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Fibrations and Yoneda structure for multicategories

*Claudio Hermida**

19 September 2002

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1 Introduction

This paper furthers the theory of multicategories from [Her00, Her01], where we take our notation and terminology from. We set up the basics of a theory of (co)fibrations for multicategories. The main difficulty is that, given the nature of 2-cells in $\mathcal{M}ulticat$ (which detect only linear morphisms) the usual representable notion of fibration in a 2-category is unsuitable in this context. The key to the development of a good theory of fibrations for multicategories is the monadic adjunction between $\mathcal{M}ulticat$ and $\mathcal{M}on\mathcal{C}at$. We give elementary definitions of fibrations for multicategories (in $\mathcal{S}et$) and then proceed to an abstract characterisation, suitable for the abstract setting of cartesian 2-monads in [Her01]. We give an indexed version in terms of homomorphisms of monoidal bicategories, which we use to construct free cofibrations and a comprehensive factorisation of morhisms of multicategories (initial, discrete cofibration). An interesting example of discrete cofibrations are *algebras for operads*. Fibrations of multicategories set up a 2-fibration over $\mathcal{M}ulticat$.

After establishing a change of base result for Yoneda structures along a left (bi)adjoint between the base 2-categories, we use the (bi)adjunction between $\mathcal{M}ulticat$ and $\mathcal{M}on\mathcal{C}at$ to obtain a Yoneda structure for multicategories. With the resulting Yoneda embedding, we get a full and faithful representation of

* CLC, Mathematics Department, IST
Lisbon, Portugal.
e-mail:chermida@math.ist.utl.pt
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a multicategory as a multicategory of modules (presheaves) and multilinear morphisms.

2 Fibrations for multicategories

2.1. Definition. Let $p : \mathbb{T} \to \mathbb{B}$ be a morphism of multicategories.

1. a morphism $f : \langle x_1 \dots x_n \rangle \to y$ in $\mathbb{T}$ is p-cartesian if for any $h : \langle z_1 \dots z_m \rangle \to y$ with a factorisation $ph = pf \circ \langle u_1 \dots u_n \rangle$, there are unique $u'_i : \vec{z}_i \to x_i$ such that $h = f \circ \langle u'_1 \dots u'_n \rangle$ ($\langle z_1 \dots z_m \rangle = \vec{z}_1 \cdots \vec{z}_n$).
2. a morphism $f : \langle x_1 \dots x_n \rangle \to y$ in $\mathbb{T}$ is v-cartesian if the same property holds but restricted to any h such that $ph = pf$.
3. The morphism $p : \mathbb{T} \to \mathbb{B}$ is a **fibration** if for every y in $\mathbb{T}$ and $u : \vec{j} \to py$ in $\mathbb{B}$ there is a p-cartesian morphism $\bar{u} : \vec{u^*}x \to y$ over u ($p\bar{u} = u$).

2.2. Proposition. *$p : \mathbb{T} \to \mathbb{B}$ is a fibration iff the following hold:*

- *for every y in $\mathbb{T}$ and $u : \vec{j} \to py$ in $\mathbb{B}$ there is a v-cartesian morphism $\bar{u} : \vec{u^*}x \to y$ over u*
- *v-cartesian morphisms are closed under (multicategory) composition*

2.3. Remark. We should distinguish between fibration of multicategories, as defined above, and *fibrations in $\mathcal{M}ulticat$* (in the representable sense), which have cartesian liftings of *linear* morphisms only.

2.4. Example. Given a fibration $q : \mathbb{E} \to \mathbb{C}$ of categories the morphism $q_{\blacktriangleright} : \mathbb{E}_{\blacktriangleright} \to \mathbb{C}_{\blacktriangleright}$ (the induced multicategories of discrete cocones [Her00, Example 2.2(2)]) is a fibration of multicategories. In particular the multicategory $\mathbb{C}_{\blacktriangleright}$ is fibred over the terminal multicategory.

Dually, we have a notion of p-cocartesian morphism and cofibration (lifting along morphisms in the base rather than against them).

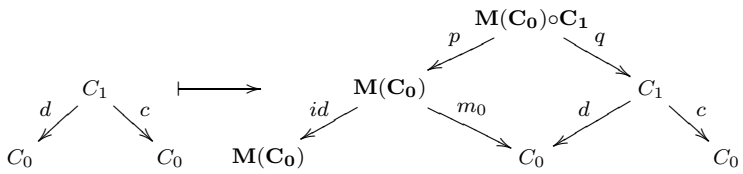
2.5. Proposition. *A multicategory $\mathbb{M}$ is representable iff the canonical morphism $! : \mathbb{M} \to \mathbf{1}$ is a cofibration.*

The above proposition strenghtens the link between representable multicategories and fibred categories outlined in [Her00]. In particular, the cocartesian morphisms for $! : \mathbb{M} \to \mathbf{1}$ are precisely the *universal* ones. We have furhtermore the following results relating representability and cofibrations:

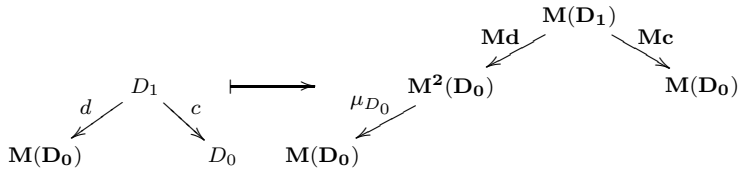
2.6. Proposition. *Let $p : \mathbb{T} \to \mathbb{B}$ be a morphism of multicategories.*

- *If both $\mathbb{T}$ and $\mathbb{B}$ are representable and p preserves universals, then p is a cofibration of multicategories iff p is a cofibration in $\mathcal{M}ulticat$.*
- *If p is a cofibration of multicategories and $\mathbb{B}$ is representable, then $\mathbb{E}$ is representable and p preserves universals*

We recall the monadic adjunction $F \dashv U : \mathcal{M}on\mathcal{C}at \to \mathcal{M}ulticat$ acts as follows: given a monoidal category $\mathbb{C}$ with objects C_0 and arrows C_1, $U\mathbb{C}$ is



where $\mathbf{M}$ is the free-monoid monad in the ambient category (in $\mathcal{S}et$, $\mathbf{MX} = \mathbf{X}^*$ the monoid of sequences under concatenation). Given a multicategory $\mathbb{D}$ with objects D_0 and arrows D_1, the free monoidal category $F\mathbb{D}$ is



Let $T : \mathcal{M}ulticat \to \mathcal{M}ulticat$ be the induced monad.

2.7. Theorem. *Let $p : \mathbb{T} \to \mathbb{B}$ be a morphism of multicategories. Tfae:*

1. *p is a fibration of multicategories.*
2. *$Fp : F\mathbb{T} \to F\mathbb{B}$ is a fibration of categories.*
3. *$Fp : F\mathbb{T} \to F\mathbb{B}$ is a fibration of monoidal categories.*
4. *$Tp : T\mathbb{T} \to T\mathbb{B}$ is a fibration in $\mathcal{M}ulticat$.*

2.8. Remark. In (3) by a fibration of monoidal categories we mean the one-object case of a 2-fibration in the sense of [Her99].

¿From the above characterisation theorem, since both F and T preserve pullbacks (because $\mathbf{M}$ is a cartesian monad), we deduce a change-of-base for fibrations of multicategories, *i.e.* they are stable under pullback. Let $\mathcal{F}ib(\mathcal{M}ulticat)$ denote the 2-category whose objects are fibrations of multicategories, morphisms are commuting squares where the top morphism preserves cartesian morphisms of the total multicategories, and the evident 2-cells.

2.9. Proposition. *The forgetful 2-functor base : $\mathcal{F}ib(\mathcal{M}ulticat) \to \mathcal{M}ulticat$ taking a fibration to its base multicategory, is a 2-fibration.*

Hence all the basic results of 2-fibrations of [Her99] (factorisation of adjunctions, construction of Kleisli objects, *etc.*) carry through to the setting of multicategories.

2.1 Coherence for fibrations of multicategories

Fixing a base multicategory $\mathbb{B}$, let $\mathcal{F}ib/\mathbb{B}$ denote the fibre over $\mathbb{B}$ of the 2-fibration $base : \mathcal{F}ib(\mathcal{M}ulticat) \to \mathcal{M}ulticat$ and similarly let $Split(\mathcal{F}ib/\mathbb{B})$ the corresponding sub-2-category of split fibrations (*i.e.* those with a choice of cartesian liftings closed under composition and identities) and morphisms between such preserving the splittings. Using the usual coherence theorem for fibrations of categories, the characterisation theorem 2.7, and the fact that the unit of the monadic adjunction $F \dashv U$ is cartesian we can deduce the following coherence result:

2.10. Theorem. *The inclusion $Split(\mathcal{F}ib/\mathbb{B}) \hookrightarrow \mathcal{F}ib/\mathbb{B}$ has a left biadjoint whose unit is a pseudo-natural equivalence. Thus, every fibration is equivalent to a split one.*

2.2 Indexed characterisation

Let $p : \mathbb{T} \to \mathbb{B}$ be a fibration of multicategories and consider the induced fibration of categories $Fp : F\mathbb{T} \to F\mathbb{B}$ (*cf.* Theorem 2.7). By making a choice of cartesian liftings (a cleavage) we obtain a pseudo-functor $\mathcal{F}_{Fp} : F\mathbb{B}^{op} \to \mathcal{C}at$. Conversely any such pseudo-functor gives rise (via the Grothendieck construction) to a fibration of categories over $F\mathbb{B}$. For a monoidal category $\mathbb{C}$ let $\mathsf{Hom}(\mathbb{C}, \mathcal{C}at)$ denote the 2-category of homomorphisms of monoidal bicategories, monoidal pseudo-natural transformations and modifications (*cf.* [DS97]). Such a homomorphism $\mathcal{F} : \mathbb{C} \to \mathcal{C}at$ amounts to a pseudo-functor together with coherent isomorphisms $\gamma_{x_1,\ldots,x_n} : \mathcal{F}x_1 \times \ldots \times \mathcal{F}x_n \xrightarrow{\sim} \mathcal{F}(x_1 \otimes \ldots \otimes x_n)$.

2.11. Theorem. *The above assignment $p \mapsto \mathcal{F}_{Fp}$ sets up a biequivalence*

$$\boxed{\mathcal{F}ib/\mathbb{B} \cong \mathsf{Hom}(\mathsf{F}\mathbb{B}^{\mathsf{op}}, \mathcal{C}at)}$$

2.3 Operads and algebras

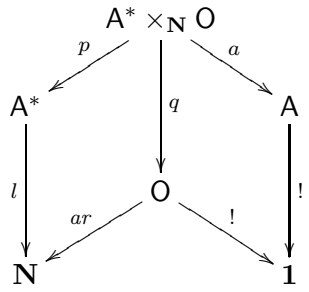
A one-object multicategory amounts to an $\mathbf{N}$-indexed family $\mathsf{O} = \{\mathsf{O_n}\}$ (elements of O_n should be thought of as n-ary operations) closed under composition (and identities).

2.12. Proposition.

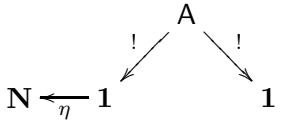
$$\boxed{\textit{one-object multicategory} \equiv \textit{(non-permutative) operad}}$$

2.13. Remark. For brevity we do not deal here with permutative operads, which we will take us into the more involved setting of lax algebras on bimodules rather than spans (as in [Her01]).

Since a multicategory, and thus an operad, is essentially a monoid O (of operations), a sensible notion of algebra for such should be an O-set, which amounts to a discrete cofibration:



where the left-hand square is a pullback. Notice that if we consider the object A as the span $A^\natural$



then an algebra structure as above amounts to a morphism $a : A^\natural \circ O \to A^\natural$ which commutes with the multicategory structure of O, that is, it is an object A together with an *O-action*. This is the (T-genereelisation) of the notion of algebra for an operad *e.g.* [KM95]. Thus we have the following identification:

2.14. Proposition.

O-*algebras* $\equiv$ *discrete cofibrations over the (one-object) multicategory* O

The usual treatment of operads deals with them in some (monoidal closed) category of (differential graded) modules $\mathcal{V}$. In such a context it is possible to construct, for a given object A, the *endomorphism operad* $\mathsf{End}(A)$, whose n-ary operations are

$$\mathsf{End}(A)_n = \mathsf{Hom}_{\mathcal{V}}(A^{\otimes n}, A)$$

In the general setting we are working in the analogue construction is possible provided $\mathbb{B}$ is locally cartesian closed:

2.15. Proposition. *If $\mathbb{B}$ is locally cartesian closed, the $\mathbf{Spn_T}(\mathbb{B})$ is right closed, i.e. for every morphism $f : X \to Y$ in $\mathbf{Spn_T}(\mathbb{B})$, the functor*

$$f\circ(_) : \mathbf{Spn_T}(\mathbb{B})(\mathbf{Y}, \mathbf{Z}) \to \mathbf{Spn_T}(\mathbb{B})(\mathbf{X}, \mathbf{Z})$$

has a right adjoint $[f, _]$.

2.16. Definition. Let $\mathbf{Spn_T}(\mathbb{B})$ be right-closed. For an object A define its *object of endomorphisms* $\mathsf{End}(A)$ as the the span $[A^\natural, A^\natural]$.

Notice that that the object of endomorphisms $\mathsf{End}(A)$ is effectively a family $p : \mathsf{End}(A) \to \mathbf{1}^* = \mathbf{N}$, since the second leg of the span is trivial.

2.17. Proposition. *Under the hypothesis of Definition 2.16, the following hold:*

1. $\mathsf{End}(A)$ *is a (one-object) multicategory (or operad).*
2. *For a given operad O, there is a one-to-one correspondence between O-algebra structures on A and morphisms of multicategories from O to* $\mathsf{End}(A)$.

3 Comprehensive factorisation

We now set out to show that, just like in $\mathcal{C}at$, any morphism of multicategories admits a functorial factorisation into an initial morphism followed by a discrete cofibration. The terminology *comprehensive factorisation* was introduced in [SW78], although the result was already present in [Par73]. Here, we follow Paré's argument as analysed in the expository note [Her98].

The argument to obtain the factorisation of a morphism $f : \mathbb{C} \to \mathbb{D}$ involves analysing the adjunction between $\mathcal{M}ulticat/\mathbb{D}$ and $DiscCofib/\mathbb{D}$; more precisely, the inclusion of $DiscCofib/\mathbb{D}$ (the category of discrete cofibrations of multicategories over $\mathbb{D}$)into $\mathcal{M}ulticat/\mathbb{D}$ has a left adjoint $\varpi : \mathcal{M}ulticat/\mathbb{D} \to DiscCofib/\mathbb{D}$. We thus have for $f = \varpi(f) \circ \eta_f$ and the problem boils down to showing that the unit η_f is initial. In order to do so, we decompose the above adjunction into two

$$\mathcal{M}ulticat/\mathbb{D} \underset{\int}{\overset{M}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{Hom}(\mathsf{F}\mathbb{D}^{\mathsf{op}}, \mathcal{C}at) \underset{\Delta^F\mathbb{D}}{\overset{\Pi_0^F\mathbb{D}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{Hom}(\mathsf{F}\mathbb{D}^{\mathsf{op}}, \mathcal{S}et)$$

Hence the unit of the adjunction is the composite of the units of both these adjunctions. We analyse them in turn.

3.1 The free cofibration on a morphism of multicategories

As we mentioned right from the outset, the fact that 2-cells in $\mathcal{M}ulticat$ refer only to linear morphisms means that the representable version of fibrations is not sufficient for our purposes. Consequently, although $\mathcal{M}ulticat$ admits comma-objects, the doctrinal characterisation of fibrations in [Str73], and in particular the construction of the free fibration on a morphism in terms of comma-objects, does not apply. But we can use the indexed characterisation of Theorem 2.11 to build the free cofibration on a morphism of multicategories. Namely, let $f : \mathbb{C} \to \mathbb{D}$ be a morphism of multicategories and consider the induced functor $Ff : F\mathbb{C} \to F\mathbb{D}$ and consider the corresponding free cofibration $Ff{\downarrow}F\mathbb{D}$. The induced pseudo-functor $Ff{\downarrow}_ : F\mathbb{D} \to \mathcal{C}at$ preserves (up to coherent isomorphism) the monoidal structure (given the nature of homs in $F\mathbb{D}$). Let $q : \int Ff{\downarrow}_ \to \mathbb{D}$ the corresponding cofibration of multicategories obtained by the dual of Theorem 2.11.

3.1. Proposition. *The morphism $q : \int Ff{\downarrow}_ \to \mathbb{D}$ is the free cofibration on $f : \mathbb{C} \to \mathbb{D}$ in $\mathcal{M}ulticat/\mathbb{D}$.*

Now the morphism $Ff : F\mathbb{C} \to F\mathbb{D}$ factorises as $Ff = Fq \circ \delta_{Ff}$. We can further show that $\delta_{Ff} = F(\theta_f)$ for a morphism of multicategories $\theta_f : \mathbb{C} \to \mathbb{D}$. Since $F(\theta_f)$ is initial, so is θ_f in accordance with the notion of initiality resulting in $\mathcal{M}ulticat$ from the Yoneda structure we set up in §4.1.

3.2. Corollary. *Any morphism of multicategories $f : \mathbb{C} \to \mathbb{D}$ factorises as $f = q \circ \theta_f$, where q is a cofibration and θ_f is initial.*

3.2 Initial - discrete cofibration factorisation of a morphism of multicategories

To conclude we only need to notice that the second adjunction above is well-defined, *i.e.* that postcomposing a homomorphism $\mathcal{F} : F\mathbb{D} \to \mathcal{C}at$ with $\Pi_0 : \mathcal{C}at \to \mathcal{S}et$ (connected components functor) yields a homomorphism again. This follows from the (frequently used) property that Π_0 preserves products. Now we can conclude (with the arguments above and the cited references):

3.3. Theorem. *$\mathcal{M}ulticat$ admits a functorial factorisation of its morphisms into (initial - discrete cofibration).*

4 Change of base for Yoneda structures along a left adjoint

The notion of *Yoneda structure* in a 2-category was introduced in [SW78] and shown to yield a satisfactory concise axiomatics for the development of elementary category theory inside a 2-category. We wish to endow $\mathcal{M}ulticat$ with such a structure. We do so by using the existing Yoneda structure on monoidal categories (which uses essentially Day's tensor product of presheaves to extend to monoidal categories the Yoneda structure of $\mathcal{C}at$). Recall that a Yoneda structure on a 2-category $\mathcal{K}$ consists of a (left-closed ideal) class of *admissible* arrows, together with the assignment to every admissible object A of a 'Yoneda embedding' $y_A : A \to PA$ satisfying certain axioms involving left extensions and liftings. The basic abstract result we obtain is the following change-of-base of Yoneda structures along a left adjoint with a fully faithful unit.

4.1. Theorem. *Let $\mathcal{L}$ be a 2-category with a Yoneda structure and consider a (bi)adjunction $F \dashv G : \mathcal{L} \to \mathcal{K}$ with a fully faithful (pseudo-natural) unit. Then $\mathcal{K}$ can be endowed with a Yoneda structure, with*

- *a morphism h in $\mathcal{K}$ is admissible iff Fh is admissible in $\mathcal{L}$, and*
- *given an admissible object X in $\mathcal{K}$, its Yoneda arrow $\bar{y}_X : X \to G(PFX)$ the adjoint transpose of the Yoneda arrow $y_{FX} : FX \to PFX$ in $\mathcal{L}$.*

4.2. Remark. The notion of admissibility for a morphism $f : X \to Y$ in a 2-category $\mathcal{K}$ for a Yoneda structure essentially requires that $Y(fx, y)$ be small. Hence, in the presence of comma-objects we can stipulate admissibility in terms of smallness.

4.1 Yoneda structure for multicategories

The monad $T : \mathcal{M}ulticat \to \mathcal{M}ulticat$ whose algebras are strict monoidal categories, has as pseudo-algebras (ordinary, infinatirily presented) monoidal categories [Her00, §9]. Moreover, the unit of the corresponding biadjunction is fully faithful [Her00, Cor.7.4]. By Theorem 4.1 $\mathcal{M}ulticat$ inherits a Yoneda structure, so that for a multicategory $\mathbb{C}$ the corresponding Yoneda object is $U[F\mathbb{C}^{op}, \mathcal{S}et]$, where the presheaf category $[F\mathbb{C}^{op}, \mathcal{S}et]$ has a monidal structure which extends that of $F\mathbb{C}$ by Kan extension along the (product of) the ordinary Yoneda embedding. Thus any multicategory has a full and faithful embedding into a multicategory $U[\mathbb{D}^{op}, \mathcal{S}et]$ for a (strict) monoidal category $\mathbb{D}$. The objects of this latter, being presheaves, could be construed as modules, while the morphisms corresponds to the linear maps classified by their (Day) tensor product in $[\mathbb{D}^{op}, \mathcal{S}et]$.

4.3. Corollary. *Any multicategory $\mathbb{C}$ admits a full and faithful embedding into a (representable) multicategory of modules and multilinear maps $U[\mathbb{D}^{op}, \mathcal{S}et]$, for a strict monoidal category $\mathbb{D}$.*

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