Delta^{n+1} \longrightarrow \Delta^n \tag{2.2.3}$$

correspond to the unique surjective map from $[n+1]$ to $[n]$ which takes the value i twice.

PROPOSITION 2.3. *The following identities hold.*

$$\partial_j^{n+1}\partial_j^n = \partial_i^{n+1}\partial_{j-1}^n \qquad i < j \tag{2.3.1}$$

$$\sigma_j^n\sigma_i^{n+1} = \sigma_i^n\sigma_{j+1}^{n+1} \qquad i \leq j \tag{2.3.2}$$

$$\sigma_j^{n-1}\delta_i^n = \begin{cases} \partial_i^{n-1}\sigma_{j-1}^{n-2} & i < j \\ 1_{\Delta^{n-1}} & i \in \{j, j+1\} \\ \partial_{i-1}^{n-1}\sigma_j^{n-2} & i > j+1 \end{cases} \tag{2.3.3}$$

The proof is straightforward.

REMARK 2.4. One can prove that the category $\mathbf{\Delta}$ is completely determined by the relations above: more precisely, it is isomorphic to the quotient by these relations of the free category generated by the oriented graph which consists of the collection of maps ∂_i^n and σ_i^n (with the $[n]$'s as vertices). In other words, a simplicial set can be described as a collection of sets X_n, $n \geq 0$, together with face operators $d_n^i = (\partial_i^n)^* : X_n \longrightarrow X_{n-1}$ for $n \geq 1$, and degeneracy operators $s_n^i = (\sigma_i^n)^* : X_n \longrightarrow X_{n+1}$ satisfying the dual version of the identities above. This pedestrian point of view is often the one taken in historical references.

NOTATION 2.5. For a simplicial set X, we shall write

$$d_n^i = (\partial_i^n)^* : X_n \longrightarrow X_{n-1} \quad \text{and} \quad s_n^i = (\sigma_i^n)^* : X_{n+1} \longrightarrow X_n$$

for the maps induced by the operators ∂_i^n and σ_i^n, respectively.

Although it follows right away from the notion of image of a map of sets, the following property is the source of many good combinatorial behaviours of the category $\mathbf{\Delta}$.

PROPOSITION 2.6. *Any morphism $f : \Delta^m \longrightarrow \Delta^n$ in $\mathbf{\Delta}$ admits a unique factorization $f = \pi\, i$, into a split epimorphism $\pi : \Delta^m \longrightarrow \Delta^p$ followed by a monomorphism (i.e. a strictly order-preserving map) $i : \Delta^p \longrightarrow \Delta^n$.*

EXAMPLE 2.7. A good supply of simplicial sets comes from the category *Top* of topological spaces (with continous maps as morphisms). For this, one defines, for each non-negative integer $n \geq 0$ the topological simplex

$$|\Delta^n| = \left\{ (x_0, \ldots, x_n) \in \mathbf{R}_{\geq 0}^{n+1} \mid \sum_{i=0}^{n} x_i \leq 1 \right\}. \tag{2.7.1}$$

Given a morphism $f : [m] \longrightarrow [n]$ in $\mathbf{\Delta}$, we get an associated continuous (because affine) map

$$|f| : |\Delta^m| \longrightarrow |\Delta^n|$$

defined by

$$|f|(x_0, \ldots, x_m) = (y_0, \ldots, y_n), \quad \text{where } y_j = \sum_{i \in f^{-1}(j)} x_i.$$

This defines a functor from Δ to *Top*. Therefore, by virtue of Theorem 1.10, we have the *singular complex functor*

$$\mathit{Top} \longrightarrow \mathit{sSet}, \quad Y \longmapsto \mathit{Sing}(Y) = \big([n] \longrightarrow \mathit{Hom}_{\mathit{Top}}(|\Delta^n|, Y)\big) \tag{2.7.2}$$

and its left adjoint, the *realization functor*

$$\mathit{sSet} \longrightarrow \mathit{Top}, \quad X \longmapsto |X|. \tag{2.7.3}$$

This example already gives an indication on the possible semantics we can apply to simplicial sets. For instance, a 0-simplex $x : \Delta^0 \longrightarrow X$ can be interpreted as a point of X, and a 1-simplex $f : \Delta^1 \longrightarrow X$, as a path in X, from the point $x = d_1^1(f)$, to the point $y = d_1^0(f)$. This is already good, but we shall take into account that the orientation of paths can be remembered. And doing so literally, this will give a semantic, in the category of simplicial sets, of the very language of category theory.

CHAPTER 3

Cellular filtrations

In this chapter, we shall review the combinatorial properties of simplicial sets which will be used many times to reduce general statements to the manipulation of finitely many operations on standard simplices. However, we shall present an axiomatized version (mainly to deal with simplicial sets over a given simplicial set X, or with bisimplicial sets, for instance). A standard source on this, in the case of simplicial sets themselves is the appropriate chapter in the book of Gabriel and Zisman [**GZ67**]. What follows consist to axiomatize the constructions and proofs of *loc. cit.* For a nice axiomatic treatment of this kind of properties, an excellent reference is Bergner and Rezk's paper [**BR13**].

Definition 3.1. An *Eilenberg-Zilber category* is a quadruple (A, A_+, A_-, d), where A is a small category, while A_+ and A_- are subcategories of A, and $d : Ob(A) \longrightarrow \mathbf{N}$ is a function with values in the set of non-negative integers, such that the following properties are verified:

- EZ0. any isomorphism of A is in both A_+ and A_-. Moreoever, for any isomorphic objects a and b in A, we have $d(a) = d(b)$;
- EZ1. if $a \longrightarrow a'$ is a morphism in A_+ (in A_-) that is not invertible, then we have $d(a) < d(a')$ (we have $d(a) > d(a')$, respectively);
- EZ2. any morphism $u : a \longrightarrow b$ in A has a unique factorization of the form $u = ip$, with $p : a \longrightarrow c$ in A_- and $i : c \longrightarrow b$ in A_+.
- EZ3. if a morphism $\pi : a \longrightarrow b$ belongs to A_- there exists a morphism $\sigma : b_t o a$ in A such that $\pi\sigma = 1_b$; moreover, for any two morphismes in A_- of the form $\pi, \pi' : a \longrightarrow b$, if π and π' have the same sections, then they are equal.

We shall say that an object a of A is of *dimension* n if $d(a) = n$.

Example 3.2. The category $\mathbf{\Delta}$ is an Eilenberg-Zilber category, with $\mathbf{\Delta}_+$ the subcategory of monomorphisms, and $\mathbf{\Delta}_-$ the subcategory of epimorphisms, and $d(\Delta^n) = n$.

Example 3.3. If A is an Eilenberg-Zilber category, then, for any presheaf X, the category A/X is an Eilenberg-Zilber category: one defines the subcategory $(A/X)_+$ (the subcategory $(A/X)_-$) as the subcategory of maps whose image in A belongs to A_+ (to A_-, respectively), and one puts $d(a, s) = d(a)$.

Example 3.4. If A and B are wo Eilenberg-Zilber categories, their product is one as well: we defines $(A \times B)_\varepsilon = A_\varepsilon \times B_\varepsilon$ for $\varepsilon \in \{+, -\}$, and one puts $d(a, b) = d(a) + d(b)$.

Let us fix an Eilenberg-Zilber category A.

DEFINITION 3.5. Let X be a presheaf over A. A section x of X over some object a of A is *degenerate*, if there exists a map $\sigma : a \longrightarrow b$ in A, with $d(b) < d(a)$, and a section y of X over b, such that $\sigma^*(y) = x$. Such a couple will be called a *decomposition of* x. A section of X in *non-degenerate* if it is not degenerate.

For any integer $n \geq 0$, we denote by $Sk_n(X)$ the maximal subpresheaf of X with the property that, for any integer $m > n$, any section of $Sk_n(X)$ over an object a of dimension m is degenerate. In other words, for any object a of A, the sections of $Sk_n(X)$ over a coincide with those of X for $d(a) \leq n$, and are the degenerate ones of X for $d(a) > n$. This construction if functorial: for any morphism of simplicial sets $f : X \longrightarrow Y$, there is a unique morphism $Sk_n(f) : Sk_n(X) \longrightarrow Sk_n(Y)$ such that the following square commutes.

$$\begin{array}{ccc} Sk_n(X) & \hookrightarrow & X \\ {\scriptstyle Sk_n(f)}\downarrow & & \downarrow{\scriptstyle f} \\ Sk_n(Y) & \hookrightarrow & Y \end{array}$$

LEMMA 3.6 (Eilenberg-Zilber). *Let $x \in X_a$ be a section of a presheaf X over A. There exists a unique decomposition (σ, y) of x, such that σ is a morphism of A_-, while y is non-degenerate.*

PROOF. There are integers m such that there exists a decomposition (σ, y) of x, where $\sigma : a \longrightarrow b$ is in A_- and $d(b) = m$ (e.g. $a = b$ and $\sigma = 1_a$). Therefore, there exists such a couple (b, y) with $d(b) = m$ minimal. If y was degenerate, this would contradict the minimality of m, hence the section y must be non-degenerate. On the other hand, if we have another decomposition (σ', y') of x, with $\sigma' : a \longrightarrow b'$ is in A_-, and $d(b') = m$, since any morphism of A_- has a section in A, we can find a section ι of σ, and we get a map $u = \sigma'\iota : b \longrightarrow b'$, such that $u^*(y) = y$. Moreover, by virtue of axiom EZ2, the morphism u has a unique factorization $u = ip$ with i in A_+ and p in A_-. But axiom EZ1, together with the minimality of m, imply that p is invertible, and so must be i as well. Since axiom EZ1 also force the isomorphisms to be ientities, we deduce that $b = b'$. We deduce from this that the two morphisms σ and σ' have the same set of sections, which implies, by virtue of axiom EZ3, that they are equal. □

NOTATION 3.7. For an object a of A, we put

$$\partial h_a = Sk_{d(a)-1}(h_a)\,.$$

This subobject is called the *boundary* of the representable presheaf h_a.

THEOREM 3.8. *Let $X \subset Y$ be presheaves over A. For any non-negative integer n, there is a canonical pushout square*

$$\begin{array}{ccc} \coprod_{y\in\Sigma} \partial h_a & \longrightarrow & X \cup Sk_{n-1}(Y) \\ \downarrow & & \downarrow \\ \coprod_{y\in\Sigma} h_a & \longrightarrow & X \cup Sk_n(Y) \end{array}$$

where Σ denotes the set of non-degenerate sections of Y of the form $y : h_a \longrightarrow Y$ which do not belong to X, and such that $d(a) = n$.

The proof is left as an exercise: the main ingredient is the preceding lemma.

Definition 3.9. A class $\mathcal{C}$ of presheaves over A is *saturated by monomorpisms* if it has the following stability properties.

(a) For any small family of presheaves $(X_i)_i$, if each X_i belongs to $\mathcal{C}$, so does the coproduct $\coprod_i X_i$.

(b) For any pushout square of presheaves

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' \end{array}$$

in which the vertical maps are monomorphisms, if X, X' and Y all are in $\mathcal{C}$, so is Y'.

(c) For any sequence of monomorphisms of presheaves

$$X_0 \hookrightarrow X_1 \hookrightarrow \cdots \hookrightarrow X_n \hookrightarrow X_{n+1} \hookrightarrow \cdots$$

in which each of the X_n's is in $\mathcal{C}$, their reunion $\varinjlim_n X_n$ belongs to $\mathcal{C}$ as well.

Corollary 3.10. *If a class of presheaves over A is saturated by monomorphisms and contains all representable presheaves, then it contains all presheaves over A.*

Proof. Let $\mathcal{C}$ be such a class. We apply Theorem 3.8 with X empty. We thus have pushout squares of the form

$$\begin{array}{ccc} \coprod_{y\in\Sigma} \partial h_a & \longrightarrow & Sk_{n-1}(Y) \\ \downarrow & & \downarrow \\ \coprod_{y\in\Sigma} h_a & \longrightarrow & Sk_n(Y) \end{array}$$

We prove by induction on n that each presheaf of the form $Sk_n(Y)$ is in $\mathcal{C}$. For $n = 0$, we see that $Sk_0(Y)$ is a small sum of representable presheaves, and therefore is in $\mathcal{C}$. For $n > 0$, the induction hypothesis means that both the domain and the codomain of the upper horizontal map of the commutative square above are in $\mathcal{C}$. Hence, using properties (a) and (b) of the definition of a saturated class by monomorphisms, we deduce that any presheaf of the form $Sk_n(Y)$ is in $\mathcal{C}$. Since the union of the $Sk_n(Y)$'s for $n \geq 0$ is Y itself, condition (c) above shows that any presheaf Y belongs to $\mathcal{C}$. □

Remark 3.11. Theorem 3.8 also implies that, for any simplicial set X, the realization $|X|$ has a natural structure of CW-complex: this comes from the fact that the realization functor preserves colimits (being a left adjoint) and that $|\partial\Delta^n| = S^{n-1}$ is the boundary of the topological simplex $|\Delta^n|$.

CHAPTER 4

Nerves

4.1. Any partially ordered set E gives rise to a category: the objects are the elements of E, and, for any couple of such elements (x, y), the set of morphisms from x to y has at most one element:

$$Hom(x,y) = \begin{cases} \{(x,y)\} & \text{if } x \leq y, \\ \varnothing & \text{if } x > y. \end{cases}$$

The letter E can safely be used to represent this category, since, for two partially ordered sets E and F there is a natural identification between the set of non-strict order preserving maps from E to F and the set of functors from the category associated to E to the one associated to F. In other words, it is a rather natural thing to see the category of partially ordered sets as a full subcategory of the category Cat of small categories (whose objects are small categories, while its morphisms are the functors between these).

In particular, we can restrict our attention to non-empty finite totally ordered sets, and we obtain a fully faithful inclusion functor

$$i : \mathbf{\Delta} \longrightarrow Cat \tag{4.1.1}$$

The *nerve* functor is defined as the evaluation at i:

$$N = i^* : Cat \longrightarrow sSet\ , \quad C \longmapsto N(C) = \big([n] \longmapsto Hom_{Cat}([n], C)\big)\ . \tag{4.1.2}$$

Hence an n-simplex of the nerve of a category C is a string of arrows of length n in C:

$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} x_n\ . \tag{4.1.3}$$

By virtue of Theorem 1.10, the nerve functor has a left adjoint

$$\tau = i_! : sSet \longrightarrow Cat\ . \tag{4.1.4}$$

The nerve of a category will serve as a paradigm to interpret the structure of a simplicial set.

Definition 4.2. Let X be a simplicial set.

An *object* of X is a 0-simplex of X, or equivalently, a map $x : \Delta^0 \longrightarrow X$.

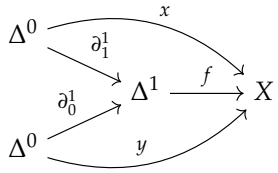
An *arrow* of X (we shall also say a *morphism* of X, or even a *map* of X) is a 1-simplex of X, that is a map $f : \Delta^1 \longrightarrow X$. Such an arrow has a source and a target, namely $f\partial_1^1$ and $f\partial_0^1$, respectively.

$$\Delta^0 \xrightarrow{\partial_1^1} \Delta^1 \xrightarrow{f} X \qquad \Delta^0 \xrightarrow{\partial_0^1} \Delta^1 \xrightarrow{f} X$$

NOTATION 4.3. A diagram of the form

$$x \xrightarrow{f} y$$

in a simplicial set X will mean that f is an arrow of X and that $x = f\partial_1^1$ is its source, while $y = f\partial_0^1$ is its target. In other words, this corresponds to a commutative diagram of simplicial sets of the following shape.



EXAMPLE 4.4. Given an object x in a simplicial set X, the *identity* of x is the morphism of X corresponding to this composition of maps of simplicial sets:

$$1_x = x\sigma_0^0 : \Delta^1 \xrightarrow{\sigma_0^0} \Delta^0 \xrightarrow{x} X\,. \tag{4.4.1}$$

4.5. Given a finite non-empty partially ordered set E, we will write

$$\Delta^E = N(E) \tag{4.5.1}$$

for its nerve. We have $\Delta^{[n]} = \Delta^{\{0,\dots,n\}} = \Delta^n$ for any integer $n \geq 0$. Any enumeration of E thus gives an isomorphism[1] of simplicial sets $\Delta^E \simeq \Delta^n$, and inclusions of finite non-empty partially ordered sets $E \subset F$ induce inclusions of simplicial sets $\Delta^E \subset \Delta^F$. For any integer $n \geq 0$, we define the *boundary* of the standard n-simplex to be

$$\partial\Delta^n = \partial h_{[n]} = \bigcup_{E \subsetneq [n]} \Delta^E \subset \Delta^n\,. \tag{4.5.2}$$

Similarly, for integers n, k, with $n \geq 1$ and $0 \leq k \leq n$, the kth *horn* of Δ^n is

$$\Lambda_k^n = \bigcup_{k \notin E \subsetneq [n]} \Delta^E \subset \Delta^n\,. \tag{4.5.3}$$

Finally, for an integer $n \geq 1$, the *spine* of Δ^n is

$$Sp^n = \bigcup_{0 \leq i < n} \Delta^{\{i,i+1\}} \subset \Delta^n\,. \tag{4.5.4}$$

We will interpret these constructions in dimension $n = 2$, using the intuition provided by the nerve of small categories. First, note that $\Lambda_1^2 = Sp^2$.

4.6. Let X be a simplicial set.

A *triangle* in X is a map $t : \partial\Delta^2 \longrightarrow X$. This can be seen as a triple (f, g, h) of morphisms in X, such that the target of f and the source of g coincide, while f and h have the same source x, and g and h have the same target z: since the boundary $\partial\Delta^2$ is the union of three copies of Δ^1, namely $\Delta^{\{i,i+1\}}$, for $0 \leq i \leq 2$, the morphism f corresponds to the map $\Delta^1 \simeq \Delta^{\{0,1\}} \subset \partial\Delta^2 \longrightarrow X$, while g

[1] Remark that the simplicial sets of the form Δ^E do not have non-trivial automorphisms, so that the existence of an isomorphism between such objects (as opposed to the specification of such an isomorphism) already is meaningful.

corresponds to the map $\Delta^1 \simeq \Delta^{\{1,2\}} \subset \partial\Delta^2 \longrightarrow X$, and h to the map $\Delta^1 \simeq \Delta^{\{0,2\}} \subset \partial\Delta^2 \longrightarrow X$. In other words, if we let y be the source of g, the triangle t can be faithfully represented as a diagram of the following shape.

$$\begin{array}{ccccc} & & y & & \\ & f \nearrow & & \searrow g & \\ x & & \xrightarrow{\quad h \quad} & & z \end{array} \tag{4.6.1}$$

Similarly, a map $\Lambda^2_1 = Sp^2 \longrightarrow X$ can be seen as a couple (f, g) of morphisms in X, such that the target of f and the source of g coincide. Indeed, Sp^2 is the union of two copies of Δ^1, namely $\Delta^{\{0,1\}}$ and $\Delta^{\{1,2\}}$, which intersects at the point $\Delta^0 \simeq \Delta^{\{1\}}$, so that f corresponds to the map $\Delta^1 \simeq \Delta^{\{0,1\}} \subset Sp^2 \longrightarrow X$, while g corresponds to the map $\Delta^1 \simeq \Delta^{\{1,2\}} \subset Sp^2 \longrightarrow X$. Such a map (f, g) thus can be represented as a diagram of the form below.

$$\begin{array}{ccccc} & & y & & \\ & f \nearrow & & \searrow g & \\ x & & & & z \end{array} \tag{4.6.2}$$

Definition 4.7. A triangle (f, g, h) in X, as in diagram (4.6.1), is *commutative* (or simply *commutes*), if there exists a morphism of simplicial sets $c : \Delta^2 \longrightarrow X$ whose restriction to the boundary coincides with (f, g, h):

$$\begin{array}{ccc} \partial\Delta^2 & \xrightarrow{(f,g,h)} & X \\ \downarrow & \nearrow_{c} & \\ \Delta^2 & & \end{array} \qquad c|_{\partial\Delta^2} = (f, g, h)\,.$$

Given a pair (f, g) of composable morphisms of X, as in diagram (4.6.2), a *composition* of f and g is a morphism h in X such that the triple (f, g, h) defines a commutative triangle in X.

Remark 4.8. If $X = N(C)$ is the nerve of a small category, the objects and arrows of X exactly are the objects and arrows of C, respectively. Furthermore, the commutative triangles of X precisely are the commutative triangles in the category C in the usual sense. A way to reformulate (and prove) this, is to say that the operation of restriction along the inclusion $Sp^2 \subset \Delta^2$ induces a bijective map

$$Hom(\Delta^2, N(C)) \xrightarrow{\sim} Hom(Sp^2, N(C))\,. \tag{4.8.1}$$

This bijection means that, in (the nerve of) a category, any couple of composable arrows has a *unique* composition. More generally, the description of the n-simplices of $N(C)$ as strings of arrows of the form (4.1.3) in C means that, for all integers $n \geq 2$, the restriction along the inclusion $Sp^n \subset \Delta^n$ induces a bijective map

$$Hom(\Delta^n, N(C)) \xrightarrow{\sim} Hom(Sp^n, N(C))\,. \tag{4.8.2}$$

These bijections express the associativity of the composition law in a category (we shall be more explicit later). This implies that one can understand maps to nerves of small categories as follows. For any simplicial set X and any small category C, a map from X to $N(C)$ is completely determined by a map $u : X_1 \longrightarrow Arr(C)$, such that the two conditions below are satisfied:

(i) for all objects x of X, $u(1_x)$ is an identity;
(ii) for any commutative triangle (f, g, h) in X, we have $u(h) = u(g) \circ u(f)$.

This implies that, in particular, we have the following property.

PROPOSITION 4.9. *For any simplicial set X, the inclusion $Sk_2(X) \subset X$ induces an isomorphism of categories $\tau(Sk_2(X)) \simeq \tau(X)$.*

4.10. The preceding remark is a description, in the language of simplicial sets, of (small) categories. To be more precise, we shall say that a simplicial set X satisfies the *Grothendieck-Segal condition* if the restriction along the inclusion map $Sp^n \subset \Delta^n$ induces a bijective map

$$Hom(\Delta^n, X) \xrightarrow{\sim} Hom(Sp^n, X)\ , \quad \text{for all integers } n \geq 2. \tag{4.10.1}$$

PROPOSITION 4.11. *The nerve functor is fully faithful: given two small categories C and D, the nerve functor defines a bijection*

$$Hom_{Cat}(C, D) \simeq Hom_{sSet}(N(C), N(D))\ .$$

Moreover, the essential image of the nerve functor precisely consists of the simplicial set which satisfies the Grothendieck-Segal condition. In other words, given a simplicial set X, the following conditions are equivalent.

(i) *There exists a small category C as well as an isomorphism of simplicial sets $X \simeq N(C)$.*
(ii) *The unit map $X \longrightarrow N(\tau(X))$ is invertible.*
(iii) *The simplicial set X satisfies the Grothendieck-Segal conditions.*

PROOF. The fully faithfulness of the nerve functor is a corollary of bijection (4.8.2) (see the description of maps toward a nerve after *loc. cit.*). Condition (ii) is a reformulation of the property of fully faithfulness. Therefore, general facts about adjoint functors imply that conditions (i) and (ii) are equivalent. We already know that condition (i) implies condition (iii). It is thus sufficient to prove that the Grothendieck-Segal condition implies (ii). We thus have to check if the Grothendieck-Segal conditions ensure that, for any non-negative integer n, the maps

$$X_n \longrightarrow N(\tau(X))_n$$

are bijective. But, for $n \leq 1$, one checks that this map always is the identity, from which one deduces, using the Grothendieck-Segal condition for $\tau(X)$, that, for $n \geq 2$, these map coincide with (4.8.2). □

REMARK 4.12. Let X be a topological space. A morphism in the simplicial set $Sing(X)$ is then a path in the space X, i.e. a continuous map from $|\Delta^1| = (0, 1)$ to X. Given two composable paths γ and γ', a composition of these in $Sing(X)$ precisely is a path λ whose starting point coincides with the one of γ, and with the same end-point as γ', such that there exists an homotopy deformation of paths

between λ and the concatenation of the paths γ and γ'. Therefore, the notion of composition of morphisms in a simplicial set encompasses both the notion of composition of morphisms in a category and the notion of composition of paths in a topological space.

A variation on the preceding proposition is the following.

PROPOSITION 4.13. *A simplicial set X satisfies the Grothendieck-Segal conditions if and only, for any integers $n \geq 2$ and $0 < k < n$, the restriction along the inclusion $\Lambda^n_k \subset \Delta^n$ induces a bijection*

$$Hom(\Delta^n, X) \xrightarrow{\sim} Hom(\Lambda^n_k, X) \tag{4.13.1}$$

We will only need to know that any simplicial set satisfying the Grothendieck-Segal condition has this property. Therefore, we will prove this fact, and let the other direction as an exercise for the reader. If X satisfies the Grothendieck-Segal condition, it is isomorphic to $N(C)$ for some small category C. Hence morphisms $Y \longrightarrow X$ always factor through $N(\tau(Y))$, and thus, by virtue of Proposition 4.9, only depends on their restriction to $Sk_2(Y)$. One checks that $Sk_2(\Lambda^n_k) = Sk_2(\Delta^n)$ whenever $n \geq 4$ because of dimension reasons. Since $\Lambda^2_1 = Sp^2$, it remains to check that $\tau(\Lambda^3_k) = \tau(\partial\Delta^3)$ for $k = 1, 2$. This is done directly, using the explicit description of τ at the end of Remark 4.8 (for a hint, see the proof of Lemma 5.9 below, in the case where $X = N(C)$ is the nerve of a small category).

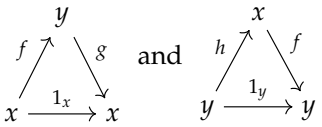
CHAPTER 5

Definition of ∞-categories

DEFINITION 5.1. An *∞-category* is a simplicial set X such that, for any integers $n \geq 2$ and $0 < k < n$, any morphism of the form $\Lambda^n_k \longrightarrow X$ extends to Δ^n. In other words, the operation of restriction along the inclusion $\Lambda^n_k \subset \Delta^n$ induces a surjection

$$Hom(\Delta^n, X) \longrightarrow Hom(\Lambda^n_k, X)\,. \tag{5.1.1}$$

A morphism $f : x \longrightarrow y$ in an ∞-category X is *invertible* if there exists two morphisms $g : y \longrightarrow x$ and $h : y \longrightarrow x$ such that 1_x is a composition of f and g, and that 1_y is a composition of h and f. In other words: both triangles



commute in X.

An *∞-groupoid* is an ∞-category in which any morphism is invertible.

A *Kan complex* is a a simplicial set X such that, for any integers $n \geq 1$ and $0 \geq k \geq n$, any morphism of the form $\Lambda^n_k \longrightarrow X$ extends to Δ^n.

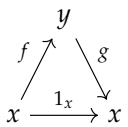
EXAMPLE 5.2. For any small category C, the nerve $N(C)$ is an ∞-category.

PROPOSITION 5.3. *Any Kan complex is an ∞-groupoid.*

PROOF. It is clear that any Kan complex is an ∞-category. If $f : x \longrightarrow y$ is a morphism in a Kan complex X, there is a unique morphism

$$\Lambda^2_0 \longrightarrow X$$

which sends the non-degenerate 1-simplex of $\Delta^{\{0,1\}}$ to f and the non-degenerate 1-simplex of $\Delta^{\{0,2\}}$ to 1_x. Its extension to Δ^2 shows the existence of a commutative triangle



in X. The existence of the other commutative triangle is proved similarly. □

EXAMPLE 5.4. For any topological space X, the singular complex $Sing(X)$ is both a Kan complex (whence an ∞-category) and an ∞-groupoid. To prove this, one checks that, if we identify $|\Delta^n|$ with the hypercube $(0,1)^n$, the topological

realization of Λ^n_k correspond to $(0,1)^{n-1} \times \{0\}$ (up to an affine automorphism of $(0,1)^n$). Using the adjunction formula

$$Hom(|K|, X) \simeq Hom(K, Sing(X))$$

thus implies that the restriction along the inclusion $\Lambda^n_k \subset \Delta^n$ induces a surjection

$$Hom(\Delta^n, Sing(X)) \longrightarrow Hom(\Lambda^n_k, Sing(X))$$

for any integers $n \geq 1$ and $0 \leq k \leq n$. Explicit inverses of paths come from the re-parametrization of the interval given by the function $t \longmapsto 1 - t$.

It was proved by Milnor that the homotopy theory of CW-complexes and the homotopy theory of Kan complexes essentially are the same (in a sense that we shall have the opportunity to make precise). We shall see later that the converse of the preceding proposition is true: any ∞-groupoid is a Kan complex. Therefore, homotopy types (of CW-complexes) will play a central role in higher category theory.

Remark 5.5. The notion of ∞-category as in definition 5.1 was discovered and introduced by Boardman and Vogt in order to understand the theory of algebraic structures up to (coherent) homotopies, under the name of *weak Kan complexes*. They were developped by Joyal under the name of *quasi-categories*, and then by Lurie under the name of ∞*-categories*. Milnor's theorem alluded to above suggests that there are many different presentations (models) of homotopy types (of CW-complexes), and this is indeed the case, with no way to consider one of them as better then the others. The same thing happens with ∞-categories.

5.6. Let $\rho : \mathbf{\Delta} \longrightarrow \mathbf{\Delta}$ be the functor defined as the identity on objects and by the formula

$$\rho(f)(i) = n - f(m - i)$$

for any map $f : [m] \longrightarrow [n]$, with $0 \leq i \leq m$. One has $\rho^2 = 1_{\mathbf{\Delta}}$ (one can check that ρ is the only non-trivial automorphism of the category $\mathbf{\Delta}$). Composing with ρ defines an automorphism of the category of simplicial sets

$$\rho^* : sSet \longrightarrow sSet\,.$$

For a simplicial set X, one defines his *opposite* simplicial set as $X^{op} = \rho^*(X)$. Given a morphism of simplicial sets $f : X \longrightarrow Y$, we also get, by functoriality, a morphism $f^{op} = \rho^*(f) : X^{op} \longrightarrow Y^{op}$.

Proposition 5.7. *For any small category C, we have a canonical identification*

$$N(C)^{op} = N(C^{op})\,.$$

Moreover, for any ∞-category X, its opposite X^{op} also is an ∞-category.

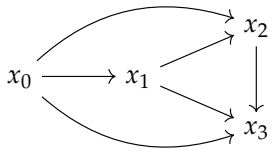
(The proof is left as an exercise.)

5.8. Let X be an ∞-category. The end of this chapter will be devoted to give an explicit description of the associated category $\tau(X)$.

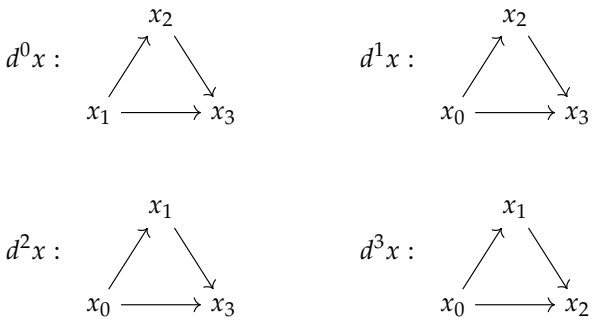
For this purpose, we will have to study maps of the form

$$x : Sk_1(\Delta^3) \longrightarrow X\,.$$

Since, by definition, the non-degenerate simplices of $Sk_1(\Delta^3)$ are of dimension ≤ 1, such a map precisely corresponds to a diagram in X of the form

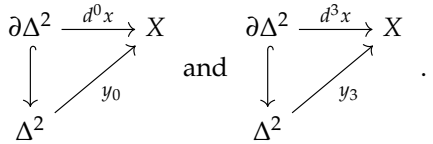


(in which none of the triangles is required to commute). There are four triangles $d^i x : \partial\Delta^2 \longrightarrow X$, corresponding to the restrictions of x to each of the subcomplexes $Sk_1(\Delta^{E_i})$, for each of the sets with three elements $E_i = \{0,1,2,3\} - \{i\}$, for $i = 0,1,2,3$.

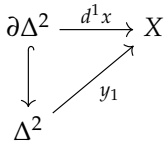


Lemma 5.9 (Coherence Lemma). *Assume that in a diagram $x : Sk_1(\Delta^3) \longrightarrow X$ as above, the two triangles d^0x and d^3x commute. Then the triangle d^1x commutes if and only if the triangle d^2x commutes.*

Proof. Since both triangles d^0x and d^3x commute, we may assume that two commutative triangles of the following form are given in the category of simplicial sets:

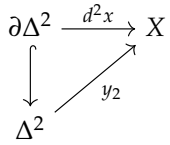


Let us assume that d^1x commutes as well. There exists a commutative diagram



and the data of y_0, y_1, y_3 defines a morphism $(y_0, y_1, y_3) : \Lambda^3_2 \longrightarrow X$. The latter extends to a 3-simplex $y : \Delta^3 \longrightarrow X$. If we put $y_2 = y\partial^3_2$, we see that the triangle

of simplicial sets



commutes, and, therefore, that the triangle d^2x commutes in X.

If d^2x commutes, then applying what precedes to the opposite ∞-category X^{op} gives the commutativity of d^1x. □

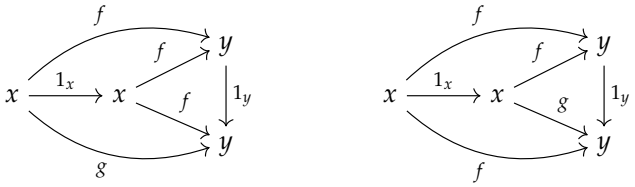
5.10. Given three morphisms f, g and h in X, we shall write $gf \sim h$ to mean that the triple (f, g, h) is a commutative diagram in X (i.e. a morphism $\partial\Delta^2 \longrightarrow X$ which can be extended to Δ^2).

Let us fix two objects x and y of X. One defines four relations on the set of morphisms of X with source x and target y:

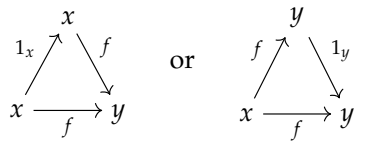
- $f \sim_1 g$ if $f1_x \sim g$;
- $f \sim_2 g$ if $1_y f \sim g$;
- $f \sim_3 g$ if $g1_x \sim f$;
- $f \sim_4 g$ if $1_y g \sim f$.

LEMMA 5.11. *The four relations $\sim_i$, $i = 1, 2, 3, 4$, are equal. Moreover, they are equivalence relations on the set of morphisms from x to y in X.*

PROOF. For any two morphisms $f, g : x \longrightarrow y$ in X, we have these two diagrams in X below.

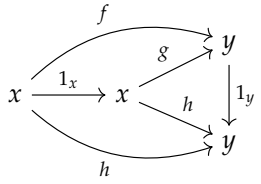


Since any triangle of the form



commutes (they are restrictions of 2-simplices of the form $\sigma^*(f)$, for appropriate surjective maps $\sigma : \Delta^2 \longrightarrow \Delta^1$), we may apply the coherence lemma (5.9) and conclude that $1_y f \sim g \Leftrightarrow f1_x \sim g$ and that $1_y f \sim g \Rightarrow g1_x \sim f$. The same argument in the opposite ∞-category thus implies that these four relations are equal. Let us put $\simeq$ for this relation. It remains to prove that this is an equivalence relation. It is clear that $f \sim_1 f$ and that $f \sim_2 g \Leftrightarrow g \sim_4 f$. Therefore, it remains to prove the property of transitivity. Assume that $f \simeq g$ and $g \simeq h$. Then we have

$g1_x \sim f$, $1_y g \sim h$ and $h1_x \sim h$. Applying the coherence lemma to the diagram



thus shows that $1_y f \sim h$. □

5.12. For two objects x and y, we define the set $Hom_{ho(X)}(x,y)$ as the quotient of the set of morphisms from x to y in X by the equivalence relation $\sim_1$ given by Lemma 5.11 above. Given a morphism $f : x \longrightarrow y$ in X, we write $[f]$ for its class in $Hom_{ho(X)}(x,y)$. Finally, we define a composition law

$$Hom_{ho(X)}(x,y) \times Hom_{ho(X)}(y,z) \longrightarrow Hom_{ho(X)}(x,z)\ , \quad ([f],[g]) \longmapsto [g] \circ [f]$$

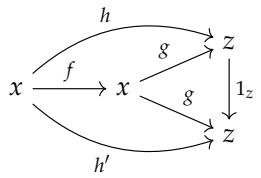
by putting $[g] \circ [f] = [h]$ whenever h is a composition of f and g.

THEOREM 5.13 (Boardman & Vogt). *The composition law constructed above is well defined, and this produces a category $ho(X)$. There is a unique morphism of simplicial sets $X \longrightarrow N(ho(X))$ which is the identity on objects and which sends a morphism $f : x \longrightarrow y$ in X to its class $[f]$ in $Hom_{ho(X)}(x,y)$. Moreover, this morphism induces an isomorphism*

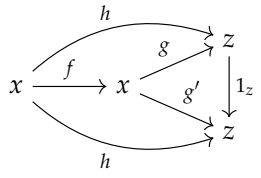
$$\tau(X) \simeq ho(X)\,.$$

.

PROOF. If ever we have the relations $gf \sim h$ and $gf \sim h'$ applying the coherence lemma to the diagram

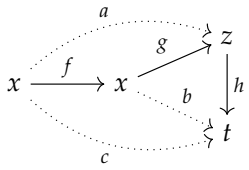


we see that we must have the relation $1_z h \sim h'$. Similarly, if $g \simeq g'$ and $gf \sim h$, applying the coherence lemma to the diagram



gives that $g'f \sim f$. The same argument in the opposite ∞-category shows that, if $f \simeq f'$ and if $gf \sim h$, then $gf' \sim h$. Therefore, the relation $gf \sim h$ only depends on the classes $[f]$, $[g]$ and $[h]$. This does prove, in particular, that the composition law is well defined. To check the associativity, we see that, for a triple of composable arrows(f,g,h), since compositions always exist in an ∞-category,

one can complete the following diagram as follows: one chooses a composition a of (f,g), a composition b of (g,h), and, finally, a composition c of (f,b).



One then deduces from the coherence lemma that c also is a composition of (a,h). In other words, this composition law is associative:

$$([h]\circ[g])\circ[f] = [b]\circ[f] = [c] = [h]\circ[a] = [h]\circ([g]\circ[f])\,.$$

The reflexivity of the equivalence relation $\sim_1{=}\sim_2$ (5.11) means in particular that it is unital as well, with units the 1_x's.

The last assertions follows right away from Remark 4.8. □

Corollary 5.14. *An ∞-category is an ∞-groupoid if and only if the associated category $\tau(X) \simeq ho(X)$ is a groupoid.*

Corollary 5.15. *A morphism $f : x \longrightarrow y$ in an ∞-category X is invertible if and only if there exists a morphism $g : y \longrightarrow x$ such that 1_x is a composition of f and g, and that 1_y is a composition of g and f.*

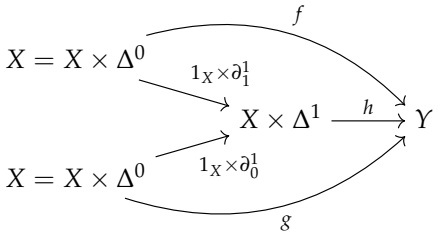
Example 5.16. For any topological space X, the ∞-groupoid $Sing(X)$ thus defines a groupoid $\pi_1(X) = ho(Sing(X))$, which is nothing else than the *Poincaré groupoid*: the objects are the points of X, and the morphims are the homotopy classes of continuous paths in X. In particular, for each point x of X, we have the *fundamental group* $\pi_1(X,x) = Hom_{ho(Sing(X))}(x,x)$.

Definition 5.17. A *functor* between ∞-categories simply is a morphism of simplicial sets.

If X and Y are ∞-categories, and if $f,g : X \longrightarrow Y$ are two functors, a *natural transformation from f to g* is a morphism $h : X \times \Delta^1 \longrightarrow Y$ such that

$$h(x,0) = f(x) \quad \text{and} \quad h(x,1) = g(x)$$

or, more formally, if the following diagram commutes.



Such a natural transformation is *invertible* if, for any object x of X, the induced morphism $f(x) \longrightarrow g(x)$ (corresponding to the restriction of h to $\Delta^1 = \{x\} \times \Delta^1$) is invertible in Y.

A functor $f : X \longrightarrow Y$ is an *equivalence of ∞-categories* if there exists a functor $g : Y \longrightarrow X$ as well as invertible natural transformations from fg to 1_Y as well as from 1_X to gf.

REMARK 5.18. All these notions extend the usual ones for ordinary categories: the ordered set $[1]$ represents the set of morphisms in the category of small categories, and the nerve of the associated category $N([1])$ is canonically isomorphic to Δ^1, whence a natural transformation between ordinary functors is essentially the same thing as a natural transformation between their nerves. Since the nerve functor is fully faithful, we see that a functor between small categories is an equivalence of categories if and only if its nerve has the same property.

In order to work with ∞-categories, as above, we would like to have usual categorical constructions: cartesian products, limits and colimits, the ∞-category of functors between two ∞-categories... If the latter exists, we would like natural transformations to be its morphisms, and the invertible natural transformations to be its invertible morphisms. This would imply that invertible natural transformations have inverses (whence their name). This is true, but non-trivial. Furthermore, we would like these categorical constructions to be compatible with equivalences of ∞-categories. As we will see in details later, all these properties (and many more) will hold, but it comes with a price: we must develop the homotopy theory of ∞-categories, in Quillen's setting of model categories (and we really must do it, since many of the properties we seek are equivalent to the verification of Quillen's axioms, at least in this particular context). The next chapters will be about the general theory of model categories, and about their constructions in the particular context of categories of presheaves.

CHAPTER 6

Factorization systems

Let us cope with some definitions.

DEFINITION 6.1. Let $i : A \longrightarrow B$ and $p : X \longrightarrow Y$ be two morphisms in a category $\mathcal{C}$.

We say that i has the *left lifting property* with respect to p, or, equivalently, that p has the *right lifting property* with respect to i, any commutative square of the form

$$\begin{array}{ccc} A & \xrightarrow{a} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow[b]{} & Y \end{array}$$

has a diagonal filler

$$\begin{array}{ccc} A & \xrightarrow{a} & X \\ {\scriptstyle i}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \xrightarrow[b]{} & Y \end{array}$$

(i.e. a morphism h such that $hi = a$ and $ph = b$).

Let F be class of morphisms in $\mathcal{C}$. A morphism has the left (right) lifting property with respect to F if it has the the left (right) lifting property with respect to any element of F.

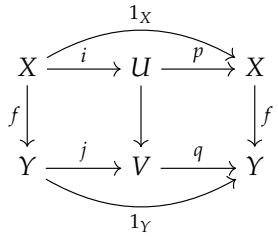
One denotes by $l(F)$ (by $l(F)$) the class of morphisms which has the left (right) lifting property with respect to F.

DEFINITION 6.2. An object X is a retract of another object U if there exists a commutative diagram of the following form

$$X \xrightarrow{i} U \xrightarrow{p} X \quad .$$

We say that a morphism $f : X \longrightarrow Y$ is a retract of a morphism $g : U \longrightarrow V$ if it so in the category of morphisms: in other words, if there exists a commutative

diagram of the form below.



A class of morphisms F is stabe under *retracts* if any morphism which is a retract of an element of F belongs to F.

A class of morphisms F is stabe under *pushouts* if, for any pushout square of the form

$$\begin{array}{ccc} X & \xrightarrow{a} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow{b} & Y' \end{array}$$

if f is in F, so is f'.

A class of morphisms F, in a category $\mathcal{C}$, is stable under *transfinite compositions* if, for any well ordered set I, with initial element 0, for any functor $X : I \longrightarrow \mathcal{C}$, if for any element $i \in I$, the colimit $\varinjlim_{j<i} X(j)$ is representable and the induced map

$$\varinjlim_{j<i} X(j) \longrightarrow X(i)$$

is in F, then the canonical morphism $X(0) \longrightarrow \varinjlim_{i\in I} X(i)$ belongs to F.

A class of morphisms is *saturated* if it is stable under retracts, under pushouts, and under transfinite composition.

Remark 6.3. If a class $\mathcal{F}$ of morphisms contains all the identities and is stable under pushouts as well as by transfinite compositions, then it is stable under small sums: for any small set I and any family of maps $u_i : X_i \longrightarrow Y_i$ in $\mathcal{F}$, indexed by $i \in I$, the induced map

$$\coprod_{i\in I} X_i \longrightarrow \coprod_{i\in I} Y_i$$

is in $\mathcal{F}$.

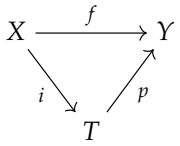
The following proposition follows straight away from the definitions.

Proposition 6.4. *Let $\mathcal{C}$ a category, together with two classes of morphisms F and F'. We have the following properties.*

a) $F \subset r(F') \Leftrightarrow F' \subset l(F)$.
b) $F \subset F' \Rightarrow l(F') \subset l(F)$.
c) $F \subset F' \Rightarrow r(F') \subset r(F)$.
d) $r(F) = r(l(r(F)))$.
e) $l(F) = l(r(l(F)))$.
f) *The class $l(F)$ is saturated.*
g) *The class $r(F)$ is co-saturated, i.e. is saturated as a class of morphisms of $\mathcal{C}^{op}$.*

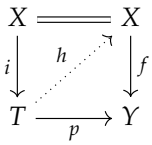
And here is a useful trick.

PROPOSITION 6.5 (Retract Lemma). *Assume that a morphism $f : X \longrightarrow Y$ can be factored into $f = pi$.*

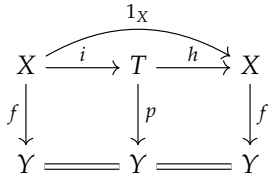


If f has the right (left) lifting property with respect to i (to p), then f is a retract of p (of i, respectively).

PROOF. If $f \in r(i)$, then the diagonal of the solid commutative square



gives rise to a commutative diagram of the form



and thus turns f into a retract of p. The respective case follows by considering what precedes in the opposite category of $\mathcal{C}$. □

EXAMPLE 6.6. For $\mathcal{C} = Set$, and $i : \varnothing \longrightarrow \{\text{point}\}$, the class $r(i)$ is the class of surjective maps, while a reformulation of the axiom of choice is the assertion that the class $l(r(i))$ is the class of injective maps. Since any small set is a small sum of sets with one element, the excluded middle principle means that the smallest saturated class of maps in *Set* which contains i is the class $l(r(i))$.

DEFINITION 6.7. A *weak factorization system* in a category $\mathcal{C}$ is a couple (A, B) of classes of morphisms satisfying the following properties:

(a) both A and B are stable under retracts;
(b) $A \subset l(B)$ ($\Leftrightarrow B \subset r(A)$);
(c) any morphism $f : X \longrightarrow Y$ of $\mathcal{C}$ admits a factorization of the form $f = pi$, with $i \in A$ and $p \in B$.

REMARK 6.8. It follows from the Retract Lemma that we must have $A = l(B)$ as well as $B = r(A)$.

We recall the following general facts (for proofs, see **[Hov99]** or **[Rie14]**, where more general assumptions will also be found).

PROPOSITION 6.9 (the small object argument). *Let $\mathcal{C}$ be a locally small category endowed with a small set or morphisms I. Assume that there exists a cardinal κ such that, for any element $i : K \longrightarrow L$ of I, the functor*

$$Hom_{\mathcal{C}}(K, -) : \mathcal{C} \longrightarrow Set$$

commutes with colimits indexed by well ordered sets of cardinal $> \kappa$. Then the couple $(l(r(I)), r(I))$ is a weak factorization system.

COROLLARY 6.10. *Let A be a small category, and I a small set of morphisms of presheaves over A. Then the couple $(l(r(I)), r(I))$ is a weak factorization system in $\widehat{A}$.*

EXAMPLE 6.11. In any category of presheaves over a small category A (or, more generally, in any Grothendieck topos), the class of monomorphisms is part of a weak factorization system. Let us call *trivial fibrations* the morphisms of presheaves which have the right lifting property with respect to monomorphisms. We may prove that any morphism of presheaves over A can be factored as a monomorphism $i : X \longrightarrow Y$ followed by a trivial fibration $p : Y \longrightarrow Z$ in two ways.

First method. We have the subobject classifyer Ω (also called the Lawvere object in Grothendieck's *Pursuing Stacks*). In other words, for any object a in A, the set Ω_a is the set of subobjects of the representable presheaf h_a (the structure of presheaf is given by pullbacks of subobjects). One checks that, for any presheaf X over A, there is a canonical bijection

$$Hom(X, \Omega) \simeq \{\text{subobjects of } X\}.$$

We put $P(X) = \underline{Hom}(X, \Omega)$. There is then a canonical embedding

$$\{.\} : X \longrightarrow P(X)$$

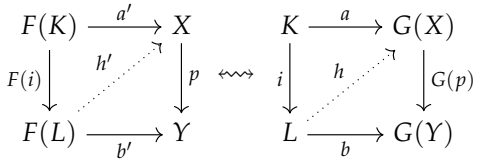
which corresponds to the diagonal of $X \times X$ (seen as a subobject). We see that $P(X)$ is an injective object (i.e. that the map from $P(X)$ to the point is a trivial fibration): this is a reformulation of the fact that, for any monomorphism $K \longrightarrow L$, any subobject of the cartesian product $K \times X$ induces a subobject of $L \times X$. In other words, we have factored the map from X to the point into a monomoprhism followed by a trivial fibration. The general case follows by applying the previous construction to the category of presheaves over A/Y, using the equivalence of categories $\widehat{A}/Y \simeq \widehat{A/Y}$, discussed in Remark 1.14.

Second method. One checks (using the axiom of choice) that the class of monomorphisms is equal to the class $l(r(I))$, for I the set of monomorphisms of presheaves of the form $K \longrightarrow L$, with L a quotient of some representable presheaf over A. One can then apply the small object argument.

A basic, although extremely useful, recognition of lifting properties is the following one.

PROPOSITION 6.12. *Let $F : \mathcal{C} \rightleftarrows \mathcal{C}' : G$ be an adjunction. Assume that $\mathcal{C}$ and $\mathcal{C}'$ are endowed with weak factorization systems (A, B) and (A', B'), respectively. Then we have $F(A) \subset A'$ if and only if $G(B') \subset B$.*

PROOF. We have, by adjunction, a natural correspondance of the form:



□

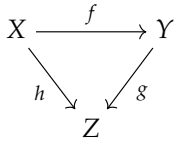
CHAPTER 7

Model Categories

The notion of model category was introduced by Quillen in [**Qui67**]. All the results of the next two chapters already are in *loc. cit.*

DEFINITION 7.1. A *model category* is a locally small category $\mathcal{C}$ endowed with three classes of morphisms W, Fib, and Cof, such that the following two properties are verified.

1. The category $\mathcal{C}$ has finite limits and finite colimits.
2. The class W has the 'two-out-of-three' property: for any commutative triangle of the form



   if two, among f, g and h, are in W, so is the third.
3. Both couples $(Cof, Fib \cap W)$ and $(Cof \cap W, Fib)$ are factorization systems.

Here is the standard terminology in the presence of such a structure.

An element of W (of Fib, of Cof) is called a *weak equivalence* (a *fibration*, a *cofibration*, respectively). A morphism which is both a weak equivalence and a fibration (and a cofibration) is called a *trivial* fibration (cofibration, respectively).

We shall often write $\varnothing$ and e for the initial and final objects of $\mathcal{C}$, respectively. An object X of $\mathcal{C}$ is *fibrant* (*cofibrant*) if the unique morphism $X \longrightarrow e$ ($\varnothing \longrightarrow X$) is a fibration (a cofibration, respectively).

REMARK 7.2. These axioms imply that any isomorphism is both a trivial fibration and a trivial cofibration (in particular, a weak equivalence). In the presence of the other axioms, one can replace axiom 2. by a weaker one: the class W is closed under compositions. The full 'two-out-of-three' property is a consequence of the definition (exercise).

REMARK 7.3. This definition is (equivalent to) the notion of *closed* model category introduced by Quillen in [**Qui67**]. Axiom (b) implies in particular that any morphism f has a factorization of the form $f = pi$ where i is a cofibration, and p a trivial fibration, as well as a factorization of the form $f = qj$, where j is a trivial cofibrations, and q a fibration.

In some modern introductions to the theory of model categories, the underlying category is assumed to have small limits and small colimits (as opposed to finite ones), and the factorizations are required to exist functorially. All the

examples we will consider here will be of this sort. There are many reasons not to rely on functorial factorizations, though[1]. In a model category, the meaningful part is the class of weak equivalences. Fibrations and cofibrations really are intermediate tools (in particular, we should always feel free to replace these at will, as far as this makes sense, of course).

EXAMPLE 7.4. For any category with finite limits and colimits, we have two model category structures for which the weak equivalences are the isomorphisms: we may either take the the cofibrations to be the isomorphisms, and the fibrations to be all morphisms, or dually, the cofibrations to be all morphisms, and the fibrations to be the isomorphisms.

EXAMPLE 7.5. It follows from Example 6.11 that, for any small category A, the category of presheaves over A has a model category structure whose class of weak equivalences is the class of all morphisms, while the cofibrations are the monomorphisms.

PROPOSITION 7.6. *The notion of model category structure is stable under the following categorical constructions.*

(a) *If $\mathcal{C}$ is a model category, so is $\mathcal{C}^{op}$: the weak equivalences are those of $\mathcal{C}$, while its fibrations (cofibrations) are the cofibrations (fibrations, respectively) of $\mathcal{C}$.*
(b) *For any object X in a model category $\mathcal{C}$, the slice category $\mathcal{C}/X$ has a natural structure of model category: the weak equivalences (the fibrations, the cofibrations) are the morphisms whose image in $\mathcal{C}$ are weak equivalences (the fibrations, the cofibrations, respectively).*
(c) *We can put the preceding two constructions together: the category of objects under X has a natural structure of model category.*

PROPOSITION 7.7 (Ken Brown's Lemma). *Let $\mathcal{C}$ be a model category, together with a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$. Assume that the category $\mathcal{D}$ is endowed with a class of weak equivalences, by which we mean a class of morphisms which contains all isomorphisms and which has the 'two-out-of-three' property. If ever the functor F sends trivial cofibrations between cofibrant objects to weak equivalences, then it sends weak equivalences between cofibrant objects to weak equivalences.*

PROOF. Let $f : X \longrightarrow Y$ be a weak equivalence between cofibrant objects. We may form the push out square

$$\begin{array}{ccc} \varnothing & \longrightarrow & Y \\ \downarrow & & \downarrow j \\ X & \xrightarrow{i} & X \amalg Y \end{array}$$

[1]These factorization are used to construct and to compute derived functors. If we only use fonctorial resolutions, we push ourselves into artificial difficulties: for instance, a soft resolution of the sheaf of locally constant complex valued functions on a manifold is given by the de Rham complex, but general soft resolutions of sheaves are not given by differential calculus. Avoiding non-functorial resolution would make us avoiding the classical proof of de Rham's theorem as part of the frame work of model categories. The other reason why it is not useful to ask for functorial factorizations, is that any proof of a general statement on model categories which uses them can be normalized into one which does not.

and see that, since both X and Y are cofibrant, the canonical maps from X and Y two the coproduct $X \amalg Y$ are cofibrations. We may factor the map $(f, 1_Y) : X \amalg Y \longrightarrow Y$ into a cofibration $k : X \amalg Y \longrightarrow T$ followed by a weak equivalence (a trivial fibration) $p : T \longrightarrow Y$. We thus have the following two commutative triangles.

$$\begin{array}{ccc} X & \xrightarrow{ki} & T \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle p} & \\ & Y & \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{kj} & T \\ & {\scriptstyle 1_Y}\searrow \quad \swarrow{\scriptstyle p} & \\ & Y & \end{array}$$

Since the map $F(p)$ has a section which is the image of a trivial cofibration between cofibrant objects, it is a weak equivalence. On the other hand the map ki is a trivial cofibration between cofibrant objects, and therefore, its image $F(ki)$ is a weak equivalence. The 'two-out-of-three' property for weak equivalences in $\mathcal{D}$ implies that $F(f)$ is a weak equivalence. □

Definition 7.8. Let $\mathcal{C}$ be category endowed with a class of morphisms W. A *localization* of $\mathcal{C}$ (by W) is a functor

$$\gamma : \mathcal{C} \longrightarrow ho(\mathcal{C}) \tag{7.8.1}$$

that sends the elements of W to isomorphisms, and which is universal for this latter property. In other words, we ask that, for any category $\mathcal{D}$, if we denote by $\underline{Hom}_W(\mathcal{C}, \mathcal{D})$ the full subcategory of the category of functors $\underline{Hom}(\mathcal{C}, \mathcal{D})$ which consists of functors sending elements of W to isomorphisms, then the operation of composing with γ

$$\gamma^* : \underline{Hom}_W(\mathcal{C}, \mathcal{D}) \longrightarrow \underline{Hom}(ho(\mathcal{C}), \mathcal{D}) \tag{7.8.2}$$

is an equivalence of categories. By abuse of terminology, we shall say that $ho(\mathcal{C})$ is the localization of $\mathcal{C}$.

Proposition 7.9. *There always exists a localization of $\mathcal{C}$ by W. Moreover, one can choose a localization $\gamma : \mathcal{C} \longrightarrow ho(\mathcal{C})$ in such a way that the map* (7.8.2) *is an isomorphism of categories. Under this more rigid constraint, the functor γ is a bijection on objects.*

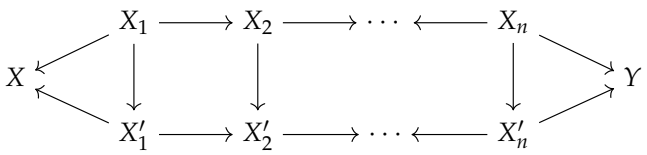
One defines the category $ho(\mathcal{C})$ as follows. The objects are those of $\mathcal{C}$, and the set of morphisms from X to Y is an appropriate quotient of the class of diagrams of the form

$$X = X_0 \longleftarrow X_1 \longrightarrow X_2 \longleftarrow \cdots \longrightarrow X_{n-1} \longleftarrow X_n \longrightarrow X_{n+1} = Y$$

where $n \geq 0$ is an integer, and where the maps of the form $\longleftarrow$ all are in W or are identities[2]. The equivalence relation we put means that we can always replace such a diagram by a bigger one by adding identities anywhere we want (and in

[2]Or one can also assume that W contains all isomorphisms, since this does not affect the categories of the form $\underline{Hom}_W(\mathcal{C}, \mathcal{D})$.

any direction we want), any two such diagrams of the same length are equivalent whenever they can be connected through a commutative diagram in $\mathcal{C}$ of the form



REMARK 7.10. There is no reason, in general, why the sets $Hom_{ho(\mathcal{C})}(X, Y)$ would be small. Therefore, for the moment, this construction only makes sense for non-necessarily locally small categories. We shall review first how to compute these sets of morphisms, and see in particular that, if W is the class of weak equivalences of a model category structure on $\mathcal{C}$, they are small sets.

From now on, let us consider a fixed model category $\mathcal{C}$.

DEFINITION 7.11. A *cylinder* of an object A is a factorization of the codiagonal of A into a cofibration followed by a weak equivalence, i.e. a commutative diagram of the form

$$A \amalg A \xrightarrow{(\partial_0, \partial_1)} IA \xrightarrow{\sigma} A, \qquad (1_A, 1_A) : A \amalg A \to A \tag{7.11.1}$$

in which (∂^0, ∂^1) is a cofibration, and σ a weak equivalence. Dually, a *cocylinder* (we also say a *path object*) of an object X is a cylinder in the opposite category, i.e. a commutative diagram of the form

$$X \xrightarrow{s} X^I \xrightarrow{(d^0, d^1)} X \times X, \qquad (1_X, 1_X) : X \to X \times X \tag{7.11.2}$$

in which the map s is a weak equivalence, and the map (d^0, d^1) is a fibration.

Let us consider two morphisms $f_0, f_1 : A \longrightarrow X$.

A *left homotopy* (a *right homotopy*) from f_0 to f_1 is given by a cylinder of A of the form (7.11.1) (a cocylinder of X of the form (7.11.2)), together with a morphism $h : IA \longrightarrow X$ (a morphism $k : A \longrightarrow X^I$), such that, for $i = 0, 1$, we have $h\partial_i = f_i$ (we have $d^i k = f_i$, respectively).

LEMMA 7.12. *Let A and X be a cofibrant object and a fibrant object, respectively. For a pair of morphisms $f_i : A \longrightarrow X$, $i = 0, 1$, the following conditions are equivalent.*

a) *There exists a left homotopy from f_0 to f_1.*
b) *There exists a right homotopy from f_0 to f_1.*
c) *For any cylinder of A of the form* (7.11.1)*, there exists a map $h : IA \longrightarrow X$ such that $h\partial_i = f_i$ for $i = 0, 1$.*
d) *For any cocylinder of X of the form* (7.11.2)*, there exists a map $k : A \longrightarrow X^I$ such that $d^i k = f_i$ for $i = 0, 1$.*

PROOF. Since we can replace the underlying category by its opposite $\mathcal{C}^{op}$, it is clearly sufficient to prove that a) implies d). Let us assume that there is a left homotopy from f_0 to f_1 given by a cylinder of A of the form (7.11.1), together with a map $h : IA \longrightarrow X$, and let us consider an arbitrary cocylinder of X, of the form (7.11.2). Since the map $\partial_1 : A \longrightarrow IA$ is a trivial cofibration (because this is a weak equivalence, and since A is cofibrant, this is also a cofibration, for this is the pushout of the cofibration $\varnothing \longrightarrow A$), and since the map (d^0, d^1) is a fibration, the following solid commutative square

$$\begin{array}{ccc} A & \xrightarrow{sf_1} & X^I \\ {\scriptstyle \partial_1}\downarrow & \overset{K}{\nearrow} & \downarrow{\scriptstyle (d^0,d^1)} \\ IA & \xrightarrow[(h,f_1\sigma)]{} & X \times X \end{array}$$

admits a filling K. Le us put $k = K\partial_0$. We then have:

$$d^1k = d^1K\partial_0 = f_1\sigma\partial_0 = f_1 1_A = f_1 \quad \text{and} \quad d^0k = d^0K\partial_0 = h\partial_0 = f_0\,.$$

In other words, condition d) is verified. □

LEMMA 7.13. *Let A and X be a cofibrant object and a fibrant object, respectively. We define an equivalence relation $\sim$ on the set of morphisms from A to X, by defining $f_0 \sim f_1$ whenever there exists a left (or right) homotopy from f_0 to f_1.*

PROOF. The reflexivity is clear: for any cylinder of the form (7.11.1), and for any map $f : A \longrightarrow X$, the morphism σf defines a left homotopy from f to itself. The equivalence between conditions a) and c) of Lemma 7.12 implies that the relation $\sim$ is symmetric: indeed, for any cylinder of A of the form (7.11.1), the diagram

$$A \amalg A \xrightarrow{(\partial_1,\partial_0)} IA \xrightarrow{\sigma} A$$

is a cylinder of A as well. Assume that we have three maps u, v and w from A to X, as well as cylinders

$$A \amalg A \xrightarrow{(\partial_0,\partial_1)} IA \xrightarrow{\sigma} A \qquad A \amalg A \xrightarrow{(\partial'_0,\partial'_1)} I'A \xrightarrow{\sigma'} A$$

and homotopies $h : IA \longrightarrow X$ and $h' : I'A \longrightarrow X$ satisfying $h\partial_0 = u$, $h\partial_1 = v = h'\partial'_0$, and $h'\partial'_1 = w$. We form the pushout square

$$\begin{array}{ccc} A & \xrightarrow{\partial_1} & IA \\ {\scriptstyle \partial'_0}\downarrow & & \downarrow{\scriptstyle e} \\ I'A & \xrightarrow{e'} & I''A \end{array}$$

There is a unique map $\sigma'' : I''A \longrightarrow A$ such that $\sigma''e = \sigma$ and $\sigma''e' = \sigma'$, and we get a cylinder of the form

$$A \amalg A \xrightarrow{(\partial''_0,\partial''_1)} I''A \xrightarrow{\sigma''} A$$

by defining $\partial''_0 = e\partial_0$ and $\partial''_1 = e'\partial'_1$. Indeed, it is clear that the map σ'' is a weak equivalence (because e is a trivial cofibration and *sigma* a weak equivalence). One thus only have to show that the map $(\partial''_0, \partial''_1)$ is a cofibration. In the commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & A \amalg A & \xrightarrow{(\partial_0,\partial_1)} & IA \\ {\scriptstyle \partial'_0}\downarrow & & {\scriptstyle 1_A \amalg \partial'_0}\downarrow & & \downarrow{\scriptstyle e} \\ I'A & \longrightarrow & A \amalg I'A & \xrightarrow{(\partial''_0,e')} & I''A \end{array}$$

the left hand square is the obvious pushout square, while the composed square is the previous pushout square, hence the right hand square is coCartesian. In particular, all the maps in the right hand square are cofibrations, and therefore, its diagonal $(\partial''_0, \partial''_1)$ is a cofibration. Finally, we define a morphism $h'' : I''A \longrightarrow X$ as the unique one such that $h''e = h$ and $h''e' = h'$. It is clear that $h''\partial''_0 = u$ and $h''\partial''_1 = w$. □

NOTATION 7.14. Under the assumptions of the preceding lemma, we write

$$[A, X] = Hom_{\mathcal{C}}(A, X)/\sim \tag{7.14.1}$$

for the quotient of the set of morphisms from A to X by the relation of left (or right) homotopy $\sim$. It is clear that the relation of left homotopy is compatible with composition on the left, while the relation of homotopy is compatible with composition on the right. In other words, we have defined a functor

$$[-,-] : \mathcal{C}_c^{op} \times \mathcal{C}_f \longrightarrow Set\,, \tag{7.14.2}$$

where $\mathcal{C}_c$ and $\mathcal{C}_f$ denote the full subcategories of $\mathcal{C}$ spanned by cofibrant objects and by fibrant objects, respectively.

THEOREM 7.15. *The inclusion $\mathcal{C}_c \longrightarrow \mathcal{C}$ induces an equivalence of categories of the form $ho(\mathcal{C}_c) \simeq ho(\mathcal{C})$ (where $ho(\mathcal{C}_c)$ denotes the localization of $\mathcal{C}_c$ by the class of weak equivalences between cofibrant objects).*

Dually, the inclusion $\mathcal{C}_f \longrightarrow \mathcal{C}$ induces an equivalence of categories of the form $ho(\mathcal{C}_f) \simeq ho(\mathcal{C})$.

This theorem is a triviality whenever the factorizations of the model structure can be obtained functorially (which will be the case in all the examples of these notes). The general case is proven in **[Qui67]**.

PROPOSITION 7.16. *The functor* (7.14.2) *is compatible weak equivalences in $\mathcal{C}_c$ and in $\mathcal{C}_f$ (i.e. sends such weak equivalences to invertible natural transformations) and thus defines a functor*

$$[-,-] : ho(\mathcal{C}_c)^{op} \times ho(\mathcal{C}_f) \longrightarrow Set\,.$$

PROOF. Let $i : A \longrightarrow B$ be a weak equivalence between cofibrant objects, and X a fibrant object. We shall prove that the induced map

$$i^* : [B, X] \longrightarrow [A, X]$$

is bijective. By virtue of Ken Brown's Lemma (7.7), it is sufficient to consider the case where i is a trivial cofibration between cofibrant objects. The surjectivity is

easy: for any morphism $f : A \longrightarrow X$, the solid commutative square

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & \overset{g}{\nearrow} & \downarrow \\ B & \longrightarrow & e \end{array}$$

admits a filler g whose homotopy class is sent by i^* to the homotopy class of f. As for the injectivity, let us consider two morphisms $f, g : B \longrightarrow X$ such that fi and gi are homotopic (i.e. their classes are equal in $[A, X]$). Then we may choose a right homotopy from if to ig, defined by a cocylinder of the form (7.11.2), together with a map $k : A \longrightarrow X^I$ such that $d^0 k = if$ and $d^1 k = ig$. The following solid commutative

$$\begin{array}{ccc} A & \xrightarrow{k} & X^I \\ {\scriptstyle i}\downarrow & \overset{K}{\nearrow} & \downarrow{\scriptstyle (d^0,d^1)} \\ B & \xrightarrow[(f,g)]{} & X \times X \end{array}$$

admits a filler: the left hand vertical map is a trivial cofibration, and the map (d^0, d^1) is a fibration. Therefore, f and g are homotopic.

Applying what precedes to $\mathcal{C}^{op}$, we see that, for any cofibrant object A and any weak equivalence between fibrant objects $p : X \longrightarrow Y$, the induced map

$$p_* : [A, X] \longrightarrow [A, Y]$$

is bijective. □

THEOREM 7.17. *For any cofibrant object A and any fibrant object X, there is a canonical bijection*

$$[A, B] \simeq Hom_{ho(\mathcal{C})}(A, X)$$

which is natural with respect to morphisms of $ho(\mathcal{C}_c)^{op} \times ho(\mathcal{C}_f)$.

COROLLARY 7.18. *Let $\mathcal{C}_{cf}$ be the full subcategory of objects which are both fibrant and cofibrant. The relation of homotopy of Lemma 7.13 defines an equivalence relation which is compatible with composition on $\mathcal{C}_{cf}$. The resulting quotient category $\pi(\mathcal{C}_{cf})$ thus has the fibrant-cofibrant objects of $\mathcal{C}$ as objects, while we have*

$$Hom_{\pi(\mathcal{C}_{cf})}(A, X) = [A, X] .$$

The inclusion functor $\mathcal{C}_{cf} \subset \mathcal{C}$ induces a canonical equivalence of categories

$$\pi(\mathcal{C}_{cf}) \simeq ho(\mathcal{C}) .$$

Theorem 7.17 is in fact a consequence of the corollary (using Theorem 7.15, we see that this is because any fibrant object X is isomorphic in $ho(\mathcal{C})$ to an object A which is both fibrant and cofibrant: simply consider a factorization of the map $\varnothing \longrightarrow X$ into a cofibration followed by a trivial fibration $A \longrightarrow X$).

Let us assume that the inclusion functor $\mathcal{C}_{cf} \subset \mathcal{C}$ induces a canonical equivalence of categories $ho(\mathcal{C}_{cf}) \simeq ho(\mathcal{C})$, where $ho(\mathcal{C}_{cf})$ stands for the localization of

$\mathcal{C}_{cf}$ by the class of weak equivalences (this is easy to check when the factorizations can be obtained functorially). We can then directly prove Corollary 7.18 as follows.

First, one checks that if two maps $f, g : A \longrightarrow X$ in $\mathcal{C}_{cf}$ are homotopic, then $\gamma(f) = \gamma(g)$ in $ho(\mathcal{C})$. Let IA be a cylinder of A as in (7.11.1), and $h : A \longrightarrow X$ be a map such that $h\partial_0 = f$ and $h\partial_1 = g$. Since $\sigma\partial_i = 1_X$ does not depend on i, and since σ becomes invertible in $ho(\mathcal{C})$, we must have $\gamma(\partial_0) = \gamma(\partial_1)$ in $ho(\mathcal{C})$. Therefore, the mophism $\gamma(g)$ must be equal to $\gamma(f)$.

As a consequence, any homotopy equivalence (i.e. any map of $\mathcal{C}_{cf}$ which becomes an isomorphism in $\pi(\mathcal{C}_{cf})$) induces an isomorphism in $ho(\mathcal{C})$. On the other hand, Theorem 7.17 implies that any weak equivalence $u : A \longrightarrow B$ in $\mathcal{C}_{cf}$ must induce bijections $[B, X] \simeq [A, X]$ for any object X in $\mathcal{C}_{cf}$, and thus, by virtue of the Yoneda Lemma for $\pi(\mathcal{C}_{cf})$, must become invertible in $\pi(\mathcal{C}_{cf})$. In other words, a morphism of $\mathcal{C}_{cf}$ is a weak equivalence if and only if it is an homotopy equivalence. Since the category $\pi(\mathcal{C}_{cf})$ is the localization of $\mathcal{C}_{cf}$ by the class of homotopy equivalences, this shows that $\pi(\mathcal{C}_{cf}) = ho(\mathcal{C})$.

COROLLARY 7.19. *Let $\mathcal{C}$ be a model category endowed with a localization functor $\gamma : \mathcal{C} \longrightarrow ho(\mathcal{C})$.*

If $\mathcal{C}$ is locally small, then so is the homotopy category $ho(\mathcal{C})$. Furthermore, a morphism of $\mathcal{C}$ is a weak equivalence if and only if its image $\gamma(f)$ is an isomorphism in $ho(\mathcal{C})$.

CHAPTER 8

Derived functors

DEFINITION 8.1. Let $\mathcal{C}$ be a model category, together with a localization functor $\gamma : \mathcal{C} \longrightarrow ho(\mathcal{C})$, as well as a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$.

A *left derived* functor of F is a functor $\mathbf{L}F : ho(\mathcal{C}) \longrightarrow \mathcal{D}$ together with a functorial morphism $a_X : \mathbf{L}F(\gamma(X)) \longrightarrow F(X)$ which turns $\mathbf{L}F$ into a right Kan extension of F along the localization functor γ. In other words, for any functor $\Phi : ho(\mathcal{C}) \longrightarrow \mathcal{D}$ and any natural morphism $\alpha_X : \Phi(\gamma(X)) \longrightarrow F(X)$, there is a unique natural morphism $f_Y : \Phi(Y) \longrightarrow \mathbf{L}F(Y)$ such that $\alpha_X = a_X f_{\gamma(X)}$ for all objects X of $\mathcal{C}$.

A *right derived* functor of F is a functor $\mathbf{R}F : ho(\mathcal{C}) \longrightarrow \mathcal{D}$ together with a functorial morphism $b_X : F(X) \longrightarrow \mathbf{R}F(\gamma(X))$ which turns $\mathbf{R}F$ into a right Kan extension of F along the localization functor γ (i.e. $\mathbf{R}F^{op}$ and b form a left derived functor of $F^{op} : \mathcal{C}^{op} \longrightarrow \mathcal{D}^{op}$).

8.2. Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor which sends trivial cofibrations between cofibrant objects to isomorphisms. By virtue of Ken Brown's Lemma, the functor F sends weak equivalences between cofibrant objects to isomorphisms. Therefore, there is a unique functor $F_c : ho(\mathcal{C}_c) \longrightarrow \mathcal{D}$ whose composition with the localization functor $\mathcal{C}_c \longrightarrow ho(\mathcal{C}_c)$ coincides with the restriction of F. Let us choose, for each object X of $\mathcal{C}$, a weak equivalence $a'_X : QX \longrightarrow X$ with QX cofibrant. If we write $i : ho(\mathcal{C}_c) \longrightarrow ho(\mathcal{C})$ for the canonical equivalence of categories induced by the inclusion $\mathcal{C}_c \subset \mathcal{C}$, there is a unique way to promote the collection of these choices to a functor

$$Q : ho(\mathcal{C}) \longrightarrow ho(\mathcal{C}_c)$$

endowed with natural isomorphisms $i(Q(X)) \simeq X$ and $Q(i(X)) \simeq X$ defined as the images of the maps a'_X. One defines a functor $\mathbf{L}F$ by the formula $\mathbf{L}F(Y) = F_c(Q(Y))$. There is a unique natural morphism $a_X : \mathbf{L}F(\gamma(X)) \longrightarrow F(X)$ which coincides with the image of a'_X.

PROPOSITION 8.3. *For any functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ which sends trivial cofibrations between cofibrant objects to isomorphisms, the pair $(\mathbf{L}F, a)$ is a left derived functor of F.*

PROOF. Let $\Phi : ho(\mathcal{C}) \longrightarrow \mathcal{D}$ be a functor, together with a natural morphism $\alpha_X : \Phi(\gamma(X)) \longrightarrow F(X)$. By definition of localizations (applied to $\mathcal{C}_c$), and by virtue of the canonical equivalence of categories $ho(\mathcal{C}_c) \simeq ho(\mathcal{C})$, there is a unique natural morphism $f_Y : \Phi(Y) \longrightarrow \mathbf{L}F(Y)$ such that $\alpha_X = a_X f_{\gamma(X)}$ for all objects X of $\mathcal{C}_c$. The latter property remains true for X running all over the class of objects of $\mathcal{C}$ simply because a, α and f are natural tranformations. □

COROLLARY 8.4. *Under the hypothesis of the previous proposition, for any functor $G : \mathcal{D} \longrightarrow \mathcal{E}$, the pair $(G\mathbf{L}F, Ga)$ is a left derived functor of GF.*

DEFINITION 8.5. Let $\mathcal{C}$ and $\mathcal{C}'$ be two model categories endowed with two localization functors $\gamma : \mathcal{C} \longrightarrow ho(\mathcal{C})$ and $\gamma' : \mathcal{C}' \longrightarrow ho(\mathcal{C}')$, respectively.

If a functor $F : \mathcal{C} \longrightarrow \mathcal{C}'$ preserves trivial cofibrations, then the composed functor $\gamma' F$ sends trivial cofibrations between cofibrant objects to isomorphisms, and therefore, by virtue of the preceding proposition, admits a left derived functor. We also denote by

$$\mathbf{L}F : ho(\mathcal{C}) \longrightarrow ho(\mathcal{C}')$$

the left derived functor of $\gamma' F$, which we call the *total left derived functor* of F.

Similarly, if a functor $F : \mathcal{C} \longrightarrow \mathcal{C}'$ preserves trivial fibrations, we denote by

$$\mathbf{R}F : ho(\mathcal{C}) \longrightarrow ho(\mathcal{C}')$$

the right derived functor of $\gamma' F$, which we call the *total right derived functor* of F.

PROPOSITION 8.6. *If $F : \mathcal{C} \longrightarrow \mathcal{C}'$ and $F'\mathcal{C}' \longrightarrow \mathcal{C}''$ both are functors between model categories which preserve cofibrant objects (fibrant objects)and trivial cofibrations (trivial fibrations), then at the level of total left derived functors (of total right derived functors), the canonical comparison map*

$$(\mathbf{L}F' \circ \mathbf{L}F)(X) \longrightarrow \mathbf{L}(F' \circ F)(X) \quad (\mathbf{R}(F' \circ F)(X) \longrightarrow (\mathbf{R}F' \circ \mathbf{R}F)(X), \textit{ respectively)}$$

is an isomorphism for all objects X of $ho(\mathcal{C})$.

PROOF. It is sufficient to consider the case of total left derived functor. One can prove this assertion using Corollary 8.4, but a very simple argument consists to remember how we constructed the total left derived functors: for a given object X of $\mathcal{C}$, we choose a weak equivalences with cofibrant domains $p : X' \longrightarrow X$ and $q : X'' \longrightarrow F(X')$. We then have $\mathbf{L}F(X) = F(X')$, $\mathbf{L}(F' \circ F)(X) = F'(F(X'))$, and $(\mathbf{L}F' \circ \mathbf{L}F)(X) = F'(X'')$. The comparison map we want to understand is then the image in $ho(\mathcal{C}'')$ of the map $F'(q)$. Since q is a weak equivalence between cofibrant objects, by virtue of Ken Brown's Lemma, such a map $F'(q)$ is a weak equivalence. □

DEFINITION 8.7. Let $\mathcal{C}$ and $\mathcal{C}'$ be two model categories.

A *Quillen adjunction* is a pair of adjoint functors

$$F : \mathcal{C} \rightleftarrows \mathcal{C}' : G$$

such that F preserves cofibrations and G preserves fibrations.

REMARK 8.8. By virtue of Remark 6.8 and of Proposition 6.12, for a pair of adjoint functors F and G as above, the following conditions are equivalent.

(i) The pair (F, G) is a Quillen adjunction.
(ii) The functor F preserves cofibrations as well as trivial cofibrations.
(iii) The functor G preserves fibrations as well as trivial fibrations.

In particular, for any Quillen adjunction (F, G), the functor F admits a total left derived functor, and the functor G a total right derived functor.

THEOREM 8.9. *Any Quillen adjunction $F : \mathcal{C} \rightleftarrows \mathcal{C}' : G$ naturally induces an adjunction*

$$\mathbf{L}F : ho(\mathcal{C}) \rightleftarrows ho(\mathcal{C}') : \mathbf{R}G$$

PROOF. It follows from Ken Brown's Lemma that, for any cofibrant object A of $\mathcal{C}$, the functor F sends cylinders of A to cylinders of $F(A)$. Since the functor G preserves fibrant objects, it follows that, for any cofibrant object A of $\mathcal{C}$ and any fibrant object X of $\mathcal{C}'$, the bijection

$$Hom_{\mathcal{C}}(A, G(X)) \simeq Hom_{\mathcal{C}'}(F(A), X)$$

is compatible with the relation of homotopy, and thus, by virtue of Proposition 7.16, induces a natural bijection

$$[A, G(X)] \simeq [F(A), X]$$

as functors from $ho(\mathcal{C}_c)^{op} \times ho(\mathcal{C}'_f)$ to the category of sets. Therefore, Theorems 7.15 and 7.17, together with the very construction of the functors $\mathbf{L}F$ and $\mathbf{R}G$, end the proof. □

Examples of useful derived functors consist to consider total derived functor of basic categorical operations, such as limits and colimits. Although we will not consider the general case here, we will need special cases which can be understood at a rather elementary level.

8.10. Let $\mathcal{C}$ be a model category.

Given a small category, let us consider the category $\mathcal{C}^I = \underline{Hom}(I, \mathcal{C})$. We define the weak equivalences of $\mathcal{C}^I$ as the morphisms $F \longrightarrow G$ such that, for any object i of I, the evaluation at i is a weak equivalence $F_i \longrightarrow G_i$ in $\mathcal{C}$.

We want to define a model structure on $\mathcal{C}^I$ by defining the class of fibrations as the one which consists of morphims $F \longrightarrow G$ such that, for any object i of I, the evaluation at i is a fibration $F_i \longrightarrow G_i$ in $\mathcal{C}$ (the cofibrations are defined by the condition of left lifting property with respect to trivial fibrations). The problem is that there is no known result asserting the existence of such a model structure, unless we make further assumptions on either I or $\mathcal{C}$. However, when such a model category structure exists, it is called the *projective model structure*. We will consider several basic examples of small categories I such that the projective model structure always exists (for any model category $\mathcal{C}$).

For instance, if I is a small discrete category, then the projective model structure exists: the cofibrations simply are the morphisms $F \longrightarrow G$ such that, for any object i of I, the evaluation at i is a fibration $F_i \longrightarrow G_i$ in $\mathcal{C}$; all the axioms simply are verified level-wise. Here are slightly less trivial examples.

PROPOSITION 8.11. *If I is the free category generated by the oriented graph $0 \longrightarrow 1$ (so that the category $\mathcal{C}$ simply is the category of arrows $X_0 \longrightarrow X_1$ in $\mathcal{C}$, with commutative squares as morphisms), then the projective model structure exists.*

PROOF. We will adopt the convention that if objects of $\mathcal{C}^I$ is denoted by uppercase letters such as X, for instance, the corresponding arrow $X_0 \longrightarrow X_1$ will be denoted by the corresponding lowercase letter $x : X_0 \longrightarrow X_1$. A morphism

$f : X \longrightarrow Y$ in $\mathcal{C}^I$ thus corresponds to a commutative square of the form

$$\begin{array}{ccc} X_0 & \xrightarrow{f_0} & Y_0 \\ \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle y} \\ X_1 & \xrightarrow{f_1} & Y_1 \end{array}$$

One defines cofibrations (trivial cofibrations) as follows: a morphism $f : X \longrightarrow Y$ is a (trivial) cofibration if the map f_0 is a (trivial) cofibration and if the canonical map $(f_1, y) : X_1 \amalg_{X_0} Y_0 \longrightarrow Y_1$ is a (trivial) cofibration. One checks that trivial cofibrations precisely are the cofibrations which are weak equivalences. The verification of the lifting properties is straightforward. To obtain factorization into a cofibration followed by a trivial fibration, we proceed as follows (the case of a trivial cofibration followed by a fibration is similar). Given a map $f : X \longrightarrow Y$ in $\mathcal{C}^I$, we factor f_0 as a cofibration $i_0 : X_0 \longrightarrow T_0$ followed by a trivial fibration $p_0 : T_0 \longrightarrow Y_0$. We then form the pushout square

$$\begin{array}{ccc} X_0 & \xrightarrow{i_0} & T_0 \\ \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle t'} \\ X_1 & \xrightarrow{i'_1} & T'_1 \end{array}$$

and we choose a factorization of the map $(f_1, p_0) : T'_1 \longrightarrow Y_1$ into a cofibration $j : T'_1 \longrightarrow T_1$ followed by a trivial fibration $p_1 : T_1 \longrightarrow Y_1$. We put $i_1 = ji'_1$. We thus have factored the map f into a cofibration $i : X \longrightarrow T$ followed by a fibration $p : T \longrightarrow Y$. □

Proposition 8.12. *In the case of the free category I generated by the oriented graph $(0,0) \longleftarrow (0,1) \longrightarrow (1,1)$ (so that the category $\mathcal{C}$ simply is the category of diagrams of the form $X_{0,0} \longleftarrow X_{0,1} \longrightarrow X_{1,1}$ in $\mathcal{C}$), the projective model structure exists. The cofibrant objects are the diagrams of the form $X_{0,0} \longleftarrow X_{0,1} \longrightarrow X_{1,1}$ in which all the objects are cofibrant and all the maps are cofibrations.*

Proposition 8.13. *For any small well ordered set I with initial element 0, the projective model structure exists. The cofibrant objects are the functors $X : I \longrightarrow \mathcal{C}$ such that, for any element $i \in I$, the map $\varinjlim_{j<i} X_j \longrightarrow X_i$ is a cofibration (in particular, all the X_i's must be cofibrant).*

The proofs of propositions 8.12 and 8.13 are left as exercises[1]

8.14. Whenever the projective model category structure exists, we then have a Quillen adjunction

$$\varinjlim_I : \mathcal{C}^I \rightleftarrows \mathcal{C} : \delta \tag{8.14.1}$$

[1]These are special cases of a more general fact: if I is a direct category, the projective model structure exists. We refer to the appropriate chapter of Hovey's book [**Hov99**] for the definition and for the proof (which involves some knowledge of these special cases anyway).

where $\delta(X) = X_I$ denotes the constant diagram indexed by I with value X. This means that, for any functor $F : I \longrightarrow \mathcal{C}$ and any object X, there is an adjunction of the form

$$Hom_{ho(\mathcal{C}^I)}(F, X_I) \simeq Hom_{ho(\mathcal{C})}(\mathbf{L}\varinjlim_I F, X) \tag{8.14.2}$$

Proposition 8.15. *Assume that the projective model category structure exists on $\mathcal{C}^I$. Any natural transformation $f : X \longrightarrow Y$ between cofibrant functors from I to $\mathcal{C}$, which is a weak equivalence $f_i : X_i \longrightarrow Y_i$ at each object i of I, induces a weak equivalence $\varinjlim_I X \longrightarrow \varinjlim_I Y$.*

Proof. Such a map $X \longrightarrow Y$ is an isomorphism in $ho(\mathcal{C})$ and therefore, its image by the functor $\mathbf{L}\varinjlim_I$ is an isomorphism of $ho(\mathcal{C})$. But, by construction of the derived functor $\mathbf{L}\varinjlim_I$ (see 8.2), this isomorphism is the image of the map $\varinjlim_I X \longrightarrow \varinjlim_I Y$. Since any map inducing an isomorphism in the homotopy category is a weak equivalence (7.19), this proves the proposition. □

Taking the case of a discrete category, as well as the examples provided by propositions 8.12 and 8.13, the preceding proposition takes the following explicit forms, respectively.

Corollary 8.16. *Let I be a small set, and $f_i : X_i \longrightarrow Y_i$ a family of weak equivalences between cofibrant objects. Then the induced map*

$$\coprod_{i\in I} f_i : \coprod_{i\in I} X_i \longrightarrow \coprod_{i\in I} Y_i$$

is a weak equivalence.

Corollary 8.17. *Consider the following commutative diagram in which all the horizontal maps are cofibrations between cofibrant objects.*

$$\begin{array}{ccccc} X' & \xleftarrow{x'} & X & \xrightarrow{x''} & X'' \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} \\ Y' & \xleftarrow{y'} & Y & \xrightarrow{y''} & Y'' \end{array}$$

If the three vertical maps are weak equivalences, then the induced morphism

$$X' \amalg_X X'' \longrightarrow Y' \amalg_Y Y''$$

is a weak equivalence.

Corollary 8.18. *Let I be a small well ordered set, and $X \longrightarrow Y$ be a natural transformation between functors from I to $\mathcal{C}$. Assume that, for any element i of I, the maps*

$$\varinjlim_{j<i} X_j \longrightarrow X_i \quad \text{and} \quad \varinjlim_{j<i} Y_j \longrightarrow Y_i$$

are cofibrations, and the map $X_i \longrightarrow Y_i$ is a weak equivalence. Then the induced morphism

$$\varinjlim_{i\in I} X_i \longrightarrow \varinjlim_{i\in I} Y_i$$

is a weak equivalence.

In the case where I is the smallest infinite ordinal, the preceding corollary takes an even more concrete form.

COROLLARY 8.19. *Consider the following commutative diagram, which consists of a morphism f of sequences of cofibrations between cofibrant objects indexed by non-negative integers.*

$$\begin{array}{ccccccccccccc} X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & \cdots & \longrightarrow & X_n & \longrightarrow & X_{n+1} & \longrightarrow & \cdots \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n+1}} & & \\ Y_0 & \longrightarrow & Y_1 & \longrightarrow & Y_2 & \longrightarrow & \cdots & \longrightarrow & Y_n & \longrightarrow & Y_{n+1} & \longrightarrow & \cdots \end{array}$$

If the vertical maps all are weak equivalences, then the induced map

$$\varinjlim_n X_n \longrightarrow \varinjlim_n Y_n$$

is a weak equivalence as well.

DEFINITION 8.20. Let $F : I \longrightarrow \mathcal{C}$ be a functor, and X an object of the model category $\mathcal{C}$. A co-cone $F \longrightarrow X_I$ (i.e. a map in $\mathcal{C}^I$) exhibits the object X as the *homotopy colimit* of F if the induced map $\mathbf{L}\varinjlim_I F \longrightarrow X$ is an isomorphism in $ho(\mathcal{C})$.

Homotopy limits are defined similarly (as homotopy colimits in $\mathcal{C}^{op}$).

EXAMPLE 8.21. For any functor $F : I \longrightarrow \mathcal{C}$ which is cofibrant in the projective model category structure on $\mathcal{C}^I$ (and provided that the latter exists), the colimit co-cone exhibits $\varinjlim_I F$ as an homotopy colimit of F. In fact, provided that the projective model category structure exists on $\mathcal{C}^I$, we can characterize homotopy colimits as follows: given a functor F from I to $\mathcal{C}$, a co-cone $F \longrightarrow X_I$ exhibits X as an homotopy colimit of F if and only if there exists a weak equivalence $F' \longrightarrow F$, with cofibrant domain in the projective model category structure, such that the induced map $\varinjlim_I F' \longrightarrow X$ is a weak equivalence.

DEFINITION 8.22. A commutative square

$$\begin{array}{ccc} X & \xrightarrow{x} & X' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow{y} & Y' \end{array}$$

is *homotopy coCartesian*, if it exhibits Y' as the homotopy pushout (i.e. the homotopy colimit), of the diagram $Y \longleftarrow X \longrightarrow X'$.

Dually, such a square is said to be *homotopy Cartesian* if it exhibits X as the homotopy pullback of the diagram $Y \longrightarrow Y' \longleftarrow X'$ (or, equivalently, if it is an homotopy coCartesian square in the opposite category).

EXAMPLE 8.23. Any coCartesian square in which all maps are cofibrations between cofibrant objects is homotopy coCartesian.

REMARK 8.24. Let $\square = [1] \times [1]$, so that functors from $\square$ to $\mathcal{C}$ precisely are the commutative squares in $\mathcal{C}$. We then have a projective model category structure on $\mathcal{C}^{\square}$ (because $\mathcal{C}^{\square} \simeq (\mathcal{C}^{[1]})^{[1]}$, so that we may apply Proposition 8.11 twice). A commutative square of $\mathcal{C}$ is homotopy coCartesian if and only if it is isomorphic in $ho(\mathcal{C})$ to a commutative square of $\mathcal{C}$ which is coCartesian and in which all the maps are cofibrations between cofibrant objects. The next two statements are direct consequences of this characterization.

PROPOSITION 8.25. *One of the two squares*

$$\begin{array}{ccc} X & \xrightarrow{x} & X' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow{y} & Y' \end{array} \qquad \text{and} \qquad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle y} \\ X' & \xrightarrow{f'} & Y' \end{array}$$

is homotopy coCartesian if and only if the other one has the same property.

PROPOSITION 8.26. *Any commutative square of the form*

$$\begin{array}{ccc} X & \xrightarrow{x} & X' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow{y} & Y' \end{array}$$

in which both x and y are weak equivalences is homotopy coCartesian.

PROPOSITION 8.27. *Consider a coCartesian square of the form below.*

$$\begin{array}{ccc} X & \xrightarrow{x} & X' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow{y} & Y' \end{array} \tag{8.27.1}$$

Assume that the morphism f is a cofibration and that both X and X' are cofibrant. If moreover the map x is a weak equivalence, so is y.

PROOF. Given any map $x : X \longrightarrow X'$, there is an adjunction of the form

$$x_! : X\backslash\mathcal{C} \rightleftarrows X'\backslash\mathcal{C} : x^!$$

where $x^!$ denotes the functor induced by composition with x:

$$x^!(f' : X' \longrightarrow Y') = (f'x : X \longrightarrow Y') .$$

The left adjoint $x_!$ is the functor which associates to any map $f : X \longrightarrow Y$ its pushout along the map x, so that we get a pushout of the form (8.27.1). The functor $x^!$ obviously preserves fibrations and cofibrations, so that we have a Quillen adjunction, and therefore, a derived adjunction

$$\mathbf{L}x_! : ho(X\backslash\mathcal{C}) \rightleftarrows ho(X'\backslash\mathcal{C}) : \mathbf{R}x^!$$

The cofibrant objects of $X\backslash\mathcal{C}$ precisely are the cofibrations $f : X \longrightarrow Y$, so that the unit of the derived adjunction evaluated at such a cofibration f is the image

of the map y in the coCartesian square (8.27.1). Therefore, if, for any cofibration $f : X \longrightarrow Y$, the map y, in diagram (8.27.1), is a weak equivalence, then the functor $\mathbf{L}x_!$ is fully faithful. By virtue of Corollary 7.19, the reverse is true: if this functor is fully faithful, then the map y is a weak equivalence for any cofibration $f : X \longrightarrow Y$. On the other hand, the functor $\mathbf{R}x^!$ is always conservative: it is sufficient to check this property for maps between objects which are both fibrant and cofibrant in $X'\backslash\mathcal{C}$, in which case this follows from the fact that, by definition, a map in $X'\backslash\mathcal{C}$ is a weak equivalence if an only if its image in $\mathcal{C}$ has the same property. Therefore, the functor $\mathbf{L}x_!$ is an equivalence of categories if and only if it is fully faithful[2]. Finally, we conclude that the functor $\mathbf{R}x^!$ is an equivalence of categories if and only if, for any cofibration $f : X \longrightarrow Y$, the associated map $y : Y \longrightarrow Y'$ is a weak equivalence. For instance, this is the case whenever the map x is a trivial cofibration (since the class of trivial cofibrations is stable under pushouts). It is time to remark that the functor $x^!$ preserves weak equivalence, from which one checks that the functors $\mathbf{R}x^!$ turn the map

$$X \longmapsto ho(X\backslash\mathcal{C})$$

into a functor from $\mathcal{C}$ to the opposite of the category of locally small categories. Since the class of equivalences of categories satisfies the 'two-out-of-three' property, by virtue of Ken Brown's lemma (7.7), we conclude that any weak equivalence between cofibrant objects $x : X \longrightarrow X'$ induces an equivalence of categories $\mathbf{R}x^!$, and this achieves the proof. □

Corollary 8.28. *Consider a coCartesian square of the form* (8.27.1)*. If the morphism f is a cofibration and if both X and X' are cofibrant, then this is an homotopy coCartesian square.*

Proof. We choose a factorization of the map x into a cofibration $i : X \longrightarrow X''$ followed by a weak equivalence $p : X'' \longrightarrow X'$. The commutative square above now is the composition of two coCartesian squares of the form below.

$$\begin{array}{ccccc} X & \xrightarrow{i} & X'' & \xrightarrow{p} & X' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow{j} & Y'' & \xrightarrow{q} & Y' \end{array}$$

The left hand commutative square is homotopy coCartesian (8.23), and, therefore, as explained in Remark 8.24, it is sufficient to prove that the induced map q is a weak equivalence. In other words, we may assume that, furthermore, the map x is a weak equivalence, and it is sufficient to prove that, under this additional assumption, the map y is a weak equivalence, which follows from the preceding proposition. □

Corollary 8.29. *In Corollary 8.17, one may only assume that all objects are cofibrant and that x' and y' are cofibrations, and still get to the same conclusion.*

[2]Given an adjunction $u : A \rightleftarrows B : v$, if the left adjoint u is fully faithful (or, equivalently, if the unit map $a \longrightarrow v(u(a))$ is invertible for any object a of A), then the functor v is a localization of B by the class of maps whose image by v is invertible. Therefore, if u is fully faithful and if v is conservative, both u and v are equivalences of categories and are quasi-inverses to each other.

CHAPTER 9

Model structures ex nihilo

Homotopical structures

9.1. As we have mentioned in Proposition 6.9, the small object argument is one of the possible tools to construct weak factorization systems. When, in a model category structure, the weak factorization systems $(Cof, W \cap Fib)$ and $(W \cap Cof, Fib)$ can be constructed out of the small object argument, we say that the model category is *cofibrantly generated*. In that case, we thus have the existence of a small set I of cofibrations, as well as of a small set J of trivial cofibrations, such that $l(r(I)) = Cof$ and $l(r(J)) = W \cap Cof$. We then say that I and J *generate* the class of cofibrations and the class of trivial cofibrations, respectively.

Example 9.2. Let A be a ring, and $\mathcal{C} = Comp(A)$ be the category of (possibly unbounded) cochain complexes of (left) A-modules. This category has a structure of cofibrantly generated model category, for which the weak equivalences are the quasi-isomorphisms and the fibrations are the epimorphims (i.e. the maps which are surjective in each degree); see the second chapter of **[Hov99]**, for instance. A generic set of cofibrations is given by all the shifts of the inclusion of the ring A (seen as a complexes of A-modules concentrated in degree zero) into the mapping cone of the identity of A. A generating set of trivial cofibration is given by all shifts of inclusion of 0 into the mapping cone of the identity of A. (This example will not be used in these notes.)

9.3. The purpose of this chapter is to explain a general procedure to construct cofibrantly generated model category structures on categories of presheaves of sets over a fixed small category, following a previous work **[Cis06]** of the author of these notes. The idea is simple: it consists to follow step by step most of the book of Gabriel and Zisman **[GZ67]** on the homotopy theory of Kan complexes, and to see that significant part of it makes sense in a wide generality. The idea of promoting the book of Gabriel and Zisman into a general way to define homotopy theories ex nihilo comes from the early work of Fabien Morel on homotopy theory of schemes **[Mor06]**.

From now on, we fix a small category A. In this chapter, all presheaves are presheaves of sets over A.

Definition 9.4. A *cellular model* is a small set M of monomorphisms of presheaves such that the class $l(r(M))$ is the class of all monomorphisms of presheaves.

Cellular models always exist: for instance, the set of monomorphisms of the form $K \longrightarrow L$, where L run over the quotients of representable presheaves, is a cellular model; see Example 6.11.

But some cellular models are nicer than others.

EXAMPLE 9.5. If A is an Eilenberg-Zilber category (e.g. $A = \mathbf{\Delta}$), it follows right away from Theorem 3.8 that the boundary inclusions $\partial h_a \longrightarrow h_a$, $a \in Ob(A)$, do form a cellular model.

DEFINITION 9.6. A cylinder of a presheaf X is a commutative diagram

$$X \amalg X \xrightarrow{(\partial_0,\partial_1)} IX \xrightarrow{\sigma} X \qquad \overset{(1_X,1_X)}{\frown} \tag{9.6.1}$$

in which the map (∂_0, ∂_1) is a monomorphism[1].

REMARK 9.7. Given a category $\mathcal{C}$, we have the category $\underline{End}(\mathcal{C})$ of endofunctors of $\mathcal{C}$. This is a monoidal category whose tensor product is defined by composition of functors. Of course, the category $\underline{End}(\mathcal{C})$ acts on the left on $\mathcal{C}$:

$$(F, X) \longmapsto F \otimes X = F(X)$$

With this convention, any natural transformation $u : F \longrightarrow G$ in $\underline{End}(\mathcal{C})$ and any morphism $p : X \longrightarrow Y$ in $\mathcal{C}$ induce a morphism

$$u \otimes p : F \otimes X \longrightarrow G \otimes X\,.$$

If we write 1 for the identity of $\mathcal{C}$, we also have $1 \otimes X = X$.

DEFINITION 9.8. A *functorial cylinder* is an endofunctor I of the category of presheaves endowed with a morphism $(\partial_0, \partial_1) : 1 \amalg 1 \longrightarrow I$ as well as a morphism $\sigma : I \longrightarrow 1$ such that, for any presheaf X, the diagram

$$X \amalg X \xrightarrow{(\partial_0\otimes 1_X,\partial_1\otimes 1_X)} I \otimes X \xrightarrow{\sigma\otimes 1_X} X \tag{9.8.1}$$

is a cylinder of X.

An *exact cylinder* is such a functorial cylinder satisfying the following properties.

DH1. The functor I commutes with small colimits and preserves monomorphisms.

DH2. For any monomorphism of presheaves $j : K \longrightarrow L$, the commutative square

$$\begin{array}{ccc} K & \xrightarrow{j} & L \\ {\scriptstyle \partial_\varepsilon\otimes 1_K}\downarrow & & \downarrow{\scriptstyle \partial_\varepsilon\otimes 1_L} \\ I \otimes K & \xrightarrow{1_I\otimes j} & I \otimes L \end{array}$$

is Cartesian for $\varepsilon = 0, 1$.

[1] This means that both maps $\partial_i : X \longrightarrow IX$ are monomorphisms and that the intersection of the image of ∂_0 and of the image of ∂_1 is empty.

REMARK 9.9. Given a Cartesian square of presheaves

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & T \end{array}$$

in which all maps are monomorphisms, the induced morphism $Y \amalg_X Z \longrightarrow T$ is a monomorphism, whose image is denoted by $Y \cup Z$. We thus have an inclusion

$$Y \cup Z \subset T\,.$$

We will write $\{\varepsilon\}$ for the subobject of I determined by the monomorphism $\partial_\varepsilon : 1 \longrightarrow I$. We thus have canonical inclusions

$$K \simeq \{\varepsilon\} \otimes K \subset I \otimes K$$

for $\varepsilon = 0, 1$. The axiom DH2 now means that we have inclusions

$$I \otimes K \cup \{\varepsilon\} \otimes L \subset I \otimes L$$

for $\varepsilon = 0, 1$.

We shall also write

$$\partial I = \{0\} \amalg \{1\}\,.$$

Since colimits are universal in any category of presheaves (i.e. pulling back along a morphism of presheaves defines a functor which preserves small colimits), the inclusion $\partial I \longrightarrow I$, together with any monomorphism of presheaves $j : K \longrightarrow L$, gives a Cartesian square made of monomorphisms

$$\begin{array}{ccc} \partial I \otimes K & \longrightarrow & I \otimes K \\ \downarrow & & \downarrow \\ \partial I \otimes L & \longrightarrow & I \otimes L \end{array}$$

and, therefore, induces a canonical inclusion

$$I \otimes K \cup \partial I \otimes L \subset I \otimes L\,.$$

EXAMPLE 9.10. If I is an interval (i.e. a cylinder object of the final presheaf), the functor $X \longmapsto I \times X$ is an elementary homotopical datum. For instance, one always can take the subobject classifier $I = \Omega$; see 6.11.

DEFINITION 9.11. Given a exact cylinder I, a class of morphisms of presheaves An is a class of *I-anodyne extensions* if it satisfies the following properties.

- An0. There exists a small set of morphisms of presheaves Λ such that $An = l(r(\Lambda))$ (in particular, the class An is saturated and is a subclass of the class of monomorphisms).
- An1. For any monomorphism of presheaves $K \longrightarrow L$, the induced morphism $I \otimes K \cup \{\varepsilon\} \otimes L \longrightarrow I \otimes L$ is in An for $\varepsilon = 0, 1$.
- An2. For any map $K \longrightarrow L$ in An, the induced map $I \otimes K \cup \partial I \otimes L \longrightarrow I \otimes L$ is in An.

An *homotopical structure* is the data of an exact cylinder together with a class of I-anodyne extensions.

Example 9.12. The class of monomorphisms is a class of I-anodyne extensions.

Example 9.13. Given an elementary homotopical datum I, together with a small set of monomorphisms of presheaves S, one can construct the smallest class of I-anodyne extensions containing S as follows. First, we choose a cellular model M, and we define the set $\Lambda_I^0(S,M)$ as:

$$\Lambda_I^0(S,M) = S \cup \{I \otimes K \cup \{\varepsilon\} \otimes L \longrightarrow I \otimes L \mid K \longrightarrow L \in M\,,\ \varepsilon = 0,1\}\,. \tag{9.13.1}$$

For $n \geq 0$, we put

$$\Lambda_I^{n+1}(S,M) = \{I \otimes K \cup \partial I \otimes L \longrightarrow I \otimes L \mid K \longrightarrow L \in \Lambda_I^n(S,M)\}\,, \tag{9.13.2}$$

and finally:

$$\Lambda_I(S,M) = \bigcup_{n \geq 0} \Lambda_I^n(S,M)\,. \tag{9.13.3}$$

Then the class $An_I(S) = l(r(\Lambda_I(S,M)))$ (i.e. the smallest saturated class of maps containing $\Lambda_I(S,M)$)) is the the smallest class of I-anodyne extensions containing S. Indeed, it is clear that any class of I-anodyne extensions containing S also contains $\Lambda_I(S,M)$), and thus the smallest saturated class of maps containing the latter, which is $An_I(S)$. Therefore, it is sufficient to check that the class $An_I(S)$ itself is a class of I-anodyne extensions. Axiom An0 is true by construction. For this we note that, since it preserves small colimits, the functor I has a right adjoint $X \longmapsto X^I$. The inclusion $\{\varepsilon\} \subset I$ induces a natural map $X^I \longrightarrow X^{\{\varepsilon\}} = X$. For any morphism of presheaves $X \longrightarrow Y$ Hence, any morphism of presheaves $X \longrightarrow Y$ defines a commutative square

$$\begin{array}{ccc} X^I & \longrightarrow & Y^I \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

and thus a canonical map $X^I \longrightarrow Y^I \times_Y X$. For any monomorphism $K \longrightarrow L$ we get a correspondence of the form:

$$\begin{array}{ccc} I \otimes K \cup \{\varepsilon\} \otimes L & \xrightarrow{a'} & X \\ \downarrow & \overset{h'}{\nearrow} & \downarrow \\ I \otimes L & \xrightarrow[b']{} & Y \end{array} \qquad \leftrightsquigarrow \qquad \begin{array}{ccc} K & \xrightarrow{a} & X^I \\ \downarrow & \overset{h}{\nearrow} & \downarrow \\ L & \xrightarrow[b]{} & Y^I \times_Y X \end{array} \tag{9.13.4}$$

In the case where the map $X \longrightarrow Y$ has the right lifting property with respect to elements of $\Lambda_I(S,M)$), we see from formula (9.13.1) and from the correspondence (9.13.4) that the map $X^I \longrightarrow Y^I \times_Y X$ has the right lifting property with respect to any element of the cellular model M, and thus, with respect to any monomorphism (since M generates the class of monomorphisms). Using correspondence (9.13.4) again, this proves Axiom An1. The proof of axiom An2 is similar (replacing (9.13.1) by (9.13.2), M by $\Lambda_I(S,M)$), and $\{\varepsilon\}$ by ∂I).

Homotopy theory of presheaves

From now on, we assume that an homotopical structure is given: an exact cylinder I together with a class An of I-anodyne extensions.

DEFINITION 9.14. Let $f_0, f_1 : K \longrightarrow X$ be two morphisms of presheaves. An *I-homotopy* from f_0 to f_1 is a map $h : I \otimes K \longrightarrow X$ such that $h(\partial_\varepsilon \otimes 1_K) = f_\varepsilon$ for $\varepsilon = 0, 1$.

We denote by $[K, X]$ the quotient of the set $Hom(K, X)$ by the smallest equivalence relation generated by the relation of being connected by an I-homotopy. Two maps $f_0, f_1 : K \longrightarrow X$ are said to be *I-homotopic* if they have the same class in $[K, X]$; equivalently, this means that there is a finite sequence of maps $h_i : I \otimes K \longrightarrow X$, $1 \leq i \leq n$, such that $h_1(\partial_0 \otimes 1_K) = f_0$ and $h_n(\partial_1 \otimes 1_K) = f_1$, and such that, for each i, $1 < i \leq n$, there exists ε and η in $\{0, 1\}$ such that $h_{i-1}(\partial_\eta \otimes 1_K) = h_i(\partial_\varepsilon \otimes 1_K)$.

The functoriality of the cylinder I ensures that this equivalence relation is compatible with composition. There is thus a well defined category of presheaves up to I-homotopy (whose objects are presheaves, and whose sets of morphisms are the quotients of the form $[K, X]$).

A morphism of presheaves is an *I-homotopy equivalence* if it defines an isomorphism in the category of presheaves up to I-homotopy.

The Yoneda Lemma applied to the category of presheaves up to I-homotopy gives the following characterization of I-homotopy equivalences.

PROPOSITION 9.15. *For a morphism of presheaves $f : X \longrightarrow Y$, the following conditions are equivalent.*

(a) *The map f is an I-homotopy equivalence.*
(b) *There exists a morphism $g : Y \longrightarrow X$ such that gf is I-homotopic to 1_X and fg is I-homotopic to 1_Y.*
(c) *For any presheaf K, the induced map $f_* : [K, X] \longrightarrow [K, Y]$ is bijective.*
(d) *For any presheaf W, the induced map $f^* : [Y, W] \longrightarrow [X, W]$ is bijective.*

DEFINITION 9.16. A morphism of presheaves $f : X \longrightarrow Y$ is a *strong deformation retract* (*the dual of a strong deformation retract*) if there exists a morphism $g : Y \longrightarrow X$ as well as a morphism $h : I \otimes Y \longrightarrow Y$ (as well as a morphism $k : I \otimes X \longrightarrow X$, respectively) such that:

(i) $gf = 1_X$ ($fg = 1_Y$, respectively);
(ii) $h(\partial_0 \otimes 1_Y) = 1_Y$ and $h(\partial_1 \otimes 1_Y) = fg$ ($h(\partial_0 \otimes 1_X) = 1_X$ and $h(\partial_1 \otimes 1_X) = gf$, respectively);
(iii) $h(1_I \otimes f) = (\sigma \otimes f)$ ($fk = \sigma \otimes f$, respectively).

Recall from Example 6.11 that we call trivial fibrations the morphisms of presheaves which have the right lifting property with respect to monomorphisms.

PROPOSITION 9.17. *Any trivial fibration is the dual of a strong deformation retract, and any section of a trivial fibration is a strong deformation retract. In particular, any trivial fibration has a section, and is an I-homotopy equivalence.*

PROOF. Let $p : X \longrightarrow Y$ be a trivial fibration. Since the map from the empty presheaf to any presheaf is a monomorphism, the following solid commutative square admits a filler.

$$\begin{array}{ccc} \varnothing & \longrightarrow & X \\ \downarrow & \overset{s}{\nearrow} & \downarrow p \\ Y & = & Y \end{array}$$

Hence the existence of sections. Let us assume that such a section s is provided. Since s must be a monomorphism, the expression $I \otimes Y \cup \partial I \otimes X$ makes sense. We thus have a solid commutative diagram

$$\begin{array}{ccc} I \otimes Y \cup \partial I \otimes X & \xrightarrow{(s(\sigma \otimes 1_Y), 1_X, sp)} & X \\ \downarrow & \overset{k}{\nearrow} & \downarrow p \\ I \otimes X & \xrightarrow{p(\sigma \otimes 1_X)} & Y \end{array}$$

which admits a filler k. □

DEFINITION 9.18. A *naive fibration* is a morphism with the right lifting property with respect to the given class An of I-anodyne extensions. A presheaf X is *fibrant* if the map from W to the final presheaf is a naive fibration.

A morphism of presheaves is a *cofibration* is it is a monomorphism.

A morphism of presheaves $f : X \longrightarrow Y$ is a *weak equivalence* if, for any fibrant presheaf W, the induced map

$$f^* : [Y, W] \longrightarrow [X, W]$$

is bijective.

THEOREM 9.19. *With this definition of weak equivalences and of cofibrations, we have a cofibrantly generated model category structure on the category of presheaves over A. Moreover, a presheaf is fibrant precisely when its map to the final presheaf is a naive fibration, and a morphism between fibrant objects is a fibration if and only if it is a naive fibration.*

REMARK 9.20. Any cofibrantly generated model category structure on the category of presheaves over A can be obtained in this way: if J is a generating set of the class of trivial cofibrations (i.e. such that $l(r(J))$ is the class of trivial cofibrations), we can take as exact cylinder the one defined as the cartesian product with the subobject classifier Ω (9.10), and consider the smallest class of Ω-anodyne maps generated by J (9.13). Using the fact that the projections $\Omega \otimes X \longrightarrow X$ are trivial fibrations, we see that the model category structure obtained by the theorem above is the one we started from.

The proof of Theorem 9.19 will require quite a few steps, and is the goal of the remaining part of this chapter.

REMARK 9.21. By virtue the characterization of I-homotopy equivalences given by condition (d) of Proposition 9.15, the class of weak equivalences contains the class of I-homotopy equivalences. Furthermore, it is clear that the class of

weak equivalences has the two-out-of-three property and is stable under retracts. Proposition 9.17 tells us that any trivial fibration is a weak equivalence. As explained in Example 6.11, any morphism can be factored into a cofibration followed by a trivial fibration (possibly using the small object argument).

REMARK 9.22. For any generating set Λ of the class of I-anodyne maps, one can apply the small object argument (6.9). In particular, any morphism of presheaves f admits a (functorial) factorization of the form $f = pi$ with i an I-anodyne map and p a naive fibration.

LEMMA 9.23. *A morphism of presheaves is a trivial fibration if and only if it has both properties of being a weak equivalence and of having the right lifting property with respect to trivial cofibrations.*

PROOF. Since any I-anodyne map is a monomorphism and since we already know that any trivial fibration is a weak equivalence, this certainly is a necessary condition. Conversely, let $p : X \longrightarrow Y$ be a morphism which is a weak equivalence and of has the right lifting property with respect to trivial cofibrations. We may choose a factorization $p = qj$ where j is a cofibration and q is a trivial fibration. The morphism j must be a weak equivalence, and, therefore, by virtue of the Retract Lemma (6.5), the morphism p is a retract of q. Hence it is a trivial fibration. □

LEMMA 9.24. *Let X and W be two presheaves, with W fibrant. Then the relation defined by the existence of an I-homotopy is an equivalence relation on the set $Hom(X, W)$.*

In particular, for any pair of morphisms $u, v : X \longrightarrow W$, if $u = v$ is the category of presheaves up to I-homotopy equivalence, then there exists an I-homotopy from u to v.

PROOF. It is clear that any map u from X to W, there is an I-homotopy from u to itself (e.g. $h = \sigma \otimes u$). Let us prove that, for three morphisms $u, v, w : X \longrightarrow W$, if there is an I-homotopy h from u to v as well as an I-homotopy k from u to w, then there exists an I-homotopy from v to w. Taking into account the identifications

$$I \otimes \partial I \otimes X \simeq I \otimes X \amalg I \otimes X \quad \text{and} \quad \{0\} \otimes I \otimes X \simeq I \otimes X\,,$$

we have a map

$$((h,k), \sigma \otimes u) : I \otimes \partial I \otimes X \cup \{0\} \otimes I \otimes X \longrightarrow W\,,$$

and the inclusion

$$I \otimes \partial I \otimes X \cup \{0\} \otimes I \otimes X \longrightarrow I \otimes I \otimes X$$

is an I-anodyne extension. Therefore, since W is fibrant, there exists a morphism $H : I \otimes I \otimes X \longrightarrow W$ whose restriction to $I \otimes X \simeq I \otimes \{0\} \otimes X$ coincides with h, whose restriction to $I \otimes X \simeq I \otimes \{1\} \otimes X$ coincides with k, and whose restriction to $I \otimes X \simeq \{0\} \otimes I \otimes X$ is the constant homotopy $\sigma \otimes u$. Let us define the morphism $\eta : I \otimes X \longrightarrow W$ by the equality $\eta = H(\partial_1 \otimes 1_I \otimes 1_X)$. We then have

$$\begin{aligned} \eta(\partial_0 \otimes 1_X) &= H(\partial_1 \otimes 1_I \otimes 1_X)(\partial_0 \otimes 1_X) \\ &= H(1_I \otimes \partial_0 \otimes 1_X)(\partial_1 \otimes 1_X) \\ &= h(\partial_1 \otimes 1_X) = v \end{aligned}$$

Similarly, one checks that $\eta(\partial_1 \otimes 1_X) = w$.

For $w = u$ and $k = \sigma \otimes u$ the constant I-homotopy from u to itself, this shows that the relation of I-homotopy is symmetric. The general case, together with the property of symmetry, proves the property of transitivity. □

PROPOSITION 9.25. *Any I-anodyne map is a weak equivalence.*

PROOF. Let $j : X \longrightarrow Y$ be an I-anodyne map (i.e. an element of the class An), and W a fibrant presheaf. We must prove that the induced map

$$j^* : [Y, W] \longrightarrow [X, W]$$

is bijective. The surjectivity is clear (W being fibrant and j an I-anodyne extension, it is already true at the level of morphisms of presheaves). Let $y_0, y_1 : Y \longrightarrow W$ be two morphisms of presheaves such that $y_0 j$ is I-homotopic to $y_1 j$. Then, by virtue of the preceding lemma, there exists an I-homotopy $h : X \longrightarrow W$ from $y_0 j$ to $y_1 j$. This defines a morphism

$$(h, (y_0, y_1)) : I \otimes X \cup \partial I \otimes Y \longrightarrow W\,.$$

On the other hand, since j is an Ianodyne extension, the induced map

$$I \otimes X \cup \partial I \otimes Y \longrightarrow I \otimes Y$$

is I-anodyne. Therefore, since W is fibrant, there exists a map $H : I \otimes Y \longrightarrow W$ which is an I-homotopy from y_0 to y_1. □

PROPOSITION 9.26. *A morphism between fibrant presheaves is a weak equivalence if and only if it is an I-homotopy equivalence.*

PROOF. This follows right away from Proposition 9.15. □

LEMMA 9.27. *A naive fibration is a trivial fibration if and only if it is the dual of a strong deformation retract.*

PROOF. We already know that this is a necessary condition (9.17). Let $p : X \longrightarrow Y$ be a naive fibration which is the dual of a strong deformation retract. We may assume that we have morphisms $s : Y \longrightarrow X$ as well as $k : I \otimes X \longrightarrow X$ such that $ps = 1_Y$, $k(\partial_0 \otimes 1_X) = 1_X$, $k(\partial_1 \otimes 1_X) = sp$, and $pk = \sigma \otimes p$. We must prove that any commutative square of the form

$$\begin{array}{ccc} K & \xrightarrow{a} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ L & \xrightarrow{b} & Y \end{array}$$

in which i is a monomorphism has a filler $l : L \longrightarrow X$. We have a morphism

$$u = (k(1_I \otimes a), sb) : I \otimes K \cup \{1\} \otimes L \longrightarrow X\,.$$

And the monomorphism i indices a I-anodyne extension

$$j : I \otimes K \otimes \{1\} \otimes L \longrightarrow I \otimes L\,.$$

Therefore, the commutative square

$$\begin{array}{ccc} I \otimes K \otimes \{1\} \otimes L & \xrightarrow{u} & X \\ {\scriptstyle j}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ I \otimes L & \xrightarrow{\sigma \otimes b} & Y \end{array}$$

admits a filler h. We put $l = h(\partial_0 \otimes 1_L)$. We then have

$$pl = ph(\partial_0 \otimes 1_L) = (\sigma \otimes b)(\partial_0 \otimes 1_L) = b$$
$$li = h(\partial_0 \otimes 1_L)i = h(1_I \otimes i)(\partial_0 \otimes 1_K) = k(1_I \otimes a)(\partial_0 \otimes 1_K) = k(\partial_0 \otimes 1_X)a = a$$

and this achieves the proof. □

Lemma 9.28. *A naive fibration with fibrant codomain is a weak equivalence if and only if it is a trivial fibration.*

Proof. Let $p : X \longrightarrow Y$ be a naive fibration, with Y fibrant, and assume that p is a weak equivalence as well. By virtue of Proposition 9.26, the map p is an I-homotopy equivalence. Furthermore, Lemma 9.24 ensures that there exists morphism $t : Y \longrightarrow X$ as well as an I-homotopy $k : I \otimes Y \longrightarrow Y$ from 1_Y to pt. In particular, we have a commutative square

$$\begin{array}{ccc} Y & \xrightarrow{t} & X \\ {\scriptstyle \partial_1 \otimes 1_Y}\downarrow & \overset{k'}{\nearrow} & \downarrow{\scriptstyle p} \\ I \otimes Y & \xrightarrow{k} & Y \end{array}$$

Let us put $s = k'(\partial_0 \otimes 1_Y)$. We then have:

$$ps = pk'(\partial_0 \otimes 1_Y) = k(\partial_0 \otimes 1_Y) = 1_Y\,.$$

Since p is an I-homotopy equivalence, using Lemma 9.24 once more, there is an I-homotopy h from the identity of X to the map sp. In particular, under the identifications

$$I \otimes X \simeq \{1\} \otimes I \otimes X \quad \text{and} \quad I \otimes X \amalg I \otimes X \simeq I \otimes \partial I \otimes X\,,$$

this defines a morphism

$$(\sigma \otimes sp, (h, sph)) : \{1\} \otimes I \otimes X \cup I \otimes \partial I \otimes X \longrightarrow X\,.$$

Finally we have a commutative square

$$\begin{array}{ccc} \{1\} \otimes I \otimes X \cup I \otimes \partial I \otimes X & \xrightarrow{(\sigma \otimes sp, (h, sph))} & X \\ \downarrow & \overset{H}{\nearrow} & \downarrow{\scriptstyle p} \\ I \otimes I \otimes X & \xrightarrow[1_I \otimes \sigma \otimes 1_X]{} I \otimes X \xrightarrow[ph]{} & Y \end{array}$$

which has a filler H. We define $K = H(\partial_0 \otimes 1_{I\otimes X})$. The identities

$$\begin{aligned} K(\partial_0 \otimes 1_X) =& H(\partial_0 \otimes 1_{I\otimes X})(\partial_0 \otimes 1_X) = h(\partial_0 \otimes 1_X) = 1_X \\ K(\partial_1 \otimes 1_X) =& H(\partial_0 \otimes 1_{I\otimes X})(\partial_1 \otimes 1_X) = sph(\partial_0 \otimes 1_X) = sp \\ pK =& pH(\partial_0 \otimes 1_{I\otimes X}) = ph(\partial_0 \otimes \sigma \otimes 1_X) = p(\sigma \otimes 1_X) \end{aligned}$$

show that p is the dual of a strong deformation retract. We conclude with Lemma 9.27. □

Corollary 9.29. *A cofibration with fibrant codomain is a weak equivalence if and only if it is an I-anodyne extension.*

Proof. We already know that any I-anodyne extension is a weak equivalence (see Proposition 9.25). Let i be a cofibration with fibrant codomain. There exists a factorization of the form $i = qj$ with j an I-anodyne map and q a naive fibration. Since j is a weak equivalence, if ever i is a weak equivalence, so will be p, and therefore, by virtue of Lemma 9.28, the morphism p is a trivial fibration. It thus follows from the Retract Lemma (6.5) that i is a retract of j and thus is I-anodyne. □

Proposition 9.30. *A cofibration is a weak equivalence if and only if it has the left lifting property with respect to the class of naive fibrations with fibrant codomain.*

Proof. Let $i : K \longrightarrow L$ be a cofibration. Let us choose an I-anodyne extension with fibrant codomain $j : L \longrightarrow L'$.

Assume that i is a trivial cofibration. For any commutative square

$$\begin{array}{ccc} K & \xrightarrow{a} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ L & \xrightarrow{b} & Y \end{array}$$

with p a naive fibration with fibrant domain, we may assume that there is a map $b' : L' \longrightarrow Y$ such that $b'j = b$. Since ji is a trivial cofibration with fibrant codomain, the previous corollary tells us that there exists a map $l' : L' \longrightarrow X$ such that $pl' = b'$ and $l'ji = a$. Therefore, the map $l = l'j$ is a filler of the square above.

Conversely, if i has the left lifting property with respect to the class of naive fibrations with fibrant codomain, we may choose a factorization of the map ji into an I-anodyne extension k followed by a naive fibration (with fibrant codomain) p. The Retract Lemma implies that ji is a retract of k, whence is I-anodyne. In particular, the map ji is a trivial cofibration. Since j has the same property, the map i itself must be a trivial cofibration as well. □

Corollary 9.31. *The class of trivial cofibrations is saturated.*

Proof. Any class of maps defined by left lifting property with respect to a given class of maps is saturated. □

The last part of the proof of Theorem 9.19 consists to exhibit a small set of generating trivial cofibrations. This requires a few more steps, including a closer

look at the proof of the small object argument in the case of monomorphisms of presheaves.

LEMMA 9.32. *Any strong deformation retract is an I-anodyne extension.*

PROOF. Let $i : K \longrightarrow L$ be a strong deformation retract. Then there is a morphism $r : L \longrightarrow K$ such that $ri = 1_K$, as well as an I-homotopy h from 1_L to ir, such that $h(1_I \otimes i) = (\sigma \otimes 1_L)(1_I \otimes i) = \sigma \otimes i$. Let us consider a commutative square of the following form

$$\begin{array}{ccc} K & \xrightarrow{a} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ L & \xrightarrow{b} & Y \end{array}$$

with p a naive fibration. We want to prove the existence of a filler $l : L \longrightarrow X$. The solid commutative square below has a filler k.

$$\begin{array}{ccc} I \otimes K \cup \{0\} \otimes L & \xrightarrow{(\sigma\otimes a, ar)} & X \\ {\scriptstyle i}\downarrow & \overset{k}{\nearrow} & \downarrow{\scriptstyle p} \\ I \otimes L & \xrightarrow{bh} & Y \end{array}$$

We define $l = k(\partial_0 \otimes 1_L)$ and check that $li = a$ and $pl = b$. □

LEMMA 9.33. *Any I-anodyne extension between fibrant presheaves is a strong deformation retract.*

PROOF. Let $i : K \longrightarrow L$ be an I-anodyne extension, with both K and L fibrant. There exists a retraction r of i. The filler h in the diagram

$$\begin{array}{ccc} I \otimes K \cup \partial I \otimes L & \xrightarrow{(\sigma\otimes i, (1_L, ir))} & L \\ {\scriptstyle i}\downarrow & \overset{h}{\nearrow} & \\ I \otimes L & & \end{array}$$

turns i into a strong deformation retract. □

PROPOSITION 9.34. *For a monomorphism $i : K \longrightarrow L$, with both K and L being fibrant presheaves, the following conditions are equivalent:*

(i) *it is a weak equivalence;*
(ii) *it is an I-anodyne extension;*
(iii) *it is a strong deformation retract.*

PROOF. The equivalence between conditions (i) and (ii) is a special case of Corollary 9.29. The equivalence between conditions (ii) and (iii) follows from Lemmas 9.32 and 9.33. □

We leave the proof of the following lemma as an exercise (it is sufficient to prove it in the category of sets to make it true of presheaves).

LEMMA 9.35. *In any category of presheaves, we have the following properties.*

(a) *For any commutative diagram*

$$\begin{array}{ccccc} X_1 & \xleftarrow{x_1} & X_0 & \xrightarrow{x_2} & X_2 \\ \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_0} & & \downarrow{\scriptstyle i_2} \\ S_1 & \xleftarrow{s_1} & S_0 & \xrightarrow{s_2} & S_2 \end{array}$$

in which the map i_2 as well as the canonical map $X_1 \amalg_{X_0} S_0 \longrightarrow S_1$ are monomorphism, the induced map

$$X_1 \amalg_{X_0} X_2 \longrightarrow S_1 \amalg_{S_0} S_2$$

is a monomorphism.

(b) *For any commutative diagram of the form*

$$\begin{array}{ccccc} X_1 & \xleftarrow{x_1} & X_0 & \xrightarrow{x_2} & X_2 \\ \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_0} & & \downarrow{\scriptstyle i_2} \\ S_1 & \xleftarrow{s_1} & S_0 & \xrightarrow{s_2} & S_2 \\ \uparrow{\scriptstyle j_1} & & \uparrow{\scriptstyle j_0} & & \uparrow{\scriptstyle j_2} \\ Y_1 & \xleftarrow{y_1} & Y_0 & \xrightarrow{y_2} & Y_2 \end{array}$$

if the vertical maps as well as the maps x_1, s_1 and y_1 all are monomorphisms, and if the commutative squares of the left column are Cartesian, then the induced map
$(X_1 \times_{S_1} Y_1) \amalg_{(X_0 \times_{S_0} Y_0)} (X_2 \times_{S_2} Y_2) \longrightarrow (X_1 \amalg_{X_0} X_2) \times_{(S_1 \amalg_{S_0} S_2)} (Y_1 \amalg_{Y_0} Y_2)$
is a isomorphism.

(c) *For any small fanily of maps of the form $X_i \longrightarrow S_i \longleftarrow Y_i$, the canonical map*

$$\coprod_i (X_i \times_{S_i} Y_i) \longrightarrow (\coprod_i X_i) \times_{(\amalg_i S_i)} (\coprod_i Y_i)$$

is invertible.

9.36. Let us recall a special case of the construction provided by the small object argument. Let us fix a set Λ of monomorphisms. We define two functors S and B as follows: for any presheaf X, we have

$$S(X) = \coprod_{i:K \longrightarrow L} Hom(K, X) \times K \quad \text{and} \quad B(X) = \coprod_{i:K \longrightarrow L} Hom(K, X) \times L$$

There is a canonical map $s_X : S(X) \longrightarrow X$ induced by the evaluation maps from $Hom(K, X) \times K$ to X, and we define $L_1(X)$ by forming the following coCartesian square.

$$\begin{array}{ccc} S(X) & \xrightarrow{s_X} & X \\ \downarrow & & \downarrow{\scriptstyle \lambda_X^{(1)}} \\ B(X) & \longrightarrow & L_1(X) \end{array}$$

Given a well orered set E, we define, for each element $e \in E$ the functor L_e, together with the natural map $\lambda_X^{(e)} : X \longrightarrow L_e(X)$, as follows. If e is the successor

of an element e_0, we put $L_e(X) = L_1(L_{e_0}(X))$, and we define the map $\lambda_X^{(e)}$ as the composition of $\lambda^{(1)}_{L_{e_0}(X)}$ with the map $\lambda_X^{(e_0)}$. If e is a limit element, we define $L_e(X)$ as the colimit of the $L_i(X)$'s for $i < e$. Finally, we define $L(X)$ as the colimit of the $L_e(X)$'s. There is a canonical embedding $X \longrightarrow L(X)$ which is a transfinite composition of pushouts of elements of Λ, and, if E is large enough, the map from $L(X)$ to the final presheaf has the right lifting property with respect to Λ.

In the case where Λ generates the given class An of I-anodyne extension, this is the way we get a functorial I-anodyne extension $X \longrightarrow L(X)$ with fibrant codomain[2].

We shall say that a presheaf X is of size $< \alpha$ if α is a cardinal which is greater than the set of morphisms of the category A/X.

LEMMA 9.37. *The functors L and I have the following properties.*

(i) *they preserve monomorphisms;*
(ii) *it preserves intersections of subobjects;*
(iii) *there exists an infinite cardinal α such that:*
  (a) *for any presheaf X, the union of of subobjects of size $< \alpha$ indexed by by a set of cardinal $< \alpha$ is of size $< \alpha$;*
  (b) *for any presheaf of size $< \alpha$, the presheaves $L(X)$ and $I \otimes X$ are of size $< \alpha$.*
  (c) *for any presheaf X, the presheaf $L(X)$ is the union of its subobjects of the form $L(V)$, where V runs over the subobjects of size $< \alpha$ in X;*
  (d) *for any well ordered set E of cardinal α and any increasing family of subobjects $(V_e)_{e\in E}$, the union of the $L(V_e)$'s canonically is isomorphic to the image by L of the union of the V_e's.*

It follows from the preceding lemma that the functor L_1 preserves monomorphisms as well as intersections of subobjects. Since filtered colimits are exact in any category of presheaves, this proves properties (i) and (ii). Property (iii) is then a lengthy but elementary exercise of set theory that we leave to the reader.

LEMMA 9.38. *There exists an infinite cardinal α such that, for any trivial cofibration $i : X \longrightarrow Y$ and any subpresheaf U of size $< \alpha$ in Y, there exists a subpresheaf V of size $< \alpha$ in Y such that the inclusion $V \cap X \longrightarrow X$ is a trivial cofibration.*

PROOF. We choose a generating set Λ of the class of I-anodyne extensions An, so that $An = l(r(\Lambda))$. We construct a functorial I-anodyne map $l_X : X \longrightarrow L(X)$ with fibrant codomain, as explained in 9.36. Finally, we choose a cardinal α such that the conclusion of Lemma 9.37 hold. Let $i : X \longrightarrow Y$ be a monomorphism of

[2]This construction provides a way to factorize any map: for a map $X \longrightarrow Y$, seen as presheaf over A/Y, we may apply this to the set Λ/Y of maps of presheaves over A/Y whose image in the category of presheaves over A belongs to Λ.

presheaves. We have a commutative square

$$\begin{array}{ccc} X & \xrightarrow{l_X} & L(X) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle L(i)} \\ Y & \xrightarrow{l_Y} & L(Y) \end{array}$$

in which both horizontal maps are I-anodyne, and therefore, are weak equivalences. Since $L(i)$ is a monomorphism between fibrant presheaves, we conclude from Proposition 9.34 that i is a weak equivalence if and only if $L(i)$ is a strong deformation retract.

From now on, we assume that $L(i)$ is a strong deformation retract. So there is a map $r : L(Y) \longrightarrow L(X)$ such that $rL(i) = 1_{L(X)}$, as well as an I-homotopy $h : I \otimes L(Y) \longrightarrow L(Y)$ from $1_{L(Y)}$ to $L(i)r$ whose restriction to $I \otimes L(X)$ is the constant homotopy $\sigma \otimes 1_{L(i)}$.

Let U be a subpresheaf of size $< \alpha$ in Y. We shall choose a well ordered set E of cardinal α and construct a increasing sequence $(V_e)_{e\in E}$ of subobjects of Y containing U, such that, for any $e \in E$, if $j_e : V_e \longrightarrow Y$ denotes the inclusion, the map h sends $I \otimes L(V_e)$ into $L(V_{e+1})$ (where $e+1$ denotes the successor of e). For $e = 0$ the initial element of E, we simply put $V_0 = U$. Assume that V_i is already constructed for all $i < e$. Let us define V'_e as the union of the V_i's for $i < e$. Since $I \otimes L(V'_e)$ is of size α, so is its image by the map h, and since $L(Y)$ is the union of its subobjects of the form $L(T)$ with $T \subset Y$ of size α, we see that there exists $V''_e \subset Y$ of size $< \alpha$ such that the image of $I \otimes L(V'_e)$ by h is contained in $L(V''_e)$. We define $V_e = V'_e \cup V''_e$, and let V be the union of the V_e's. Since the union of the $L(V_e)$'s canonically is isomorphic to $L(V)$, we check that r maps $L(V)$ to $L(V \cap X)$ and that the restriction of the map h to $I \otimes L(V)$ turns the inclusion $L(V \cap X) \longrightarrow L(V)$ into a strong deformation retract. Therefore, the inclusion $V \cap X \longrightarrow V$ is a trivial cofibration. □

Lemma 9.39. *There exists a small set J of trivial cofibrations such that the class $l(r(J))$ coincides with the class of trivial cofibrations.*

Proof. Let us choose a cardinal α such that the conclusion of of the preceding lemma holds. We define J to be the set of trivial cofibrations $U \longrightarrow V$ with V of size $< \alpha$.

Since the class of trivial cofibrations is saturated (Corollary 9.31), it is clear that any element of the smallest saturated class of maps containing J (i.e. any element of $l(r(J))$) is a trivial cofibration. It is thus sufficient to prove that any trivial cofibration belongs to $l(r(J))$.

Let $i : X \longrightarrow Y$ a be a trivial cofibration. Since any totally ordered set has a cofinal well ordered subset, by Zorn's Lemma, there exists a maximal subobject W of Y containing X such that the map $X \longrightarrow W$ belongs to $l(r(J))$. Let a be a representable presheaf, and $y \in Y_a$. The image U of the map $y : h_a \longrightarrow Y$ is of size $< \alpha$, and therefore, there exists a subobject V of Y of size $< \alpha$ such that the inclusion $V \cap W \longrightarrow V$ belongs to J. Its pushout along $V \cap W \longrightarrow W$, namely the map $W \longrightarrow V \cup W$, thus belongs to $l(r(J))$. Since W is maximal, this means that $W \cup V = W$, and thus that $y \in W_a$. In other words, we have $W = Y$. □

This achieves the proof of Theorem 9.19: once a cellular model has been chosen, the small object argument and Lemma 9.23 show that cofibrations and fibrations which are weak equivalences do form a weak factorization system; the small object argument and Lemma 9.39 show that trivial cofibrations and fibrations form a weak factorization system; proposition 9.30 also asserts that fibrations between fibrant objects are nothing else than naive fibrations between fibrant presheaves.

PROPOSITION 9.40. *Assume that $\mathcal{C}$ is a model category, and let $F : \widehat{A} \longrightarrow \mathcal{C}$ be a colimit preserving functor which sends monomorphisms to cofibrations. Then F is a left Quillen functor (i.e. F and its right adjoint form a Quillen adjunction) if and only if F sends I-anodyne extensions to trivial cofibrations. If Λ is a generating set of the specified class An of I-anodyne extensions, then this property holds whenever F sends Λ into the class of trivial cofibrations.*

PROOF. Since the class of trivial cofibrations always is saturated, and since F preserves small colimits, the class of morphisms of presheaves which are sent to a trivial cofibration by F is saturated. Therefore, if F sends a generating set of the class of An into the class of trivial cofibrations, then it sends any element of An to a trivial cofibration. If this is the case, for any morphism of presheaves $f : X \longrightarrow Y$, there is a commutative square

$$\begin{array}{ccc} X & \longrightarrow & L(X) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle L(f)} \\ Y & \longrightarrow & L(Y) \end{array}$$

in which the horizontal maps are I-anodyne. Therefore, the map $F(f)$ is a weak equivalence if and only if $F(L(f))$ has the same property. Furthermore, if f is a monomorphism, we may assume that $L(f)$ is a monomorphism between fibrant presheaves. Therefore, $L(f)$ is a weak equivalence if and only if it is an I-anodyne extension. Thus, if f is a trivial cofibration, $L(f)$ is an I-anodyne extension and this implies that $F(f)$ must be a weak equivalence. □

REMARK 9.41. It follows from the proof of the preceding proposition that inverting An inverts all weak equivalences. However, it might be the case that the lass An is strictly smaller than the class of trivial cofibrations. We shall see natural examples in both cases: where these two notions coincide, and where they do not.

REMARK 9.42. Lemmata 9.37 and 9.38 simply are pedestrian ways to prove a particular case of Jeff Smith's Theorem for constructing combinatorial model categories[3]. Note however that Theorem 9.19 gives more than the mere existence of a model category structure: since it relies on the description of fibrations with

[3]For the reader who already knows what this means: it is easy to see that the class of I-homotopy equivalences is accessible and that the functor L is accessible, so that the class of weak equivalence is accessible, so that, by virtue of Corollary 9.31, we may apply Smith's Theorem, and thus get a proof of Theorem 9.19

fibrant codomain in terms of a class which might be smaller than the class of trivial cofibrations, it gives the possibility to describe such fibrations in several different ways. This will be useful when we will start doing such constructions in families (i.e. working over a presheaf X which is allowed to vary).

CHAPTER 10

Absolute weak equivalences

In this chapter, we still consider a fixed small category A, and we will work in the category of presheaves of sets over A. We also assume that an exact cylinder I is given, together with a given class An of I-anodyne maps.

10.1. For each presheaf S, the category of presheaves over A/S is canonically equivalent to the slice category $\widehat{A}/S$.

On defines an exact cylinder I_S on the category of presheaves over A/S by describing its effect on morphisms $f : T \longrightarrow S$, seen as objects of $\widehat{A}/S$. The object $I_S \otimes (X, f)$ is the object corresponding to the map

$$\sigma \otimes f = f(\sigma \otimes 1_T) : I \otimes X \longrightarrow S\,.$$

Similarly, one defines An_S as the class of morphisms in $\widehat{A}/S$ whose image in $\widehat{A}$ belong to An. If Λ is a set of generators of An, then we have $An_S = l(r(\Lambda_S))$, where Λ_S stands for the morphisms of $\widehat{A}/S$ of the form

$$\begin{array}{ccc} K & \xrightarrow{u} & L \\ & {\scriptstyle a}\searrow \quad \swarrow{\scriptstyle b} & \\ & S & \end{array}$$

such that $u \in \Lambda$. The class An_S is a class of I_S-anodyne extensions in $\widehat{A}/S$. Therefore, applying Theorem 9.19 to the category of presheaves over A/S with the cylinder I_S and the class An_S gives a unique model category structure whose cofibrations are the monomorphisms and whose fibrant objects are the naive fibrations of codomain S.

Definition 10.2. A morphism of presheaves over S is an *S-weak equivalence* if it is a weak equivalence in the model category structure associated to I_S and An_S.

A morphism of presheaves $f : X \longrightarrow Y$ (over A) is an *absolute weak equivalence* if, for any presheaf S and any morphism $b : Y \longrightarrow S$ the map

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle a=bf}\searrow \quad \swarrow{\scriptstyle b} & \\ & S & \end{array}$$

is an S-equivalence.

PROPOSITION 10.3. *A monomorphism is in An if and only if it is an absolute weak equivalence. Similarly, a morphism of presheaves is both a naive fibration and an absolute weak equivalence if and only if it is a trivial fibration.*

PROOF. It is clear that any I-anodyne extension is an absolute weak equivalence. For the converse, let $i : X \longrightarrow Y$ be a monomorphism which is an absolute weak equivalence. Then, in particular, it is a trivial cofibration with fibrant codomain in $\widehat{A}/X$. By virtue of Corollary 9.29, this proves that i is an I-anodyne extension.

It is obvious that any trivial fibration is both a naive fibration and an absolute weak equivalence. Conversely, let $p : X \longrightarrow Y$ be a morphism which is both a naive fibration and an absolute weak equivalence. Then one can consider f as a naive fibration with fibrant codomain in $\widehat{A}/Y$. Therefore, the map f is a trivial fibration in $\widehat{A}/Y$, and thus in $\widehat{A}$. □

PROPOSITION 10.4. *The class of absolute weak equivalences W^a is the smallest class* C *satisfying the following conditions.*

(a) *the class* C *is stable under composition;*
(b) *for any pair of composable morphisms $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$, if both f and gf are in* C*, so is g;*
(c) *$An \subset$* C*.*

Moreover, a morphism of presheaves is an absolute weak equivalence if and only if it admits a factorization into an I-anodyne extension followed by a trivial fibration.

PROOF. Let us prove that the class of absolute weak equivalences satisfies these three properties. Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be a pair of composable morphisms, and let us assume that f is an absolute weak equivalence. Then, for any morphism $Z \longrightarrow S$, we may see the maps g and gf as maps in $\widehat{A}/S$, and it is clear that g is an S-equivalence if and only if gf is an S-equivalence. This proves that W^a satisfies properties (a) and (b). Property (c) is already known.

Let us prove the last assertion. Since trivial fibrations and I-anodyne extensions are in W^a, it follows from (a) that any composition of such morphisms is in W^a. Conversely, if f is an absolute weak equivalence, then it admits a factorization of the form $f = pi$ with i an I-anodyne extension, and p a naive fibration. Condition (b) ensures that p is an absolute weak equivalence as well, and thus the second assertion of Proposition 10.3 implies that p is a trivial fibration.

To finish the proof, let C be a class of morphisms satisfying the conditions (a), (b) and (c). To prove that $W^a \subset$ C, it is sufficient to prove that any trivial fibration is in C. But this follows from the fact that any trivial fibration has a section and that any such section is I-anodyne (being a strong deformation retract). □

COROLLARY 10.5. *Let us consider a pair of composable monomorphisms $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$, and assume that f is an I-anodyne extension. Then the map g is an I-anodyne extension if and only if the map gf has the same property.*

PROOF. This follows right away from Propositions 10.3 and 10.4. □

Recall that a supply of I-anodyne extensions comes from strong deformation retracts (Lemma 9.32). The following statement might thus be meaningful when it comes to prove that a map is an absolute weak equivalence.

PROPOSITION 10.6. *For any Cartesian square*

$$\begin{array}{ccc} X' & \xrightarrow{i} & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ Y' & \xrightarrow{j} & Y \end{array}$$

in which p is a naive fibration and j is a strong deformation retract, the map i is a strong deformation retract.

PROOF. Let us choose a retraction $r : Y \longrightarrow Y'$ of j as well as an homotopy $h : I \otimes Y \longrightarrow Y$ from 1_Y to jr, such that $h(1_I \otimes j) = \sigma \otimes j$. The commutative square below then has a filler.

$$\begin{array}{ccc} I \otimes X' \cup \{0\} \otimes X & \xrightarrow{(\sigma\otimes i, 1_X)} & X \\ \downarrow & \overset{k}{\nearrow} & \downarrow{\scriptstyle p} \\ I \otimes X & \xrightarrow{h(1_I \otimes p)} & Y \end{array}$$

The maps $u = k(\partial_1 \otimes 1_X) : X \longrightarrow X$ and $v = rp : X \longrightarrow Y'$ satisfy the relations $pu = jv$ (because $h(\partial_1 \otimes 1_X) = jr$) and thus define a unique map $s : X \longrightarrow X'$ such that $qs = v$ and $is = u$. We have $si = 1_{X'}$ because we check that $qsi = q$ and $isi = i$. Hence i is a strong deformation retract. □

CHAPTER 11

Kan fibrations

The simplicial set Δ^1 has an obvious structure of interval, so that the Cartesian product functor $\Delta^1 \times (-)$ defines an exact cylinder on the category of simplicial sets.

DEFINITION 11.1. An *anodyne extension* is an element of the smallest class of $\Delta^1 \times (-)$-anodyne maps (see Example 9.13).

PROPOSITION 11.2 (Gabriel and Zisman). *The following three classes of morphisms of simplicial sets are equal:*

(a) *the class of anodyne extensions;*

(b) *the smallest saturated class of maps containing inclusions of the form*

$$\Delta^1 \times \partial\Delta^n \cup \{\varepsilon\} \times \Delta^n \longrightarrow \Delta^1 \times \Delta^n \text{ for } n \geq 0 \text{ and } \varepsilon = 0, 1;$$

(c) *the smallest saturated class containing inclusions of the form*

$$\Lambda^n_k \longrightarrow \Delta^n \text{ for } n \geq 1 \text{ and } 0 \leq k \leq n.$$

PROOF. We know that the set M of boundary inclusions $\partial\Delta^n \longrightarrow \Delta^n$, $n \geq 0$, is a cellular model. With the notations introduced in Example 9.13, the class of anodyne extensions is the class $l(r(\Lambda_I(\varnothing, M)))$ with $I = \Delta^1 \times (-)$. The class described in (b) above is the class $l(r(\Lambda^0_I(\varnothing, M)))$. To prove that these two classes coincide, since, by construction, $\Lambda^0_I(\varnothing, M) \subset \Lambda_I(\varnothing, M)$, it is sufficient to prove that any element of $\Lambda_I(\varnothing, M)$ belongs to $l(r(\Lambda^0_I(\varnothing, M)))$. The explicit description of $\Lambda_I(\varnothing, M)$ shows that it is sufficient to prove that, for any monomorphism of simplicial sets $K \longrightarrow L$ and for $\varepsilon = 0, 1$, the inclusion

$$\Delta^1 \times K \cup \{\varepsilon\} \times L \longrightarrow \Delta^1 \times L$$

belongs to the class $l(r(\Lambda^0_I(\varnothing, M)))$. Using the correspondence (9.13.4), we see that the class of maps $K \longrightarrow L$ having this property is saturated. Therefore, it is sufficient to check this property for the inclusions $\partial\Delta^n \longrightarrow \Delta^n$, in which case this holds by definition. The equality between the classes (b) and (c) follows from the lemma below (and the dual version obtained by applying the auto-equivalence $X \longmapsto X^{op}$), which is a slightly more precise version of the proposition. □

LEMMA 11.3. *The following two classes of morphisms are equal:*

(a) *the smallest saturated class of maps containing inclusions of the form*

$$\Delta^1 \times \partial\Delta^n \cup \{1\} \times \Delta^n \longrightarrow \Delta^1 \times \Delta^n \text{ for } n \geq 0;$$

(b) *the smallest saturated class containing inclusions of the form*

$$\Lambda^n_k \longrightarrow \Delta^n \text{ for } n \geq 1 \text{ and } 0 < k \leq n.$$

PROOF. We define two maps

$$s : [n] \longrightarrow [1] \times [n] \quad \text{and} \quad r : [1] \times [n] \longrightarrow [n]$$

as follows. We define

$$s(i) = \begin{cases} (0,i) & \text{if } i < k, \\ (1,i) & \text{else.} \end{cases}$$

We also put:

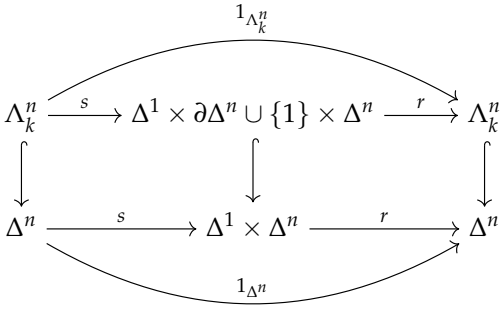
$$r(0,i) = \begin{cases} i & \text{if } i \leq k, \\ k & \text{else} \end{cases} \quad \text{and} \quad r(1,i) = \begin{cases} k & \text{if } i \leq k, \\ i & \text{else.} \end{cases}$$

Since the nerve functor preserves products and $N([m]) = \Delta^m$, these maps define morphisms

$$s : \Delta^n \longrightarrow \Delta^1 \times \Delta^n \quad \text{and} \quad r : \Delta^1 \times \Delta^n \longrightarrow \Delta^n .$$

The image of Λ^n_k by s must be in the union $\Delta^1 \times \partial\Delta^n \cup \partial\Delta^1 \times \Delta^n$ (for cardinality reasons), and none of the faces of Λ^n_k can be in the component $\{0\} \times \Delta^n$ (because $k > 0$), so that $s(\Lambda^n_k)$ fits in $\Delta^1 \times \partial\Delta^n \cup \{1\} \times \Delta^n$. On the other hand, the image of the union $\Delta^1 \times \partial\Delta^n \cup \{1\} \times \Delta^n$ by r fits in Λ^n_k. To see this, recall that Λ^n_k is the union of the images of the face maps δ^n_i for $i \neq k$. This means that the non-degenerate simplices of Λ^n_k correspond to the maps $u : [n-1] \longrightarrow [n]$ whose image contains k. Since $k > 0$, the image of $\{1\} \times \Delta^n$ is in the image of δ^n_0, and thus fits in Λ^n_k. It remains to prove that r sends $\{1\} \times \partial\Delta^n$ into Λ^n_k. This means that for any injective map $u : [n-1] \longrightarrow [n]$, there exists an x in $[n-1]$ such that $u(x) \leq k$. For $u = \delta^n_i$ with $i \neq k$, it is clear that k is in the image of u, while for $u = \delta^n_k$, we must have $u(0) = 0 < k$ (because u must preserve the ordering and reach all the possible values except k itself). The composed map $r \circ s$ is the identity of Δ^n. Therefore, we have a commutative diagram of the following form.

(11.3.1)



This proves that the class (b) is contained in the class (a). For the converse, we define, for each integer i, $0 \leq i \leq n$, the map $c_i : [n+1] \longrightarrow [1] \times [n]$ as the unique strictly increasing map which reaches both values $(0,i)$ and $(1,i)$. We write $c_i : \Delta^{n+1} \longrightarrow \Delta^1 \times \Delta^n$ for the induced morphisms of simplicial sets, and C_i for its image. We define a finite filtration

$$\Delta^1 \times \partial\Delta^n \cup \{1\} \times \Delta^n = A_{-1} \subset A_0 \subset A_1 \subset \ldots \subset A_n = \Delta^1 \times \Delta^n \tag{11.3.2}$$

by the formula $A_i = A_{i-1} \cup C_{n-i}$ for $0 \leq i \leq n$. The isomorphism $\Delta^{n+1} \simeq C_{n-i}$ defined by c_{n-i} induces an isomorphism $\Lambda^{n+1}_{n-i+1} \simeq C_{n-i} \cap A_{i-1}$. Let us check that,

for any face map $\delta_j^{n+1} : \Delta^n \longrightarrow \Delta^{n+1}$, with $j \neq n-i+1$, the image of $c_i \circ \delta_j^{n+1}$ is in A_{i-1}. If $j \neq n-i$, the image is in $\Delta^1 \times \partial\Delta^n$, while, if $n-i=j$, we have two cases: for $n-i=j=0$, the image precisely is $\{1\} \times \Delta^n$, while for $n > i$, it is in C_{n-i+1}. Conversely, since C_{n-i} is not contained in A_{i-1}, we must have $c_{n-i}^{-1}(C_{n-i} \cap A_{i-1}) \subset \partial\Delta^{n+1}$. This means that the simplices of $C_{n-i} \cap A_{i-1}$ must all factor through a simplex of dimension at most n. One checks that any injective map $\Delta^m \longrightarrow C_{n-i} \cap A_{i-1}$ (with $m \leq n$) must factor through the image of Λ_{n-i+1}^{n+1}. In conclusion, we have a biCartesian square of the following form.

$$\begin{array}{ccc} \Lambda_{n-i+1}^{n+1} & \longrightarrow & A_{i-1} \\ \downarrow & & \downarrow \\ \Delta^{n+1} & \xrightarrow{c_{n-i}} & A_i \end{array} \tag{11.3.3}$$

This shows that the generators of the class (b) all belong to the class (a) and thus achieves the proof of the lemma. □

Corollary 11.4. *For any anodyne extension $K \longrightarrow L$ and any monomorphism $U \longrightarrow V$, the induced inclusion $K \times V \cup L \times U \longrightarrow L \times V$ is an anodyne extension.*

Definition 11.5. A *Kan fibration* is a morphism of simplicial sets with the right lifting property with respect to the inclusions of the form $\Lambda_k^n \longrightarrow \Delta^n$, for $n \geq 1$ and $0 \leq k \leq n$.

A *Kan complex* is a simplicial set X such that the morphism from X to the final simplicial set is a Kan fibration.

Corollary 11.6. *For any monomorphism $i : U \longrightarrow V$ and for any Kan fibration $p : X \longrightarrow Y$, the canonical map*

$$(i^*, p_*) : \underline{Hom}(V, X) \longrightarrow \underline{Hom}(U, X) \times_{\underline{Hom}(U,Y)} \underline{Hom}(V, Y)$$

is a Kan fibration.

Corollary 11.7. *For any anodyne extension $i : K \longrightarrow L$ and for any Kan fibration $p : X \longrightarrow Y$, the canonical map*

$$(i^*, p_*) : \underline{Hom}(L, X) \longrightarrow \underline{Hom}(K, X) \times_{\underline{Hom}(L,Y)} \underline{Hom}(K, Y)$$

is a trivial fibration (i.e. has the right lifting property with respect to monomorphisms).

Theorem 11.8. *There is a unique model category structure on the category of simplicial sets whose class of cofibrations is the class of monomorphisms, and whose fibrant objects are the Kan complexes. Moreover, any anodyne extension is a trivial cofibration, and the fibrations between fibrant objects exactly are the Kan fibrations between Kan complexes.*

Proof. By virtue of Proposition 11.2, this follows from Theorem 9.19 applied to the category of simplicial sets for the exact cylinder $\Delta^1 \times (-)$ and for the class *An* of anodyne extensions. □

DEFINITION 11.9. The *Kan-Quillen model category structure* is the model category structure on the category of simplicial sets obtained by Theorem 11.8. The weak equivalences of this model category structure are called the *weak homotopy equivalences*.

COROLLARY 11.10. *The class of weak homotopy equivalences is closed under finite products.*

PROOF. This assertion is equivalent to the property that, for any simplicial set Y, the functor $X \longmapsto X \times Y$ preserves weak equivalences. Since all simplicial sets are cofibrant, this can be reformulated by saying that it is part of a Quillen adjunction from the Kan-Quillen model category structure to itself. This functor obviously preserves cofibrations, so that it is sufficient to prove that it preserves trivial cofibrations. By virtue of Proposition 9.40, it is sufficient to prove that it preserves anodyne extensions, which is a particular case of Corollary 11.4. □

LEMMA 11.11. *Let $\mathcal{C}$ be a category in which a commutative square*

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle l} \\ C & \xrightarrow{k} & D \end{array}$$

is given. Assume that the maps j, k and l have retractions p, q and r, respectively, and that $rk = ip$. Then this square is absolutely Cartesian (i.e. its image by any functor of domain $\mathcal{C}$ is Cartesian).

PROOF. It is sufficient to prove that such a square is Cartesian. Let $u : X \longrightarrow B$ and $v : X \longrightarrow C$ be two morphisms such that $kv = lu$. We must prove that there is a unique map $w : X \longrightarrow A$ such that $iw = u$ and $jw = v$. Since $pj = 1_A$, we must have $w = pjw = pu$, so that the unicity is clear. For the existence, we put $w = pu$. Since both k and l are monomorphisms, it is sufficient to check that $kju = kv$ and that $liu = lv$, which one sees right away. □

LEMMA 11.12. *For $n \geq 2$ and $0 \leq i < j \leq n$, the following square is absolutely Cartesian.*

$$\begin{array}{ccc} \Delta^{n-2} & \xrightarrow{\delta_i^{n-1}} & \Delta^{n-1} \\ {\scriptstyle \delta_{j-1}^{n-1}}\downarrow & & \downarrow{\scriptstyle \delta_j^n} \\ \Delta^{n-1} & \xrightarrow{\delta_i^n} & \Delta^n \end{array}$$

PROOF. It is sufficient to consider the case where $i \neq n-1$ (otherwise, we apply the functor $X \longmapsto X^{op}$ and get back to this case). We define $q = \sigma_i^n$, and, then, have two cases:

- if $i < j-1$, we put $p = \sigma_j^{n-1}$ and $r = \sigma_{j-1}^{n-2}$;
- if $j = n$ and $i < j-1$, we put $p = \sigma_{j-1}^{n-1}$ and $r = \sigma_{j-2}^{n-2}$.

The morphisms p, q and r are retractions which satisfy the assumptions of the preceding lemma. □

PROPOSITION 11.13. *Let C be a small category and $A : \mathbf{\Delta} \longrightarrow \widehat{C}$ a functor. Then the induced colimit preserving functor $A_! : sSet \longrightarrow \widehat{C}$ preserves monomorphisms if and only if the map*

$$(A(\delta_0^1), A(\delta_1^1)) : A^0 \amalg A^0 \longrightarrow A^n = A_!(\Delta^n)$$

is a monomorphism.

PROOF. The class of monomorphisms of simplicial sets is the smallest saturated class which contains the inclusions $\partial\Delta^n \longrightarrow \Delta^n$. Therefore, the functor $A_!$ preserves monomorphisms if and only if, for any integer $n \geq 0$, it sends $\partial\Delta^n \longrightarrow \Delta^n$ to a monomorphism. We shall see that the map

$$A_!(\partial\Delta^n) \longrightarrow A_!(\Delta^n) = A^n$$

always is a monomorphism for $n \neq 1$, which will prove the proposition. The case where $n = 0$ comes from the fact that, since the functor $A_!$ preserves colimits, it sends the empty simplicial set to the empty presheaf over C. Let us assume that $n \geq 2$. The boundary $\partial\Delta^n$ is the union of representable subpresheaves of the form $F_i = Im(\delta_i^n)$, $0 \leq i \leq n$, and we have Cartesian squares of subobjects of A^n of the form

$$\begin{array}{ccc} A_!(F_i \cap F_j) & \hookrightarrow & A_!(F_i) \\ \downarrow & & \downarrow \\ A_!(F_j) & \hookrightarrow & A^n \end{array}$$

(because either $F_i \cap F_j = \varnothing$, either this square is the image by $A_!$ of an absolute Cartesian square provided by Lemma 11.12). In other words, $A_!(\partial\Delta^n)$ is canonically isomorphic to the union of the $A_!(F_i)$'s, and thus, in particular, is a subobject of A^n. □

PROPOSITION 11.14. *Let $A, B : \mathbf{\Delta} \longrightarrow \mathcal{C}$ be two functors with values in a model category. We denote by $A_!$ and $B_!$ their extensions by colimits, respectively, and we assume that both of them send monomorphisms to cofibrations. If a natural transformation $u : A \longrightarrow B$ induces a weak equivalence $A^n \longrightarrow B^n$ for all $n \geq 0$, then, for any simplicial set X, the map*

$$u_X : A_!(X) \longrightarrow B_!(X)$$

is a weak equivalence.

PROOF. It follows from corollaries 8.16, 8.18 and 8.29 that the class of simplicial sets X such that the map u_X is a weak equivalence is saturated by monomorphisms (see Definition 3.9). Since the category of simplices is an Eilenberg-Zilber category, we may apply Corollary 3.10 for $A = \mathbf{\Delta}$, and conclude that the smallest class of simplicial sets which is saturated by monomorphisms and which contains all representable simplicial sets is the class of all simplicial sets. Therefore, since the map u_X is a weak equivalence for $X = \Delta^n$, it must have the same property for any X. □

11.15. A *bisimplicial set* is a presheaf over the product $\mathbf{\Delta} \times \mathbf{\Delta}$. For a bisimplicial set X, its evaluation at $([m], [n])$ is denoted by $X_{m,n}$.

For two simplicial sets X and Y, we write $X \boxtimes Y$ for the bisimplicial set defined by

$$(X \boxtimes Y)_{m,n} = X_m \times Y_n \,.$$

Note that this operation preserves reresentable presheaves: for $m, n \geq 0$, we have:

$$\Delta^m \boxtimes \Delta^n = h_{([m],[n])} \,.$$

For a bisimplicial set X and a simplicial set K, we write X^K for the simpicial set defined by

$$(X^K)_n = \varprojlim_{\Delta^m \longrightarrow K} X_{m,n} \,.$$

In other words, the functor $K \longmapsto X^K$ is the extension by colimits of the functor $\mathbf{\Delta} \longrightarrow sSet^{op}$ which sends $[m]$ to the simplicial set $X^{\Delta^m} = ([n] \longmapsto X_{m,n})$.

Finally, for a biimplicial set X, we write $diag(X)$ for the simplicial set defined by

$$diag(X)_n = X_{n,n} \,.$$

THEOREM 11.16. *If a morphism of bisimplicial sets $X \longrightarrow Y$ is a level-wise weak homotopy equivalence (i.e. induce a weak homotopy equivalence $X^{\Delta^m} \longrightarrow Y^{\Delta^m}$ for all $m \geq 0$), then the diagonal map*

$$diag(X) \longrightarrow diag(Y)$$

is a weak homotopy equivalence.

PROOF. It follows from Theorem 3.8, applied to $A = \mathbf{\Delta} \times \mathbf{\Delta}$ (see 3.4), that a cellular model for the category of bisimplicial sets is given by the maps of the form

$$\Delta^m \boxtimes \partial\Delta^n \cup \partial\Delta^m \boxtimes \Delta^n \longrightarrow \Delta^m \boxtimes \Delta^n$$

for $m, n \geq 0$ (because the domain of this embedding is the boundary of the codomain in the sense of 3.7). In other words, in the context of bisimplicial sets, the trivial fibrations are the maps $X \longrightarrow Y$ such that the induced morphism of simplicial set

$$X^{\Delta^m} \longrightarrow Y^{\Delta^m} \times_{Y^{\partial\Delta^m}} X^{\partial\Delta^m} \tag{11.16.1}$$

is a trivial fibration for all $m \geq 0$.

There is an exact cylinder I defined as the cartesian product with the interval $\Delta^0 \boxtimes \Delta^1$. Let An be the smallest class of I-anodyne extensions containing the maps of the form

$$\Delta^m \boxtimes \Lambda^n_k \longrightarrow \Delta^m \boxtimes \Delta^n \tag{11.16.2}$$

for $m, n \geq 0$ and $0 \leq k \leq n$. Using the explicit construction of the class An given in Example 9.13 together with Proposition 11.2, we see that the corresponding naive fibrations precisely are the morphisms $X \longrightarrow Y$ such that, for any $m \geq 0$, the map (11.16.1) is a Kan fibration. In the sequel of this proof, we will consider the category of bisimplicial sets as endowed with the model category structure associated to I and An by Theorem 9.19.

On the other hand, the class of morphisms of bisimplicial sets whose image by the diagonal functor $diag$ is a trivial cofibration of the Kan-Quillen model structure is saturated. Since the diagonal of any map of the form (11.16.2) is a weak

homotopy equivalence, we deduce that the functor *diag* sends any element of *An* to a weak homotopy equivalence. Therefore, by virtue of Proposition 9.40, the functor *diag* sends weak equivalences of the model category structure associated to I and *An* to weak homotopy equivalences.

Since level-wise trivial cofibrations also form a saturated class of morphisms of bisimplicial sets and since the maps of the form (11.16.2) have this property, we also know that any element of *An* is a level-wise weak homotopy equivalence. For a general morphism of bisimplicial sets $f : U \longrightarrow V$, we can form a commutative diagram of shape

$$\begin{array}{ccc} U & \xrightarrow{i} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} \\ V & \xrightarrow{j} & Y \end{array}$$

in which i and j are I-anodyne, while p is a naive fibration with fibrant codomain. If f is a level-wise weak equivalence, then so is p. Applying Proposition 11.14 for $\mathcal{C} = sSet^{op}$, we deduce that, for any simplicial set K, the induced map $X^K \longrightarrow Y^K$ is a weak homotopy equivalence between Kan complexes. Using the fact that pulling back weak equivalences between fibrant objects along fibrations gives weak equivalences, together with the two-out-of-three property for weak equivalences, we deduce that the map (11.16.1) is a trivial fibration for all $m \geq 0$. In particular, f is a weak equivalence, and thus $diag(f)$ is a weak homotopy equivalence. □

11.17. Given a partially ordered set E, we write $s(E)$ for the set of finite non-empty totally ordered subsets of E, ordered by inclusion. This defines a functor from the category of partially ordered sets to the category of small categories: given a morphism $f : E \longrightarrow F$, the induced map $s(f) : s(E) \longrightarrow s(F)$ simply is defined by

$$s(f)(U) = f(U) = \{f(x) \mid x \in U\}\,.$$

The (barycentric) *subdivision* functor

$$Sd : sSet \longrightarrow sSet$$

is the extension by colimits of the functor $[n] \longmapsto s([n])$. It has a right adjoint

$$Ex : sSet \longrightarrow sSet$$

defined by $Ex(X)_n = Hom(Sd(\Delta^n), X)$. Taking the maximal element of non-empty subsets of totally ordered finite sets define a functorial order preserving map

$$a_n : s([n]) \longrightarrow [n]\,, \quad U \longmapsto max(U)\,.$$

This extends to a natural transformation

$$a_X : Sd(X) \longrightarrow X\,,$$

which, by transposition, defines an embedding

$$b_X : X \longrightarrow Ex(X)$$

(obtained by composing the Yoneda isomorphism $X_n \simeq Hom(\Delta^n, X)$ with the map $a_n^* : Hom(\Delta^n, X) \longrightarrow Hom(Sd(\Delta^n), X)$).

PROPOSITION 11.18. *The functor Sd preserves monomorphisms as well as anodyne extensions.*

PROOF. The fact that this functor preserves monomorphisms is a direct application of Proposition 11.13.

For each $n \geq 0$, the simplicial set $Sd(\Delta^n)$ is the nerve of a partially ordered set with a final element (namely $[n]$) and the inclusions $S \subset [n]$ define a Δ^1-homotopy from the identity of $Sd(\Delta^n)$ to the constant map with value $[n]$. In particular, the map $\omega : \Delta^0 \longrightarrow Sd(\Delta^n)$ corresponding to the final element is a strong deformation retract, and thus is an anodyne extension (see Lemma 9.32).

Let us prove that any point $\eta : \Delta^0 \longrightarrow Sd(\Delta^n)$ is an anodyne extension. There is an homotopy $h : \Delta^1 \longrightarrow Sd(\Delta^n)$ such that $h(0) = \eta$ and $h(1) = \omega$. By virtue of Proposition 10.4, we see that h and η must be absolute weak equivalences and thus that η must be an anodyne extension.

Using Proposition 10.4 again, we deduce that, for any morphism $u : \Delta^m \longrightarrow \Delta^n$ the map $Sd(u) : Sd(\Delta^m) \longrightarrow Sd(\Delta^n)$ is an absolute weak equivalence (because we may always choose a base point).

Let W be the class of morphisms whose image by Sd becomes an absolute weak equivalence. We want to prove that all inclusions of the form $\Lambda^n_k \longrightarrow \Delta^n$ belong to W, knowing that all the maps $\Delta^m \longrightarrow \Delta^n$ are in W. We proceed by induction on $n \geq 1$. The case $n = 1$ is easy: we have to check that the maps $\Delta^0 \longrightarrow \Delta^1$ are in W, which we already know. For a subset $I \subset [n]$, we put

$$\Lambda^n_I = \bigcup_{i \notin I} Im(\delta^n_i)\,.$$

Hence $\Lambda^n_{\{k\}} = \Lambda^n_k$. We shall prove that, for any non-empty proper subset I, the map $\Lambda^n_I \longrightarrow \Delta^n$ is in W. We proceed by induction on the number c of elements in the complement of I. If $c = 1$, this means that we have to prove that any face map $\Delta^{n-1} \longrightarrow \Delta^n$ is in W, which is clear. If $c > 1$, we pick an element $k \notin I$, and, using Lemma 11.12, we see that there is a unique non-empty proper subset I' of $[n-1]$ such that we have biCartesian squares of the following form.

$$\begin{array}{ccc} \Lambda^{n-1}_{I'} & \longrightarrow & \Lambda^n_{I\cup\{k\}} \\ \downarrow & & \downarrow \\ \Delta^{n-1} & \xrightarrow{\delta^n_k} & \Lambda^n_I \end{array}$$

Therefore, by induction on n, the inclusion $\Lambda^n_{I\cup\{k\}} \longrightarrow \Lambda^n_I$ is in W. Since the inclusion $\Lambda^n_{I\cup\{k\}} \longrightarrow \Delta^n$ is in W by induction on c, this proves that the inclusion $\Lambda^n_k \longrightarrow \Delta^n$ is in W. □

PROPOSITION 11.19. *For any simplicial set X, the map $a_X : Sd(X) \longrightarrow X$ is a weak homotopy equivalence.*

PROOF. By virtue of Proposition 11.14, we are reduced to prove that the map a_X is a weak homotopy equivalence for $X = \Delta^n$, $n \geq 0$. Therefore, using the two-out-three property for weak homotopy equivalences we obtain the property we seek. □

LEMMA 11.20. *Let $f, g : K \longrightarrow L$ be two morphisms of simplicial sets, such that there exists an homotopy $h : \Delta^1 \times K \longrightarrow L$ from f to g. Then, for any simplicial set X, there exists an homotopy from f^* to g^*, where $f^*, g^* : \underline{Hom}(L, X) \longrightarrow \underline{Hom}(K, X)$ are the morphisms induced by f and g, respectively.*

PROOF. The map h defines a morphism

$$h^* : \underline{Hom}(K, X) \longrightarrow \underline{Hom}(\Delta^1 \times K, X) \simeq \underline{Hom}(\Delta^1, \underline{Hom}(K, X))$$

which induces, by transposition, a morphism

$$\tilde{h} : \Delta^1 \times \underline{Hom}(L, X) \longrightarrow \underline{Hom}(K, X)\,.$$

The latter is an homotopy from f^* to g^*. □

PROPOSITION 11.21 (Kan). *For any simplicial set X, the map $b_X : X \longrightarrow Ex(X)$ is a weak homotopy equivalence.*

PROOF. In the particular case where X is a Kan complex, the assertion follows from Proposition 11.19. Indeed, this proposition means that the pair Sd and Ex form a Quillen adjunction, which induces a derived adjunction $\mathbf{L}Sd$ and $\mathbf{R}Ex$, and that the total left derived functor $\mathbf{L}Sd$ is isomorphic to the identity via the natural maps a_X. By transposition, this means that the functor $\mathbf{R}Ex$ is isomorphic to the identity, which precisely mean that b_X is a (weak) homotopy equivalence for any Kan complex X. For the general case, we consider an anodyne extension $i : X \longrightarrow Y$ with Y a Kan complex, and we consider the commutative diagram below.

$$\begin{array}{ccc} X & \xrightarrow{\;i\;} & Y \\ {\scriptstyle b_X}\downarrow & & \downarrow{\scriptstyle b_Y} \\ Ex(X) & \xrightarrow{\;Ex(i)\;} & Ex(Y) \end{array}$$

The maps i and b_Y are weak homotopy equivalences. To finish the proof, it is thus sufficient to prove that the maps $Ex(i)$ is a weak homotopy equivalence. We will now prove that the functor Ex preserves weak homotopy equivalences, which will allow to conclude.

For each simplicial set X we define a bisimplicial set $E(X)$ by the formula

$$E(X)_{m,n} = Hom(\Delta^m \times Sd(\Delta^n), X)\,.$$

The projections $\Delta^m \longleftarrow \Delta^m \times Sd(\Delta^n) \longrightarrow Sd(\Delta^n)$ induce functorial morphisms of bisimplicial sets

$$X \boxtimes \Delta^0 \longleftarrow E(X) \longrightarrow \Delta^0 \boxtimes Ex(X)\,.$$

For any simplicial set K such that the map $K \longrightarrow \Delta^0$ is an homotopy equivalence, it follows from the preceding lemma that the induced map $X \longrightarrow \underline{Hom}(K, X)$ is an homotopy equivalence. In particular, the map $X \longrightarrow \underline{Hom}(Sd(\Delta^n), X)$ is an homotopy equivalence for all $n \geq 0$. Applying Theorem 11.16 (possibly after permuting the factors in $\mathbf{\Delta} \times \mathbf{\Delta}$), we conclude that the induced map

$$X \simeq diag(X \boxtimes \Delta^0) \longrightarrow diag(E(X))$$

is a weak homotopy equivalence.

Similarly, the map $X \longrightarrow \underline{Hom}(\Delta^m, X)$ is an homotopy equivalence, and thus, since the functor Ex commutes with finite products, the map

$$Ex(X) \simeq (\Delta^0 \boxtimes Ex(X))^{\Delta^m} \longrightarrow Ex(\underline{Hom}(\Delta^m, X)) = E(X)^{\Delta^m}$$

is an homotopy equivalence with respect to the cylinder associated to the interval $Ex(\Delta^1)$. The map $b_{\Delta^1} : \Delta^1 \longrightarrow Ex(\Delta^1)$ turns any $Ex(\Delta^1)$-homotopy into a Δ^1-homotopy. Applying Theorem 11.16 once more, we deduce that the functorial map

$$Ex(X) \simeq diag(\Delta^0 \boxtimes Ex(X)) \longrightarrow diag(E(X))$$

is a weak homotopy equivalence. For any morphism $f : X \longrightarrow Y$, we now have the commutative diagram below, in which all horizontal maps are weak homotopy equivalences.

$$\begin{array}{ccccc} X & \longleftarrow & diag(E(X)) & \longrightarrow & Ex(X) \\ {\scriptstyle f}\downarrow & & {\scriptstyle diag(E(f))}\downarrow & & \downarrow{\scriptstyle Ex(f)} \\ Y & \longleftarrow & diag(E(Y)) & \longrightarrow & Ex(Y) \end{array}$$

Therefore, the map f is a weak homotopy equivalence if and only if the map $Ex(f)$ has this property. □

11.22. For any simplicial set X, and for any integer $n \geq 1$, we define Ex^n by induction:

$$Ex^{n+1}(X) = Ex(Ex^n(X))\,. \tag{11.22.1}$$

It follows right away from Proposition 11.18 that the functors Ex^n all preserve Kan fibrations as well as trivial fibrations. We have natural morphisms

$$b_{Ex^n(X)} : Ex^n(X) \longrightarrow Ex^{n+1}(X) \tag{11.22.2}$$

and thus a sequence of morphisms

$$X \longrightarrow Ex(X) \longrightarrow Ex^2(X) \longrightarrow \cdots \longrightarrow Ex^n(X) \longrightarrow Ex^{n+1}(X) \longrightarrow \cdots \tag{11.22.3}$$

One defines

$$Ex^\infty(X) = \varinjlim_{n \geq 0} Ex^n(X)\,. \tag{11.22.4}$$

By virtue of Proposition 11.21, the canonical map

$$\beta_X : X \longrightarrow Ex^\infty(X) \tag{11.22.5}$$

is a trivial cofibration.

Proposition 11.23. *The functor Ex^∞ preserves Kan fibrations as well as trivial fibrations.*

Proof. Since the analogous property is known for Ex^n with n finite, this follows from the next lemma. □

Lemma 11.24. *The classes of Kan fibrations and of trivial fibrations are closed under filtered colimits.*

PROOF. For any set J of morphisms between finite simplicial sets (i.e. simplicial sets which are finite colimits of representable presheaves, or equivalently, which have finitely many non-degenerate simplices), the class of morphisms with the right lifting property with respect to J is closed under filtered colimits. Indeed, for any finite simplicial set K, the functor $Hom(K, -)$ commutes with filtered colimits (being a finite limit of such things), and a morphism $p : X \longrightarrow Y$ has the right lifting property with respect to a map $u : K \longrightarrow L$ if and only if the canonical

$$(u^*, p_*) : Hom(L, X) \longrightarrow Hom(K, X) \times_{Hom(K,Y)} Hom(L, Y)$$

is surjective. With u a fixed map between finite simplicial sets, the formation of such a map (u^*, p_*) commutes with filtered colimits of the p's, and this proves our assertion. □

LEMMA 11.25. *For any partially ordered set E there is a canonical isomorphism*

$$Sd(N(E)) \simeq N(s(E)) .$$

PROOF. There is a canonical functorial map $Sd(N(E)) \longrightarrow N(s(E))$ which is characterized by being the identity for $E = [n]$, $n \geq 0$. One checks that

$$\varinjlim_{P \in s(E)} N(P) \simeq N(E) .$$

In other words, the nerve of E is the union of the subobjects of the form $N(P)$ (one sees this thanks to the fact that, for two finite non-empty totally ordered subsets P and Q in E, the intersection $N(P) \cap N(Q)$ is either empty either the nerve of a finite non-empty totally ordered subset). Therefore, it is sufficient to check that the natural map

$$\varinjlim_{P \in s(E)} N(s(P)) \longrightarrow N(s(E))$$

is an isomorphism. Since the funtor s preserves monomorphisms, intersection of subobjects, and since $s(\varnothing) = \varnothing$, we see that the left hand side is the union, in the right hand side, of the subobjects of the form $N(s(P))$ for $P \in s(E)$. Therefore, it is sufficient to check the surjectivity, which is obvious: any finite sequence $P_0 \subset P_1 \subset \ldots \subset P_n$ in $s(E)$ can be seen as a sequence in $s(P_n)$. □

LEMMA 11.26. *There are canonical isomorphisms*

$$Sd(\Lambda^n_k) \simeq N(\Phi^n_k) \quad \textit{and} \quad Sd(\partial\Delta^n) \simeq N(\partial\Phi^n)$$

where Φ^n_k denotes the set of non-empty subsets of $[n]$ which do not contain the complement of $\{k\}$, and $\partial\Phi^n$ is the set of proper non-empty subsets of $[n]$ (ordered by inclusion).

PROOF. We already know that $Sd(\Lambda^n_k)$ is a subobject of $Sd(\Delta^n)$, and it is clear, by virtue of the functoriality of the isomorphism provided by the preceding lemma, that it is contained in $N(\Phi^n_k)$. Any m-simplex of $N(\Phi^n_k)$ is of the form $P_0 \subset P_1 \subset \ldots \subset P_m$ and thus defines an m-simplex of $N(s(P_m))$. Since the inclusion $N(P_m) \subset \Delta^n$ factors through Λ^n_k, this shows that $Sd(\Lambda^n_k) = N(\Phi^n_k)$. The case of $Sd(\partial\Delta^n)$ is proven similarly. □

THEOREM 11.27 (Kan). *For any simplicial set X, we have a Kan complex $Ex^\infty(X)$.*

PROOF. We define a morphism $\psi_k^n : s(s([n])) \longrightarrow \Phi_k^n$ by the formula

$$\psi_k^n(P) = \{c_k^n(S) \mid S \in P\},$$

where, for a non-empty subset $S \subset [n]$ we put

$$c_k^n(S) = \begin{cases} max(S) & \text{if } S \in \Phi_k^n, \\ k & \text{else.} \end{cases}$$

By virtue of the two preceding lemmata, the nerve of ψ_k^n defines a morphism

$$u_k^n = N(\psi_k^n) : Sd^2(\Delta^n) \longrightarrow Sd(\Lambda_k^n),$$

and we check that the following triangle commutes

$$\begin{array}{ccc} Sd^2(\Lambda_k^n) & \xrightarrow{Sd(a_{\Lambda_k^n})} & Sd(\Lambda_k^n) \\ \downarrow & \nearrow_{u_k^n} & \\ Sd^2(\Delta^n) & & \end{array}$$

Let $x : \Lambda_k^n \longrightarrow Ex^\infty(X)$ be a morphism. Since the horn Λ_k^n is a finite simplicial set, there exists $m \geq 1$ such that the map x factors through $Ex^m(X)$. The map x corresponds by adjunction to a map $\tilde{x} : Sd(\Lambda_k^n) \longrightarrow Ex^{m-1}(X)$. The morphism $\tilde{x}u_k^n$ defines a morphism $y : \Lambda_k^n \longrightarrow Ex^{m+1}(X)$ such that the following diagram commutes.

$$\begin{array}{ccccc} \Lambda_k^n & \xrightarrow{x} & Ex^m(X) & \longrightarrow & Ex^\infty(X) \\ \downarrow & & \downarrow{\scriptstyle b_{Ex^m(X)}} & \nearrow & \\ \Delta^n & \xrightarrow{y} & Ex^{m+1}(X) & & \end{array}$$

This proves that $Ex^\infty(X)$ is a Kan complex. □

COROLLARY 11.28. *For any Cartesian square of simplicial sets*

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ Y' & \xrightarrow{g} & Y \end{array}$$

in which p is a Kan fibration, if g (or p) is a weak homotopy equivalence, so is the map f (or p', respectively).

PROOF. Applying the functor Ex^∞ to this square, we obtain a Cartesian square whose vertices all are Kan complexes, and in which the map $Ex^\infty(p)$ is a Kan fibration. If the map $Ex^\infty(g)$ is a weak equivalence. By virtue of the dual version of Proposition 8.27 (i.e. applying the latter for $\mathcal{C} = (sSet)^{op}$), we see that the map $Ex^\infty(f)$ is a weak equivalence. If the map $Ex^\infty(p)$ is a weak equivalence, then it is a trivial fibration (because it is a naive fibration between fibrant object, hence a fibration). The existence of the natural weak equivalence from the identity to Ex^∞ (11.22.5) thus implies that f (resp. p) is a weak homotopy equivalence. □

THEOREM 11.29 (Quillen). *A morphism of simplicial sets is a fibration of the Kan-Quillen model category structure if and only if it is a Kan fibration. A morphism of simplicial sets is a trivial cofibration of the Kan-Quillen model category structure if and only if it is an anodyne extension.*

PROOF. It is sufficient to prove the second assertion. Factorizing such a trivial cofibration into an anodyne extension followed by a Kan fibration, we see from the Retract Lemma that it is sufficient to prove that any Kan fibration which is a weak homotopy equivalence is a trivial fibration. Since there is a cellular model which consists of monomorphisms with representable codomain, and since, by virtue of Corollary 11.28, the class of Kan fibrations which are weak homotopy equivalences is closed under pullbacks, it is sufficient to prove the special case where the codomain is representable. Let $p : X \longrightarrow \Delta^n$ be a Kan fibration which is a weak homotopy equivalence. Let F be the fibre of p at 0. Then, by virtue of Proposition 10.6, the inclusion $F \longrightarrow X$ is a strong deformation retract, and, therefore, is an anodyne extension (see Lemma 9.32). The map $F \longrightarrow \Delta^0$ is a weak homotopy equivalence and a fibration, which implies that it is a trivial fibration, hence an absolute weak equivalence. It follows from Proposition 10.4 that the map p is an absolute weak equivalence, from which we deduce, thanks to Proposition 10.3, that it is a trivial fibration. □

CHAPTER 12

Inner anodyne extensions

DEFINITION 12.1. An *inner anodyne extension* is an element of the smallest saturated class of morphisms of simplicial sets which contains the set of inclusions of the form $\Lambda^n_k \longrightarrow \Delta^n$ with $n \geq 2$ and $0 < k < n$.

We already used the following lemma in the case where $m = 1$.

LEMMA 12.2. *For any non-negative integers m and n, the simplicial set $\Delta^m \times \Delta^n$ is the union of its subobjects of the form $N(P)$, where P runs over the set of totally ordered subsets of cardinal $m + n + 1$.*

PROPOSITION 12.3 (Joyal). *The following three classes of morphisms of simplicial sets are equal:*

(a) *the class of inner anodyne extensions;*
(b) *the smallest saturated class of maps containing inclusions of the form*

$$\Delta^2 \times \partial\Delta^n \cup \Lambda^2_1 \times \Delta^n \longrightarrow \Delta^2 \times \Delta^n \text{ for } n \geq 0;$$

(c) *the smallest saturated class containing inclusions of the form*

$$\Delta^2 \times K \cup \Lambda^2_1 \times L \longrightarrow \Delta^2 \times L \text{ for any monomorphism } K \longrightarrow L.$$

PROOF. The fact that the classes (b) and (c) coincide is a formal consequence of the fact that the smallest saturated class of maps containing the boundary inclusions $\partial\Delta^n \longrightarrow \Delta^n$, for $n \geq 0$, is the class of all monomorphisms, and from the fact that the class of monomorphisms $K \longrightarrow L$ such that the induced map $\Delta^2 \times K \cup \Lambda^2_1 \times L \longrightarrow \Delta^2 \times L$ is inner anodyne is saturated (because it can be characterized in terms of left lifting property with respect to a certain class of morphisms).

Let us prove that the class (a) is in the class (b). For $n \geq 2$ and $0 < k < n$, we define morphisms

$$s : [n] \longrightarrow [2] \times [n] \quad \text{and} \quad r : [2] \times [n] \longrightarrow [n]$$

by the formulas

$$s(j) = \begin{cases} (0,j) & \text{if } j < k, \\ (1,j) & \text{if } j = k, \\ (2,j) & \text{if } j > k, \end{cases} \quad \text{and} \quad r(i,j) = \begin{cases} inf\{j,k\} & \text{if } i = 0, \\ k & \text{if } i = 1, \\ max\{j,k\} & \text{if } i = 2, \end{cases}$$

We clearly have $rs = 1_{[n]}$, and we denote by the same letters

$$s : \Delta^n \longrightarrow \Delta^2 \times \Delta^n \quad \text{and} \quad r : \Delta^2 \times \Delta^n \longrightarrow \Delta^n$$

the corresponding morphisms of simplicial sets. We claim that these maps s and r induce a commutative diagram of the form below.

$$\begin{array}{ccccc} \Lambda^n_k & \xrightarrow{s'} & \Delta^2 \times \Lambda^n_k \cup \Lambda^2_1 \times \Delta^n & \xrightarrow{r'} & \Lambda^n_k \\ \downarrow & & \downarrow & & \downarrow \\ \Delta^n & \xrightarrow{s} & \Delta^2 \times \Delta^n & \xrightarrow{r} & \Delta^n \end{array}$$

To prove that the map s' is well defined, we have to check that the image of any composition of the form $s\delta^n_i$, $i \neq k$, is in $\Delta^2 \times \Lambda^n_k \cup \Lambda^2_1 \times \Delta^n$. Such an image must land in $\Delta^2 \times \partial\Delta^n \cup \partial\Delta^2 \times \Delta^n$ for cardinality reasons. Therefore, it is sufficient to check that the image is not contained in $Im(\delta^2_1) \times \Delta^n$ nor in $\Delta^2 \times Im(\delta^n_k)$, which follows, in both cases, from the fact that the point $(1,k)$ must be reached. To prove that r' is well defined, we have several cases to consider. The image of $\Delta^{\{0,1\}} \times \Delta^n$ by r is contained in $\Delta^{\{0,\dots,k\}} \subset \Lambda^n_k$, and, dually, the image of $\Delta^{\{1,2\}} \times \Delta^n$ is contained in $\Delta^{\{k,\dots,n\}} \subset \Lambda^n_k$. It remains to prove that r sends $\Delta^2 \times \Lambda^n_k$ in Λ^n_k. But, for $0 \leq l \leq n$, $l \neq k$, we have $r(i,j) \neq l$ for any $j \neq l$. In other words, the image of $\Delta^2 \times Im(\delta^n_l)$ by r does not reach the value i, and is thus contained in $Im(\delta^n_i) \subset \Lambda^n_k$.

For $0 \leq i \leq j < n$, we let $U_{i,j}$ be the image of the map

$$u_{i,j} : \Delta^{n+1} \longrightarrow \Delta^2 \times \Delta^n$$

defined as the nerve of the map

$$k \longmapsto \begin{cases} (0,k) & \text{if } 0 \leq k \leq i, \\ (1,k-1) & \text{if } i < k \leq j+1, \\ (2,k-1) & \text{else.} \end{cases}$$

For $0 \leq i \leq j \leq n+1$, we let $V_{i,j}$ be the image of the map

$$v_{i,j} : \Delta^{n+2} \longrightarrow \Delta^2 \times \Delta^n$$

defined as the nerve of the map

$$k \longmapsto \begin{cases} (0,k) & \text{if } 0 \leq k \leq i, \\ (1,k-1) & \text{if } i < k \leq j+1, \\ (2,k-2) & \text{else.} \end{cases}$$

We define

$$X(-1,-1) = \Delta^2 \times \partial\Delta^n \cup \Lambda^2_1 \times \Delta^n$$

and, for $n > j \geq 0$ and $0 \leq i \leq j$, we put

$$X(i,j) = X(j-1,j-1) \cup \bigcup_{0 \leq l \leq i} U_{i,j}\,.$$

Using the fact that $u^{-1}_{i,j}(\Delta^{\{0,2\}} \times \Delta^n) = \Delta^{\{0,\dots,n\}-\{i+1,\dots,j+1\}}$, we see that we have $u^{-1}_{0,j}(\Delta^{\{0,2\}} \times \Delta^n) = Im(\delta^{n+1}_{j+1})$, and that $u^{-1}_{i,j}((\Delta^{\{0,2\}} \times \Delta^n) \cup X(i,j+1)) = Im(\delta^{n+1}_{i+1})$,

from which we deduce biCartesian squares of the form below.

$$\begin{array}{ccc} \Lambda^{n+1}_{j+1} & \longrightarrow & X(j-1,j-1) \\ \downarrow & & \downarrow \\ \Delta^{n+1} & \xrightarrow{u_{0,j}} & X(0,j) \end{array} \qquad \begin{array}{ccc} \Lambda^{n+1}_{i+1} & \longrightarrow & X(i,j) \\ \downarrow & & \downarrow \\ \Delta^{n+1} & \xrightarrow{u_{i+1,j}} & X(i+1,j) \end{array}$$

In particular, the inclusion $X(-1,-1) \longrightarrow X(n,n)$ is an inner anodyne extension. Similarly, we define

$$Y(-1,-1) = X(n,n)$$

and, for $n+2 > j \geq 0$ and $0 \leq i \leq j$, we put

$$Y(i,j) = Y(j-1,j-1) \cup \bigcup_{0 \leq l \leq i} V_{l,j}\,.$$

One then checks as above that each inclusion of the form $Y(j-1,j-1) \longrightarrow Y(0,j)$ as well as each inclusion of the form $Y(i,j) \longrightarrow Y(i+1,j)$ is a pushout of a horn inclusion of type $\Lambda^n_k \longrightarrow \Delta^n$ for $0 < k < n$, and, since $Y(n+2,n+2) = \Delta^2 \times \Delta^n$, this proves that the class (b) is contained in the class (a). □

Corollary 12.4. *For any inner anodyne extension $K \longrightarrow L$ and any monomorphism $X \longrightarrow Y$, the induced inclusion*

$$K \times Y \cup L \times X \longrightarrow L \longrightarrow Y$$

is an inner anodyne extension.

Definition 12.5. An *inner (Kan) fibration* is a morphism of simplicial sets which has the right lifting property with respect to the class of inner anodyne extensions.

Example 12.6. A simplicial set X is an ∞-category if and only if the morphism $X \longrightarrow \Delta^0$ is an inner Kan fibration.

Example 12.7. For any integers $n \geq 2$ and $0 < k < n$, it follows from Proposition 4.13 that the natural map $\tau(\Lambda^n_k) \longrightarrow \tau(\Delta^n)$ is an isomorphism of categories. This implies right away that, for any functor between two small categories $C \longrightarrow D$, the induced morphism of simplicial sets $N(C) \longrightarrow N(D)$ is an inner Kan fibration. This gives an even larger supply of examples: more generally, for any ∞-category X and any small category C, any morphism $X \longrightarrow N(C)$ is an inner Kan fibration.

Corollary 12.8. *For a morphism of simplicial sets $p : X \longrightarrow Y$, the following conditions are equivalent.*

(i) *The morphism p is an inner Kan fibration.*
(ii) *For any inner anodyne extension $i : K \longrightarrow L$, the restrition along i induces a trivial fibration*

$$\underline{Hom}(L,X) \longrightarrow \underline{Hom}(K,X) \times_{\underline{Hom}(K,Y)} \underline{Hom}(L,Y)\,.$$

(iii) *The restriction along the inclusion of* Λ^2_1 *into* Δ^2 *induces a trivial fibration*

$$\underline{Hom}(\Delta^2, X) \longrightarrow \underline{Hom}(\Lambda^2_1, X) \times_{\underline{Hom}(\Lambda^2_1, Y)} \underline{Hom}(\Delta^2, Y)\,.$$

(iv) *For any monomorphism* $i : K \longrightarrow L$*, the restrition along* i *induces an inner fibration*

$$\underline{Hom}(L, X) \longrightarrow \underline{Hom}(K, X) \times_{\underline{Hom}(K, Y)} \underline{Hom}(L, Y)\,.$$

COROLLARY 12.9. *A simplicial set* X *is an* ∞*-category if and only if the canonical map*

$$\underline{Hom}(\Delta^2, X) \longrightarrow \underline{Hom}(\Lambda^2_1, X)$$

is a trivial fibration.

COROLLARY 12.10. *For any* ∞*-category* X *and any simplicial set* A*, the simplicial set* $\underline{Hom}(A, X)$ *of morphisms from* A *to* X *is an* ∞*-category. In particular, the functors from an* ∞*-category* A *to an* ∞*-category* X *do form an* ∞*-category.*

CHAPTER 13

The Joyal model category structure

13.1. Let X be a simplicial set and $f : x \longrightarrow y$ a morphism in X. A *left inverse* of f is a morphism $g : y \longrightarrow x$ such that the triangle

$$\begin{array}{ccc} & y & \\ {\scriptstyle f}\nearrow & & \searrow{\scriptstyle g} \\ x & \xrightarrow{1_x} & x \end{array}$$

commutes. One can freely add a left inverse as follows. One first adds a map $g : y \longrightarrow x$ by forming the pushout below.

$$\begin{array}{ccc} \partial\Delta^1 & \xrightarrow{(y,x)} & X \\ \downarrow & & \downarrow \\ \Delta^1 & \xrightarrow{g} & X[g] \end{array}$$

Finally, we form the following pushout.

$$\begin{array}{ccc} \partial\Delta^2 & \xrightarrow{(f,g,1_x)} & X[g] \\ \downarrow & & \downarrow \\ \Delta^2 & \xrightarrow{G} & X[gf = 1] \end{array}$$

We still denote by f the image of f in $X[gf = 1]$. Applying this procedure to X^{op} gives a way to freely add a right inverse, which gives a simplicial set $X[fh = 1]$ together with a 2-simplex H witnessing that the new 1-simplex h is a right inverse of f, and we also denote by f the image of f in $X[fh = 1]$.

Therefore, we have defined a procedure to freely invert a morphism f: we define

$$X[f^{-1}] = (X[gf = 1])[fh = 1]\,.$$

The canonical inclusion $X \longrightarrow X[f^{-1}]$ has the following universal property: For any morphism of simplicial sets $u : X \longrightarrow Y$, and for any pair of 2-simplices of $G', H' : \Delta^2 \longrightarrow Y$ whose boundaries respectively are triangles of the form

$$\begin{array}{ccc} & u(y) & \\ {\scriptstyle u(f)}\nearrow & & \searrow{\scriptstyle g'} \\ u(x) & \xrightarrow{1_{u(x)}} & u(x) \end{array} \quad \text{and} \quad \begin{array}{ccc} & u(x) & \\ {\scriptstyle h'}\nearrow & & \searrow{\scriptstyle u(f)} \\ u(y) & \xrightarrow{1_{u(y)}} & u(y) \end{array}$$

in Y, there exists a unique morphism $v : X[f^{-1}] \longrightarrow Y$ whose restriction to X is u, and such that $v(G) = G'$ and $v(H) = H'$.

PROPOSITION 13.2. *The inclusion map $X \longrightarrow X[f^{-1}]$ is an anodyne extension.*

PROOF. Since the functor $X \longmapsto X^{op}$ preserves anodyne extensions, it is sufficient to prove that the inclusion $X \longrightarrow X[gf = 1]$ is an anodyne extension. An alternative construction of $X[fg = 1]$ is the following. One first form the pushout below.

$$\begin{array}{ccc} \Delta^1 & \xrightarrow{f} & X \\ {\scriptstyle \delta_2^2}\downarrow & & \downarrow \\ \Delta^2 & \xrightarrow{s} & X' \end{array}$$

Finally, we see that $X[gf = 1]$ naturally fits in the following coCartesian square.

$$\begin{array}{ccc} \Delta^1 & \xrightarrow{s\delta_1^2} & X' \\ \downarrow & & \downarrow \\ \Delta^0 & \xrightarrow{g} & X[gf = 1] \end{array}$$

The map $X \longrightarrow X'$ is an anodyne extension because it is the pushout of such a thing. By virtue of Proposition 8.27, the map $X' \longrightarrow X[gf = 1]$ is a weak homotopy equivalence because it is the pushout of the weak equivalence $\Delta^1 \longrightarrow \Delta^0$ along the cofibration $s\delta_1^2$. Therefore, the map $X \longrightarrow X[gf = 1]$ is both a monomorphism and a weak homotopy equivalence, whence an anodyne extension. □

DEFINITION 13.3. We define the interval J as

$$J = \Delta^1[f^{-1}],$$

where $f : 0 \longrightarrow 1$ is the unique non-trivial map in Δ^1 (corresponding to the identity of Δ^1). The class of *categorical anodyne extensions* is the smallest class of J-anodyne extensions containing the set of inner horn inclusions $\Lambda_k^n \longrightarrow \Delta^n$, for $n \geq 2$ and $0 < k < n$ (see Example 9.13).

REMARK 13.4. The class of categorical anodyne extensions is the smallest saturated class of morphisms of simplicial sets containing the inner horn inclusions as well as the inclusion maps of the form

$$J \times \partial\Delta^n \cup \{\varepsilon\} \times \Delta^n \longrightarrow J \times \Delta^n$$

for $n \geq 0$ and $\varepsilon = 0, 1$. This follows from Proposition 12.3. One deduces the next proposition right away.

PROPOSITION 13.5. *For any categorical anodyne extension $K \longrightarrow L$ and any monomorphism $X \longrightarrow Y$, the induced map*

$$K \times Y \cup L \times X \longrightarrow L \times Y$$

is a categorical anodyne extension.

PROPOSITION 13.6. *Any inner anodyne extension is a categorical anodyne extension. Any categorical anodyne extension is an anodyne extension.*

PROOF. The first assertion is obvious. The second one follows from Corollary 11.4 and Proposition 13.2. □

DEFINITION 13.7. The *Joyal model category structure* is the model category structure obtained by applying Theorem 9.19 to the exact cylinder defined as the cartesian product of J and to the class of categorical anodyne extensions. The weak equivalences of this model category structure are called the *weak categorical equivalences*.

REMARK 13.8. A reformulation of the explicit description of a generating family of the saturated class of categorical anodyne extensions given in Remark 13.4 a morphism of simplicial sets $p : X \longrightarrow Y$ has the right lifting property with respect to the class of categorical anodyne extensions if and only if the operation of evaluation at ε (i.e. of restriction along $\{\varepsilon\} \longrightarrow J$), $\varepsilon = 0, 1$, induces trivial fibrations

$$\underline{Hom}(J, X) \longrightarrow X \times_Y \underline{Hom}(J, Y),$$

and the operation of restriction along $\Lambda^2_1 \longrightarrow \Delta^2$ induces trivial fibrations

$$\underline{Hom}(\Delta^2, X) \longrightarrow \underline{Hom}(\Lambda^2_1, X) \times_{\underline{Hom}(\Lambda^2_1, Y)} \underline{Hom}(\Delta^2, Y).$$

One can check that there is a unique automorphism of J which exchanges the objects 0 and 1, so that it is sufficient to consider $\varepsilon = 0$ in these lifting properties.

In particular, the fibrant objects of the Joyal model category structure are the simplicial sets X such that the two restriction maps

$$\underline{Hom}(J, X) \longrightarrow \underline{Hom}(\{0\}, X) = X \quad \text{and} \quad \underline{Hom}(\Delta^2, X) \longrightarrow \underline{Hom}(\Lambda^2_1, X)$$

are trivial fibrations.

It is clear that any fibrant of object of the Joyal model category structure is an ∞-category. We will prove, among other things, that the converse is true: the fibrant objects of this model category precisely are the ∞-categories. The proof of this fact will require quite a few intermediate results which will be meaningful in their own right in our understanding of the theory of ∞-categories as a semantic of the language of category theory. More generally, we will also characterize the fibrations between fibrant objects of the Joyal model category structure.

In order to do this, it is enlightening to compare the Joyal model category structure with the usual homotopy theory of categories (which consists to invert the class of equivalences of categories).

DEFINITION 13.9. A functor $p : X \longrightarrow Y$ is an *isofibration* if, for any object x_0 in X and any invertible map of the form $g : y_0 \longrightarrow y_1$ in Y with $p(x_0) = y_0$, there exists an invertible map $f : x_0 \longrightarrow x_1$ in X such that $p(f) = g$.

THEOREM 13.10. *The category Cat of small categories admits a cofibrantly generated model category structure whose weak equivalences are the equivalences of categories, whose cofibrations are the functors which induce an injective map at the level of objects, and whose fibrations are the isofibrations.*

PROOF. Here are several functors of interest. The inclusion functor $\eta : \{0\} \longrightarrow \tau(J)$ (which picks the object 0 in the unique category whose set of objects is $\{0,1\}$ and which is equivalent to the final category; the functor $i : \varnothing \longrightarrow *$, from the empty category to the final category; the functor $j : \{0,1\} \longrightarrow [1]$; the unique functor $k : S \longrightarrow [1]$ which is the identity on objects, where S is the free category whose set of objects is $\{0,1\}$ with two parallel maps from 0 to 1. The class of isofibrations is the class of morphism with the right lifting property with respect to $\{\eta\}$. The morphisms with the right lifting property with respect to $\{i,j,k\}$ precisely are the functors which induce a surjective map at the level of objects and which are fully faithful. Conversely, one checks that the morphisms with the left lifting property with respect to the class of equivalences of categories which are surjective on objects precisely is the class of functors which are injective on objects. It is clear that any equivalence of categories which is surjective on objects is an isofibration, and that any isofibration which is an equivalence of categories is surjective on objects. Using the small object argument for each sets of maps $\{\eta\}$ and $\{i,j,k\}$ provide the existence of expected factorizations. □

DEFINITION 13.11. The model category structure provided by the preceding proposition is called the *canonical model category structure*.

LEMMA 13.12. *The left adjoint of the nerve functor sends inner anodyne extensions to isomorphisms.*

PROOF. The class of morphisms of simplicial sets whose image by the functor τ is an isomorphism of categories is saturated. Therefore, it is sufficient to check that τ sends each inclusion $\Lambda^n_k \longrightarrow \Delta^n$ to an invertible map of *Cat* whenever $n \geq 2$ and $0 < k < n$. But this latter property is a reformulation of Proposition 4.13. □

LEMMA 13.13. *The functor $\tau : sSet \longrightarrow Cat$ commutes with finite products.*

PROOF. Let X and Y be two simplicial sets. We choose two inner anodyne extensions $X \longrightarrow X'$ and $Y \longrightarrow Y'$ such that both X' and Y' are ∞-categories. Then the product map $X \times Y \longrightarrow X' \times Y'$ is an inner anodyne extension (this is a consequence of Corollary 12.4). Therefore, by virtue of Lemma 13.12, we have canonical isomorphisms $\tau(X) \simeq \tau(X')$, $\tau(Y) \simeq \tau(Y')$ and $\tau(X \times Y) \simeq \tau(X' \times Y')$. The explicit description of $\tau(W)$, for any ∞-category W, provided by the theorem of Boardman and Vogt (5.13), implies that the natural map from $\tau(X' \times Y')$ to $\tau(X')$ to $\tau(Y')$ is an isomorphism. □

PROPOSITION 13.14. *The adjunction*

$$\tau : sSet \rightleftarrows Cat : N$$

is a Quillen adjunction from the Joyal model category structure to the canonical model category structure.

PROOF. Since, for any simplicial set X, the set of objects of the associated category $\tau(X)$ coincides with X_0, the functor τ sends monorphisms to functors which are injective on objects. In other words, it preserves cofibrations. For any simplicial set X, the functor τ sends the inclusion $X = \{0\} \times X \longrightarrow J \times X$ to the inclusion $\tau(X) = \{0\} \times \tau(X) \longrightarrow \tau(J) \times \tau(X)$. Since $\tau(J)$ is equivalent

to the final category the explicit description of the class of categorical anodyne extensions given in Remark 13.4 and Lemma 13.12 imply that τ sends categorical anodyne extensions to trivial cofibrations. This proves that τ and N define a Quillen adjunction; see Proposition 9.40. □

DEFINITION 13.15. A morphism of simplicial sets $p : X \longrightarrow Y$ is an *isofibration* if it is an inner fibration, and if, for any object x_0 in X and any invertible map of the form $g : y_0 \longrightarrow y_1$ in Y with $p(x_0) = y_0$, there exists an invertible map $f : x_0 \longrightarrow x_1$ in X such that $p(f) = g$.

REMARK 13.16. Lemma 13.12 mean that the nerve of any functor between small categories is an inner fibration. One easily checks that a functot $p : X \longrightarrow Y$ is an isofibration if and only if its nerve $N(p) : N(X) \longrightarrow N(Y)$ is an isofibration. We will prove later that a functor between ∞-categories is an isofibration if and only if it is a fibration of the Joyal model category structure.

REMARK 13.17. Any fibration of the Joyal model category structure is an isofibration. This is a consequence of the fact that, by construction, a map $J \longrightarrow W$ is the same thing as a morphism $f : x \longrightarrow y$ in W equipped with a proof that it is invertible.

PROPOSITION 13.18. *A morphism of ∞-categories $p : X \longrightarrow Y$ is an isofibration if and only if it is an inner fibration and if, for any object x_1 in X and any invertible map of the form $g : y_0 \longrightarrow y_1$ in Y with $p(x_1) = y_1$, there exists an invertible map $f : x_0 \longrightarrow x_1$ in X such that $p(f) = g$.*

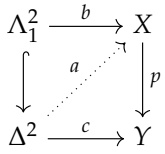
PROOF. Assume that p has the property stated in the definition, and let us prove that it is an isofibration. Let us chose an object x_0 in X and an invertible map $g : y_0 \longrightarrow y_1$ such that $p(x_0) = y_0$. We chose a map $g' : y_1 \longrightarrow y_0$ which is a left inverse of g. Then, by assumption on p, we can find an invertible morphism $f' : x_1 \longrightarrow x_0$ in X such that $p(f') = g'$. Let us chose a map $\tilde{f} : x_0 \longrightarrow x_1$ which is a right inverse of f'. Then, $p(\tilde{f})$ and g coincide in $\tau(X)$, and thus, by virtue of Theorem 5.13, there exists a map $c : \Delta^2 \longrightarrow Y$ whose boundary is the following triangle.

$$c_{|\partial\Delta^2} = \quad \begin{array}{ccc} & & y_1 \\ & {\scriptstyle p(\tilde{f})} \nearrow & \quad \searrow {\scriptstyle 1_{y_1}} \\ y_0 & \xrightarrow{\quad g \quad} & y_1 \end{array}$$

There is a unique morphism $b : \Lambda^2_1 \longrightarrow X$ corresponding to the diagram below.

$$b = \quad \begin{array}{ccc} & & x_1 \\ & {\scriptstyle \tilde{f}} \nearrow & \quad \searrow {\scriptstyle 1_{x_1}} \\ x_0 & & x_1 \end{array}$$

Since p is an inner fibration, the solid commutative square



has a filler a. For $f = a\delta^2_1$, the image of the map $f : x_0 \longrightarrow x_1$ is equal to g.

Conversely, applying what precedes to $X^{op} \longrightarrow Y^{op}$ shows that any isofibration satisfies the assumption of the proposition. □

Corollary 13.19. *A morphism of ∞-categories $X \longrightarrow Y$ is an isofibration if and only if the induced morphism $X^{op} \longrightarrow Y^{op}$ is an isofibration.*

CHAPTER 14

Left or right fibrations, joins and slices

DEFINITION 14.1. A *left anodyne extension* (a *right anodyne extension*) is an element of the smallest saturated class of morphisms of simplicial sets containing the horn inclusions of the form $\Lambda^n_k \longrightarrow \Delta^n$ for $n \geq 1$ and $0 \leq k < n$ (and $0 < k \leq n$, respectively).

A *left (Kan) fibration* (a *right (Kan) fibration*) is a morphism of simplicial sets with the right lifting property with respect to the class of left (right) anodyne extensions.

REMARK 14.2. The functor $X \longmapsto X^{op}$ maps the horn Λ^n_k to Λ^n_{n-k}. Therefore, it sends right fibrations to left fibrations and right anodyne maps to left anodyne maps, et vice versa. In particular, any statement about right fibrations and right anodyne extensions always has a counterpart in terms of left fibrations and left anodyne extensions. The following three propositions are interpretations of Lemma 11.3 in terms of right anodyne extensions and right Kan fibration.

PROPOSITION 14.3. *For any left (right) anodyne extension $K \longrightarrow L$ and any monomorphism $X \longrightarrow Y$, the induced inclusion*

$$L \times X \cup K \times Y \longrightarrow L \times Y$$

is a left (right) anodyne extension.

PROPOSITION 14.4. *Let $X \longrightarrow Y$ be a left (right) fibration. If the map $K \longrightarrow L$ is a monomorphism (a left anodyne extension), then the induced morphism*

$$\underline{Hom}(L, X) \longrightarrow \underline{Hom}(K, X) \times_{\underline{Hom}(K,Y)} \underline{Hom}(L, Y)$$

is a left fibration (a trivial fibration, respectively).

PROPOSITION 14.5. *A morphism of simplicial sets $X \longrightarrow Y$ is a left fibration (a right fibration) if and only if the evaluation at 1 (the evaluation at 0, respectively) induce a trivial fibration of the form*

$$\underline{Hom}(\Delta^1, X) \longrightarrow X \times_Y \underline{Hom}(\Delta^1, Y)\,.$$

DEFINITION 14.6. A morphism of simplicial sets $p : X \longrightarrow Y$ is *conservative* if, for any morphism $f : x \longrightarrow x'$ in X, if $p(f) : p(x) \longrightarrow p(x')$ is invertible in Y, so is f in X.

REMARK 14.7. If $p : X \longrightarrow Y$ is a functor between ∞-categories, the explicit desciption of $\tau(X)$ and $\tau(Y)$ given by the theorem of Boardman and Vogt (5.13) implies that p is conservative if and only if the functor $\tau(p) : \tau(X) \longrightarrow \tau(Y)$ is conservative.

PROPOSITION 14.8. *Let $p : X \longrightarrow Y$ be a morphism of ∞-categories. If p is either a left fibration or a right fibration, then it is a conservative isofibration.*

PROOF. It is sufficient to prove the case of a left fibration. Let us prove first that p is conservative. It is sufficient to prove that Let $x : x_0 \longrightarrow x_1$ be a morphism in X whose image $p(x) = y : y_0 \longrightarrow y_1$ is invertible. This means that we have a map $c : \Delta^2 \longrightarrow Y$ whose restriction to $\partial\Delta^2$ corresponds to a triangle of the form below.

$$\begin{array}{ccc} & & y_1 \\ & \overset{y}{\nearrow} & \\ y_0 & \xrightarrow{1_{y_0}} & y_0 \end{array} \qquad \begin{array}{c} \\ \searrow^{g} \\ \\ \end{array}$$

Let $b : \Lambda^2_0 \longrightarrow X$ be the morphism corresponding to the diagram

$$\begin{array}{ccc} & & x_1 \\ & \overset{x}{\nearrow} & \\ x_0 & \xrightarrow{1_{x_0}} & x_0 \end{array}$$

in X. The solid commutative square

$$\begin{array}{ccc} \Lambda^2_0 & \xrightarrow{b} & X \\ \downarrow & \nearrow & \downarrow{\scriptstyle p} \\ \Delta^2 & \xrightarrow{c} & Y \end{array}$$

has a filler whose restriction to $\partial\Delta^2$ provides a commutative triangle of the form

$$\begin{array}{ccc} & & x_1 \\ & \overset{x}{\nearrow} & \\ x_0 & \xrightarrow{1_{x_0}} & x_0 \end{array} \qquad \begin{array}{c} \\ \searrow^{f} \\ \\ \end{array}$$

Since $p(x)$ is invertible, so is $p(f)$, and we can repeat the procedure above to get a commutative triangle of the form below.

$$\begin{array}{ccc} & & x_0 \\ & \overset{f}{\nearrow} & \\ x_1 & \xrightarrow{1_{x_1}} & x_1 \end{array} \qquad \begin{array}{c} \\ \searrow^{f'} \\ \\ \end{array}$$

In other words, the morphism f is invertible. This implies that the morphism corresponding to x in $\tau(X)$ is invertible, and thus that x itself is invertible.

Since p is conservative and has the right lifting property with respect to the inclusion $\{0\} \longrightarrow \Delta^1$, it must be an isofibration. □

14.9. Let $\mathbf{\Delta}_+$ be the category whose objects are the ordered sets $[n] = \{0, \ldots, n\}$ for $n \geq -1$, and whose morphisms are the (non-strictly) increasing maps. In other words, the category $\mathbf{\Delta}_+$ is the category obtained from the usual category of

simplices $\mathbf{\Delta}$ by adjoining an intial object $[-1]$ (the empty ordered set). One can see a presheaf over the category $\mathbf{\Delta}_+$ as a triple (X, a, E), where X is a simplicial set, E is a set, and a is an augmentation from X to E (i.e. a morphism from X to the constant simplicial set with value E). We denote by $Set^+_{\mathbf{\Delta}}$ the category of presheaves over $\mathbf{\Delta}_+$, which we shall call the *augmented simplicial sets*. The operation of composition with the inclusion functor $i : \mathbf{\Delta} \longrightarrow \mathbf{\Delta}_+$ induces a functor

$$i^* : Set^+_{\mathbf{\Delta}} \longrightarrow sSet$$

which can be described as $(X, E, a) \longmapsto X$ in the language of augmented simplicial sets. This functor has a left adjoint $i_!$ and a right adjoint i_*; the left adjoint sends a simplicial set X to the triple $(X, p_X, \pi_0(X))$, where $\pi_0(X)$ is the set of connected components of X (i.e. the set of simplicial homotopy classes of maps from Δ^0 to X), and p_X is the canonical map[1]. The functor i_* sends a simplicial set X to the triple (X, p_X, Δ^0).

14.10. The category $\mathbf{\Delta}_+$ is equipped with a monoidal structure induced by the sum of ordinals

$$\begin{aligned} \mathbf{\Delta}_+ \times \mathbf{\Delta}_+ &\longrightarrow \mathbf{\Delta}_+ \\ ([m], [n]) &\longmapsto [m + 1 + n] \end{aligned}$$

(the unit object thus is $[-1]$). By a repeated use of Theorem 1.10, there is a unique monoidal structure on the category of augmented simplicial sets whose tensor product commutes with small colimits in each variables, which extends the sum of ordinals in $\mathbf{\Delta}_+$. The tensor product of two augmented simplicial sets X and Y is denoted by $X * Y$, so that, in particular, $h_{[m]} * h_{[n]} = h_{[m+1+n]}$. The unit object is the representable presheaf $h_{[-1]} = i_*(\varnothing)$. However, we also have an explicit description of this tensor product.

Proposition 14.11. *For any augmented simplicial sets X and Y, and for any integer $n \geq -1$, there is a canonical identification*

$$(X * Y)_n = \coprod_{i+1+j=n} X_i \times Y_j \,.$$

Proof. For $X = h_{[p]}$ and $Y = h_{[q]}$, since we restricts ourselves to consider maps which never decrease, one has:

$$Hom([n], [p + 1 + q]) = \coprod_{i+1+j=n} Hom([i], [p]) \times Hom([j], [q]) \,.$$

We then extend these identifications by colimits. □

14.12. For two simplicial sets X and Y, we define the *join* of X and Y as:

$$X * Y = i^*(i_*(X) * i_*(Y)) \,.$$

[1]One can also describe $\pi_0(X)$ as the colimit of X in the category of sets. Indeed, since each simplex Δ^n is simplicially contactible, one can describe $\pi_0(X)$ as the set of simplicial homotopy classes of maps from Δ^n to X. This provides a canonical family of surjective maps $X_n \longrightarrow \pi_0(X)$ which define a cocone from X to $\pi_0(X)$ and exhibit $\pi_0(X)$ as the colimit of X in the category of sets.

In other words, for each $n \geq 0$, the simplicial set $X * Y$ evaluated at n is:

$$(X * Y)_n = \coprod_{i+1+j=n} X_i \times Y_j \,.$$

This defines a monoidal structure on the category of simplicial sets whose unit object is the empty simplicial set $\varnothing = i^*(h_{[-1]})$. Therefore, there is a functorial injective map

$$X \amalg Y \longrightarrow X * Y$$

induced by adjunction from the inclusions $\varnothing \longrightarrow W$ for $W = X, Y$. Given a fixed simplicial set T, we thus get two functors:

$$\begin{aligned} (-) * T : sSet &\longrightarrow T\backslash sSet \\ X &\longmapsto (T = \varnothing \amalg T \longrightarrow X * T) \end{aligned}$$

$$\begin{aligned} T * (-) : sSet &\longrightarrow T\backslash sSet \\ X &\longmapsto (T = T \amalg \varnothing \longrightarrow T * X) \end{aligned}$$

Proposition 14.13. *Both functors $(-) * T$ and $T * (-)$ commute with small colimits.*

Proof. These functors both preserves the initial objects. It is thus sufficient to prove that the join operation preserves connected colimits in each variable. For this, since the tensor product preserves small colimits in each variables in the category of augmented simplicial sets, it is sufficient to prove that the functor i_* commutes with small connected colimit. Since it obviously preserves filtered colimits, we only need to check that it preserves coCartesian squares, which follows right away from its explicit description. □

14.14. Let $t : T \longrightarrow X$ be a morphism of simplicial sets. We shall write X/t or X/T ($t\backslash X$ or $T\backslash X$, respectively) the image of (X, t) by the right adjoint of the functor $(-) * T$ (of the functor $T * (-)$, respectively), which exists by virtue of the preceding proposition.

Remark 14.15. The join operation is compatible with the opposite operations as follows:

$$(T * S)^{op} = S^{op} * T^{op} \,.$$

Therefore, we have:

$$(X/T)^{op} = T^{op}\backslash X^{op} \,.$$

Since the join operation is associative, we also have the following formulas:

$$X/(S * T) \simeq (X/S)/T \qquad \text{and} \qquad (S * T)\backslash X \simeq S\backslash(T\backslash X) \,.$$

Proposition 14.16. *For any monomorphisms of simplicial sets $U \longrightarrow V$ and $K \longrightarrow L$, the induced commutative square*

$$\begin{array}{ccc} K * U & \hookrightarrow & K * V \\ \downarrow & & \downarrow \\ L * U & \hookrightarrow & L * V \end{array}$$

is Cartesian, and all its maps are monomorphisms. We thus have a canonical inclusion map

$$L * U \cup K * V \longrightarrow L * V\,.$$

Proof. This follows right away from Proposition 14.11 and the analogous property for the binary Cartesian product of sets. □

The following statement is obvious.

Proposition 14.17. *As subobjects of $\Delta^{m+1+n} = \Delta^m * \Delta^n$, we have the following identities.*

$$\partial\Delta^m * \Delta^n \cup \Delta^m * \partial\Delta^n = \partial\Delta^{m+1+n}$$
$$\Lambda^m_k * \Delta^n \cup \Delta^m * \partial\Delta^n = \Lambda^{m+1+n}_k$$
$$\partial\Delta^m * \Delta^n \cup \Delta^m * \Lambda^n_k = \Lambda^{m+1+n}_{m+1+k}.$$

Theorem 14.18 (Joyal). *Let X and Y be ∞-categories, and $p : X \longrightarrow Y$ an inner fibration. We consider given a commutative square of the form*

$$\begin{array}{ccc} \Lambda^n_0 & \xrightarrow{a} & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ \Delta^n & \xrightarrow{b} & Y \end{array} \qquad n \geq 2\,,$$

and we assume that the morphism $\alpha : a_0 \longrightarrow a_1$, defined by the restriction of a on $\Delta^{\{0,1\}}$, is invertible in X. Then there exists a morphism $h : \Delta^n \longrightarrow X$ such that the restriction of h to Λ^n_0 is equal to a, and such that $ph = b$.

By contemplating the identification $\Lambda^1_0 * \Delta^{n-2} \cup \Delta^1 * \partial\Delta^{n-2} = \Lambda^n_0$, this theorem is a particular case of the following one.

Theorem 14.19 (Coherence Theorem). *Let $p : X \longrightarrow Y$ an inner fibration, and assume that Y is an ∞-category. We consider a monomorphism $S \longrightarrow T$ as well as a commutative square of the form*

$$\begin{array}{ccc} \{0\} * T \cup \Delta^1 * S & \xrightarrow{a} & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ \Delta^1 * T & \xrightarrow{b} & Y \end{array}$$

and we assume that the morphism $\alpha : a_0 \longrightarrow a_1$, obtained as the restriction of a on $\Delta^1 * \varnothing \subset \Delta^1 * S$, is invertible in X. Then the square above has a filler $c : \Delta^1 * T \longrightarrow X$.

The proof of this theorem will require a few preliminary results. Remark that, for a morphism of simplicial sets $S \longrightarrow T$ and a map $T \longrightarrow X$, there is an induced map $X/T \longrightarrow X/S$ which is fonctorial in X. Therefore, if $X \longrightarrow Y$ is a

morphism of simplicial sets, we obtain a canonical commutative square

$$\begin{array}{ccc} X/T & \longrightarrow & Y/T \\ \downarrow & & \downarrow \\ X/S & \longrightarrow & Y/S \end{array}$$

and thus a canonical map $X/T \longrightarrow X/S \times_{Y/S} Y/T$. We leave the next lemma as an exercise for the reader.

LEMMA 14.20. *Let $i : A \longrightarrow B$ and $j : S \longrightarrow T$ be two monomorphisms. For any morphism of simplicial sets $p : X \longrightarrow Y$, we have the following correspondence of lifting properties.*

$$\begin{array}{ccc} A * T \cup B * S & \xrightarrow{a} & X \\ \downarrow & \nearrow_{h} & \downarrow p \\ B * T & \xrightarrow[b]{} & Y \end{array} \quad \leftrightsquigarrow \quad \begin{array}{ccc} A & \xrightarrow{\tilde{a}} & X/T \\ {\scriptstyle i}\downarrow & \nearrow_{\tilde{h}} & \downarrow \\ B & \xrightarrow[\tilde{b}]{} & X/S \times_{Y/S} Y/T \end{array}$$

LEMMA 14.21. *Let $p : X \longrightarrow Y$ be an inner fibration. Given any integers $n \geq 1$ and $0 \leq k < n$ and any morphism $t : \Delta^n \longrightarrow X$, the induced map*

$$X/\Delta^n \longrightarrow X/\Lambda^n_k \times_{Y/\Lambda^n_k} Y/\Delta^n$$

is a trivial fibration.

PROOF. Since $0 < m + 1 + k < m + 1 + n$, the third equality in Proposition 14.17 and the preceding lemma explain everything we need to know. □

THEOREM 14.22. *Let $p : X \longrightarrow Y$ be an inner Kan fibration. We consider a monomorphism $S \longrightarrow T$, together with a morphism $t : T \longrightarrow X$. Then the canonical projection*

$$X/T \longrightarrow X/S \times_{Y/S} Y/T$$

is a right fibration. Furthermore, if ever Y is an ∞-category, then so are X/T as well as the fibre product $X/S \times_{Y/S} Y/T$.

PROOF. Let $n \geq 1$ and $0 < k \leq n$. We then have the following correspondence of lifting problems.

$$\begin{array}{ccc} \Lambda^n_k * T \cup \Delta^n * S & \xrightarrow{a} & X \\ \downarrow & \nearrow_{h} & \downarrow p \\ \Delta^n * T & \xrightarrow[b]{} & Y \end{array} \quad \leftrightsquigarrow \quad \begin{array}{ccc} \Lambda^n_k & \xrightarrow{\tilde{a}} & X/T \\ {\scriptstyle i}\downarrow & \nearrow_{\tilde{h}} & \downarrow \\ \Delta^n & \xrightarrow[\tilde{b}]{} & X/S \times_{Y/S} Y/T \end{array}$$

Similarly, we have a correspondence of the form below.

$$\begin{array}{ccc} \Lambda^n_k * T \cup \Delta^n * S & \xrightarrow{a} & X \\ \downarrow & \nearrow_{h} & \downarrow p \\ \Delta^n * T & \xrightarrow[b]{} & Y \end{array} \quad \leftrightsquigarrow \quad \begin{array}{ccc} S & \xrightarrow{a'} & \Delta^n \backslash X \\ {\scriptstyle i}\downarrow & \nearrow_{h'} & \downarrow \\ T & \xrightarrow[b']{} & \Lambda^n_k \backslash X \times_{\Lambda^n_k \backslash Y} \Delta^n \backslash Y \end{array}$$

Therefore, it is sufficient to prove that the map $\Delta^n\backslash X \longrightarrow \Lambda^n_k\backslash X \times_{\Lambda^n_k\backslash Y} \Delta^n\backslash Y$ is a trivial fibration. We even may check this property after applying the functor $W \longmapsto W^{op}$, so that the first assertion follows from lemma 14.21. In the case where Y is a final object, we obtain that the map $X/T \longrightarrow X/S$ is a right fibration, and the case where S is empty tells us that the map $X/T \longrightarrow X$ is a right fibration. Since right fibrations are inner fibrations, we deduce that X/T and X/S always are ∞-categories. The projection of $X/S \times_{Y/S} Y/T$ on X/S is a right fibration as well (as the pullback of the right fibration $Y/T \longrightarrow Y/S$), from which we see that $X/S \times_{Y/S} Y/T$ is an ∞-category. □

Proof of the Coherence Theorem (14.19). Let $p : X \longrightarrow Y$ be an inner fibration between ∞-categories, $S \longrightarrow Y$ a monomorphism, and a commutative square

$$\begin{array}{ccc} \{0\} * T \cup \Delta^1 * S & \xrightarrow{a} & X \\ \downarrow & & \downarrow p \\ \Delta^1 * T & \xrightarrow{b} & Y \end{array}$$

such that the morphism $\alpha : a_0 \longrightarrow a_1$, given by the restriction of a on $\Delta^1 * \varnothing$, is invertible in X. By virtue of Lemma 14.20, it is sufficient to prove that the induced commutative square

$$\begin{array}{ccc} \{0\} & \longrightarrow & X/T \\ \downarrow & & \downarrow \\ \Delta^1 & \xrightarrow{\tilde{\alpha}} & X/S \times_{Y/S} Y/T \end{array}$$

has a filler, where $\tilde{\alpha}$ is the morphism induced by α by transposition. But Theorem 14.22 ensures that the right hand vertical map is a right fibration between ∞-categories, and thus, by virtue of Proposition 14.8, is an isofibration. Therefore, we are reduced to prove that the morphism $\tilde{\alpha}$ is invertible in $X/S \times_{Y/S} Y/T$. On the other hand, by virtue of Theorem 14.22, the canonical morphism $X/S \times_{Y/S} Y/T \longrightarrow X$ is a right fibration between ∞-categories, whence, applying Proposition 14.8 again, is conservative. In other words it is now sufficient to prove that this map sends the morphism $\tilde{\alpha}$ to an invertible morphism in X. But, by definition, it sends $\tilde{\alpha}$ to α, which is invertible by assumption. □

CHAPTER 15

Invertible natural transformations

THEOREM 15.1. *An ∞-category is a Kan complex if and only if it is an ∞-groupoid.*

PROOF. If X is an ∞-category, then the map $X \longrightarrow \Delta^0$ has the right lifting property with respect to horns of the form $\Lambda^n_k \longrightarrow \Delta^n$ with $n = 1$ or $0 < k < n$. This theorem thus follows from applying Theorem 14.18 to the ∞-category X as well as to its opposite X^{op}. □

15.2. Let *Gpd* be the full subcategory of the category of small categories whose objects are the groupoids. The inclusion functor *Gpd* $\longrightarrow$ *Cat* has a left adjoint π_1 and a right adjoint k. For a small category C, the groupoid $\pi_1(C)$ is obtained as the localization of C by all its morphisms, while $k(C)$ is the groupoid whose objects are those of C, and whose morphisms are the invertible morphisms of C. The latter construction can be extended to ∞-categories as follows. For an ∞-category X, we form the pullback square below.

$$\begin{array}{ccc} k(X) & \longrightarrow & X \\ \downarrow & & \downarrow \\ N(k(\tau(X))) & \longrightarrow & N(\tau(X)) \end{array}$$

Remark that the inclusion $k(\tau(X)) \longrightarrow \tau(X)$ is an isofibration, hence so is its nerve. Therefore, the morphism $k(X) \longrightarrow X$ is an inner fibration and $k(X)$ is an ∞-category. It immediately follows from Theorem 5.13 that the map $k(X) \longrightarrow N(k(\tau(X)))$ induces an isomorphism after we apply the functor τ. By virtue of the preceding theorem, the ∞-category $k(X)$ is a Kan complex.

COROLLARY 15.3. *The ∞-category $k(X)$ is the largest Kan complex contained in the ∞-category X.*

In other words, the functor k is a right adjoint of the inclusion functor from the category of Kan complexes to the category of ∞-categories.

COROLLARY 15.4. *For any ∞-category X, the canonical map $k(X) \longrightarrow X$ is a conservative isofibration.*

PROPOSITION 15.5. *Any left (right) fibration whose codomain is a Kan complex is a Kan fibration between Kan complexes.*

PROOF. Indeed, such a map is conservative (Proposition 14.8), so that it is a morphism of Kan complexes. By virtue of Theorem 14.18 (and of its dual version, replacing Λ^n_0 by $\Lambda^n_n = (\Lambda^n_0)^{op}$), any inner fibration between Kan complexes has

the right lifting property with respect to horns of the form $\Lambda^n_k \longrightarrow \Delta^n$ with $n \geq 2$ and $0 \leq k \leq n$. Since left (right) fibrations also are isofibrations, they have the right lifting property with respect to the inclusion $\{\varepsilon\} \longrightarrow J$ for $\varepsilon = 0, 1$. But, for any Kan complex W, any morphism $\Delta^1 \longrightarrow W$ factors through J, and therefore, left (right) fibrations has the right lifring property with respect to $\Lambda^1_k \longrightarrow \Delta^1$ for $k = 0, 1$. □

Corollary 15.6. *Let $p : X \longrightarrow Y$ be a left (or right) fibration. For any object y of Y, the fibre of p at y is a Kan complex.*

Proof. The fibre of p at y is the pull back of p along the map $y : \Delta^0 \longrightarrow Y$ and thus is a left (or right) fibration of the form $q : F \longrightarrow \Delta^0$. Since Δ^0 obviously is a Kan complex, the previous proposition applied to q implies that F must have the same property. □

15.7. Let A be a simplicial set. We write $Ob(A)$ for the constant simplicial set whose set of object is A_0. There is a unique morphism

$$Ob(A) \longrightarrow A$$

which is the identity on set of objects. Let X be an ∞-category. By virtue of Corollary 12.8, the restriction map

$$\begin{aligned} \underline{Hom}(A, X) &\longrightarrow \underline{Hom}(Ob(A), X) = X^{A_0} \\ F &\longmapsto (F(a))_{a \in A_0} \end{aligned}$$

is an inner fibration between ∞-categories.

One defines the ∞-category $k(A, X)$ by forming the the following Cartesian square.

$$\begin{array}{ccc} k(A, X) & \longrightarrow & \underline{Hom}(A, X) \\ \downarrow & & \downarrow \\ k(\underline{Hom}(Ob(A), X)) & \longrightarrow & \underline{Hom}(Ob(A), X) \end{array} \tag{15.7.1}$$

Note that $k(\underline{Hom}(Ob(A), X)) = k(Ob(A), X) = k(X)^{A_0}$ is the maximal Kan complex in X^{A_0}. The ∞-category $k(A, X)$ is the ∞-category of functors $F : A \longrightarrow X$, with morphisms $F \longrightarrow G$ those natural transformations such that, for any object a of A, the induced map $F(a) \longrightarrow G(a)$ is invertible in X. More precisely, a simplex $x : \Delta^n \longrightarrow \underline{Hom}(A, X)$ belongs to $k(A, X)$ if and only if, for any object a of A, the map $x(a) : \Delta^n \longrightarrow X$, obtained by composing x with the evaluation at a, factors through $k(X)$.

For a simplicial set B, we introduce the simplicial set $h(B, X)$ as the subobject of $\underline{Hom}(B, X)$ whose simplices correspond to maps $\Delta^n \longrightarrow \underline{Hom}(B, X)$ such that the associated morphism $B \longrightarrow \underline{Hom}(\Delta^n, X)$ factor through $k(\Delta^n, X)$.

One checks that the bijections $Hom(A, \underline{Hom}(B, X)) \simeq Hom(B, \underline{Hom}(A, X)$ induce canonical bijections

$$Hom(A, h(B, X)) \simeq Hom(B, k(A, X)) . \tag{15.7.2}$$

15.8. Let $p : X \longrightarrow Y$ be an inner fibration between ∞-categories. Let us consider the morphism

$$ev_1 : h(\Delta^1, X) \longrightarrow X \times_Y h(\Delta^1, Y) \tag{15.8.1}$$

induced by the inclusion $\{1\} \subset \Delta^1$.

We remark that the morphism p has the right lifting property with respect to $\partial\Delta^0 \longrightarrow \Delta^0$ (i.e. is surjective on objects) if and only if p is an isofibration. Indeed, we have a correspondence of the following form:

(15.8.2)
$$\begin{array}{ccc} \varnothing & \longrightarrow & h(\Delta^1, X) \\ \downarrow & \nearrow & \downarrow{\scriptstyle ev_1} \\ \Delta^0 & \longrightarrow & X \times_Y h(\Delta^1, Y) \end{array} \quad \leftrightsquigarrow \quad \begin{array}{ccc} \{1\} & \longrightarrow & X \\ \downarrow & \overset{x}{\nearrow} & \downarrow{\scriptstyle p} \\ \Delta^1 & \longrightarrow & Y \end{array} \text{ with } x \text{ invertible.}$$

THEOREM 15.9. *For any inner fibration between ∞-categories $p : X \longrightarrow Y$, the evaluation at 1 map $ev_1 : h(\Delta^1, X) \longrightarrow X \times_Y h(\Delta^1, Y)$ has the right lifting property with respect to inclusions of the form $\partial\Delta^n \longrightarrow \Delta^n$, $n > 0$.*

PROOF. For any map $A \longrightarrow B$ inducing a bijection $A_0 \simeq B_0$ and any ∞-category W, the obvious commutative square

$$\begin{array}{ccc} k(B, W) & \longrightarrow & \underline{Hom}(B, W) \\ \downarrow & & \downarrow \\ k(A, W) & \longrightarrow & \underline{Hom}(A, W) \end{array}$$

is Cartesian. Since the inclusion $\partial\Delta^n \longrightarrow \Delta^n$ induces a bijection on objects, we deduce from this property that there is a correspondence between lifting problems of shape

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & h(\Delta^1, X) \\ \downarrow & \nearrow & \downarrow{\scriptstyle ev_1} \\ \Delta^n & \longrightarrow & X \times_Y h(\Delta^1, Y) \end{array}$$

and lifting problems of shape

$$\begin{array}{ccc} \Delta^1 \times \partial\Delta^n \cup \{1\} \times \Delta^n & \xrightarrow{a} & X \\ \downarrow & \overset{c}{\nearrow} & \downarrow{\scriptstyle p} \\ \Delta^1 \times \Delta^n & \xrightarrow[b]{} & Y \end{array}$$

in which b correspond to an element of $k(\Delta^n, Y)_1$ and the restriction of a to $\Delta^1 \times \partial\Delta^n$ to an element of $k(\partial\Delta^n, X)_1$ (the lifting c, if ever it exists, will always correspond to an element of $k(\Delta^n, X)_1$ thanks to the Cartesian square above for $X = W$, $A = \partial\Delta^n$ and $B = \Delta^n$).

Let us prove the existence of a lifting c (in the presence of the extra hypothesis on a and b as above). We recall that there is a finite filtration of the form

$$\Delta^1 \times \partial\Delta^n \cup \{1\} \times \Delta^n = A_{-1} \subset A_0 \subset \ldots \subset A_n = \Delta^1 \times \Delta^n \,,$$

each step $A_{i-1} \longrightarrow A_i$ being obtained through a biCartesian square of the form

$$\begin{array}{ccc} \Lambda^{n+1}_{i+1} & \xrightarrow{c'_i} & A_{i_1} \\ \downarrow & & \downarrow \\ \Delta^{n+1} & \xrightarrow{c_i} & A_i \end{array}$$

where c_i is induced by the unique order preserving map $[n+1] \longrightarrow [1] \times [n]$ which reaches both points $(0,i)$ and $(1,i)$. In particular, the map $A_{-1} \longrightarrow A_{n-1}$ is an inner anodyne extension, so that we may assume that the map a is the restriction of a map $a' : A_{n-1} \longrightarrow X$ such that pa' equals the restriction of b to A_{n-1}. It is thus sufficient to prove that the commutative square

$$\begin{array}{ccc} \Lambda^{n+1}_{n+1} & \xrightarrow{a'c'_n} & X \\ \downarrow & & \downarrow p \\ \Delta^{n+1} & \xrightarrow[b]{} & Y \end{array}$$

admits a filler. The map $\alpha \in X_1$, corresponding to the image of $\Delta^{\{n-1,n\}}$ by $c'_n a'$ is the image of the map $(0,n) \longrightarrow (1,n)$ by a; in particular, it is invertible. Therefore, by virtue of Theorem 14.18 (applied to $X^{op} \longrightarrow Y^{op}$), the expected filler $\Delta^{n+1} \longrightarrow X$ exists. □

Corollary 15.10. *An inner fibration between ∞-categories $p : X \longrightarrow Y$ is an isofibration if and only if the evaluation at 1 map $ev_1 : h(\Delta^1, X) \longrightarrow X \times_Y h(\Delta^1, Y)$ is a trivial fibration.*

Proof. This follows from (15.8.2) and from Theorem 15.9. □

Theorem 15.11. *Let $p : X \longrightarrow Y$ be an isofibration between ∞-categories. For any monomorphism of simplicial sets $i : A \longrightarrow B$, the induced map*

$$(i^*, p_*) : k(B, X) \longrightarrow k(A, X) \times_{k(A,Y)} k(B, Y)$$

is a Kan fibration between Kan complexes.

Proof. We will prove first that the map $q = (i^*, p_*)$ is a left Kan fibration. More precisely, we will prove that it has the right lifting property with respect to the maps

$$\Delta^1 \times \partial\Delta^n \cup \{1\} \times \Delta^n \longrightarrow \Delta^1 \times \Delta^n \,, \quad n \geq 0 \,.$$

Let us fist remark that the correspondence (15.7.2) induces the following one.

$$\begin{array}{ccc} \{1\} & \longrightarrow & k(B,X) \\ \downarrow & \nearrow & \downarrow q \\ \Delta^1 & \longrightarrow & k(A,X) \times_{k(A,Y)} k(B,Y) \end{array} \qquad \leftrightsquigarrow \qquad \begin{array}{ccc} A & \longrightarrow & h(\Delta^1,X) \\ \downarrow & \nearrow & \downarrow ev_1 \\ B & \longrightarrow & X \times_Y h(\Delta^1,Y) \end{array}$$

Therefore, by virtue of Theorem 15.9, the case where $n = 0$ is done. We may now focus on the case where $n > 0$.

Let us consider a commutative square of the form

$$\begin{array}{ccc} \Delta^1 \times \partial\Delta^n \cup \{1\} \times \Delta^n & \longrightarrow & k(B,X) \\ \downarrow & & \downarrow q \\ \Delta^1 \times \Delta^n & \longrightarrow & k(A,X) \times_{k(A,Y)} k(B,Y) \end{array}$$

It gives rise to a commutative solid square of the form

$$\begin{array}{ccc} B \times \partial\Delta^n \cup A \times \Delta^n & \longrightarrow & h(\Delta^1,X) \\ \downarrow & \overset{l}{\nearrow} & \downarrow ev_1 \\ B \times \Delta^n & \longrightarrow & X \times_Y h(\Delta^1,Y) \end{array}$$

which admits a lift l by Theorem 15.9. It remains to check that the morphism $\tilde{l} : \Delta^1 \times \Delta^n \longrightarrow \underline{Hom}(B,X)$, induced by l, factors through $k(B,X)$. But, by assumption, the restriction of $\tilde{l}$ to $\Delta^1 \times \partial\Delta^n \cup \{1\} \times \Delta^n$ factors through $k(B,X)$. This means that, for each object b of B, in the commutative diagram of $\tau(X)$

$$\tilde{l}(b) = \begin{array}{ccccccc} \tilde{l}_{0,0}(b) & \longrightarrow & \tilde{l}_{0,1}(b) & \longrightarrow & \cdots & \longrightarrow & \tilde{l}_{0,n}(b) \\ \downarrow & & \downarrow & & & & \downarrow \\ \tilde{l}_{1,0}(b) & \longrightarrow & \tilde{l}_{1,1}(b) & \longrightarrow & \cdots & \longrightarrow & \tilde{l}_{1,n}(b) \end{array}$$

all the vertical maps as well as all the maps of the second line are invertible. Therefore, since $n > 0$, all the maps of the first line are invertible, which readily implies that $\tilde{l}$ factors through $k(B,X)$, as required.

The map q is a Kan fibration: we have seen that it is a left fibration, and applying what precedes to $X^{op} \longrightarrow Y^{op}$ shows that q also is a right fibration. The case $A = \varnothing$ and $Y = \Delta^0$ shows that $k(A,X)$ is a Kan complex. The Cartesian square

$$\begin{array}{ccc} k(A,X) \times_{k(A,Y)} k(B,Y) & \longrightarrow & k(B,Y) \\ \downarrow & & \downarrow \\ k(A,X) & \longrightarrow & k(A,Y) \end{array}$$

thus implies that $k(A,X) \times_{k(A,Y)} k(B,Y)$ is a Kan complex. □

COROLLARY 15.12. *For any ∞-category X and any simplicial set A, we have an equality: $k(A,X) = k(\underline{Hom}(A,X))$. In other words, a natural tranformation is invertible if and only if it is object-wise invertible.*

More generally, for any isofibration between ∞-categories $p : X \longrightarrow Y$ and any monomorphism of simplicial sets $A \longrightarrow B$, we have

$$k(A,X) \times_{k(A,Y)} k(B,Y) = k\big(\underline{Hom}(A,X) \times_{\underline{Hom}(A,Y)} \underline{Hom}(B,Y)\big)\,.$$

PROOF. We obviously have $k(\underline{Hom}(A,X)) \subset k(A,X)$. Since, by virtue of the preceding theorem, $k(A,X)$ is a Kan complex, we conclude with Corollary 15.3.

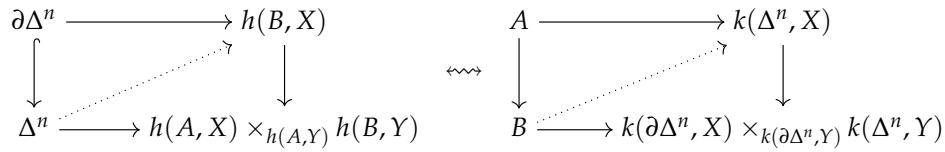
The functor $k(-)$ sends ∞-categories to Kan complexes and isofibration between Kan complexes to Kan fibration. But it is also a right adjoint, hence it preserves the representable limits (such as pullbacks of maps between ∞-categories along isofibrations). In particular it sends $\underline{Hom}(A,X) \times_{\underline{Hom}(A,Y)} \underline{Hom}(B,Y)$ to $k(A,X) \times_{k(A,Y)} k(B,Y)$. □

COROLLARY 15.13. *Let $p : X \longrightarrow Y$ be an isofibration between ∞-categories. For any anodyne extension $A \longrightarrow B$, the induced map*

$$h(B,X) \longrightarrow h(A,X) \times_{h(A,Y)} h(B,Y)$$

is a trivial fibration.

PROOF. We have the correspondence of lifting problems below.



Therefore, we conclude with Theorem 15.11 □

CHAPTER 16

More on the Joyal model category structure

THEOREM 16.1 (Joyal). *A simplicial set is a fibrant object of the Joyal model category structure if and only if it is an ∞-category. A morphism of ∞-categories is a fibration of the Joyal model category structure if and only if it is an isofibration.*

PROOF. For any ∞-category X, any map $J \longrightarrow X$ factor through $k(X)$, because $\tau(J)$ is the contractible groupoid with objects 0 and 1. Therefore, we have

$$\underline{Hom}(J, X) = h(J, X)\,.$$

On the other hand, by Proposition 13.2, the inclusion $\{\varepsilon\} \longrightarrow \Delta^1 \longrightarrow J$ are anodyne extensions for $\varepsilon = 0, 1$. Hence Corollary 15.13 implies that the evaluation at $\varepsilon = 0, 1$

$$ev_\varepsilon : \underline{Hom}(J, X) \longrightarrow X \times_Y \underline{Hom}(J, Y)$$

is a trivial fibration. By virtue of Remark 13.8, this proves the theorem. □

PROPOSITION 16.2. *The class of weak categorical equivalences (i.e. of weak equivalences of the Joyal model category structure) is the smallest class of maps of simplicial sets W satisfying the following conditions.*

(i) *The class W has the two-out-of-three property.*
(ii) *Any inner anodyne extension is in W.*
(iii) *Any trivial fibration between ∞-categories is in W.*

PROOF. Let W be such a class of maps, and let us show that any weak categorical equivalence is in W. Let $f : X \longrightarrow Y$ be a morphism of simplicial sets. We can form a commutative diagram of the form

$$\begin{array}{ccc} X & \xrightarrow{i} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow{j} & Y' \end{array}$$

in which i and j are inner anodyne morphisms, and X' and Y' both are ∞-categories. The proof of Ken brown's lemma (7.7), applied to $\mathcal{C} = (sSet)^{op}$, shows that f' has a factorization of the form $f = kq$, where k is the section of a trivial fibration p, while q is an isofibration (i.e. a fibration of the Joyal model category structure). Therefore, the map f is a weak categorical equivalence. if and only if q is a trivial fibration. Hence, $f \in W$ whenever f is weak categorical equivalence. □

COROLLARY 16.3. *The class of weak categorical equivalences is closed under finite products.*

PROOF. Since the cartesian product is symmetric and since weak categorical equivalences are stable under composition, it is sufficient to prove that the cartesian product with a given simplicial set A preserves weak categorical equivalence. Let us consider the class of maps W which consists of the morphisms $X \longrightarrow Y$ such that $A \times X \longrightarrow A \times Y$ is a weak categorical equivalence. Then W has the two-out-of-three property, and it contains inner anodyne extensions (as a particular case of Corollary 12.4) as well as trivial fibrations (because the pullback of a trivial fibration is a trivial fibration). Hence it contains the class of weak categorical equivalences. □

COROLLARY 16.4. *For any monomorphisms of simplicial sets $i : K \longrightarrow L$ and $j : U \longrightarrow V$, if either i or j is a weak categorical equivalence, so is the induced map $K \times V \cup L \times U \longrightarrow L \times V$.*

For any trivial cofibration (cofibration) $K \longrightarrow L$, and for any fibration $X \longrightarrow Y$ of the Joyal model category structure, the induced map

$$\underline{Hom}(L, X) \longrightarrow \underline{Hom}(K, X) \times_{\underline{Hom}(K,Y)} \underline{Hom}(L, Y)$$

is a trivial fibration (a fibration, respectively)

PROOF. The second assertion is a direct consequence of the first, which itself follows right away fron the preceding corollary (and from the fact that trivial cofibrations are closed under pushouts). □

REMARK 16.5. In Definition 5.17, one can find the notions of natural transformation and of equivalence of categories. It is clear that natural transformations are the morphisms in the ∞-categories of the form $\underline{Hom}(A, X)$ (with X an ∞-category) and it follows from Corollary 15.12 that the invertible natural transformations precisely are the invertible maps in $\underline{Hom}(A, X)$. Note finally that a natural transformation $\Delta^1 \longrightarrow \underline{Hom}(A, X)$ is invertible if and only if it factors through a map $J \longrightarrow \underline{Hom}(A, X)$, which corresponds to an homotopy $J \times A \longrightarrow X$. In other words, two functors $u, v : A \longrightarrow X$ are related by an invertible natural transformation if and only if they are J-homotopic. Hence the equivalences of ∞-categories precisely are the J-homotopy equivalences.

COROLLARY 16.6. *A functor between ∞-categories is an equivalence of ∞-categories if and only if it is a weak equivalence of the Joyal model category structure.*

PROOF. By virtue of Corollary 7.18 since any simplicial set is cofibrant, and since ∞-categories are fibrant, the weak equivalences between ∞-categories for the Joyal model category structures precisely are the homotopy equivalences. And by virtue of Lemma 7.12, we can pick any of our favourite cylinder to define the notion of homotopy; e.g. the cartesian product with J. In that case, the preceding remark explains why the notion of equivalence of ∞-categories is nothing else than the notion of homotopy equivalence between cofibrant-fibrant object in the Joyal model category structure. □

THEOREM 16.7. *Let $f : X \longrightarrow Y$ be a morphism of simplicial sets. The following conditions are equivalent.*

(i) *The map f is a weak categorical equivalence.*

(ii) *For any ∞-category W, the map $f^* : \underline{Hom}(Y, W) \longrightarrow \underline{Hom}(X, W)$ is an equivalence of ∞-categories.*
(iii) *For any ∞-category W, the functor $f^* : \tau\underline{Hom}(Y, W) \longrightarrow \tau\underline{Hom}(X, W)$ is an equivalence of categories.*
(iv) *For any ∞-category W, the map $f^* : k(Y, W) \longrightarrow k(X, W)$ is an equivalence of ∞-groupoids (or, equivalently, a weak homotopy equivalence).*

Proof. By definition of the Joyal model category structure, and, by virtue of Theorem 16.1, the map f is a weak categorical equivalence if and only if, for any ∞-category W, the induced map $f^* : [Y, W] \longrightarrow [X, W]$ is bijective, where $[X, W]$ is the set of maps from X to W up to J-homotopy equivalence. Since $\underline{Hom}(X, W)$ is an ∞-category, the explicit description of $\tau\underline{Hom}(X, W)$ given by Theorem 5.13 shows that $[X, W]$ can alternatively be described as the set of isomorphism classes of objects in the category $\tau\underline{Hom}(X, W)$. Remark 16.5 explains why a third description of the set $[X, W]$ is the one given by the set of connected components of the Kan complex $k(X, W)$. Therefore, it is clear that either of conditions (ii), (iii) or (iv) implies condition (i). By virtue of Proposition 13.14, the functor τ sends weak categorical equivalences (in particular, equivalences of ∞-categories) to equivalences of categories. Hence it is clear that condition (ii) implies condition (iii). Let us prove that condition (i) implies condition (ii): by virtue of Kan Brown's Lemma 7.7 and of Corollaries 16.6 and 16.4, the functor $\underline{Hom}(-, W)$ sends weak categorical equivalences to ∞-categories. By virtue of Corollary 15.12, it remains to prove that condition (ii) implies condition (iii). It is sufficient to prove that the functor k sends equivalences of ∞-categories to equivalences of ∞-groupoids. For this, using Ken Brown's Lemma, it is sufficient to prove that the functor k sends trivial fibrations between ∞-categories to trivial fibrations between Kan complexes. Let $p : X \longrightarrow Y$ be a trivial fibration, with Y an ∞-category. Then X is an ∞-category, and since p obviously is conservative (because it is an equivalence of ∞-categories), the commutative square

$$\begin{array}{ccc} k(X) & \longrightarrow & X \\ {\scriptstyle k(p)}\downarrow & & \downarrow{\scriptstyle p} \\ k(Y) & \longrightarrow & Y \end{array}$$

is cartesian, which implies that the induced map $k(X) \longrightarrow k(Y)$ is a trivial fibration. □

With similar arguments, one proves:

Theorem 16.8. *Let $f : X \longrightarrow Y$ be a functor between ∞-categories. The following conditions are equivalent.*

(i) *The functor f is an equivalence of ∞-categories.*
(ii) *For any simplicial set A, the functor $\underline{Hom}(A, X) \longrightarrow \underline{Hom}(A, Y)$ is an equivalence of ∞-categories.*
(iii) *For any simplicial set A, the functor $\tau\underline{Hom}(A, X) \longrightarrow \tau\underline{Hom}(A, Y)$ is an equivalence of categories.*

(iv) *For any simplicial set A, the functor $k(A, X) \longrightarrow k(A, Y)$ is an equivalence of ∞-groupoids.*

CHAPTER 17

The Boardman-Vogt construction, revisited

17.1. Let X be an ∞-category. Given two objects x and y in X, we form the ∞-category $X(x,y)$ of maps from x to y by considering the following pulback square, in which s and t are the evaluation maps at 0 and 1, respectively.

$$\begin{array}{ccc} X(x,y) & \longrightarrow & \underline{Hom}(\Delta^1, X) \\ \downarrow & & \downarrow{\scriptstyle (s,t)} \\ \Delta^0 & \xrightarrow{(x,y)} & X \times X \end{array} \tag{17.1.1}$$

By definition of $k(\Delta^1, X)$, and by virtue of Corollary 15.12, We have in fact two pullback squares

$$\begin{array}{ccccc} X(x,y) & \longrightarrow & k(\underline{Hom}(\Delta^1, X)) & \hookrightarrow & \underline{Hom}(\Delta^1, X) \\ \downarrow & & \downarrow{\scriptstyle (k(s),k(t))} & & \downarrow{\scriptstyle (s,t)} \\ \Delta^0 & \xrightarrow{(x,y)} & k(X) \times k(X) & \hookrightarrow & X \times X \end{array} \tag{17.1.2}$$

in which the middle vertical map is a Kan fibration (hence the left vertical map as well). In other words, we have defined an ∞-groupoid of maps from x to y.

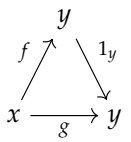
Proposition 17.2. *There is a canonical bijection $\pi_0(X(x,y)) \simeq Hom_{ho(X)}(x,y)$ (where the right hand side is the set of maps in the Boardman-Vogt homotopy category of X; see Theorem 5.13).*

Proof. The canonical morphism $X \longrightarrow N(\hom(X))$ induces a canonical morphism $X(x,y) \longrightarrow N(\hom(X))(x,y) = Hom_{ho(X)}(x,y)$ (where we identify any set with the corresponding constant simplicial set), whence a canonical map

$$X(x,y)_0 \longrightarrow Hom_{ho(X)}(x,y)\,.$$

The domain of this map is the set of maps from x to y in X, and this map simply is the quotient map defining $Hom_{ho(X)}(x,y)$. In particular, this map is surjective and obviously factors through $\pi_0(X(x,y))$ (since it comes from a morphism of simplicial sets). In other words, it is sufficient to check that, for any maps $f, g : x \longrightarrow y$ in X, if f and g agree in $ho(X)$, then they belong to the same path-component of the Kan complex $X(a,b)$. Let us choose a morphism $t : \Delta^2 \longrightarrow X$

whose restriction to $\partial\Delta^2$ correspond to the following diagram.



Let $p : \Delta^1 \longrightarrow \Delta^1 \longrightarrow \Delta^2$ be the nerve of the surjective morphism which sends both $(0,0)$ and $(0,1)$ to 0. Then the composition $tp : \Delta^1 \times \Delta^1 \longrightarrow X$ induces by transposition a map $h : \Delta^1 \longrightarrow X(x,y) \subset \underline{Hom}(\Delta^1, X)$ which connects the points f and g. □

17.3. To understand the composition law better, we need the Grothendieck-Segal condition to hold for ∞-categories. Recall that, for $n \geq 2$, the spine Sp^n is the union of the images of the maps $u_i : \Delta^1 \longrightarrow \Delta^n$ which sends 0 to i and 1 to $i+1$, for $0 \leq i < v$.

Proposition 17.4. *The inclusion $Sp^n \longrightarrow \Delta^n$ is a weak categorical equivalence.*

For the proof, we will need the following lemma.

Lemma 17.5. *Let S be a subset of $\{0, \ldots, n\}$, $n \geq 0$. Assume that the complement of S is not an interval (i.e. that there exists a and c not in S and $b \in S$ such that $a < b < c$). Then the inclusion $\Lambda^n_S \longrightarrow \Delta^n$ is an inner anodyne extension (where Λ^n_S is the union of the images of the faces δ^n_i for $i \notin S$).*

Proof. We proceed by induction on n. For $n = 2$, we must have $S = \{1\}$, in which case the inclusion of $\Lambda^2_S = \Lambda^2_1$ into Δ^2 obviously is an inner anodyne extension. Assume that $n > 2$. We now proceed by induction on the cardinal of S. If S only has one element k, then we have $\Lambda^n_S = \Lambda^n_k$, and we are done (because, for the complement of $\{k\}$ not to be an interval, we need $0 < k < n$). Otherwise, let us choose an element $i \in S$. We shall write T for the set of elements j of $[n-1]$ such that $\delta^n_i(j) \in S$. Then, we have a biCartesian square of the following form.

$$\begin{array}{ccc} \Lambda^{n-1}_T & \hookrightarrow & \Delta^{n-1} \\ \downarrow & & \downarrow{\scriptstyle \delta^n_i} \\ \Lambda^n_S & \hookrightarrow & \Lambda^n_{S-\{i\}} \end{array}$$

Since, by induction on S, the map $\Lambda^n_{S-\{i\}} \longrightarrow \Delta^n$ is an inner anodyne extension, it is sufficient to check that, for an appropriate choice of element i in $[n]$, the map $\Lambda^{n-1}_T \longrightarrow \Delta^{n-1}$ is an inner anodyne extension. By induction on n, it is sufficient to prove that the complement of T in $[n-1]$ is not an interval. Assume that there exists a and c not in S and $b \in S$ such that $a < b < c$. Then we may choose $i \in S$ with $i \neq b$ (our inductive assumptions implies that the cardinal of S is > 1). Then, all elements distinct from i are in the image of δ^n_i. Let α, β and γ be the elements which are sent to a, b and c, respectively. We have $\alpha < \beta < \gamma$ with $\beta \in T$, but α and γ are not in T. □

PROOF OF PROPOSITION 17.4. We first prove by induction that the inclusion $Im(\delta_0^n) \cup Sp^n \longrightarrow \Delta^n$ is an inner anodyne extension for $n \geq 2$. For $n = 2$, we have: $Im(\delta_0^n) \cup Sp^n = \Lambda_1^2$. For $n > 2$, the square

$$\begin{array}{ccc} Im(\delta_0^{n-1}) \cup Sp^{n-1} & \longrightarrow & Im(\delta_0^n) \cup Sp^n \\ \downarrow & & \downarrow \\ \Delta^{n-1} & \xrightarrow{\delta_n^n} & Im(\delta_0^n) \cup Im(\delta_n^n) \end{array}$$

is Cartesian, hence also coCartesian. Therefore, it is sufficient to prove that the inclusion of $Im(\delta_0^n) \cup Im(\delta_n^n)$ into Δ^n is an inner anodyne extension. Since, for $n > 1$, the subset $\{0, n\}$ obviously is not an interval in $[n]$, this readily follows from Lemma 17.5.

By duality (i.e. using the fact that the functor $X \longmapsto X^{op}$ preserves inner anodyne extensions), we also proved that the inclusion $Im(\delta_n^n) \cup Sp^n \longrightarrow \Delta^n$ is an inner anodyne extension for $n \geq 2$.

We now can prove the proposition by induction on n. The case $n = 2$ is obvious, and we may thus assume that $n > 2$. The Cartesian square

$$\begin{array}{ccc} Sp^{n-1} & \longrightarrow & Sp^n \\ \downarrow & & \downarrow \\ \Delta^{n-1} & \xrightarrow{\delta^n - n} & \Delta^n \end{array}$$

implies that the inclusion $Sp^n \longrightarrow Im(\delta_n^n) \cup Sp^n$ is the pushout of the inclusion map $Sp^{n-1} \longrightarrow \Delta^{n-1}$, whence is an inner anodyne extension. Therefore, the composition of the maps

$$Sp^n \longrightarrow Im(\delta_n^n) \cup Sp^n \longrightarrow \Delta^n$$

is an inner anodyne extension. □

COROLLARY 17.6. *For any isofibration between ∞-categories $X \longrightarrow Y$, and for any integer $n \geq 2$, the canonical map*

$$\underline{Hom}(\Delta^n, X) \longrightarrow \underline{Hom}(Sp^n, X) \times_{\underline{Hom}(Sp^n, Y)} \underline{Hom}(\Delta^n, Y)$$

is a trivial fibration. In particular, for any ∞-category X, the restriction map

$$\underline{Hom}(\Delta^n, X) \longrightarrow \underline{Hom}(Sp^n, X)$$

is a trivial fibration.

PROOF. The first assertion is the particular case of Corollary 16.4 obtained from the trivial cofibration of Proposition 17.4. The second assertion is the particular case of the first when $Y = \Delta^0$. □

17.7. Let us consider an ∞-category X, together with a $n+1$-tuple of objects $(x_0, \dots, x_n)$, for $n \geq 2$. We can then form the following pullback squares.

$$\begin{array}{ccccc} X(x_0,\dots,x_n) & \longrightarrow & k(\underline{Hom}(\Delta^n, X)) & \hookrightarrow & \underline{Hom}(\Delta^n, X) \\ \downarrow & & \downarrow & & \downarrow \text{evaluation} \\ \Delta^0 & \xrightarrow{(x_0,\dots,x_n)} & k(X)^{n+1} & \hookrightarrow & X^{n+1} \end{array} \tag{17.7.1}$$

The ∞-groupoids $X(x_0, \dots, x_n)$ is the space of sequences of maps of the form $x_0 \longrightarrow \cdots \longrightarrow x_n$ in X. Remark that we also have a pullback square of the form

$$\begin{array}{ccccc} \prod_{0\leq i<n} X(x_i, x_{i+1}) & \longrightarrow & k(\underline{Hom}(Sp^n, X)) & \hookrightarrow & \underline{Hom}(Sp^n, X) \\ \downarrow & & \downarrow & & \downarrow \text{evaluation} \\ \Delta^0 & \xrightarrow{(x_0,\dots,x_n)} & k(X)^{n+1} & \hookrightarrow & X^{n+1} \end{array} \tag{17.7.2}$$

This implies that the diagram

$$\begin{array}{ccccc} X(x_0,\dots,x_n) & \hookrightarrow & k(\underline{Hom}(\Delta^n, X)) & \hookrightarrow & \underline{Hom}(\Delta^n, X) \\ \downarrow & & \downarrow & & \downarrow \\ \prod_{0\leq i<n} X(x_i, x_{i+1}) & \hookrightarrow & k(\underline{Hom}(Sp^n, X)) & \hookrightarrow & \underline{Hom}(Sp^n, X) \end{array} \tag{17.7.3}$$

is made of Cartesian squares in which, by virtue of Corollary 17.6, all the vertical maps are trivial fibrations. The unique map $\gamma : \Delta^1 \longrightarrow \Delta^n$ which sends 0 to 0 and 1 to n induces a morphism

$$X(x_0, \dots, x_n) \longrightarrow X(x_0, x_n)\,. \tag{17.7.4}$$

The *choice* of a section of the trivial fibration $X(x_0, \dots, x_n) \longrightarrow \prod_{0\leq i<n} X(x_i, x_{i+1})$ composed with the map (17.7.4) thus provide a composition law

$$\prod_{0\leq i<n} X(x_i, x_{i+1}) \longrightarrow X(x_0, x_n)\,. \tag{17.7.5}$$

Applying the functor π_0 to the map (17.7.5) gives a composition law

$$\prod_{0\leq i<n} Hom_{ho(X)}(x_i, x_{i+1}) \longrightarrow Hom_{ho(X)}(x_0, x_n) \tag{17.7.6}$$

which is nothing else than the composition law of Boardman and Vogt's homotopy category $ho(X)$.

CHAPTER 18

Serre's long exact sequence

18.1. Let X be an ∞-groupoid. For any object $x \in X_0$, we define the *loop space* of X at x as

$$\Omega(X,x) = X(x,x)\,. \tag{18.1.1}$$

The *fundamental group* of X at the point x is $\pi_1(X,x) = \pi_0(\Omega(X,x))$ (the group structure is obtained, for instance, from the identification of $\pi_0(\Omega(X,x))$ with the group $Hom_{ho(X)}(x,x)$ given by Proposition 17.2). There is a canonical element of $\Omega(X,x)_0$ which we write, by abuse, anf for historical reasons x (but that really is 1_x). We thus can iterate this construction: we write $\Omega^1(X,x) = \Omega(X,x)$, and, for $n \geq 1$, we define

$$\Omega^{n+1}(X,x) = \Omega(\Omega^n(X,x))\,. \tag{18.1.2}$$

In particular, for each $n \geq 1$, we have defined a group

$$\pi_n(X,x) = \pi_0(\Omega^n(X,x)) \tag{18.1.3}$$

called the *nth homotopy group of X at the point x*.

Proposition 18.2. *For any integer $n \geq 2$, the group $\pi_n(X,x)$ is abelian.*

Proof. It is sufficient to prove the case where $n = 2$. As a particular case of the map (17.7.5), we have a composition law

$$c_X : \Omega(X,x) \times \Omega(X,x) \longrightarrow \Omega(X,x)$$

which, by functoriality of the loop space construction, gives a composition law

$$\Omega(c_X) : \Omega^2(X,x) \times \Omega^2(X,x) \longrightarrow \Omega^2(X,x)\,.$$

On the other hand, we have the composition law

$$c_{\Omega^2(X,x)} : \Omega^2(X,x) \times \Omega^2(X,x) \longrightarrow \Omega^2(X,x)\,.$$

Let us define

$$a \bullet b = \pi_0(\Omega^2(c_X))(a,b) \quad \text{and} \quad a \circ b = \pi_0(c_{\Omega^2(X,x)})(a,b)\,.$$

The pairing $a \circ b$ is the group law on $\pi_2(X,x)$, while $a \bullet b$ is another pairing which is a morphism of groups from $\pi_2(X,x) \times \pi_2(X,x)$ to $\pi_2(X,x)$. In particular, we have:

$$(a \bullet b) \circ (c \bullet d) = (a \circ c) \bullet (b \circ d) \quad \text{and} \quad a \bullet 1_x = a = 1_x \bullet a\,.$$

Hence we have

$$a \bullet b = (a \circ 1_x) \bullet (1_x \circ b) = (a \bullet 1_x) \circ (1_x \bullet b) = a \circ b$$

and also

$$a \circ b = (1_x \bullet a) \circ (b \bullet 1_x) = (1_x \circ b) \bullet (a \circ 1_x) = b \bullet a$$

from which we deduce that $a \bullet b = b \bullet a$. □

PROPOSITION 18.3. *Let us consider a commutative triangle of simplicial sets.*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & S & \end{array}$$

(1) *If, for any map $\Delta^n \longrightarrow S$, the induced morphism $\Delta^n \times_S X \longrightarrow \Delta^n \times_S Y$ is a weak homotopy equivalence, then, for any map $S' \longrightarrow S$, the induced morphism $X' = S' \times_S X \longrightarrow Y' = S' \times_S Y$ is a weak homotopy equivalence (in particular, f itself is a weak homotopy equivalence).*
(2) *If p and q are Kan fibrations, the converse is true: if f is a weak homotopy equivalence, so are the maps $\Delta^n \times_S X \longrightarrow \Delta^n \times_S Y$ for any map $\Delta^n \longrightarrow S$.*

PROOF. Since the functors of the form $(-) \times_S X$ commutes with small colimits and preserves cofibrations, by virtue of Corollaries 8.16, 8.18 and 8.29, the class of maps $S' \longrightarrow S$ such that the induced morphism $S' \times_S X \longrightarrow S' \times_S Y$ is a weak homotopy equivalence is saturated by monomorphisms in the sense of Definition 3.9 (when considered as a class of presheaves over the small category $\mathbf{\Delta}/S$). Corollary 3.10 for $A = \mathbf{\Delta}/S$ tells us that if ever it contains all maps of the form $\Delta^n \longrightarrow S$, this class must be the one of all maps $S' \longrightarrow S$. This proves the first assertion[1].

Let us assume that both p and q are Kan fibrations. Then, for any map of the form $\Delta^n \longrightarrow S$, we may consider the Cartesian squares

$$\begin{array}{ccc} \Delta^n \times_S X & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ \Delta^n & \longrightarrow & S \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \Delta^n \times_S Y & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle q} \\ \Delta^n & \longrightarrow & S \end{array}$$

and see that both of them are homotopy Cartesian (by virtue of Corollary 8.28 and of Proposition 11.23 they have this property after we apply the finite limits preserving functor Ex^∞, and therefore, the functorial trivial cofibration (11.22.5) gives our claim). This implies that the commutative square

$$\begin{array}{ccc} \Delta^n \times_S X & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ \Delta^n \times_S Y & \longrightarrow & Y \end{array}$$

is homotopy Cartesian. Therefore, the left hand vertical map of the latter is a weak homotopy equivalence whenever f has this property. □

[1]Remark that we hardly used any specific property of the Kan-Quillen model category structure: the same proof applies to any model category structure on the category of presheaves of sets over an Eilenberg-Zilber category whose class of cofibrations is the class of monomorphisms

COROLLARY 18.4. *Let us consider a commutative square of simplicial sets*

$$\begin{array}{ccc} X' & \xrightarrow{u} & X \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ Y' & \xrightarrow{v} & Y \end{array}$$

in which the maps p and q are supposed to be Kan fibrations. Such a square is homotopy Cartesian if and only if, for any point $y' \in Y'_0$, if we put $y = v(y')$, the induced morphism between the fibres $X'_{y'} \longrightarrow X_y$ is a weak homotopy equivalence.

PROOF. Since the map p is a Kan fibration the homotopy pullback of Y' and X over Y simply is the ordinary pullback $Y' \times_Y X$. Therefore, this square is homotopy Cartesian if and only if the induced map $f : X' \longrightarrow Y' \times_Y X$ is a weak homotopy equivalence. By virtue of the preceding proposition, this property is thus equivalent to the assertion that, for any map of the form $s' : \Delta^n \longrightarrow Y'$, the induced map

$$\Delta^n \times_{Y'} X' \longrightarrow \Delta^n \times_{Y'} (Y' \times_Y X) \simeq \Delta^n \times_Y X$$

is a weak homotopy equivalence. If we let y' and y be the image of 0 in Y' and Y respectively, we have a commutative square of the form

$$\begin{array}{ccc} X'_{y'} & \longrightarrow & X_y \\ \downarrow & & \downarrow \\ \Delta^n \times_{Y'} X' & \longrightarrow & \Delta^n \times_Y X \end{array}$$

in which the vertical maps are strong deformation retracts (see Proposition 10.6). Therefore, it is sufficient consider pullbacks along maps of the form $\Delta^0 \longrightarrow Y'$ only. □

COROLLARY 18.5. *For any equivalence of ∞-groupoids $f : X \longrightarrow Y$ and any point $x \in X_0$, if we put $y = f(x)$, the induced map $\pi_n(X, x) \longrightarrow \pi_n(Y, y)$ is an isomorphism of groups for $n \geq 1$.*

PROOF. We have a commutative square of the form

$$\begin{array}{ccc} \underline{Hom}(\Delta^1, X) & \xrightarrow{f_*} & \underline{Hom}(\Delta^1, Y) \\ \downarrow{\scriptstyle (s,t)} & & \downarrow{\scriptstyle (s,t)} \\ X \times X & \xrightarrow[f \times f]{} & Y \times Y \end{array}$$

in which the horizontal maps are equivalences of ∞-groupoids (by virtue of Ken Brown's Lemma, it sufficient to check this when is a trivial fibrations, in which case thse maps are trivial fibrations as well). Therefore, this commutative square is homotopy Cartesian. Hence the preceding corollary implies that it induces weak homotopy equivalences on fibres of the vertical maps. There latter weak homotopy equivalences precisely are the maps $\Omega(X, x) \longrightarrow \Omega(Y, y)$ for all $x \in X_0$. Iterating the process gives weak equivalences $\Omega^n(X, x) \longrightarrow \Omega^n(Y, y)$ for all

$n \geq 1$. Since the functor π_0 sends weak homotopy equivalences to bijections, this achieves the proof. □

18.6. For an arbitrary simplicial set X, we have a canonical trivial cofibration $X \longrightarrow Ex^\infty(X)$ which induces a bijection $X_0 \simeq Ex^\infty(X)_0$ (because $Sd(\Delta^0) \simeq \Delta^0$). By virtue of the preceding corollary, if we define

$$\pi_n(X,x) = \pi_n(Ex^\infty(X),x) \tag{18.6.1}$$

we extend the definition of homotopy groups to arbitrary simplicial sets in a way which is compatible with the definition given at the beginning of this chapter. Since the functor Ex^∞ sends weak homotopy equivalences to equivalences of ∞-groupoids, we can reformulate the previous corollary as follows.

COROLLARY 18.7. *For any weak homotopy equivalence of simplicial sets $f : X \longrightarrow Y$ and any point $x \in X_0$, if we put $y = f(x)$, the induced map $\pi_n(X,x) \longrightarrow \pi_n(Y,y)$ is an isomorphism of groups for $n \geq 1$.*

18.8. For any simplicial set X, one can freely add a base point by forming the pointed simplicial set $X_+ = X \amalg \Delta^0$ (where the base point corresponds to the new copy of Δ^0). This defines a left adjoint to the forgetful functor from pointed simplicial sets to simplicial sets.

Given two pointed simplicial sets X and Y, with base points x and y, respectively, The simplicial set of pointed maps from X to Y, denoted by $\underline{Hom}_*(X,Y)$ is defined as the pullback

$$\begin{array}{ccc} \underline{Hom}_*(X,Y) & \longrightarrow & \underline{Hom}(X,Y) \\ \downarrow & & \downarrow{\scriptstyle x^*} \\ \Delta^0 & \xrightarrow{\quad y \quad} & Y \end{array} \tag{18.8.1}$$

(where x^* is the evaluation at x).

We also define the smash-product $X \wedge Y$ as the pushout

$$\begin{array}{ccc} X \vee Y & \hookrightarrow & X \times Y \\ \downarrow & & \downarrow \\ \Delta^0 & \longrightarrow & X \wedge Y \end{array} \tag{18.8.2}$$

where $X \vee Y$ is the union of $X \times \{y\}$ and $\{x\} \times Y$ in the cartesian product $X \times Y$. For any three pointed simplicial sets X, Y and Z, we have a natural bijection

$$Hom_*(X \wedge Y, Z) \simeq Hom_*(X, \underline{Hom}_*(Y,Z))$$

induced by the natural bijection $Hom(X \times Y, Y) \simeq Hom(X, \underline{Hom}(Y,Z))$ (where Hom_* denotes the set of morphisms of pointed simplicial sets). Moreover the operation $\wedge$ defines a symmetric monoidal structure on the category of pointed simplicial sets with unit object $S^0 = (\Delta^0)_*$. In particular we have the following canonical isomorphisms (for all pointed simplicial sets X, Y and Z).

$$X \wedge (Y \wedge Z) \simeq (X \wedge Y) \wedge Z \qquad X \wedge Y \simeq Y \wedge X \qquad S^0 \wedge X \simeq X \tag{18.8.3}$$

One defines the *simplicial circle* S^1 with the following coCartesian square.

$$\begin{array}{ccc} \partial\Delta^1 & \hookrightarrow & \Delta^1 \\ \downarrow & & \downarrow \\ \Delta^0 & \longrightarrow & S^1 \end{array} \tag{18.8.4}$$

Then, for any integer $n \geq 2$ we define the *simplicial n-sphere* as:

$$S^n = S^1 \wedge S^{n-1}\,. \tag{18.8.5}$$

One checks directly that, for each pointed Kan complex X we have

$$\Omega^n(X,x) = \underline{Hom}_*(S^n, X) \tag{18.8.6}$$

(and we really mean an equality here).

On the other hand, for any pointed simplicial set A and any pointed Kan complex X, the set $\pi_0(\underline{Hom}_*(A,X))$ is canonically identified with the set of maps from A to X in the homotopy category $ho(\Delta^0\backslash sSet)$ (with respect to the Kan-Quillen model category structure): indeed, this is a particular case of Theorem 7.17, because, using Corollary 8.17, we see right away that $A \wedge (\Delta^1)_+$ is a cylinder of A in the Kan-Quillen model category structure. In particular, we get another proof of Corollary 18.5, but we also have the following property.

PROPOSITION 18.9. *For any pointed weak homotopy equivalence $A \longrightarrow B$ and any pointed Kan complex X, the induced map $\pi_0(\underline{Hom}_*(B,X)) \longrightarrow \pi_0(\underline{Hom}(A,X))$ is bijective.*

PROPOSITION 18.10. *Let X be an ∞-groupoid, and $x \in X_0$ a point. For any integer $n \geq 1$, and for any choice of base point for $\partial\Delta^n$, there is a canonical bijection*

$$\pi_0(\underline{Hom}_*(\partial\Delta^{n+1}, X)) \simeq \pi_n(X,x)$$

such that the constant map with value x corresponds to the unit 1_x of the homotopy group.

PROOF. It is sufficient to prove that $\partial\Delta^{n+1}$ and S^n are isomorphic in the *unpointed* homotopy category of the Kan-Quillen model category structure. Indeed, if this is the case, there will exists a weak homotopy equivalence from $\partial\Delta^{n+1}$ to $Ex^\infty(S^n)$. Since S^n has exactly one 0-simplex (object) and since the trivial cofibration $S^n \longrightarrow Ex^\infty(S^n)$ is bijective on objects, these maps will always be compatible with any choice of base point we make for $\partial\Delta^{n+1}$. We will conclude with the preceding proposition.

We prove this assertion by induction on n. If $n = 0$, this is clear: $S^0 = \partial\Delta^1$. Hence we may assume that $n > 0$. We have the (homotopy) coCartesian square below.

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & \Lambda^{n+1}_{n+1} \\ \downarrow & & \downarrow \\ \Delta^n & \xrightarrow{\delta^{n+1}_{n+1}} & \partial\Delta^{n+1} \end{array}$$

Note that $X \wedge \Delta^0 = \Delta^0$ whatever X is. Therefore, we also have the following (homotopy) coCartesian square.

$$\begin{array}{ccc} S^{n-1} \wedge S^0 & \hookrightarrow & S^{n-1} \wedge \Delta^1 \\ \downarrow & & \downarrow \\ S^{n-1} \wedge \Delta^0 & \hookrightarrow & S^{n-1} \wedge S^1 \end{array} \quad = \quad \begin{array}{ccc} S^{n-1} & \hookrightarrow & S^{n-1} \wedge \Delta^1 \\ \downarrow & & \downarrow \\ \Delta^0 & \hookrightarrow & S^n \end{array}$$

By virtue of Corollary 8.17, the map $X \wedge \Delta^1 \longrightarrow X \wedge \Delta^0 = \Delta^0$ is a weak homotopy equivalence for any pointed simplicial set X. In conclusion, the simplicial sets $\partial\Delta^{n+1}$ and S^n are homotopy pushouts of diagrams of the form

$$\Delta^0 \longleftarrow \partial\Delta^n \longrightarrow \Delta^0 \quad \text{and} \quad \Delta^0 \longleftarrow S^{n-1} \longrightarrow \Delta^0 \,,$$

respectively. By induction, we see that these two diagrams are weakly equivalent, and therefore, so are their homotopy colimits. □

Proposition 18.11. *Let X be a simplicial set. The map $X \longrightarrow \Delta^0$ is a weak homotopy equivalence if and only if the set $\pi_0(X)$ exactly has one element and if there exists a point $x_0 \in X_0$ such that, for any integer $n \geq 1$, the homotopy group $\pi_n(X, x_0)$ is trivial.*

Proof. This clearly is a necessary condition, by Corollary 18.7. For the converse, we may assume that X is a Kan complex. Let us assume that X has exactly one path-component as well as trivial homotopy groups $\pi_n(X, x_0)$ in any degree $n > 0$, for a specified point x_0. We first prove that, for any point x of X, the homotopy groups $\pi_n(X, x)$ are trivial. Indeed, since X is path-connected (and a Kan complex), there exists a map $h : \Delta^1 \longrightarrow X$ such that $h(0) = x_0$ and $h(1) = x$. We can form the following (homotopy) Cartesian square.

$$\begin{array}{ccc} E & \longrightarrow & \underline{Hom}(\Delta^1, X) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle (s,t)} \\ \Delta^1 & \xrightarrow{(h,h)} & X \times X \end{array}$$

Since pulling back a weak homotopy equivalence along a Kan fibration is a weak homotopy equivalence, the two inclusions $\Omega(X, x_0) \longrightarrow E \longleftarrow \Omega(X, x)$ are weak homotopy equivalences. Iterating the process gives a zig-zag of weak homotopy equivalences relating the iterated loop spaces $\Omega^n(X, x_0)$ and $\Omega^n(X, x)$, hence isomorphisms $\pi_n(X, x_0) \simeq \pi_n(X, x)$. By virtue of Proposition 18.10, this means that, for $n > 0$ and for any map $a : \partial\Delta^n \longrightarrow X$, there exists a morphism $h : \Delta^1 \times \Delta^n \longrightarrow X$ which is an homotopy from a to the constant map with value x. If we still denote by $x : \Delta^n \longrightarrow X$ the constant with value x, we thus get a morphism $(h, x) : \Delta^1 \times \partial\Delta^n \cup \{1\} \times \Delta^n \longrightarrow X$. Since X is a Kan complex, this morphism is the restriction of some morphism $k : \Delta^1 \times \Delta^n \longrightarrow X$. The restriction of k to $\Delta^n \simeq \{0\} \times \Delta^n$ defines a morphism $b : \Delta^n \longrightarrow X$ such that $b_{|\partial\Delta^n} = a$. Since $X \neq \varnothing$, this means that $X \longrightarrow \Delta^0$ is a trivial fibration, hence a weak homotopy equivalence. □

THEOREM 18.12. *Let $p : X \longrightarrow Y$ be a Kan fibration. We consider given a point $x \in X_0$ and $y = p(x)$, and we let F be the fibre of p at y. Then there is a canonical long exact sequence pointed sets (of groups, if we restrict to the part of degree ≥ 1) of the form*

$$\cdots \longrightarrow \pi_n(F,x) \longrightarrow \pi_n(X,x) \longrightarrow \pi_n(Y,y) \rightarrowtail \pi_{n-1}(F,x) \longrightarrow \cdots \longrightarrow \pi_1(Y,y) \rightarrowtail \pi_0(F) \longrightarrow \pi_0(X) \longrightarrow \pi_0(Y)$$

PROOF. We first prove that, for any homotopy Cartesian square of simplicial sets of the form

$$\begin{array}{ccc} F & \longrightarrow & E \\ \downarrow & & \downarrow \pi \\ P & \longrightarrow & B \end{array}$$

in which P is weakly contractible (i.e. the map $P \longrightarrow \Delta^0$ is a weak homotopy equivalence), for any point x of E which is sent to the connected component of Y receiving P, we have a short exact sequence of pointed sets:

$$\pi_0(F) \longrightarrow \pi_0(E) \longrightarrow \pi_0(B)$$

(where the base points correspond to the connected component containing (the image of) x). Indeed, we may assume that π is a Kan fibration between Kan complexes, and, by Corollary 18.4, we may also assume that $P = \Delta^0$ (because we can replace F by the fibre of $F \longrightarrow P$). We then have to prove that if an object a of the ∞-groupoid X is sent into the connected component of y in Y, then there is a morphism $a \longrightarrow b$ in X such that $p(b) = y$. This immediately follows from the fact that the map p has the right lifting property with respect to maps of the form $\{0\} \longrightarrow \Delta^1$.

Therefore, we already have a short exact sequence of pointed sets

$$\pi_0(F) \longrightarrow \pi_0(X) \longrightarrow \pi_0(Y)\,.$$

Remark that the loop space $\Omega(X,x)$ fits in a diagram made of (homotopy) Cartesian squares of the form

$$\begin{array}{ccccc} \Omega(X,x) & \longrightarrow & P(X) & \longrightarrow & \underline{Hom}(\Delta^1, X) \\ \downarrow & & \downarrow & & \downarrow (s,t) \\ \Delta^0 & \xrightarrow{x} & X & \xrightarrow{(1_X,x))} & X \times X \\ & & \downarrow & & \downarrow pr_2 \\ & & \Delta^0 & \xrightarrow{x} & X \end{array}$$

Since the map $pr_2(s,t) = t$ is a trivial fibration, so is the map $P(X) \longrightarrow \Delta^0$. This means that $\Omega(X,x)$ is the homotopy limit of the diagram $\Delta^0 \longrightarrow X \longleftarrow \Delta^0$.

Using the fact that homotopy Cartesian squares are closed under composition, this implies that there is a canonical homotopy Cartesian square of the form

$$\begin{array}{ccc} \Omega(Y,y) & \longrightarrow & P \\ \downarrow & & \downarrow \\ F & \hookrightarrow & X \end{array}$$

where P is weakly contractible. Therefore, we also have a short exact sequence of pointed sets

$$\pi_1(Y,y) \longrightarrow \pi_0(F) \longrightarrow \pi_0(X)\,.$$

Applying what precedes to the Cartesian squares

$$\begin{array}{ccc} \Omega^n(F,x) & \longrightarrow & \Omega^n(X,x) \\ \downarrow & & \downarrow \\ \Delta^0 & \hookrightarrow & \Omega^n(Y,y) \end{array}$$

for $n \geq 1$ gives the rest of the long exact sequence. □

Remark 18.13. The natural identification comparing the homotopy pullback of the diagrams

$$\Delta^0 \longrightarrow X \longleftarrow P(X) \quad \text{and} \quad P(X) \longrightarrow X \longleftarrow \Delta^0$$

induces a bijection $\pi_1(X,x) \simeq \pi_1(X,x)$ which is not the identity: this is the inverse map $g \longmapsto g^{-1}$. This is because this inversion can be seen in the construction of $\Omega(X,x) = \underline{Hom}_*(\Delta^1/\partial\Delta^1, X)$: it corresponds to the unique isomorphism $\Delta^{1^{op}} \simeq \Delta^1$ which induces an isomorphism $S^{1^{op}} \simeq S^1$ and thus an isomorphism of groups

$$\pi_1(X,x) \simeq \pi_1(X^{op},x) = \pi_1(X,x)^{op}$$

which is the canonical one. (where, for a group G, the opposite G^{op} is the group defined by the pairing $(g,h) \longmapsto h^{-1}g^{-1}$.

Corollary 18.14. *A morphism of simplicial sets $f : X \longrightarrow Y$ is a weak homotopy equivalence if and only if, it induces a bijection $\pi_0(X) \simeq \pi_0(Y)$ and if, for any integer $n \geq 1$ and any $x \in X_0$, if we put $y = f(x)$, the induced map $\pi_n(X,x) \longrightarrow \pi_n(Y,y)$ is an isomorphism of groups.*

Proof. This is a necessary condition, by Corollary 18.7. Conversely, let us assume that the map f induces a bijection at the levels of π_0 and of homotopy groups, for all base points. Since any map can be factored into a weak homotopy equivalence followed by a Kan fibration, Corollary 18.7 shows that it is sufficient to prove the case where f is a Kan fibration. The preceding theorem then ensures that the fibres of f are Kan complexes F such that $\pi_0(F)$ exactly has one elements and whose homotopy groups all are trivial. By virtue of Proposition 18.11, all the fibres of f are (weakly) contractible. Since strong deformation retracts are stable under pullbacks along Kan fibrations (see Proposition 10.6), we deduce that, for any map $\Delta^n \longrightarrow Y$, the induced map $\Delta^n \times_Y X \longrightarrow \Delta^n$ is a weak equivalence. Therefore, Proposition 18.3 tells us that f is a weak homotopy equivalence. □

CHAPTER 19

Fully faithful and essentially surjective functors

LEMMA 19.1. *Let X be an ∞-category. The formation of the ∞-groupoids $k(A, X)$ satisfies the following operations.*

(1) *For any coCartesian square of simplicial sets*

$$\begin{array}{ccc} A & \xrightarrow{a} & A' \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i'} \\ B & \xrightarrow{b} & B' \end{array}$$

in which i is a monomorphism, the induced square

$$\begin{array}{ccc} k(B', X) & \longrightarrow & k(B, X) \\ \downarrow{\scriptstyle i'^*} & & \downarrow{\scriptstyle i^*} \\ k(A', X) & \longrightarrow & k(A, X) \end{array}$$

is Cartesian and its vertical maps are Kan fibrations.

(2) *For any sequence of monomorpims of simplicial sets*

$$A_0 \longrightarrow A_1 \longrightarrow \cdots \longrightarrow A_n \longrightarrow A_{n+1} \longrightarrow \cdots$$

the induced transition maps $k(A_{n+1}, X) \longrightarrow k(A_n, X)$ are Kan fibrations and the comparision map

$$k(\varinjlim_n A_n, X) \longrightarrow \varprojlim_n k(A_n, X)$$

is an isomorphism.

(3) *For any small family of simplicial sets $(A_i)_{i \in I}$, the induced map*

$$k(\coprod_i A_i, X) \longrightarrow \prod_i k(A_i, X)$$

is an isomorphism.

PROOF. We know that $k(A, X) = k(\underline{Hom}(A, X))$ is the maximal Kan complex contained in the ∞-category $\underline{Hom}(A, X)$. On the other hand, we know from Theorem16.1 and Corollary 16.4 that the functor $\underline{Hom}(-, X)$ sends monomorphisms to isofibrations between ∞-categories. We also know from Theorem 15.11 that the functor k sends isofibrations between ∞-categories to Kan fibrations. On the other hand, since the functor k is right adjoint of the inclusion functors of the category of ∞-groupoids into the category of ∞-categories, it preserves the limits which are representable in the category of ∞-categories. In conclusion, the

properties listed in this lemma are direct consequences of the fact that the functor $\underline{Hom}(-, X)$ preserves small limits. □

THEOREM 19.2. *Let $f : X \longrightarrow Y$ be a functor of ∞-categories. The following conditions are equivalent.*

(i) *The induced maps $k(\Delta^n, X) \longrightarrow k(\Delta^n, Y)$ are equivalences of ∞-groupoids for $n = 0$ and $n = 1$.*
(ii) *The induced maps $k(\Delta^n, X) \longrightarrow k(\Delta^n, Y)$ are equivalences of ∞-groupoids for all non-negative integers n.*
(iii) *The map f is an equivalence of ∞-categories.*

PROOF. The dual versions of Corollaries 8.16, 8.18 and 8.29 (applied to the opposite of the category of simplicial sets, endowed with the Kan-Quillen model category structure) together with the preceding lemma shows that, given such a functor $f : X \longrightarrow Y$, the class of simplicial sets A such that the map $k(A, X) \longrightarrow k(A, Y)$ is an homotopy equivalence between Kan complexes is saturated by monomorphisms. Therefore, applying Corollary 3.10 (for $A = \mathbf{\Delta}$) and Theorem 16.8, we see that condition (ii) is equivalent to condition (iii). Therefore, it is sufficient to prove that condition (i) implies condition (ii). For $n > 1$, we have coCartesian squares of the form

$$\begin{array}{ccc} \Delta^0 & \longrightarrow & Sp^{n-1} \\ \downarrow & & \downarrow \\ \Delta^1 & \longrightarrow & Sp^n \end{array}$$

(where we put $Sp^1 = \Delta^1$). Hence condition (i) and the first assertion of the preceding lemma imply that, for any integer $n > 1$, the induced map $k(Sp^n, X) \longrightarrow k(Sp^n, Y)$ is a weak homotopy equivalence. On the other hand, it follows from Proposition 17.4 that the inclusions $Sp^n \longrightarrow \Delta^n$ induce a commutative diagram

$$\begin{array}{ccc} k(\Delta^n, X) & \longrightarrow & k(\Delta^n, Y) \\ \downarrow & & \downarrow \\ k(Sp^n, X) & \longrightarrow & k(Sp^n, Y) \end{array}$$

in which the vertical maps are trivial fibrations (they are Kan fibrations by Theorem 15.11, as well as weak homotopy equivalences by Theorem 16.7). Hence, condition (ii) is fulfilled. □

DEFINITION 19.3. Let $f : X \longrightarrow Y$ be a functor between ∞-categories.

We say that f is *fully faithful* if, for any objects x and y in X, the induced map

$$X(x, y) \longrightarrow Y(f(x), f(y))$$

is an equivalence of ∞-groupoids.

We say that f is *essentially surjective* if, for any object y in Y, there exists an object x in X as well as an invertible morphism from x to y in Y.

REMARK 19.4. The explicit description of $\tau(X)$ given by Boardman and Vogt (5.13) implies that the functor f is essentially surjective if and only if the induced functor $\tau(f) : \tau(X) \longrightarrow \tau(Y)$ is essentially surjective. Proposition 17.2 also implies that the functor $\tau(f)$ is fully faithful whenever f has this property.

PROPOSITION 19.5. *A functor between ∞-categories $f : X \longrightarrow Y$ is fully faithful if and only if the induced commutative square of ∞-groupoids*

$$\begin{array}{ccc} k(\Delta^1, X) & \longrightarrow & k(\Delta^1, Y) \\ {\scriptstyle (s,t)}\downarrow & & \downarrow{\scriptstyle (s,t)} \\ k(X) \times k(X) & \longrightarrow & k(Y) \times k(Y) \end{array}$$

is homotopy Cartesian.

PROOF. Since both vertical maps are Kan fibrations between Kan complexes (Theorem 15.11), this is a particular case of Corollary 18.4. □

THEOREM 19.6. *A functor between ∞-categories is an equivalence of ∞-categories if and only if it is fully faithful and essentially surjective.*

PROOF. Let $f : X \longrightarrow Y$ be such a functor.

If f is an equivalence of ∞-categories, then the functor $\tau(f) : \tau(X) \longrightarrow \tau(Y)$ is an equivalence of categories (this is implied by Proposition 13.14), hence f is essentially surjective. Moreover, the commutative square of Proposition 19.5 obviously is homotopy Cartesian because its horizontal maps are weak homotopy equivalences (Theorem 16.8), hence f is fully faithful.

Conversely, let us assume that f is fully faithful and essentially surjective. To prove that f is an equivalence of ∞-categories, by virtue of Theorem 19.2 and of Proposition 19.5, it is sufficient to prove that the induced morphism of Kan complexes $k(X) \longrightarrow k(Y)$ is a weak homotopy equivalence. We have a bijection $\pi_0(k(X)) \longrightarrow \pi_0((k(Y))$ because we have an equivalence of categories $\tau(X) \simeq \tau(Y)$ whence a bijection at the level of sets of isomorphism classes of objects. Therefore, by virtue of Corollary 18.14, it is sufficient to prove that, for any object x in X, if we put $y = f(x)$, the induced map $\Omega(k(X), x) \longrightarrow \Omega(k(Y), y)$ is a weak homotopy equivalence. Since f is conservative (because $\tau(f)$ is, being an equivalence of categories), the commutative square

$$\begin{array}{ccc} k(X) & \longrightarrow & k(Y) \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

is Cartesian. In particular, we also have a Cartesian square in the category of ∞-groupoids

$$\begin{array}{ccc} \Omega(k(X),x) & \longrightarrow & \Omega(k(Y),y) \\ \downarrow & & \downarrow \\ X(x,x) & \xrightarrow{f} & Y(y,y) \end{array}$$

in which the vertical maps are both monomorphisms and Kan fibrations: the Kan complex $\Omega(k(X),x)$ is the union of the connected components of the Kan complex $X(x,x)$ corresponding to the elements of $\pi_0(X(x,x)) = Hom_{\tau(X)}(x,x)$ which are the invertible automorphisms of x in the category $\tau(X)$. Since the map $X(x,x) \longrightarrow Y(y,y)$ is a weak homotopy equivalence, so is its (homotopy) pullback $\Omega(k(X),x) \longrightarrow \Omega(k(Y),y)$. □

Corollary 19.7. *The class of weak categorical equivalences is stable under small filtered colimits.*

Proof. Let us check first that the class of homotopy equivalences between Kan complexes is closed under filtered colimits. This makes sense because the class of Kan complexes is stable under filtered colimits (Lemma 11.24). For any pointed finite simplicial set K (such as Δ^1 or $\partial\Delta^1$), the functor $\underline{Hom}_*(K,-)$ preserves filtered colimits, whence the formation of iterated loop spaces $\Omega(X,x)$ is compatible with filtered colimits. Since the functor π_0 obviously preserves filtered colimits as well, this proves that the formation of the groups $\pi_n(X,x)$ is compatible with filtered colimits. The characterization of weak equivalences between Kan complexes given by Corollary 18.14 thus implies that weak equivalences between Kan complexes as closed under filtered colimits.

The proof of Lemma 11.24 also shows that ∞-categories are closed under filtered colimits. Since the functor $\underline{Hom}(\Delta^1,-)$ preserves filtered colimits and since filtered colimits commute with finite limits in sets, the formation of ∞-groupoids of morphisms $X(x,y)$ is compatible with filtered colimits of ∞-categories. The stability of equivalences of ∞-groupoids by filtered colimits implies the stability of fully faithful functors between ∞-categories by filtered colimits. It is an easy exercise to check that essentially surjective functors are stable under filtered colimits. Therefore, by virtue of Theorem 19.6, the class of equivalences of ∞-categories is closed under filtered colimits.

Finally, since the functor $Hom(K,-)$ commutes with filtered colimits for any finite simplicial set K, the explicit construction of the small object argument shows that there is a functorial inner anodyne extension $X \longrightarrow L(X)$ such that the functor take its values in the category of ∞-categories and commutes with filtered colimits. Therefore, the class of weak categorical equivalences is the class of maps whose image by L is an equivalence of ∞-categories. This implies the theorem. □

CHAPTER 20

Catégories fibrées en ∞-groupoïdes

20.1. Let C be a small category. The category of presheaves of sets over C can be embedded in the category Cat/C of categories over C as follows. For a presheaf $X : C^{op} \longrightarrow Set$ one considers the category of elements C/X (see Definition 1.7), which comes equiped with a canonical projection map $C/X \longrightarrow C$. This defines a functor

$$\widehat{C} \longrightarrow Cat/C\,, \qquad X \longmapsto C/X\,. \tag{20.1.1}$$

This functor is fully faithful and admits a left adjoint as well as a right adjoint. The left adjoint is easy: it associates to a functor $F : I \longrightarrow C$ the colimit of the functor

$$I \xrightarrow{F} C \xrightarrow{h} \widehat{C}$$

(where h is the Yoneda embedding). The right adjoint is less enlightening (at least for the authors of this notes) but has the merit of being explicit: it associates to a functor $F : I \longrightarrow C$ the set of functors $u : C/c \longrightarrow I$ such that $Fu : C/c \longrightarrow C$ is the canonical projection.

Therefore, there is a way to think of presheaves as a full subcategory of the category Cat/C. The aim of this section is to introduce a candidate for an analogous subcategory of the category of ∞-categories over a given ∞-category C. Well, in fact over *any* object of the Joyal model category structure, i.e., over any simplicial set.

20.2. Let C be a simplicial set. We denote by $P(C)$ the full subcategory of $sSet/C$ whose objets are the right Kan fibrations of the form $p_F : F \longrightarrow C$. The objects of $P(C)$ will be called the *presheaves over* C. Most of the time, the structural map p_F will be implicitly given, and we shall speak of the presheaf F over C. A morphism of presheaves $F \longrightarrow G$ is a *fibrewise equivalence* if, for any object c of C, the induced map $F_c \longrightarrow G_c$ is an equivalence of ∞-groupoids. Here, F_c is the pullback of the map $c : \Delta^0 \longrightarrow C$ along the structural map p_F; such a fibre F_c always is an ∞-groupoid, by Corollary 15.6.

Theorem 20.3 (Joyal). *There is a unique model category structure on the category $sSet/C$ whose cofibrations are the monomorphisms, and whose fibrant objects are the presheaves over C. Moreover, a morphism of presheaves over C is a fibration if and only if it is a right fibration.*

The proof of the theorem will be given below. This model category structure will be called the *contravariant model category structure*.

20.4. We shall study another presentation of the Kan-Quillen model category structure. Let us consider the interval J' defined as the nerve of the contractible groupoid with objects 0 and 1. This is a contractible Kan complex: this is a Kan complex because this is an ∞-groupoid in which all morphisms are invertible, and it is contractible because the nap $J' \longrightarrow \Delta^0$ is a simplicial homotopy equivalence. Therefore, the projection map $J' \times X \longrightarrow X$ is a trivial fibration for any simplicial set X. The interval J' thus defines an exact cylinder $X \longmapsto J \times X$. Let us consider the smallest class $An^{right}_{J'}$ of J'-anodyne extensions which contains the inclusions of the form $\Lambda^n_k \longrightarrow \Delta^n$ with $n \geq 1$ and $0 < k \leq n$; see Example 9.13. It coincides with the smallest saturated class of morphisms of simplicial sets containing the following two kinds of maps.

(1) Inclusions of the form $J' \times \partial\Delta^n \cup \{\varepsilon\} \times \Delta^n \longrightarrow J' \times \Delta^n$ for $n \geq 0$ and $\varepsilon = 0, 1$.
(2) Inclusions of the form $\Lambda^n_k \longrightarrow \Delta^n$ with $n \geq 1$ and $0 < k \leq n$.

However, according to the next proposition, this description can be dramatically simplified.

Proposition 20.5. *The class $An^{right}_{J'}$ is the class of right anodyne extensions.*

Proof. It is clear that any right anodyne extension belongs to $An^{right}_{J'}$. To prove the converse, it is sufficient to prove that any right fibration has the right lifting property with respect to the generators of the class $An^{right}_{J'}$. These generators are all of the form $i : A \longrightarrow B$, where B is the nerve of small category (in particular, an ∞-category). Let $p : X \longrightarrow Y$ be a right fibration. To solve lifting problems of the form

$$\begin{array}{ccc} A & \xrightarrow{a} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{b} & Y \end{array}$$

we may factor the given commutative square into

$$\begin{array}{ccccc} A & \xrightarrow{\alpha} & B \times_Y X & \xrightarrow{\alpha} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ B & = & B & \xrightarrow{b} & Y \end{array}$$

(where the right hand square is Cartesian). The map q is now a right fibration whose codomain is an ∞-category. In other words, it is sufficient to prove that any right fibration whose codomain is an ∞-category has the right lifting property with respect to the generators of the class $An^{right}_{J'}$. And here is a final translation of the latter problem: we must prove that, for any right fibration $p : X \longrightarrow Y$ with Y an ∞-category, the evaluation at ε map

$$\underline{Hom}(J', X) \longrightarrow X \times_Y \underline{Hom}(J', Y)$$

is a trivial fibration for $\varepsilon = 0, 1$. But such morphism p alway is an isofibration between ∞-categories (Proposition 14.8) hence a fibration of the Joyal model category structure (Theorem 16.1). In particular, it must have the required lifting property: since J' is an ∞-groupoid, we have $h(J', C) = \underline{Hom}(J', C)$ for any ∞-category C, so that we can apply Corollary 15.13. □

Proof of Theorem 20.3. Thanks to Proposition 20.5, this is a particular case of the construction of 10.1, which allows to apply Theorem 9.19 for $A = \mathbf{\Delta}/C$. □

In fact, Proposition 20.5 has several other consequences.

Definition 20.6. A morphism of simplicial sets $u : A \longrightarrow B$ is *final* if, for any simplicial set C and any morphism $p : B \longrightarrow C$, the map $u : (A, pu) \longrightarrow (B, p)$ is a weak equivalence of the contravariant model category structure.

Corollary 20.7. *A monomorphism of simplicial sets is a right anodyne extension if and only if it is a final map. A morphism of simplicial sets is final if and only if it admits a factorization into a right anodyne extension followed by a trivial fibration.*

The class of final maps is the smallest class C *of morphisms of simplicial sets satisfying the following properties:*

(a) *the class* C *is closed under composition;*
(b) *for any pair of composable morphisms* $f : X \longrightarrow Y$ *and* $g : Y \longrightarrow Z$*, if both* f *and* gf *are in* C*, so is* g*;*
(c) *any right anodyne extension belongs to* C*.*

Proof. This is a particular case of Propositions 10.3 and 10.4. □

Corollary 20.8. *Let us consider a pair of composable monomorphisms* $f : X \longrightarrow Y$ *and* $g : Y \longrightarrow Z$*, and assume that* f *is a right anodyne extension. Then the map* g *is a right anodyne extension if and only if the map* gf *has the same property.*

20.9. Let $p_F : F \longrightarrow C$ and $p_G : G \longrightarrow C$ be two morphisms of simplicial sets. We define $[F, G]_C$ by forming the following Cartesian square.

$$\begin{array}{ccc} [F,G]_C & \hookrightarrow & \underline{Hom}(F,G) \\ \downarrow & & \downarrow{\scriptstyle (p_G)_*} \\ \Delta^0 & \xrightarrow{p_F} & \underline{Hom}(F,C) \end{array} \tag{20.9.1}$$

If ever G is a presheaf over C, then $[F, G]_C$ is the fibre of a right fibration (by Proposition 14.5), and therefore is a Kan complex, by Corollary 15.6. In the case of an object c of C, if we write $h(c)$ for the image of the corresponding morphism $c : \Delta^0 \longrightarrow C$, the canonical isomorphism $\underline{Hom}(\Delta^0, C) \simeq C_0$ induces a canonical isomorphism

$$[h(c), G]_C \simeq G_c \tag{20.9.2}$$

Proposition 20.10. *For any monomorphism* $F \longrightarrow F'$ *of simplicial sets over* C*, and for any presheaf* G *over* C*, the induced map* $[F, G]_C \longrightarrow [F', G]_C$ *is a Kan fibration.*

PROOF. Since the source and target of this maps are Kan complexes, by virtue of Proposition 15.5 it is sufficient to prove that it is a right Kan fibration. Since right fibrations are stable under base change, by virtue of Proposition 14.5, the Cartesian square

$$\begin{array}{ccc} [F,G]_C & \hookrightarrow & \underline{Hom}(F,G) \\ \downarrow & & \downarrow \\ [F',G]_C & \hookrightarrow & \underline{Hom}(F',G) \times_{\underline{Hom}(F',C)} \underline{Hom}(F,C) \end{array} \tag{20.10.1}$$

thus implies this proposition. □

PROPOSITION 20.11. *For any weak equivalence of the contravariant model category structure $F \longrightarrow F'$ over C, and for any presheaf G over C, the induced map*

$$[F,G]_C \longrightarrow [F',G]_C$$

is an equivalence of ∞-groupoids.

PROOF. By a variation on the proof of the preceding proposition, one checks that, for any presheaf G over C, the functor $[-,G]_C$ sends right anodyne extensions to trivial fibrations. By virtue of Propositions 9.40 and 20.5, this implies that the functor $[-,G]_C$ is a left Quillen functor from the the contravariant model category structure to opposite of the Kan-Quillen model category structure. In particular, this functor preserves weak equivalences between cofibrant objects (which are all simplicial sets over C), whence the proposition. □

LEMMA 20.12. *For any integer $n \geq 0$, the inclusion $\{n\} \longrightarrow \Delta^n$ is a right anodyne extension.*

PROOF. Let $h : \Delta^1 \times \Delta^n \longrightarrow \Delta^n$ be the morphism induced by the map

$$(\varepsilon, x) \longrightarrow \begin{cases} n & \text{if } \varepsilon = 1, \\ x & \text{else.} \end{cases}$$

Identifying Δ^n with $\{0\} \times \Delta^n$, we have the following commutative diagram

$$\begin{array}{ccccc} \{n\} & \longrightarrow & \Delta^1 \times \{n\} \cup \{1\} \times \Delta^n & \longrightarrow & \{n\} \\ \downarrow & & \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & \Delta^1 \times \Delta^n & \xrightarrow{h} & \Delta^n \end{array}$$

which turns the inclusion $\{n\} \longrightarrow \Delta^n$ into a retract of the right anodyne extension $\Delta^1 \times \{n\} \cup \{1\} \times \Delta^n \longrightarrow \Delta^1 \times \Delta^n$ (see Proposition 14.3). This proves the lemma. □

THEOREM 20.13. *Let $\varphi : F \longrightarrow G$ be a morphism of presheaves over C. The following conditions are equivalent.*

(i) *The morphism φ is a weak equivalence of the contravariant model category structure.*
(ii) *The morphism φ is a fibrewise equivalence.*

(iii) *For any simplicial set X over C, the induced map*

$$[X, F]_C \longrightarrow [X, G]_C$$

is an equivalence of ∞-groupoids.

Proof. By virtue of the identification (20.9.2), it is clear that condition (iii) implies condition (ii). The Cartesian square shows that the functors $[X, -]_C$ preserve trivial fibrations, whence weak equivalences between fibrant objects. In other words, condition (i) implies condition (iii). One deduces from Propositions 20.10 and 20.11 that, for any simplicial set X over C and any presheaf F over C, there is a canonical identification between the set $\pi_0([X, F]_C)$ and the set of homotopy classes of maps from X to F: this is because the functor $[-, F]$ will send any cylinder of X (in the sense of the contravariant model category structure) into a path object of the Kan complex $[X, F]$. This implies that condition (iii) implies condition (i). It now remains to prove that condition (ii) implies condition (iii). For any integer $n \geq 0$, by virtue of the preceding lemma and of Proposition 20.11, for any map $s : \Delta^n \longrightarrow C$, there is a commutative square of the form

$$\begin{array}{ccc} [(\Delta^n, s), F]_C & \longrightarrow & F_{s(n)} \\ \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \varphi_{s(n)}} \\ [(\Delta^n, s), G]_C & \longrightarrow & G_{s(n)} \end{array}$$

in which the horizontal maps are equivalences of ∞-groupoids. Therefore, condition (ii) is equivalent to the condition that, for any map $s : \Delta^n \longrightarrow C$, the induced morphism

$$\varphi_* : [(\Delta^n, s), F]_C \longrightarrow [(\Delta^n, s), G]_C$$

is an equivalence of ∞-groupoids. On the other hand, since the functors of the form $[-, F]_C$ send small colimits of simplicial sets over C to limits of simplicial sets, Proposition 20.10 and Corollaries 8.16, 8.18 and 8.29 (all three applied to the opposite of the Kan-Quillen model category) imply that the class of simplicial sets X over C such that the induced map

$$[X, F]_C \longrightarrow [X, G]_C$$

is an equivalence of ∞-groupoids is saturated by monomorphisms. Applying Corollary 3.10 in the case of $A = \mathbf{\Delta}/C$ thus proves that condition (ii) implies condition (iii). □

Corollary 20.14. *The class of weak equivalences of the contravariant model category structure over C is closed under small filtered colimits.*

Proof. The proof of Lemma shows that the class of right fibrations is closed under small filtered colimits, and that the fibrant resolution functor of the contravariant model category structure over C preserves small filtered colimits. Therefore, it is sufficient to prove that the class of weak equivalences between fibrant objects of the contravariant model category structure over C is closed under small filtered colimits. By virtue of the preceding theorem, this amounts to prove that fibrewise equivalences are closed under filtered colimits. Since filtered colimits

are exact, they are compatible with the formation of fibres of right fibrations. Finally, we simply have to check that the class of equivalences of ∞-groupoids is closed under small filtered colimits, which follows right away from Corollary 19.7 (and was in fact used in is proof). □

CHAPTER 21

Mapping spaces as fibres of slices

21.1. For two simplicial sets X and Y, one defines $X \diamond Y$ by the following push out square

$$\begin{array}{ccc} X \times \partial\Delta^1 \times Y & \longrightarrow & X \amalg Y \\ \downarrow & & \downarrow \\ X \times \Delta^1 \times Y & \longrightarrow & X \diamond Y \end{array} \tag{21.1.1}$$

where the upper horizontal map is the disjoint union of the two canonical projections $X \times Y \longrightarrow X$ and $X \times Y \longrightarrow Y$ interpreted through the identification

$$X \times \partial\Delta^1 \times Y \simeq (X \times Y) \amalg (X \times Y).$$

One checks that the functor $sSet \longrightarrow Y\backslash sSet$, $X \longmapsto (Y \longrightarrow X \diamond Y)$ has a right adjoint $Y\backslash sSet \longrightarrow sSet$, $(t : Y \longrightarrow W) \longmapsto W//t$. Indeed, the functor $(-) \diamond Y$ preserves connected colimits (because one directly checks that it preserves pushouts as well as filtered colimits). The object $W//t$ has to be distinguished from the slice W/t that was introduced earlier (see 14.14). However, this chapter is all about comparing them. If there is no ambiguity on t, we will also write $W//T = W/t$.

Remark that we have $\Delta^1 = \Delta^0 \diamond \Delta^0 = \Delta^0 * \Delta^0$, so that there are canonical maps $X \diamond Y \longrightarrow \Delta^1$ and $X * Y \longrightarrow \Delta^1$. For either of these maps, the fibre at 0 is X, and the fibre at 1 is Y.

Proposition 21.2. *There is a unique natural map $\gamma_{X,Y} : X \diamond Y \longrightarrow X * Y$ such that the diagram below commutes.*

$$\begin{array}{ccc} X \amalg Y & \longrightarrow & X * Y \\ \downarrow & \overset{\gamma_{X,Y}}{\nearrow} & \downarrow \\ X \diamond Y & \longrightarrow & \Delta^1 \end{array}$$

Proof. Recall that, for $n \geq 0$, we have:

$$(X * Y)_n = \coprod_{i+1+j=n} X_i \times Y_j.$$

A map $\Delta^n \longrightarrow X \times \Delta^1 \times Y$ consists of elements $x \in X_n$ and $y \in Y_n$ together with an integer i, which correspond to the map $u : \Delta^n \longrightarrow \Delta^1$ (the integer i is the smallest integer such that $u(i+1) = 1$). There is a unique $\xi : \Delta^i \longrightarrow X$ which is the restriction of x to Δ^i, as well as a unique $\eta : \Delta^j \longrightarrow Y$, with $j = n - i - 1$,

which corresponds to the restriction of y to $\Delta^{\{i+1,\dots,n\}} \simeq \Delta^j$. One associates to such a triple (x,i,y) the element (ξ,η). This defines a commutative square

$$\begin{array}{ccc} X \times \partial\Delta^1 \times Y & \longrightarrow & X \amalg Y \\ \downarrow & & \downarrow \\ X \times \Delta^1 \times Y & \xrightarrow{c_{X,Y}} & X * Y \end{array}$$

over Δ^1, and thus a map $\gamma_{X,Y} : X \diamond Y \longrightarrow X * Y$. This proves the existence. The unicity is clear. □

Proposition 21.3. *The map $X \diamond Y \longrightarrow X * Y$ is a weak categorical equivalence.*

Proof. The map $\gamma_{X,Y}$ is the identity whenever X or Y is empty. Since the functors $(-) \diamond Y$ and $(-) * Y$ preserve connected colimits as well as a monomorphisms, we deduce from Corollaries 8.16, 8.18 and 8.29, that the class of simplicial sets X such that the map $\gamma_{X,Y}$ is a weak categorical equivalence is saturated by monomorphisms. Similarly for the class of Y's such that $\gamma_{X,Y}$ is a weak categorical equivalence. By Corollary 3.10, it is thus sufficient to prove the case where $X = \Delta^m$ and $Y = \Delta^n$. In this case, there is a section s of the map $c = c_{\Delta^m,\Delta^n} : \Delta^m \times \Delta^1 \times \Delta^n \longrightarrow \Delta^m * \Delta^n = \Delta^{m+1+n}$ above Δ^1, defined as follows.

$$s(x) = \begin{cases} (x,0,0) & \text{if } x \leq m, \\ (m,1,x-m-1) & \text{if } x > m. \end{cases}$$

This defines a section σ of the map $\gamma = \gamma_{\Delta^m,\Delta^n}$. We also define a morphism

$$p : \Delta^m \times \Delta^1 \times \Delta^n \longrightarrow \Delta^m \times \Delta^1 \times \Delta^n$$

by the formula $p(x,u,y) = (x,u,uy)$. There is a natural transformation from p to the identity and a natural transformation from p to sc. The map p induces a map $\pi : X \diamond Y \longrightarrow X \diamond Y$, and in $\underline{Hom}(\Delta^m \diamond \Delta^n, \Delta^m \diamond \Delta^n)$, we have a morphism h from π to the identity and a morphism k from π to $\sigma\gamma$. These natural transformations are the identity on the image of s. Therefore, after applying the functor τ, they are equal to a single invertible natural transformations from $\tau(\pi)$ to the identity (which is equal to $\tau(\sigma)\tau(\gamma)$ because $\tau(\gamma)$ is an isomorphism). Let $u : \Delta^m \diamond \Delta^n \longrightarrow C$ be a weak categorical equivalence with C an ∞-category. Then h and k induce invertible morphisms from $u\pi$ to u, and from $u\pi$ to $u\sigma\gamma$, respectively, in the ∞-category $\underline{Hom}(\Delta^m \diamond \Delta^n, C)$. Therefore, the maps $u\pi$, u and $u\sigma\gamma$ must be J-homotopic. Since u is a weak equivalence and $\gamma\sigma$ is the identity, this proves that γ is invertible in the homotopy category of the Joyal model category, and thus is a weak equivalence. □

Proposition 21.4. *The functors $X \diamond (-)$ and $(-) \diamond Y$ preserve weak categorical equivalences.*

Proof. This follows right away from Corollary 8.29, applied to the coCartesian squares of the form (21.1.1). □

Corollary 21.5. *The functors $X * (-)$ and $(-) * Y$ preserve weak categorical equivalences.*

PROOF. This follows right away from the preceding two propositions. □

REMARK 21.6. On can prove the preceding corollary without using Proposition 21.2. Here is an other proof. By Ken Brown's Lemma, it is sufficient to prove that these functors preserve trivial cofibrations of the Joyal model structure. Let $K \longrightarrow L$ be such a trivial trivial cofibration. To check that $K * Y \longrightarrow L * Y$ is a trivial cofibration, it is sufficient to prove that it has the right lifting property with respect to isofibrations between ∞-categories (see Proposition 9.30). But, for such an isofibration between ∞-categories $E \longrightarrow B$, for any map $Y \longrightarrow E$, the induced map $E/Y \longrightarrow E \times_B B/Y$ is a right fibration between ∞-categories (Theorem 14.22) whence an isofibration (Proposition 14.8). Since such an isofibrations have the right lifting property with respect to the trivial cofibrations of the Joyal model category structure (Theorem 16.1), we conclude from Lemma 14.20 that the map $K * Y \longrightarrow L * Y$ has the right lifting property with respect to $E \longrightarrow B$.

The case of the functor $X * (-)$ is deduced from the case of $(-) * X^{op}$.

PROPOSITION 21.7. *Let $p : X \longrightarrow Y$ be a fibration of the Joyal model category (e.g., an isofibration between ∞-categories). We consider a monomorphism $S \longrightarrow T$, together with a morphism $t : T \longrightarrow X$. Then the canonical projection*

$$X//T \longrightarrow X//S \times_{Y//S} Y//T$$

is a right fibration. Furthermore, if ever Y is an ∞-category, then so are $X//T$ as well as the fibre product $X//S \times_{Y//S} Y//T$.

PROOF. As in the proof of Theorem 14.22, it is sufficient to prove that, for any right anodyne extension $U \longrightarrow V$, the induced inclusion

$$V \diamond S \cup U \diamond T = V \diamond S \amalg_{U \diamond S} U \diamond T \longrightarrow V \diamond T$$

is a trivial cofibration of the Joyal model category structure. But we have the commutative square

$$\begin{array}{ccc} V \diamond S \cup U \diamond T & \hookrightarrow & V \diamond T \\ \downarrow & & \downarrow \\ V * S \cup U * T & \hookrightarrow & V * T \end{array}$$

of which the vertical maps are weak categorical equivalences, by a repeated use of Proposition 21.3 (and by Corollary 8.29). Therefore, it is sufficient to prove that the lower horizontal map of the commutative square above is a trivial cofibration of the Joyal model category structure. But, by virtue of Lemma 14.20 and Theorem 14.22, we know that it has the left lifting property with respect to any inner fibration (hence with respect to any fibration of the Joyal model category structure). □

21.8. For Y fixed, the comparison map γ_X of Proposition 21.2 is a natural transformation between left adjoints and thus induce a natural transformation of their right adjoints

$$X/T \longrightarrow X//T\,.$$

PROPOSITION 21.9. *For any ∞-category X and any map $t : T \longrightarrow X$, the canonical comparison map $X/T \longrightarrow X//T$ is an equivalence of ∞-categories.*

PROOF. Both functors $(-) \diamond T$ and $(-) * T$ are left Quillen functors (they preserve monomorphisms as well as weak equivalences by Proposition 21.4 and Corollary 21.5) and the natural transformation $\gamma_{(-),T}$ is a termwise weak equivalence, by Proposition 21.3. Therefore, their right adjoints are right Quillen functors and the corresponding transposed map from $(-)/T$ to $(-)//T$ is a weak equivalence on fibrant objects. This is because the total left derived functors of $(-) \diamond T$ and $(-) * T$ are isomorphic, so that their right adjoints, the total right derived functors of $(-)/T$ and $(-)//T$, must be isomorphic as well. □

COROLLARY 21.10. *Let X be an ∞-category, and $x, y : \Delta^0 \longrightarrow X$ two objects. There is a canonical equivalence of ∞-groupoids from the fibre of the right fibration $X/y \longrightarrow X$ over x to the ∞-groupoid of maps $X(x, y)$.*

PROOF. Since both maps $X/y \longrightarrow X$ and $X//y \longrightarrow X$ are right fibrations, by virtue of Theorem 20.13, the weak equivalence of Proposition 21.9 induces equivalences of ∞-groupoids on their fibres. Therefore, we can replace X/y by $X//y$. By definition, the simplicial set $X//y$ fits in the Cartesian square below.

$$\begin{array}{ccc} X//y & \hookrightarrow & \underline{Hom}(\Delta^1, X) \\ \downarrow & & \downarrow{\scriptstyle ev_1} \\ \Delta^0 & \longrightarrow & X \end{array}$$

Since the square

$$\begin{array}{ccc} X & \xrightarrow{(1_X, y)} & X \times X \\ \downarrow & & \downarrow{\scriptstyle pr_2} \\ \Delta^0 & \xrightarrow{y} & X \end{array}$$

is Cartesian, we thus have a Cartesian square of the following form.

$$\begin{array}{ccc} X//y & \hookrightarrow & \underline{Hom}(\Delta^1, X) \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \times X \end{array}$$

This immediately implies, by construction of $X(x, y)$, that the fibre at x of the right fibration $X//Y \longrightarrow X$ is $X(x, y)$. □

PROPOSITION 21.11. *Let X be an ∞-category endowed with an object x. For any simplicial set A, the canonical functor $\underline{Hom}(A, X/x) \longrightarrow \underline{Hom}(A, X)/x$ is an equivalence of ∞-categories (where we also denote by x the constant functor $A \longrightarrow X$ with value x).*

PROOF. The canonical functor $\underline{Hom}(A, X/x) \longrightarrow \underline{Hom}(A, X)$ sends 1_x to x and thus induces a canonical functor $\underline{Hom}(A, X/x) \longrightarrow \underline{Hom}(A, X)/x$. We have

a commutative square

$$\begin{array}{ccc} \underline{Hom}(A, X/x) & \longrightarrow & \underline{Hom}(A, X)/x \\ \downarrow & & \downarrow \\ \underline{Hom}(A, X//x) & \longrightarrow & \underline{Hom}(A, X)//x \end{array}$$

whose vertical maps are equivalences of ∞-categories (Theorem 16.8 and Proposition 21.9). But we also have the Cartesian square

$$\begin{array}{ccc} X//x & \hookrightarrow & \underline{Hom}(\Delta^1, X) \\ \downarrow & & \downarrow{\scriptstyle ev_1} \\ \Delta^0 & \longrightarrow & X \end{array}$$

and since $\underline{Hom}(A, \underline{Hom}(B, X)) \simeq \underline{Hom}(B, \underline{Hom}(A, X))$ for all simplicial sets A and B, we conclude that the canonical map

$$\underline{Hom}(A, X//x) \longrightarrow \underline{Hom}(A, X)//x$$

is an isomorphism. □

CHAPTER 22

Final objects

DEFINITION 22.1. An object x of a simplicial set X is *final* if the corresponding morphism $x : \Delta^0 \longrightarrow X$ is final (see Definition 20.6).

REMARK 22.2. An object x of X is final if and only if the map $x : \Delta^0 \longrightarrow X$ is a right anodyne extension (this is a particular case of the first assertion of Corollary 20.7).

PROPOSITION 22.3. *Let $f : X \longrightarrow Y$ be a morphism of simplicial sets, and assume that X has a final object x. Then the morphism f is final if and only if the object $f(x)$ is final in Y.*

PROOF. This is a special case of properties (a) and (b) of Corollary 20.7. □

22.4. When we see the category of simplical sets as moinoidal category with the join operation $X * Y$, it acts naturally on the category of pointed simplicial sets as follows. Given a pointed simpliical set (X, x) and a simplicial set S, we define the pointed simplicial set $X *_x S$ by forming the pushout square below.

$$\begin{array}{ccc} \Delta^0 * S & \longrightarrow & \Delta^0 \\ {\scriptstyle x*1_S}\downarrow & & \downarrow{\scriptstyle x'} \\ X * S & \longrightarrow & X *_x S \end{array}$$

The fact that this actually defines an action (in particular, that this operation is associative up to a coherent isomorphism) follows from the property of the join operation $(-) * S$ of commuting with connected colimits.

We define

$$C(X) = X *_x \Delta^0 .$$

LEMMA 22.5. *For any pointed simplicial set (X, x), the object x' is final in $C(X)$.*

PROOF. The map $x' : \Delta^0 \longrightarrow C(X)$ is the push out of the map of joins $x * 1_{\Delta^0} : \Delta^0 * \Delta^0 \longrightarrow X * \Delta^0$. It is thus sufficient to prove that the latter is a right anodyne extension. A morphism $p : A \longrightarrow B$ is a right fibration if and only if, for any object a in A, the induced map $A/a \longrightarrow B/p(a)$ is a trivial fibration: this is because, for any integer $n \geq 0$, we have $\partial\Delta^n * \Delta^0 = \Lambda^{n+1}_{n+1}$ as subobjects of $\Delta^n * \Delta^0 = \Delta^{n+1}$. In particular, for any monomorphism $i : X' \longrightarrow X$, the induced inclusion from $X' * \Delta^0$ into $X * \Delta^0$ is a right anodyne extension. The particular case where $X' = \Delta^0$ thus proves the lemma. □

22.6. For a pointed simplicial set (Y, y), the map $1_y : \Delta^1 = \Delta^0 * \Delta^0 \longrightarrow Y$ defines an object $1_y : \Delta^0 \longrightarrow Y/y$. The association $(Y, y) \longrightarrow (Y/y, 1_y)$ is functorial and actually defines a right adjoint to the functor C. In other words, pointed maps $X \longrightarrow Y/y$ correspond to pointed maps $C(X) \longrightarrow Y$.

PROPOSITION 22.7. *Let X be a simplicial set, and x an object of X. If the canonical map $X/x \longrightarrow X$ has a section s such that $s(x) = 1_x$, then x is a final object of X. In the case where X is an ∞-category, the converse is true.*

PROOF. If there is such a section, by transposition, the inclusion $X \subset C(X)$ has a retracttion $r : C(X) \longrightarrow X$ such that $r(x') = x$. This means that the map $x : \Delta^0 \longrightarrow X$ is a retract of the final map $x' : \Delta^0 \longrightarrow C(X)$, whence that x is a final object.

If X is an ∞-category and if x is is a final object of X, then the commutative square

$$\begin{array}{ccc} \Delta^0 & \xrightarrow{1_x} & X/x \\ {\scriptstyle x}\downarrow & & \downarrow \\ X & = & X \end{array}$$

has a filler because $X/x \longrightarrow X$ is a right fibration, while the left hand vertical map is a right anodyne extension (see Remark 22.2). □

COROLLARY 22.8. *For any object x in a simplicial set X, the object 1_x is final in the slice X/x.*

PROOF. The object Δ^0 is a unital associative monoid with respect to the tensor product defined by the join operation: the unit is $\varnothing \longrightarrow \Delta^1$, while the multiplication is the unique map $\Delta^1 \longrightarrow \Delta^0$. Since the operation $X *_x S$ defines a unital and associative action of simplicial sets, this implies that the functor C has a natural structure of a monad. By transposition, its right adjoint as a structure of a comonad. The map $(X/x)/1_x \longrightarrow X/x$ is the comultiplication of this comonad, and thus has a pointed section s (i.e. a section such that $s(1_x) = 1_{1_x}$). Therefore, the preceding proposition implies that 1_x is a final object in X/x. □

We just proved the following important theorem (one of the many incarnations of the Yoneda Lemma).

THEOREM 22.9. *Let x be an object in an ∞-category X. The map $\Delta^0 \longrightarrow X$, seen as an object of $sSet/X$, has a canonical fibrant replacement in the contravariant model category structure over X, namely the canonical right fibration $X/x \longrightarrow X$. In particular, if y is another object of X, there is a canonical equivalence of ∞-groupoids*

$$[X/x, X/y]_X \longrightarrow X(x, y)\,.$$

PROOF. The first assertion directly follows from the previous corollary. This implies that the map $1_x : \Delta^0 \longrightarrow X/x$ induces a trivial fibration from $[X/x, X/y]$ to the fiber of $X/y \longrightarrow X$ at x (see Propositions 20.10 and 20.11). We conclude with Corollary 21.10. □

THEOREM 22.10. *Let X be an ∞-category. We consider an object ω in X, and write $\pi : X/\omega \longrightarrow X$ for the canonical projection. The following conditions are equivalent.*

(i) *The object ω is final in X.*
(ii) *For any object x in X, the ∞-groupoid of maps $X(x, \omega)$ is contractible.*
(iii) *The map $\pi : X/\omega \longrightarrow X$ is a trivial fibration.*
(iv) *The map $\pi : X/\omega \longrightarrow X$ is an equivalence of ∞-categories.*
(v) *The map $\pi : X/\omega \longrightarrow X$ has a section which sends ω to 1_ω.*
(vi) *Any morphism $\partial\Delta^n \longrightarrow X$, such that $n > 0$ and $u(n) = \omega$, is the restriction of a morphism $v : \Delta^n \longrightarrow X$.*

PROOF. We already know that conditions (i) and (v) are equivalent, by Proposition 22.7. Since π is a fibration between fibrant objects of the Joyal model category structure, it is a weak equivalence of this model structrure if and only if it is a trivial fibration. Therefore, conditions (iii) and (iv) are equivalent. On the other hand, the map π is also a fibration between fibrant objects of the contravariant model category structure over X. Hence it is a trivial fibration if and only if it is a weak equivalence of the contravariant model category structure. By virtue of Theorem 20.13, and of Corollary 21.10, conditions (ii) and (iii) tous are equivalent. Condition (vi) is equivalent to condition (iii): this is because $\partial\Delta^n = \Delta^{n-1} * \varnothing \cup \partial\Delta^{n-1} * \Delta^0$, using Lemma 14.20.

If π is a trivial fibration, then it is final, and since final maps are stable by composition, the map $\omega : \Delta^0 \longrightarrow X$ is final, by Corollary 22.8. Conversely, if ω is a final object, then condition (vi) holds, which implies that the map π has a section which sends ω to a final object (namely 1_ω, by Corollary 22.8). By virtue of Proposition 22.3, such a section is a right anodyne extension. This means that π is a weak equivalence of the contravariant model category structure over X, hence a trivial fibration as well. Therefore, conditions (i) and (iii) are equivalent. □

COROLLARY 22.11. *If ω is a final object in an ∞-category X, then it is a final object in the category $\tau(X)$.*

PROOF. This follows right away from condition (ii) above and from Proposition 17.2. □

COROLLARY 22.12. *The final objects of an ∞-category X form an ∞-groupoid which is either empty or equivalent to the point.*

PROOF. Let F be the full subcategory of X which consists of final objects: this is the subobject of the simplicial set X whose simplices are the maps $s : \Delta^n \longrightarrow X$ such that $s(i)$ is a final object for $0 \leq i \leq n$. Condition (ii) says that the map $F \longrightarrow \Delta^0$ is fully faithful. Therefore, if F is not empty, such a functor is an equivalence of ∞-categories. □

COROLLARY 22.13. *Let x be a final object in an ∞-category. For any simplicial set A, the constant functor $A \longrightarrow X$ with value x is a final object of $\underline{Hom}(A, X)$.*

PROOF. This follows from Proposition 21.11, by using the characterization of final objects given by condition (iv) of Theorem 22.10. □

LEMMA 22.14. *Let X be a simplicial set such that, for any finite partially ordered set E, any map $N(E) \longrightarrow X$ is Δ^1-homotopic to a constant map. Then the map $X \longrightarrow \Delta^0$ is a weak homotopy equivalence.*

PROOF. It follows from Lemmas 11.26 and 11.26 that, for any integers $n \geq 0$ and $i > 0$, the iterated subdivision $Sd^i(\Delta^n)$ is the nerve of a partially ordered set. Therefore, any map $\partial\Delta^n \longrightarrow Ex^\infty(X)$ is homotopic to a constant map: indeed, such a map factors through a map $\partial\Delta^n \longrightarrow Ex^i(X)$ with $i > 0$, which corresponds by adjunction to a map of the form $Sd^i(\Delta^n) \longrightarrow X$. In particular, $\pi_0(X)$ has exactly one element[1]. This also implies that the homotopy groups $\pi_n(X, x)$ are all trivial (they do not depend on the base point, since all objects are isomorphic in the groupoid $\tau(Ex^\infty(X))$). Therefore, the Kan complex $Ex^\infty(X)$ is contractible, by Propositions 18.10 and 18.11. Since we have a canonical trivial cofibration $X \longrightarrow Ex^\infty(X)$, this achieves the proof. □

THEOREM 22.15. *Let ω be an object in an ∞-category X. The following conditions are equivalent.*

(i) *The object ω is final.*
(ii) *For any simplicial set A, the constant functor from A to X with value ω is a final object in the category $\tau(\underline{Hom}(A, X))$.*
(iii) *For any simplicial set A which is isomorphic to the nerve of a finite partially ordered set, the constant functor from A to X with value ω is a final object in the category $\tau(\underline{Hom}(A, X))$.*

PROOF. Corollaries 22.11 and 22.13 show that condition (i) implies condition (ii). Since condition (iii) is obviously a particular case of condition (ii), it remains to prove that condition (iii) implies that ω is a final object of X. Let us assume that ω (or the constant functor with value x) is a final object of $\tau(\underline{Hom}(A, X))$ for any A of the form $A = N(E)$, with E a finite partially ordered set. We want to prove that, for any object x in X, the ∞-groupoid $X(x, \omega)$ is equivalent to the point. For any simplicial set A, since it commutes with limits and preserves Kan fibrations as well as trivial fibrations, the functor $\underline{Hom}(A, -)$ commutes with the formation of homotopy fibres of Kan fibrations between Kan complexes. Therefore, it follows from Corollary 21.10 that the Kan complex $\underline{Hom}(A, X(x, \omega))$ is the homotopy fibre of the map

$$\underline{Hom}(A, X/\omega) \longrightarrow \underline{Hom}(A, X)\,.$$

Hence, by virtue of Proposition 21.11, the Kan complex $\underline{Hom}(A, X(x, \omega))$ is homotopy equivalent to the Kan complex $\underline{Hom}(A, X)(x, \omega)$ (where x and ω also denote the corresponding constant functors). Therefore, the set $\pi_0(\underline{Hom}(A, X(x, \omega)))$ has exactly one element (being the set of maps from x to ω in $\tau(\underline{Hom}(A, X))$). The preceding lemma implies that $X(x, \omega)$ is contractible. We conclude thanks to Theorem 22.10. □

[1]The map $u : \varnothing \longrightarrow Ex^\infty X$ is homotopic to a constant map. This implies that there exists an element x_0 of X_0 such that $u(x) = x_0$ for any element x of $\varnothing$. In particular, X is not empty.

CHAPTER 23

Base change formulas

DEFINITION 23.1. A morphism of simplicial sets $p : A \longrightarrow B$ is *proper* if, for any Cartesian squares of the form

$$\begin{array}{ccccc} A'' & \xrightarrow{u'} & A' & \xrightarrow{u} & A \\ \downarrow{\scriptstyle p''} & & \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ B'' & \xrightarrow{v'} & B' & \xrightarrow{v} & B \end{array} \tag{23.1.1}$$

if v' is final, so is u'.

The proof of the following proposition is an easy exercise left to the reader.

PROPOSITION 23.2. *The class of proper morphisms is stable under composition and under base change.*

LEMMA 23.3. *Let $i : K \longrightarrow L$ be a monomorphism of simplicial sets. Assume that there exists a retraction $r : L \longrightarrow K$ of i as well as a simplicial homotopy $h : \Delta^1 \times L \longrightarrow L$ which is constant on K, from 1_L to ir (i.e., such that $h(1_{\Delta^1} \times i)$ is the composition of the projection from $\Delta^1 \times K$ to K with the map i, and such that $h(\{0\} \times L) = 1_L$ and $h(\{1\} \times L) = ir$). Then the map i is a right anodyne extension.*

PROOF. We have a commutative diagram

$$\begin{array}{ccccc} K & \xrightarrow{u} & \Delta^1 \times K \cup \{1\} \times L & \xrightarrow{k} & K \\ {\scriptstyle i}\downarrow & & \downarrow & & \downarrow{\scriptstyle i} \\ L & \xrightarrow{v} & \Delta^1 \times L & \xrightarrow{h} & L \end{array}$$

in which v corresponds to the identification of L with $\{0\} \times L$, and u and k are the restrictions of v and h, respectively. The map i thus is a retract of the middle vertical inclusion, which is a right anodyne extension, by Proposition 14.3. □

LEMMA 23.4. *For any $n \geq 1$ and $0 < k \leq n$, the inclusion of $\{n\}$ in Λ^n_k is a right anodyne extension.*

PROOF. The homotopy $h : \Delta^1 \times \Delta^n \longrightarrow \Delta^n$ which sends $(1, x)$ to n and $(0, x)$ to x restricts to an homotopy $\Delta^1 \times \Lambda^n_k \longrightarrow \Lambda^n_k$ which is constant on $\{n\}$, from the constant map with value n to the identity of Λ^n_k. We may thus apply the previous lemma. □

PROPOSITION 23.5. *For a morphism of simplicial sets $p : A \longrightarrow B$, the following conditions are equivalent.*

(i) *The morphism p is proper.*
(ii) *For any Cartesian squares of the form* (23.1.1)*, if B'' has a final object whose image by v' is a final object of B', the map u' if final.*
(iii) *For any Cartesian squares of the form* (23.1.1)*, if $B'' = \Delta^0$ and if v' is a final object of B', the map u' if final.*
(iv) *For any Cartesian squares of the form* (23.1.1)*, if v' is a right anodyne extension, so is u'.*
(v) *For any Cartesian squares of the form* (23.1.1) *in which the map v' is the canonical inclusion of Λ^n_k into Δ^n, for $n \geq 1$ and $0 < k \leq n$, the map u' is a right anodyne extension.*

PROOF. A monomorphism is final if and only if it is a right anodyne extension (see Corollary 20.7). Therefore, condition (i) implies condition (iv). We also know that a morphism of simplicial sets is final if and only if it admits a factorization into a right anodyne extension followed by a trivial fibration. Since the class of trivial fibrations is stable under base change, one deduces that condition (iv) implies (i). It is clear that condition (iv) implies condition (v). Since any map of the form $1_b : \Delta^0 \longrightarrow B/b$ is final (Corollary 22.8), we see that condition (iii) is equivalent to condition (ii) as follows. It is clear that condition (ii) implies condition (iii). Conversely, for any Cartesian squares of the form 23.1.1, if B'' has a final object ω whose image by v' is a final object of B', if we write A''_ω for the fibre of p'' at ω, there is a commutative triangle of the form

$$\begin{array}{ccc} & A'_\omega & \\ {\scriptstyle i'}\swarrow & & \searrow{\scriptstyle i} \\ A'' & \xrightarrow{u'} & A' \end{array}$$

in which, assuming (iii), the maps i and i' are right anodyne extensions. Corollary 20.7 thus implies that u' is a right anodyne extension. Condition (ii) implies (iv): by virtue of the preceding lemma, for $n \geq 1$ and $0 < k \leq n$, the object n is final in both Λ^n_k and Δ^n. Since condition (i) clearly implies condition (iii). It is now sufficient to prove that condition (v) implies condition (i).

Let C of maps $v' : B'' \longrightarrow B'$ such that, for any map $v : B' \longrightarrow B$, if we form the Cartesian squares as in diagram (23.1.1), the induced map $u' : A'' \longrightarrow A'$ is final. Since pulling back along p preserves monomorphisms and commutes with colimits over A, we see that the class of monomorphisms which are in C is saturated. Assuming condition (v), the class C contains all inclusions of the form Λ^n_k into Δ^n, for $n \geq 1$ and $0 < k \leq n$. Therefore, it contains all right anodyne extensions. The class C is closed under composition, and, for any pair of morphisms $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$, if both f and gf are in C, so is g. These latter conditions are verified because the class of final maps has these properties. And since the class of final maps is the smallest class with these stability properties containing right anodyne extensions, the second part of Corollary 20.7 implies that any final map is in C. Hence the map p is proper. □

23.6. For any morphism of simplicial sets $p : A \longrightarrow B$, the functor

$$p_! : sSet/A \longrightarrow sSet/B\,, \quad (X \longrightarrow A) \longmapsto (X \longrightarrow B) \tag{23.6.1}$$

has a right adjoint

$$p^* : sSet/B \longrightarrow sSet/A\,, \quad (X \longrightarrow B) \longmapsto (X \times_B A \longrightarrow A)\,. \tag{23.6.2}$$

Since it commutes with small colimits, the latter functor also has a right adjoint

$$p_* : sSet/A \longrightarrow sSet/B\,. \tag{23.6.3}$$

Applying Proposition 9.40 twice, we obtain the following two statements.

Proposition 23.7. *For any morphism of simplicial sets $p : A \longrightarrow B$, the pair $(p_!, p^*)$ is a Quillen adjunction for the contravariant model category structures.*

Proposition 23.8. *If a morphism of simplicial sets $p : A \longrightarrow B$ is proper, the pair (p^*, p_*) is a Quillen adjunction for the contravariant model category structures.*

23.9. Let $RFib(A)$ be the homotopy category of the contravariant model category structure on $sSet/A$. As seen in the proof of Theorem 20.13, it can be described as the category whose objects are the presheaves over A, and such that, for two such presheaves F and G,

$$Hom_{RFib(A)}(F, G) = \pi_0([F, G]_A)\,. \tag{23.9.1}$$

For any morphism of simplicial sets $p : A \longrightarrow B$, we thus have a derived adjunction

$$p_! = \mathbf{L}p_! : RFib(A) \rightleftarrows RFib(B) : \mathbf{R}p^*\,. \tag{23.9.2}$$

For any map $f : X \longrightarrow A$, we have $p_!(X \longrightarrow A) = (X \longrightarrow B)$, and, for any right fibration $g : Y \longrightarrow B$, we have $\mathbf{R}p^*(Y \longrightarrow X) = (A \times_B Y \longrightarrow A)$.

If the map p is proper we also have a derived adjunction

$$p^* = \mathbf{L}p^* : RFib(B) \rightleftarrows RFib(A) : \mathbf{R}p_*\,, \tag{23.9.3}$$

where $p^* = \mathbf{L}p^* = \mathbf{R}p^*$. Proposition 23.8 can be interpreted as a base change formula. Indeed, for any commutative square of simplicial sets

$$\begin{array}{ccc} A' & \xrightarrow{u} & A \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow{v} & B \end{array} \tag{23.9.4}$$

there is a canonical base change map

$$\mathbf{L}q_!\mathbf{R}u^*(X) \longrightarrow \mathbf{R}v^*\mathbf{L}p_!(X) \tag{23.9.5}$$

defined as follows, for any object X of $RFib(A)$. By adjunction, it corresponds to a map

$$\mathbf{R}u^*(X) \longrightarrow \mathbf{R}q^*\mathbf{R}v^*\mathbf{L}p_!(X) \simeq \mathbf{R}(vq)^*\mathbf{L}p_!(X) \simeq \mathbf{R}(pu)^*\mathbf{L}p_!(X) \simeq \mathbf{R}u^*\mathbf{R}p^*\mathbf{L}p_!(X)$$

which is the pull back along u of the unit map $X \longrightarrow \mathbf{R}p^*\mathbf{L}p_!(X)$. Similarly, there is a base change map

$$\mathbf{L}u_!\mathbf{R}q^*(Y') \longrightarrow \mathbf{R}p^*\mathbf{L}v_!(Y') \tag{23.9.6}$$

for any object Y' of $RFib(B')$.

COROLLARY 23.10. *For any Cartesian square of the form* (23.9.4) *in which the map p is proper, the base change map* (23.9.6) *is an isomorphism for all objects Y' of $RFib(B')$.*

PROOF. Since $\mathbf{L}p^* = \mathbf{R}p^*$ whenever p is proper, since the formation of total left derived functors commutes with composition of left Quillen functors, it is sufficient to prove the analogue of this assertion in the non derived case, which consists essentially to see that Cartesian squares are stable by composition. □

A good supply of proper morphisms comes from the following two propositions.

PROPOSITION 23.11. *Any left fibration is proper.*

PROOF. Let C be the class of monomorphisms $j : Y' \longrightarrow Y$ wuch that, for any Cartesian square of the form

$$\begin{array}{ccc} X' & \xrightarrow{i} & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ Y' & \xrightarrow{j} & Y \end{array}$$

in which p is a left fibration, the map i is a right anodyne extension. This class C is saturated, and for any monomorphisms $f : Y' \longrightarrow Y$ and $f' : Y'' \longrightarrow Y'$ such that f' is in C, we have f' in C if and only if ff' is in C. Using Lemma 11.3 and Corollary 20.8, we see that it is sufficient to prove that the class C contains the inclusions of $\{1\} \times Y$ into $\Delta^1 \times Y$ for $Y = \partial\Delta^n$ or $Y = \Delta^n$. Let us consider a Cartesian square of the form

$$\begin{array}{ccc} X' & \xrightarrow{i} & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{j} & \Delta^1 \times Y \end{array}$$

in which p is a left fibration, and j is the inclusion of $\{1\} \times Y$ into $\Delta^1 \times Y$. Let $r : \Delta^1 \times Y \longrightarrow Y$ be the projection, and $h : \Delta^1 \times \Delta^1 \times Y \longrightarrow \Delta^1 \times Y$ be the product of Y with the homotopy from the identity of Δ^1 to the constant map with value 1. The proof of Proposition 10.6 can be repeated *mutatis mutandis* and we see that the map i satisfies the hypothesis of Lemma 23.3, hence is right anodyne. □

PROPOSITION 23.12. *For any simplicial sets A and B, the canonical projection map $A \times B \longrightarrow B$ is proper.*

PROOF. Since the class proper morphisms is stable under pull back, we may assume, without loss of generality, that $B = \Delta^0$. Using property (v) of Proposition 23.5, it is sufficient to prove that, for any $n \geq 1$ and $0 < k \leq n$, the induced map $A \times \Lambda^n_k \longrightarrow A \times \Delta^n$ is a right anodyne extension, by Proposition 14.3. □

In order to go further, we need to consider the dual version of the notions of final maps and of proper maps.

DEFINITION 23.13. A morphism of simplicial sets $X \longrightarrow Y$ is *cofinal* if the induced morphism $X^{op} \longrightarrow Y^{op}$ is final. An object x of a simplicial set X is *initial* if the corresponding map $x : \Delta^0 \longrightarrow X$ is cofinal.

A *copresheaf* over a simplicial set C is a left fibration of the form $F \longrightarrow C$, seen as an object of $sSet/C$. A morphism of copresheaves $F \longrightarrow G$ over C is a *fibrewise equivalence* if, for any object c of C, the induced maps on the fibers $F_c \longrightarrow G_c$ is an equivalence of ∞-groupoids.

THEOREM 23.14 (Joyal). *There is a unique model category structure on the category $sSet/C$ whose cofibrations are the monomorphisms, and whose fibrant objects are the copresheaves over C. Moreover, a morphism of copresheaves over C is a fibration if and only if it is a right fibration. Finally, a morphism between copresheaves is a weak equivalence if and only if it is a fibrewise equivalence.*

This model category structure will be called the *covariant model category structure over C*.

PROOF. This comes from translating Theorems 20.3 and 20.13 through the equivalence of categories $X \longmapsto X^{op}$. □

DEFINITION 23.15. A morphism of simplicial sets $p : A \longrightarrow B$ is *smooth* if the induced morphism $A^{op} \longrightarrow B^{op}$ is proper.

REMARK 23.16. The dual version of 23.5 holds; we leave to the reader the task of translating the precise statement and proof. Similarly, the dual versions of Propositions 23.11 and 23.12 give us examples of smooth maps: any right fibration is smooth, and so is any Cartesian projection $A \times B \longrightarrow B$.

PROPOSITION 23.17. *For any morphism of simplicial sets $p : A \longrightarrow B$, the pair $(p_!, p^*)$ is a Quillen adjunction for the covariant model category structures.*

PROPOSITION 23.18. *If a morphism of simplicial sets $p : A \longrightarrow B$ is smooth, the pair (p^*, p_*) is a Quillen adjunction for the covariant model category structures.*

23.19. Let $LFib(A)$ be the homotopy category of the covariant model category structure on $sSet/A$. Since the functor $X \longmapsto X^{op}$ is a Quillen equivalence from the contravariant model structure over A^{op} to the covariant model category structure over A, we have a canonical equivalence of categories

$$RFib(A^{op}) \simeq LFib(A)\,. \tag{23.19.1}$$

The constructions of paragraph 23.9 correspond to the following ones. For any morphism of simplicial sets $p : A \longrightarrow B$, we thus have a derived adjunction

$$p_! = \mathbf{L}p_! : LFib(A) \rightleftarrows LFib(B) : \mathbf{R}p^*\,. \tag{23.19.2}$$

If the map p is smooth we also have a derived adjunction

$$p^* = \mathbf{L}p^* : RFib(B) \rightleftarrows RFib(A) : \mathbf{R}p_*\,, \tag{23.19.3}$$

where $p^* = \mathbf{L}p^* = \mathbf{R}p^*$.

For any commutative square of simplicial sets

$$\begin{array}{ccc} A' & \xrightarrow{u} & A \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow{v} & B \end{array} \tag{23.19.4}$$

there is a canonical base change map

$$\mathbf{L}q_!\mathbf{R}u^*(X) \longrightarrow \mathbf{R}v^*\mathbf{L}p_!(X) \tag{23.19.5}$$

for any object X of $LFib(A)$. There is also a base change map

$$\mathbf{L}u_!\mathbf{R}q^*(Y') \longrightarrow \mathbf{R}p^*\mathbf{L}v_!(Y') \tag{23.19.6}$$

for any object Y' of $LFib(B')$.

Corollary 23.20. *For any Cartesian square of the form* (23.19.4) *in which the map v is smooth, the base change map* (23.19.6) *is an isomorphism for all objects X of $LFib(A)$.*

23.21. For a simplicial set X and an object F of $LFib(X)$, we define

$$\int_A F = \mathbf{L}p_!(F) \tag{23.21.1}$$

where $p : A \longrightarrow \Delta^0$ denotes the structural map. We define the category $ho(S)$ as the homotopy category of the Kan-Quillen model category structure. In other words, we have

$$ho(S) = LFib(\Delta^0) = RFib(\Delta^0)\,. \tag{23.21.2}$$

The object $\int_A F$ is thus an object of $ho(S)$.

Lemma 23.22. *Let A be a simplicial set endowed with a final object a. Then there is a canonical isomorphism $\mathbf{R}a^*(F) \simeq \int_A F$ in the homotopy category $ho(S)$.*

Proof. Up to isomorphism, we may assume that F is represented by a left fibration $p : X \longrightarrow A$ whose fibre over a will be written X_a. Since, by virtue of Proposition 23.11, p is proper, and since $a : \Delta^0 \longrightarrow A$ is a right anodyne extension, the induced map $X_a \longrightarrow X$ is a right anodyne extension, hence an anodyne extension, and therefore defines an invertible morphism from X_a to X in $ho(S)$. But, but, by construction, we have $\int_A F = X$ and $\mathbf{R}a^*(F) = X_a$. □

Lemma 23.23. *Let $p : X \longrightarrow A$ be a proper morphism, and a an object of A. For any object F of $LFib(X)$, the base change morphism*

$$\int_{X_a} F_{|X_a} \longrightarrow \mathbf{R}a^*\mathbf{L}p_!(F)$$

is invertible, where $F_{|X_a} = \mathbf{R}i^*(F)$, with $i : X_a \longrightarrow X$ the embedding of the fibre of p over a.

Proof. Let us choose a fibrant replacement $\tilde{a} : \tilde{A}_{/a} \longrightarrow A$ of $a : \Delta^0 \longrightarrow A$ in the contravariant model category structure over A. In particular, the map a

factors through a final map $\ell_a : \Delta^0 \longrightarrow \tilde{A}_{/a}$. We form the following Cartesian squares.

$$\begin{array}{ccccc} X_a & \xrightarrow{i} & \tilde{A}_{/a} \times_A X & \xrightarrow{\xi} & X \\ \downarrow & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ \Delta^0 & \xrightarrow{\ell_a} & \tilde{A}_{/a} & \xrightarrow{\tilde{a}} & A \end{array}$$

Since $\tilde{a}$ is a right fibration, it is smooth. Therefore, the base change map

$$\mathbf{L}q_!\mathbf{R}\xi^*(F) \longrightarrow \mathbf{R}\tilde{a}^*\mathbf{L}p_!(F)$$

is invertible in $LFib(\tilde{A}_{/a})$. Hence we have an invertible map in $ho(S)$ of the form:

$$\int_{\tilde{A}_{/a}\times_A X} \mathbf{R}\xi^*(F) \simeq \int_{\tilde{A}_{/a}} \mathbf{L}q_!\mathbf{R}\xi^*(F) \longrightarrow \int_{\tilde{A}_{/a}} \mathbf{R}\tilde{a}^*\mathbf{L}p_!(F)\,.$$

The preceding lemma provides a canonical isomorphism

$$\mathbf{R}a^*\mathbf{L}p_!(F) \simeq \mathbf{R}\ell_a^*\mathbf{R}\tilde{a}^*\mathbf{L}p_!(F) \longrightarrow \int_{\tilde{A}_{/a}} \mathbf{R}\tilde{a}^*\mathbf{L}p_!(F)\,.$$

On the other hand, we may assume that F is represented by a left fibration of the form $\pi : Y \longrightarrow X$. Then $p\pi : Y \longrightarrow A$ is proper (as the composition of two proper maps), which implies that the inclusion $X_a \times_X Y \longrightarrow \tilde{A}_{/a} \times_A Y$ is a (right) anodyne extension, hence an invertible map in $ho(S)$. This map thus provides an invertible map of the form

$$\int_{X_a} F_{|X_a} \longrightarrow \int_{\tilde{A}_{/a}\times_A X} \mathbf{R}\xi^*(F)$$

in $ho(S)$. Since the square

$$\begin{array}{ccc} \int_{X_a} F_{|X_a} & \xrightarrow{\sim} & \int_{\tilde{A}_{/a}\times_A X} \mathbf{R}\xi^*(F) \\ \downarrow & & \downarrow{\scriptstyle \wr} \\ \mathbf{R}a^*\mathbf{L}p_!(F) & \xrightarrow{\sim} & \int_{\tilde{A}_{/a}} \mathbf{R}\tilde{a}^*\mathbf{L}p_!(F) \end{array}$$

commutes in $ho(S)$ (because it is the image of a commutative square in *sSet*), this proves the lemma. □

THEOREM 23.24. *For any Cartesian square of the form* (23.19.4) *in which the map p is proper, the base change map* (23.19.5) *is an isomorphism for all objects X of $LFib(A)$.*

PROOF. A map in $LFib(B')$ is invertible if and only if its image by the functor $\mathbf{R}x^*$ is invertible in $LFib(\Delta^0) = ho(S)$ for any object x of B'. Indeed, we may assume that such a map comes from a morphism of copresheaves over B', and we see that we are reformulating the last assertion of Theorem 23.14. Therefore, to prove that the base change map (23.9.6) is an isomorphism, it is sufficient to prove that, for any object x in B', the induced map

$$\mathbf{R}x^*\mathbf{L}q_!\mathbf{R}u^*(X) \longrightarrow \mathbf{R}x^*\mathbf{R}v^*\mathbf{L}p_!(X) \simeq \mathbf{R}v(x)^*\mathbf{L}p_!(X)$$

is invertible. But, applying the preceding lemma twice, we see that this map is isomorphic to the identity of $\int_{A_{v(x)}} X_{|A_{v(x)}}$. □

COROLLARY 23.25. *Let $p : A \longrightarrow B$ be a proper morphism, and consider a commutative square of simplicial sets of the form*

$$\begin{array}{ccc} X & \xrightarrow{\phi} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{\psi} & B \end{array}$$

in which both ϕ and ψ are left fibrations and f is a cofinal map. Then, for any map $v : B' \longrightarrow B$, the induced map

$$g : X' = B' \times_B X \longrightarrow Y' = B' \times_B Y$$

is cofinal.

PROOF. We form the pullback square

$$\begin{array}{ccc} A' & \xrightarrow{u} & A \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow{v} & B \end{array}$$

and then apply the preceding theorem: the base change map

$$\mathbf{L}q_!\mathbf{R}u^*(X) \longrightarrow \mathbf{R}v^*\mathbf{L}p_!(X)$$

is an isomorphism in $LFib(B')$. Unpacking the construction of the involved derived functors, we see that this base change map is the image in $LFib(B')$ of the map g. Therefore, the morphism g is a weak equivalence of the covariant model category structure over B'. If we choose a factorization of g into a cofinal map i followed by a left fibration π, we get that π is a trivial fibration (it is a left fibration between copresheaves, hence a fibration and a weak equivalence of the covariant model structure), so that g is a cofinal map (applying the second assertion of Corollary 20.7 to g^{op}). □

Dually, the have the following two statements.

THEOREM 23.26. *For any Cartesian square of the form* (23.9.4) *in which the map v is smooth, the base change map* (23.9.5) *is an isomorphism for all objects X of $RFib(A)$.*

COROLLARY 23.27. *Let $p : A \longrightarrow B$ be a smooth morphism, and consider a commutative square of simplicial sets of the form*

$$\begin{array}{ccc} X & \xrightarrow{\phi} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{\psi} & B \end{array}$$

in which both ϕ and ψ are right fibrations and f is a final map. Then, for any map $v : B' \longrightarrow B$, the induced map

$$g : X' = B' \times_B X \longrightarrow Y' = B' \times_B Y$$

is final.

23.28. Let A be a simplicial set A. Given two objects E and F of $sSet/A$, corresponding to maps $p : X \longrightarrow A$ and $q : Y \longrightarrow A$, respectively, we define

$$\langle E, F \rangle = X \times_A Y . \tag{23.28.1}$$

This defines a functor

$$\langle -, - \rangle : sSet/A \times sSet/A \longrightarrow sSet . \tag{23.28.2}$$

If E is a right fibration, then the functor $\langle E, - \rangle$ is a left Quillen functor from the covariant model category structure over A to the Kan-Quillen model category structure; similarly, if F is a left fibration, the functor $\langle -, F \rangle$ is a left Quillen functor from the contravariant model category structure over A to the Kan-Quillen model category structure. In particular, the functor (23.28.2) preserves weak equivalences between fibrant objects on the product $sSet/A \times sSet/A$, where we consider the contravariant model category structure on the first factor, and the covariant model structure on the second factor. Therefore, we have a total right derived functor

$$\mathbf{R}\langle -, - \rangle : RFib(A) \times LFib(A) \longrightarrow ho(S) . \tag{23.28.3}$$

When either E is a right fibration or F is a left fibration, we have a canonical isomorphism

$$\langle E, F \rangle \simeq \mathbf{R}\langle E, F \rangle \tag{23.28.4}$$

in $ho(S)$. Given an object a of A and an object F of $LFib(A)$, we define

$$F_{/a} = \mathbf{R}\langle (\Delta^0, a), F \rangle . \tag{23.28.5}$$

Dually, for an object E of $RFib(A)$, we define

$$E_{a/} = \mathbf{R}\langle E, (\Delta^0, a) \rangle . \tag{23.28.6}$$

In other words, we may construct $F_{/a} = \tilde{A}_{/a} \times_A Y$ and $E_{a/} = X \times_A \tilde{A}_{a/}$, where $\tilde{A}_{/a} \longrightarrow A$ and $\tilde{A}_{a/} \longrightarrow A$ are fibrant replacements of $a : \Delta^0 \longrightarrow A$ in the contravariant and covariant model category structure over A, respectively.

Proposition 23.29. *Let us consider a morphism $\varphi : F \longrightarrow G$ in $LFib(A)$. The following conditions are equivalent.*

(i) *The morphism φ is an isomorphism in $LFib(A)$.*
(ii) *For any object E of $RFib(A)$, the induced morphism $\mathbf{R}\langle E, F \rangle \longrightarrow \mathbf{R}\langle E, G \rangle$ is an isomorphism in $ho(S)$.*
(iii) *For any object a of A, the induced map $\varphi_{/a} : F_{/a} \longrightarrow G_{/a}$ is an isomorphism in $ho(S)$.*

is an isomorphism if and only if, for any object a of A, the induced map $\varphi_{/a} : F_{/a} \longrightarrow G_{/a}$ is an isomorphism in $ho(S)$.

PROOF. Let F be an object of $LFib(A)$. We assume that F is equiped with an isomorphism with (Y, q), where $q : Y \longrightarrow A$ is a left fibration. Let us consider the Cartesian squares of simplicial sets

$$\begin{array}{ccccc} Y_a & \longrightarrow & \tilde{A}_{/a} \times_A Y & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow q \\ \Delta^0 & \longrightarrow & \tilde{A}_{/a} & \longrightarrow & A \end{array}$$

in which the lower horizontal line corresponds to the factorization of a map of the form $a : \Delta^0 \longrightarrow A$ into a final map followed by a right fibration. Since the map q is proper, the inclusion $Y_a \longrightarrow \tilde{A}_{/a} \times_A Y$ is final as well, hence a weak equivalence of the Joyal model structure. In other words, we have a canonical isomorphism of the form $Y_a \simeq F_{/a}$ in $ho(S)$, for any object a of A. The equivalence between conditions (i) and (iii) is thus a reformulation of Theorem 20.13. Since the implications (i)⇒(ii)⇒(iii) are obvious, this proves the proposition. □

COROLLARY 23.30. *A functor between ∞-categories $u : A \longrightarrow B$ is cofinal if and only if, for any object b of B, the ∞-category $A/b = B/b \times_B A$ has the weak homotopy type of the point.*

PROOF. Let F be the object of $LFib(B)$ defined as $F = (A, u)$. The map u is final if and only if F is a final object of $LFib(B)$. By virtue of the preceding proposition, this is equivalent to say that $F_{/b}$ is a final object of $ho(S)$ for any object b of B. But, by virtue of Theorem 22.9, a canonical fibrant replacement of (Δ^0, b) in the contravariant model category structure over B is the canonical map $B/b \longrightarrow B$. Therefore, we have a canonical isomorphism $A/b = F_{/b}$ in the homotopy category $ho(S)$. □

COROLLARY 23.31 (Quillen's Theorem A). *Let $u : A \longrightarrow B$ a functor between small categories. Assume that, for any object b of B, the nerve of $A/b = B/b \times_B A$ has the weak homotopy type of the point. Then the nerve of u is a weak homotopy equivalence.*

PROOF. We have $N(B)/b = N(B/b)$. Therefore, the hypothesis implies that $N(u) : N(A) \longrightarrow N(B)$ is a cofinal map, hence a weak equivalence of the Kan-Quillen model structure. □

PROPOSITION 23.32. *Let $u : A \longrightarrow B$ be a morphism of simplicial sets. For any objects E in $RFib(A)$ and F in $LFib(B)$, there is a canonical isomorphism*

$$\mathbf{R}\langle E, \mathbf{R}u^*(F)\rangle \simeq \mathbf{R}\langle \mathbf{L}u_!(E), F\rangle$$

in the homotopy category $ho(S)$. Similarly, for any object F' in $RFib(B)$ and any object E' in $LFib(A)$, there is a canonical isomorphism

$$\mathbf{R}\langle F', \mathbf{L}u_!(E')\rangle \simeq \mathbf{R}\langle \mathbf{R}u^*(F'), E'\rangle .$$

PROOF. Cartesian diagrams of the form

$$\begin{array}{ccccc} X \times_B Y & \longrightarrow & A \times_B Y & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow q \\ X & \xrightarrow{p} & A & \xrightarrow{u} & B \end{array}$$

show that, for $E = (X, p)$ and $F = (Y, q)$, we have

$$\langle E, u^*(F) \rangle = \langle u_!(E), F \rangle .$$

When q is a left fibration, this proves the first isomorphism. Using the identification

$$\langle E, F \rangle^{op} \simeq \langle F^{op}, E^{op} \rangle ,$$

where $E^{op} = (X^{op}, p^{op})$, this also implies the second one. □

PROPOSITION 23.33. *Let $p : A \longrightarrow B$ be a morphism of simplicial sets. The following conditions are equivalent.*

(i) *The morphism p is proper.*

(ii) *For any pull back squares of the form*

$$\begin{array}{ccccc} A'' & \xrightarrow{u'} & A' & \xrightarrow{u} & A \\ \downarrow{\scriptstyle p''} & & \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ B'' & \xrightarrow{v'} & B' & \xrightarrow{v} & B \end{array} \tag{23.33.1}$$

and for any object F of $LFib(A')$, the base change map

$$\mathbf{L}p''_! \mathbf{R}u'^*(F) \longrightarrow \mathbf{R}v'^* \mathbf{L}p'_!(F)$$

is an isomorphism in $LFib(B'')$.

(iii) *For any pullback squares of the form* (23.33.1)*, and for any object E in $RFib(B'')$, the base change map*

$$\mathbf{L}u'_! \mathbf{R}p''^*(E) \longrightarrow \mathbf{R}p'^* \mathbf{L}v'_!(E)$$

is an isomorphism in $RFib(A')$.

PROOF. For E and F corresponding to a right fibration $X \longrightarrow A$ and a left fibration $Y \longrightarrow A$, respectively, we have the following commutative diagram.

$$\begin{array}{ccc} \mathbf{R}\langle E, \mathbf{L}p''_! \mathbf{R}u'^*(F) \rangle & \longrightarrow & \mathbf{R}\langle E, \mathbf{R}v'^* \mathbf{L}p'_!(F) \rangle \\ \wr \downarrow & & \downarrow \wr \\ \mathbf{R}\langle \mathbf{L}u'_! \mathbf{R}p''^*(E), F \rangle & \longrightarrow & \mathbf{R}\langle \mathbf{R}p'^* \mathbf{L}v'_!(E), F \rangle \end{array}$$

Using Propositions 23.29 and 23.32, we see easily that conditions (ii) and (iii) are equivalent. We already know that condition (i) implies conditions (ii) (by Corollary 23.10) and (iii) (by Theorem 23.24)[1]. It is thus sufficient to prove that condition (ii) implies condition (i).

[1] Remark that (i)⇒(iii)⇒(ii) gives another proof of Theorem 23.24.

Let us consider Cartesian squares of the form (23.33.1) in which we have $B'' = \Delta^0$ with v' final. It is sufficient to prove that the map u is final, by Proposition 23.5. In other words, we must show that the map u is a weak equivalence of the contravariant model category structure over A'. The dual version of Proposition 23.29 shows that is is sufficient to prove that, for any object a' of A', if $\tilde{A}'_{a'/} \longrightarrow A'$ denotes a fibrant replacement of $a' : \Delta^0 \longrightarrow A'$ with respect to the covariant model category structure over A', then the projection from $A''_{a'/} = A'' \times_{A'} \tilde{A}'_{a'/}$ to $\tilde{A}'_{a'/}$ is a weak equivalence of the Kan-Quillen model category structure. Let F be the object of $LFib(A')$ corresponding to the map $\tilde{A}'_{a'/} \longrightarrow A'$. Then there is a cannical isomorphism $\int F \simeq \int \mathbf{L}p'_!(F)$, and, since v' is final, Lemma 23.22 ensures that we have a canonical isomorphism $\mathbf{R}v'^*\mathbf{L}p'_!(F) \simeq \int F$. On the other hand, assuming condition (ii), we have a base change isomorphism in $ho(S)$:

$$A''_{a'/} \simeq \int \mathbf{R}u'^*(F) = \mathbf{L}p''_!\mathbf{R}u'^*(F) \longrightarrow \mathbf{R}v'^*\mathbf{L}p'_!(F) \simeq \int F \simeq \tilde{A}'_{a'/}$$

This ends the proof. □

23.34. Let $p : A \longrightarrow B$ be a morphism of simplicial sets. We consider a morphism $v : b_0 \longrightarrow b_1$ in B, as well as an object a_0 of A such that $p(a_0) = b_0$. We may see the map v as a morphism from v to 1_{b_1} in the slice B/b_1, hence as a morphism of simplicial sets $v : \Delta^1 \longrightarrow B/b_1$. We then form the following pull back squares.

$$\begin{array}{ccccccc} A_{b_1} & \hookrightarrow & A' & \longrightarrow & A/b_1 & \longrightarrow & A \\ \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle p} \\ \{1\} & \hookrightarrow & \Delta^1 & \xrightarrow{v} & B/b_1 & \longrightarrow & B \end{array} \tag{23.34.1}$$

We denote by (v, b_0) the object of B/b_1 corresponding to $v : \Delta^0 * \Delta^0 = \Delta^1 \longrightarrow B$, and we write $(v, a_0) = ((v, b_0), a_0)$ for the corresponding object in the fibre product $B/b_1 \times_B A = A/b_1$. Similarly, we have the object $a' = (0, a_0)$ of $A' = \Delta^1 \times_B A$. Since the map u sends a' to (v, a_0), we have a canonical commutative square of the following form.

$$\begin{array}{ccc} a'\backslash A' & \longrightarrow & (v,a_0)\backslash(A/b_1) \\ \downarrow & & \downarrow \\ 0\backslash\Delta^1 & \longrightarrow & (v,b_0)\backslash(B/b_1) \end{array} \tag{23.34.2}$$

This square is Cartesian, since the functor $(X, x) \longmapsto x\backslash X$ is right adjoint to the functor $C \longmapsto \Delta^0 * C$, and, therefore, commutes with small limits. Since we also have Cartesian squares of the form

$$\begin{array}{ccccccc} A_{b_1} & \hookrightarrow & 0\backslash A' & \longrightarrow & (v,b_0)\backslash(A/b_1) & \longrightarrow & A/b_1 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \{1\} & \xrightarrow{0<1} & 0\backslash\Delta^1 & \longrightarrow & (v,b_0)\backslash(B/b_1) & \longrightarrow & B/b_1 \end{array} \tag{23.34.3}$$

we deduce that there is the Cartesian square below.

$$\begin{array}{ccc} a'\backslash A' & \longrightarrow & (v,a_0)\backslash(A/b_1) \\ \downarrow & & \downarrow \\ 0\backslash A' & \longrightarrow & (v,b_0)\backslash(A/b_1) \end{array} \tag{23.34.4}$$

In particular, the fibres of the two vertical maps of the latter are isomorphic, which implies that there is a canonical isomorphism

$$a'\backslash A_{b_1} = A_{b_1} \times_{A'} (a'\backslash A') \simeq A_{b_1} \times_{A/b_1} ((v,a_0)\backslash(A/b_1)) = (v,a_0)\backslash A_{b_1}\,. \tag{23.34.5}$$

THEOREM 23.35. *Let $p : A \longrightarrow B$ be an inner fibration between ∞-categories. The following conditions are equivalent.*

(i) *The functor p is proper.*
(ii) *For any object b of B, the canonical map $A_b \longrightarrow A/b$ is final.*
(iii) *For any Cartesian squares of the form*

$$\begin{array}{ccccc} A'' & \xrightarrow{u'} & A' & \xrightarrow{u} & A \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle p} \\ \{1\} & \hookrightarrow & \Delta^1 & \xrightarrow{v} & B \end{array}$$

the functor u' is final.
(iv) *For any Cartesian squares of the form*

$$\begin{array}{ccccc} A'' & \xrightarrow{u'} & A' & \xrightarrow{u} & A \\ \downarrow{\scriptstyle p''} & & \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ \Delta^0 & \xrightarrow{v'} & B' & \xrightarrow{v} & B \end{array}$$

in which B' is an ∞-category, if v' is final, so is u'.

PROOF. Proposition 23.5 shows that condition (i) implies both conditions (ii) and (iii): the map u' is the pull back along p of the final map $\{1\} \longrightarrow \Delta^1$, while $A_b \longrightarrow A/b$ is the inverse image by p of the final map $b : \Delta^0 \longrightarrow B/b$. Let us check that conditions (ii) and (iii) are equivalent.

Let $v : \Delta^1 \longrightarrow B$ be a morphism from b_0 to b_1 in B. An object a' of $A' = \Delta^1 \times_B A$ is a pair (ε, a) where a is an object of A, and $\varepsilon = 0, 1$, such that $p(a) = b_\varepsilon$. The object a' belongs to $A'' = A_{b_1}$ if and only if $\varepsilon = 1$. For any object a such that $p(a) = b_1$, since $a' = (1, a)$ belongs to A'', the fibre product $A'' \times_{A'} a'\backslash A'$ is isomorphic to the slice $a'\backslash A''$, and thus has an initial object. This proves in particular that the ∞-category $A'' \times_{A'} (a'\backslash A')$ has an initial object and thus is weakly equivalent to the point. In other words, by virtue of Corollary 23.30, the map $A'' \longrightarrow A'$ is final if and only if, for any object a_0 of A such that $p(a_0) = b_0$, if we put $a' = (0, a_0)$, the ∞-category $A'' \times_{A'} a'\backslash A'$ is weakly contractible. The pair (v, b_0) can be seen as an object of $A/b_1 = B/b_1 \times_B A$. Therefore, condition (ii) is equivalent to the property that the ∞-category $A_{b_1} \times_{A/b_1} ((v,b_0)\backslash(A/b_1))$

is weakly contractible for any v and a_0 as above. The isomorphism (23.34.5) thus proves that conditions (ii) and (iii) are equivalent.

We now prove that condition (iii) implies condition (iv). Let us consider Cartesian squares of the following form.

$$\begin{array}{ccccc} A'' & \xrightarrow{u'} & A' & \xrightarrow{u} & A \\ \downarrow{\scriptstyle p''} & & \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ \Delta^0 & \xrightarrow{v'} & B' & \xrightarrow{v} & B \end{array}$$

in which B' is assumed to an ∞-category, with v' final. If p satisfies condition (iii), so does p'. Therefore, the functor p' also satisfies condition (ii). In other words, the map $A'' = A'_v \longrightarrow A'/v$ is final. Since the map $B'/v \longrightarrow B'$ is a trivial fibration, so is the induced map $A'/v = B'/v \times_{B'} A' \longrightarrow A'$. As any trivial fibration is final, the composed map u' is final.

To finish the proof, it remains to prove that condition (iv) implies that p is proper. Let C be the class of monomorphisms $v' : B'' \longrightarrow B'$ such that, for any map $v : B' \longrightarrow B$, the induced map $u' : B'' \times_B A \longrightarrow B' \times_B A$ is a right anodyne extension. The class C is saturated. Moreover, for any pair of monomorphisms $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$, if both f and gf are in C, so is g: this follows right away from the fact that the class of right anodyne extensions has this property; see Corollary 20.8. In particular, condition (iv) implies that the class C contains any monomorphism $v' : B'' \longrightarrow B'$ such that both B' and B'' are ∞-categories, and such that there exists a final object in B'' whose image by v is a final object of B'. Therefore, it contains all the inclusions of the form $\{1\} \times \Delta^n \subset \Delta^1 \times \Delta^n$, for $n \geq 0$. This implies that it contains all inclusions of the form $\{1\} \times X \longrightarrow \Delta^1 \times X$, for any simplicial set X. To prove this, by Corollary 3.10, it is sufficient to prove that the class of such simplicial sets is saturated by monomorphisms. The stability by small sums is obvious. Let us consider a coCartesian square of the form

$$\begin{array}{ccc} X_0 & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & X_3 \end{array}$$

and assume that the map $\{1\} \times X_i \longrightarrow \Delta^1 \times X_i$ is in C for $i = 0, 1, 2$. For any map $\Delta^1 \times X_3 \longrightarrow B$, by virtue of Corollary 8.17, the induced map

$$(\{1\} \times X_3) \times_B A \longrightarrow (\Delta^1 \times X_3) \times_B A$$

is a trivial cofibration of the contravariant model category structure over the fibre product $(\Delta^1 \times X_3) \times_B A$, and is thus a final monomorphism, hence a right anodyne extension. The case of a countable unions is proved similarly.

For any integer $n \geq 0$, the inclusion of $\{1\} \times \Delta^n$ into $\Delta^1 \times \partial\Delta^n \cup \{1\} \times \Delta^n$ is in C, because it is the push out of the inclusion of $\{1\} \times \partial\Delta^n$ into $\Delta^1 \times \partial\Delta^n$. Since $\{1\} \times \Delta^n \subset \Delta^1 \times \Delta^n$ is in C, this shows that the inclusion of $\Delta^1 \times \Delta^n \cup \{1\} \times \Delta^n$ into $\Delta^1 \times \Delta^n$ is in C. Lemma 11.3 thus implies that inclusions $\Lambda^n_k \longrightarrow \Delta^n$ are all

in C for $0 < k \leq n$. This shows that p satisfies condition (v) of Proposition 23.5, hence is proper. □

REMARK 23.36. The proof that conditions (i) and (iv) of Theorem 23.35 are equivalent does not require that A and B are ∞-category. It would deserve to be inserted in Proposition 23.5.

CHAPTER 24

Fully faithful and essentially surjective functors, revisited

PROPOSITION 24.1. *Let $u : A \longrightarrow B$ be a morphism of simplicial sets. The following conditions are equivalent.*

(i) *For any objects E in $RFib(A)$, the unit map $E \longrightarrow \mathbf{R}u^*\mathbf{L}u_!(E)$ is an isomorphism.*
(ii) *For any object a of A, the unit map $h_a \longrightarrow \mathbf{R}u^*\mathbf{L}u_!(h_a)$ is an isomorphism in $RFib(A)$, where $h_a = (\Delta^0, a)$.*
(iii) *For any objects F in $LFib(A)$, the unit map $F \longrightarrow \mathbf{R}u^*\mathbf{L}u_!(F)$ is an isomorphism.*

PROOF. We shall use Propositions 23.29 and 23.32 repeatedly. Condition (i) is equivalent to the condition that, for any objects E in $RFib(A)$ and F in $LFib(A)$, the induced map

$$\mathbf{R}\langle E, F\rangle \longrightarrow \mathbf{R}\langle \mathbf{R}u^*\mathbf{L}u_!(E), F\rangle \simeq \mathbf{R}\langle \mathbf{L}u_!(E), \mathbf{L}u_!(F)\rangle \simeq \mathbf{R}\langle E, \mathbf{R}u^*\mathbf{L}u_!(F)\rangle$$

is invertible in $ho(S)$. This is equivalent to the assertion that, for any object a of A and any object F in $LFib(A)$, the induced map

$$\mathbf{R}\langle h_a, F\rangle \longrightarrow \mathbf{R}\langle h_a, \mathbf{R}u^*\mathbf{L}u^*(F)\rangle \simeq \mathbf{R}\langle \mathbf{R}u^*\mathbf{L}u_!(h_a), F\rangle$$

is invertible. □

PROPOSITION 24.2. *Let $u : A \longrightarrow B$ be a functor of ∞-categories. The following conditions are equivalent.*

(i) *The functor u is fully faithful.*
(ii) *The functor $\mathbf{L}u_! : RFib(A) \longrightarrow RFib(B)$ is fully faithful.*
(iii) *The functor $\mathbf{L}u_! : LFib(A) \longrightarrow LFib(B)$ is fully faithful.*

PROOF. We already know from Proposition 24.1 that conditions (ii) and (iii) are equivalent. To prove that condition (i) is equivalent to condition (ii), we shall verify that u is fully faithful if and only if condition (ii) of Proposition 24.1 is satisfied. Let a be an object of A. Then $\mathbf{L}u_!(h_a) = h_{u(a)}$. Since A and B are ∞-categories, fibrant replacements of h_a and $h_{u(a)}$ are given by A/a and $B/u(a)$ in the contravariant model category structures over A and B, respectively. The map $h_a \longrightarrow \mathbf{R}u^*\mathbf{L}u_!(h_a)$ is thus the natural morphism $A/a \longrightarrow A \times_B B/u(a)$. By corollary 21.10, for any object x of A, the induced map on the homotopy fibers is the natural morphism $A(x,a) \longrightarrow B(u(x), u(a))$. Therefore, u is fully faithful if and only if the functor $\mathbf{L}u_!$ is fully faithful on $RFib(A)$. □

REMARK 24.3. Let A be a simplicial set, and a an object of A. For any object F of $RFib(A)$, there is a canonical isomorphism

$$\mathbf{R}\langle F, h_a\rangle \simeq \mathbf{R}a^*(F) \tag{24.3.1}$$

in $ho(S)$. Indeed, it is sufficient to prove this for F fibrant, i.e., for F corresponding to a right fibration $p : X \longrightarrow A$. In this case, $\mathbf{R}a^*(F) = X_a = \langle F, h_a\rangle$ is simply the fibre of X over a.

Any map $u : a \longrightarrow b$ in A induces a natural morphism $u^* : \mathbf{R}b^*(F) \longrightarrow \mathbf{R}a^*(F)$. Indeed, for F fibrant as above, one can also describe $\mathbf{R}a^*(F)$ as $[h_a, X]_A \simeq X_a$, where $h_a = (\Delta^0, a)$. Since the inclusion $\{1\} \longrightarrow \Delta^1$ is a right anodyne extension, the canonical map $[(\Delta^1, u), X)]_A \longrightarrow X_b$ is a trivial fibration. The choice of a section of the latter, composed with the map $[(\Delta^1, u), X)]_A \longrightarrow X_a$, defines a morphism

$$u^* : \mathbf{R}b^*(F) \simeq [h_b, X] \longrightarrow [h_a, X] \simeq \mathbf{R}a^*(F) \tag{24.3.2}$$

in $ho(S)$.

LEMMA 24.4. *Let us assume that A is an ∞-category. If the map $u : a \longrightarrow b$ is invertible in A, for any object F of $RFib(A)$, the induced map $u^* : \mathbf{R}b^*(F) \longrightarrow \mathbf{R}a^*(F)$ is an isomorphism in $ho(S)$.*

PROOF. Since $\Delta^2 = \Delta^1 * \Delta^0 = \Delta^0 * \Delta^1$, the map

$$v : \Delta^2 \xrightarrow{\sigma_0^1} \Delta^1 \xrightarrow{u} A$$

can be interpreted in two ways: as a map $\tilde{u} : \Delta^1 \longrightarrow A/b$, and as a map $\tilde{a} : \Delta^0 \longrightarrow A/u$. We then have the following commutative diagram.

$$\begin{array}{ccccc} & & \{0\} & \xrightarrow{\tilde{a}} & A/u \longrightarrow A/a \\ & & \downarrow & & \downarrow \\ \{1\} & \hookrightarrow & \Delta^1 & \xrightarrow{\tilde{u}} & A/b \end{array}$$

The map $A/u \longrightarrow A/a$ is a trivial fibration because the inclusion $\{0\} \longrightarrow \Delta^1$ is a left anodyne extension; see Lemma 14.21. Therefore, $\tilde{a}$ is a final object of A/u, because its image in A/a is final. Similarly, the map $\tilde{u}$ is final because it sends the final object 1 of Δ^1 to the final object 1_b of A/b. If is thus sufficient to prove that the canonical map $A/u \longrightarrow A/b$ has contractible fibres over A (this maps is a morphism of right fibrations over A, so that this will ensure that it is a weak equivalence of the contravariant model category structure over A, by Theorem 20.13, and this will allow to apply Proposition 20.11). Proposition 21.9 shows that we may as well consider the fibres of the canonical map $A//u \longrightarrow A//b$ over A. For each object x of A we have a trivial fibration $A(x,a) \longleftarrow (A//u)_x$, and therefore, a map

$$Hom_{ho(A)}(x,a) = \pi_0(A(x,a)) \simeq \pi_0((A//u)_x) \longrightarrow \pi_0(A(x,b)) = Hom_{ho(A)}(x,b)$$

which is nothing else than the composition by f in $ho(A) \simeq \tau(A)$. Since u is invertible, the latter map is bijective. But, since the image of an invertible map is invertible, we also know that u, seen as a morphism of constant diagrams,

is invertible in the ∞-category $\underline{Hom}(K, A)$ for any simplicial set K. Therefore, since $\underline{Hom}(K, A)(x, a) = \underline{Hom}(K, A(x, a))$ for x and a as above, seen as constant diagrams, what precedes also proves that the induced map

$$\pi_0(\underline{Hom}(K, A(x, a))) \simeq \pi_0\underline{Hom}(K, (A//u)_x)) \longrightarrow \pi_0(\underline{Hom}(K, A(x, b)))$$

is bijective for any simplicial set K. For any simplicial set K and any Kan complex T, there is a canonical bijection $Hom_{ho(S)}(K, T) \simeq \pi_0(\underline{Hom}(K, T))$. Therefore, the Yoneda Lemma applied to the homotopy category $ho(S)$ implies that the map $(A//u)_x \longrightarrow A(x, b)$ is a simplicial homotopy equivalence. □

Proposition 24.5. *If a functor of ∞-categories $u : A \longrightarrow B$ is essentially surjective, then the derived pull back functor $\mathbf{R}u^* : RFib(B) \longrightarrow RFib(A)$ is conservative.*

Proof. If a map $E \longrightarrow F$ in $RFib(A)$ induces an isomorphism $\mathbf{R}u^*(E) \simeq \mathbf{R}u^*(F)$, then it induces an isomorphism

$$\mathbf{R}u(a)^*(E) \simeq \mathbf{R}a^*\mathbf{R}u^*(E) \simeq \mathbf{R}a^*\mathbf{R}u^*(F) \simeq \mathbf{R}u(a)^*(F)$$

for any object a of A. But, since u is essentially surjective, any object b of B is equivalent to an object of the form $u(a)$. Hence the functor $\mathbf{R}b^*$ is isomorphic to $\mathbf{R}u(a)^*$, by the preceding lemma. Therefore, Theorem 20.13 implies that the map $E \longrightarrow F$ is an isomorphism. The functor $\mathbf{L}u_!$ being fully faithful with a conservative right adjoint, it is an equivalence of categories. □

Corollary 24.6. *If any equivalence of ∞-categories $u : A \longrightarrow B$ induces equivalences of categories*

$$\mathbf{L}u_! : RFib(A) \longrightarrow RFib(B) \quad \textit{and} \quad \mathbf{L}u_! : LFib(A) \longrightarrow LFib(B)\,.$$

Proof. If u is an equivalence of ∞-categories, it is fully faithful and essentially surjective. Therefore, by virtue of Propositions 24.2 and 24.5, the functor $\mathbf{L}u_!$ is fully faithful on $RFib(A)$, and its right adjoint $\mathbf{R}u^*$ is conservative, hence both functors are equivalences of categories and are quasi-inverse to each other. The other equivalence comes from the natural identification $RFib(A) \simeq LFib(A^{op})$. □

Remark 24.7. There is a much more direct way to prove Corollary 24.6: one proves rather easily that any trivial fibration of the Joyal model ategory structure $u : A \longrightarrow B$ induces an equivalence of categories $\mathbf{L}u_! : RFib(A) \simeq RFib(B)$ and we use Brown's Lemma (Proposition 7.7) for the Joyal model category structure.

CHAPTER 25

Locally constant presheaves and Quillen's Theorem B

For a simplicial set X, we have three model category structures on the category $sSet/X$: the contravariant model category structure, the covariant model category structure, and the model category structure induced by the Kan-Quillen model category structure. Since we want to interpret the contravariant model category structure as a theory of presheaves over X, and since any Kan fibration of codomain X is a fibrant object in the contravariant model category structure, it is natural to ask what is the meaning of Kan fibrations in this semantic of the theory of presheaves. This is what we shall investigate in this chapter.

PROPOSITION 25.1. *Let W be a class of morphisms of simplicial sets. We assume that W has the following properties.*

(i) *For any morphisms of the form $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$, with f in W, then g is in W if and only if gf is in W.*
(ii) *The class of monomorphisms which are in W is saturated.*
(iii) *Any trivial fibration is in W.*
(iv) *For any integer $n \geq 0$, any map $\Delta^0 \longrightarrow \Delta^n$ is in W.*

Then any weak homotopy equivalence is in W.

PROOF. We first prove that any inclusion of the form $\Lambda^n_k \longrightarrow \Delta^n$, $n \geq 1$, $0 \leq k \leq n$, is in W. In fact, we already saw a proof: we just have to use the same combinatorial arguments as in the proof of Proposition 11.18 (this only requires conditions (i), (ii) and (iv)). Condition (ii) thus implies that any anodyne extension is in W. Since any weak homotopy equivalence is the composition of an anodyne extension with a trivial fibration, condition (iv) implies that any weak homotopy equivalence is in W. □

PROPOSITION 25.2. *Let $p : X \longrightarrow Y$ be a morphism of simplicial sets. The following conditions are equivalent.*

(i) *For any Cartesian squares of the form*

$$\begin{array}{ccccc} X'' & \xrightarrow{u'} & X' & \xrightarrow{u} & X \\ \downarrow{\scriptstyle p''} & & \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ Y'' & \xrightarrow{v'} & Y' & \xrightarrow{v} & Y \end{array} \tag{25.2.1}$$

if v' is a weak homotopy equivalence, so is u'.
(ii) *As in (i), but only when $Y'' = \Delta^0$.*
(iii) *As in (ii), but only when $Y' = \Delta^n$.*

(iv) *As in (i), but only when v' is an inclusion of the form $\Lambda^n_k \longrightarrow \Delta^n$ for $n \geq 1$ and $0 \leq k \leq n$.*

(v) *Any Cartesian square of the form*

$$\begin{array}{ccc} X' & \xrightarrow{u} & X \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ Y' & \xrightarrow{v} & Y \end{array} \tag{25.2.2}$$

is homotopy Cartesian in the Kan-Quillen model category structure.

Proof. We prove first that conditions (i) and (v) are equivalent. If condition (v) is verified, then any Cartesian squares of the form (25.2.1) are homotopy Cartesian, and, since weak equivalences are stable under homotopy pull backs; this proves that (v)⇒(i). For the converse, we first notice that any Cartesian square of the form (25.2.2) in which v is a Kan fibration is a homotopy Cartesian: since the functor Ex^∞ commutes with finite limits and preserves Kan fibrations, and since there is a functorial weak homotopy equivalence $W \longrightarrow Ex^\infty(W)$ for any simplicial set W, it is sufficient to prove this when X, Y and Y' are Kan complexes, which follows from the dual version of Corollary 8.28. Let us assume condition (i). Given any Cartesian square of the form (25.2.2), we choose a factorization of the map v into an anodyne extension $i : Y' \longrightarrow T$ followed by a Kan fibration $s : T \longrightarrow Y$, and we form the Cartesian squares below.

$$\begin{array}{ccccc} X' & \xrightarrow{i} & U & \xrightarrow{r} & X \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle p} \\ Y'' & \xrightarrow{j} & T & \xrightarrow{s} & Y \end{array}$$

Then both maps i and j are weak homotopy equivalences, and the right hand square is homotopy Cartesian, which implies that the commutative square (25.2.2) has the same property.

Let W be the class of morphisms of simplicial sets $A \longrightarrow B$ such that, for any map $B \longrightarrow Y$, the induced map $A \times_Y X \longrightarrow B \times_Y X$ is an anodyne extension. Since pulling back along p preserves monomorphisms and colimits, we check that the class of monomorphisms which are in W is saturated. Finally, the class W satisfies the two out of three property and contains the class of trivial fibrations. Since any weak homotopy equivalence is the composition of an anodyne extension and of a trivial fibration, condition (i) is equivalent to the assertion that the pull back of any anodyne extension along p is a trivial cofibration of the Kan-Quillen model category structure. Since, by definition, the smallest saturated class of morphisms of simplicial sets which contains forns $\Lambda^n_k \longrightarrow \Delta^n$ for $n \geq 1$ and $0 \leq k \leq n$ is the class of anodyne extensions, we see that conditions (i) and (iv) are equivalent.

It is clear that we have (i)⇒(ii)⇒(iii). Proposition 25.1 shows that condition (iii) implies condition (iv). □

Definition 25.3. A morphism of simplicial sets is *locally constant* if it satisfies the equivalent conditions of Proposition 25.2.

PROPOSITION 25.4. *A proper morphism of simplicial sets $p : X \longrightarrow Y$ is locally constant if and only if, for any map $v : \Delta^n \longrightarrow Y$, the inclusion $X_{v(0)} \longrightarrow \Delta^n \times_Y X$ is a weak homotopy equivalence.*

PROOF. This is clearly a necessary condition. For the converse, we shall prove that a proper morphism p satisfying this extra condition satisfies condition (iii) of Proposition 25.2. Let us consider Cartesian squares of the form (25.2.1) such that $Y'' = \Delta^0$. We want to prove that the map u' is a weak equivalence. We proceed by induction on n. Since the case $n = 0$ is trivial, we may assume that $n > 0$. If v' is initial, the property we seek is true by assumption. If not, we may assume that v' factors through the final map $w : \Delta^{n-1} \longrightarrow \Delta^n$ defined by $w(i) = i + 1$. Since p is proper, the induced map $\Delta^{n-1} \times_Y X \longrightarrow \Delta^n \times_Y X$ is final, hence a weak homotopy equivalence, and the induced map $\Delta^0 \times_Y X \longrightarrow \Delta^{n-1} \times_Y X$ is a weak homotopy equivalence by induction on n. Therefore, u' is the composition of two weak homotopy equivalences. □

COROLLARY 25.5. *Any morphism of simplicial sets which is both smooth and proper is locally constant.*

PROOF. If $p : X \longrightarrow Y$ is smooth then, for any map $v : \Delta^n \longrightarrow Y$, the map $X_{v(0)} \longrightarrow \Delta^n \times_Y X$ is initial, hence a weak homotopy equivalence. Therefore, if p is also proper, the preceding proposition ensures that it is locally constant. □

COROLLARY 25.6. *Any Kan fibration is locally constant.*

PROOF. Any right fibration is smooth, and any left fibration is proper. □

REMARK 25.7. The proof above is a little pedantic, since, for Kan fibrations, one can check several of the conditions of Proposition 25.2 directly. For instance, as explained in the proof of loc. cit. condition (v) follows right away from the good properties of the functor Ex^∞. Another quite natural approach would be to prove condition (iii), but this would go through the proof of Proposition 23.11, as well as its dual version, and therefore would be a variation on the pedantic version after all.

DEFINITION 25.8. Let A be a simplicial set. An object F of $RFib(A)$ is *locally constant* if, for any map $u : a \longrightarrow b$ in A, the induced map $u^* : \mathbf{R}b^*(F) \longrightarrow \mathbf{R}a^*(F)$ (as defined in Remark 24.3) is an isomorphism in $ho(S)$.

LEMMA 25.9. *Let $p : X \longrightarrow A$ be a right fibration, and $u : a \longrightarrow b$ be a morphism in A. We form the following Cartesian squares.*

$$\begin{array}{ccccc} X_b & \xrightarrow{i} & X_u & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow p \\ \Delta^0 & \xrightarrow{1} & \Delta^1 & \xrightarrow{u} & A \end{array}$$

If $F = (X, p)$ denotes the object of $RFib(A)$ associated to p, then there are a canonical isomorphisms

$$X_b \simeq \mathbf{R}b^*(F) \quad \textit{and} \quad X_u \simeq \mathbf{R}a^*(F)$$

in $ho(S)$. Furthermore, under these identifications, the map i corresponds in $ho(S)$ to the canonical map $\mathbf{R}b^*(F) \longrightarrow \mathbf{R}a^*(F)$ of Remark 24.3.

PROOF. Since 0 is an initial object of Δ^1 and since p is smooth, the inclusion $j : X_a \longrightarrow X_u$ is initial, hence a weak homotopy equivalence. It remains to check the compatibility of the map i with the construction of Remark 24.3. We want to check that the diagram

$$\begin{array}{ccc} X_b & \xrightarrow{u^*} & X_a \\ & {\scriptstyle i}\searrow \quad \swarrow{\scriptstyle j} & \\ & X_u & \end{array}$$

commutes in $ho(S)$.

In other words, we want to check that, for any simplicial set K, the diagram

$$\begin{array}{ccc} \pi_0(\underline{Hom}(K, X_b)) & \xrightarrow{\qquad u^* \qquad} & \pi_0(\underline{Hom}(K, X_a)) \\ & {\scriptstyle i}\searrow \quad \swarrow{\scriptstyle j} & \\ & \pi_0(\underline{Hom}(K, X_u)) & \end{array}$$

commutes in the category of sets.

For a given simplicial set K, let $q : K \times A \longrightarrow A$ be the second projection. since q is proper, the pair (q^*, q_*) is a Quillen adjunction for the contravariant model category structures, and we write $p^K : X^K \longrightarrow A$ for the map whose corresponding object of $sSet/A$ is $q_*q^*(X, p)$. There is a Cartesian square of simplicial sets of the form

$$\begin{array}{ccc} X^K & \longrightarrow & \underline{Hom}(K, X) \\ {\scriptstyle p^K}\downarrow & & \downarrow{\scriptstyle p_*} \\ A & \longrightarrow & \underline{Hom}(K, A) \end{array}$$

where the lower horizontal map is the one induced by $K \longrightarrow \Delta^0$. Remark that, for any map $w : A' \longrightarrow A$, if $p' : X' = A' \times_A X \longrightarrow A'$ denotes the pull back of p along w, then we have a canonical Cartesian square of the following form.

$$\begin{array}{ccc} X'^K & \longrightarrow & X^K \\ {\scriptstyle p'^K}\downarrow & & \downarrow{\scriptstyle p^K} \\ A' & \longrightarrow & A \end{array}$$

In particular, the fibre of p^K at $a \in A_0$ is $\underline{Hom}(K, X_a)$. We also have, for any simplicial set E over A, a canonical isomorphism of simplicial sets

$$\underline{Hom}(K, [E, X]_A) \simeq [E, X^K]_A .$$

This shows that the construction of the map u^* is compatible with the operation $X \longmapsto X^K$. We remark finally that the map $X_u^K \longrightarrow \underline{Hom}(K, X_u)$ is a weak homotopy equivalence: this is the inverse image of the inclusion $\Delta^1 \longrightarrow \underline{Hom}(K, \Delta^1)$

along the right fibration $\underline{Hom}(K, X_u) \longrightarrow \underline{Hom}(K, \Delta^1)$. Since any right fibration is smooth, it is sufficient to prove that $\Delta^1 \longrightarrow \underline{Hom}(K, \Delta^1)$ is initial. This follows from the fact that the object 0 is initial in both Δ^1 and $\underline{Hom}(K, \Delta^1)$. This shows that, up a canonical weak homotopy equivalence, the formation of the maps i and j is compatible with the operation $X \longmapsto X^K$.

In other words, we may replace X by X^K, and it is sufficient to prove that $\pi_0(j)\pi_0(u^*) = \pi_0(i)$. In fact, both $\pi_0(u^*)$ and $\pi_0(j)^{-1}\pi_0(i)$ have the following explicit description. Given a connected component C of X_b, we choose an element $x \in C$. We then have the solid commutative square below

$$\begin{array}{ccc} \{1\} & \xrightarrow{x} & X \\ \downarrow & \overset{h}{\nearrow} & \downarrow p \\ \Delta^1 & \xrightarrow{u} & A \end{array}$$

a dotted filler of which always exists (because 1 is final in Δ^1 and p is a right fibration), and the image of C in $\pi_0(X_a)$ is the connected component of $h(0)$. Each of the map $\pi_0(u^*)$ and $\pi_0(j)^{-1}\pi_0(i)$ is determined by a choice of a point x in C, and of such a lift h. Nevertheless, the filler h is unique in the following sense. If we endow the category of pointed simplicial sets over A with the model structure induced by the contravariant model category structure over A, the map h is a morphism from the cofibrant object $(\Delta^1, 1, u)$ to the fibrant object (X, x, p). Since $(\Delta^1, 1, u)$ is weakly equivalent to the initial object, such a map h is unique up to homotopy in the category of pointed simplicial sets over A. In particular, the connected component of $h(0)$ in the fibre X_a does not depend on the choice of the lift h. It remains to check that it does not depend on the choice of the point x in C. Since C is the connected component of a Kan complex, if there are two points x and x' in C, there is a map $\xi : x' \longrightarrow x$ in C. The pair (ξ, h) can be seen as a diagram of the form

$$\begin{array}{ccc} x' & & \\ & \searrow^{\xi} & \\ h(0) & \xrightarrow{h} & x \end{array}$$

whence as a map $(\xi, h) : \Lambda^2_2 \longrightarrow X$. The obvious commutative triangle

$$\begin{array}{ccc} & b & \\ {\scriptstyle u}\nearrow & & \searrow{\scriptstyle 1_b} \\ a & \xrightarrow{u} & b \end{array}$$

given by the map $u\sigma_2^1 : \Delta^2 \longrightarrow A$ thus provides a solid commutative square of simplicial sets

$$\begin{array}{ccc} \Lambda_2^2 & \xrightarrow{(\xi,h)} & X \\ \downarrow & \overset{t}{\nearrow} & \downarrow p \\ \Delta^2 & \xrightarrow{\sigma_2^1 u} & A \end{array}$$

which admits a filler t. The restriction of t to $\Delta^{\{0,1\}}$ defines a map $h' : h(0) \longrightarrow x'$ in X such that $p(h') = u$, which shows that the choice of a point of C does not affect the final result in $\pi_0(X_a)$. □

THEOREM 25.10. *Let $p : X \longrightarrow A$ be a right fibration, and $F = (X, p)$ the corresponding object in $RFib(A)$. The following conditions are equivalent.*

(i) *The map p is locally constant.*
(ii) *The object F of $RFib(A)$ is locally constant.*
(iii) *The map p is a Kan fibration.*

PROOF. We already know that (iii)⇒(i). It follows right away from Lemma 25.9 that (i)⇒(ii). It remains to prove that (ii)⇒(iii).

Let us assume that F is locally constant. For any map $u : \Delta^n \longrightarrow A$, and any element i of $[n]$, the induced map

$$[(\Delta^n, u), F]_A \longrightarrow [(\Delta^0, u(i)), F]_A = X_{u(i)}$$

is a trivial fibration. Indeed, by Proposition 20.10, this map is always a Kan fibration, and, by Proposition 20.11, the map

$$[(\Delta^n, u), F]_A \longrightarrow [(\Delta^0, u(n)), F]_A = X_{u(n)}$$

is a trivial fibration. Under the latter identification, the corresponding map

$$X_{u(n)} = \mathbf{R}u(n)^*(F) \longrightarrow \mathbf{R}u(i)^*(F) = X_{u(i)}$$

in $ho(S)$ is the canonical map induced by the map $u(i) \longrightarrow u(n)$ in A. Let W be the class of maps $u : K \longrightarrow L$ such that, for any map $v : L \longrightarrow A$, the induced morphism

$$u^* : [(L, v), F]_A \longrightarrow [(K, vu), F]_A$$

is an equivalence of ∞-groupoids. This class satisfies the hypothesis of Proposition 25.1 and thus contains all weak homotopy equivalences. In particular, for any anodyne extension $u : K \longrightarrow L$ and any map $v : L \longrightarrow A$, the map

$$u^* : [(L, v), F]_A \longrightarrow [(K, vu), F]_A$$

is a trivial fibration. Since any trivial fibration is surjective on objects, this implies that p is a Kan fibration. □

PROPOSITION 25.11. *Let $u : A \longrightarrow B$ be a morphism of simplicial sets. The following conditions are equivalent.*

(i) *The pair (A, u) is locally constant as an object of $RFib(B)$.*

(ii) *For any Cartesian squares of the form*

$$\begin{array}{ccccc} A'' & \xrightarrow{f'} & A' & \xrightarrow{f} & A \\ \downarrow{\scriptstyle u''} & & \downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle u} \\ B'' & \xrightarrow{g'} & B' & \xrightarrow{g} & B \end{array} \tag{25.11.1}$$

if g and gg' are left fibrations and if both B' and B'' have initial objects, then the map f' is a weak homotopy equivalence.

(iii) *Any Cartesian square of the form*

$$\begin{array}{ccc} A' & \xrightarrow{f} & A \\ \downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle u} \\ B' & \xrightarrow{g} & B \end{array} \tag{25.11.2}$$

in which the map g is a left fibration is homotopy Cartesian in the Kan-Quillen model category structure.

Proof. In a Cartesian square of the form (25.11.2), if g is a left fibration and if B' has an initial object whose image in B is denoted by b, then $\mathbf{R}b^*(A,u)$ is canonically isomorphic to A' in $ho(S)$. Using Lemma 25.9, one sees that conditions (i) and (ii) are equivalent. Since the homotopy pull back of a weak equivalence is a weak equivalence, condition (iii) implies condition (ii). Let $i : A \longrightarrow C$ be a right anodyne extension followed by a right fibration $p : C \longrightarrow B$ such that $u = pi$. Since (A,u) and (C,p) are isomorphic in $RFib(B)$, one is locally constant if and only if the other has the same property. By virtue of Theorem 25.10, condition (i) is equivalent to the condition that p is a Kan fibration. Let us consider a Cartesian square of the form (25.11.2). If we put $C' = B' \times_B C$, the map $i' : A' \longrightarrow C'$ induced by i is a right anodyne extension because it is the pull back of i by the left fibration g, hence by a proper map. Since C' is the homotopy pull back of C and B' over B (because p is a Kan fibration), this proves that the square (25.11.2) is homotopy Cartesian. □

Corollary 25.12 (Quillen's Theorem B). *Let $u : A \longrightarrow B$ be a functor between small categories. Assume that, for any map $b_0 \longrightarrow b_1$ in B, the nerve of the induced functor $b_1\backslash A \longrightarrow b_0\backslash A$ is a weak homotopy equivalence. Then, for any object b of B, the Cartesian square*

$$\begin{array}{ccc} N(A\backslash b) & \longrightarrow & N(A) \\ \downarrow & & \downarrow{\scriptstyle N(u)} \\ N(b\backslash B) & \longrightarrow & N(B) \end{array}$$

is homotopy Cartesian in the Kan-Quillen model category structure.

Proof. For any left fibration $B' \longrightarrow N(B)$, if B' has an initial object b' whose image in B is denoted by b, the canonical maps $b'\backslash B' \longrightarrow N(b\backslash B) = b\backslash N(B)$ and $b'\backslash B' \longrightarrow B'$ are trivial fibrations. Therefore, the pull backs of any of these maps

along any morphism $X \longrightarrow N(B)$ will remain a weak homotopy equivalence. One deduces from there that the nerve of u satisfies condition (ii) of the previous Proposition, hence also condition (iii). $\square$

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