Journal of Philosophical Logic (2006) 35: 621–651
DOI: 10.1007/s10992-006-9028-9
© Springer 2006

GEOFFREY HELLMAN

MATHEMATICAL PLURALISM: THE CASE OF SMOOTH INFINITESIMAL ANALYSIS

Received 1 September 2005

ABSTRACT. A remarkable development in twentieth-century mathematics is smooth infinitesimal analysis ('SIA'), introducing nilsquare and nilpotent infinitesimals, recovering the bulk of scientifically applicable classical analysis ('CA') without resort to the method of limits. Formally, however, unlike Robinsonian 'nonstandard analysis', SIA conflicts with CA, deriving, e.g., 'not every quantity is either = 0 or not = 0.' Internally, consistency is maintained by using intuitionistic logic (without the law of excluded middle). This paper examines problems of interpretation resulting from this 'change of logic', arguing that standard arguments based on 'smoothness' requirements are question-begging. Instead, it is suggested that recent philosophical work on the logic of vagueness is relevant, especially in the context of a Hilbertian structuralist view of mathematical axioms (as implicitly defining structures of interest). The relevance of both topos models for SIA and modal-structuralism as appled to this theory is clarified, sustaining this remarkable instance of mathematical pluralism.

KEY WORDS: infinitesimal analysis, synthetic differential geometry, intuitionistic logic, Kock-Lawrence Axiom, nilsquare infinitesimal, pluralism, structuralism, modal-structuralism, toposes, vagueness

1. INTRODUCTION

A remarkable twentieth century development in mathematics has been the construction of smooth inifintesimal analysis ('SIA') and its extension, synthetic differential geometry ('SDG'), realizing a non-punctiform conception of continua in contrast with the dominant classical (set-theoretic) conception.[1] Here one is concerned with 'smooth worlds' in which all functions (on or between spaces) are continuous and have continuous derivatives of all orders. In this setting, the once discredited notion of 'infinitesimal quantity' is admitted and placed on a rigorous footing, reviving intuitive and effective methods in analysis prior to the 19th century development of the limit method. The infinitesimals introduced, however – unlike the invertible ones of Robinsonian non-standard analysis – are nilsquare and nilpotent: While they themselves are not

provably identical to zero, their squares or higher powers are set = 0. As will be explained, contradictions are avoided by retreating to intuitionistic logic, giving up the law of excluded middle ('LEM') in its full generality (whereas the logic underlying non-standard analysis is fully classical). That this weakening of the logic suffices for consistency (at least relative to classical analysis ('CA')) is demonstrated in the framework of category and topos theory which have been deployed to define models of these theories. Toposes themselves are categories fulfilling generalizations of rich universes of sets, e.g., closure under products and 'exponentiation'; in particular, the machinery for defining 'subobjects' allows for 'multiple truth values' beyond just 'true' and 'false' and gives rise to an internal logic which in general is intuitionistic, just as SIA and SDG require.

Despite the internal, formal coherence of all this, on reflection these developments present an interesting challenge to philosophy of mathematics and logic. What precisely is the relationship between SIA and CA? Are they even mutually consistent? *Prima facie*, one would say not, as SIA derives theorems which formally are negations of classical ones, notoriously:

$$\neg\forall x[x = 0 \vee \neg(x = 0)]. \tag{1}$$

In this respect, the situation is *formally* analogous to that arising from Brouwerian intuitionistic analysis, which, in contrast to the (logically) conservative constructive analysis of Bishop, also derives theorems which look like contradictories of classical laws (indeed, for example, the very one displayed, with the variable ranging over infinitely proceeding sequences of rationals). Now, there is a well-known reconciliation between intuitionistic analysis and classical analysis based on simply keeping track of radically distinct meanings given to the logical and mathematical symbols involved.[2] When this is done, for example, the intuitionistic statement of (propositional) 'LEM' really asserts that a method is available for either proving or refuting any mathematical statement, something no one would want built into one's logic. Similarly, the intuitionistic version of the universal formula negated in the display above asserts that a method is available for proving, of any presented real number generator, either that it converges to 0 or that that leads to absurdity, something that, not surprisingly, is itself refutable in the intuitionistic framework. Again, the classicist cheerfully accepts all of this. The question then naturally arises, whether anything along these lines applies in relation to SIA and CA? If not (as will be argued and as seems to be recognized by those well acquainted with SIA), what

alternative reconciliation might there be? And, more pressing, what reasons *can* be given for renouncing LEM if 'meaning change' isn't an option? As Bell brings out, one can construct arguments for limiting LEM from various smoothness requirements, and these will be examined below (and found wanting). Closely related and more generally, how should we interpret SIA? While topos theory provides models of SIA, these are highly abstract, whereas SIA has its roots in geometry and intuitive ideas about continuity. Can SIA be interpreted on its own, independently of topos models? How is its subject matter related to that of CA, identical or distinct, and if the latter, how? If '=' really is logical identity, how can there be objects, nilsquares, whose identity or distinctness vis-à-vis 0 is metaphysically (as opposed to epistemically) indeterminate? As will be explained, in SIA, it cannot be proved that 0 is the *only* nilsquare, but neither can it be proved that nilsquares *other than* ($\neq$) 0 do exist. One speaks of these things as having 'only a potential existence.' (Cf. Bell, 7.) What sense can be made of this? As will be brought out below, there is an interesting connection with recent philosophical literature on *vagueness*, in particular the question whether the notion of 'vague objects' – whose status as 'vague' is not traceable to use of language – is coherent. Finally, can a structuralist interpretation of SIA be given, and are any of the ideas of modal-structuralism relevant? The 'points' of classical punctiform analysis and geometry are treated as 'merely possible' according to this view; how then would nilsquare or nilpotent infinitesimals differ in their 'merely potential' status?

Let us make clear that these questions are raised in searching for a substantive interpretation of SIA, one which respects its geometric roots and regards it as having a subject matter ('smooth worlds'), even if only in a hypothetical, idealized sense (as we would argue is also the case with respect to classical analysis and set theory). We are explicitly *not* suggesting that, in the absence of such an interpretation, somehow SIA would be illegitimate *as mathematics*. On the contrary, we are in sympathy with Carnap's remark that "in logic [and, we would add, in mathematics] there are no morals" – meaning that these subjects must be free to be developed without *a priori* constraints beyond minimal requirements of clarity and rigor.[3] The results can then be judged according to their fruitfulness along a variety of dimensions, i.e., 'no morals' certainly does not mean 'no standards,' especially *after* new developments! In the case of SIA and SDG, it is already clear that these theories are presented with as much clarity and rigor as one finds in standard mathematics, and at least they present sufficient conditions for a self-consistent use of infinitesimals in place of the standard limit method

in analysis, and that is already historically and philosophically interesting. (Whether these theories will help solve new problems in analysis or differential geometry remains, I believe, to be determined.) Even if it should prove impossible to provide a realist or structuralist interpretation, there remains a fallback 'formalist' or 'instrumentalist' stance: These theories develop various formal methods for solving problems, in many cases much more simply than their classical counterparts (replacing limit computations with straightforward algebra). Of course, one would like more. Our overarching question can be put: Can one obtain it?

In the next section, we present a brief overview of the basics of SIA (following Bell's *Primer*), concentrating on those aspects most relevant to the problems of interpretation highlighted here, in particular the essential role of intuitionistic logic. In subsequent sections we take up in turn the question of justification for this restriction and the problem of interpretation, the relevance of work on vagueness, and the bearing of structuralism.

2. Leading Ideas of Smooth Infinitesimal Analysis

Here we concentrate on just the basic system of analysis on the real line as presented in (Bell, 1998). The basic motivations are geometric and go back to ancient times. Two traditional notions of 'infinitesimal' are brought together in the notion of *nilsquare infinitesimal*: One is the idea of linear infinitesimal 'segments' of a curve, along which the length between two given points is thought to be measured as a sum of 'microsegments' which are part of the curve but actually straight. The second notion relates to the familiar Archimedean method of defining e.g., the area under a curve (bounded say by the x-axis and two vertical lines) as a sum of infinitesimally thin (along x) rectangles. The first notion embodies the (informal) *Principle of Microstraightness*,

> Given any smooth curve C and any point P on it, there is a nondegenerate microsegment about P which is straight.

This leads to the central (formal) axiom of SIA (called the *Principle of Microaffineness*, below), and it turns out to be sufficient for the Archimedean method. Given a point P on a curve C, the Principle of Microstraightness implies that an infinitesimal segment about P along C is straight, and then the 'area defect,' ∇, between that of a true (micro) rectangle, $\square$, under the curve about P and the true area is proportional to ε^2, where ε is the base of the rectangle along x (See Figure 1). For the method to work, this latter length must not be identified with 0

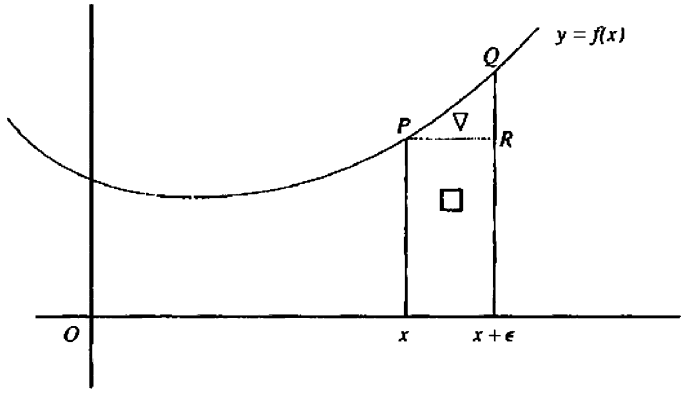


Figure 1. Motivating $\varepsilon^2 = 0$

whereas the area defect should $= 0$, i.e., we should have $\varepsilon^2 = 0$. That is we require that *the nilsquares*, $\triangle =^{df}$ *the set of magnitudes* ε *such that* $\varepsilon^2 = 0$, *not reduce to* $\{0\}$. But this too is guaranteed by the Principle of Microstraightness: Just apply it to the curve $y = x^2$ at the origin (Figure 2).

The formal development of SIA proceeds with algebraic ordered field axioms governing the smooth or affine line R, familiar from standard axioms or theorems for the classical real line, $\mathbb{R}$, except that, along with LEM, the principle of trichotomy, "$x < y \vee x = y \vee x > y$," is avoided, as is 'decidability of identity,' "$x = y \vee x \neq y$." Instead, SIA gets by with

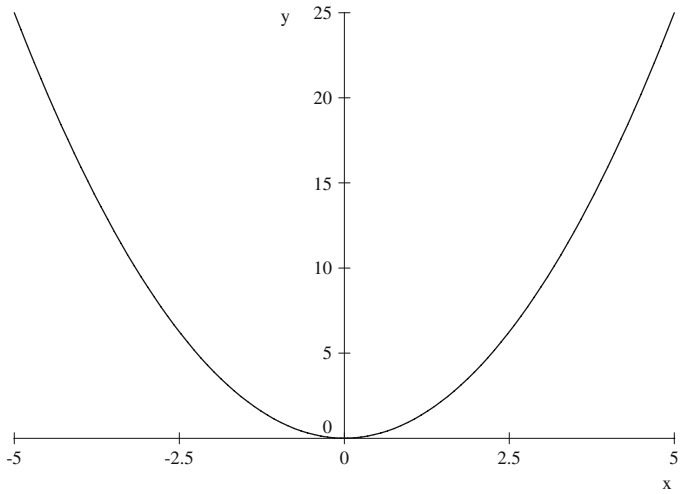


Figure 2. $y = x^2$

the axioms, "$0 < x \vee x < 1$," and "$x \neq y \rightarrow x < y \vee y < x$." Also, the axiom governing multiplicative inverses, "$x \neq 0 \rightarrow x/x = 1$," exempts nilsquares from invertibility since, as will emerge, the antecedent in such a case cannot be demonstrated. Functions from R (or a closed interval) to R are introduced and associated with curves as graphs in $R \times R$ in the usual way. Informally, one now assumes that the Principle of Microstraightness applies to all curves (i.e., in a 'smooth world') determined by functions $f : R \rightarrow R$. This means that all functions behave locally like polynomials $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$, in which case $f(\varepsilon) = a_0 + a_1\varepsilon$, for ε in $\triangle$ (squares and higher powers of which vanish). This means that the graph of f restricted to $\triangle$ is a unique microstraight segment about $(0, f(0))$ on the curve and tangent to it at that point, i.e., that the restriction g of f to $\triangle$ is *affine on* $\triangle$. This is summarized formally in the

PRINCIPLE OF MICROAFFINENESS: *For any map* $g : \triangle \rightarrow R$, *there exists a unique* b *in* R *such that for all* ε *in* $\triangle$,

$$g(\varepsilon) = g(0) + b\varepsilon.$$

That is, the graph of g satisfies the equation of straight line with slope b passing through $(0, g(0))$. This is the central axiom of SIA, also known as the Kock–Lawvere Axiom ('KL'). It says, in effect (as Bell puts it), that the microneighborhood $\triangle$ of 0 can be translated and rotated but it cannot be bent. It behaves as an infinitesimal 'rigid rod' or generic tangent vector to curves in R^2.

The unique existence of the slope b stipulated by the KL Axiom yields three immediate and fundamental consequences:

1. The nilsquares-microneighborhood, $\triangle$, of 0 does not reduce to $\{0\}$. (Apply the axiom to the identity function, $g(\varepsilon) = \varepsilon$. If ε could only take the value 0, then the displayed equation would be satisfied with $b = 0$ or 1, contrary to uniqueness. ($1 \neq 0$ is axiomatic.)) Thus, it is proved that

$$\neg\forall\varepsilon \text{ in } \triangle(\varepsilon = 0).$$

2. *Principle of Microcancellation*: *For any x,y in R, if* $\varepsilon x = \varepsilon y$ *for all* ε *in* $\triangle$, *then* $x = y$. (Simply apply the axiom to the function $g(\varepsilon) = \varepsilon x$.)
3. As expected, the slope b of the axiom allows a simple definition of the derivative of a function. Given $f : R \rightarrow R$, fix x in R and define $g_x : \triangle \rightarrow R$ by $g_x(\varepsilon) = f(x + \varepsilon)$. By KL, $f(x + \varepsilon) = g_x(\varepsilon) = g_x(0) +$

$b_x\varepsilon = f(x) + b_x\varepsilon$, where b_x is unique. This allows introduction of $f'(x)$ defined by

$$f(x+\varepsilon) = f(x) + \varepsilon f'(x).$$

Concerning the first point, the restriction to intuitionistic logic is essential: We *cannot* go on to infer

$$\exists \varepsilon \text{ in } \triangle(\varepsilon \neq 0)$$

for, by the second of the order axioms noted above, this would allow a nilsquare to be ordered with respect to 0, i.e., we would have an ε such that $\varepsilon < 0$ or $\varepsilon > 0$, and either alternative implies $\varepsilon^2 > 0$, contradicting the definition of $\triangle$. With LEM, the KL Axiom immediately leads to inconsistency. Thus, SIA actually *refutes* the universal 'decidability of identity,' i.e., we have

$$\vdash \neg\forall x(x = 0 \vee x \neq 0),$$

as claimed above.

Concerning the second and third points, these are the key tools in developing the differential calculus bypassing limit arguments. The essential idea can be illustrated in about the simplest possible case, the polynomial rule in the case of $f = cx^2$. To find f', substitute in the defining equation:

$c \cdot (x+\varepsilon)^2 = cx^2 + \varepsilon(cx^2)'$; then
$c \cdot (x^2 + 2x\varepsilon + \varepsilon^2) = cx^2 + c2x\varepsilon$; equating the right sides, simplifying, and microcancelling then yield

$(cx^2)' = c2x$, the familiar result. This is typical. In particular, note that we reason with 'ε' standing for an *arbitrary* microquantity; which then drops out on microcancellation. We never single out a particular element of $\triangle$ (with the exception of 0, i.e., '0' still behaves as a name). This will be important below.

The only other axiom needed for carrying out differential and integral calculus is an

INTEGRATION PRINCIPLE: *Given any (smooth)* $f : [0,1] \to R$, *there exists a unique (smooth)* $g : [0,1] \to R$, *such that* $g' = f$ *and* $g(0) = 0$.[4]

This suffices for the introduction of definite integrals, for the first and second fundamental theorems of calculus, and, with standard machinery for dealing with Cartesian products, the development of multivariate

calculus and complex analysis, with standard applications to geometry (areas, volumes, arc lengths, etc.) and physics. (For details, see Bell (1998)).

We close this section by calling attention to one further highly non-classical consequence of SIA, the *indecomposability of the smooth line R*. Call a part J of R 'microstable' if it is closed under addition of nilsquares. From the Integration Principle (and other basic results, e.g. the sum rule for differentiation), one derives the

CONSTANCY PRINCIPLE: *Let* $f : J \to R$, J *microstable, such that* $f' = 0$ *identically. Then* f *is constant.*

Now call a part (subset) U of R '*detachable*' just in case, for all x in R, $x \in U \vee x \notin U$. Then it follows from the combination of the KL Axiom and the Constancy Principle that *the only detachable parts of R are the empty part and R itself*. This is thought to express a fundamental aspect of smoothness. Note that, although the result depends on limiting the LEM, and that a similar sounding result is a feature of intuitionistic analysis, the proof here turns, not on an analysis of 'constructive real number', but on the completely different concept of nilsquare infinitesimal and the notion of derivative it provides, ideas as foreign to intuitionism as they are to classical analysis.

3. The Problem of Interpretation

As we have seen, it is essential that the Law of Excluded Middle be restricted if SIA is to avoid outright self-contradiction. But this in itself can hardly qualify as a sufficient justification for restricting or 'giving up' a logical rule, at least if we are trying to conceive of SIA as more than a calculational device (going beyond the 'fallback formalist' position). We don't normally have the option of saving a theory from inconsistency merely by declaring some logical principle inapplicable. How does one answer the critic who says, "SIA really *is* self-contradictory. Its practitioners delude themselves into thinking otherwise simply by not themselves drawing the conclusions implicit in their assumptions. That, however, doesn't in any way alter the logical facts, i.e., that contradictions *are* implied by the assumptions. Merely declaring that only the rules of intuitionistic logic are to be used in this case is like sticking one's head in the sand!"?

As indicated in the introduction, the defender of intuitionistic mathematics (à la Brouwer, Heyting, Dummett, et al) has a ready answer to such a critic: The logical connectives and quantifiers in that context have

distinctive constructivist meanings, quite different from the classical ones, and these constructivist meanings both support the logical axioms and rules that are accepted and disqualify those that are not accepted, LEM and related principles, such as the law of double negation and proof of existence by *reductio ad absurdum*. For instance, if asserting a 'disjunction' (symbolized with '$\vee$') is taken to imply availability (to an idealized human mathematician) of a method of determining which disjunct holds, and if asserting a 'negation' ('$\neg$') is understood as implying availability of a method of refuting what is negated, then *of course* one should not build LEM ($p \vee \neg p$) into one's logic. (Classicists agree completely with intuitionists about *this*!) And if the formula

$$\forall \sigma(\sigma = 0 \vee \sigma \neq 0) \tag{2}$$

is understood as asserting the availability of a method of proving of any presented choice sequence (of rationals) that it constructively converges to 0 or that this is refutable, then all parties should agree that this ought not to be a theorem and that it might well itself be refutable (as indeed it is in intuitionistic mathematics)![5] Thus, a fully explicit formalism should use different symbols from the classical ones to indicate the different operative meanings; then it would be clear that it is not strictly accurate to speak, e.g., of '*the* law of excluded middle,' but rather that we have to deal with different statements of similar formal shape, a classical one (LEM_{Cl}, say) and an intuitionistic one (LEM_{Int}). The former is guaranteed by classical truth tables, whereas the latter, as just observed, is not generally correct for the intended meanings.[6]

Unfortunately, as we have already suggested, no such neat resolution is available in the case of SIA in comparison with CA. Not only do the inventors and expositors of SIA not offer any such justification for restricting the logic to be intuitionistic;[7] it is reasonably evident that such a justification is not in the cards. For starters, consider the status of the nilsquares, members of $\triangle$. None other than 0 is explicitly recognized let alone 'constructed' in any mathematical sense. Formally, as already noted above, the 'existence of non-zero nilsquares' is actually *refutable*, i.e.,

$$\vdash \neg\exists \varepsilon \in \triangle(\varepsilon \neq 0)$$

but in the background one assumes their 'possibility' in making sense of the whole theory. So clearly this possibility could not rest on any claim of 'constructibility,' which would lead directly to contradiction! Rather this 'possibility' is simply *assumed* in the background as an expression of a geometric intuition (as in the Principle of Microstraightness). One

could stipulate that the 'possibility' in question just means that it is refutable that all nilsquares equal zero, i.e.,

$$\vdash \neg\forall\varepsilon \text{ in } \triangle(\varepsilon = 0),$$

which is true, whereas the direct statement of existence is not.[8] But this is a *consequence* of the combination of the KL Axiom and the restriction to intuitionistic logic, and cannot serve as a justification or grounding for either.

What of the KL Axiom itself? Here the problem is even more stark, for the axiom is an assertion of existence, viz. of the slope b of the 'linelet' determined by the given function $f : \triangle \to R$. A constructive justification for this would consist in some explicit procedure for passing from (a presentation of) an arbitrary (smooth) function f to a b together with a proof that b uniquely satisfies

$$f(\varepsilon) = f(0) + b\varepsilon,$$

for arbitrary ε in $\triangle$. But it is hard to see in what such a procedure could possibly consist that wouldn't already be a method of finding the derivative of f at 0. But we cannot at this stage presuppose such a method, even if we take as given the restriction to smooth functions, for we don't yet know what 'derivative' means. (Remember, we are supposed to be building up an independent concept of derivative based on the KL Axiom, and cannot presuppose standard notions based on the limit concept.) We conclude that, in contrast to principles of intuitionistic mathematics, the KL Axiom is no expression of constructive meanings and must be understood differently.[9]

Although a 'meaning change' argument based on intuitionist/constructivist meanings is not available for SIA, that is not to say that no meaning change argument is possible. We shall return to this idea in connection with 'identity' in the next section, but let us now consider other strategies.

Indeed there is an alternative route to intuitionistic logic peculiar to SIA presented by its expositors. That route appeals to *smoothness requirements* governing 'smooth worlds' that, in some sense, SIA is taken as describing. Bell presents two distinct arguments along these lines. The first seeks to establish the following:

(*) *If LEM holds, then discontinuous functions arise, contrary to the background assumption that (in a smooth world) all functions are continuous*.

The argument is almost trivial: Given LEM, we have for every x in R, either $x = 0$ or $x \neq 0$. Then we can introduce f defined as taking the value 1, say, at $x = 0$, and taking the value 0 everywhere else. (This is called the 'blip' function.). f is defined everywhere but is obviously discontinuous, qed.

But is it just the assumption of LEM and the restriction to smooth functions that leads to contradiction? Surely not, for the classicist certainly can and does reason consistently about the class of C^∞ functions on $\mathbb{R}$. The further assumption, taken for granted in this argument, is that in a 'smooth world', any functions that are *definable* from functions already given in such a world must also be smooth, in brief, such worlds are *closed under definability*. Only when this is combined with (*) does it follow that LEM 'fails in smooth worlds.'

Now of course, one is free to stipulate whatever one likes as to the sort of worlds one wants to describe or investigate mathematically, and the fact that classes of smooth functions classically conceived (as arising in the topos of discrete sets, as category theorists would put it) are not closed under definability does not prevent one from exploring 'alternative worlds'. But the question that then arises, and that a sceptic would pose, is whether there *are*, mathematically speaking, any such alternatives! Normally, if a closure condition leads to a logical contradiction, we conclude that the condition cannot be fulfilled (despite our wishes); we don't normally weaken our logic to accommodate it. (Thus, I cannot insist that the domain of an unbounded linear Hermitian operator with domain included in infinite-dimensional Hilbert space be extended to the whole space by any kind of adjunction of new elements. It is a theorem that any linear Hermitian operator defined on the whole space is bounded.) The assumption that there *are* smooth worlds in the stipulated sense is precisely what the adherent of classical logic will challenge, and that assumption is nowhere discharged in the argument but rather plays an essential role. As it stands, the argument begs the question.

Let us consider the second argument from smoothness. This has to do with 'subobject classification' in smooth worlds, that is, with the machinery for determining, given e.g., a collection of objects in such a world, what subcollections exist. Classical set theory handles this in a familiar way susceptible to generalization in topos theory, viz. by identifying a subcollection with its characteristic function, a mapping from the original collection to a two-element set coding the truth values, 'true' and 'false,' typically taken as the set, $2 =^{df} \{1, 0\}$. The classical power set of a set X is thus identified with the object 2^X, the set of

functions from X into 2. Now above we saw that the smooth line R is indecomposable, the only detachable parts being the empty set and R itself. This illustrates the problem with identifying the 'truth-values object' Ω with the discrete set 2. In a smooth world, the only functions that could represent subobjects of R with this choice of Ω are constant, hence the only subobjects are trivial. To get around this, one must allow Ω to 'expand' (ultimately to a Heyting lattice of open sets of a topological space, when this is formalized in category theory). This forces us to give up bivalence and LEM.

Unlike the first argument, this one does not appeal to definitional closure. It does appear in this case that the mere demand that continua have non-trivial parts together with classical logic and the restriction to smooth functions lead to contradiction. This is true but only when the latter requirement, the 'restriction to smooth functions,' is understood in a particularly strong sense, namely as applying not just to the mathematical subject matter of, say, smooth manifolds, but also to *the internal representation of logic* afforded by suitably rich mathematical machinery, available in set theory and topos theory. That is, *if* it is required that subobject classification be part of a smooth world rather than handled by logical machinery external to such a world, then indeed LEM cannot be correct in the internal logic if there are to be non-trivial subobjects of continua. But, a critic might ask, just because classical set theory is able to represent internally what we take to be genuine logical operations systematized in external systems and governing our ordinary reasoning in presenting SIA, why should we expect internal representability – in putative 'smooth worlds' – of 'genuine logic' in this sense, as applied to the subject of smooth manifolds? Put another way, how do we assure ourselves that there *are* 'smooth worlds' in the sense of meeting the strong requirement that logic applicable to reasoning about its objects, including of course non-trivial subobjects, be internally representable, i.e., by smooth logical valuation functions? Topos models do show that, mathematically, 'smooth worlds' exist but in a weaker sense: An internal 'smooth' logic arises with a rich 'truth-values' object, Ω, obeying the laws of intuitionistic logic, and subobject classification is highly non-trivial. However, no reference to external or 'genuine' logic is involved in this characterization of 'smooth world'; there is no guarantee that the 'truth-values' occurring in Ω are really worthy of the name, as opposed to merely fulfilling a formal role.[10] Nor is there any guarantee that the logical connectives and quantifiers of the internal logic genuinely correspond to, or represent, the connectives and quantifiers with which we actually reason externally with the notions of SIA,

which, as we have said, is thought to stand on its own prior to topos modelling. In particular, topos modelling does not tell us whether the consistency established for SIA is merely a formal artifact of suppressing inferences that, from an external point of view, are still perfectly justified by prior meanings associated with the symbols, e.g., 'or', 'not', '=', etc. Thus, once again, the argument from smoothness is inconclusive regarding the sense in which 'LEM' is 'given up' and regarding how a genuine 'revision of logic' might be justified if that is really what is involved.[11]

An elementary example of toposes with 'extra truth values' leading to intuitionistic logic is instructive. (See Mac Lane, 1986, 402–404.) This involves 'sets varying in time' (functions between pairs of indexed sets, e.g., X_0 and X_1, Y_0 and Y_1, as the objects, with certain pairs of functions running from the X_i and the Y_i, as the morphisms), and the 'internal truth values' just correspond to more possibilities of 'membership' than usually arise simply because a set may 'change from one time to another.' Those internal truth values, just three in this example, are even given temporal names, 'now,' 'eventually,' and 'never,' and it is crystal clear that there is no conflict whatsoever with an external description whose background logic is classical, with the familiar two truth values. Closer to home in the present context of SIA, consider the relation of the internal notion of 'truth' to external of a well-adapted topos model **E** of SIA (described by Bell, 1998, pp. 115–116). As Bell puts it, "If we examine the meaning of any statement of the internal language of **E** containing a variable over R, we find that it is true in **E** if and only if the corresponding statement in **Man** (the category of classical manifolds) is, in addition to being true for all points of $\mathbb{R}$, also *locally true* for all smooth maps to $\mathbb{R}$." For example, a smooth map $f : M \to \mathbb{R}$ (M a classical manifold) may have $f(a) = 0$ but not be constantly 0 on any neighborhood of a, so that neither $f = 0$ nor $f \neq 0$ is locally true, nor therefore their disjunction. Clearly, there is complete compatibility here between the 'non-classical' internal logic's treatment of the formula and the external classical treatment. These examples are thus unproblematic. The problematic case is where the external reasoning involves the highly non-classical nilsquares object $\triangle$, i.e., reasoning of SIA prior to topos modelling. It cannot be classical on pain of contradiction, but topos modelling in itself does not tell us why we should not think that, e.g., what Equation 1 negates is 'really true,' even though formally SIA is cleverly devised to avoid recognizing it. (The cartoon character who walks off the cliff with his eyes staring straight ahead never falls!)

Now the alert reader may well object to this objection, asking "Doesn't all this reference to 'genuine logic' conflict with the Carnapian

pluralist stance espoused above?" Not if it is properly understood, I would say. Logical particles (connectives, quantifiers, etc.) in use have meanings (surely a good, Carnapian assumption, one that even Quine had trouble questioning (cf. Hellman, 1986)), and, to a large extent, these determine which rules are sound. Tolerance allows one to use different particles with different meanings (perhaps determined by stipulated rules which, of course, may differ from the ones we're used to), and tells us then to evaluate the results in light of the relevant goals. But tolerance has its limits. We don't tolerate, e.g., mistakes in subtraction by accountants on tax forms even if defended by appeal to some new rules governing '$-$' (perhaps even when they work out in our favor)! Either we say there is a contradiction with established usage, or we find a new meaning for the new usage which resolves the contradiction under a translation back to the old, in which case we fire the accountant for a confusing and misleading choice of symbols, something even worse than a simple error in subtraction (for which we know how to check)!

In the present case, the problem is to reconcile (1) with what we normally understand by the symbols involved, especially '='. If we are not using the (other) logical symbols constructively, how can what (1) negates fail to hold, whatever the objects in the range of the quantifier might be? Once we are speaking of *objects* x, y, *at all*, are we not committed to (a) its making sense to write '$x = y$' and '$x \neq y$' and to (b) the truth of their disjunction, that one or the other is in fact the case whether or not we have any method of ever determining which it is? An affirmative answer is captured in Quine's famous slogan: "No entity without idendity!" Even if we are only entertaining objects as mere possibilities, we seem committed to (a) and to a modal form of (b), that the disjunction *would hold* in any situation in which the object(s) existed. Let us call the conjunction of (a) and (b) (or its modal version) the 'Determinateness of Identity,' 'DI.' SIA accepts (a) but negates (b), which is (1).[12] A substantive interpretation of SIA (transcending formalism) must make sense of this remarkable exception to DI. In the next section, we take up some strategies for doing this.

4. Reinterpretative Strategies: Does '=' Mean Identity?

Even without constructivist readings of logical vocabulary, there are other routes leading to truth-value gaps thereby limiting bivalence and the LEM. Predicates may, for example, have limited domains over which they are well-defined so that, outside such a domain, atomic sentences and their negations formed with such predicates are taken as

'meaningless' in the sense of simply lacking truth value (ordinary 'true' and 'false'). Ordinary usage is a source of many examples, e.g., "My cat is (or isn't) > 17," "The number 7 is (or isn't) pink," etc. Vagueness offers another common route: "This pebble [near the base of Mt. Kilimanjaro] is (or isn't) part of the mountain" is naturally taken as indeterminate, simply because prior usage has never reflected the conventions that would be required for such refinements. In the first sort of cases – limited domains of definition – while there is indeed a sense of (total) exclusion negation which renders the negative cases of such examples true, we also make room for 'choice negation' leading to gaps, especially in the colloquial formulations given. (To call attention to exclusion negation, we would naturally appeal to the formulation, "It is *not* the case that 7 is pink.") In the second sort of cases – vagueness – even appeal to exclusion negation is insufficient (for we simply don't think it is determinate e.g., whether or *not* the pebble is part of the mountain), and to restore bivalence we have recourse to 'definitely' as in "That pebble is not definitely part of (or not part of) the mountain," but of course now we're dealing with a different predication. In any case, these familiar sources leading to relaxation of bivalence and LEM remind us that classical logic is an idealization: Bivalence is best viewed as a *presupposition* of applicability of classical systems, not as an assertion about any particular applications (or even an assertion that such applications exist!); and when the presupposition is met, LEM_{Cl} of course is satisfied. When it is not met, there is then the challenge to come up with a satisfactory logical system that works. In the case of vagueness, especially, this is a daunting one, full of complexity and plagued by paradox (the *sorites*), and many proposals and theories have been put forward. In the present context, however, we have a system, SIA, which is precisely formulated and which works beautifully (on its own terms). Our question here and in the next section is whether its logic – obtained from classical by simply omitting LEM and suitably weakening the order axioms – can be better understood in light of these familiar phenomena leading to truth value gaps.

The key relations affected by the weakening of the logic are the ordering, <, and, of course, =. Concentrating first on <, note that nilsquares are not ordered with respect to 0 or with respect to one another. We cannot prove of any hypothetical nilsquare ε that $\varepsilon < 0$ or that $\varepsilon > 0$ (written as a notational variant of $0 < \varepsilon$), for in either case we would have $\varepsilon^2 > 0$, a contradiction. (An axiom insures that < is irreflexive, so < implies $\neq$.) Thus we *do* prove $\neg(\varepsilon < 0)$ and $\neg(\varepsilon > 0)$. Thus, these formulas do not present examples of truth-value gaps. Likewise with

formulas such as $\varepsilon < \delta$, for arbitrary elements of $\triangle$. (If $\varepsilon < \delta$, then it would be provable that $\varepsilon < 0 \vee 0 < \delta$, whence $0 < \varepsilon^2 \vee 0 < \delta^2$, contradicting the definition of $\triangle$.Thus we actually prove $\neg(\varepsilon < \delta)$ for arbitrary ε, δ in $\triangle$.) It follows that non-identity inside $\triangle$ is always *refutable*, i.e., SIA proves

$$\forall \varepsilon, \delta \in \triangle \neg(\varepsilon \neq \delta),$$

(where $\neq$ as usual abbreviates $\neg(_ = _)$) but, lacking the law of double negation, this does not collapse to $\varepsilon = \delta$. Thus, we have the curious situation that non-identity within $\triangle$ is determinately false, whereas identity remains indeterminate, as reflected in (1). Still, the notion that predicates have limited domains of application may help answer the question *how it can be that parts of a linear continuum are not ordered.* The notion of linear ordering applies to points any two of which mark a *finite* separation from one another. It is no violation of entrenched *a priori* principle to withhold this ordering from infinitesimals which are conceived as *not* separated by any finite distance, no matter how small (a fact that can be expressed and derived in SIA with a natural numbers object, cf. Bell, 1998, Exercise 8.1, p. 108).

This brings us back to the real culprit, =: How can it be that determinateness, DI, fails if we are speaking of objects at all and if '=' really means 'is identical with'? We cannot simply restrict the range of application of =: Even SIA proves '$\forall x(x = x)$,' for example, including '$\forall \varepsilon \in \triangle(\varepsilon = \varepsilon)$.' What we do do, however, is restrict the range of *true* application of $\neq$: In SIA, this can only obtain for objects which are finitely separated. This includes elements of $\triangle$ in relation to other elements of R, e.g., we do have $\varepsilon \neq 10^{-40}$, $\varepsilon \neq 10^{-40} - \delta$, etc. (for arbitrary ε, δ in $\triangle$). But as already noted, we never have $\varepsilon \neq \delta$ or $\varepsilon \neq 0$, and indeed have the negations of these, but without that implying that they are 'one and the same object'! How can this be? Perhaps, although '=' really means 'is identical with,' '$\neq$' really means (or should be interpreted as) something other than 'is not identical with.' Bell in his presentation seems to suggest this, calling two 'points,' a, b (in an extended sense of 'point' to allow for infinitesimals) 'distinguishable' when they are 'not identical' ($\neq$), and calling them 'indistinguishable' in case $\neg(a \neq b)$, so that the latter does not imply that they are identical, $a = b$, in conformity with giving up the law of double negation. We do prove, for arbitrary ε in $\triangle$, that $\neg\neg(\varepsilon = 0)$, so that nilsquares are, according to this usage, 'indistinguishable from 0.' This is an attractive reading, but, unfortunately, it is problematic. For, since the combination '$\neq$' is not a new primitive but just abbreviates '$\neg(_ = _)$', it suggests a

special, non-classical meaning of '¬', along constructivist lines. But we have already seen the problems with attempting to assign constructivist meanings systematically to the logical symbols of SIA, and it is unclear what anything short of a systematic constructivist interpretation would look like and whether it could be made coherent.

It is worth mentioning that, in intuitionist analysis, a new 'distinguishability' relation, $\sharp$, is introduced (defined), read 'apartness', applied to choice sequences eventually bounded away from each other (by a finite amount, of course). But this has no bearing on the present case of infinitesimals in SIA, which of course have nothing to do with sequences of rationals. Moreover, it turns out that, in intuitionist analysis, one proves that $\neg(x \sharp y) \rightarrow x = y$. (This follows since reals x, y are taken as (equivalence classes of) Cauchy sequences of rationals and the ordering of rationals is decidable. Cf. Dummett, 1977, pp. 40–41, Theorem 3.) Again, the ideas of intuitionist analysis are not helpful in interpreting SIA.

Perhaps '=' could be interpreted as some equivalence relation broader than strict identity. That would still leave the problem of making sense of the theorem (1), but perhaps that would be possible along the lines of 'truth-value gaps,' more so than in the case of strict identity in that there is nothing in the mere supposition that we are speaking of objects (in the plural) that requires them to fall within the range of application of 'not $\approx$,' i.e., the idea of choice negation might hold up more intuitively than in the case of strict identity. Or perhaps theorem (1) and non-classical theorems like it need not be upheld at all. Carrying out such an idea, however, can be expected to be a complicated affair, since elsewhere in SIA, especially in recovering classical theorems, presumably '=' should retain its role as true identity.[13] It is interesting therefore to note that a systematic treatment of SIA along these lines has actually been worked out in detail (Giordano, 2001). This is, in fact, a *classical approximation* to SIA in which the nilsquares D (the surrogate for $\triangle$) is introduced via the definition:

$$D = \{h \in \mathbb{R}^{\bullet} | h \approx 0\},$$

where $\mathbb{R}^{\bullet}$, the extended (classical) reals, is introduced as a quotient of sequences of continuous functions from $\mathbb{R} \rightarrow \mathbb{R}$ under an equivalence relation $\sim$ (viz. that for some m and all $n \geq m$, $(x_n(t) - y_n(t))/t \rightarrow 0$ as $t \rightarrow 0$)) and $\approx$ is a further equivalence relation on ($\sim -$ equivalence classes of) such sequences, viz.

$$\exists m \forall n \geq m : \limsup_{t \rightarrow 0} \left| \frac{x_n(t) - y_n(t)}{t} \right| < +\infty$$

read 'x is close to y.' $\mathbb{R}^\bullet$ is a ring and D is an ideal in it satisfying $h^2 = 0$. (Remember here $=$ is in the quotient space.) Based on these constructions, a restricted version of the KL Axiom is derived as a theorem. There are, however, a number of differences from SIA:

1. The principal restriction in deriving the analogue of the KL Axiom is to a class of functions satisfying a Lipschitz condition (not a very severe restriction). In this case, the unique 'slope' b of the microtangent to f at 0 given by the KL Axiom is proved to coincide with the derivative of f at 0 in the usual sense, and this generalizes to arbitrary arguments x of f in $\mathbb{R}$. (Cf. Giordano, Theorem 1.4, pp. 81–82.)
2. Instead of the Microcancellation principle there is a stronger cancellation principle: For x in $\mathbb{R}^\bullet$, $x \neq 0$, and r, s in $\mathbb{R}$: $x \cdot r = x \cdot s \Rightarrow r = s$.
3. Unlike SIA where for ε, δ in $\triangle$ we cannot prove $\varepsilon \cdot \delta = 0$, and indeed can refute the universal generalization, in Giordano's reconstruction, for h, k in D, we do have $h \cdot k = 0$.
4. An ordering $\leq$ on the extended reals is introduced which is antisymmetric (unlike SIA, where $a \leq b$ means '$\neg\ b < a$', and where for ε in $\triangle$, we have both $\varepsilon \leq 0$ and $0 \leq \varepsilon$ but of course cannot prove $\varepsilon = 0$).
5. Most fundamentally, since the logic is classical, no theorems formally inconsistent with classical mathematics, such as (1), arise.

Note that this latter point holds regardless of what '=' means in (1). LEM is still available even where an equivalence relation is what matters in relating nilsquares to 0. That is just what one would expect from a classical reconstruction.

Because of these and related differences from SIA, we should not regard this as an 'interpretation' of SIA but rather as a classical reconstruction or approximation thereof. Certainly it is, in any case, interesting in its own right, for it shows that the key idea of SIA of doing analysis and differential geometry with nilpotent infinitesimals is indeed coherent even from a wholly classical perspective. Although Giordano's reconstruction is abstract and somewhat elaborate, topos models are also abstract, built on certain function classes obeying ring axioms. Although the basic ideas of SIA are intuitive and straightforwardly presentable without such models (as Bell's *Primer* demonstrates), they are invoked to allay doubts about formal consistency that naturally arise due to the non-standard logic and axioms. Still, it seems clear that no classical reconstruction or approximation can itself answer our question of interpretation of the original, genuine article.

5. Vague Objects?

This brings us to consider some recent philosophical literature on vagueness, specifically the debate over whether it even makes sense to speak of inherently 'vague objects,' objects whose identity conditions are 'objectively indeterminate,' that is, indeterminate but in such a way that the source of indeterminateness cannot be located in our linguistic descriptions or other symbolic modes of representation.

A strong tradition going back at least to Russell (1923) has held that objective indeterminateness of identity really makes no sense, that all vagueness of identity statements must be traceable to vagueness in language. If there is indeterminateness of "Kilimanjaro = Kilimanjaro + that small pebble at the base of the mountain," it is because the precise boundaries of the designator 'Kilimanjaro' are not fixed by anything established in usage, a common form of linguistic vagueness.[14] We can also say that 'part of a mountain,' as well as 'mountain' itself, are vague terms. Indeed, most would agree that 'small,' 'pebble,' 'base,' i.e., all the terms in that sentence – with the exception of '=' itself – are vague, as are most terms in most ordinary sentences, although of course the indeterminateness of any particular sentence may be due to vagueness of some terms occurring in it rather than others. (In this case, even if we pretend that 'small pebble' is absolutely precise, the indeterminateness is unaffected.) But identity itself is thought to be different. What counts as an object may well be indeterminate in certain ways, but once we are speaking of objects at all, according to this 'standard view,' we are *ipso facto* committed to recognizing objective facts as to non-identity of one from another. Ordinary use of the plural commits us to 'more than just one,' and, although we may have no method whatsoever of saying '*how* many' and indeed may not even make sense of that question, still the idea of 'many' carries with it the idea of cases of $\neq$. Thus, we may not make sense of "How many mountains are there in the Himalayan range," but we insist that the problem of counting is due to the vagueness of the terms 'mountain,' and 'Himalayan range'; once that is (sufficiently) resolved, however arbitrarily, cases of $\neq$ will be determinately true and enable us to answer the question (perhaps better, 'a related question'). And even if, as in set theory, we recognize no cardinality at all of a mathematically precise notion, e.g., of 'all von Neumann ordinals,' still we recognize 'many' cases of $\neq$ (indeed, as one says, 'too many'!).

In this tradition, there is even a(n) (in)famous *proof*, due to Gareth Evans, that there cannot be 'vague objects,' i.e., that indeterminateness of =, if treated objectively, independent of linguistic sources, leads to

contradiction. Using ∇ as a sentential operator meaning 'it is indeterminate that' and λ-abstract notation to designate properties expressed by formulas, the argument assumes given designators a, b:

1.	$\nabla(a = b)$.	Hypothesis
2.	$\lambda x[\nabla(x = b)](a)$	1, λ -abstraction
3.	$\neg\nabla(b = b)$	Law of self-identity
4.	$\neg\lambda x[\nabla(x = b)](b)$	3, λ -abstraction
5.	$\exists P[P(a) \wedge \neg P(b)]$	2, 4 Existential Generalization
6.	$\neg(a = b)$	Leibniz principle (non-identity of discernibles)

6. contradicts 1., that '$a = b$' is indeterminate. In words, if the latter, then the property 'indeterminately $= b$' holds of a but not of b, so this property discriminates a from b, so they are determinately non-identical after all.

Now if all vagueness is due to language, to the imprecision with which our symbols attach to parts of reality, then this reasoning can be blocked by pointing to a fallacy in passing from 1. to 2. associated with a kind of referential opacity – failure of substitutivity of equals preserving truth-value – of contexts involving ∇, "it is indeterminate that. . . ." This prevents us from moving from a sentence of the form

$$\nabla(\ldots a \ldots)$$

to one of the form

$$a \text{ is such that } \lambda x[\nabla(\ldots x \ldots) \text{ holds of it,}$$

as symbolized in line 2. Lewis (1988) diagnoses this fallacy pointing to an analogy with "it is contingent that" in modal logic: We cannot infer from, e.g., "It is contingent that the number of planets = 9" that "The number of planets is such that it is contingent whether it (that number) = 9." For another analogy, consider the operator, "It is uncertain whether. . ." ('U'). Under a natural reading of this, we have (adapting a nice example of referential opacity of Quine's),

$$U[\text{Tegucigalpa} = \text{the capital of Nicaragua}]$$

but surely not

$$U[\text{The capital of Honduras} = \text{the capital of Nicaragua}]!$$

On the 'linguistic doctrine' of vagueness, the Evan's argument is blocked by noting the opacity of contexts created by ∇, arising from sensitivity

to the particular information conveyed by the designating terms employed. On the other hand, if we are dealing with purported cases of entirely objective indeterminateness, there would seem to be no fallacy in the argument.

However, in making a persuasive case for the cogency of objective indeterminateness of =, Parsons and Woodruff (1996) find that the principle of $\lambda-$abstraction combined with EG as in step 5 in the Evans argument is essentially question-begging. If, as they argue, objective indeterminateness of identity is coherent, then we simply cannot assume that $\lambda-$abstraction leads to designation of a genuine property open to existential quantification as in step 5.[15] They go on to develop a hypothetical example of how this could fail, and then they show how the linguistic doctrine surprisingly leads to the counterintuitive conclusion that virtually no ordinary designation can be definite. We need not examine their example and arguments here. Rather we want simply to suggest that SIA itself actually provides a striking example of the possibility they promote and nicely illustrates their criticism of the Evans argument.

Consider that argument in the context of SIA. As presented, it cannot even get started for the simple reason that it presupposes individual designators *a*, *b*, whereas in the case of nilsquares, there are no such designators (with the sole exception of terms for 0)! As mentioned in our overview above, reasoning with nilsquares never invokes terms for particular items in $\triangle$ (other than 0) but uses variables ε, δ, etc., for *arbitrary elements of* $\triangle$. Precisely because identity 'inside $\triangle$' is indeterminate, there can be no basis in the SIA framework for introducing individual designators for infinitesimals. That framework is a radical enough departure from ordinary usage so that Evans' starting point simply begs the question.

Now it might be suggested that the Evans argument be reformulated so as to use variables only. Here one must be careful. We cannot start by presupposing an existential claim such as,

$$\exists \varepsilon, \delta \text{ in } \triangle[\Phi(\varepsilon, \delta) \wedge \varepsilon \neq 0],$$

as this is refutable in SIA, as we have seen, as is

$$\exists \varepsilon, \delta \neg[\varepsilon = \delta \vee \varepsilon \neq \delta].$$

The closest we come is the SIA theorem,

$$\neg \forall \varepsilon, \delta[\varepsilon = \delta \vee \varepsilon \neq \delta],$$

but this cannot lie behind step 1 in the Evans argument even if read with variables. We are not here requiring that SIA prove 1, which it cannot for lack of the operator ∇. We are allowing that the argument takes place formally outside SIA. Still its steps have to be grounded in the theory. Suppose then, that we simply allow step 1 as it stands but treat a, b, as variables (writing u, v), so that the whole argument would establish the universal conditional,

$$\forall u, v[\nabla(u = v) \rightarrow \neg(u = v)],$$

whence,

$$\forall u, v[\nabla(u = v) \rightarrow \neg(\nabla(u = v))],$$

i.e.,

$$\forall u, v[\neg(\nabla(u = v))].$$

Even on this formulation, however, we surely distort the theory rather badly at step 2 when we introduce '$\lambda x[\nabla(x = v)]$' and reason as though this picked out a property that can be predicated of a particular nilsquare (existentially generalized in step 5). Indeed, if λ-abstraction were not restricted, we would be committed to properties corresponding to terms such as $\lambda x(x = v)$, individual properties ('haecceities') which are just as problematic in SIA as individual designation (within $\triangle$ apart from '0'). λ-abstraction in this situation is highly suspect, for, as we have seen, in SIA we treat nilsquares entirely generically. Precisely because of indeterminate identity conditions, we refrain from recognizing properties that would imply non-identity and lead to contradiction. Thus, SIA actually provides a fairly clear example from mathematics itself of a logical possibility Parsons and Woodruff consider in the abstract.

To illustrate further SIA's radical departure from classical set-theoretic ideas, especially relevant to talk of 'properties' of its objects, consider the question of *the cardinality* of $\triangle$. Here our intuitive ideas and pictures can be misleading. Depicting or thinking of the micro-neighborhood $\triangle$ as a tiny stretch about 0 leads us to expect it to be uncountable, like an ordinary (finite) stretch. But the concept of 'cardinality,' of 'how many,' is intimately bound up with the concepts of '$=$' and '$\neq$', and as we know these are not treated standardly in SIA. In a topos setting, as in set theory, the cardinality of a collection C corresponds to the totality of maps from a terminal object 1 to C. (By definition, an object 1 is *terminal* in a category just in case, for any object O in the category, there is exactly one map from O to 1. In the

topos **Set** any singleton is terminal.) Applying this to $\triangle$, we have at least one map $f : 1 \to \triangle$ since 0 may be assigned. Are there any others? If $\exists g : 1 \to \triangle$ and $g \neq f$, then we infer $\exists \varepsilon$ in $\triangle$ such that $\varepsilon \neq 0$, which as we know leads to contradiction, i.e., we have proved

$$\neg[\exists g : 1 \to \triangle \wedge g \neq f],$$

whence

$$\forall g : 1 \to \triangle \Rightarrow \neg(g \neq f),$$

which we can take to express 'f is *weakly unique*,' correspondingly that '$\triangle$ is weakly of cardinality 1.' But we cannot go on to infer 'strong uniqueness,' i.e.,

$$\forall g : 1 \to \triangle \Rightarrow g = f,$$

or that '$\triangle$ is strongly of cardinality 1.' Thus, we can say that, in a weak sense, $\triangle$ has cardinality 1, but in a strong sense, it is indeterminate as to cardinality. In the strong ('un-double-negated') sense, the classical dichotomy, "Either a non-empty totality has exactly one member or it has more than one," is not available in SIA.

The situation may be summed up by saying that SIA replaces traditional, vague talk and imprecise treatments of 'infinitesimal quantities' with a precise treatment of vague objects. (In this it is reminiscent of the old caricature of Impressionism as 'a clear picture of a fog.') Limitations on the application of LEM arise from the fact that an entity, $\triangle$, posited in the theory is conceived as consisting of entities indefinite in $=$-conditions, as in principle not susceptible of any particular, *individual* reference or property ascription whatsoever. Ordinary linguistic or other symbolic resources are simply *stipulated* to be inapplicable to 'items in $\triangle$,' including even this oblique use of a plural construction were it to be taken literally in its ordinary sense. Yet, remarkably, SIA manages to permit *quantification* over $\triangle$ without ever implying any disallowed $=$-conditions, and the results are actually interesting and useful! Its surprisingly simple rules manage to allow it to say just enough – but not too much!

6. The Bearing of Structuralism: Peaceful Coexistence

Although we lack a clear resolution of the apparent conflicts between SIA and classical mathematics by appeal to constructivist meanings, our discussion suggests quite a different resolution along structuralist lines: *Despite the large measure of shared terminology, axioms, and appli-*

cations, SIA and CA are really postulating and theorizing about distinct mathematical structures, each having good claims to model aspects of our experience of continuity and continuous phenomena. Structural descriptions of classical continua are very familiar (carried out in set theory, or in a second-order (modal) logic, or in category theory). But structures for SIA (and SDG) can also be described, most readily in the terms of topos theory, and this in turn can be given a modal-structural description. (Hellman, 2003) Non-classical objects arise, viz. micro-neighborhoods, obeying special axioms and restricted logical rules, viz. the KL Axiom and the limitations on LEM reflecting indeterminateness of $=$, intrinsic vagueness of infinitesimals. The key points of this structuralist conception are the following:

1. The axioms of SIA, like those of CA and for other structures in mathematics generally, serve a dual function: They are of course the starting points of proofs, but also they are *defining conditions*, stipulating the sort of mathematical structure or structures the theory purports to investigate. As individual 'statements', the axioms need not be actually true (as Hilbert insisted, against Frege's traditional pre-structuralist 'direct description' view of axioms); it suffices if they are coherent. At most what must be true are claims of coherence, or the possibility of satisfaction, hence consistency, of the system as a whole. Furthermore, the axioms need not be 'analytic' in the sense of 'guaranteed true entirely by meanings of constituent symbols.'
2. The framework of category theory is especially useful in describing 'smooth structures' for SIA, as its 'arrows only' formulations (without set-membership) characterize objects only 'up to isomorphism,' bringing out their functional-structural roles in abstraction from the intrinsic nature of objects. Indeed, questions about 'the nature of nilpotent infinitesimals' can be completely bypassed in favor of an algebraic representation of the maps comprising $R^{\triangle}$. And, as already explained, the appropriate, restricted logic arises naturally through topos-theoretic generalization of classical subobject classification.
3. A modal-structural (MS) presentation of all this would distinguish *internal* satisfaction conditions, based on intuitionistic logic, from *external* logic, which may remain classical. Utilizing the algebraic representation alluded to in 2., one thereby avoids saying things like, "It is possible that there are nilsqures other than 0 itself." This would actually land us in contradiction. Rather one claims that a structure of the appropriate type, $\langle R, \triangle, <, 0, 1\rangle$, obeying the axioms and rules of SIA, is possible. Thus, talk of nilpotent infinitesimals as 'merely

potential entities' is not to be understood in the way in which we understand talk of mathematical structures generally, e.g., the classical continuum, as 'merely possible.'

The first of these points has been set out in detail elsewhere and need not be elaborated upon here.[16] It should be stressed, however, that, on this Hilbertian structuralist approach, *stipulations* which would be quite out of place in a direct description of actual phenomena, are perfectly in order. If, for instance, SIA were taken as a theory of the actual constitution of ordinary space, the burden of justification would be far heavier than on a structuralist approach, for one would confront such questions as, "Why should we believe there actually are such things as nilsquares, not identifiable with 0 but at the same time *not* $\neq 0$?", or "Why should we think space or time or space-time actually contains objects that are in principle indistinguishable from one another, not even not = to one another, but still are somehow 'many'?" If all one need care about in pursuing the mathematics of SIA is the possibility or coherence of entertaining of such things, this being claimed sufficient for the theory's applicability, then such questions can be bypassed.

Concerning point 2: The construction of topos models for SIA begins with the category **Man** of smooth manifolds (whose arrows are smooth maps between smooth manifolds), and adds to this enough structure to incorporate the mathematical role of microobjects such as $\triangle$ (speaking here in the language of SIA prior to modelling). This is accomplished through a representation of the 'coordinate ring' $\mathbb{R}^{\triangle}$ of functions from $\triangle$ to the (classical) real line, $\mathbb{R}$. This arises as follows: The KL Axiom induces a correspondence between pairs of reals (a, b) and maps $\varphi_{a,b} : \triangle \rightarrow \mathbb{R}$ defined by $\varphi_{a,b}(\varepsilon) = a + b\varepsilon$. The Axiom implies that this correspondence, φ, sending (a, b) to $\varphi_{a,b}$ is a bijection between $\mathbb{R} \times \mathbb{R}$ and $\mathbb{R}^{\triangle}$. Now the latter is a ring with the operations $+, \cdot$ defined pointwise in the expected way $((f + g)(\varepsilon) = f(\varepsilon) + g(\varepsilon)$ and $(f \cdot g)(\varepsilon) = f(\varepsilon) \cdot g(\varepsilon))$. If we now introduce operations $\oplus$ and $\odot$ on $\mathbb{R} \times \mathbb{R}$ via $(a, b) \oplus (c, d) = (a + b, c + d)$ and $(a, b) \odot (c, d) = (ac, ad + bc)$, respectively, then we have that the correspondence φ is a ring isomorphism between $\mathbb{R}^{\triangle}$ and the ring $(\mathbb{R} \times \mathbb{R}, \oplus, \odot)$. (Note that in the definition of $\odot$, the expected algebraic 'fourth term' for bd in the expansion of $f(\varepsilon) \cdot g(\varepsilon)$ is 'missing' since this is a term with $\varepsilon^2 = 0$.) Thus, the latter ring provides all the relevant information of $\mathbb{R}^{\triangle}$ and it, along with similar structures for coordinate rings of microobjects for other manifolds, can be added to the coordinate rings of manifolds to obtain an algebraic structure that behaves 'as if' microobjects were present. (This is not yet a topos, but the further constructions for that

need not concern us here. See Bell (1998), *Appendix*, for a sketch, and Moerdijk and Reyes (1991) for full details.) In this way, topos modelling sidesteps metaphysical questions about 'the nature' of nilsquare and nilpotent infinitesimals. All that matters for mathematics, whether it is SIA or CA or whatever, are the 'structural roles' of its objects, and topos modelling realizes this insight.

Concerning point 3: It might then seem that the apparatus of modal-structuralism is not needed in telling this story. However, as we have argued elsewhere (2003), there is no generally agreed upon starting point or background framework – with an explicit logic along with comprehension and other mathematical existence axioms – for category theory and topos theory, conceived as autonomous from set theory. Modal-structuralism provides such a background: It too lacks set-membership as a primitive, although the essential mathematical content of $\in$ can be recovered through a combination of mereology and plural quantification (providing the strength of second-order logic, hence a full theory of relations); as a result, the notions of 'category,' 'morphism,' 'functor,' need not be taken as primitive. Modality enters to avoid commitments to actual infinities, and to avoid commitments to maximal, inextendable totalities (avoiding thereby the embarrasments of proper classes, the 'absolute infinite', etc.).

When it comes to non-standard mathematics such as SIA, however, care must be taken in applying modal-structural ideas. The point is elementary. In normal cases of classical theories, there is no need to distinguish internal from external logic, as the MS framework is fully classical (although certain crucial restrictions on modal-comprehension axioms must be observed). Thus, in asserting, e.g., the possibility of structures for classical number theory or analysis, one can simply write out suitable (second-order) axioms characterizing (standard) structures for those theories, relativized to a hypothetical domain X along with 'functions and relations' over X realizing the conditions expressed in those axioms. (Officially, functions and relations are explained away in the mereological-plurals reduction, but this justifies the use of function and relation *variables*.) That is, one asserts the possibility of the relevant structures in direct, mathematical–logical terms, without the semantic ascent involved in model theory via the use of 'satisfaction'. Now clearly, in the case of SIA, this procedure won't work, due to the restrictions on LEM built into SIA. That is, if we attempt to assert the possibility of an SIA structure in the usual, direct way, we would be saying in effect that it is possible there is a $\langle R, \triangle, <, 0, 1\rangle$ obeying the KL Axiom, Integration Principle, etc., but this implies the possibility that

$\neg\forall x[x = 0 \vee x \neq 0]$ which directly contradicts the (MS-demonstrable necessitation of the) classical logical theorem, $\forall x[x = 0 \vee x \neq 0]$! To avoid this, either we must restrict LEM in the background logic of MS, or we must enforce a distinction between external and internal logics by introducing *satisfaction* as a relation between structures and formulas of SIA, framed so as to respect intuitionistic internal logic. Call this relation $\models_{Int}$. Then, we end up saying (in fewer words): "An SIA-structure bears $\models_{Int}$ to '$\neg\forall x[x = 0 \vee x \neq 0]$' (NB the quotes!), even though in our classical background, everything is either equal to or not equal to 0," and there is no inconsistency in this.[17] This latter route is the natural one to take, since we are attempting to apply a general MS framework to a wide variety of mathematical theories, and we have no reason to impose logical restrictions in this general setting. Rather we are trying to assert the possibility of structures which themselves allow 'indeterminate objects,' reflected in limitations on LEM internal to those structures. But with these adjustments, this can be accomplished as indicated, and the result is only somewhat more complicated than the MS treatment of classical analysis.[18]

This leaves us finally with the question of what sense can be made of the notion that nilsquare and nilpotent infinitesimals 'exist only in a potential sense.' It should be clear from the last paragraph that this cannot be understood simply as "possible existence in a model of SIA": It is manifestly *impossible* that

$$\exists x[x \neq 0 \wedge x \in \triangle],$$

in a model of the SIA axioms! 'Merely potential existence' cannot mean 'possible existence in a model.' What then can it mean? The best I can think of is simply 'not excluded,' where this in turn is spelled out as *the possibility that*

$$\neg\forall x[x \in \triangle \rightarrow x = 0].$$

This as we know is a consequence of the KL Axiom but, without LEM, it doesn't yield the existential formula. Well, that's all consistent, but the phrase 'exists only in a potential sense' now plays no explanatory role at all, but merely refers to this logical situation.

In summation: A structuralist approach to SIA and SDG, through topos theory in a modal-structural setting sustains these theories as consistent alternatives to classical analysis and differential geometry and, moreover, does so while subtly sidestepping metaphysical puzzles about

'the nature of infinitesimals' that threaten to derail these theories. In this arena as in so many others, the utility and flexibility of the Hilbertian conception of mathematical axiomatics proves its worth, even in the absence of combinatorial consistency proofs. There is, however, this compromise with respect to 'geometric intuition': While it can certainly play a motivating and even guiding role in coming up with axioms (e.g., the KL Axiom), when it comes to questions as to the coherence and consistency of the system as a whole, intuitions run out of steam, and we appeal to model-theoretic constructions. At the same time, questions as to 'the nature of the objects' fall by the wayside. As in the case of complex numbers and Hamilton's reduction to $\mathbb{R} \times \mathbb{R}$ with specially defined algebraic operations, we bypass such questions in favor of a structuralist treatment. (At a deeper level, much the same can be said with regard to $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{N}$ themselves, as we have maintained.) In the end, the algebraic stuctures of topos models seem as far removed from geometric intuition of continua as does the set-theoretic continuum at the hands of Dedekind. But precisely because we are dissatisfied with appeals to intuition as a response to questions of consistency, these models assume an unexpected, but it seems, unavoidable importance. Ironically, this lesson of modern logic and mathematics is, if anything, reinforced by 'synthetic geometry.'

ACKNOWLEDGMENTS

Support of this work by the National Science Foudation, Award SES-0349804, is gratefully acknowledged. I am also grateful to Steve Awodey, John Bell, Colin McLarty, Stephen Read, and Dana Scott for helpful discussion and correspondence, and to an anonymous referee for useful comments.

NOTES

[1] These theories originate in the work of Lawvere (1979).

[2] See e.g., Hellman (1989b).

[3] This famous remark of Carnap's occurs in his (1937), Section 17, entitled "The Principle of Tolerance in Syntax," p. 52. Carnap's comment appended to this section makes it clear that he thinks the attitude of tolerance is applicable to mathematics and is "tacitly shared by the majority of mathematicians" (p. 53).

[4] Here there appears to be a trade-off *vis-à-vis* classical analysis, where the integration principle is derived using the method of limits.

[5] Cf. e.g., Dummett (1977), p. 84, where the statement is even more elementary, viz. that σ either consists of all zeroes or not.

[6] A presupposition of particular *applications* of classical logic (not of the logic itself!) is that the reasoning in question can be represented as involving bivalent statements or propositions, i.e., that the classical truth tables are applicable. In the case of pure mathematics, it is this assumption that intuitionism typically rejects (for various reasons that have been offered). *That* is the *locus* of the dispute with classicism (standard practice), not the correctness of a 'law of logic.' Sometimes this is understood but, in the interests of brevity, one tolerates talk of 'giving up LEM,' where this is taken in a purely formal sense, i.e., no axiom or rule of this syntactic shape is adopted in a particular system or context.

[7] See e.g., Kock (2003).

[8] Interestingly, we *cannot* construe the 'possibility of non-zero nilsquares' as the claim that the double negation of existence is provable, i.e., that

$$\vdash \neg\neg\exists\varepsilon \text{ in } \triangle(\varepsilon \neq 0),$$

for then SIA would be inconsistent, as we do have

$$\text{whence} \quad \vdash \forall\varepsilon \text{ in } \triangle\neg\neg(\varepsilon = 0),$$

$$\vdash \neg\exists\varepsilon \text{ in } \triangle(\varepsilon \neq 0)$$

as already indicated.

[9] A further consideration reinforcing our assessment (pointed out by an anonymous referee) is that topos theory, the general background theory in which SIA and SDG and models theoreof are fully developed, is itself not a constructive theory.

[10] An analogy may be helpful here: In presenting semantics for paraconsistent logics, one has recourse to 'truth-value gluts', assignments of 'both T and F' to sentences, as well as gaps. But one can use such a semantics for formal logical purposes and even claim that this is necessary for a good systematization of certain informal reasoning patterns – i.e., one can 'be a paraconsistentist' – without becoming a full-fledged 'dialethist' who believes that, e.g., paradoxical sentences are genuinely both true and false.

[11] It should be noted that there is significant variety among topos models, e.g., Archimedean and non-Archimedean, and, among the latter some admitting invertible infinitesimals over and above nilpotents (cf. Bell, 1998, pp. 106, ff.). Especially relevant to questions of logic is the fact that some topos models actually respect LEM for (closed) sentences, i.e., the propositional law $p \vee \neg p$ is validated, although instances involving free variables are not, e.g.,

$$\forall x\forall y[x = y \vee x \neq y],$$

as they cannot be, the negation being provable. Such models reinforce our points above on the inapplicability of intuitionist meaning analysis to SIA, for intuitionistic meanings lead decisively to rejection of propositional LEM as well as instances with free variables.

[12] In the sketch of SIA above we called what (1) negates the 'decidability of identity,' which is familiar terminology in constructive mathematics based on intuitionistic logic. But at this point, we do not want to identify the logical connectives of SIA with those of intuitionism or constructivism, at least semantically speaking. Thus, 'determinateness of identity' seems a more appropriate term to use.

[13] Indeed, if we naively try reading '=' as '= or only infinitesimally different', in accordance with the idea that '$\neq$' behaves like 'finitely separated,' we run into the problem that now, at least in the presence of the Archimedean axiom (with which SIA is compatible) double negation should hold, i.e., we should have $\neg\neg(\varepsilon = 0) \Rightarrow \varepsilon = 0$, and SIA would become inconsistent.

[14] For an interesting discussion of such examples and their ubiquity, see e.g., McGee and McLaughlin (2000).

[15] They also examine the original presentation by Evans which appears to bypass reference to properties, moving directly from lines 2 and 4 to 6, citing 'Leibniz' Law,' which Parsons and Woodruff take to be the perfectly correct 'indiscernibily of identicals': From '$a = b$' and '$\ldots a \ldots$' to infer '$\ldots b \ldots$'. Here, as they point out, the problem is that it is really the *contrapositive* of this law that is needed, and this is not generally available when there are truth-value gaps, so that, on this presentation as well, the question is begged.

[16] E.g., Hellman (1989a, 1996, 2003), Shapiro (1997), Awodey (1996).

[17] There is no general license for *disquotation*, that is; we cannot in general pass from truth of a sentence in a model to truth *simpliciter* or truth in our background framework. A helpful analogy might be the Skolem 'paradox': In our classical set-theoretic background we recognize both that 'The classical reals are uncountable' holds inside a countable model of the relevant axioms *and* that those reals are really countable as seen in our background framework.

[18] There are some delicate logical questions about just what background strength is required to insure a model of SIA. For a topos model, the 'theory of large domains' sketched in Hellman (2003) would certainly suffice, but much of the topos machinery is 'overkill' for a mere model of the basic system of SIA (analysis on the smooth line R), where a fragment of second-order number theory ought to suffice. Whether the background logic actually needs to be classical, or whether intuitionistic logic would suffice, is a further question, certainly relevant to a better understanding of the relation between SIA and CA, but one that need not be resolved for our purposes in this paper.

REFERENCES

Awodey, S. (1996): Structure in mathematics and logic: A categorical perspective, *Philos. Math.* **4**, 209–237.

Bell, J. L. (1998): *A Primer of Infinitesimal Analysis*, Cambridge University Press, Cambridge.

Carnap, R. (1937): *The Logical Syntax of Language*, Routledge & Kegan Paul, London.

Dummet, M. (1977): *Elements of Intuitionism*, Oxford University Press, Oxford.

Evans, G. (1978): Can there be vague objects? *Analysis* **38**, 208, reprinted in R. Keefe and P. Smith (eds.), *Vagueness: A Reader*, MIT, Cambridge, Massachusetts 1996, p. 317.

Giordano, P. (2001): Nilpotent infiitesimals and synthetic differential geometry in classical logic, in U. Berger, H. Osswald and P. Schuster (eds.), *Reuniting the Antipodes–Constructive and Nonstandard Views of the Continuum*, Kluwer, Dordrecht, pp. 75-92.

Hellman, G. (1986): Logical truth by linguistic convention, in L. E. Hahn and P. A. Schilpp (eds.), *The Philosophy of W. V. Quine*, Open Court, LaSalle, pp. 189–205.

Hellman, G. (1989a): *Mathematics without Numbers: Towards a Modal-Structural Interpretation*, Oxford University Press, Oxford.

Hellman, G. (1989b): Never say 'never'! On the communication problem between intuitionism and classicism. *Philos. Top.* **17**(2), 47–67.
Hellman, G. (1996): Structuralism without structures. *Philos. Math.* **4**, 100–123.
Hellman, G. (2003): Does category theory provide a framework for mathematical structuralism? *Philos. Math.* **11**, 129–157.
Kock, A. (1981): *Synthetic Differential Geometry*, London Mathematical Society Lecture Note Series 51, Cambridge University Press, Cambridge.
Kock, A. (2003): Differential calculus and nilpotent real numbers. *Bull. Symb. Log.* **9**, 225–230.
Lawvere, F. W. (1979): Categorical dynamics, *Topos Theoretic Methods in Geometry* (Aarhus University), pp. 1–28.
Lewis, D. (1988): Vague identity: Evans misunderstood. *Analysis* **48**, 128–130, reprinted in R. Keefe and P. Smith, (eds.), *op. cit.* (Evans), pp. 318-320.
Mac Lane, S. (1986): *Mathematics: Form and Function*, Springer, Berlin Heidelberg New York.
McGee, V. and McLaughlin, B. (2000): The lessons of the many. *Philos. Top.* **28**, 128–151.
Moerdijk, I. and Reyes, G. E. (1991): *Models for Smooth Infinitesimal Analysis*, Springer, Berlin Heidelberg New York.
Parsons, T. and Woodruff, P. (1996): Worldly Indeterminacy of Identity, in R. Keefe and P. Smith (eds.), *op. cit.* (Evans), pp. 321–337.
Russell, B. (1923): Vagueness. *Australasian Journal of Philosophy and Psychology* **1**, 84–92, reprinted in R. Keefe and P. Smith, (eds.), *op. cit.* (Evans), pp. 61–68.
Shapiro, S. (1997): *Philosophy of Mathematics: Structure and Ontology*, Oxford University Press, New York.

University of Minnesota,
Minneapolis, MN, USA
E-mail: hellm001@umn.edu