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HERBART'S INFLUENCE ON BERNHARD RIEMANN

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SUMMARIES

B. Riemann (1826-1866) knew a great deal of the thought of the German philosopher J. Fr. Herbart (1776-1841). During his studies of the philosopher's work he copied out numerous excerpts and made a few notes which are preserved (at least partially) in the Riemann Archiv *at Göttingen. This material reveals that Herbart influenced Riemann much more in his epistemology and the comprehension of science than in his particular philosophy of space and spatial thinking. Thus the relationship between Herbart and Riemann has to be looked upon as an example of an influence of German* Bildungsphilosophie *on the mathematics of the 19th century.*

Bekanntlich befasste sich B. Riemann (1826-1866) gründlich mit der Philosophie J. Fr. Herbarts (1776-1841). Er fertigte während seiner Herbart-Studien umfangreiche Exzerpte und einige Notizen an, die (zumindest teilweise) im Göttinger Riemann Archiv erhalten sind. Dem Inhalt dieser Materialien nach zu schließen lag der Schwerpunkt des Herbartschen Einflusses auf Riemann nicht so sehr in seiner Philosophie des Raumes und des räumlichen Denkens als vielmehr im Bereich der Erkenntnistheorie und Wissenschaftsauffassung. Gerade darin stellt sich aber die Beziehung zwischen Herbart und Riemann als Fallbeispiel eines Einflusses der deutschen Bildungsphilosophie auf die Mathematik des 19. Jahrhunderts dar.

Bernhard Riemann was one of the most influential mathematicians of the last century, working in mathematical fields as different as complex function theory, partial differential equations, foundations of Fourier analysis, topology, differential geometry, birational geometry, and number theory [Dieudonné

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1978, Klein 1926, Kline 1972]. His work, although small in volume, had a strong impact on later researchers. Because of its deep conceptual penetration of the subject matter in question, it became a source of inspiration when the tendency of mathematics toward a clarification of the underlying main concepts became stronger in the second half of the 19th century [1]. Thus it is of interest to investigate the background of Riemann's research related to problems of conceptual structure.

Riemann's published works contain philosophical fragments [1892a, 509 ff.] which shed some light upon his reflections about science. They also provide evidence that Riemann was strongly influenced by the philosopher Johann Friedrich Herbart, who had been teaching at Königsberg and Göttingen until his death in 1841 (see Appendix 1a). Riemann stated:

> *The author* [Riemann himself] *is a Herbartian in psychology and epistemology (methodology and eidolology); in most cases, however, he cannot agree with Herbart's natural philosophy and the metaphysical disciplines (ontology and synechology) referring to it.* [1892a, 508]

This statement has been cited often, and so some influence of Herbart on Riemann is generally assumed in the literature [Russell 1956, 62 f.; Torretti 1978, 107 f.; Scholz 1980, 94-95]. But there have been differences of opinion over which of Herbart's ideas were most influential. Russell believed that it was Herbart's philosophy (and psychology) of space which made a deep impact on Riemann. That would not correspond to the contents of the citation above, however, because "synechology" [*Synechologie*] was Herbart's name for the philosophical foundation of the concept of space, and this was exactly one of the points Riemann said had failed to convince him. But this alone will not justify rejecting Russell's statement without further consideration, for it gains some plausibility from a similarity between Riemann's explanation of the concept of manifold and Herbart's "serial forms" (see below). Torretti's discussion of the topic agrees essentially with that of Russell, although he does not find all of it acceptable. In (implicit) contrast to this, Scholz [1980] looks for the link between Herbart and Riemann much more on the general level of epistemology, which, again, played a distinct role in Riemann's formation of the concept of manifold.

But all of these estimations suffer from the limitation of the material on which they are founded. On the basis of the published works of Riemann and Herbart one could do no more than try to make reasonable guesses. But the Riemann *Nachlass* at Göttingen University library contains excerpts and notes from Riemann's studies of Herbart, which give a clearer picture than before of Herbart's philosophy and how it was of particular

interest to Riemann. This article presents this previously unpublished material, discusses the most essential points of Herbart's philosophy to which Riemann responded, and finally offers a short evaluation which might point to questions for further investigations.

THE MATERIAL

Folder 18 of Riemann's *Nachlass* contains 203 sheets with manuscripts classified as "Fragmente naturphilosophischen Inhalts"; but there are at least 12 sheets with notes and excerpts from Riemann's studies of Herbart (numbers 57, 59, 60, 64, 66, 70, 72, 73, 90, 140, 141, 171) [2]. The excerpts are taken from Herbart's works on metaphysics and psychology [Herbart 1796, 1799, 1811, 1812, 1822, 1824, 1825, 1828]. In addition, there is a note (R.177) [3] with references to [Herbart 1807] and another (R.141) with references to [Herbart 1851].

Herbart distinguished two large fields of philosophy, metaphysics and aesthetics and practical philosophy. Metaphysics, according to Herbart, contained four disciplines: Methodology, eidolology [*Eidolologie*], ontology, and synechology [*Synechologie*].

Trained in the philosophy of Kant and Fichte, he saw the goal of philosophy and the sciences as advancing from contradictory sense perceptions to concepts of the underlying reality. This advancement he believed should be governed by sound principles of *methodology*. His thorough studies of Fichte's philosophy of the self led Herbart to the insight that the self was one of the most subtle concepts of philosophy, endowed with a multitude of contradictions (see Appendix 1a). To show this, and to come to grips with it, was the task of the discipline he called *eidolology*. Methodology and eidolology together can be looked upon (as did Riemann in the citation above) as forming Herbart's epistemology. The most general categories of reality (for Herbart: being [*Sein*], quality, inherence, and change) formed the subject matter of his *ontology*. However, this was not yet sufficient as a foundation for scientific investigation and had to be completed by generation of the concepts of space, time, number, and matter, which was done in the discipline of *synechology* [Weiss 1928].

Thematically, all four disciplines of Herbart's metaphysics are covered by Riemann's excerpts, whereas aesthetics and practical philosophy are touched upon only marginally (e.g., R.60^{r}). Table 1 provides a list of the most important of these excerpts with reference to topic, source (in Herbart's *Werke*), discipline in Herbart's classification of metaphysics, and folio number

TABLE 1

Survey of Riemann's excerpts from Herbart in folder 18 of Riemann's *Nachlass, Handschriftenabteilung Universitätsbibliothek Göttingen*

Subject matter	Excerpt from Herbart	Folio (folder 18)
(i) Methodology		
Philosophical concepts as changing products of thought	[1825, 198/199]	59^{v}
Reality lying behind perceptions	[1825, 199/200]	59^{v}
Reality lying behind perceptions	[1828, 123/124]	64^{r}
Method of relations	[1851, 594-596]	141^{r}
(ii) Eidolology		
Schelling's dialectics of the self	[1796, 30/31]	64^{r}
Herbart's explanation of the self	[1799, 108-110]	64^{v}
Herbart's criticism of Fichte's self	[1822, 107/108]	64^{v}
(iii) Ontology		
Things as bundles of properties	[1825, 193]	59^{r}
Substance as understood by Leibniz, Locke, and Kant	[1825, 193/194, 196]	59^{v}
Farreaching character of change of chemical substances	[1828, 130/131]	66^{r}
Difficulties of Greek philosophers with contradictions of change	[1828, 134/135, 330-334, 337, 340]	$66^{r,v}$
Causality	[1825, 115, 424/425]	$70^{r,v}$
Causality	[1828, 122]	64^{r}
(iv) Synechology		
Serial forms	[1825, 192/193]	59^{r}
Serial forms	[1824, 428, 421]	72^{r}, 73^{v}
Objective basis of space imagination	[1824, 425]	70^{v}
Criticism of Kant's idea of space	[1824, 428]	72^{r}

in Riemann's *Nachlass*, folder 18. Apart from the excerpts there are three notes by Riemann referring to Herbart (R.140, 141, 177). They are documented in Appendix 2 and note 17.

An analysis of the excerpt material and the additional notes shows that there are certain points in Herbart's philosophy which seemed of particular importance to Riemann: elements of dialectics, methodology, spatial concepts, and the orientation of mathematical research. These points are discussed in the following sections, in order to draw a picture of Herbart's influence on Riemann.

ELEMENTS OF DIALECTICS

Philosophy of the Self (eidolology)

Riemann came into contact with Fichte's and Schelling's philosophy of the self by way of the early works of Herbart [1796, 1799, R.64]. At the beginnings of his studies (1794), Herbart had been one of Fichte's main disciples and was fascinated by the idea of the "absolute self" as the basis for cognition of all of reality [Asmus 1968, 98 ff.]. But already during the final phase of his studies (1795/1796) he developed--in his criticism of Fichte's idea--the foundations of his own philosophy of the self, which he later called eidolology.

Herbart's argument was based on the observation that the conception of the self always contained in itself the contradiction between the act of reflection and the reflected, and so in Fichte's terminology of the self and the not-self [*Ich und Nicht-Ich*]. This contradiction, however, served him as proof that the self was no simple idea; on the contrary, it was one of the richest and most concrete concepts, and so could not be the starting point for philosophy. This led him to regard Fichte's "absolute self" as an idealist fiction [Herbart 1822, 107-108*; 1824, 237 ff.] [4]. The self seemed to him to be a bundle of properties [*Complexion von Merkmalen*] [1828, 209/210] and, according to Herbart, it contained a fluctuating structure of ideas or presentations [*Vorstellungen*] opposed to each other and impeding one another [1799, 108 f.*]. This appeared to him to be important because he regarded the process of gaining knowledge as governed not only by the principles of methodology but also by the laws of the movement and impediments of presentations.

Riemann excerpted the essential points of Herbart's argumentation, was obviously convinced by it, and adopted the basic features of the philosopher's eidolology/psychology. This, already stated in the citation above, is made completely clear by a note in which Riemann summarized Herbart's position in his

own words, expressing what he felt were its most important aspects (R.140; see Appendix 2):

1. Presentations are the elementary forces of the soul. They oppose and impede each other. The resulting antagonisms generate the movement of thought.
2. Images of the outside world are generated by negations of the given perceptions. Negation itself has to be learned from the experience of cancellation of presentations.
3. Negation is canceled and a new position gained, when a connection with the already formed and interdependent images/concepts is attained.

Riemann thus came to know and accept certain elements of Herbart's dialectics in his reception of such eidolology: opposition of concepts/presentations, negation, and position as canceled negation.

In his published philosophical fragments we find a passage in which he experimented with a conceptual structure of thesis/antithesis [1892a, 518 ff.]. Previously it was unclear what his background was for expressing this idea. Now I think an answer can be given: Riemann's knowledge of certain aspects of Fichte's and Schelling's dialectical philosophy came through his studies of Herbart, although Riemann was not directly influenced by their philosophy. Nevertheless, he adopted basic features of a dialectical character from Herbart's philosophy, from eidolology in particular. It was German philosophy of the first half of the 19th century, which Riemann encountered by way of his Herbart studies, that set the background for his own dialectical arguments.

The Problem of Change and the Structure of the Real

Riemann studied Herbart's historical treatise on metaphysics from the Greeks to Schelling and Fries [Herbart 1828] with interest and made long excerpts from this work (R.64, 66), including a long passage about Plato (R.66). Herbart explained that Plato, once he stumbled upon the contradiction lying in change as a form of experience, made a strange [*ungereimt*] distinction between the world of knowledge containing eternal ideas and the world of opinions dealing with changing phenomena.

According to Herbart, Plato tried to mitigate the strangeness of the sharp opposition of these two worlds by assuming a substance without any qualities, which, shaped by ideas, would generate the changing things of the sensual world [Herbart 1828, 330 f.*]. Herbart, and with him Riemann, traced a line of followers who somehow adopted this conception, although with modificatons, which ran from Aristotle [Herbart 1828, 134 f.*] to the scholastics [1828, 323 f.*], Spinoza, and even to Schelling [1828, 135*].

In contrast to these philosophers Herbart saw a much closer connection between changing phenomena and the real. He was sure that criticism of the phenomena could lead to knowledge about "the real" [*das Reale*] [5], and he set himself the task of structuring that knowledge according to the connections of the phenomena. This was the responsibility of metaphysics as Herbert understood it, but metaphysics had to do more. Once it had penetrated into the depths of "the real," it had to rise again, in order to distinguish "the given" (the phenomena) from "the real" as far as possible. This movement of thought, compared by him to an arc, ought to be governed by the principles of methodology [1829, 14].

Riemann was obviously interested in this type of argument. He copied out the following explanations of Herbart point by point:

In the first place, according to Herbart, experience shows us properties and bundles [*Complexionen*] of properties, the underlying reality of which must first be sought in *things* to which the properties are ascribed [1825, 199*]. But the next step of the analysis, so he continued, led investigators to the *elements of Greek philosophy,* from which the *things* now obtained a "borrowed reality" [*geliehene Realität,* 1825, 199]. Later the scientists' analysis led to the introduction of *chemical elements* from which now the old elements (at least water and air, which Herbart explicitly referred to) were derived. Finally idealist philosophy came, according to Herbart, and reduced even chemical elements, as well as properties, things, and old elements, to intuition [*Anschauung*] and thought, concluding that the idea of the *self* underlay all other concepts and "lent reality" to them [Herbart 1825, 200] [6].

We see from this argument that Herbart did not restrict his reflections to metaphysics proper but also included science. This goes hand in hand with his opinion concerning the relationship between philosophy and science, which rejected a sharp dividing line between the two.

From Riemann's selection of passages we may conclude that he was particularly interested in the question of how the problem of change and its consequences for the structure of reality had been tackled in metaphysics [7]. He noted Herbart's conceptions of things as bundles [*Complexionen*] of properties (R.59), behind which lay the *real* [Herbart 1828, 199-200]. But the specific construction of "the real" as proposed by Herbart is not refleced in Riemann excerpts. This corresponds perfectly well with his declaration that he did not agree with the philosopher's ontology. The distinction, however, between the phenomena and a more stable underlying reality with an intense relationship between both became an essential point in Riemann's own reflections about the epistemology of science [1892b].

THE PROCESS OF KNOWLEDGE (METHODOLOGY)

Riemann adopted Herbart's idea of the two-phase movement ("arc") for the advancement of knowledge (see above), although he preferred a terminology which was closer to that of scientific research than to that of metaphysics. So in a note which can be looked upon as an intermediate paper coming somewhere between the pure Herbart study and his own epistemological essay [1892b], he characterized these two phases in the following way:

1. *Formation of concepts from perceptions*
   --abstraction and induction, synthesis a posteriori
2. *Generation of the perceived from the concepts*
   --synthesis a priori. [R.141; see Appendix 2]

Obviously Riemann did not mean what Kant did by *synthesis a priori*. He had a *relative a priori* in mind, which, serving as a conceptual framework for experience, presupposed the other half of the whole process of knowledge (abstraction and induction). Thus the historical change of concepts was not only possible but was to be achieved as follows:

3. *Change (or completion) of the concepts as small as possible, where the perceived is impossible or unlikely according to the concepts.* [R.141]

It seemed to him that Herbart's "method of relations" which dealt with the dialectics of reason and consequence (see Appendix 1b) was useful for accomplishing this change.

This was not the only reference to Herbart, showing that Riemann's ideas about change of concepts were partially due to the philosopher. He also referred to Sections 139-145 of Herbart's "Science of Psychology" [1825] from which he made long excerpts (R.59). There he noted in particular that Herbart looked upon changes of ideas about reality resulting from the solution of contradictions as conceptual revolutions similar to those in astronomical knowledge [Herbart 1825, 198*], thus considering the process of knowledge as a stepwise acquisition of levels in the intellectual evolution of man [*Bildungsstufen ..., welche successiv erreicht werden*, Herbart 1825, 199*] [8].

It seems clear, then, that two important features of Riemann's developmental dialectics as proposed in his published epistemological fragment [1892b] were due to Herbart:

--a distinction between the phenomena and the underlying reality with a corresponding difference between the perception

and the conceptual acquisition of reality; the latter had "to go back behind the phenomena" and according to Riemann could thus serve as a base for an explanation of the former;

--a belief in progress of knowledge by "transformation of older conceptual systems" because of contradictions on the level of the explanations or between concepts and phenomena.

But Riemann's presentation differed in three points from that of Herbart [Scholz 1980, 94-96]:

--Riemann referred mainly to the sciences, whereas Herbart had investigated both metaphysics and science, with a clear emphasis on the former.

--Riemann was therefore able to integrate the materialist criterion of truth, corresponding to the tradition of science, much more clearly into his essay. According to Riemann knowledge was true if the connection of the concepts corresponded to the connection of things, which again was to be deciphered from the connection of the percieved phenomena [Riemann 1892b, 523].

--Finally, Herbart had considered the historical change of concepts (mainly in ontology) to have been a chain of errors [1825, 198 ff., partially cited in note [8]]. Riemann, looking at the development of scientific knowledge, expressed a different point of view. He declared that the relationship of new knowledge about a certain sector of reality to older knowledge of the same sector was not necessarily that of correction and error. He stated that modification of conceptual structure was also possible, which only *refined* the conceptual elements of the old system without falsification of part or all of them [9].

Thus, when Riemann called himself a "Herbartian in epistemology," he meant that he had not only adopted certain central features of Herbart's epistemology, but that he had also assimilated the latter into the tradition of science, which enriched it by a scientific materialist component.

SPATIAL CONCEPTS

Kant's conception of fixed and a priori deducible categories of space and time were criticized by later philosophers of German idealism (Fichte, Schelling, Hegel, Schleiermacher). In this respect Herbart was no different. "Historical change of concepts" was used by Herbart as a strong argument against the opinion that the fundamental concepts of reality were inborn concepts or categories, an argument obviously directed against Kantian philosophy. To the contrary he stressed that concepts were

changing products of thought [1825, 198*], including those that served as "forms of experience" and as such had the very function of Kant's categories [Herbart 1829, 21].

Riemann agreed with Herbart's refutation of Kant's epistemology [1892b, 522], although we find little evidence in his excerpts of a specific interest in Herbart's explicit discussions of Kant. But in one of them (R.72) we find a short polemical argument in that direction. In it the idea of space and time as "empty vessels ... in which the senses ought to pour their perceptions" was criticized as "a completely shallow, meaningless, and inappropriate [*völlig gehaltlose, nichtssagende, unpassende*] hypothesis" [Herbart 1824, 428*]. It seems that Riemann thought the question was settled on this point, and was not very much interested in the detailed philosophical argument that dealt with Kant.

Herbart claimed that, like all concepts which served as "forms of experience," spatial concepts had their origin in experience, but then had to be shaped and developed by philosophical and scientific thinking. Spatial presentations are formed on this level by the perceptions of things due to the "mobility of man in his neighbourhood" [1824, 425*]. So "... the series of presentations [Vorstellungsreihen], which eventually form, order, and connect themselves and in which the order of perceptions is contained [aufbewahrt]," come into being [10].

Space and time were Herbart's starting point for generating more general "continuous serial forms" [continuierliche Reihenformen] of concepts. The explanation of the latter was a very complicated procedure, part of the discipline of metaphysics that he called synechology [Herbart 1829, 110-158; Weiss 1928, 50-57]. Riemann's excerpts suggest that he did not bother about specific procedures to generate "serial forms," although he was interested in how all of this related to Herbart's geometrical thinking, because the very general idea made it possible to transfer spatial concepts into a nongeometric context.

Vaguely speaking, a continuous "serial form" is produced when a specified class of presentations undergoes a "graded fusion" [*abgestufte Verschmelzung*] through which the corresponding presentations are ordered, so that one cannot but unite them in a spatial mode [Herbart 1825, 192*] [11].

Consequently, Space did not exist for Herbart; instead there was a *collection of spaces* for which the modes of existence were completely different [1812, 207]. His two main examples were the "line of sound" [*Tonlinie*] and the color triangle with blue, red, and yellow at the corners and the mixing colors in the two-dimensional continuum in between [1825, 193*] [12]. Similarly he considered any thing as a "bundle [*Complexion* of properties," each property of which ought to be thought of as lying in a different "qualitative continuum" [1825, 193*].

In this respect Herbart showed a strong affinity to the scholastic philosophers, in particular to Nicole Oresme [Clagett 1968]. Moreover, the idea of geometrizing any "thing" in that way put Herbart (like Oresme) on the very fringe of higher-dimensional thinking, because there was no intrinsic reason to geometrize only those "things" which could be analyzed by a collection of no more than three properties. Herbart stopped exactly at this barrier and claimed a *limitation of spatial thinking to three dimensions* [1812, 208]. In this respect he did not go beyond scholastic philosophy.

Because such general aspects of Herbart's ideas about serial forms are reflected in Riemann's excerpts (R.59, 73), it is important to determine *how important* Herbart's transfer of spacelike ideas into nongeometric concepts was for Riemann's inclination to make a similar transfer *inside mathematics*.

To start with, it is worth noting a difference of opinion between Riemann and Herbart. When Herbart analyzed things as bundles of properties, he emphasized that *any* such property could be understood as a "qualitative continuum" [Herbart 1825, 193*] (cited in R.59). The background for the transfer of geometric thinking to other concepts was a very broad, nearly universal one for him [13]. Riemann stated exactly the contrary in his inaugural lecture, namely, that "in general life" concepts of continuous magnitudes (which would correspond to Herbart's continuous serial froms) were encountered *only very rarely*, but that in mathematics they arose often [Riemann 1892a, 274].

The latter remark refers to a tendency in mathematics of the first part of the 19th century to transfer geometrical language to algebraic or analytical systems of several variables, a tendency which was at least partially known to Riemann via Gauss [Scholz 1980, 15 ff., 53 ff.] and which was independent of Herbartian philosophy. From Riemann's very clear remark about the dominance of mathematics in the total range of extension for the concept of continuous magnitude (manifold), we may draw the conclusion that again *it was this mathematical tendency* to transfer geometric thinking to nongeometric fields which was the dominant background of his concept formation.

This conclusion is corroborated by another difference between Herbart and Riemann with respect to the connotation (the "intension" according to Wussing [1967]) of their spatial concepts. Herbart adhered strictly to the traditional dimensional barrier (three), whereas for Riemann's concept of manifold, multidimensionality was one of the most essential innovations. This feature of Riemann's thought was again closely linked to the mathematical tendency to geometrize nongeometric algebraic and analytic systems [Scholz 1980, 15 ff., 88 ff.], and stands in strict opposition to Herbart's opinions.

Thus Herbart's *direct* influence on Riemann's formation of the concept of manifold does not appear to have been very strong.

However, Riemann must have found Herbart's philosophical reflections about serial forms of presentations stimulating. He may have received very general hints of how to proceed in the formation of the concept, fostered by a structural similarity between Herbart's geometric arguments and the mathematical tendency "to geometrize." Moreover, he took up Herbart's idea of an objective foundation for the formulation of spatial concepts and sharpened it from the point of view of the scientist. He did so in proposing to formulate the structure of spatial concepts as closely as possible according to experimental physical evidence. But the essential points of Riemann's concept of manifold (multidimensionality, opposition between locally simple and globally complex behavior, separation of qualitative aspects of extended magnitides from quantitative ones, and the separation and interdependence of structures on it [Scholz 1980, 30 ff., 88 ff.]) had no connection with Herbart's geometric thoughts, and Riemann had to elaborate these from the mathematical material.

PHILOSOPHICAL STUDIES OF SCIENCE

The preceding section leads to the conclusion that Herbart's direct influence on Riemann's formulation of the concept of manifold was of rather limited importance. We must not conclude, however, that his overall influence on Riemann was similarly restricted. We have already found a much closer connection between the two on the level of epistemology (see above), which played a role in Riemann's mathematics as much as it dealt with the question of how to proceed in research. It might be worthwhile to ask whether Herbart had some influence on Riemann's perception of the goal of mathematics or the task of mathematical research.

Some answer may be formulated from the notes Riemann made (R.177) in a short evaluation of Herbart's article about philosophical studies [Herbart 1807]. The philosophical studies Herbart had in mind involved the elaboration of unity in diversity; for this he proposed working out a central concept [*Hauptbegriff*] in which this unity was to be reflected for each field of study [1807, 231, 238].

Whereas the sciences developed only central concepts linked to their specific subject matter, it seemed necessary for Herbart to form unifying concepts transcending a specific context, and this led from philosophical studies of the sciences to philosophy proper [*Philosophie als eigene Wissenschaft*]. The task of professional philosophy was not just to group the most general unifying concepts worked out by the sciences neatly together, but to analyze them and to resolve their intrinsic difficulties [Herbart 1807, 235]. To do this he thought it

important that philosophy avoid the mistakes of both empiricism (as in the philosophies of Hume and Locke) and rationalism (Plato, Descartes, Kant, Fichte). He also rejected the conception of philosophy as a dominating power "from afar," and had a much more dialectical relationship between science and philosophy in mind: "[philosophy] does not lie outside of other knowledge, but constitutes itself *with* and *in* the same" [14].

Consequently, philosophy and the sciences had to work hand in hand. Their mutual aim was to generate many-sidedness of education [*Bildung*] in the "higher leading class." Philosophy always depended for its educational influence on "other sciences which stand nearer to the professions" [15].

The generation of philosophy proper was to be achieved by *speculation* [Herbart 1807, 237], which Herbart understood in a wider sense as meaning "any endeavour to make the way for transitions" between concepts [1807, 275]. In a more restricted sense speculation ought to show only the necessary connections between concepts. In any case, Herbart saw the representation of "the real" as the final goal of speculation. So philosophical activity (speculation) had a characteristic feature; it had *concepts as its objects*.

This, so it seemed to Herbart, was the essential difference between philosophy and science which dealt with "the given." Such distinctions even held for most of mathematics up to his time: "Philosophically treated, it [mathematics] becomes part of philosophy which had to create a science of quantity [*Größenlehre*] for its own necessities, if one did not already exist," [1807, 275]. For Herbart mathematics stood in even closer relationship to philosophy than to the rest of the sciences [16].

When Riemann made his notes on this treatise [Herbart 1807], he summed up the main ideas about "speculation" with a few catchwords:

... Speculation = endeavour [*Streben*] toward resolution of problems
... Demonstration of a necessary connection between concepts
A problem of further specul[-ation]
... Philosophy as a science
General character that it is generated by speculation
That it takes concepts as its object ... (R.177) [17].

It is likely that Riemann wrote these notes during the preparation of his inaugural lecture [18], which must have been a very important period for his interpretation of the methods and goals of mathematics and mathematical sciences. It would therefore be interesting to know why Riemann extracted just these features from Herbart's article, especially if the reason is related to some structural relationship between Herbart's characterization of philosophy and Riemann's own thoughts.

The similarity, in fact, of the note and Riemann's perception of science and mathematics is striking. Riemann characterized science as "the attempt to perceive nature through accurate concepts" [1892a, 521], which proceeded gradually through the resolution of problems resulting from contradictions in concepts or between concepts and experience. In his opinion mathematics had an important share in the formation, development, and extension of scientific concepts, resembling very much the task Herbart gave philosophy in relation to science. Riemann opened his inaugural lecture with questions concerning the "possibility" and "necessity" of connections between the basic concepts of geometry [1892a, 272], and he hoped that his new concept of manifold would be particularly helpful.

Moreover the idea of clarifying conceptual structures stood at the center of Riemann's investigations throughout his mathematical work, whether it was complex function theory, geometry, or integration. This was true to such an extent that one might be tempted to read Herbart's note as providing a characterization of *mathematics* as Riemann himself would have given: A science dealing with concepts generated to solve problems arising in the attempt to gain knowledge and to clarify connections between already established knowledge ("speculation" in Herbart's language).

Riemann's understanding of the task of mathematics shows a striking similarity to the aims of philosophy as understood by Herbart, and the latter's characterization of the relationship between science and philosophy resembles very much the relationship between natural science and mathematics as seen by Riemann [Scholtz 1980, 96 ff.]. In fact, Riemann's interest in Herbart's article [1807] seems to have been the result of his desire to *clarify his own perception of mathematics in the mirror of philosophy*.

This is all the more likely since a very close relation between mathematics and philosophy was suggested by Herbart in the very article summarized by Riemann. According to Herbart, mathematics became a part of philosophy if dealt with philosophically [1807, 275], and at the beginning of his article, he proposed something very much along this line: The mathematician feels the call [*Beruf*] to unveil the spirit of his ingenious [*geistvoll*] formulas [19].

It would be difficult to imagine a better characterization of Riemann's way of doing mathematics [20].

Riemann's views on mathematics seem to have been deepened and clarified by his extensive studies of Herbart's philosophy. Moreover, without this orientation, Riemann might never have formulated his profound and innovative concept of manifold. This represents an indirect but nevertheless effective influence of Herbart on Riemann's mathematical and (in particular) his geometrical thinking.

CONCLUSIONS

The foregoing evaluation makes it possible to characterize more precisely the influence of Herbart's philosophy on Riemann. The assumption that Herbart's philosophy of space had an important impact on Riemann's formulation of the manifold concept cannot be confirmed. On the other hand Herbart's epistemology and his ideas on the relationship between philosophy and science do seem to have influenced Riemann and thus Riemann's perception of the task of mathematics.

In the center of Herbart's philosophizing stood the practical problem of the education [*Bildung*] of man. In this respect he was one of the German *"Bildungsphilosophen"* of the early 19th century, most of whom tended toward an idealist and dialectical philosophy. Starting from similar origins Herbart shared a number of beliefs and goals with the philosophers of German idealism (Fichte, Schelling, Humboldt, Schleiermacher, Hegel):

--the prominent role given to education,
--the serious attempt to overcome the division between empiricism and traditional rationalism,
--the central parts contradiction and change played as motivation for progress in philosophy.

Nevertheless, Herbart's philosophy had its own particular features which distinguished it from that of mainstream German idealism in a number of general ways:

--He assigned an auxiliary role to philosophy with respect to the sciences and valued highly the empirical and the mathematical sciences.
--His philosophy was of a moderately dialectical realism and therefore contrasted sharply with the strictly dialectical idealism common in Germany of the time.

The *realism* appeared in Herbart's epistemology insofar as knowledge was to proceed from experience ("the given") via conceptual clarification of the underlying reality to the explanation of the phenomena. Contradictions played an essential role in the progress of thought, but Herbart's goal was a conceptual system without contradiction [21]. In this respect his philosophy was *"moderately dialectical,"* in contrast to stricter dialectics which would admit contradiction, not only as a driving force for progress, but as a structural element in the developed conceptual system as well.

These three distinguishing features, the auxiliary role of philosophy, realism, and moderate dialectics, may have given Herbart's thought a greater affinity to science and mathematics

than was the case for German idealist philosophy proper. His proposal to do mathematics "philosophically" as a science dealing with concepts agreed with tendencies in contemporary mathematics, and could therefore serve to stimulate Riemann in clarifying his ideas about mathematics. Riemann's views about the process of gaining knowledge were elaborated to a high degree as a result of his studies of Herbart, and were influenced by the dialectical elements in Herbart's epistemology. Moreover, they had a specific influence on his mathematics. A perception of mathematics as a science which has to penetrate its object by a clear conceptual approach runs throughout Riemann's works like a *leitmotif*. Herbart's proposal to work out a central concept [*Hauptbegriff*] for any field of studies (see above) was realized by Riemann most clearly in complex function theory and in geometry. He started to reorganize the latter field around the concept of manifold, which he also thought to be appropriate for a new foundation of physical geometry.

If it could be shown that Riemann had adapted intensional (connotational) features of the concept of manifold from Herbart's philosophy of geometry, this would offer direct evidence of an "influx" of concrete conceptual elements into mathematics from outside, and the influence of philosophy on mathematics would have been, in this instance, an "external factor" of concept formation. But the situation is different. Herbart's philosophical conceptions of science and mathematics were assimilated by Riemann on a very general level, influencing his ideas as to how and in which direction mathematical research ought to proceed. In this way he transformed certain features of Herbart's philosophy into guiding principles of mathematics which then operated in his own work. This was exactly what Herbart had expected philosophy could do in relation to science. He rejected philosophy as "light coming from afar" and wanted to develop it not outside but in "a thoroughly immanent relationship to" other knowledge [1807, 230].

It might be worthwhile to investigate this type of influence of philosophy on mathematical and scientific research in a broader scope. In fact, Albert Lewis has already given a detailed study of a similar relation between a philosopher and a mathematician, one very close to Herbart/Riemann in time and space, namely, Schleiermacher's influence on H. Grassmann [Lewis 1977]. Of course, the situations are different. Insofar as Grassmann was an outsider, he was hardly recognized by his contemporaries and remained unappreciated for most of the 19th century, whereas Riemann was very much in the center of mathematical activity. But the similarities are too striking to be overlooked, and it would be a mistake to regard the two cases as separate and exceptional examples of the influence of philosophy on the orientation of mathematical research [22]. Perhaps there is a more general relationship underlying the influence of Herbart on Riemann or of Schleiermacher on Grassmann.

In an investigation of such a relationship we must keep in mind that German philosophy of the time was not only a question of pure thought. In an attempt to give an intellectual answer to questions arising from the confrontation of the results of the French revolution with the social and cultural reality of Germany at the beginning of the 19th century [23], the problem of education [*Bildung*] had been posed by German intellectuals, and it stood at the very center of contemporary philosophy, along with the idealist facing of the development of the self, the idea, or the spirit. And this had an influence on the contents and the structure of their philosophy.

Herbart's philosophy of education, for example, was of crucial importance for the particular relationship philosophy should have to the sciences and mathematics (see above). His developmental dialectics in epistemology was part of the same context. And there seems to have been a common root of Herbart's philosophizing and that of the idealists, underlying all their specific differences, in the commitment to the problem of education [*Bildung*].

As this commitment referred directly to the research and teaching activities of the professors at the reformed German universities, it is better to pose the question of a possible relationship between philosophy and mathematics during the first part of the 19th century (at least in Prussia and the other German states) in an extended and perhaps more fundamental form: Could it be that philosophical conceptions, worked out to some extent as an attempt to master intellectually the social challenges of the early 19th century, have been incorporated into mathematics (and science more broadly) as guiding principles of research, thus constituting a specific link in the transformation of social change into intrinsic developmental factors of the scientific process?

Were this the case, then the relationship between Herbart and Riemann would have to be considered as a primary example of significant social and philosophical influence on the mathematical development of the 19th century.

APPENDIX 1: JOHANN FRIEDRICH HERBART (1776-1841)

(a) Biographical Note

Herbart studied law, philosophy, literature, and mathematics at Jena from 1794 to 1796. At the time, intellectual circles in Germany were heavily influenced by the experience of the French revolution. Attempts were made to draw consequences adapted to the backward economic and social conditions, as well as the philosophical and cultural context in Germany [Lassahn 1978, 31 ff.]. Herbart was deeply impressed by Fichte's phil-

osophy and became for a while one of his main disciples [Asmus 1968, 98-107]. But by the end of his studies he had developed important differences from the position of his teacher.

After his studies Herbart worked as a private tutor for an aristocratic family at Bern (1797-1800). There he came into contact with Pestalozzi whose pedagogical teachings he later helped to propagate [Weiss 1928, 161]. Upon his return to Germany, Herbart spent some time at Bremen, completed his doctorate and his *Habilitation* at Göttingen, and started to teach pedagogy and philosophy. He soon laid down the foundations for his metaphysics and his practical philosophy, and as a result, he was offered the chair of philosophy formerly held by Kant at the University of Königsberg (in 1809).

Despite his new position, Herbart did not confine himself to philosophical teaching. He also founded a pedagogical seminar which he combined with an experimental school. Between 1813 and 1815 Herbart was a member of the scientific deputation [*wissenschaftliche Deputation*] at Königsberg, one of three regional scientific advisory boards for educational reform in Prussia (the other two boards were at Breslau and Berlin; Schleiermacher was a prominent member of the latter). In this way Herbart was active in the theory and practice of educational reform [24].

He was convinced that improvement of scientific, philosophical, and cultural education was to play an important role for broader political and scoial reforms, because he thought theoretical education of the higher state officials to be a precondition for any appropriate reform activity of the state. Moreover, education of the citizens was necessary, because only thus would they be able to understand fully the meaning of state reform and the new opportunities opened by it.

Herbart was a conservative reformer, as were many other German intellectuals of the time, including Goethe. They expected a self-reform of the monarchy's state power, which could only be prompted by cultural and educational activities. For the most part, however, they rejected the idea of social pressure for reform (while Herbart was in Switzerland, e.g., he criticized the democratic radical movement harshly). In Herbart's opinion the role of the citizen was to support the reform measures applied by the state in agreement with their own values prepared by *Bildung*, and thus in a spirit of "inner freedom" [*innere Freiheit*].

In 1834 Herbart took over a chair at Göttingen University where he taught philosophy and pedagogy until his death in 1841. As early as the 1820s, however, the reform in Prussia and the other German states had been confronted with the forces of restoration and had gradually come to an end. In the 1830s and 1840s, after a short interlude of the aborted 1830 demo-

cratic revolutionary movement, the states of the German Confederation adopted a number of laws and measures to repress liberal, constitutional, or democratic aspirations. The last remnants of liberty of the press were abolished, the secret police controlled public life, and even the universities were subjected to rigorous political control. In this situation reform intentions could no longer be articulated in public, but were forced to take on the form of activities in conspiratorial circles or groups which in fact were organized in several universities in Germany. Of course Herbart, following his convictions, did not take part in these activities. The time of his involvement in reform activities had ended with the phase of state reform from above.

The situation was aggravated in 1837 when Ernst August, King of Hannover, to which Göttingen belonged, autocratically suspended constitutional rights which had been promised in a declaration of 1833. Seven professors (including Gauss' coworker Wilhem Weber) criticized the measure in an open letter and were suspended from their positions. This resulted in broadly articulated student protests and lively faculty discussions. Herbart, however, declined to take a public position on the question, and preferred to keep the university out of political struggle. This was characteristic of Herbart's increasing alienation from his students in the last part of his life, resulting from his continuing rejection of reform activities from below, as well as the fact that his hopes for reforms from above had reached a dead end.

After Herbart's death (1841) his pedagogical views were diffused widely in Germany and abroad [Lassahn 1978]. In discussions of pedagogy he is still mentioned, but he is relatively neglected today in philosophy. A historical reason for this might lie in the fact that Herbart's philosophy led him to the fringe of mainstream German idealism which shaped the philosophical thinking of his time. This does not, however, explain his relative neglect in philosophy today.

(b) Herbart's Method of Relations [*Methode der Beziehungen*]

Herbart considered contradictions in experience and in already constituted concepts as the motive force behind their creation and modification [1829, 23, ff.]. Contradictions had to be resolved by intellectual work resulting in the advancement of knowledge. Any progress was a transition from a reason (e.g., the discovery of a contradiction) to a conclusion [*Grund und Folge*]. The relationship between reason and conclusion, however, itself contained a contradiction. The conclusion was to be a consequence of the reason and therefore to this extent was contained in it. On the other hand it was to be something new; otherwise it would only be part of the reason and would fail to give any new insight.

Herbart described his solution to this problem as the "method of relations" [*Methode der Beziehungen*] [Herbart 1829, 24 ff.; Weiss 1928, 37 ff.]. In short, he believed that the reason in the wider sense was no simple idea. The reason in the narrower sense (designated *g*) had to be linked to elements of knowledge not explicitly considered as part of it, thereby giving a system of knowledge (denoted *G*). The conclusion (denoted *f*) then followed from *G*. The conclusion *f* formed a part of *G* from which it had to be separated. This having been done, *f* contained something new with respect to *g*.

This line of thought he held particularly useful for the resolution of contradictions in the conceptual system or between concepts and experience. In this situaiton the reason was the contradiction and the conclusion was to be the result of an appropriate change of the concepts. Now he considered the dual aspect of the old concepts (*g*) and the more involved contextual situation which gave rise to the open contradiction.

According to this general idea Herbart proposed a methodological procedure to resolve the contradiction in which the old conceptual system was enlarged ("blown up"), thus taking into account the more involved contextual situation. From this extended structure (*G*) the essential parts (*f*) had to isolated for a conceptual representation of the distinction which gave rise to the contradiction. Thus the contradiction was resolved and it led to a changed conceptual system (*f*) as conclusion. The contradiction in the relation between reason and conclusion, Herbart argued, was thus a result of the contradictory idea that the reason itself is both *g* and *G*: "Therefore we say: *the reason is a contradiction.* To blunt the sharpness of this proposition means to remove all the force from the reason" [25].

APPENDIX 2: DOCUMENTATION OF FOLIOS 140^r AND 141^r OF FOLDER 18 FROM THE RIEMANN NACHLASS

(a) Folio 140^r

Herbart ist ausgegangen von der Untersuchung über das Ph(ilosophieren). Hieraus ergaben sich ihm folgende psychologische Postulate, wozu die Erklärungen gesucht werden mußten in der nothwendig vorauszusetzenden--und eben dadurch zu erkennenden--Beschaffenheit und Folge der Vorstellungen:

1. Gegensatz und Ausschließungskraft der Vorstellungen untereinander. --Dieser Begriff der Vorstellungen selbst als Kräfte (statt aller vermeintlichen Gemüthskräfte, welche nichts anderes sind als allgemeine Namen für Gruppen ähnlicher Phänomene) muß als die Grundlage der gesammtem Psychologie angesehen werden.-- Es gehört dazu das Auftreten der Zeitfolge der

Vorstellungen [also auch ... (unreadable, E. S.)] als Bedingung der Ichheit; weil sonst nur ein stetiges Gleichgewicht aller unter einander statt haben könnte.

2. Anheftung des Begriffs der Negation an diejenigen Vorstellungen, welche *als* Bilder gesetzt werden sollen. Aber der Begriff der Negation ist, so wenig wie irgend ein anderer Begriff, ursprünglich in Bereitschaft: Er muß erst erzeugt werden. (Das *allgemeine* Negieren muß entstehen aus den mancherlei Aufhebungen der Vorstellungen unter einander.)

3. Anheftung neuer Position, oder des Seins, an die Bilder als Bilder (als des inneren Prinzips ihrer Regsamkeit).

4. Auffindung dieses Seins der Bilder in der Reihe des Übrigen, das da sei, und abgebildet wurde; zum Behuf der Subsumtion.

(b) Folio 141^{r}

1. Bildung von Begriffen aus den Wahrnehmungen. (Abstraction u. Induction. Synthesis a posteriori.)

2. Erzeugung der Wahrnehmungen/des Wahrgenommenen aus den Begriffen. (Synthesis a priori.)

3. Möglichst geringe Veränderung (oder Ergänzung) der Begriffe, wo das Wahrgenommene nach den Begriffen unmöglich oder unwahrscheinlich ist.

In Bezug auf die Synthesis a priori zu scheiden: Woher der Antrieb? Woher die Wahrscheinlichkeit? (Bürgschaft)

Der Antrieb, nach einer Erklärung zu suchen, liegt in den vorgefundenen Begriffen.

Die Wahrscheinlichkeit beruht auf der Bestätigung durch die Erfahrung.

H.W.Bd.4, 594. Die Methode der Beziehungen ist die Methode der kleinsten Veränderungen .... "Wenn die Erfahrung sich selbst Lügen straft, so müssen wir sie ganz verlassen, nicht aber uns rühmen, ihr so nahe als möglich zu bleiben" [26].

Wir brauchen uns gar nicht ernstlich mit der Erfahrung zu entzweien, wenn es sich zeigen lässt, dass diejenige *Veränderung* der Erfahrungsformen, worauf die Methode der Beziehungen uns hinweist, überall nicht über die Grenzen eines solchen *Fehlers der Auffassung* hinwegführt, den wir der Erfahrung nach *gemeiner* psychologischer Beobachtung füglich zutrauen können [27].

In Bezug hierauf Ps. Bd.2 §139-145 "Von unserer Auffassung der Welt und den damit verbundenen Täuschungen" auszuziehen [28] und die kritischen Einflechtungen auszuschildern.

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NOTES

1. Examples of this tendency can be found in the investigations of Wussing [1969] for the concept of group and Mehrtens [1979] for the case of Dedekind.

2. There is other matherial that would fit the classification less well (e.g., notes to number theory on folios 182-186; summation of infinite series, perhaps in preparation for the inaugural thesis, folio 208).

3. "R.n" refers to Riemann Nachlass, *Göttingen Universitäts bibliothek,* folder 18, folio n.

4. The * in references to Herbart indicates that the corresponding section was studied by Riemann (the appropriate folio in question can be found in Table 1).

5. "The real" [*das Reale*] in Herbart's usage is determined by the fundamental categories "being, quality, inherence, and change" [*Sein, Qualität, Inhärenz, Veränderung*], as developed in his ontology. It is not identical with common-sense reality.

6. Auf den gegenseitigen Hemmungen der Vorstellungen unter einander beruhen die Negationen, und die Zweifel, ob auch das Wahrgenommene sey oder nicht sey; endlich die Unterscheidungen der Eigenschaften, denen nur ein inhärentes Seyn, und eben darum *kein wahres* Seyn zugeschrieben wird, von den *Sachen,* in welche die Realität der Eigenschaften (des ersten Positiven) zurück verlegt wird.

Die Wanderung der Realität aus den Eigenschaften in die Sachen ist nur der erste Schritt zu einer weiteren Reise. Auf höheren Bildungsstufen entsteht die Frage nach der Einfachheit der Stoffe. Wie vorhin den Eigenschaften die Sachen, so werden jetzt den Sachen die *Elemente* entgegengesetzt; *diese* sind nun das wahre Reale; von ihnen haben die Sache eine *geliehene* Realität, nicht anders als vorhin die Eigenschaften von den Sachen.

Die Elemente, Feuer, Wasser, Luft, Erde--müssen sich weiterhin die Versuche der Chemiker gefallen lassen. Nun werden Sauerstoff, Wasserstoff, Stickstoff, das Reale; hingegen Wasser

und Luft, vorhin Elemente, haben nur noch eine geliehene, das heißt, keine wahre Realität. Jedoch auch hiebey bleibt es nicht, sondern:

Der Idealist findet, daß, wie die Eigenschaften, so die Sachen, die Elemente, die Grundstoffe des Chemikers, nur Anschauungen und Gedanken sind. Dahinter das Ich, welches dem Nicht-Ich Realität verleiht [Herbart 1825, 199-200; R.59^{V}, italics in the original].

7. This is corroborated by the contents of another passage of folio 66. Here Herbart criticizes Leibniz' explanations of change by movements of monads, giving the counterargument that changes in chemistry go much deeper than would be possible through changes of mere spatial relations (of monads) [Herbart 1828, 130*].

8. Denn die *Mannigfaltigkeit* der Irrthümer über Substanzen und Kräfte beweis't faktisch, *daß die Begriffe hiervon im menschlichen Geiste nicht vest stehn, daß sie keinesweges Kategorien oder angeborene Begriffe sind, sondern wandelbare Erzeugnisse eines durch die Erfahrung aufgeregten, durch allerley Meinungen umhergeworfenen, Nachdenkens, welches nur dann erst in eine sichere und bleibende Überzeugung übergehn wird, wenn die Wissenschaft, Metaphysik genannt, zur Reife gelangt.* Wie die astronomische Betrachtung, die in die Weiten des Weltbaues hinausgeht, so muß auch die metaphysische Forschung, welche in die Tiefen der Natur hineindringt, mancherley Revolutionen durchlaufen, ehe sie so glücklich ist, solche Begriffe zu *erzeugen,* welche der Erscheinung genugthun, und mit sich selbst zusammenstimmen....

Ist meine Theorie [of substance--of ontology] unrichtig: so bestätigt sie meine jetzige Behauptung, daß die Begriffe ein noch unvollendetes Werk sind; an welchem der menschliche Geist fortdauernd arbeitet; sie bestätigt meinen Satz: *daß die menschliche Auffassung der Welt im Werden begriffen ist.*

Daraus folgt dann sogleich, *daß auch die Täuschungen, die in diesem Werden nach einander entstehen, sehr mannigfaltig; daß sie den verschiedenen Bildungsstufen angemessen sind, welche successiv erreicht werden;* daß sie also *in kein Register,* etwa von Antinomien der reinen Vernunft, *sich einschließen lassen* [Herbart 1825, 198-199; R.59^{V}, italics in the original].

9. Die Wahrheit des Bildes ist unabhängig von dem Grade der Feinheit des Bildes; sie hängt nicht davon ab, ob die Elemente des Bildes größere oder kleinere Mengen des Realen repräsentieren. Aber die Verbindungen müssen einander entsprechen; es darf nicht im Bilde eine unmittelbare Wirkung zweier Elemente aufeinander angenommen werden, wo in Wirklichkeit nur eine mittelbare stattfindet. In diesem Fall würde das Bild falsch sein und der Berichtigung bedürfen; wird dagegen ein Element des Bildes durch eine Gruppe von feineren Elementen ersetzt, so dass seine Eigenschaften theils aus einfacheren

Eigenschaften der feineren Elemente, theils aber aus ihrer Verbindung sich ergeben und also zum Theil begreiflich werden, so wächst daraus zwar unsere Einsicht in den Zusammenhang der Dinge, aber ohne dass die frühere Auffassung für falsch erklärt werden müsste [Riemann 1892b, 523].

10. Ein zweyter Umstand, den wir aus unserem Verhältnisse zur Außenwelt hervorheben müssen, ist die *Beweglichkeit des Menschen in seiner Umgebung*. Ohne diese würden die Anschauungen der Dinge stets für die Dinge selbst gehalten werden; dadurch aber, daß der Mensch einen Unterschied des Abwesenden und des Gegenwärtigen faßt, lernt er, daß den Gegenständen ihr Erscheinen oder Nicht-Erscheinen zufällig ist. Die Gegenstände bekommen, so fern sie vest stehen, auch veste Plätze in seinen sich allmählig bildenden, ordnenden, und verknüpfenden Vorstellungsreihen, worin die Reihenfolge der Anschauungen aufbewahrt wird [Herbart 1824, 425; R.70^{v}, italics in the original].

11. Wie der Raum auf abgestuften Verschmelzungen beruht: (...) so erzeugen sich die Vorstellungen von ähnlichen Continuen allemal unter ähnlichen Umständen. Es sey demnach eine gewisse Klasse von Vorstellungen so beschaffen, daß, wenn viele derselben zugleich im Bewußtsein sind, alsdann aus ihrer Qualität bestimmte Abstufungen ihres Verschmelzens erfolgen müssen: so ordnen sich unfehlbar diese Vorstellungen dergestalt *neben* und *zwischen* einander, daß man sie nicht anders als auf räumliche Weise zusammenfassen ... kann [Herbart 1825, 192; R.141^{r}, italics in the original].

12. In [1811, 102] Herbart discussed the one-dimensional character of the line of sound by stressing the fact that in it "only a single transition" between two states [*nur ein einziger Uebergang durch die sämmtlichen dazwischenliegenden*] is possible, an explanation which greatly resembles Riemann's own approach to the idea of a one-dimensional manifold [*Nachlass*, folder 16, folios 13-14] (compare [Scholz 1983]).

13. Russell's statement that Herbart had a preference for the discrete over the continuous, and thereby had an influence on Riemann, is plainly misleading. See [Russell 1956, 62] and Torretti's criticism of this point [Torretti 1978, 107].

14. Diejenige Philosophie, um die es uns zu thun ist, liegt gar nicht *außer* dem übrigen Wissen, sondern sie erzeugt sich *mit* und *in* demselben, als dessen unabtrennlicher Bestandteil; sie had zu demselben ein ganz und gar *immanentes* Verhältnis [Herbart 1807, 230, italics in the original].

15. Ist in der Gesellschaft die vielseitige Kultur *zerstreut*, so kann die richtige Zusammenwirkung der verschiedenen Gebildeten nur dadurch gesichert werden, wenn in der höhern leitenden Klasse viel Einzelne sind, deren jeder diese Mannigfaltigkeit und Bildung in sich selbst besitzt, überschaut, beherrscht und in der Gesellschaft zu beherrschen weiß. Aber das innere Beherrschen der eigenen Vielseitigkeit, die letzte Bestimmung und Temperatur kann nur durch die *Philosophie* bewirkt werden....

Die rechte Wirkung der Philosophie auf die Gesellschaft ist diejenige, welche durch andere Wissenschaften, die den Berufsgeschäften näher stehen, hindurchgeht. Die Philosophie, der im Grunde kein Stoff eigentümlich zugehört, hat eben darum eine Art von wissenschaftlicher Allgegenwart [Herbart 1807, 300-301, italics in the original].

16. Es ist nämlich die Eigenthümlichkeit dieser unserer Wissenschaft: daß sie *Begriffe* zu ihren *Gegenstande* macht. Dagegen sind die übrigen Disciplinen vertieft im Auffassen dessen, was entweder gegeben ist, oder doch gegeben werden könnte. Selbst die Mathematik (denn von dem historischen Wissen ist nicht nöthig zu reden), so wie sie pflegt behandelt zu werden, denkt sich ihre abstracten Formeln immer als Formeln für mögliche Fälle, und symbolisiert sehr gern ihre Functionen durch die Gestalt von Curven, wie sie überhaupt den Raum nur verläßt, um reicher an Mitteln zur Herrschaft in ihn zurückzukehren. Auch kann sie nur in dieser formellen Hinsicht von der Philosophie geschieden werden. Philosophisch behandelt, wird sie selbst ein Theil der Philosophie, die sich für ihr eigenes Bedürfniß eine Größenlehre würde schaffen müssen, wenn noch keine vorhanden wäre [Herbart 1807, 275, italics in the original]

17. The first part of folio 177^{V} (folder 18 of Riemann's *Nachlass*) reads as follows:

Philosophie = Untersuchung der Begriffe
Phil. Studien; Bd.I pag. 556 (...)
(This is the same as [Herbart 1807], E. S.)

I. Phil(osophische) Ansichten
II. Speculation = Streben zur Auflösung der Probleme (...)
Nachweisung eines nothwendigen Zusammenhangs unter Begriffen
Problem weiterer Specul(-ation) als Bemühung zwischen den Begriffen die gehörigen Übergänge zu bahnen.
III. Philosophie als Wissenschaft
Allgemeiner Charakter, daß sie durch Speculation zu Stande kommt; dass sie Begriffe zu ihrem Gegenstande macht.

The following part of the folio contains catchwords from the index of [Herbart 1807].

18. On the same page we find a preliminary draft of a letter to his father, obviously written shortly after his inaugural colloquium. "... Mein Colloquium habe ich am Sonnabend Vormittag glücklich abgemacht...." It is probable that this refers to Riemann's inaugural lecture, since (1) it took place on Saturday morning [Riemann 1892a, 548], and (2) another remark on the same page refers to Leiden bottles, with which Riemann worked a short while before and after that lecture.

19. Der Mathematiker fühlt den Beruf, uns den Geist seiner geistreichen Formeln zu enthüllen [Herbart 1807, 231].

20. Of course Riemann is not the only mathematician for whom this characterization would hold (e.g., for Dedekind [Mehrtens 1979]). Riemann, however, is a particularly outstanding example. His influence, perhaps even to a certain degree on Dedekind, was often substantial.

21. Herbart's ontology postulated a concept of "the real" free from contradictions and change.

22. Thomas Kuhn's discussion of philosophical influences on the formation of the concept of energy [1977] might indicate that in the first half of the 19th century there was possibly an even wider range of influence of philosophical conceptions on science.

23. For the importance of the French revolution for the discussions of the Jena circle to which the young Herbart belonged, see [Asmus 1968, 68 ff.]; also see Appendix 1.

24. As Schubring [1981, 19] explains, Herbart concentrated in pedagogy on individual education (as was important for private tutors) rather then on group education (as was important for the public school system). So his pedagogical ideas contain a tension between a sense of reform and conservatism.

25. Offenbar fordern wir von dem Grunde, daß, *indem er die Folge erzeugt,* er selbst sich *ändert.* Seine Materie soll sich verwandeln in die neue Materie der Folge. Hier kann nicht Wahrheit an Wahrheit geknüpft werden, sondern, damit die Folge Wahrheit enthalte, muß der Grund das Gegentheil davon sein. Seine Verwandlung darf nicht ein Verlust an Wahrheit seyn; nur ein Irrthum, der sich in nothwendiger Besserung befindet, kann hier den Grund abgeben....

Darum sagen wir: *der Grund ist ein Widerspruch.* Die Schärfe dieser Behauptung abstumpfen, heißt, dem Grunde seine Kraft benehmen. Denn die vollkommene Nothwendigkeit, im Denken vorwärts zu gehen, findet sich nur da, wo das, was man schon denkt, sich selbst aufhebt [Herbart 1829, 36, italics in the original].

26. This is part of a fictitious citation which Herbart used to characterize possible criticism of his method of relations that he anticipated from Platonists or Eleatics, etc.

27. This passage is excerpted from [Herbart 1851, 594-596].

28. This refers to [Herbart 1825, 191 ff.], excerpted by Riemann on folio 59.

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