Available at
www.ElsevierComputerScience.com
POWERED BY SCIENCE DIRECT
Theoretical Computer Science 311 (2004) 71–103

Theoretical Computer Science

www.elsevier.com/locate/tcs

Paracategories II: adjunctions, fibrations and examples from probabilistic automata theory

Claudio Hermida[1], P. Mateus*,[2]

Departamento de Matematica, Instituto Superior Technico (IST), Center for Logic and Computation, Av. Rovisco Pais, Lisboa 1049-001, Portugal

Received 13 September 2002; received in revised form 19 February 2003; accepted 23 February 2003
Communicated by D. Sannella

Abstract

In this sequel to Hermida and Mateus (Paracategories I: internal paracategories and saturated partial algebras, Theoret. Comput. Sci., in press), we explore some of the global aspects of the category of paracategories. We establish its (co)completeness and cartesian closure. From the closed structure we derive the relevant notion of *transformation* for paracategories. We set up the relevant notion of *adjunction* between paracategories and apply it to define (co)completeness and cartesian closure, exemplified by the paracategory of bivariant functors and dinatural transformations. We introduce *partial multicategories* to account for partial tensor products. We also consider fibrations for paracategories and their indexed-paracategory version. Finally, we instantiate all these concepts in the context of probabilistic automata.
© 2003 Elsevier B.V. All rights reserved.

Keywords: Paracategories; Partial multicategories; Probabilistic automata

1. Introduction

Paracategories were originally proposed by Freyd [7]. A paracategory can be understood as a category in which the composition of morphisms is a *partial* operation. Freyd produced an *enveloping category* construction, which allows us to understand

* Corresponding author.
E-mail address: pmat@math.ist.utl.pt (P. Mateus).
[1] Partially supported by *Fundação* para a Ciência e a Tecnologia, FCT project Praxis XXI - 18976 - 98.
[2] Partially supported by *Fundação* para a Ciência e a Tecnologia, FCT PRAXIS XXI Project PRAXIS/P/MAT/10002/1998 Problog, FCT Project FEDER POCTI/2001/MAT/37239 Fiblog and FCT grant SFRH/BPD/5625/2001.

0304-3975/$ - see front matter © 2003 Elsevier B.V. All rights reserved.
doi:10.1016/S0304-3975(03)00317-7

paracategories as categories with a distinguished collection of morphisms. A motivating example is that of dinatural transformations [19, Section IX.4], whose pointwise composition is not necessarily dinatural.

In [12] we reexamined Freyd's notion (which is presented as a Horn theory) and produced a different (but equivalent) axiomatisation. First, using the monoid classifier Δ, we recasted paramonoids as *strong monoidal saturated lax functors* $\mathsf{M} : \Delta \to \mathsf{Ptl}_{\mathscr{M}}(\mathbb{B})$ (with non-empty support) into a bicategory of partial maps. This notion gave us a suitable internal version of paramonoid/paracategory. With further assumptions on the ambient category $\mathbb{B}$, we reelaborated paramonoids as *saturated partial algebras* $x : \mathsf{M}x \to x$ for the free-monoid monad M. We also established the basic result on paramonoids at the level of partial algebras, namely, the existence of the enveloping algebra construction and the fact that saturated partial algebras can be recovered from their enveloping ones.

In this paper we continue our study of paracategories by exploring the global structure of $\mathscr{P}ar\mathscr{C}at$, the category of paracategories and functors. For this, we rely on the reflectivity of $\mathscr{P}ar\mathscr{C}at$ in $\mathscr{C}at_P$, the category of categories-with-distinguished-subcollection-of-morphisms. This latter is seen to be (co)complete and cartesian closed (as it is fibred over $\mathscr{C}at$). An important property of the enveloping category construction is that it preserves finite products (Proposition 3), so that the reflection from $\mathscr{C}at_P$ into $\mathscr{P}ar\mathscr{C}at$ implies that the latter is cartesian closed as well. This closed structure yields a natural notion of *transformation* for paracategories.

Although $\mathscr{P}ar\mathscr{C}at$ with this two-dimensional structure does not form a 2-category (its Hom's are only paracategories), we can nevertheless use it to obtain a meaningful notion of *adjunction*. The traditional definition of adjunction between categories [19, Section IV.1] is that of a natural bijection

$$\theta_{X,Y} : \ \mathbb{C}(FX, Y) \cong \mathbb{B}(X, GY)$$

for functors $F : \mathbb{B} \to \mathbb{C}$ and $G : \mathbb{C} \to \mathbb{B}$. After [17], where comma-categories were introduced, we can equally understand the above natural bijection as an isomorphism of comma-categories θ: $F \downarrow \mathbb{C} \cong \mathbb{B} \downarrow G$ compatible with the projections to $\mathbb{B}$ and $\mathbb{C}$. This approach is applicable in our present context: we can sensibly define *comma-paracategories* and demand an isomorphism as above. This definition yields the usual unit/counit data and the triangular identities, but it is stronger in that it guarantees that enough composites are defined so as to perform adjoint-transposition.

Since adjoints are unique *up to isomorphism*, we analyse in Section 5 the appropriate notion of isomorphism in the context of partial composition. To this end, we introduce the notion of *tightness* to single out those morphisms whose presence does not affect the compositionality of sequences of morphisms on their source/target, and contend that isomorphisms in paracategories should be tight. Tightness conditions also apply to universal morphisms, for instance, in the context of partial multicategories and fibred paracategories, which we introduce in Sections 7 and 8, respectively.

Similarly to the comma-paracategory construction, we also have at our disposal *cotensoring* of paracategories by categories (inherited via the reflection from $\mathscr{C}at_P$). Thus we can define the notion of *limit*/*colimit* for paracategories. In particular, we can make sense of *products* in a paracategory. Assuming these, we can further demand the

relevant adjoints so as to have *cartesian closed* paracategories (Section 6.2). In fact, with these definitions we can accommodate Freyd's intended example of the cartesian closed paracategory of bivariant functors $T:\mathbb{C}^{\mathrm{op}}\times\mathbb{C}\to\mathbb{D}$ (with $\mathbb{D}$ a cartesian closed category) and dinatural transformations between these (Example 21).

In [20], the combination of probabilistic automata involves a certain 'product without diagonals'. Following [11], we introduce *partial multicategories* (which is an instance of the abstract notion of paramonoid or saturated partial algebra from [12]) and define the appropriately restricted notion of *representability*. This framework allows us to exhibit the above 'product without diagonals' as the *tensor product* corresponding to the representability of a suitable partial multicategory.

We then examine the relevant notion of *fibration* of paracategories. We show that the traditional Grothendieck correspondence between fibrations and contravariant functors into $\mathscr{C}at$ applies in this context (Proposition 33), once again appealing to the enveloping category construction.

We finally consider, as an extended example of the notions so far introduced, their application in the context of *probabilistic automata*. The study of this 'category with partial composites' of probabilistic automata was the motivation behind Mateus' et al. [20,21] preliminary study of the notion of *precategory*. A precategory is defined similarly to a paracategory but only with a *binary* partial composition. But once composition is partial, it is rather cumbersome to define a single n-ary composite out of the binary ones, thus the paracategory approach is to consider all partial n-ary composites at once, with the associativity/saturation relation among them. Thus, we have rephrased (and extended) the (pre)categorical material in *ibid.* in the context of paracategories.

Although all the concepts developed for paracategories here make sense at the internal level (as in [12]), at present our examples correspond to the ordinary, $\mathscr{S}et$-based scenario. Therefore we have chosen to present matters concretely.

2. Freyd's paracategories

We recall from [7] the elementary definitions of paramonoid and paracategory and their corresponding morphisms.

- A *paramonoid* consists of a set M and n-ary partial operations $\otimes_n : M^n \rightharpoonup M$, which we write indistinctly as $[_]$ for any arity. These operations are subject to the following axioms:
  (1) ${}_0\otimes : M^0(=1) \rightharpoonup M$ is total.
  (2) ${}_1\otimes = (id, id) : M \rightharpoonup M$.
  (3) If $[\vec{y}]$ is defined then $[\vec{x}\,[\vec{y}]\,\vec{z}] = [\vec{x}\vec{y}\vec{z}]$,
  where the equality in the last axiom above is Kleene equality (if either side is defined so is the other and then they are equal).
- A *paracategory* $\mathbb{C}$ consists of a directed graph $C_0 \overset{d}{\leftarrow} C_1 \overset{c}{\rightarrow} C_0$ and partial n-ary operations $\circ_n : C_n \rightharpoonup C_1$, where

$$C_n = C_1 \times_{C_0} \cdots \times_{C_0} C_1 = \{(f_1, \ldots, f_n) | c f_i = d f_{i+1},\ 1 \leqslant i \leqslant n-1\}$$

is the set of composable n-tuples of morphisms. They are subject to the following axioms

(1) $\circ_0 = \iota : C_0 \rightharpoonup C_1$ is total. This yields *identity morphisms* $id_A : A \to A$ for every object $A \in C_0$.
(2) $\circ_1 = (id, id) : C_1 \rightharpoonup C_1$.
(3) If $\circ_n \vec{y}$ is defined, then

$$\circ_{m+1+n}(\vec{x}, \circ_k(\vec{y}), \vec{z}) = \circ_{m+k+n}(\vec{x}, \vec{y}, \vec{z}),$$

where $\vec{x} \in C_m$, $\vec{y} \in C_k$ and $\vec{z} \in C_n$.

- A *functor* between paracategories $\mathbb{C}$ and $\mathbb{D}$ is a morphism of graphs

$$(f_0 : C_0 \to D_0,\ f_1 : C_1 \to D_1)$$

such that if $[\vec{x}]$ is defined, then $f_1[\vec{x}] = [\vec{f_1 x}]$ (notice that this entails preservation of identities). The functor is called a *Kleene functor* if $[\vec{f_1 x}] = f_1 y$ implies $[\vec{x}] = y$.
- A *subparacategory* of a paracategory $\mathbb{C}$ is a subgraph such that the inclusion is a Kleene functor.

A category $\mathbb{C}$ and a subset of morphisms $P \subseteq C_1$ (including the identity morphisms) determines a subparacategory, to wit, that where $[\vec{x}]$ is defined whenever the composite of the tuple $\vec{x}$ in $\mathbb{C}$ belongs to P. Similarly a functor between categories $F : \mathbb{C} \to \mathbb{D}$ with distinguished subsets of morphisms $P \subseteq C_1$ and $Q \subseteq D_1$ such that $f_1(P) \subseteq Q$ determines a functor between the induced paracategories (see [12] for a more general version of this construction in the setting of partial algebras for a monad). Let $\mathscr{P}ar\mathscr{C}at$ denote the category of paracategories and functors and $\mathscr{C}at_P$ the category of categories with distinguished subsets of morphisms (containing the identities) and functors compatible with such subsets. The construction above yields a functor $U : \mathscr{C}at_P \to \mathscr{P}ar\mathscr{C}at$.

Proposition 1 (Enveloping Category [7]). *The functor* $U : \mathscr{C}at_P \to \mathscr{P}ar\mathscr{C}at$ *admits a* (*fully faithful*) *left adjoint* EC.

Proof. Given a paracategory $\mathbb{C}$ define a category $\mathsf{EC}(\mathbb{C})$ as follows: let $\mathsf{M}(\mathbb{C})$ be the free category on the underlying graph of $\mathbb{C}$ and let $\mathsf{EC}(\mathbb{C})$ be the quotient of $\mathsf{M}(\mathbb{C})$ by the relation

$$\langle x_1, \ldots, x_n \rangle \sim \langle [x_1, \ldots, x_n] \rangle$$

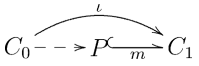
whenever $[x_1, \ldots, x_n]$ is defined. The distinguished subset of morphisms $P_{\mathsf{EC}(\mathbb{C})}$ consists of the (equivalence classes of) singletons. This construction extends in an obvious manner to functors of paracategories to yield the desired left adjoint. See [12] for further details, as well as the general result at the level of partial algebras. □

One important consequence of the above enveloping category construction is that it explains precisely how every paracategory arises, namely by specifying a collection of morphisms in a given category (the left adjoint EC being fully faithful is equivalent to the unit $\eta : \mathbb{C} \to U\mathsf{EC}(\mathbb{C})$ being an isomorphism [19, Section IV.3, Theorem 1]).

3. The global structure of the category of paracategories

Given the enveloping construction of Proposition 1, we can deduce properties of $\mathscr{P}ar\mathscr{C}at$ from the corresponding ones in $\mathscr{C}at_P$. So we analyse this latter category first.

First of all, let us be more precise about $\mathscr{C}at_P$. Its objects are small categories, with graph $C_0 \overset{d}{\leftarrow} C_1 \overset{c}{\rightarrow} C_0$ and a subset $m : P \hookrightarrow C_1$ such that the identities $\iota : C_0 \to C_1$ belong to it:



Notice that this includes the empty category with empty distinguished subobject of morphisms. There is an evident forgetful functor $\underline{U} : \mathscr{C}at_P \to \mathscr{C}at$ (throw away P).

Proposition 2. *The category $\mathscr{C}at_P$ is (co)complete and cartesian closed.*

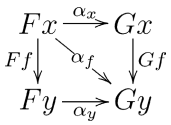
Proof. We spell out the cartesian closed structure of $\mathscr{C}at_P$ at the level of objects, the verification of the relevant universal properties being straightforward: given categories $\mathbb{C}$ (with a subset of morphisms C) and $\mathbb{D}$ (with a subset of morphisms D) we have:

product: $\mathbb{C} \times \mathbb{D}$ with subset of morphisms $C \times D$.

internal-hom: $[\mathbb{C}, \mathbb{D}]$, the category whose objects are those functors from $\mathbb{C}$ to $\mathbb{D}$ preserving distinguished morphisms, and whose arrows are natural transformations. The distinguished subset of morphisms (natural transformations) is

$$C \Rightarrow D = \{\alpha : F \Rightarrow G : \mathbb{C} \to \mathbb{D} \mid \forall (f : x \to y) \in C \Rightarrow (\alpha_f : Fx \to Gy) \in D\},$$

where α_f is the diagonal of the naturality square:



Notice that, since we assume that the subsets of morphisms include the identities, the components $\alpha_x = \alpha_{id_x}$ of the transformations in the subset $C \Rightarrow D$ lie in D. □

We now would like to transfer the above structure to $\mathscr{P}ar\mathscr{C}at$. The forgetful functor $U : \mathscr{C}at_P \to \mathscr{P}ar\mathscr{C}at$, being a right adjoint, preserves products. Since $\mathscr{C}at_P$ is a cartesian closed category, we would seek a similar structure on $\mathscr{P}ar\mathscr{C}at$ in such a way that U preserves it, thereby relating the enrichment of both categories. Thus, just like we obtain the partial composites in a paracategory by restriction of the composition in the ambient category, we would proceed in the same way for whatever additional structure this latter might carry.

Given a coreflection $J \dashv U : \mathbb{D} \to \mathbb{C}$ where both categories admit finite products and $\mathbb{D}$ is cartesian closed, $\mathbb{C}$ is cartesian closed whenever the embedding J preserves finite products [5,24]. This is indeed the case for the embedding $\mathsf{EC} : \mathscr{P}ar\mathscr{C}at \to \mathscr{C}at_P$

(Proposition 2.1 shows it is full and faithful). We show this for paramonoids to simplify the presentation, the extension to paracategories being entirely straightforward.

Proposition 3. *The enveloping monoid functor* $\mathsf{E} : \mathscr{P}ar\mathscr{M}on \to \mathsf{Mon}_P$ *preserves finite products.*

Proof. Recall from the proof of Proposition 1 above that the enveloping monoid $\mathsf{E}(M)$ of a paramonoid M is obtained as a quotient of the free monoid on M, $\mathsf{M}(M)$.

Given paramonoids M_1, M_2, their product is given by the product of their carriers and the corresponding tupling of partial compositions. The free-monoid monad $\mathsf{M} : \mathscr{S}et \to \mathscr{S}et$ preserves pullbacks (in fact, this is true in any elementary topos with a natural-numbers-object [4]). Thus we have a pullback

$$\begin{array}{ccc} \mathsf{M}(M_1 \times M_2) & \xrightarrow{\mathsf{M}\pi'} & \mathsf{M}M_2 \\ {\scriptstyle \mathsf{M}\pi}\downarrow & & \downarrow{\scriptstyle \mathsf{M}!} \\ \mathsf{M}M_1 & \xrightarrow[\mathsf{M}!]{} & \mathsf{M}\mathbf{1} \end{array}$$

where $\mathsf{M}\mathbf{1} = \mathsf{N}$ is the natural-numbers-object of $\mathscr{S}et$. Since $\mathsf{M}(M_1 \times M_2)$ consists of *lists of pairs*, the above square identifies

lists of pairs = *pairs of lists of the same length*

via the canonical comparison morphism $\langle \mathsf{M}\pi, \mathsf{M}\pi' \rangle : \mathsf{M}(M_1 \times M_2) \to \mathsf{M}M_1 \times \mathsf{M}M_2$.

Using the units of the monoids, we can transform any pair of lists into an equivalent pair whose lists have the same length: define $\phi : \mathsf{M}M_1 \times \mathsf{M}M_2 \to \mathsf{M}(M_1 \times M_2)$ by

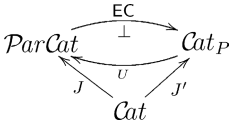
$$\phi(\vec{x}, \vec{y}) = (\vec{x}.[e_1]^{n-|\vec{x}|}, \vec{y}.[e_2]^{n-|\vec{y}|}),$$

where $n = \mathsf{max}(|\vec{x}|, |\vec{y}|)$ and e_1 and e_2 are the units of M_1 and M_2, respectively. Since upon passage to the quotient $(\vec{x}, \vec{y}) \sim \phi(\vec{x}, \vec{y})$, we conclude $\mathsf{E}(M_1 \times M_2) \cong \mathsf{E}(M_1) \times \mathsf{E}(M_2)$. □

Corollary 4. (1) *The category* $\mathscr{P}ar\mathscr{C}at$ *is* (*co*)*complete and cartesian closed.*

(2) *The functors* $U : \mathscr{C}at_P \to \mathscr{P}ar\mathscr{C}at$ *and* $\mathsf{EC} : \mathscr{P}ar\mathscr{C}at \to \mathscr{C}at_P$ *preserve the closed structure.*

(3) *The embeddings* $J : \mathscr{C}at \to \mathscr{P}ar\mathscr{C}at$ *and* $J' : \mathscr{C}at \to \mathscr{C}at_P$ (*which sends a category to itself with all the morphisms as distinguished subset*) *preserve the cartesian closed structure and the following diagram commutes*:



The cartesian closed nature of $\mathscr{P}ar\mathscr{C}at$ means that we obtain a meaningful notion of transformation for paracategories, by restricting the transformations given in the

internal-hom of $\mathscr{C}at_P$ to the corresponding functors induced by the envelope functor EC:

Definition 5. Given functors $F, G : \mathbb{C} \to \mathbb{D}$, a *transformation* $\alpha : F \Rightarrow G$ consists of a collection of morphisms $\{\alpha_x : Fx \to Gx\}_{x \in \mathbb{C}}$ of morphisms of $\mathbb{D}$ (indexed by the objects of $\mathbb{C}$) such that both composites in the square

$$\begin{array}{ccc} Fx & \xrightarrow{\alpha_x} & Gx \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ Fx & \xrightarrow[\alpha_y]{} & Gy \end{array}$$

are defined and equal for every morphism $f : x \to y$ in $\mathbb{C}$. Thus $\mathscr{P}ar\mathscr{C}at(\mathbb{C}, \mathbb{D})$ denotes the paracategory of functors and transformations between them, with partial compositions defined pointwise in $\mathbb{D}$: $(\beta \circ \alpha)_x = \beta_x \circ \alpha_x$. Thus, $\beta \circ \alpha$ is defined when *all* such composites *and the corresponding diagonal morphisms* are defined.

Notice that if $\mathbb{D}$ is a category, so is $\mathscr{P}ar\mathscr{C}at(\mathbb{C}, \mathbb{D})$. Just like in $\mathscr{C}at$, there are horizontal composites of transformations, obtained by vertical composites and (pre)composition of functors with transformations.

Remark 6.

- The cartesian-closure-preserving embedding $J' : \mathscr{C}at \to \mathscr{C}at_P$ is right adjoint to the evident forgetful functor $\underline{U} : \mathscr{C}at_P \to \mathscr{C}at$.
- The definition of transformations in $\mathscr{P}ar\mathscr{C}at$ is forced upon us by regarding $\mathscr{P}ar\mathscr{C}at$ as a $\mathscr{P}ar\mathscr{C}at$-category. It agrees with the definition given in [22] in the context of precategories.
- Although we have formulated the above definitions and statements for ordinary paracategories, the whole matter internalises straightforwardly, assuming the ambient category $\mathbb{B}$ is cartesian closed, in addition to the requirements in [12].

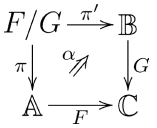
Before passing to specific constructions in $\mathscr{P}ar\mathscr{C}at$ and $\mathscr{C}at_P$, let us notice that the usual *dual category* construction, obtained by formally reversing the orientation of the arrows in a category, applies literally to this setting. Thus we can speak of the dual paracategory $\mathbb{C}^{\mathrm{op}}$.

4. Comma-paracategories

In order to produce an appropriate notion of adjunction for paracategories, we show that the usual comma-category construction applies in the context of paracategories

Definition 7. Given functors $F : \mathbb{A} \to \mathbb{C}$ and $G : \mathbb{B} \to \mathbb{C}$, its *comma-paracategory* F/G is the paracategory whose objects are triples $(x, a : Fx \to Gy, y)$ and whose morphisms $(f, g) : (x, a : Fx \to Gy, y) \to (x', a' : Fx' \to Gy', y')$ are pairs of morphisms $f : x \to x'$ in $\mathbb{A}$ and $g : y \to y'$ in $\mathbb{B}$ such that $a' \circ Ff = Gg \circ a$ with both sides defined. The partial composites are inherited componentwise from $\mathbb{A}$ and $\mathbb{B}$. We have thus a 'universal

transformation'



where α has components $\alpha_{(x,a:Fx\to Gy,y)}=a$. The comma paracategory is characterised by the following universal property:

$$\mathscr{P}ar\mathscr{C}at(\mathbb{X},F/G)\cong\mathscr{P}ar\mathscr{C}at(\mathbb{X},F)/\mathscr{P}ar\mathscr{C}at(\mathbb{X},G)$$

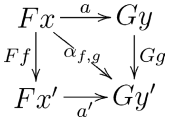
naturally in (the paracategory) $\mathbb{X}$, which means that the *lax square* above is the universal one with respect to F and G.

When $G=id_{\mathbb{C}}$, we write $F/\mathbb{C}$ for F/G and similarly when $F=id_{\mathbb{C}}$.

An analogous construction of the *comma-object* (in the spirit of 2-category theory [23]) can be carried out in $\mathscr{C}at_P$: given functors $F:(\mathbb{A},A)\to(\mathbb{C},C)$ and $G:(\mathbb{B},B)\to(\mathbb{C},C)$ their comma-object F/G in $\mathscr{C}at_P$ is the category $C\cap(F/G)$ with

objects: Triples $(x,a:Fx\to Gy,y)$ with $a\in C$.

morphisms: $(f,g):(x,a:Fx\to Gy,y)\to(x',a':Fx'\to Gy',y')$ given by $f:x\to x'$ in $\mathbb{A}$ and $g:y\to y'$ in $\mathbb{B}$ such that the outer square



commutes. the distinguished collection of morphisms in $C\cap(F/G)$ is the set

$$\{(f,g):(x,a:Fx\to Gy,y)\to(x',a':Fx'\to Gy',y')\mid f\in A\wedge g\in B\wedge\alpha_{f,g}\in C\}.$$

Thus, the 'universal 2-cell' $\alpha:F\pi\Rightarrow G\pi'$ is a 'distinguished' transformation (as it should be).

We should point out that the correspondence of the notions of transformations in $\mathscr{P}ar\mathscr{C}at$ and $\mathscr{C}at_P$ (Definition 5 and the distinguished subset of the internal-hom $C\Rightarrow D$ of Proposition 2, respectively) entails that of comma-objects: [3]

Proposition 8. *Given functors $F:\mathbb{A}\to\mathbb{C}$ and $G:\mathbb{B}\to\mathbb{C}$ in $\mathscr{P}ar\mathscr{C}at$*

$$\mathsf{EC}(F/G)\equiv\mathsf{EC}(F)/\mathsf{EC}(G).$$

[3] Although we have not introduced equivalences of objects in $\mathscr{C}at_P$ the notion should be evident: an adjoint equivalence of the underlying categories preserving distinguished morphisms, with 'distinguished' natural isomorphisms as unit/counit.

Thus, in both the $\mathscr{P}ar\mathscr{C}at$ and $\mathscr{C}at_P$ context comma-objects classify transformations: to give a transformation $\alpha : FX \Rightarrow GY$ (for functors $X : \mathbb{D} \to \mathbb{A}$ and $Y : \mathbb{D} \to \mathbb{B}$) amounts to giving a functor $\hat{\alpha} : \mathbb{D} \to F/G$ such that

$$\pi\hat{\alpha} = X \quad \pi'\hat{\alpha} = Y.$$

5. Tight and total morphisms

There are morphisms in a category that play a special role, notably *isomorphisms*, which allow us to interchange their source and target in just about every context (universally characterised objects are unique only up to a canonical isomorphism between any two of them) and *cartesian* morphisms in a fibred category, which establish an essentially *virtual* connection between their source and target (in view of the Grothendieck construction, we can rightfully consider them as *virtual fillers*). In the context of paracategories, these kind of morphisms should satisfy a further property if there are to serve the same role as in ordinary categories, namely, they should not affect the *composability* of sequences of morphisms into either their source or target objects. This leads us to introduce the following notions:

Definition 9. (1) A morphism $f : x \to y$ of a paracategory $\mathbb{C}$ is called *left tight* if for every sequence of morphisms $\vec{h} : x' \to x$ on its source,

$$[f, \vec{h}] \text{ is defined iff } [\vec{h}] \text{ is defined.}$$

We also have the attendant dual notion of *right tightness* (left tightness in the dual paracategory $\mathbb{C}^{\mathrm{op}}$).

(2) A morphism $f : x \to y$ of a paracategory $\mathbb{C}$ is called *left total* if for every sequence of morphisms $\vec{h} : x' \to x$ on its source,

$$[f, \vec{h}] \text{ is defined if } [\vec{h}] \text{ is defined}$$

with the attendant dual notion of *right totality*.

In diagrammatic terms, the left/right tightness of f amounts to the following diagrams being pullbacks (for all n):

$$\begin{array}{ccc} D_n & \dashrightarrow & D_{n+1} \\ {\scriptstyle d_n}\downarrow & & \downarrow{\scriptstyle d_{n+1}} \\ C_n & \xrightarrow[[f,_]]{} & C_{n+1} \end{array} \qquad \begin{array}{ccc} D_n & \dashrightarrow & D_{n+1} \\ {\scriptstyle d_n}\downarrow & & \downarrow{\scriptstyle d_{n+1}} \\ C_n & \xrightarrow[[_,f]]{} & C_{n+1} \end{array}$$

respectively, where $d_n : D_n \hookrightarrow C_n$ is the domain of definition of the n-ary partial composition of $\mathbb{C}$.

Notice that left totality for $f : x \to y$ amounts to the totality of the functions $[f, _] : \mathbb{C}(y,z) \to \mathbb{C}(x,z)$ for every object z.

Given an isomorphism, i.e. morphisms $f : x \to y$ and $g : y \to x$ such that $f \circ g = id$ and $g \circ f = id$, we have that f is left/right tight iff g is, hence there is no ambiguity in referring to left/right tight isomorphisms.

The counterpart of the notions of tightness and totality in categories with a distinguished subset of morphisms $(\mathbb{C}, C)$ simply replaces "is defined" by "belongs to the distinguished subset C". For instance, $f : x \to y$ is *right tight* iff for all $h : x' \to x$

$$(f \circ h) \in C \quad \Leftrightarrow \quad h \in C.$$

In particular, f itself is in C. We will see that with all the notions introduced in this paper, it is usually simpler to reason with paracategories in the context of $\mathscr{C}at_P$ appealing to the enveloping construction of Proposition 1. For tight/total morphisms, we have the following correspondence:

Proposition 10. *Given a paracategory $\mathbb{C}$, a morphism $f : x \to y$ is (left/right) tight/total iff its image $[\langle f \rangle]$ is so in $(\mathsf{EC}(\mathbb{C}), P_{\mathsf{EC}(\mathbb{C})})$.*

For our treatment of adjunctions in Section 6 we need to relativise the notions of tightness/totality, by extending these notions to objects of comma-paracategories:

Definition 11. Given functors $F : \mathbb{A} \to \mathbb{C}$ and $G : \mathbb{B} \to \mathbb{C}$, an object $(x, f : Fx \to Gy, y)$ of F/G is *left total* if for every (compatible) sequence of morphisms $\vec{k}$ in $\mathbb{B}$ with source y

$$[f, G\vec{k}] \text{ is defined whenever } [\vec{k}] \text{ is defined}$$

and (x, f, y) (or simply f) is *left tight* if

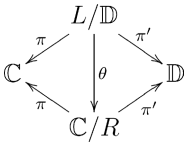
$$[f, G\vec{k}] \text{ is defined iff } [\vec{k}] \text{ is defined.}$$

6. Adjunctions of paracategories

In [22] the authors introduce a notion of transformation between functors of 'precategories' which amounts to the usual data for a natural transformation, subject to the requirement that the relevant composites in the naturality squares are defined, in accordance to our *derived* definition of transformation of paracategories above. The authors further analyse several elementary definitions of adjointness, and observe the important role of comma-categories in this setting.

As we explained in the introduction, we adopt the point of view of [17] and formulate the notion of adjunction in terms of isomorphism of comma-paracategories: [4]

Definition 12. A functor $L:\mathbb{C}\to\mathbb{D}$ is *left adjoint* to $R:\mathbb{D}\to\mathbb{C}$ if there exists an isomorphism of comma-paracategories $\theta:L/\mathbb{D}\xrightarrow{\sim}\mathbb{C}/R$ such that



Given an adjunction, which we write in the customary way $L\dashv R$, we obtain transformations $\eta:id\Rightarrow RL$ (the *unit*, $\eta=\theta(id_{L_})$) and $\varepsilon:LR\Rightarrow id$ (the *counit*, $\varepsilon=\theta^{-1}(id_{R_})$) verifying the *triangle identities*

$$\varepsilon R\circ R\eta=id,\quad L\varepsilon\circ\eta L=id.$$

It is important to emphasise that the existence of the isomorphism θ is a *stronger* requirement than that of the existence of η and ε satisfying the above equations. Namely, the fact that η is a transformation of paracategories ensures that for any $f:x\to y$ in $\mathbb{C}$, the composite $RL(f)\circ\eta_x$ is defined. The existence of θ demands more: that all composites of the form $R(g)\circ\eta_x$ (the *adjoint transpose* of g) exist, for given $g:Lx\to y$ in $\mathbb{D}$, the previous condition being the special case $g=Lf$. In fact, this extra composability condition (with the corresponding dual one on the counit) is enough to recover an adjunction from the unit/counit data satisfying the triangle identities, as we will show in Proposition 14 below.

In practice, we verify the *existence* of a left adjoint to a given functor $R:\mathbb{D}\to\mathbb{C}$ by means of a universal property: for every object $c\in\mathbb{C}$, there exists an object Lc of $\mathbb{D}$ and a universal morphism $\eta_c:c\to RLc$. This leads us to consider the notion of universal morphism in a paracategory:

Definition 13 (Universal morphisms). Given a functor of paracategories $R:\mathbb{D}\to\mathbb{C}$ and an object $c\in\mathbb{C}$, a *universal morphism* from c to R amounts to a pair $(d,u:c\to Rd)$ (d an object of $\mathbb{D}$) such that

- For every morphism $v:c\to Rd'$, there exists a unique $\hat{v}:d\to d'$ satisfying $R\hat{v}\circ u=v$ (which implies that the left-hand composite must be defined).
- u is *left tight* (as an object of $\mathbb{C}/R$) and *right total*.

The requirement that u be left tight means that u should set up a one-to-one correspondence between *composable sequences* of morphisms out of d (into some d') and such sequences between c and Rd'. The right totality means that any morphism 'reaching' c should be able to 'reach' Rd by postcomposing with u. Thus, u is in some sense 'virtual', preserving the composability properties of 'ambient' morphisms with respect to c and R.

[4] A similar approach is taken in [22] for adjunctions of precategories.

Now we can sum up the correspondences between our given notion of adjunction and its attendant alternatives:

Proposition 14. *Given a functor of paracategories* $R : \mathbb{D} \to \mathbb{C}$, *tfae*:
(1) *There exists a functor* $L : \mathbb{C} \to \mathbb{D}$ *left adjoint to* R (*via an isomorphism* $\theta : L/\mathbb{D} \xrightarrow{\sim} \mathbb{C}/R$ *as in Definition* 12).
(2) *There exists a functor* $L : \mathbb{C} \to \mathbb{D}$ *and transformations* $\eta : id \Rightarrow RL$ (*left total*) *and* $\varepsilon : LR \Rightarrow id$ (*right total*) *verifying the following triangle identities*

$$\varepsilon R \circ R\eta = id, \quad L\varepsilon \circ \eta L = id.$$

(3) *For every object* $c \in \mathbb{C}$ *there exists a universal morphism from* c *to* R, $(\tilde{c}, \eta_c : c \to R\tilde{c})$.

Proof. (1) $\Rightarrow$ (2): This direction was spelled out immediately after Definition 12. Left totality of the (components of) the unit is easily verified: for $f : Lc \to d$, $[\eta_c, Rf] = \theta(f) : c \to Rd$.

(2) $\Rightarrow$ (3): We claim that $(c, \eta_c : c \to RLc)$ is a universal morphism: it is right total by the naturality of η. To verify it is left tight, its left totality gives us one direction, so we must show that if $[\eta_c, R\vec{k}]$ is defined so is $\vec{k}$ (for a suitable sequence $\vec{k}$). We have that $[L[\eta_c, R\vec{k}], \varepsilon_y]$ is defined because ε_y is right total, and

$$[L\eta_c, LR\vec{k}, \varepsilon_y] = [L\eta_c, \varepsilon_{Lc}, \vec{k}] = [id, \vec{k}]$$

and thus $[\vec{k}]$ is defined as required. The one-to-one correspondence follows from the usual argument [19, Section IV.1, Theorem 2]).

(3) $\Rightarrow$ (1): Here once again the usual argument [19, Section IV.1, Theorem 2] applies to yield a morphism of graphs $L : \mathbb{C} \to \mathbb{D}$: for a morphism $f : c \to c'$, $Lc = \tilde{c}$ and $Lf = [\widehat{f, \eta_{c'}}]$, where $[f, \eta_{c'}]$ is defined by right totality of universals. Uniqueness of the mediating morphisms between universals ensure preservation of identities, while preservation of composites follows from uniqueness and *totality*. We have thus a functor L. The collection of universal morphisms η_c organise themselves into a transformation, which being left tight is therefore left total. The other transformation is obtained in the usual way: $\varepsilon_d = \widehat{id_{Rd}} : LRd \to d$. □

Remark 15.
- In our second formulation of adjunction in the above proposition (2), we could have equivalently required only that the unit be left tight: if the unit is left total, it is left tight iff the counit is right total.
- We direct the reader's attention to the fact that *reflecting composability* is crucial to guarantee the functoriality of the construction of a left adjoint out of collection of universal morphisms in the proof above. This is what motivated us to introduce *tightness* as a combination of preservation/reflection of composability suitable for use in this context. We will see another instance of this phenomenon in the transformation of fibrations into indexed paracategories in Proposition 33.

The same (equivalent) definitions of adjunction apply to $\mathscr{C}at_P$, namely:

- Isomorphism of comma-objects $\theta : L/(\mathbb{D}, D) \xrightarrow{\sim} (\mathbb{C}, C)/R)$. This means that we have isomorphisms

$$(\mathbb{D}(LX, Y) \cap D) \cong (\mathbb{C}(X, RY) \cap C)$$

  natural in X and Y.
- Transformations $\eta : id \Rightarrow RL$ (left total) and $\varepsilon : LR \Rightarrow id$ (right total) satisfying the triangle identities.
- The provision of a universal morphism (with the same definition as above for paracategories) $(d, \eta : c \to Rd)$ for every object $c \in \mathbb{C}$.

Just as expected, we do get a precise correspondence between adjunctions in $\mathscr{P}ar\mathscr{C}at$ and $\mathscr{C}at_P$ (bearing in mind the correspondence of comma-objects in both cases, Proposition 8):

Proposition 16 (Correspondence of adjunctions). (1) $L \dashv R : \mathbb{C} \to \mathbb{D}$ *in* $\mathscr{P}ar\mathscr{C}at$ *iff* $\mathsf{EC}L \dashv \mathsf{EC}R : \mathsf{EC}\mathbb{C} \to \mathsf{EC}\mathbb{D}$ *in* $\mathscr{C}at_P$.

(2) $L \dashv R : (\mathbb{C}, C) \to (\mathbb{D}, D)$ *in* $\mathscr{C}at_P$ *iff* $UL \dashv UR : U(\mathbb{C}, C) \to U(\mathbb{D}, D)$ *in* $\mathscr{P}ar\mathscr{C}at$.

(3) *A functor* $F : \mathbb{C} \to \mathbb{D}$ *in* $\mathscr{P}ar\mathscr{C}at$ *admits a right*(/*left*) *adjoint iff the functor* $\mathsf{EC}F : \mathsf{EC}\mathbb{C} \to \mathsf{EC}\mathbb{D}$ *admits a right*(/*left*) *adjoint in* $\mathscr{C}at_P$.

Just like in the case of ordinary of adjunctions between categories, an adjoint to a given functor of paracategories is unique up to canonical isomorphism whenever it exists. This is a straightforward consequence of the corresponding result for universal morphisms:

Lemma 17. *Given a functor* $R : \mathbb{D} \to \mathbb{C}$ *and an object* $c \in \mathbb{C}$, *and any two universal morphisms* $(d, u : c \to Rd)$ *and* $(d', v : c \to Rd')$, *there is a canonical isomorphism* $\hat{v} : d \to d'$, *left tight as a morphism of* $\mathbb{C}/R$.

Corollary 18 (Uniqueness of adjoints). *Given a functor* $R : (\mathbb{C}, C) \to (\mathbb{D}, D)$ *and two left adjoints* $L, L' : (\mathbb{D}, D) \to (\mathbb{C}, C)$, *there is a canonically determined natural isomorphism* $\phi : L \xrightarrow{\sim} L'$, *pointwise left tight in* $\mathbb{C}/R$.

6.1. Limits and colimits

Using the embedding of $\mathscr{C}at$ in $\mathscr{P}ar\mathscr{C}at$ and the closed structure of the latter, we may speak of *cotensors* in $\mathscr{P}ar\mathscr{C}at$ paraphrasing the enriched-category notion [15]. Thus for a paracategory $\mathbb{C}$ and a category X, the *cotensor* $\{X, \mathbb{C}\}$ is given by $\mathscr{P}ar\mathscr{C}at(J(X), \mathbb{C})$ which has the following universal property:

$$\mathscr{P}ar\mathscr{C}at(\mathbb{A}, \{X, \mathbb{C}\}) \cong \mathscr{P}ar\mathscr{C}at(J(X), \mathscr{P}ar\mathscr{C}at(\mathbb{A}, \mathbb{C})),$$

naturally on the paracategory $\mathbb{A}$. We have the corresponding diagonal functor $\delta : \mathbb{C} \to \{X, \mathbb{C}\}$, induced by the functor $x \mapsto id$ in $\mathscr{C}at(X, \underline{U}\mathsf{EC}(\mathscr{P}ar\mathscr{C}at(\mathbb{C}, \mathbb{C})))$. We can thus adopt the adjoint characterisation of (co)limits [19, Section IV.2] as our definition in the paracategorical context:

Definition 19. Let X be a small category. A paracategory $\mathbb{C}$ admits X-limits (colimits) if the diagonal $\delta : \mathbb{C} \to \{X, \mathbb{C}\}$ has a right (left) adjoint.

Applying Proposition 16(3) we can characterise (co)limit structure for a paracategory either in $\mathscr{P}ar\mathscr{C}at$ or $\mathscr{C}at_P$.

Corollary 20.
- *Given a category* $\mathbb{D}$ *with a subset of morphisms* D, $(\mathbb{D}, D)$ *has* X*-(co)limits* (δ *has an adjoint in* $\mathscr{C}at_P$) *iff* $U(\mathbb{D}, D)$ *has* X*-(co)limits.*
- *Given a paracategory* $\mathbb{C}$, $\mathbb{C}$ *has* X*-(co)limits iff* $\mathsf{EC}(\mathbb{C})$ *has* X*-(co)limits* (*in* $\mathscr{C}at_P$).

In elementary terms, the above means that if we consider a paracategory $\mathbb{C}$ arising from a category $\mathbb{D}$ and a subset of morphisms D, $\mathbb{C}$ will have X-limits if $\mathbb{D}$ has a cone (the counit of the adjunction) and diagonals (the unit of the adjunction) with components in D. Moreover given another cone with components in D, there must be a unique mediating morphism (in D) to the given (counit) cone. Notice that this property does not entail the existence of ordinary limits in $\mathbb{D}$. See Section 9.4 where the analogous situation with tensor products appears.

6.2. Cartesian closed paracategories

Given a paracategory $\mathbb{C}$ with finite products, every object $X \in \mathbb{C}$ determines a functor $X \times _ : \mathbb{C} \to \mathbb{C}$. The object X is *exponentiable* if this functor admits a right-adjoint $X \Rightarrow _ : \mathbb{C} \to \mathbb{C}$. The paracategory $\mathbb{C}$ is *cartesian-closed* if every object is exponentiable. We have thus a transformation $\varepsilon^X : X \times (X \Rightarrow _) \Rightarrow id$ satisfying the following universal property: given $f : X \times Y \to Z$, there is a unique morphism $\hat{f} : Y \to X \Rightarrow Z$ such that

$$\begin{array}{ccc} X \times Y & \xrightarrow{X \times \hat{f}} & X \times (X \Rightarrow Z) \\ & \searrow^{f} & \downarrow{\scriptstyle \epsilon^X_Z} \\ & & Z \end{array}$$

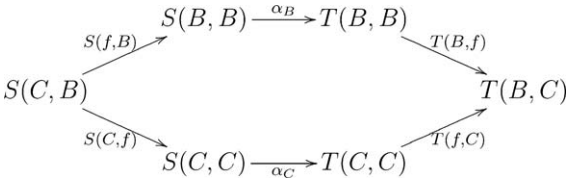
Furthermore, the components ε^X_Z being couniversal morphisms, they are left total and right tight. We also have the unit transformation $\eta^X : id \Rightarrow X \Rightarrow (X \times _)$ which satisfies the triangular identities with ε:

$$\varepsilon^X_{X \Rightarrow _} \circ (X \times \eta^X) = id, \quad (X \Rightarrow \varepsilon^X) \circ \eta^X_{X \Rightarrow _} = id.$$

Just like in the case of categories, the collection of $X \Rightarrow _$ organise themselves into a bivariant functor $_ \Rightarrow _ : \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbb{C}$. Bearing in mind the connection between cartesian closed categories and simply typed lambda calculus [13,16], we see that bivariant functors arise naturally in the categorical semantics of such calculi.

Example 21. Consider a (small) cartesian closed category $\mathbb{D}$ and the collection of bivariant functors $T : \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbb{D}$ from a given small category $\mathbb{C}$. Except for rather uninteresting $\mathbb{C}$, the category $\mathscr{C}at(\mathbb{C}^{\mathrm{op}} \times \mathbb{C}, \mathbb{D})$ will *not* be cartesian closed with the pointwise structure in $\mathbb{D}$. However, the situation becomes more interesting if we

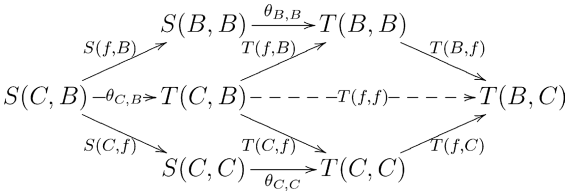
consider *dinatural transformations* rather than only natural ones. Recall (from [19, Section IX.4]) that given bivariant functors $S, T : \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbb{D}$, a *dinatural transformation* $\alpha : S \Rightarrow T$ is given by a collection of morphisms $\{\alpha_C : S(C,C) \to T(C,C)\}_{C \in \mathbb{C}}$ in $\mathbb{D}$ such that for every morphism $f : B \to C$ in $\mathbb{C}$ we have the following commutative hexagon in $\mathbb{D}$:



A problem arises: the componentwise composition of two dinatural transformations is not necessarily dinatural. Hence with this composition we get only a paracategory $Dinat(\mathbb{C}^{\mathrm{op}} \times \mathbb{C}, \mathbb{D})$ of bivariant functors and dinatural transformations.

The relationship between natural and dinatural transformations is the following:

- Given a natural transformation $\theta : S \Rightarrow T$, the collection of morphisms $\{\theta_{C,C} : S(C,C) \to T(C,C)\}_{C \in \mathbb{C}}$ is a dinatural transformation:



- Natural transformations act by composition on dinatural ones: given dinatural transformations $\alpha : S' \Rightarrow S$ and $\beta : T \Rightarrow T'$, the collections

$$\{\theta_{C,C} \circ \alpha_C : S'(C,C) \to T(C,C)\}_{C \in \mathbb{C}}$$

and

$$\{\beta_C \circ \theta_{C,C} : S(C,C) \to T'(C,C)\}_{C \in \mathbb{C}}$$

are dinatural.

Given $S, T : \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbb{D}$, the bivariant functor $(S \times T)(B,C) = S(B,C) \times T(B,C)$ together with the dinatural transformations $\pi_{C,C} : (S \times T)(C,C) \to S(C,C)$ and $\pi'_{C,C} : (S \times T)(C,C) \to T(C,C)$ form a product in $\mathscr{D} = Dinat(\mathbb{C}^{\mathrm{op}} \times \mathbb{C}, \mathbb{D})$. More generally, $\mathscr{D}$ inherits (co)completeness from $\mathbb{D}$, pointwise, since the relevant natural (co)limit cones are also dinatural and suitably closed under (pre)composition with dinaturals as indicated above.

For exponentials in $\mathscr{D}$, define

$$(S \Rightarrow T)(B,C) = S(C,B) \Rightarrow T(B,C).$$

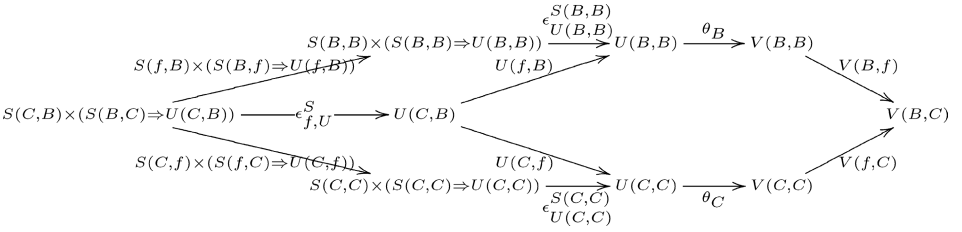
We have an associated dinatural transformation with components

$$\varepsilon^{S(C,C)}_{T(C,C)} : S(C,C) \times (S(C,C) \Rightarrow T(C,C)) \Rightarrow T(C,C).$$

This yields a transformation $\varepsilon : S \times (S \Rightarrow \pi') \Rightarrow \pi'$ in the paracategory sense: given a dinatural transformation $\theta : U \Rightarrow V$ we must show that both composites $\theta \circ \varepsilon$ and $\varepsilon \circ S \times (S \Rightarrow \theta)$ are defined and equal. Since $\mathbb{D}$ is cartesian closed, we work with the simply typed lambda-calculus which is its internal language [16] (or if the reader prefers, pretend $\mathbb{D}$ is $\mathscr{S}et$ and work with elements). So, $\varepsilon^{S(C,C)}_{T(C,C)}(s,\phi) = ev \circ \langle \phi, s \rangle$ (evaluate function ϕ at argument s). Given a morphism $f : B \to C$ in $\mathbb{C}$, define $\varepsilon^S_{f,T} : S(C,B) \times (S(B,C) \Rightarrow T(C,B)) \to T(C,B)$ as

$$\varepsilon^S_{f,T}(s,\phi) = ev \circ \langle \phi, S(\,f, f\,)s \rangle$$

so that $\varepsilon^{S(C,C)}_{T(C,C)} = \varepsilon^S_{id,T}$. We have the following commutative diagram:



so that $\theta \circ \varepsilon$ is a dinatural transformation. The equality $\varepsilon \circ S \times (S \Rightarrow \theta) = \theta \circ \varepsilon$ is easily verified, so that the left-hand side is a dinatural transformation as well. Similarly, the unit of the exponential adjunction in $\mathbb{D}$ becomes a dinatural transformation with components

$$\eta^{S(C,C)}_{T(C,C)} : T(C,C) \to S(C,C) \Rightarrow (S(C,C) \times T(C,C)).$$

In order to verify that we get a well-defined transformation $\eta^S : \pi' \Rightarrow S \Rightarrow (S \times \pi')$, define $\eta^S_{f,T} : T(B,C) \to S(C,B) \Rightarrow (S(B,C) \times T(B,C))$ (for $f : B \to C$ in $\mathbb{C}$) as

$$\eta^S_{f,T} t = \lambda s.\langle S(\,f, f\,)s, t \rangle,$$

so that $\eta^{S(C,C)}_{T(C,C)} = \eta^S_{id,T}$. A similar diagram chase like the one above for $\varepsilon^S_{f,T}$ shows that $\eta \circ \theta$ is a dinatural transformation whenever θ is. With all these data we can finally assert that every bivariant functor $S : \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbb{D}$ is exponentiable:

$$\mathscr{D}(S \times T, U) \cong \mathscr{D}(T, S \Rightarrow U).$$

Indeed, we are only left to verify that given a dinatural transformation $\theta : S \times U \Rightarrow T$, its adjoint transpose $\hat{\theta} = (S \Rightarrow \theta) \circ \eta$ is a dinatural transformation $\hat{\theta} : U \Rightarrow (S \Rightarrow T)$, which follows easily 'chasing elements' in λ-notation: given an element $u \in U(C,B)$ the composite

$$(S(\,f, B) \Rightarrow T(B, f\,)) \circ \hat{\theta}_B \circ U(\,f, B)$$

corresponds to the term

$$u : U(C,B) \mid \lambda s : S(C,B).T(B, f\,)(\theta_B \langle S(\,f, B)s, U(\,f, B)u)$$

while the composite

$$(S(C,f) \Rightarrow T(f,C)) \circ \hat{\theta}_C \circ U(C,f)$$

corresponds to the term

$$u : U(C,B) \mid \lambda s : S(C,B).T(f,B)(\theta_C \langle S(C,f)s, U(C,f)u \rangle)$$

and the bodies of these λ-abstractions are identified by dinaturality of θ. Thus the assignment $\theta \mapsto ((S \Rightarrow \theta) \circ \eta)$ is an isomorphism, so that by the argument in the proof of Proposition 14, the triangle identities hold, η is left total and ε is right total (the totality requirements on the transformations amount to the totality of both directions of the 'hom' isomorphism, the right-to-left direction boiling down to a calculation as simple as the one above), setting up the required adjunction. □

7. Partial multicategories

In [12] we have shown that the notions of paramonoid and paracategory are instances of that of *saturated partial algebra*. Here we present another useful instance of this latter notion, namely *partial multicategories*.[5] Such structures allow us to capture certain 'tensor products' which arise in the context of probabilistic automata (Section 9).

Our reference for the theory of multicategories is [11]. As indicated there, an internal multicategory with object-of-objects C_0 is an *internal monoid* in $\mathbf{Spn}_{\mathsf{M}}(\mathbb{B})(C_0, C_0)$. Hence a *partial multicategory* is an internal paramonoid in $\mathbf{Spn}_{\mathsf{M}}(\mathbb{B})(C_0, C_0)$. We will not indulge here in this internal version of partial multicategories and give instead a concrete description below. We do mention this abstract version though as it yields a neat application of the theory of partial algebras for a monad developed in [12]. Since the iterated composites of (multi)morphisms involve a rather heavy syntax, it would be easier to work with T-$\mathscr{A}lg_P$ in the terminology of [12], that is, regarding partial multicategories as determined by a multicategory together with a specified subset of morphisms (or multiarrows) to give a full-fledged explicit definition.

Let us recall the basic definition of a *multicategory* $\mathbb{M}$ in $\mathscr{S}et$: it consists of a set of objects, ranged over by $x, y, \ldots$, and a set of morphisms, ranged over by $f, g, \ldots$. Every morphism is endowed with *source-target* information, displayed as $f : \vec{x} \to y$, where $\vec{x}$ is a list $\langle x_1, \ldots, x_n \rangle$ of objects. These data constitute a *multigraph* $\mathsf{M}(C_0) \xleftarrow{d} C_1 \xrightarrow{c} C_0$. In addition, there are *identity* morphisms $id_x : \langle x \rangle \to x$ (for every object x) and an associative *composition*, which takes a morphism $f : \langle x_1, \ldots, x_n \rangle \to y$ and a sequence of morphisms

$$f_1 : \langle x_{11}, \ldots, x_{1m_1} \rangle \to x_1, \ldots, f_n : \langle x_{n1}, \ldots, x_{nm_n} \rangle \to x_n$$

and produces their composite $f\langle f_1, \ldots, f_n \rangle$ with source-target as displayed:

$$f\langle f_1, \ldots, f_n \rangle : \langle x_{11}, \ldots, x_{1m_1}, \ldots, x_{n1}, \ldots, x_{nm_n} \rangle \to Y.$$

[5] The term *paramulticategory* is doubly incovenient for being a mouthful and causing havoc with line breaks.

The point of view of [11] is that multicategories arise as auxiliary structures to axiomatise *monoidality* or *tensor products* as universal constructions. Here too, we adopt partial multicategories to axiomatise the corresponding notion of *tensor products*.

Definition 22. A *partial multicategory* consists of a multigraph $\mathsf{M}(C_0) \xleftarrow{d} C_1 \xrightarrow{c} C_0$ and partial n-ary composition operations $\circ_n : C_1^n \rightharpoonup C_1$ where C_1^n is the nth tensor power of the multigraph as object in the monoidal category $\mathbf{Spn}_{\mathsf{M}}(\mathbb{B})(C_0, C_0)$. More explicitly, C_1^n consists of 'trees of composable morphisms of height n', so that C_1^2 consists of the data above for composition in a multicategory. These partial operations are subject to the axioms that rule the partial compositions in a paracategory, suitably modifying the third one to accommodate multicomposition.

More simply we can construe a partial multicategory as specified by a multicategory with a distinguished subset of morphisms $D \subseteq C_1$ (including the identities). The corresponding partial composites are those induced by the multicategory composites whenever the result lies in D. We write $(\mathbb{M}, D)$ for such data.

The evident notion of functor between partial multicategories is similarly obtained from that of functor between multicategories (morphism of multigraphs preserving composites) which is compatible with the partiality information: whenever the composite in the source multicategory is defined, so is the composite of its image in the target one.

Definition 23. Given a partial multicategory $(\mathbb{M}, D)$, its *underlying paracategory* $\bar{\mathbb{M}}$ has the same objects of $\mathbb{M}$ while its morphisms are those with a singleton source $f : \langle x \rangle \to y$. The partial composites of $\mathbb{M}$ restrict to give the corresponding ones in $\bar{\mathbb{M}}$.

Using the second version of partial multicategory, we can easily describe the corresponding notion of *representability*. Recall that a multicategory $\mathbb{M}$ is representable whenever we have, for every list of objects $\vec{x}$, a v-universal morphism $\pi_{\vec{x}} : \vec{x} \to \otimes\vec{x}$, such that every morphism $f : \vec{x} \to z$ factors uniquely through $\pi_{\vec{x}}$, and such v-universals are closed under (multicategory) composition. In the case of partial multicategories, we restrict such universality condition to morphisms in D:

Definition 24. A partial multicategory $(\mathbb{M}, D)$ is *representable* whenever for every list of objects $\vec{x}$ there is a *universal morphism* $\pi_{\vec{x}} : \vec{x} \to \otimes\vec{x} \in D$ (*right total* and *left tight*), so that every morphism $f : \vec{x} \to z \in D$ factors uniquely through $\pi_{\vec{x}}$, and the factors belong to D, and vice versa. Thus, precomposition with $\pi_{\vec{x}}$ sets up an isomorphism

$$_ \circ \pi_{\vec{x}} : \bar{\mathbb{M}}(\otimes\vec{x}, z) \cap D \xrightarrow{\sim} \mathbb{M}(\vec{x}, z) \cap D.$$

Furthermore, universal morphisms must be *closed under composition*.

We write $\mathscr{RM}$ for the category of representable partial multicategories and functors between them which preserve universal morphisms.

As we mentioned earlier, the purpose of representable multicategories is to axiomatise 'tensor products' by means of universal properties; they provide an alternative (but equivalent) formulation of *monoidal categories* c.f. [11, Section 9]. We briefly outline the 'monoidal paracategory' counterpart of representable partial multicategories, indicating how the equivalence in [11] translates to the partial context. Of course, a full fledged presentation along the lines of [11] falls outside the scope of this paper, where

we only give the bare bones of the theory and introduce some interesting examples, which we hope would prompt further developments.

Recall [11, Definition 6.7] that a multicategory $\mathbb{M}$ has an underlying category $\bar{\mathbb{M}}$ of 'linear morphisms' and that representability of $\mathbb{M}$ endows $\bar{\mathbb{M}}$ with a monoidal structure. In order to state this correspondence at the partial level, we should specify what we mean by monoidal structure on a paracategory. The simplest way is to consider the strict version and regard a monoidal category as an internal monoid in $\mathscr{C}at$ (see [19, Section VII]). Since $\mathscr{P}ar\mathscr{C}at$ has finite products, we can equally well consider monoids in it. Furthermore, the coherence theorem for monoidal categories (see e.g. [14]) says that every monoidal category is equivalent to a strict one, and thus we can construe a monoidal category $\mathbb{V}$ as a category with a given equivalence to a strict monoidal category $\underline{\mathbb{V}}$, and use this procedure to define monoidal paracategories. The notion of *equivalence* in the paracategory context is the same as for categories: an adjunction whose unit and counit transformations are isomorphisms. [6]

Definition 25. A *strict monoidal paracategory* is a monoid in $\mathscr{P}ar\mathscr{C}at$, that is, a paracategory $\mathbb{C}$ endowed with a unit $I \in \mathbb{C}$ and a multiplication $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, associative and unitary. A *monoidal paracategory* is a paracategory $\mathbb{C}$ together with an equivalence $e : \mathbb{C} \to \underline{\mathbb{C}}$, where $\underline{\mathbb{C}}$ is a strict monoidal paracategory. A *strong monoidal functor* between monoidal paracategories is thus given by a pair of functors $(F : \mathbb{C} \to \mathbb{D}, \underline{F} : \underline{\mathbb{C}} \to \underline{\mathbb{D}})$ and a *left tight* natural isomorphism $\phi : \underline{F}e \Rightarrow e'F$, such that $\underline{F}$ preserves the strict monoidal structure (on the nose). We have thus the category $\mathscr{M}on(\mathscr{P}ar\mathscr{C}at)$ of monoidal paracategories and strong monoidal functors.

Remark 26. In the context of $\mathscr{C}at_P$, it is clear that monoidal structure is the same as for ordinary categories, with the additional proviso that the coherent isomorphisms (built up from associativity and unit isomorphisms via composition and tensoring) are left tight, since they should arise as canonical isomorphisms between universals (tensor products). Translating this situation to paracategories, a monoidal paracategory in the sense above amounts to a paracategory together with the usual tensor and unit functors and coherent (left tight) isomorphisms: all the relevant composites (required to state that 'all diagrams commute' in the sense of [19, Section VII.2, Theorem 1]) are defined since left tight isomorphisms compose (and their composites are left tight, of course).

Using the techniques for the correspondence between monoidal categories and representable multicategories of [11, Theorem 9.8], we would obtain the analogous one at the partial level:

$$\mathscr{RM} \equiv \mathscr{M}on(\mathscr{P}ar\mathscr{C}at).$$

Either side of this 'equivalence' is a suitable axiomatisation for the notion of tensor product for paracategories. Despite the restriction on the morphisms, we do get the

[6] For an equivalence, the inverse of the unit would constitute a *left tight* counit and dually for the inverse of the counit. Thus, for an adjoint equivalence what matters is the provision of invertible transformations satisfying the triangle identities, with one of them being left tight/total.

usual properties of tensor products in this context, namely associativity: $\otimes\langle x, \otimes\langle y, z\rangle\rangle \cong \otimes\langle\otimes\langle x, y\rangle, z\rangle$, canonically.

Remark 27. The precise statement of the above correspondence would involve making explicit the two-dimensional structure implicit on both sides, as we only have a 'biequivalence'. We have refrained from indulging in such technicalities in this presentation.

We conclude this brief foray into partial multicategories pointing out that the construction of the free representable multicategory of [11, Theorem 7.2] would translate straightforwardly to the partial context (keeping track of the distinguished morphisms). Hence, every partial multicategory would be fully and faithfully embedded into its free representable one.

In Section 9.4 we illustrate the above notions setting up representable partial multicategories $(\mathscr{B}or^{\eth}_{\blacktriangleright}, D)$ and $(\mathscr{M}\mathscr{P}aut^{\eth}_{\blacktriangleright}, \underline{D})$; the induced tensor on the latter case accounts for the (free) aggregation of probabilistic automata.

8. Fibrations of paracategories

[20] considers fibrations for *precategories*. The relevant notion of fibration translates literally to the context of paracategories. Given a functor of paracategories $p : \mathbb{E} \to \mathbb{B}$, for every object X we have a *fibre paracategory* $\mathbb{E}_X$ (subparacategory of $\mathbb{E}$) consisting of those objects sent by p to X, and the morphisms sent by p to id_X, which we refer to as *vertical* morphisms.

Definition 28. Given a functor $p : \mathbb{E} \to \mathbb{B}$:

(1) A morphism $f : x \to y$ in $\mathbb{E}$ is (*strong*) *p-cartesian* if for any other morphism $h : z \to y$ with a factorisation $ph = pf \circ u$ in $\mathbb{B}$, there exists a unique factorisation $h = f \circ \tilde{u}$ with $p\tilde{u} = u$. Furthermore, we require that f be *right tight* and *left total* with respect to *vertical* morphisms.

(2) The functor p is a *fibration of paracategories* if for every object x in $\mathbb{E}$ and every morphism $v : i \to px$ in $\mathbb{B}$, there exists a (strong) p-cartesian morphism $\bar{v} : v^*(x) \to x$ such that $p\bar{v} = v$. The cartesian morphism $\bar{v}$ is called a *cartesian lifting* of v at x.

(3) A fibration is called *split* if there is a (given) *choice* of cartesian liftings such that

$$\overline{id} \quad \bar{v} \circ \bar{u} = \overline{v \circ u}$$

for composable v, u.

Remark 29.

- The composability requirements on cartesian morphisms are exactly dual to those on *universal morphisms* in Definition 13, so that cartesian morphisms are 'couniversal'. In fact, the adjoint characterisation of (ordinary) fibrations in [23] shows that cartesian morphisms become the instances of the counit of the relevant adjunction.

In the dual notion of *cofibration* (or covariant fibration), the cocartesian morphisms must be left tight and right total.

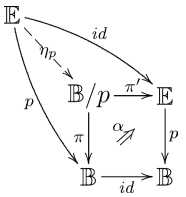
- It is equivalent to require for p to be a fibration the weaker property of existence of p-cartesian morphisms, which are universal only with respect to morphisms $h : z \to y$ with $ph = pf$ provided that composable p-cartesian morphisms do compose whenever their images in the base category do, and their composite is p-cartesian as well. Of course, they would still be required to be right tight and left total with respect to vertical morphisms.

For a functor of paracategories $p : \mathbb{E} \to \mathbb{B}$, with $\mathbb{B}$ a *category*, there is a simple adjoint characterisation of the property of being a fibration, analogous to the one for fibrations of categories [23].

Proposition 30. *Given a functor $p : \mathbb{E} \to \mathbb{B}$ with $\mathbb{B}$ a category, the following are equivalent*:

(1) *p is a fibration of paracategories.*

(2) *The functor $\eta_p : \mathbb{E} \to \mathbb{B}/p$ induced as depicted below*:



$\eta_p : x \mapsto (px, id : px \to px, x)$ has a right adjoint right inverse, i.e. with identity counit.

Remark 31. The reason we must restrict ourselves to $\mathbb{B}$ being a category is the following: in the ordinary categorical setting the functor $\pi : \mathbb{B}/p \to \mathbb{B}$ is a fibration. In fact, it is the *free fibration over* p [23]. But if $\mathbb{B}$ is only a paracategory, the relevant cartesian liftings:

$$f : x \to y, \quad a : y \to pz \mapsto a \circ f : x \to pz$$

may not be defined. This is also the reason why the isomorphism θ in Definition 12 cannot be said to be an isomorphism of *bimodules*, as it is the case for categories.

These considerations suggest we might adopt a more relaxed notion of fibration, with *partial* actions, so as to accommodate π. At present though, the situation in which we apply fibrations in Section 9 fits our restricted formulation above.

For categories, there is a well-known alternative formulation of the notion of fibration (appealing to the axiom of choice), namely, that of a *pseudo-functor* or *indexed category* $F : \mathbb{B}^{\mathrm{op}} \to \mathscr{C}at$. The Grothendieck construction produces a fibration (with base $\mathbb{B}$) out of such a pseudo-functor and sets up a (bi)equivalence between the two notions. This correspondence restricts to an equivalence between (ordinary) functors and *split* fibrations. We consider only this latter (strict) version of the correspondence for paracategories, for simplicity.

Definition 32. Given a paracategory $\mathbb{C}$ and a functor $\mathscr{F}:\mathbb{C}^{\mathrm{op}}\to\mathscr{P}ar\mathscr{C}at$, define its *Grothendieck paracategory* $\int\mathscr{F}$ as follows:

objects: Pairs (X,x) with $X\in\mathbb{C}$ and $x\in\mathscr{F}X$.

morphisms: $(b,f):(X,x)\to(Y,y)$ is a pair $b:X\to Y$ in $\mathbb{C}$ and $f:x\to(\mathscr{F}b)y$ in the fibre $\mathscr{F}X$.

identities: $(id,id):(X,x)\to(X,x)$.

composition:

$$[(b_1,f_1),\ldots,(b_n,f_n)]=([b_1,\ldots,b_n],[f_1,(b_1)^*f_2,\ldots,b_1^*\ldots(b_{n-1}^*f_n)]),$$

where we have written $b^*f=(\mathscr{F}b)f$ and the second composition on the right takes place in the fibre $\mathscr{F}X_1$, with $b_1:X_1\to X_2$. Notice that the codomain of the second component (the vertical morphism) is well-defined: whenever $[b_1,\ldots,b_n]$ is defined $b_1^*\ldots(b_n^*y)=[b_1,\ldots,b_n]^*(y)$.

There is an evident functor $\pi_{\mathscr{F}}:\int\mathscr{F}\to\mathbb{C}$ taking (X,x) to X.

We want to relate this construction to the usual Grothendieck construction for contravariant functors into $\mathscr{C}at$. Given a functor $\mathscr{F}:\mathbb{C}^{\mathrm{op}}\to\mathscr{P}ar\mathscr{C}at$, since $\mathscr{P}ar\mathscr{C}at$ is a category, we get a functor $\hat{\mathscr{F}}:\mathsf{EC}(\mathbb{C})^{\mathrm{op}}\to\mathscr{P}ar\mathscr{C}at$, which we postcompose with $\underline{U}\mathsf{EC}:\mathscr{P}ar\mathscr{C}at\to\mathscr{C}at$ thus obtaining a contravariant $\mathscr{C}at$-valued functor $\bar{\mathscr{F}}\equiv\underline{U}\mathsf{EC}\hat{\mathscr{F}}$, and consider its associated split fibration $\pi_{\bar{\mathscr{F}}}:\int\bar{\mathscr{F}}\to\mathsf{EC}(\mathbb{C})$. Notice that $\int\bar{\mathscr{F}}$ carries a distinguished set of morphisms

$$\textstyle\int P=\{(f,v)\mid f:X\to Y\in P_{\mathsf{EC}(\mathbb{C})}\wedge v\in P_{\mathsf{EC}(\mathscr{F}X)}\}$$

and $\pi_{\bar{\mathscr{F}}}:(\int\bar{\mathscr{F}},\int P)\to(\mathsf{EC}(\mathbb{C}),P_{\mathsf{EC}(\mathbb{C})})$ in $\mathscr{C}at_P$.

Proposition 33. *Given a functor $\mathscr{F}:\mathbb{B}^{\mathrm{op}}\to\mathscr{P}ar\mathscr{C}at$, the following hold*:

(1) $\pi_{\mathscr{F}}:\int\mathscr{F}\to\mathbb{C}$ *is a split fibration of paracategories, and every split fibration so arises*.

(2) $\mathsf{EC}(\pi_{\mathscr{F}})\simeq\pi_{\bar{\mathscr{F}}}$.

Proof. (1) A cartesian lifting for $v:I\to X$ at (X,x) is $(v,id_{v^*x}:(I,v^*x\to(X,x)$. Such liftings obviously verify the composionality conditions for a (split) fibration, and are easily seen to be right tight and left total, because of the neutral nature of identity morphisms. Conversely, given a split fibration $p:\mathbb{E}\to\mathbb{B}$, we get a functor $\mathscr{F}_p:\mathbb{B}^{\mathrm{op}}\to\mathscr{P}ar\mathscr{C}at$ assigning $X\mapsto\mathbb{E}_X$. On morphisms $f:X\to Y$, given a (vertical) morphism $v:x\to y$ in $\mathbb{E}_Y$, we obtain $f^*v:f^*x\to f^*y$ as the unique vertical morphism filling the diagram

$$\begin{array}{ccc} f^*x & \overset{f^*v}{\dashrightarrow} & f^*y \\ {\scriptstyle\bar{f}_x}\downarrow & & \downarrow{\scriptstyle\bar{f}_y} \\ x & \xrightarrow[v]{} & y \end{array}$$

where the vertical morphisms are cartesian. Notice that $v\circ\bar{f}_x$ is defined because of f_x's left totality with respect to verticals. Right tightness and universality of cartesian morphisms ensures that $f^*v\circ f^*w=f^*(v\circ w)$ whenever $v\circ w$ is defined. The composionality conditions of the chosen cartesian liftings assemble these f^*'s into the required

functor $\mathscr{F}_p : \mathbb{B}^{\mathrm{op}} \to \mathscr{P}ar\mathscr{C}at$. The verification that this construction yields an equivalence between split fibrations and contravariant functors into $\mathscr{P}ar\mathscr{C}at$ is routine.

(2) Routine calculation. □

Finally, to establish the relationship between fibrations of paracategories and that of categories, we must specify these latter in the context of $\mathscr{C}at_P$.

Definition 34. Given objects $(\mathbb{E}, E)$ and $(\mathbb{B}, B)$ of $\mathscr{C}at_P$, a functor $p : \mathbb{E} \to \mathbb{B}$ is a *fibration in* $\mathscr{C}at_P$ if it is an ordinary fibration in $\mathscr{C}at$ such that p-cartesian liftings of morphisms in B belong to E and are right tight with respect to vertical morphisms.

Proposition 35.

- *If* $p : \mathbb{E} \to \mathbb{B}$ *is a fibration of paracategories, then* $\mathsf{EC}\,p : \mathsf{EC}(\mathbb{E}) \to \mathsf{EC}(\mathbb{B})$ *is a fibration in* $\mathscr{C}at_P$.
- *If* $p : (\mathbb{C}, C) \to (\mathbb{D}, D)$ *is a fibration in* $\mathscr{C}at_P$, *then* $U(p) : U(\mathbb{C}, C) \to U(\mathbb{D}, D)$ *is a fibration of paracategories.*

Proof. If $p : \mathbb{E} \to \mathbb{B}$ is a fibration of paracategories, we get cartesian liftings of (equivalence classes of) sequences of morphisms in $\mathsf{EC}(\mathbb{B})$ elementwise. The closure of cartesian morphisms under composition ensures that whenever a sequence is equivalent to a singleton (a distinguished morphism) so is its lifting.

In the converse direction, the requirement that cartesian liftings of morphisms in D belong to C ensure the closure of cartesian liftings under composition in $U(\mathbb{C}, C)$.

The tightness and totality conditions translate trivially in both directions. □

As a particular instance of the above indexed-family-of-paracategories/fibration correspondence, notice that for presheaves

$$\mathscr{P}ar\mathscr{C}at(\mathbb{C}, \mathscr{S}et) \simeq \mathscr{C}at_P(\mathsf{EC}(\mathbb{C}), \mathscr{S}et) \simeq \mathscr{C}at(\underline{U}\mathsf{EC}(\mathbb{C}), \mathscr{S}et),$$

thus a paracategory $\mathbb{C}$ and its envelope do have the same 'modules' indeed.

9. Examples from probabilistic automata theory

Several results on automata theory can be elegantly explained using categorical language [2,3], since an automaton is essentially an action over a monoid [6]. Bearing this in mind, it is easy to explain combination and realization of automata as universal constructions [9,25].

Most of these results can be extended to richer structures, such as linear, internal and non-deterministic automata [1]. However, probabilistic automata tend to be an exception to this elysian setting, specially when we need to restrict the probabilistic transitions of an auotmaton [8]. The reason for this inconvenience lies in the fact that there is no good notion of morphism for probability spaces such that *conditional probability* would correspond to *subobjects* and *measure preserving transformations* would correspond to some kind of *epimorphisms*. As we shall see, the natural notion

of morphism for probabilistic automata fails to be closed under composition, yielding a proper paracategory.

Since probability theory can be formulated with Borel measure spaces, we start by organising these within a convenient paracategory $\mathscr{B}or$. This paracategory serves as a basis for more elaborated structures: random variables, stochastic processes and probabilistic automata, which provide examples of fibrations and pullbacks in $\mathscr{P}ar\mathscr{C}at$. We show the central result of the behaviour/realisation adjunction of probabilistic automata as an example of a fibred adjunction of paracategories (Theorem 42).

Finally, we show how the aggregation of automata from [20] can be construed in terms of representability for a suitable partial multicategory of cones (Corollary 46).

9.1. Borel probability measures

Given a topological space T, its *Borel algebra* $\mathscr{B}T$ is the σ-algebra[7] generated by the open subsets of T. Measures, and probabilities in particular, can be defined over Borel sets, with the usual requirement that T be Hausdorff, locally compact and with a countable base, in order to adapt results smoothly from the classical Borel algebra over the real numbers [10]. Here we consider all topological spaces.

A well established result asserts that given a probability measure $\mathscr{P}$ over a Borel algebra on T and a (special kind of) continuous map $f : T' \to T$, we can, canonically, obtain a probability measure on T'. We will set out this result as a fibration on paracategories, but first we provide some basic background.

- A *Borel probability measure* $\mathscr{P}$ is a topological space T endowed with a probability measure over the Borel σ-algebra of T. It is useful to consider the *degenerated* $\mathscr{P}^\infty$ Borel probability measure over some topological space T, with value $\mathscr{P}^\infty(A) = \infty$ (this is not even a measure since $\mathscr{P}^\infty(\emptyset) = \infty$);
- A *measure preserving transformation* between $\mathscr{P}$ and $\mathscr{P}'$ is a continuous map $t : T \to T'$ between the underlying topological spaces such that $P(t^{-1}(B')) = P'(B')$, for all $B' \in \mathscr{B}T'$.

The above notion of measure preserving transformation is often too strict to be useful in practice. In general, when we deal with probability spaces we need to consider subobjects so that the inclusion $\imath : \mathscr{P} \hookrightarrow \mathscr{P}'$ should be a morphism. This requirement is important, for instance, when we combine/restrict probabilistic automata or stochastic processes [20]. We are thus led to consider the following notion of morphism between probability measures:

- A *subspace* $\mathscr{P}$ of a Borel probability measure $\mathscr{P}'$ is the Borel probability measure over the relative topology of T in T' where $P(B) = P'(B|T) = P'(B)/P'(T)$. Note that the if $P(T) = 0$ we obtain, by convention, the degenerated probability measure over T. So the empty set (with its topology) endowed with the degenerated measure is always a subspace of any Borel probability measure.

[7] A σ-algebra $\mathscr{F}$ over a set Ω is a collection of subsets of Ω closed under complements and countable (including empty) unions and disjunctions.

- A *morphism* between Borel probability measures $f:\mathscr{P}\to\mathscr{P}'$ is a measure preserving transformation between $\mathscr{P}$ and a subspace of $\mathscr{P}'$. More precisely:
  - $f:T\to T'$ is continuous;
  - $f(T)$ is a Borel set of T' [8];
  - $P(f^{-1}(B'))=P'(B'|f(T))$ for all Borel sets B' of T'.
- Borel probability measures endowed with their morphisms constitute the (proper) paracategory $\mathscr{B}or$, with partial composites induced by function composition in $\mathscr{S}et$ (or $\mathscr{T}op$).

The following proposition allows us to change the underlying topological space of a probability measure.

Proposition 36. *Given a continuous map $f:T\to T'$ and Borel probability measure $\mathscr{P}'$ over T' such that*:
(1) *the topology of T is $\{f^{-1}(O')\}_{O'\in T}$*;
(2) *$f(T)$ is a Borel set of T'*;
there exists a unique Borel measure $\mathscr{P}$ over T such that $f:\mathscr{P}\to\mathscr{P}'$ is a morphism in $\mathscr{B}or$.

Proof. Note that for a given Borel set B of T we can find another Borel set $B'\subseteq f(\Omega)$ such that $B=f^{-1}(B')$ (this result follows from (1) and by noticing that f^{-1} preserves unions, intersections and complements). Hence, the required unique Borel measure is such that $P(B)=P(B')/P(f(\Omega))$. It remains to show that $P(B')=P(B'')$ whenever $B',B''\subseteq f(\Omega)$ and $B=f^{-1}(B')=f^{-1}(B'')$. By naive Set Theory it is easy to show that $B'=B''$ since $B',B''\subseteq f(\Omega)$, which concludes the proof. □

Remark 37. A continuous map $f:T\to T'$ satisfying condition (1) above is precisely a *cartesian morphism* of the fibration [9] $U:\mathscr{T}op\to\mathscr{S}et$, which forgets the topology of a space.

Corollary 38. *Let $\mathscr{T}op_C$ be the broad* [10] *subcategory of $\mathscr{T}op$ whose morphisms satisfy conditions* (1) *and* (2) *of Proposition* 36 *and let $\mathscr{B}or_C$ be the analogous broad subparacategory of $\mathscr{B}or$. The forgetful functor $U:\mathscr{B}or_C\to\mathscr{T}op_C$ is a fibration of paracategories.*

Proof. We have to show the universal property of the morphism $f:\mathscr{P}'\to\mathscr{P}$ presented in Proposition 36. Let $g:\mathscr{P}''\to\mathscr{P}$ and $h:T''\to T'$ be morphisms such that $g=fh$ in $\mathscr{T}op_C$. We only prove that $h:\mathscr{P}''\to\mathscr{P}'$ is a morphism in $\mathscr{B}or_C$. First, observe that $h(\Omega'')=f^{-1}fh(\Omega'')$. Indeed, since $h(\Omega'')$ is a Borel set and f is in $\mathscr{T}op_C$ there is a Borel set A such that $h(\Omega'')=f^{-1}(A)$, and therefore $h(\Omega'')=f^{-1}fh(\Omega'')$.

[8] Whenever T is locally compact, has a countable base and T' is Hausdorff, $f(T)$ is a Borel set of T'.
[9] Given a topological space $(\Omega,\mathscr{U})$ and a function $f:\Omega'\to\Omega$, its U-cartesian lifting is $\bar{f}:(\Omega',\mathscr{U}')\to(\Omega,\mathscr{U})$ where $\mathscr{U}'$ is the topology given by the inverse images of opens in Ω, $f^{-1}(\mathscr{U})$. This topology is the smallest one making f continuous.
[10] Having the same objects as the ambient category.

Finally, we check that h is a morphism in $\mathscr{B}or_C$. Let $B' \subseteq h(\Omega'')$ be a Borel set and B another Borel set such that $B' = f^{-1}(B)$. We then have

$$P''(h^{-1}(B')) = P''(h^{-1}(f^{-1}(B))) = P(B|g(\Omega'')) = \frac{P(B \cap fh(\Omega''))}{P(fh(\Omega''))}$$

$$= \frac{P'(f^{-1}(B) \cap f^{-1}fh(\Omega''))}{P'(f^{-1}fh(\Omega''))} = \frac{P'(B' \cap h(\Omega''))}{P'(h(\Omega''))} = P(B'|h(\Omega)).$$

Since it is enough to check this property for the Borel sets contained in $h(\Omega'')$, we conclude that h is a morphism in $\mathscr{B}or_C$. □

As an immediate consequence topological subspaces (which are measurable) lift to probability subspaces (conditional probabilities), which was one of our desiderata.

9.2. Random variables and probabilistic automata

Random variables and stochastic processes are defined in terms of Borel probability measures. They provide examples of fibrations of paracategories. We build them in indexed terms, using the following basic concepts:

- A *measure space* is a set Ω endowed with a σ-algebra $\mathscr{F}$ over it. A *measurable map* between measure spaces $m : \langle \Omega, \mathscr{F} \rangle \to \langle \Omega', \mathscr{F}' \rangle$ is a function $m : \Omega \to \Omega'$ such that $m^{-1}(B') \in \mathscr{F}$ for all $B' \in \mathscr{F}'$. Measurable set and measurable maps constitute the category $\mathscr{M}es$.
- Each Borel probability measure $\mathscr{P}$ becomes a measure space $\mathscr{B}_{\mathscr{P}}$ via its Borel σ-algebra, while a morphism $f : \mathscr{P} \to \mathscr{P}'$ induces a measurable map $f : \mathscr{B}_{\mathscr{P}} \to \mathscr{B}_{\mathscr{P}'}$. We get thus a functor $M : \mathscr{B}or \to \mathscr{M}es$.
- A *random variable* X over a Borel probability measure $\mathscr{P}$ is a measurable map $X : M(\mathscr{P}) \to \mathbb{R}$ (that is, an element of $\mathscr{M}es(M(\mathscr{P}), \mathbb{R})$).
- A *random quantity* Q is an object of $M/\mathscr{M}es$, that is, an element of $\mathscr{M}es(M(\mathscr{P}), S)$ for some $\mathscr{P}$ and S. The measure space S is called the *state space of* Q.
- A *stochastic process* is a family of random variables $\{X_i\}_{i \in I}$ over some probability space $\mathscr{P}$. I is called the *index parameter*. When I is a total order, it is called the *time structure*, otherwise it is called the *spatial structure*. The range of the random variable is called the *state space* of the process.
- A *general stochastic process* is a family of random quantities $\{Q_i\}_{i \in I}$ over some probability space $\mathscr{P}$. The process is called *space stable* iff all random quantities range over the same state space S. The terminology on the index parameter I is similar to the previous case. We are interested in state stable processes with spatial index structure.

We define the paracategories of random variables $\mathscr{R}ndV$ and random quantities $\mathscr{R}ndQ$ as the following comma-paracategories:

- $\mathscr{R}ndV \equiv M/\mathbb{R}$;
- $\mathscr{R}ndQ \equiv M/\mathscr{M}es$.

The functor that associates each random variable to the underlying Borel space $P : \mathscr{R}ndV \to \mathscr{B}or$ is a fibration, with corresponding indexed paracategory $D : \mathscr{B}or^{\mathrm{op}} \to \mathscr{P}ar\mathscr{C}at$

where $D(\mathscr{P}) = \mathscr{R}ndV_{\mathscr{P}}$. We are now able to define the paracategory of stochastic processes $\mathscr{S}tcP$:

- $\mathscr{S}tcP$ is the Grothendieck paracategory $\int F$, where $F : (\mathscr{B}or \times \mathscr{S}et)^{\mathrm{op}} \to \mathscr{P}ar\mathscr{C}at$ is given as $F(\mathscr{P}, I) = \{I, \mathscr{R}ndV_{\mathscr{P}}\}$ (cotensor by the discrete category I).

Similarly, the functor that associates to a random quantity its underlying Borel space $P : \mathscr{R}ndQ \to \mathscr{B}or$ is a fibration. Thus, we obtain the Grothendieck paracategory $\mathscr{G}\mathscr{S}tcP = \int G$, where $G : (\mathscr{B}or \times \mathscr{S}et)^{\mathrm{op}} \to \mathscr{P}ar\mathscr{C}at$ is given as $G(\mathscr{P}, I) = \{I, \mathscr{R}ndQ(\mathscr{P})\}$.

The paracategory of space stable processes is introduced by constraining $\mathscr{G}\mathscr{S}tcP$ to families of random quantities with the same space state. Hence, $\mathscr{S}sp$ is given by the following pullback in $\mathscr{P}ar\mathscr{C}at$:

$$\begin{array}{ccc} \mathcal{S}sp & \dashrightarrow & \mathcal{G}StcP \\ \downarrow & & \downarrow{\scriptstyle St} \\ \mathcal{S}et \times \mathcal{M}es & \xrightarrow[Cn]{} & \int In \end{array}$$

where $In : \mathscr{S}et^{\mathrm{op}} \to \mathscr{P}ar\mathscr{C}at$ is $In(I) = \mathscr{M}es^I$, St maps each process to its family of state spaces and $Cn(I, S)$ is the constant I-indexed family with measurable space S.

Proposition 39. *The functor $\pi_G : \mathscr{S}sp \hookrightarrow \mathscr{G}\mathscr{S}tcP \to \mathscr{B}or \times \mathscr{S}et$ is a fibration.*

Proof. Since $\pi_G : \mathscr{G}\mathscr{S}tcP \to \mathscr{B}or \times \mathscr{S}et$ is a fibration, for any object $T = (\mathscr{P}, I, \{Q_i : \Omega \to D\}_{i \in I})$ in $\mathscr{S}sp$ and morphism $(p, f) : (\mathscr{P}', I') \to \pi_G T$ there is a strong π_G-cartesian morphism $\overline{(p, f)}$ in $\mathscr{G}\mathscr{S}tcP$ such that $\pi_G \overline{(p, f)} = (p, f)$. Since $dom\overline{(p, f)} = (\mathscr{P}', I', \{Q'_{i'} : \Omega \to D\}_{i' \in I'})$ where $Q'_{i'} = Q_{f(i')}$, we conclude that $dom\overline{(p, f)}$ is stable and so $\overline{(p, f)} = (p, f, \{id_D\}_{i \in I}) \in \mathscr{S}sp$.

It remains to show the universal property of $\overline{(p, f)}$ in $\mathscr{S}sp$. Consider morphisms $(q, g, c) : (\mathscr{P}'', I'', \{Q''_{i''} : \Omega'' \to D''\}_{i'' \in I''}) \to T$ and $(r, h) : (\mathscr{P}'', I'') \to (\mathscr{P}', I')$ such that

$$\begin{array}{ccc} (\mathcal{P}', I') & \xrightarrow{(p,f)} & \pi_G T \\ & {\scriptstyle (r,h)} \nwarrow \quad \nearrow {\scriptstyle (q,g)} & \\ & (\mathcal{P}'', I'') & \end{array}$$

There exists a unique morphism $\widetilde{(r, h)} = (r, h, c)$ which makes the following diagram commute in $\mathscr{S}sp$:

$$\begin{array}{ccc} (\mathcal{P}', I', \{Q'_{i'} \to D\}_{i' \in I'}) & \xrightarrow{\overline{(p,f)}} & T \\ & {\scriptstyle \widetilde{(r,h)}} \nwarrow \quad \nearrow {\scriptstyle (q,g,c)} & \\ & (\mathcal{P}'', I'', \{Q''_{i''} : \Omega'' \to D''\}_{i'' \in I''}) & \end{array}$$

□

A probabilistic automaton with input set I and state space $S \in \mathscr{M}es$ is a state stable process indexed by $S \times I$ and with state space S (this process is usually called the

probability transition function) plus an element from S (the *initial state*). Therefore, we obtain the paracategory $\mathscr{P}aut$ of probabilistic automata as the following pullback in $\mathscr{P}ar\mathscr{C}at$:

$$\begin{array}{ccc} \mathcal{P}aut & \dashrightarrow & \mathcal{S}et \times \mathcal{M}es_* \\ \downarrow & & \downarrow X \\ \mathcal{S}sp & \xrightarrow[\langle Ind, St\rangle]{} & \mathcal{S}et \times \mathcal{M}es \end{array}$$

where X is such that $X(I,S) = (I, S \times I)$ (imposing the index parameter to be a product of the state space with a set), *Ind* is the functor that associates each state stable process to its index parameter and Mes_* is the category of pointed measure spaces such that for the point $s \in S$ its singleton $\{s\}$ is measurable.

Finally, in order to augment probabilistic automata with outputs (in Moore style [18]), we consider an *output labeling function* which assigns to each state an output, that is, an object Λ of $S/\mathscr{M}es$. The *set of outputs* of the automata is the codomain of Λ. We obtain then the paracategory of *Moore probabilistic automata* $\mathscr{M}\mathscr{P}aut$ as the following pullback:

$$\begin{array}{ccc} \mathcal{M}\mathcal{P}aut & \dashrightarrow & \mathcal{M}es^{\rightarrow} \\ \downarrow & & \downarrow Dom \\ \mathcal{P}aut & \xrightarrow[St]{} & \mathcal{M}es \end{array}$$

More explicitly, $\mathscr{M}\mathscr{P}aut$ is consists of:

objects: Tuples of the form $(\mathscr{P}, S, I, O, \delta, s_0, \Lambda)$ where:

- $\mathscr{P} \in \mathscr{B}or$; $S, O \in \mathscr{M}es$; $I \in Set$;
- $\delta = \{\delta_{si}\}_{si \in S \times I}$, δ_{si} a random quantity over $\mathscr{P}$ with state space S;
- $s_0 \in S$;
- $\Lambda : S \to O$ is a morphism in $\mathscr{M}es$.

morphisms: Tuples of the form (p, f, g, h) where:

- $p \in \mathscr{B}or(\mathscr{P}, \mathscr{P}')$; $f \in \mathscr{M}es(S, S')$; $g \in \mathscr{S}et(I, I')$; $h \in \mathscr{M}es(O, O')$;

such that the following diagram commutes in $\mathscr{M}es$ for all $si \in S \times I$

$$\begin{array}{ccccccc} 1 & \xrightarrow{s_0} & S & \quad M(\mathcal{P}) & \xrightarrow{\delta_{si}} & S & \xrightarrow{\Lambda} & O \\ & \searrow^{s'_0} & \downarrow f & M(p) \downarrow & & \downarrow f & & \downarrow h \\ & & S' & M(\mathcal{P}'_{\delta}) & \xrightarrow[f(s)g(i)]{} & S' & \xrightarrow[\Lambda']{} & O' \end{array}$$

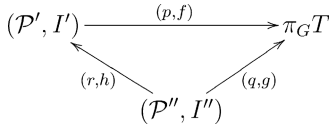
composition: Inherited from $\mathscr{B}or \times \mathscr{M}es \times Set \times \mathscr{M}es$.

We conclude this section by pointing out that $\langle P, In\, p\rangle : \mathscr{M}\mathscr{P}aut \to \mathscr{B}or \times \mathscr{S}et$ is a fibration:

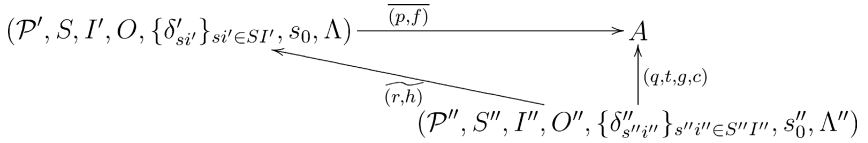
Proposition 40. *The functor* $\langle P, In\, p\rangle : \mathscr{M}\mathscr{P}aut \to \mathscr{B}or \times \mathscr{S}et$ *is a fibration.*

Proof. Given an automaton $A = (\mathscr{P}, S, I, O, \{\delta_{si}\}_{si \in SI}, s_0, \Lambda)$ and a morphism $(p, f) : (\mathscr{P}', I') \to \langle P, In\, p\rangle A$ its $\langle P, In\, p\rangle$-cartesian lifting is $\overline{(p, f)} = (p, id_S, f, id_O) : (\mathscr{P}', S, I', O, \{\delta'_{si'}\}_{si' \in SI'}, s_0, \Lambda) \to A$, with $\delta'_{si'} = \delta_{sf(i)}$.

To verify its universal property, given another morphism $(q,t,g,c):(\mathcal{P}'',S'',I'',O'',\{\delta''_{s''i''}\}_{s''i''\in S''I''},s''_0,\Lambda'')\to A$ and a factorisation



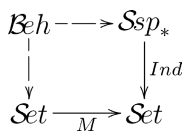
it lifts to a factorisation with $\widetilde{(r,h)}=(r,t,h,c)$. □



The above fibration allows us to change the probabilities of the transition process of an automaton canonically, along a morphism in $\mathcal{B}or$. In particular, the cartesian lifting of a monomorphism in $\mathcal{B}or$ provides us with our required restriction of an automata to a subset of outcomes.

9.3. Behaviour and realisation

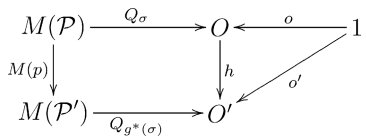
The behaviour of a Moore probabilistic automaton with input set I and output set O is a state stable process indexed by I^* and with state space O. The process gives the probability of obtaining some output o after performing a sequence of actions $\sigma\in I^*$. We define the paracategory of behaviours $\mathcal{B}eh$ via the following pullback:



where $M(I)=I^+$, the set of non-empty sequences of I, and $\mathcal{S}sp_*$ is the subparacategory of $\mathcal{S}sp$ where the state space is pointed (and the morphisms preserve the points). More explicitly, $\mathcal{B}eh$ consists of:

objects: Tuples $(\mathcal{P},I,O,o,\{Q_\sigma\}_{\sigma\in I^+})$.

morphisms: Tuples (p,g,h) such that the following diagram commutes in $\mathcal{M}es$ for all $\sigma\in I^*$:



composition: Inherited from $\mathcal{B}or\times Set\times\mathcal{M}es$.

Just as for automata, we have the following result:

Proposition 41. *The functor* $\langle P, In\, p\rangle : \mathscr{B}eh \to \mathscr{B}or \times \mathscr{S}et$ *is a fibration.*

Proof. We note that $\mathscr{B}eh$ can be identified with the pullback

$$\begin{array}{ccc} \mathscr{B}eh & \dashrightarrow & \mathscr{S}sp_* \\ {\scriptstyle \langle P, Inp\rangle}\downarrow & & \downarrow{\scriptstyle \langle P, Ind\rangle} \\ \mathscr{B}or \times \mathscr{S}et & \xrightarrow[id_{\mathscr{B}or}\times M]{} & \mathscr{B}or \times \mathscr{S}et \end{array}$$

Adapting Proposition 39, we see that $\langle P, Ind\rangle : \mathscr{S}sp_* \to \mathscr{B}or \times \mathscr{S}et$ is a fibration, and hence, $\langle P, In\, p\rangle : \mathscr{B}eh \to \mathscr{B}or \times \mathscr{S}et$ is a fibration as well, since these are stable under pullback (*change of base*). □

Given a Moore probabilistic automaton $(\mathscr{P}, I, S, \delta, s_0, \Lambda)$ we can extract its *behaviour* $(\mathscr{P}, I, O, o, \{Q\}_{\sigma\in I^+})$, where:

- $o = \Lambda(s_0)$;
- $Q_\sigma(\omega) = \Lambda(\delta^*_{s_0\sigma}(\omega))$;
- δ^* is a state stable process over $\mathscr{P}$, with state space S, indexed by $S \times I^*$, with $\delta^*_{s\varepsilon}(\omega) = s$ and $\delta^*_{s\sigma i}(\omega) = \delta_{\delta^*_{s\sigma}(\omega)i}(\omega)$.

This construction can be extended in the obvious way to a (Kleene) functor $B : \mathscr{M}\mathscr{P}aut \to \mathscr{B}eh$.

Theorem 42. *The functor B has a full and faithful left adjoint* (*the free realisation*), *fibred over* $\mathscr{B}or \times \mathscr{S}et$:

$$\begin{array}{ccc} & \mathscr{B}or \times \mathscr{S}et & \\ {\scriptstyle \langle P, Ind\rangle}\nearrow & & \nwarrow{\scriptstyle \langle P, Ind\rangle} \\ \mathscr{M}Paut & \underset{B}{\overset{R}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} & \mathscr{B}eh \end{array}$$

Proof. Set $R(\mathscr{P}, I, O, o, \{Q\}_{\sigma\in I^+}) = (\mathscr{P}, S, I^+, \delta, (\varepsilon, o), \Lambda)$, where

- S is the smallest measure space containing the space $M(\mathscr{P}) \times I^+ \times O$ and $\{(\varepsilon, o)\}$, where I^+ is the free measure space over the non-empty sequences of I;
- $\delta_{(\omega,\sigma,o)i}(\omega') = (\omega', \sigma i, Q_{si}(\omega'))$;
- $\Lambda(\omega, \sigma, o) = o$.

The counit is $\varepsilon_{(\mathscr{P}, S, I, O, \{\delta_{si}\}_{si\in SI}, s_0, \Lambda)} = (id_{\mathscr{P}}, f, id_I, id_O)$ with

- $f(\varepsilon, o) = s_0$;
- $f(\omega, \sigma, u) = \delta^*_{s_0\sigma}(\omega)$.

while the unit is $\eta_{(\mathscr{P}, I, O, o, \{Q_\sigma\}_{\sigma\in I^+})} = (id_{\mathscr{P}}, id_I, id_O)$ (hence R is full and faithful). □

Remark 43. When B is restricted to the fibre over $\langle \mathscr{P}, Inp, Out\rangle$ (*Out* being the functor that associates to an automaton its output set), it has a right adjoint called the *Nerode* or *minimal realisation* functor.

9.4. Tensor product of probabilistic automata

In this section we show that the free aggregation of probabilistic automata in [20] arises as a tensor product associated to a representable partial multicategory. First, we set up the corresponding structure at the level of probability spaces $\mathscr{B}or^{\eth}_{\blacktriangleright}$ and define the relevant multicategory of probabilistic automata $\mathscr{MP}aut^{\eth}_{\blacktriangleright}$ upon it.

9.4.1. Tensor product of probability spaces

We consider the full subparacategory $\mathscr{B}or^{\eth}$ of $\mathscr{B}or$ whose underlying topological spaces are locally compact, with a countable base and Hausdorff.

Define a multicategory $\mathscr{B}or^{\eth}_{\blacktriangleright}$ as follows:

objects: Those of $\mathscr{B}or^{\eth}$.

morphisms: $\langle f_1,\dots f_n\rangle : \langle \mathscr{P}_1,\dots,\mathscr{P}_n\rangle \to \mathscr{P}$ where $f_i : \mathscr{P} \to \mathscr{P}_i$ is a continuous map between the underlying topological spaces.

composition: Pointwise composition of cones:

$$\langle f_1,\dots,f_n\rangle\langle\langle f_{11},\dots,f_{1m_1}\rangle\dots\langle f_{n1},\dots,f_{nm_n}\rangle\rangle = \langle f_{11}f_1,\dots,f_{nm_n}f_n\rangle.$$

Consider the following class of morphisms in $\mathscr{B}or^{\eth}_{\blacktriangleright}$:

$$D = \left\{ \langle f_1,\dots f_n\rangle : \langle \mathscr{P}_1,\dots,\mathscr{P}_n\rangle \to \mathscr{P} \mid \forall_{i\in 1,\dots,n}\forall_{B_i\in\mathscr{F}_i} P\left(\bigcap_{i=1}^{n} f_i^{-1}(B_i)\right) = \prod_{i=1}^{n} P_i(B_i|f_i(\Omega_i))\right\}.$$

Theorem 44. *The partial multicategory* $(\mathscr{B}or^{\eth}_{\blacktriangleright}, D)$ *is representable.*

Proof. Set $\otimes_{i=1}^{n}\mathscr{P}_i = (\prod_{i=1}^{n}\Omega_i, \mathscr{F}, P)$ where

- $\mathscr{F}$ is the σ-algebra generated by $\mathscr{S} = \mathscr{F}_1 \times \cdots \times \mathscr{F}_n$.
- P is the unique measure such that $P(B_1\times,\cdots,\times B_n) = \prod_{i=1}^{n} P_i(B_i)$, which exists because $\mathscr{S}$ is a semi-algebra for $\mathscr{F}$.

The cone of projections out of $(\prod_{i=1}^{n}\Omega_i, \mathscr{F}, P)$ is the required universal morphism. Further details can be found in [20]. □

We have thus endowed the paracategory $\mathscr{B}or^{\eth} = \overline{(\mathscr{B}or^{\eth}_{\blacktriangleright}, D)}$ with a monoidal structure.

Remark 45. The multicategory $\mathscr{B}or^{\eth}_{\blacktriangleright}$ is essentially the same construction as the multicategory of discrete (co)cones $\mathbb{C}_{\blacktriangleright}$ associated to any category $\mathbb{C}$ in [11, Example 2.2(2)], whose representability corresponds to the existence of (co)products in $\mathbb{C}$.

9.4.2. Aggregation of probabilistic automata

We consider the full subparacategory $\mathscr{MP}aut^{\eth}$ of $\mathscr{MP}aut$ whose underlying probability spaces are in $\mathscr{B}or^{\eth}$.

Define a multicategory $\mathscr{MP}aut^{\eth}_{\blacktriangleright}$ as follows:

objects: Those of $\mathscr{MP}aut^{\eth}$;
morphisms: $\langle f_1,\dots f_n\rangle : \langle M_1,\dots,M_n\rangle \to M$
where $f_i = (f_i^1, f_i^2, f_i^3, f_i^4) : M = (\mathscr{P}, S, I, O, \delta, s_0, \Lambda) \to M_i$ is such that

- f_i^1 is a continuous map between the underlying topological spaces of $\mathscr{P}$ and $\mathscr{P}_i$;
- f_i^2, f_i^3, f_i^4 are like in $\mathscr{MP}aut$.

composition: Pointwise composition of cones as in $\mathscr{B}or^{\eth}_{\blacktriangleright}$.

Consider the following class of morphisms in $\mathscr{MP}aut^{\eth}_{\blacktriangleright}$:

$$\underline{D} = \{\langle f_1,\dots f_n\rangle : \langle M_1,\dots,M_n\rangle \to M \mid \langle f_1^1,\dots f_n^1\rangle \in D\}.$$

Corollary 46. *The partial multicategory* $(\mathscr{MP}aut^{\eth}_{\blacktriangleright}, \underline{D})$ *is representable.*

We have thus produced yet another monoidal paracategory $\mathscr{MP}aut^{\eth} = \overline{(\mathscr{MP}aut^{\eth}_{\blacktriangleright}, \underline{D})}$.

Acknowledgements

The authors gratefully acknowledge the prompt and comprehensive reviews by the referees, whose sharp comments led us to substantial improvements in our presentation.

References

[1] J. Adámek, Realization theory for automata in categories, J. Pure Appl. Algebra 9 (3) (1976) 281–296.
[2] J. Adámek, V. Trnková, Automata and Algebras in Categories, Kluwer Academic, Dordrecht, 1991.
[3] M. Arbib, E. Manes, Machines in a category: an expository introduction, SIAM Rev. 16 (2) (1974) 163–192.
[4] J. Bénabou, Some remarks on free monoids in a topos, in: A. Carboni, M.C. Pedicchio, G. Rosolini (Eds.), Category Theory 90, Lecture Notes in Mathematics, Vol. 1488, Springer, Berlin, 1990, pp. 20–25. Como.
[5] B. Day, A reflection theorem for closed categories, J. Pure Appl. Algebra 2 (1972) 1–11.
[6] S. Eilenberg, J. Wright, Automata in general algebras, Inform. and Control 11 (1967) 452–470.
[7] P.J. Freyd, Paramathematics, unpublished manuscript, Talk delivered at XII MFPS, Boulder, CO, 1996.
[8] R. van Glabbeek, S. Smolka, B. Steffen, C. Tofts, Reactive, generative, and stratified models for probabilistic processes, Inform. and Comput. 121 (1) (1995) 59–80.
[9] J. Goguen, Realization is universal, Math. Systems Theory 6 (1972) 359–374.
[10] P. Halmos, Measure Theory, van Nostrand, New York, 1969.
[11] C. Hermida, Representable multicategories, Adv. in Math. 151 (2000) 164–225, available at http://www.cs.math.ist.utl.pt/s84.www/cs/claudio.html.
[12] C. Hermida, P. Mateus, Paracategories I: internal paracategories and saturated partial algebras, Theor. Comput. Sci. (2003) to appear.
[13] B. Jacobs, Categorical Logic and Type Theory, in: Studies in Logic and the Foundations of Mathematics, Vol. 141, North Holland, Amsterdam, 1999.

[14] A. Joyal, R. Street, Braided monoidal categories, Adv. in Math. 102 (1993) 20–78.
[15] G.M. Kelly, Basic Concepts of Enriched Category Theory, in: London Mathematical Society Lecture Notes Series, Vol. 64, Cambridge University Press, Cambridge, 1982.
[16] J. Lambek, P.J. Scott, Introduction to Higher-Order Categorical Logic, in: Cambridge Studies in Advanced Mathematics, Vol. 7, Cambridge University Press, Cambridge, 1986.
[17] F.W. Lawvere, The category of categories as a foundation for mathematics, in: Proc. Conf. Categories of Algebra, La Jolla, 1966, pp. 1–20.
[18] K. Leeuw, E. Moore, C. Shannon, N. Shapiro, Computability by probabilistic machines, Ann. Math. Stud. 34 (1956) 183–212.
[19] S. MacLane, Categories for the Working Mathematician, 2nd ed., Springer, Berlin, 1998.
[20] P. Mateus, Interconnection of probabilistic systems, Ph.D. Thesis, IST, Universidade Técnica de Lisboa, 2000.
[21] P. Mateus, A. Sernadas, C. Sernadas, Precategories for combining probabilistic automata, Electronic Notes in Theoretical Computer Science, Vol. 29, Elsevier, Edinburgh, 1999.
[22] L. Schröder, P. Mateus, Universal aspects of probabilistic automata, Math. Struct. Comput. Sci. 12 (4) (2002) 481–512.
[23] R. Street, Fibrations and Yoneda's lemma in a 2-category, in: G.M. Kelly (Ed.), Category Seminar, Lecture Notes in Mathematics, Vol. 420, Springer, Berlin, 1973.
[24] R. Street, Cosmoi of internal categories, Trans. Amer. Math. Soc. 258 (1980) 271–318.
[25] G. Winskel, M. Nielsen, Models of concurrency, in: D. Gabbay, S. Abramsky, T. Maibaum (Eds.), Handbook of Logic in Computer Science, Vol. 4, Oxford Science, Oxford, 1995, pp. 1–48.