DENIS HIGGS

A Universal Characterization of $[0,\infty]$*

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1. INTRODUCTION

F. Faltin, N. Metropolis, B. Ross, and G.-C. Rota [3] (see also [9], [10]) and N. G. de Bruijn [2] have recently used the 'decimal' expansions (to some base) of real numbers to give a direct construction of the reals, direct in that the rationals do not constitute an intervening stage. Here we exhibit the interval $[0, \infty]$ as the free algebra on one generator in a certain easily-described equational class of universal algebras, the motivating idea again being decimal expansions (to base 2).

2. MAGNITUDE MODULES

A *magnitude module* is a set M together with a nullary operation 0, a unary operation h, and an ω-ary operation $\sum$ satisfying the following identities:

(i) $$\sum\left(\sum(x_{00}, x_{01}, x_{02}, \ldots), \sum(x_{10}, x_{11}, x_{12}, \ldots), \ldots\right) = \sum\left(\sum(x_{00}, x_{10}, x_{20}, \ldots), \sum(x_{01}, x_{11}, x_{21}, \ldots), \ldots\right),$$

(ii) $$\sum(0, \ldots, 0, x, 0, \ldots) = x,$$

(iii) $$h\left(\sum(x_0, x_1, x_2, \ldots)\right) = \sum(h(x_0), h(x_1), h(x_2), \ldots),$$

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(iv) $\sum(h(x), h^2(x), h^3(x), \ldots) = x.$

(In (iv), $h^2(x) = h(h(x))$, etc.).

The following are some examples of magnitude modules.

(2.1) $M = [0, \infty]$ with $0 =$ the usual 0, $h(x) = \frac{1}{2}x$, $\sum(x_0, x_1, x_2, \ldots) = x_0 + x_1 + x_2 + \ldots$ (where $\frac{1}{2}\infty = \infty$ and divergent series have sum equal to ∞).

Since magnitude modules form an equational class, we have free magnitude modules (and will show in § 4 that $[0, \infty]$ is the free magnitude module on one generator), submodules, product modules, and quotient modules. Amongst products we have in particular

(2.2) $M = [0, \infty]^X$, the module of all $[0, \infty]$-valued functions on a set X (pointwise operations), together with various submodules thereof. For example if X is a measurable space ([1], p. 35) then the $[0, \infty]$-valued measurable functions on X form a magnitude module (and the morphisms from this module to $[0, \infty]$ are precisely the countably-additive $[0, \infty]$-valued integrals on X). If $X = [0, \infty]$, various standard types of function from X to $[0, \infty]$ form magnitude modules: continuous increasing functions, analytic functions $\sum a_n x^n$ with the coefficients a_n in $[0, \infty]$ (this is the submodule of $[0, \infty]^{[0,\infty]}$ generated by $\{1, x, x^2, \ldots\}$), etc..

(2.3) $M =$ any poset in which all countable subsets have joins, with $0 = \vee\phi$, $h(x) = x$, $\sum(x_0, x_1, x_2, \ldots) = x_0 \vee x_1 \vee x_2 \vee \ldots$. It is not difficult to show, using the results obtained below, that such magnitude modules are precisely those satisfying the additional identity $h(x) = x$ and that they form the only equational class of magnitude modules besides the classes of all and of one-element magnitude modules.

3. ELEMENTARY PROPERTIES OF MAGNITUDE MODULES

(v) $\sum(x_{\pi(0)}, x_{\pi(1)}, x_{\pi(2)}, \ldots) = \sum(x_0, x_1, x_2, \ldots)$ where π is any permutation of ω.

This follows by (ii) on equating, as we may by (i), the $\sum$'s of the column and row $\sum$'s in the $\omega \times \omega$ matrix with $x_{\pi(j)}$ in the $(\pi(j), j)$th position, $j \in \omega$, and 0's elsewhere.

(vi) $\sum(0, \ldots, 0, x_0, 0, \ldots, 0, x_1, 0, \ldots, 0, x_2, 0, \ldots) = \sum(x_0, x_1, x_2, \ldots)$. Here the matrix

$$\begin{bmatrix} 0 \ldots 0\ x_0\ 0 \ldots & & & \\ 0 \ldots \ldots \ldots 0\ x_1\ 0 \ldots & & \\ 0 \ldots \ldots \ldots \ldots 0\ x_2\ 0 \ldots \\ \ldots \ldots \ldots \ldots \ldots \ldots \end{bmatrix}$$

does the job.

(vii) $\sum(\sum(x_{00}, x_{01}, x_{02}, \ldots), \sum(x_{10}, x_{11}, x_{12}, \ldots), \ldots)$
$= \sum(x_{00}, x_{01}, x_{10}, x_{02}, x_{11}, x_{20}, \ldots).$

Applying (i) and (vi) appropriately to the three matrices

$$\begin{bmatrix} x_{00} & x_{01} & x_{02} & \ldots \\ x_{10} & x_{11} & x_{12} & \ldots \\ x_{20} & x_{21} & x_{22} & \ldots \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \quad \begin{bmatrix} x_{00} & x_{01} & x_{02} & \ldots & \\ 0 & x_{10} & x_{11} & x_{12} & \ldots \\ 0 & 0 & x_{20} & x_{21} & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \quad \begin{bmatrix} x_{00} & 0 & 0 & \ldots \\ 0 & x_{01} & 0 & \ldots \\ 0 & x_{10} & 0 & \ldots \\ 0 & 0 & x_{02} & \ldots \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

gives this result.

It follows from these identities that the customary manipulations made on convergent series of non-negative terms are valid for arbitrary $\sum$'s in a magnitude module. In particular, if we define the binary operation $+$ by $x+y = \sum(x, y, 0, 0, \ldots)$ then

(viii) $+$ is associative and commutative with 0 as identity element,

$$x_0 + x_1 + \ldots + x_n = \sum(x_0, x_1, \ldots, x_n, 0, 0, \ldots),$$
$$x_0 + \sum(x_1, x_2, \ldots) = \sum(x_0, x_1, x_2, \ldots), \text{ and}$$
$$\sum(x_0, x_1, x_2, \ldots) + \sum(y_0, y_1, y_2, \ldots) = \sum(x_0 + y_0, x_1 + y_1, x_2 + y_2, \ldots).$$

Furthermore we have

(ix) $h(x+x) = x,\ h(0) = 0,\ h(x+y) = h(x) + h(y).$

PROOF. $h(x+x) = h(x + \sum(h(x), h^2(x), \ldots))$ by (iv)
$= h(\sum(x, h(x), h^2(x), \ldots)$ by (viii)
$= \sum(h(x), h^2(x), h^3(x), \ldots)$ by (iii)
$= x$ by (iv) again.

Then $h(0) = h(0+0) = 0$, and $h(x+y) = h(x) + h(y)$ is immediate from (iii) together with this.

Although we shall only use it in discussing the relationship of magnitude modules to cardinal algebras (§ 7), it is convenient to define here the natural order relation on a magnitude module M: for x, y in M, $x \leqslant y$ if and only if there exists z in M such that $x + z = y$.

(x) $\leqslant$ is a partial order with 0 as least element,
$x_0 \leqslant y_0, x_1 \leqslant y_1, x_2 \leqslant y_2, \ldots$ imply $\sum(x_0, x_1, x_2, \ldots) \leqslant \sum(y_0, y_1, y_2, \ldots)$,
and $x \leqslant y$ implies $h(x) \leqslant h(y)$.

PROOF. The only fact which is not immediate is the antisymmetry of $\leqslant$. Suppose that $x \leqslant y$ and $y \leqslant x$, say $x + u = y$, $y + v = x$. Then $x = x + u + v$ so that $x = x + (u+v) + \ldots + (u+v)$ with 2^n $u+v$'s. By (ix) this gives

$h^n(x) = h^n(x) + (u+v)$. Summing over $n = 1, 2, 3, \ldots$ and applying (iv) gives

$$\begin{aligned} x &= x + (u+v) + (u+v) + \ldots \\ &= x + u + (v+u) + (v+u) + \ldots \\ &= x + (u+v) + (u+v) + \ldots + u \\ &= x + u = y. \end{aligned}$$

4. $[0, \infty]$ IS THE FREE MAGNITUDE MODULE ON ONE GENERATOR, BEING FREELY GENERATED BY 1

PROOF. Let F be the free magnitude module on a single generator x and let $f: F \to [0, \infty]$ be the unique morphism from F to $[0, \infty]$ such that $f(x) = 1$. Since 1 generates $[0, \infty]$ (every element of $[0, \infty]$ can be written as a sum, in general infinite, of $1/2^n$'s), f is certainly onto. To show that f is $1-1$, and thus an isomorphism from F to $[0, \infty]$, we first observe that every non-zero element of F can be written in the form $\sum (h^{n_0}(x), h^{n_1}(x), \ldots)$: by (iii) and (vii), the h and $\sum$ respectively of such elements is again of the same form, and since x itself can be written in this form by (iv), the set of such elements, together with 0, forms a submodule of F containing x and is thus all of F (one uses $h(0) = 0$ and (vi) to deal with 0). (The same argument shows that in general every magnitude module term in variables x_i, $i \in I$, is either equivalent to 0 or to a term of the form $\sum (h^{n_0}(x_{i_0}), h^{n_1}(x_{i_1}), \ldots)$.) Now suppose that a and b are elements of F such that $f(a) = f(b) = \lambda$ say. If $\lambda = 0$ then necessarily $a = b = 0$; if $\lambda \neq 0$ then $a, b \neq 0$ and by the observation just made we can write

$$a = \sum (h^{m_0}(x), h^{m_1}(x), \ldots), \quad b = \sum (h^{n_0}(x), h^{n_1}(x), \ldots)$$

with $1/2^{m_0} + 1/2^{m_1} + \ldots = 1/2^{n_0} + 1/2^{n_1} + \ldots = \lambda$. Let us partition the interval $[0, \lambda]$ in the two ways indicated by these expressions for λ:

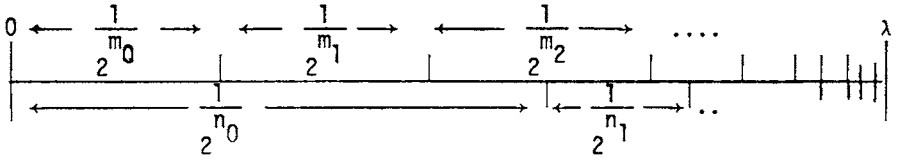


Fig. 1

The common refinement of these two partitions leads to a third partition of the same type if we divide up each portion common to a pair of overlapping intervals, say of lengths $1/2^{m_i}$ and $1/2^{n_j}$ respectively, into intervals of length $1/2^k$ where $k = \max(m_0, \ldots, m_i, n_0, \ldots, n_j)$. Let $1/2^{k_0} + 1/2^{k_1} + \ldots = \lambda$ be the resulting expression for λ and let $c = \sum (h^{k_0}(x), h^{k_1}(x), \ldots)$ be the corresponding element of F. It is then clear from the properties of $+$, h, and $\sum$ obtained in § 3 (in particular the identity $x = h(x) + h(x)$) that $a = c$ and $b = c$; thus $a = b$ again, as required.

5. FREE MAGNITUDE MODULES IN GENERAL

It is routinely verified that the coproduct of a family of magnitude modules M_i, $i \in I$, is the submodule of their product consisting of the elements in the product having countable support (support of $f = \{i; f(i) \neq 0\}$). It follows from this and the fact that $[0, \infty]$ is the free magnitude module on one generator that the free magnitude module on a set X consists of the functions in $[0, \infty]^X$ with countable support, under the pointwise operations.

As a consequence we see that $[0, \infty]$ generates the class of all magnitude modules as an equational class: every magnitude module is a homomorphic image of a submodule of a power of $[0, \infty]$. One is tempted to ask whether every magnitude module is actually a submodule of a power of $[0, \infty]$ but this seems to be most doubtful in view of the existence of ω-complete boolean algebras admitting no countably additive real-valued measures ([5], Theorem 3.2). A related question, also unresolved, is the following. Let X be a measurable space and let $\mathcal{M}$ be the set of measurable subsets of X. Then, besides the submodule M_1 of $[0, \infty]^X$ generated by the characteristic functions of the members of $\mathcal{M}$ (incidentally, it follows from standard results – see [1], Sect. 16, Theorem 4, for example – that M_1 is the module of all $[0, \infty]$– valued measurable functions on X mentioned in § 2), one can also consider the free magnitude module M_2 with the non-empty elements of $\mathcal{M}$ as generators subject to the relations $\sum(A_0, A_1, A_2, \ldots) = B$ whenever $A_0, A_1, A_2, \ldots, B \in \mathcal{M}$ are such that B is the disjoint union of the A_n's. The question is whether M_1 and M_2 are isomorphic. An answer, preferably positive, would be helpful in setting up a universal algebra approach to integration theory via magnitude modules[1]; in this connection it is appropriate to refer to Linton's functorial measure theory [8] which employs ω-complete boolean algebras, and to quote Fremlin's remark that "One of the oddities of measure theory is the lack of a completely convincing class of morphisms" ([4], p. 150).

6. SCALAR MULTIPLICATION, MAGNITUDE ALGEBRAS

Since $[0, \infty]$ is freely generated by 1, every magnitude module M (in particular $[0, \infty]$ itself) admits a scalar multiplication $[0, \infty] \times M \to M$: for $\lambda \in [0, \infty]$ and $x \in M$, λx is the image of λ under the unique morphism $\hat{x}$ from $[0, \infty]$ to M which maps 1 to x. Besides thus being a morphism in its first argument (with the second argument kept fixed) and satisfying $1x = x$, scalar multiplication also satisfies $(\lambda\mu)x = \lambda(\mu x)$: both sides are morphisms in λ (μ and x kept fixed) which agree at $\lambda = 1$. It follows that $[0, \infty]$ is a monoid under scalar multiplication – but here we have just the usual multiplication on $[0, \infty]$ (with $\lambda\infty = \infty\lambda = \infty$ for $\lambda \neq 0$, $0\infty = \infty 0 = 0$) since both multiplications, considered as functions of their first argument, are module morphisms agreeing at 1.

[1] (Added in proof). The answer is positive.

Scalar multiplication is actually a morphism in its second argument too, this being a consequence of identities (i) and (iii) (which express the fact that the algebraic theory of magnitude modules is autonomous [7] or commutative [12], Chapter 12): by (i) and (iii), the set Hom (M_1, M_2) of morphisms from a magnitude module M_1 to another, M_2, is a submodule of the power $M_2{}^{|M_1|}$ and in particular, for each $\hat{x}$ in our module M, $h(\hat{x})$, like $\hat{x}$, is a *morphism* from $[0, \infty]$ to M; but $h(\hat{x})(1) = h(\hat{x}(1)) = h(x) = \widehat{h(x)}(1)$ and hence $h(\hat{x})(\lambda) = \widehat{h(x)}(\lambda)$ – that is, $h(\lambda x) = \lambda h(x)$ – for all λ in $[0, \infty]$; similarly $\sum(\lambda x_0, \lambda x_1, \ldots) = \lambda \sum(x_0, x_1, \ldots)$, as required. We remark that a direct verification of the commutativity of multiplication on $[0, \infty]$ is now immediate (handy if one were to use magnitude modules to develop the reals ab initio): both $\lambda\mu$ and $\mu\lambda$, considered as functions of λ with μ fixed, are endomorphisms of $[0, \infty]$ agreeing at $\lambda = 1$ ($\mu 1 = \mu$ by the definition of scalar multiplication).

Using scalar multiplication as a primitive notion we can give an alternative (and somewhat pedestrian) definition of a magnitude module as a set M with a nullary operation 0, an ω-ary operation $\sum$, and an action $[0, \infty] \times M \to M$ satisfying the previous (i) and (ii) and also the laws

$$(\lambda\mu)x = \lambda(\mu x),\ 1x = x,\ 0x = 0$$
$$(\lambda_0 + \lambda_1 + \ldots)x = \sum(\lambda_0 x, \lambda_1 x, \ldots),\ \lambda \sum(x_0, x_1, \ldots) = \sum(\lambda x_0, \lambda x_1, \ldots).$$

Hence the name 'magnitude module', $[0, \infty]$ being the semiring of 'magnitudes' as in Huntingdon [6] (where, however, 0 and ∞ are excluded).

A magnitude *algebra*, in contrast, is a magnitude module M together with a multiplication $M \times M \to M$ which is a bimorphism, that is, a morphism in each argument separately (and which will usually be associative with identity element 1). For instance $[0, \infty]$ is a (commutative) magnitude algebra and so are many modules of $[0, \infty]$– valued functions, such as the specific examples mentioned in (2.2) above; if M is any magnitude module then End (M) is a magnitude algebra with multiplication = composition.

As in every equational class, the tensor product $M \otimes N$ of two magnitude modules M and N can be formed so as to represent all bimorphisms out of $M \times N$ by morphisms out of $M \otimes N$. The usual adjointness relation to the internal Hom functor holds, namely

$$\text{Hom}\,(L \otimes M, N) \cong \text{Hom}\,(L, \text{Hom}\,(M, N)).$$

7. RELATIONSHIP TO CARDINAL ALGEBRAS, ETC

The notion of a magnitude module owes a lot to the theory of cardinal algebras, as developed in Tarski's book [11]. In [11], pp. 11 and 12, Tarski gives an axiomatization of cardinal algebras in terms of the infinite addition operation $\sum$ only, from which (i) and (ii) of our axioms are taken (to be precise, (ii) replaces Tarski's two axioms for zero). In cardinal

algebras, two further conditions are imposed on $\sum$: the refinement and remainder conditions, neither of which necessarily hold in magnitude modules; again, in a cardinal algebra, every infinite sum is the least upper bound of its finite partial sums ([11], Theorem 2.21) and this need not be the case in a magnitude module (see below for counterexamples). On the other hand, cardinal algebras do not in general admit a halving operation – equivalently, scalar multiplication by $[0, \infty]$ – but when they do (a possibility discussed by Tarski, [11], p. 38) they are magnitude modules [1].

The posets in figs. 2 and 3 give magnitude modules of the type considered in (2.3) which fail the refinement and remainder conditions respectively (in fig. 3, $a_n = b_n + a_{n+1}$ for all n but there is no element c such that $a_n = c + b_n + b_{n+1} + \ldots$ for all n).

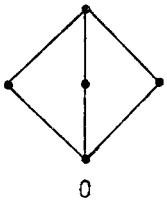


Fig. 2

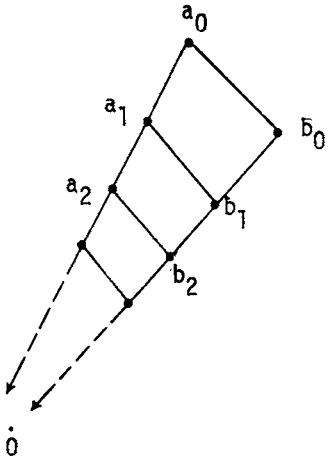


Fig. 3

To see that an infinite sum in a magnitude module is not always the least upper bound of its finite partial sums, posets are no use and we consider the free magnitude module M on generators

$$a_0, a_1, a_2, \ldots, b_0, b_1, b_2, \ldots, c$$

subject to the relations

$$c = a_0 + b_0 = a_0 + a_1 + b_1 = a_0 + a_1 + a_2 + b_2 = \ldots,$$

so that we have $a_0, a_0 + a_1, a_0 + a_1 + a_2, \ldots \leqslant c$ in M. It is easily verified that the only relations of the form $c = \sum \alpha_n a_n + \sum \beta_n b_n + \gamma c$ holding in

[1] It should be remarked that Chuaqui (Trans. Amer. Math. Soc. 142 (1969), 61–79; Fund. Math. 71 (1971), 77–84) has applied cardinal algebras specifically to measure theory and integration.

M are those with

$$\alpha_0 = \beta_0 + \beta_1 + \beta_2 + \ldots, \quad \alpha_1 = \beta_1 + \beta_2 + \beta_3 + \ldots, \text{ etc.},$$
$$\beta_0 + \beta_1 + \beta_2 + \ldots + \gamma = 1.$$

It follows that no relation of the form $c = \sum a_n + d$ holds in M, so that $\sum a_n \leqslant c$.

In conclusion, I would like to say that the stimulus for the introduction of magnitude modules was a question of Lawvere as to whether a direct definition of the continuum, appropriate for use in a topos, could be given. The present magnitude module definition does not really fit the bill, there being too much digitry in the reliance on the natural numbers whereas the continuum is a more immediate intuition (from which, granted, 0, 1, 2, 3, ... may be extracted). Nevertheless, in a topos with a natural numbers object, the free magnitude module on 1 will exist and one can ask what it looks like: it appears to correspond to the less interesting real numbers object obtained by Cantor's construction using Cauchy sequences (in the topos of sheaves on a topological space X, it would be given by the locally constant $[0, \infty]$-valued functions on X). Perhaps more interesting would be to take the free magnitude module on Ω subject to the relation false $= 0$ (for a space X, we presumably now obtain the sheaf of lower semicontinuous $[0, \infty]$-valued functions on X).

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