C.S Peirce's Early Developments in Linear Logic

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Abstract. Early developments of linear logic can be traced back to Barr's *-autonomous categories and Lambek's bilinear logic. We show here that C.S. Peirce's early work on the logic of relations should be placed within this tradition. Peirce understood linear distributivity and the linear negation laws, understood linear implication in the form of residuation, and emphasized the dialogical nature of the linear connectives. Much of this can be found in Peirce's early algebraic work on the study of relations going back as early as the 1880s. Peirce eventually went on to develop a diagrammatic calculus — what he called the Existential Graphs — he thought better suited for the purpose. We go on to show graphs corresponding to these notions and confirm that many of Peirce's later studies in the graphs employ these concepts. The result is a dramatic revision of our understanding of the Existential Graphs, as well as Peirce's place in the logic tradition.

Keywords: Linear Logic · Bilinear Logic · Cyclic Linear Logic · Relation Algebra · Existential Graphs · Charles Peirce · History of Logic

Linear logic is one of the most general logics we have. De Morgan duality, often associated with classical logic, is restored in the linear case, while intuitionistic logic, associated in turn with constructive mathematics and the lambda calculus, can be recovered as an important fragment [51]. Further, the ability of linear logic to keep track of resources makes it an attractive logic for explorations in computation and proof theory.

Relations and relation algebra provide a significant setting for linear logic. Relation algebra has a history going back to Tarski's paper on the subject [75] — a paper that, and we come back to this point below, is really a return to the work of C.S. Peirce — and has been developed in many contexts, including the foundation of mathematics [76] and programming theory [68,18,9], and continues in the present day in relational mathematics [69,67], relational algebraic theories [15,29], and in other (relational) categorical settings like allegories [30].

The explicit connection between linear logic and the relational model goes back at least to Pratt [61,62,63], who noticed — and again, it should be noted, after returning to the work of C.S. Peirce — that the combined complement-converse relation is analogous to linear negation and that residuation is analogous to linear implication. The relational model is also given as a key example in Lambek's presentation of bilinear logic [42] and is the motivating example behind Cockett et al.'s introduction of linear bicategories [24]. In such examples one

needs to express the dual of relational composition, which importantly serves as the par'd ($\parr$) context in the linear case. The connection between relation algebra (RA) and linear logic (LL) is summarized by Desharnais et al. in Figure 1 [27]. Of note, the additives correspond to the Boolean or non-relative terms, while

LL	RA
1	$\mathbb{I}$
$\bot$	$\breve{\mathbb{I}}$
0	$\perp\!\!\!\perp$
$\top$	$\mathbb{T}$

LL	RA
$r^{\bot}$	$\breve{r}$
$!r$	$!r$
$?r$	$?r$

LL	RA
$r \otimes s$	$r ; s$
$r \parr s$	$r \dagger s$
$r \multimap s$	$r \backslash s$

LL	RA
$r \oplus s$	$r \sqcup s$
$r \& s$	$r \sqcap s$
$r \circ\!\!- s$	r / s

Fig. 1: Correspondence between linear connectives and relation algebra.

the multiplicatives correspond to the relative terms.[1] The multiplicative units include the identity relation ($\mathbb{I}$) as the unit of relational composition and its linear dual ($\mathbb{I}^{\bot}$, which is the complement-converse of $\mathbb{I}$). The latter corresponds to the *difference* or *diversity* relation, which we will come back to below. The additive units for union ($\sqcup$) and intersection ($\sqcap$) are $\perp\!\!\!\perp$ and $\mathbb{T}$, respectively.

A further key feature about the relational model is that it provides a natural setting for non-commutative variants of linear logic [27]. Two points are worth highlighting about this setting. The first is that the sequentiality of relational composition is naturally non-commutative. Swapping the order of composed relations in general leads to a different relation, and the resulting left and right residuals — serving as they do as linear implications — have this directionality and non-commutativity built in. Certain other relational operations, like union and intersection, are commutative. The result is that relation algebra offers a natural setting for combining both commutative and non-commutative operations. The second point is that while sequential composition is generally non-commutative, certain cyclic permutations within inclusions are allowed so long as an overall ordering is preserved. This rule is the significant feature behind what is called *cyclic linear logic* and early examples include the presentation by Yetter [78] and Abrusci [2,3]. The relational model also provides a natural setting for studying this cyclicity.

While precursors of these views can be found earlier, we note that these directions only began to see systematic development in the 1990s. The non-commutative setting, often taken to be more involved (Girard even calling it a 'far-west' [31]) is still arguably finding the adherents and developments it deserves.

[1] Unfortunately, almost all of the sources cited here use different notations and we continue to do so. We tend to follow the categorically-minded presentations of Lambek and Cockett et al., but are still left choosing the relational symbols. Where appropriate we follow Peirce, otherwise we follow Schmidt in [67].

It is remarkable then that C.S. Peirce's studies in the logic of relations as far back as the 1880s led him to understand and present the essential rules for (cyclic) bilinear logic. This includes recognizing the importance of the dual to relational composition, presenting linear distributivity and the linear negation laws, and also recognizing the further cyclicity condition. The subject of this paper is to situate Peirce's early developments and studies in cyclic bilinear logic.

We note upfront that the major developments in this direction are already present in Peirce's 'Note B' [57] from 1883. Proper recognition of this fact requires a dramatic revision of Peirce's subsequent understanding of the logic of relations. Peirce later went on to develop a graphical calculus — what he termed the *Existential Graphs* — that generalized the theory of relations given in 'Note B'. We similarly argue, and this paper goes on to demonstrate the point, for a dramatic revision of Peirce's subsequent graphical calculus.

There are numerous further ramifications for Peirce studies and for his place in the logic tradition. We finish this introduction section by listing three broad themes:

- Historically, this work places the developments of bilinear logic — and so also key aspects of linear logic — significantly earlier than recognized precursors to the view,
- Peirce's early insights into bilinear logic occurred prior to the *Existential Graphs* (EGs) and our readings of the graphs must likewise be revised: we cite, as key examples, an awareness in the graphs of the dual of relational composition, of linear distributivity and linear negation, as well as renewed significance in regarding Peirce's sign of the 'double cut' or 'scroll' as an involution,
- Finally, in terms of contemporary graphical developments, we note that the corresponding EGs still compete favourably with modern graphical presentations of these rules, such as Cockett et al.'s circuit diagrams.

Subsequent sections contain lengthier discussions of these points.

While we give the historical and logical significance of Peirce's developments here, we note that a modern algebraic formulation has been given — with a sound and complete axiomization for full first order-logic — and can be found in [14]. The modern account does not discuss the details of Peirce's work and the historical importance, which is our primary aim here. Neither does that account compare these developments to the cyclic bilinear case, a connection we find important to stress as we find it to be perhaps the closest extant theory to Peirce's own. All of this is to say that Peirce's work continues to reveal itself as remarkably ahead of its time.

As an overview, in Section 1 we give a background of bilinear logic and the connection to Peirce's Existential Graphs. Section 2 introduces the graphical syntax and the corresponding linear connectives in the graphs. Section 3 focuses more specifically on linear implication. Discussion of linear implication shows that Peirce's 'scroll' is an efficient presentation of the equivalences that characterize the bilinear operations. In Section 4 we compare the resulting graphical

presentation with Cockett et al.'s circuit diagrams in the literature. The 'scroll' can be seen as providing — as a type of dual presentation — the 'inner workings' of the nodes in a circuit diagram. Some of Peirce's derivations of the linear laws above were motivated by studies of intuitionistic negation, as well as a concern for the refutation clauses of the linear connectives. In Section 5 we give an example of the refutation clauses of the connectives and how some of the linear equivalences are directly captured in the graphs. Finally, in Section 6 we give a few examples of resource sensitivity and the linear modalities.

The work here is just the beginning of this undertaking. Following Tarski and Pratt, we hope this return to Peirce renews interest in this direction of work. The last paragraph of Tarski's seminal paper on relation algebra is still as fitting today as 80 years ago when it was written:

> The aim of this paper has been, not so much to present new results, as to awaken interest in a certain neglected logical theory, and to formulate some new problems concerning this theory. I do believe that the calculus of relations deserves much more attention than it receives. For, aside from the fact that the concepts occurring in this calculus possess an objective importance and are in these times almost indispensable in any scientific discussion, the calculus of relations has an intrinsic charm and beauty which makes it a source of intellectual delight to all who become acquainted with it. [75, p. 89]

1 Bilinear Logic: History, Contemporary Context, and Relation to EGs

While the first significant early presentation of linear logic is attributed to Girard's paper from 1987 [32], other forerunners to linear logic can be found in *-autonomous categories [6,7] and the Lambek Calculus [40,41]. *-autonomous categories arose from a categorical study of an involution that behaves like negation does in the classical case. *-autonomy corresponds to the multiplicative fragment of linear logic, which is generally taken to be the most novel and interesting in the linear case. The Lambek calculus is the (non-commutative) logic of linear implication. It is a substructural logic with only residuation.

The connection between *-autonomy and linear logic was not known immediately, as the connection between linear distributivity and linear logic took time to spell out. Cockett and Seely's work on linearly distributive categories (LDCs) in [25] (and again with the addition of Blute and Trimble in [12]) is perhaps the first systematic treatment. The intuitive idea is that linear distributivity mediates between the two multiplicative connectives by giving a rule of inference that preserve linear sequence and resources, i.e. it does not duplicate a premise.

In addition to linear distributivity, the linear negation laws are perhaps just as essential to the characterization of linear logic. The intuitive idea here is that these rules enforce that every introduction of a logical term contains a positive and negative context that keeps track of how the term can be 'used up' as a resource is used up.

While one could take linear distributivity as the more primitive concept, as in Cockett and Seely's early work cited above, linear negation should arguably be taken to be just as logically essential. One reason for this is that while linearly distributive categories can be discussed without linear negation laws, it is known that the latter can be added as a conservative extension [12]. The intuitive (again) idea here is that categorical composition can be broken down into a two step process of linear distributivity followed by a linear negation, so adding linear negation when composition is already presumed adds no further expressivity. A second reason comes from the move to proof theory, where the linear negation laws find renewed significance. This includes the importance such rules have found within deep inference [28] and the calculus of structures [34]. In short, linear distributivity and the linear negations laws are increasingly seen as key ingredients to linear logic.

Following Lambek in [42], we give a summary of the key morphisms characteristic of bilinear logic. We show a first key set of rules in Figure 2. These correspond to the key morphisms of linearly distributive categories with negation and to *-autonomous categories [12]. The rules include (i) associativity and unit

$$\begin{array}{lll}
\text{(i)} & (R \otimes S) \otimes T \iff R \otimes (S \otimes T) & (R \oplus S) \oplus T \iff R \oplus (S \oplus T) \\
& \mathbb{I} \otimes R \iff R \iff R \otimes \mathbb{I} & \mathbb{d} \oplus R \iff R \iff R \oplus \mathbb{d} \\
\text{(ii)} & R \otimes (S \oplus T) \implies (R \otimes S) \oplus T & (R \oplus S) \otimes T \implies R \oplus (S \otimes T) \\
\text{(iii)} & R^{\perp} \otimes R \implies \mathbb{d} & \mathbb{I} \implies R \oplus R^{\perp} \\
& R \otimes R^{\perp} \implies \mathbb{d} & \mathbb{I} \implies R^{\perp} \oplus R
\end{array}$$

Fig. 2: Key Rules for LDC with Negation

laws for both tensor and co-tensor, (ii) the left and right linear distributivity laws, and (iii) the additional linear negation laws.

We take the additional linear negation laws to be of prime importance here. We add that the aim — or at least our aim in doing so — is not so much to emphasize negation but to emphasize linear implication. The linear negation laws can be thought of as a linear implication introduction rule along with a (linear) contrapositive elimination rule that leads into the dualizing object.

The resulting logical theory, emphasizing as it does both linear implication and its dual, goes by the name *bilinear logic*. The key rules for the two linear implications of bilinear logic are given in Figure 3. Cockett and Seely return to linear negation and these bilinear operations — and now preferably so — in later generalizations of linearly distributive categories into linear bicategories in [24] and for developing the corresponding proof theory in [26]. Lambek similarly comes back to linear negation laws in a later axiomization of the bilinear case in the system BL2 [42].

To these Lambek discusses further rules for cyclic linear logic, which characterizes his system BL3 [42]. The additional rules are given in Figure 4.

$$R \otimes S \implies T \quad \text{iff} \quad R \implies T/S \quad \text{iff} \quad S \implies R\backslash T$$

and

$$T \implies R \oplus S \quad \text{iff} \quad T \oslash S \implies R \quad \text{iff} \quad R \oslash T \implies S$$

Fig. 3: Key Rules for Bilinear Operations

$$A \otimes B \implies \mathbb{d} \quad \text{iff} \quad B \otimes A \implies \mathbb{d}$$

and

$$\mathbb{I} \implies A \oplus B \quad \text{iff} \quad \mathbb{I} \implies B \oplus A$$

Fig. 4: Key Rules for Cyclic Conditions

The rules given in Figures 2-4 will be our main concern here. We refer throughout to this key set of rules as *bilinear logic* and then stress the added *cyclicity* condition when called upon.

We note this system above crops up in close connection to quantales [66], polycategorical composition [74], in characterizing adjunctions [72], in the evolution of quantum systems [13,73], and in linguistics [21]. We believe a strong case can be made that this forms the basis of an extremely important logical system.

The relational setting is often cited as a key example of bilinear logic.[2] In this setting the tensor is relational composition and the dual to relational composition serves as the par or cotensor. Suggestively, we have already written the unit of the tensor above as $\mathbb{I}$ for the identity relation and the unit of the cotensor as $\mathbb{d}$ for the diversity relation. Linear distributivity and the linear negation laws can then be clearly stated via relational inclusion, as can the linear implications in the form of residuation.

Given the emphasis on linear distributivity and linear negation has only seen consistent development since the 1990s, it is (again) remarkable that C.S. Peirce presented all of these rules in his early work on the logic of relations. The key presentation is found in Peirce's 'Note B' from 1883 [57], which also served as the principle inspiration behind Tarski's presentation of relation algebra [75] (for historical context, see [46]). But while Pierce stressed the importance of both linear distributivity and linear negation — going so far to state that they are "highly important" and "so constantly used that hardly anything can be done without them" [57, p. 192 & 190, respectively] — these rules were not emphasized in Tarski's presentation and much of the work that followed. Who knows how the history of logic may have differed without this omission.

Around the same time as the studies around 'Note B,' Peirce expressed linear (and intuitionistic) intuitions about the linear connectives, including game-style semantics of the connectives, and studied linear implication. We show these, almost all for the first time, within the linear context.

[2] See Section 12 in Lambek's [42] and Section 1 in [24].

Peirce continued his work on the logic and algebra of relations for almost thirty years after his original presentation in 'Note B.' Peirce was dissatisfied with his presentation in 'Note B' as it only dealt with binary relations, and went on to develop his *Existential Graphs* (EGs) as an alternative for a calculus of relations. EGs employ a graphical syntax akin to contemporary string diagrams in category theory that emphasize triadic relations over binary relations [36]. EGs employ an additional sign — a circle called the 'cut' — that can surround a graph and that serves as a complement or negation operation. A question is whether Peirce continued to discuss and present these linear rules in his later work on the graphs.

A preliminary response would answer in the negative. Peirce does not mention the linear rules in his most well known descriptions of the inference rules for the Existential Graphs. This is also the predominant story told in the literature. Robert's original presentation on EGs focuses on the classical case of first-order logic with the 'cut' as negation [64], and almost all of the subsequent literature has followed suit (such as in [79,19,70,45] and the overview in [54]). Peirce's presentation of *modus ponens*, which is one of the most common examples Peirce discusses in EGs, also begins by duplicating a premise — and in so doing seemingly goes against the linear rules — and then requires an extra erasure (i.e. projection) to end up with the required result. Finally, many of Peirce's own descriptions take negation or complement as primitive and leave out the more nuanced, linear case.

We show here, however, that this is not the case. The traditional reading of the EGs with 'cut-as-negation' is misguided. Peirce continued to insist on the importance of the linear rules after 'Note B' and continued to develop his understanding of the linear connectives. This becomes clear when we look at Peirce's later EGs, which contain — and even predominately so — linear rules in the form of linear implication and its (linear) dual.

Our judgment of Peirce's later studies in EGs is that, analogous to the importance one might place on the implicational fragment in the classical case, Peirce appears to have made considered effort to emphasize and develop the *linear* implicational fragment. We present for the first time these linear rules in Peirce's later graphs and show corresponding graphs drawn by Peirce. All of this should make us see Peirce's work and his Existential Graphs in a new light.

Several contextual remarks are in order. Much as implication can be seen as the most primitive inferential connective, Peirce insisted on the importance of the 'scroll' — imagine two circles or 'cuts' nested inside the other — as the most primitive connective. We note that this emphasis is often lost in the single 'cut-as-negation' story. In fact, if one only employs the inference rules and graph rewrites as Peirce describes, then one can seemingly never write a single cut! Adding to this perspective here, we return the 'scroll' to its position of prime importance.

Some work has tried to extend Peirce's EGs with 'cut-as-negation' towards the intuitionistic case and towards *-autonomy and LDCs. Extending EGs to the intutionistic case can be found in [48,52]. The work presented here flips this

direction around, as the intuitionistic case can be seen as being recovered from the linear one and not as having to extend the classical one. Residuation, as with Peirce's understanding of the sequentiality of negation, already captures the intuitionistic case. We save this development of the intuitionsitic case for future work, but the bilinear case presented here is the groundwork for this approach.

A second noteworthy approach by Brady and Trimble [16,17], and one motivated by categorical considerations, suggests that Peirce's inference rules can be construed by linear strengths (like LDCs) and by an involution along the lines of *-autonomy. The work here affirms this direction and shows that Pierce himself had a much more extensive knowledge of these directions than previously known. We elaborate Peirce's historical developments and connect these to areas in contemporary logic like LDCs and to circuit diagrams.

In terms of other connections to linear logic, we give an example from Peirce of how the 'scroll' mediates between the proof and refutation cases of the connectives. As another example, we show how the linear equivalences are straightforwardly captured — just like the De Morgan duality in propositional EGs — in the graphs.

Finally, we stress that while Peirce appears to have foreseen much of the structure that is cyclic bilinear logic, there are aspects of full linear logic that fall outside this scope. For example, while Peirce seems to have understood some of the key topological intuitions behind the !-exponential (see Section 6), we as of yet know of no place where Peirce draws attention to what would be the significant developments that follow from the linear modalities and to other intuitionistic developments that have been so significant over the last 100+ years. In the end much further study is needed.

Neither do we want to suggest that Peirce understood further aspects of category theory or that motivate linearly distributive categories or the theory of adjunctions. Peirce was concerned with compositionality and took it to be one of the main puzzles EGs helped resolve.[3] Given that category theory is so concerned with compositionality, and that other approaches to category theory, such as allegories, rely so much on relational operations, we find that Peirce's view is surprisingly modern and prescient. A good example is the symmetric quotient and the straightness condition given in Section 3, which do not appear in modern literature until the early 1990s. Our takeaway of Peirce's studies is that it offers a good reminder of where an interest in relational operations, and a concern for both *better syntax* and a better understanding of *topological features*, can lead.

2 Linear Distributivity and Linear Negation

*-autonomous categories provide the semantics for the multiplicative fragment of linear logic. Cockett and Seely showed that *-autonomy can be characterized

[3] See 'Recent Developments of Existential Graphs and their Consequences for Logic' [56, Selection 45].

by linear distributivity and linear negation laws [25,12]. Peirce clearly stated both in his original presentation of a logic of relations in 'Note B' [57] and goes on to give numerous derivations that we now recognize as corresponding to the Lambek Calculus and to the logic of residuation. We discuss this connection to linear implication in the next section and in this section introduce rules for linear distributivity and the linear negation laws. In this section we follow closely the order of Peirce's presentation in 'Note B.' The appropriate conclusion is that Peirce's studies in the logic of relations led him, as far back as 1883, onto the rules that correspond to the multiplicative fragment of linear logic.

We also use this section to introduce the graphical syntax. The syntax is the same as Peirce's Existential Graphs, but the exposition will emphasize relation algebraic operations and the linear laws. This follows the work found in [36] and [35], where the compositional features of the graphs are emphasized. This is a departure from most introductions to Peirce's EGs in terms of the 'cut' as negation and Peirce's descriptions of the graphs and rules in terms of natural language. It has the advantage of being more algebraic and more compositional, as each graph can be decomposed into smaller relational components, e.g. relational composition, intersection, union, etc.

We assume some basic familiarity with EGs as can be found in [36] and [35] (with further background found in [64]). It should (again) be noted that this syntax and corresponding presentation have recently been formalized as a calculus of generalized relations in [14].[4] Though the formal theory mentioned above in [14] extends to n-ary relations, all of our examples here are, for matter of economy, taken from the setting of binary relations. Readers interested in more algebraic treatments are encouraged to look in the citations above.

To begin, a relation is represented as in Figure 5 with a corresponding ingoing and outgoing wire representing the domain and codomain of the relation. The leftmost expression is a relation, between say x and y, such that xRy. We use

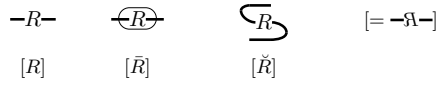


Fig. 5: Primitive (Unary) Relational Operations.

brackets when stating the equivalent relational expression or equivalent graphical expressions. The further relational expressions in Figure 5 show the complement ($^-$) relation, $x\bar{R}y$ that uses the single 'cut' (graphically drawn as ◯ around the relation) and the converse ($^\smile$) relation, $y\breve{R}x$. In regards to the converse, one could wrap the 'wires' around the other way in a 'z'-shape instead of an 's'-shape and

[4] A further difference is worth stating. The general calculus given in [36] uses 'dots' along the wires to allow the treatment of 'free' vs 'bound' variables. The distinction matters little for our purposes here.

achieve the same purpose. We often write a shorthand version of converse that takes the mirror (left-right) image of R as in the rightmost figure. If R is the 'loves' relation then the converse can be read as the 'loved by' relation, reading now the rightmost wire first, which has been wrapped around to the leftmost position, as in 'y is loved by x'. We often omit the typing information (x, y) and just read the relational operations.

There are several graphs we stress that are specific to the linear case. In place of the single 'cut' we emphasize the 'double cut' or 'scroll' in Figure 6. Peirce suggests the 'scroll' as a means of writing continuously and in one motion

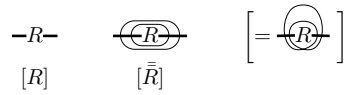


Fig. 6: Involution in EGs and Peirce's scroll.

two nested 'cuts', as shown in the rightmost graph of Figure 6. The 'scroll' corresponds to an involution — the same motivation and the key feature behind the development of *-autonomous categories — and so it should perhaps come as little surprise that these are related. While many introductions to Peirce's EGs introduce the 'cut' as a key element of the graphical syntax, Peirce himself often states that the 'scroll' or 'double-cut' is of more fundamental importance.[5]

A variant of the single 'cut' remains meaningful in the linear case. The combined complement-converse relation ($\breve{\bar{R}}$) is shown in Figure 7. This corresponds to the linear negation of R, or $R^{\perp}$. The primacy of such an operation over

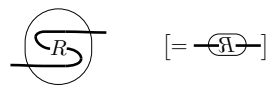


Fig. 7: Linear Negation

compliment or the 'cut' alone may seem surprising. We return to motivate this graphically in Section 3. Related to the comment regarding Figure 5, there is no difference between an 's' or 'z' shaped bending of the wires. We also note that

[5] One such key discussion occurs in 'Selection 8: On Logical Graphs [Euler and EGs] (R 481)' in [54].

whether the converse is outside or inside the cut similarly has not effect, i.e. $(\bar{R})^{\smile}$ is the same as $(\check{R})^{-}$.[6]

Where a relation is represented with corresponding ingoing and outgoing wires, relational composition (;) is represented in the string diagrammatic syntax by connecting the wire with a shared codomain-domain, such as in Figure 8. As

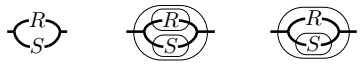


[R;S]

Fig. 8: Relational Composition

is usual with relational composition, there is an implicit existential quantification over the shared domain wire. We point this out as it will be recalled in the dual case.

We find it convenient to represent graphs of the Booleans vertically, in contrast to the horizontal (sequential) composition of relations above. The way to



$[R \sqcap S]$ $[R \sqcup S]$ $[R \sqsubseteq S]$

Fig. 9: Boolean Operations

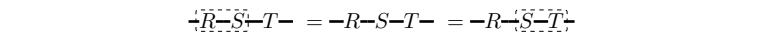
read these parallel terms in Figure 9 is that the domain on the left and codomain on the right are shared, i.e. they are the same type. The nexus of the shared branch is Peirce's *tri*-identity or *teridentity* relation (), which ensures that the shared (co)domain are the same. Sans cut, the way to read the parallel term in $\sqcap$, is that both R and S hold together (i.e. conjunctively), as in $xRy \wedge xSy$. The reading of the dual, given as $\sqcup$, is that both R and S hold disjunctively, as in $xRy \vee xSy$.[7]

[6] This is all that is needed for the relational setting described in this paper. We note that these follow, as shown in [36], from the lines of identity obeying the laws of a special Frobenius algebra. In [36] it is also shown that the bending of the wires can freely pass through a 'cut'. Of further note, these laws also allow for commutativity in the parallel connectives but not necessarily the sequential ones, and allow us, as we will observe in Section 5, to derive the linear equivalences.

[7] We note that Peirce includes the lattice operations in 'Note B' on p. 109 and that these correspond to the additive or non-relative terms in the linear case. Our discussion will mostly focus on the multiplicatives or relative terms, however, as these tend to be considered most interesting for the linear case.

Much like how implication can be seen as the most primitive inferential connective, the 'scroll' serves the same purpose in Peirce's EGs. The rightmost graph of relational inclusion ($\sqsubseteq$) therefore deserves special mention. Note that the shape of inclusion (omitting the R in Figure 9, for example) is the same shape as the 'scroll'. This means that the involution $\textcircled{\textcircled{R}}$ is the same as the blank inclusion $\textcircled{\textcircled{\mathrm{R}}}$.[8] That the 'scroll' manages to be both an involution and an inclusion at one and the same time is perhaps not emphasized enough. We will see that in the linear case this takes on a further meaning, as the 'scroll' mediates between the linear contexts and connectives.

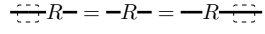
The graphical syntax automatically captures associativity of composition and the unit of composition, which is the identity relation ($\mathbb{I}$) and gives meaning directly to Peirce's expression 'the line of identity'. Notice the meaning of the center graph in Figure 10 is the same irrespective of the order of parenthesis (graphically depicted as dashed boxes) used for ordering the composition. The



$[(R;S);T]$ $[R;S;T]$ $[R;(S;T)]$

Fig. 10: Associativity of Relational Composition

identity relation, i.e. the unit of relational composition, is captured directly by the wire as seen in Figure 11. Again, the center graph is the same irrespective of ordering composition with the unit.



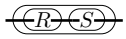
$[\mathbb{I};R]$ $[R]$ $[R;\mathbb{I}]$

Fig. 11: Unit of Relational Composition

The two tensors forming the multiplicatives correspond in our relational setting to relational composition and its De Morgan dual, an operation Peirce called *relative sum*. We sometimes, following the linear logic tradition, also refer to this as the par'd (as in '⅋') context. Relational composition in EGs is well known, though it is not (perhaps) the way Peirce's graphs are typically presented. On the other hand, the dual to relational composition is virtually absent in the literature in Peirce's graphs. This is unfortunate.

[8] One may read the blank in the outer scroll as $\top$, or 'True', where a scroll around a relation means primitively that the relation follows from 'True' or from whatever follows, as Peirce would say, from the blank that is the sheet of assertion.

Relative sum, the De Morgan dual of relational composition, is represented in the graphs by adding a 'cut' around each sub-composed term and a 'cut' around the whole expression as in Figure 12. We prefer to see this as a primitive

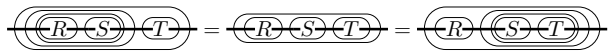


$[R \dagger S]$

Fig. 12: Relative Sum

n-ary scroll (as in [48]), with multiple inner cuts.[9] While relative composition implicitly existentially quantifies over the shared domain-codomain, relative sum implicitly *universally* quantifies over the shared domain-codomain. One familiar with Peirce's EGs will recognize this reading and will also recognize that relative sum shares the shape as disjunction.[10] Rather than disjunction in parallel, relative sum is disjunction in *sequential* or *compositional* order. The meaning of the expression in Figure 12 is thus: a relation from x to y, such that $\forall z(xRz \vee zSy)$. Just as one can choose to read off an implication from the form of a disjunction, if one has negation then one can also read off a (sequential) implication from the relative sum. An alternative reading of relative sum is then the following: a relation from x to y, such that $\forall z(xR\bar{z} \Rightarrow zSy)$ or a relation from x to y, such that $\forall z(xRz \Leftarrow z\bar{S}y)$.

Like relational composition, the dual is both associative and has a unit. Here we see another hint of the good behavior of the 'double cut' or 'scroll', which captures associativity and a nested hierarchy of the par'd context. As the 'double cut' is an involution it (as usual) is similarly extraneous, and we can capture the meaning by the one graph in the center of Figure 13. As in the case of relational composition, associativity of relative sum is similarly absorbed by the graphical syntax.



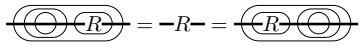
$[(R \dagger S) \dagger T]$ $[R \dagger S \dagger T]$ $[R \dagger (S \dagger T)]$

Fig. 13: Associativity of Relative Sum

[9] The main difference is that we are in the Beta variant of the graph with lines of identity, and we connect n-ary scrolls to the par'd context from multiplicative linear logic. We go further in the next section to draw *generalized entailment relations* consisting in a list of composed terms in the antecedent of the inclusion and a list of dually composed, i.e. par'd, terms in the consequent of the inclusion.

[10] Consider the intersection and join operations, where the latter is De Morgan dual to the former and is read disjunctively as in Figure 9.

Less obvious, but fitting, is that the 'double cut' also makes explicit the unit of relative sum. While the unit of relational composition is simply the wire itself,



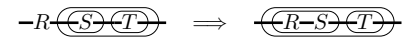
$[\mathbb{d} \dagger R] \qquad [R] \qquad [R \dagger \mathbb{d}]$

Fig. 14: Unit of Relative Sum

the unit of relative sum is what Peirce calls the *diversity* or *difference* relation (⊖) as seen in Figure 14. Following Peirce's naming we use $\mathbb{d}$ for this relation, as it asserts that two things are not the same. We conclude that the 'scroll' as an involution mediates between the twin tensors: it makes explicit a nesting for the associativity of par and makes explicit the par unit. The 'scroll' also re-associates in the implicational context, a point we return to in the next section.

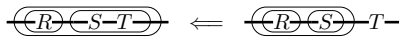
We have now introduced relational composition and its dual. While Peirce stresses the operations in 'Note B' and states their importance, we know of no emphasis given to the par'd context in the literature on EGs. As far as we know, the observation that the 'scroll' serves as a par-unit introduction and to reassociate within the par'd context has also not been made before.

Once we have relational composition and its dual we can express linear distributivity and the linear negation laws. Peirce does so on p. 190 and p. 192 in 'Note B'. We express these rules here for the first time in the syntax of EGs. Given relational composition and its dual, left and right linear distributivity are expressed in Figure 15:



(left linear distributivity)

$[R; (S \dagger T) \sqsubseteq (R; S) \dagger T]$



(right linear distributivity)

$[R \dagger (S; T) \sqsupseteq (R \dagger S); T]$

Fig. 15: Equations for linear distributivity.

Peirce sometimes shaded positive and negative regions of the graphs. We show linear distributivity in this way in Figure 16. It perhaps helps to show that linear distributivity takes the outermost composed relation and pushes it inwards to 'tunnel' into an inner par'd context.

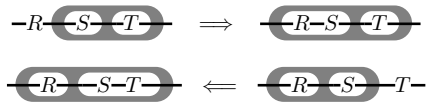


Fig. 16: Equations for linear distributivity with shaded regions.

In addition to the linear distributive laws, Peirce states what we now recognize as the linear negation laws. We first give the introduction rule for left and right linear implication in Figure 17. We call these *introduction* rules because

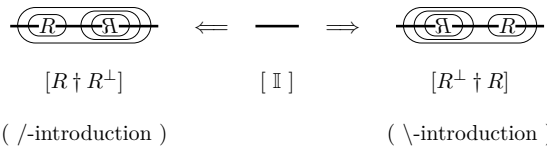


Fig. 17: Equations for linear implication introduction.

taken out of the par'd context they express the residual, which corresponds to linear implication. We show the equivalent residuals in the coloured form in Figure 18. We return to Peirce's study of residuation in the next section. Lastly,

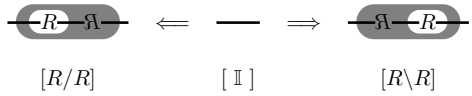
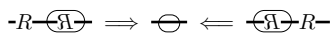


Fig. 18: Equations for linear implication introduction with shaded regions.

we give the corresponding elimination rules that follow directly as the (linear) contrapositive in Figure 19.

Peirce goes on to give a number of examples of the rules (pp. 195-198), all of which we now recognize as examples from categorial grammar and the Lambek calculus. Peirce also demonstrates some of the requisite morphisms in the commuting diagrams for LDCs. Before moving to discuss residuation and linear implication, we give in Figure 20 one paradigmatic example highlighting how linear negation can interact with linear distributivity. This is known as one of the 'zig-zag' equations in categorical logic. Peirce was both aware of this connection, as he employs the steps often in derivations, but also draws explicit mention to it in 'Note B.' Following linear distributivity, he notes that when the



$[R; R^{\perp}]$ $[\mathbb{d}]$ $[R^{\perp}; R]$

($\oslash$-elimination) ($\otimes$-elimination)

Fig. 19: Additional equations for linear negation.

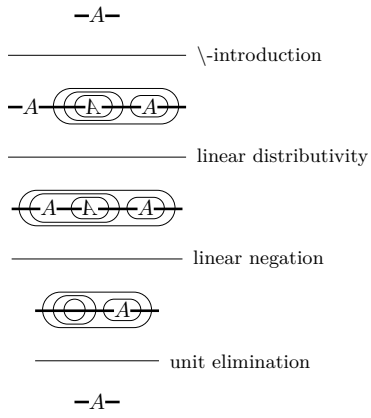


Fig. 20: The 'zig-zag' equation in EGs.

relative to be eliminated has been replaced by a unit then it "can often be got rid of" (pp. 193-194).

Finally, we add that Peirce was also aware of admissible cyclic permutations. He observes on p. 194 that the following permutations of the sequent in Figure 21 are equivalent. This is the key feature of the further rule emphasized in cyclic

$$\mathbb{I} \sqsubseteq R \dagger S \quad \Longleftrightarrow \quad \mathbb{I} \sqsubseteq S \dagger R$$

Fig. 21: Allowed (Cyclic) Permutations in the †-context

linear logic in Figure 4. Immediately proceeding this passage, he points out that such cyclicity is not possible when relational composition is in the conclusion of the sequent, for which case only the regular converse operation applies. We will return to these cyclic permutations in the context of the graphs in the next section.

The rules discussed so far — in particular relational composition and its dual, associativity and corresponding units, and left and right linear distributivity — form the foundation of Cockett and Seely's linear distributive categories (LDCs). Adding the linear negation laws, in turn, form the foundation of *-autonomous categories. As mentioned in Section 1, Peirce's characterization of the logic of relations via these rules in 'Note B' places historically the awareness of these rules much earlier.

We also reiterate that Peirce's previous algebraic work on the logic of relations in 'Note B', and in particular his awareness of relative sum and of linear distributivity and linear negation above, has not be been made or addressed in subsequent studies of the graphs. Peirce himself seems to have made little mention of these rules in his later algebraic work (though exceptions can be found, such as in [54, p. 240 & p. 281], where he refers, as linear distributivity is sometimes referred to today, as reassociation.)[11] In subsequent sections we show, however, that many of the graphs Peirce went on to draw employ these concepts.

3 Residuation and Linear Implication

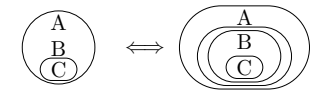
After Peirce introduces linear distributivity and the linear negation laws, he then proceeds in 'Note B' to give a number of examples. Many of these examples show the interplay of linear implication. As has been pointed out in [35], Peirce systematically studied linear implication in the form of residuation and drew corresponding graphs of residuals in EGs. The standard account of the development of residuation traces it back to Dilworth [77] and Birkhoff [10], then

[11] In the latter passage Peirce also presents the rule in EGs and also adds a further operation, graphically motivated, called *fusion*, which adds the 'double-cut' to the outer term first and then 'fuses' the two 'scrolls' together.

on to Lambek [40] and Grishin [33]. The concept of residuation sees substantial development through Galois connections and adjunctions in category theory.

We begin this section with a brief discussion of residuation in the context of conjunction, then move to instances with composition that correspond to the (multiplicative) linear case. The conclusion to draw is that Peirce's developments of residuation and his emphasis on linear implication (along with its dual, both forming the bilinear operations) were far more extensive than previously thought.

As has been pointed out before in [45], Peirce may have been the first to write about residuation in connection to conjunction. In the propositional case, residuation comes from adding a 'double cut' or 'scroll' inside the inclusion as shown in Figure 22. This is perhaps the simplest expression of 'currying' and 'uncurrying'.[12] For future discussion, we also note in Figure 23 the similar



$A \wedge B \to C$ $\qquad$ $A \to (B \Rightarrow C)$

Fig. 22: Intuitionistic Entailment in Alpha

connection between the operation and intuitionistic negation. The move ensures



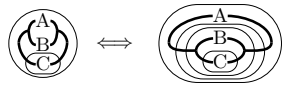
$\neg A$ $\qquad$ $A \Rightarrow \bot$

Fig. 23: Intuitionistic Negation in Alpha

that the intuitionistic implication is analogous to the standard 'cut' as negation.

As we will see, Peirce was explicitly aware of this notion of intuitionistic negation. We add to this story here by noting that the intuitionistic case employing the 'double cut' was actually Peirce's motivation for the single 'cut' as negation and not the other way around. In one of several passages drafted for the purpose, and a passage worth quoting in full, Peirce makes this clear.

[12] A Beta variant is easily given. The difference in the lines of identity is a simple application of associativity.



> [The failing... lies in its] encouraging the idea that negation, or denial, is a relatively simple concept, and that the concept of Consequence is a special composite of two negations. In opposition to that, all my writings upon formal logic have been based on the belief that the idea of sequence in reasoning and in judgment, whether conditional or categorical, could in no wise be replaced by any composition of ideas. Now this view inevitably leads to a negative predication, say "is not P" being regarded as the assertion that upon the supposition of the affirmation "is P" there would be sequent the essence of falsity; and I regard this essence of falsity to consist in permitting the interpreter to opine whatever he may choose. I thus analyze the negation of P into a positing of P as a mere idea together *with the assertion that falsity is sequent upon it.* As a matter of fact, this idea was the starting-point in my mind of the notion of logical graphs;—not merely those of the existential kind, but also of the earlier entitative kind. *The simple Cut is a scroll...* [56, p. 353n10, emphasis added]

The "essence of falsity" Peirce refers to above is the notion of absurdity familiar to us from the intuitionistic case (where anything follows, or, as Peirce says, where the asserter is allowed to "opine whatever").[13] Peirce goes on to suggest two ways this can be interpreted (given the position with respect to the scroll) and to express doubts about how best to iconically capture the process diagrammatical in EGs.

We take the above passage as substantial evidence that the intuitionistic negation is not a further development needing to be added to the theory of EGs, but is rather at the heart of Peirce's understanding of the graphs and has been since the beginning.[14] Further, in the passage Peirce makes clear his belief that negation (at least when taken as a primitive) is not compositional. Regardless, Peirce is emphatic that the single 'cut' really is a 'scroll' and that $\neg P$ is really $P \Rightarrow \bot$.

A natural question is whether and to what extent Peirce understood what we know about intuitionistic negation and its role in constructive mathematics. Further work is needed to answer this question but given that the linear case is more general, and Peirce appears to have had a surprising understanding of the linear case, we suspect the answer may also surprise us.

Peirce also understood residuation with respect to relational composition. A preliminary discussion of residuation in this context, i.e. in the Beta variant of EGs, and an overview of Peirce's historical developments along these lines can be found in [35]. We extend this development in three ways here. First, the motivation for adding converse inside the scroll in [35] was given by topological considerations governing the line of identity: a line of identity can freely

[13] See also Peirce's discussion of the 'the blot' in [59].

[14] See similar passages in [54, p. 169, p. 288, & p. 582] and [56, p.356]. Another noteworthy passage occurs in [CP 4.564] where Peirce describes the single 'cut' as a degenerate implication that lacks a specified consequent. Further connections between Peirce's views and constructivism more generally can be found in [60].

be deformed (into an 's' or 'z') and doing so makes the graphs topologically equivalent. Here we give a derivation that shows how these equivalences can also be motivated directly by linear distributivity and the linear negation laws given above. Second, we show that Peirce was aware of coresiduation and so to the operations that give rise to bilinear logic and the multiplicative fragment of non-commutative linear logic. To this we highlight the ability of the 'scroll' to represent, in an effective way, these bilinear operations. Third, we add further instances of graphs Peirce drew of residuation and identify them with contemporary notions in the literature.

The important change in the move to the Beta graphs is keeping track of lines of identity for relational composition. One can summarize the equivalences for residuation in EGs with the Schröder equivalences shown in Figure 24.



$[S \sqsubseteq R\backslash T]$ $[R; S \sqsubseteq T]$ $[R \sqsubseteq T/S]$

Fig. 24: Schröder Equivalences in EGs

Note that the left and right residuals can be expressed as in Figure 25. These are the expression in the consequent location — i.e. read from the inner most, concluding 'cut' — within the 'scroll' from Figure 24. $R\backslash T$ is a relation from



$[R\backslash T]$ $[T/S]$

Fig. 25: Left and Right Residuals

x to y, such that $\forall z(zRx \Rightarrow zTy)$. Peirce sometimes refers to residuation as "the relation of inclusion of correlates". In his own words (and adapted to our example) "it implies that everything that R stands in any fixed relation to is included among the things to which T stands in that same relation" [55, p. 286].

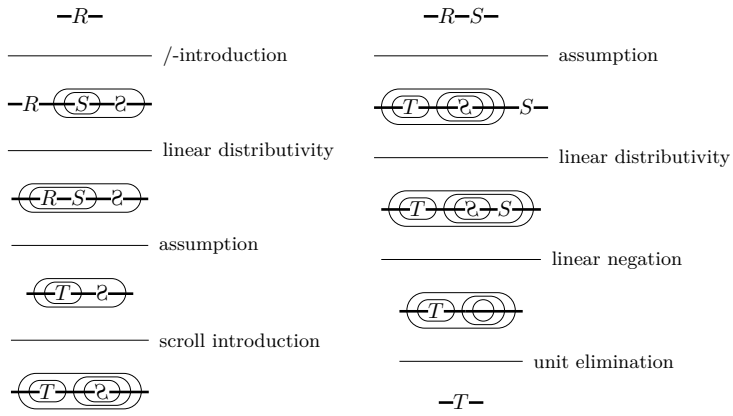
Peirce was not only aware of the above equivalences but also stated a method for arriving at different permutations in 1882:

> Hence the rule is that having a formula of the form $[R; S \sqsubseteq T]$, the three letters may be cyclically advanced one place in the order of writing, those which are carried from one side of the copula to the other being both negatived and converted. [53, p. 341]

The citation is from around the same time as 'Note B' and is before Peirce substantially develops and presents the graphs. Given that linear negation is the

combined complement-converse relation (see Figure 7), Peirce is stating exactly the well known rule from linear logic that one can move a term from one side of the entailment to the other by adding a linear negation.

In Figure 26 we show that the equivalences can be derived from linear distributivity and linear negation. We derive both sides of the right-most equivalence in Figure 24. The other side follows from symmetry.



(a) If $R; S \sqsubseteq T$ then $R \sqsubseteq T \dagger S^{\perp}$

(b) If $R \sqsubseteq T \dagger S^{\perp}$ then $R; S \sqsubseteq T$

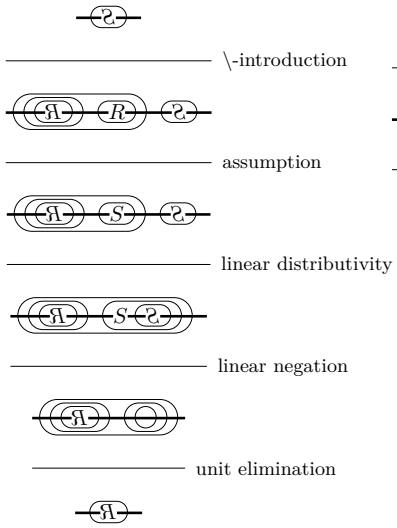
Fig. 26: Derivations of one side of the Schröder Equivalences

We add two additional derivations in Figure 27 to show how these rules work in Peirce's EGs. The first is the linear contrapositive, i.e. If $R \sqsubseteq S$ then $S^{\perp} \sqsubseteq R^{\perp}$. We also derive, as an example of a key inference rule, a modus ponens-like inference rule for the residual, i.e. $R; R\backslash S \sqsubseteq S$.

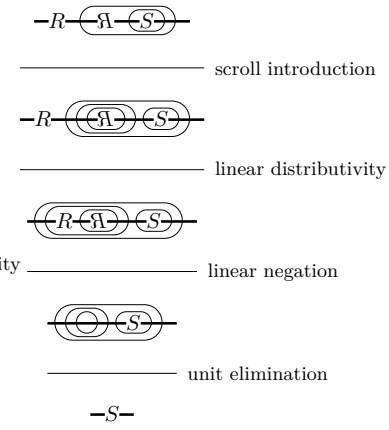
Peirce himself tended to write residuation vertically, as in the right side of Figure 28. Turning back to show the full converse relation, it is clear that these are topologically the same.

Once aware of residuation in the graphs, we find examples often in his later work. In a draft of Lowell Lecture V from 1903, Peirce derives the transitivity of residuation, i.e. $A\backslash C$ from $A\backslash B$ and $B\backslash C$.[15] At one point, Peirce refers to the residual as the "graph of inclusion" and says it has "the shape of necessary reasoning," for necessary reasoning "is that whose conclusion is true of whatever

[15] Reprinted in [55, p.305-307]. We note that Peirce previously derives the transitivity of linear implication in his earlier algebraic studies in [CP 4:94].

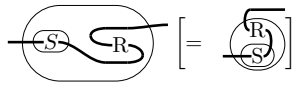


(a) If $R \sqsubseteq S$ then $S^{\perp} \sqsubseteq R^{\perp}$



(b) $R; R\backslash S \sqsubseteq S$

Fig. 27: Further Derivations in EGs



$[S/R]$

Fig. 28: Horizontal and Vertical Presentations of the Residual

state of things there may be in which the premise is true" [55, p. 287].[16] The



Fig. 29: The Graph of Inclusion in Peirce's hand.

graph, reprinted from a copy in Peirce's hand, is given in Figure 29. In fact, many of the more complicated graphs given by Peirce in Lowell Lecture III and V consist in and can be expressed in terms of residuals.

As a final example, we draw attention to the first three graphs Pierce describes in Lowell Lecture III, here reprinted again in Peirce's hand in Figure 30.[17] The first graph states the existence of a residual, the second is the dual of what

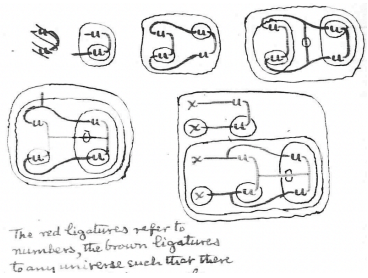


Fig. 30: Graphs Containing Residuals.

is called the symmetric quotient [8] and the third is the straightness condition found in Freyd and Scedrov's "Categories and Allegories" [30]. The symmetric quotient is $(R\backslash S)\sqcap(R\backslash S)$ and the straightness condition is $(R/R)\sqcap(R/R)^{\smile}\sqsubseteq\mathbb{I}$.[18]

[16] Rendering taken from Jukka Nikulainen and can be found in the same section Lowell Lectures V cited above.

[17] Image taken from MS 464-645 as part of the Jeffrey Downard's 'Scalable Peirce Interpretation Network' (SPIN) project at: `http://fromthepage.com/collection/show?collection_id=16`.

[18] When analyzing the conception of quantity in [CP 4:96], Pierce states an analogous straightness condition and offers the following (contrapositive) reading: "if A and B are not identical, either A can do something B cannot or B can do something A cannot."

Both play roles in domain constructions and, in the latter case, moving from a division allegory towards a power allegory.

As far as we know, Peirce presentation of these relations precedes any other presentation of them in the literature — work on the symmetric quotient in the above works not appearing until the late 1980s. While Peirce describes the meaning of the graphs in the Lowell Lectures, he unfortunately does not describe his means of arriving at them. We are left wondering how he came to place so much importance on the five graphs in Figure 30. More work needs to be done to characterize these and the remaining two graphs.

The logic of residuation, which forms the core of the Lambek calculus, plays a key role in categorial grammar and other non-commutative logics. The non-commutativity comes from having distinct left and right residuals. The intuitive idea behind categorial (or type) grammar is that the words that go on to form a complete sentence do so by either becoming 'more complete' when combined with further words on the left or becoming 'more complete' when combined with further words on the right. Like a chemical molecule, nouns and other phrases are 'built up' into submolecules, and a sentence is complete if the submolecules combine to form a larger compound in the right way. Lambek did not seem to be aware of Peirce's developments early in his studies but eventually cites in [39] features of Peirce's *valental* account of relations, which Peirce based on the same chemical analogy [54, pp. 212-7], as anticipating aspects of these type grammars. This section shows the connection to be much stronger as Peirce understood residuation — a fact almost certainly not known by Lambek at the time — and already drew many graphs and derivations employing residuation.

Finally we show the dual to residuation, what is sometimes called coresiduation, in Figure 31.[19] Whereas the starting point for residuation is when composed relations fall under (i.e. are $\sqsubseteq$) a 'shortcut' relation, the starting point for coresiduation is when a relation falls under dually composed relations. As



$[R \oslash T \sqsubseteq S]$ $[T \sqsubseteq R \dagger S]$ $[T \oslash S \sqsubseteq R]$

Fig. 31: Coresiduation Equivalences

residuation is the right adjoint to relational composition, coresiduation is left adjoint to relative sum. Looking back at Figure 30, one sees that these graphs really contain instances of residuals and coresiduals.

The rule Peirce gives for permuting the terms in the inclusion captures both the case of residuation and of coresiduation. After Peirce states the cyclic rule

[19] As of yet there seems to be no standard name or symbol given to the dual of residuation. Moortgat calls them *right* and *left difference* in [49], which is where we adopt the symbols $\oslash$ and $\oslash$.

given above, he goes on to list over a dozen further examples of which some include the coresiduation case. We also find examples where Peirce specifically studied the par'd context that is the basis for coresiduation. We reproduce two sample graphs from R488 in Figure 32.[20] Given the 'lover of' $(-l-)$, 'servant



Fig. 56 **Fig. 57**

Fig. 32: Sample graphs from R488

of' $(-s-)$, and 'benefactor of' $(-b-)$ relations, in Peirce's words, sub-Figure 56 reads: "Whoever loves any benefactor of any man serves that man" or "Every lover of any man serves everybody benefitted by that man", or "Any benefactor of any man is loved only by servants of that man". The sub-Figure 57 is the center graph that is the start of coresiduation. In Peirce's words the meaning of the graph reads: "Whoever serves any man loves everybody but the benefactors of that man" or "Any man benefits every man served only by lovers of him". The multiple readings of these graphs shows an awareness that one can freely interpret the orientation within the 'scroll', and shows clearly that Pierce studied the par'd context inside the 'scroll'. These graphs are also related to Cockett et al.'s circuit diagrams that we discuss in the next section.

Coresiduation plays an important role in Moortgat's categorial grammar [49] and in the bilinear case emphasized by Cockett and Seely [26]. It is indeed this second linear implication that is the characteristic feature of bilinear logic. Lambek originally notes coresiduation and the par'd context in the presentation of bilinear logic, but goes on to downplay its linguistic importance. He writes:

> the distinction between the tensor product and its de Morgan dual, called par by Girard, seemed to be irrelevant for linguistic purposes. We therefore decided to drop it (see [22]) and turn to what is now known as compact bilinear logic and its algebraic presentation in the form of pregroups. [43, p. 672]

While it does not add expressivity over the linear implication with linear negation — it is in fact the linear dual of linear implication, i.e. $R \oslash T = (T \backslash R)^{\perp}$ — it does aide as an important connective for the proof theory (e.g. proof nets) and for characterizing inference rules. It appears that Peirce, contra Lambek, would side decisively with Moortgat here.

[20] Reprinted in [54, p. 266]. See also R480 (and again in [54, p. 270].

A question was posed in the last Section 2 about why the combined complement-converse relation (the $\breve{\bar{R}}$ or $R^{\perp}$ shown in Figure 7) could have more fundamental importance than complement itself. These bilinear operations give the answer: taking the inclusion as prime importance, the bilinear operations produce equivalences within the inclusion that induce the combined complement-converse relation.

We add two further points about Pierce's understanding of residuation. First, we repeat a significant passage where Peirce stresses the importance of residuation.

> Yet really, the form $l \dagger \breve{\bar{l}}$ is all-important, inasmuch as it is the basis of all quantitative thought. For the relation expressed by it is a transitive relation...[It] is not only a transitive relation, but it is one which includes identity under it. That is, $\mathbb{I} \prec l \dagger \breve{\bar{l}}$. But it is further demonstrable that every transitive relation which includes identity under it is of the form $l \dagger \breve{\bar{l}}$. [CP 4:94]

Peirce goes on to demonstrate the last claim. The passage shows that Peirce understood that any derivation from the identity can be put in this form.[21] Peirce makes a similar claim on p. 194 in 'Note B':

> When we have only to deal with universal propositions, it will be found convenient so to transpose everything from subject to predicate as to make the subject $\mathbb{I}$. Thus, if we have been given $l \prec b$ we may relatively add $\breve{\bar{l}}$ to both sides; whereupon we have $\mathbb{I} \prec l \dagger \breve{\bar{l}} \prec b \dagger \breve{\bar{l}}$. Every proposition will then be in one of the forms: $\mathbb{I} \prec b \dagger l$ or $\mathbb{I} \prec b; l$.

Note, first, that Peirce is (again) making a claim analogous to moving to the one-sided sequent calculus. Further, Peirce is aware that in this calculus both relational composition and the dual is sufficient to express all the 'universal' propositions. We believe these passages demonstrate a turn towards proof theory, as they suggest i) placing importance on those constructions only from the unit/identity, ii) placing importance, not just on the linear negation rules given above that are so crucial to proof theory, but suggest an awareness of the sufficiency of these rules for such an undertaking.

We summarize to this point. Peirce understood linear distributivity and linear negation laws and states them clearly as far back as 1883 in 'Note B'. At around the same time, Peirce carried out a systematic study of residuation that includes a rule for permuting terms to arrive at various equivalencies. This rule works

[21] The omitted line in the '...' above is also of interest: "By a transitive relation, we mean a relation like that of the copula. If A be so related to B, and B be so related to C, then A is so related to C. The 'copula' is Peirce's sign for illation or inference — it is formally like that of implication — but the passage is significant in that Peirce sides decisively with the linear case and that the sign of inference ought to be based on sequence and not just inclusion. We come back to the importance Peirce places on sequence in Section 6.

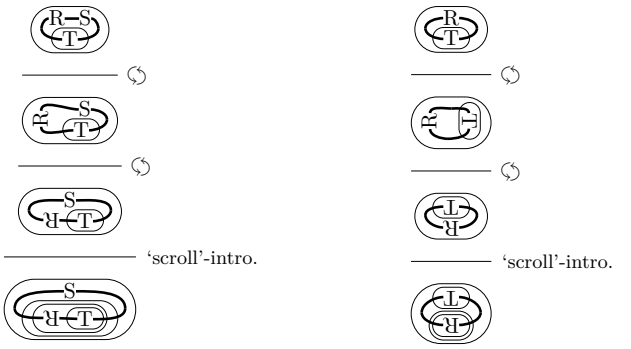
for both residuation and coresiduation and, in an example we return to later in Figure 35, captures the cyclicity condition behind cyclic linear logic. Peirce also states the cyclicity condition in 'Note B' and goes on to give in 'Note B' and elsewhere many examples employing these rules. We also have given numerous examples demonstrating that Peirce continued to develop and draw graphs of residuation. We conclude that Peirce presented and understood the same relation algebraic structure that models cyclic bilinear logic.

We now turn to his Existential Graphs and situate the resulting diagrammatic syntax in more contemporary terms.

4 Existential Graphs and Circuit Diagrams

We showed in the last section how Pierce had an understanding of the bilinear operations corresponding to linear implication and its linear dual. Because the 'scroll' also serves as an inclusion, we suggestively showed — primarily in Figure 24 and Figure 31 — how these equivalences are captured graphically inside the 'scroll'. The residuation equivalences given in [35] were motivated by topological considerations governing Peirce's 'lines of identity'. We go further in this section and make explicit how these equivalences can arise directly from topological features of the 'scroll' itself. As a key source of examples, we go on to compare Peirce's notation to Cockett et al.'s circuit diagrams in the literature.

To begin, we note that the 'scroll' allows the bilinear operations to be efficiently presented as topological moves. Two examples in Figure 33 show how relations can be freely rotated inside the inclusion. These are the same deriva-



(a) \-introduction (b) the linear contrapositive

Fig. 33: Topological derivations inside the 'scroll'

tions as those given in Figure 24(a) and Figure 27(a). The added 'scroll' in each of the last steps makes explicit the change in direction of the inclusion.

Rather than present converse as mirror image (-Я-), in these examples the the relation is 'wrapped around' so that it becomes inverted as in -Ʀ-. One can similarly imagine rotating the page itself and looking at the graphs upside down. In the case of binary relations, this inverted relation is topologically equivalent to the converse in Figure 5 (it is as if one grabs both outermost wires and pulls, leaving the left and rightmost wires the same).[22] Given the derivations from linear distributivity and linear negation above in Figure 26, we can freely associate these re-orientations with derivations. Regardless, we note that Peirce's 'scroll' presents various ways of capturing these logical operations via topological moves.

With these topological considerations, it is worth comparing the ease of presenting the Schröder equivalences with similar diagrammatic approaches in the literature found in [67, p. 159] and [30, p. 259] (and the FOL derivation found, for example, on [67, p. 42]). We find the cyclic presentation within the 'scroll' to be a simple and elegant means of capturing the underlying logical relationship: the 'scroll' allowing clear presentations of the tensor'd and par'd context, along with the variety of linear implications that mediate them in the bilinear context. Given it is well understood that linear sequents can be so permuted, it is perhaps a wonder why a graph with these cyclic connections has not been so used for the purpose.

We summarize all of the relevant operations to this point. Figure 34 offers a shorthand notation for representing residuation and coresiduation via rotations inside the 'scroll'. These moves make clear the various introductions (or

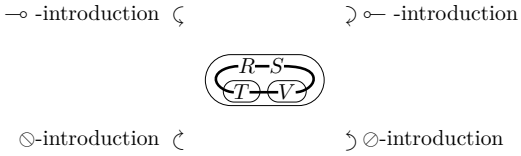


Fig. 34: Summary of Bilinear Operations

eliminations) of bilinear terms while preserving the overall cyclic order.

As a final example, we note that one can use the residuation and coresiduation rules to perform a 'full cycle' and move a par'd context from one side of the sequent to the other side as in the cyclic rules in Figure 4 from Section 1. We show the rule in EGs in Figure 35. At first glance this may look like

[22] One must be more careful in the n-ary case, where the overall top-to-bottom ordering matters. In this more general case it is perhaps better to see the 'scroll' as a sphere, like an 8-ball, where the antecedent is brought around from the back of the 8-ball to the front.

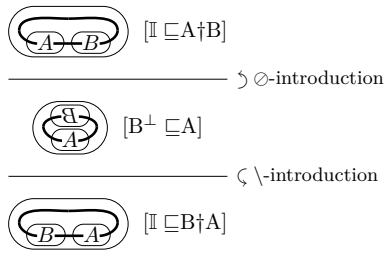


Fig. 35: Cyclic Law via Bilinear Operations

commutativity, as it appears analogous to a (binary) swapping of the order of the consequent. The empty antecedent, however, ensures that the total ordering is (cyclically) preserved. Graphically, this rule corresponds to taking a par'd expression from the bottom right of a 'scroll' and moving it to the upper part (using coresiduation), and then moving it down again to the lower left (now using residuation).[23] This was, as stated in Section 2 and Figure 21, known by Peirce and given algebraically in 'Note B.'

We claim that Peirce's Existential Graphs can represent the 'geometric' graphs of the residuation laws and their respective equivalences in [4] and [5]. It would also be worth comparing the presentation in EGs to the version with quantification in [50]. We will save these developments for another time.

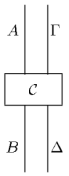
With these topological features of the 'scroll' now emphasized, we return to the contemporary significance of these graphs and moves. Presently we compare the resulting graphs to Cockett et al.'s circuit diagrams for the bilinear case as found in [26].[24]

The key to reading a Cockett et al.'s circuit diagram is that the topmost, incoming wires are implicitly tensored and the output, bottom-most wires are implicitly par'd. See the comparison in Figure 36. In a 'scroll', the inputs are captured by composed terms in the antecedent position, and the outputs are captured by terms in the dual composition, which are captured by separate 'cut' contexts inside the inclusion.

A further, telling comparison is given in Figure 37. Here a standard deduction takes the generic morphism from Figure 36 and introduces a linear implication (think, again, of the example of currying) to the output port. Notice the addition

[23] An involution is introduced during the last step with the residuation introduction, but in our setting the involution is again superfluous. One can also express this rule with left and right (linear) negations. We come back to this point at the end of this section.

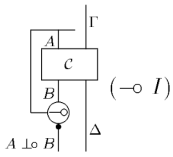
[24] Further summaries can be found in [23] and [24].



$A \otimes \Gamma \vdash B \oplus \Delta$



Fig. 36: A comparison of a generic morphism in Cockett et al.'s circuit diagrams (left) and Peirce's EGs (above)



$\Gamma \vdash (A \multimap B) \oplus \Delta$

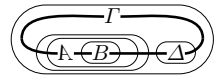


Fig. 37: Another comparison of a morphism in Cockett et al.'s circuit diagrams (left) and Peirce's EGs (above)

of a 'scope box' on the left side in the circuit diagram that keeps track of the original position and type of the wire.

The logical conception that motivated Cockett et al.'s circuit diagrams — almost certainly without any awareness of Peirce — and Peirce's analogous presentation in EGs is remarkable. Even more striking, perhaps, is the description Cockett et al. give of the scope box. They write that "the 'opaque' side of the box ought to be regarded as the wire A bent to join the ⊸ node at the bottom of the box" [24, p. 18]. This is given explicit meaning in the topological features of Peirce's 'scroll'. We give an example of the linear negations below, but we note that each of the component circuits given by Cockett et al. in [26] can be given correspondingly simple expressions in terms of Peirce's 'scroll'.

We draw a further connection to circuit diagrams. Peirce was aware that inclusion graphs like the the 'scroll' can be composed using inference rules for EGs. We give an example in Figure 38 that follows Pierce's graphical depiction of the process in Lowell Lecture II from 1903. [55, p. 201-2]. The steps are as follows. A 'scroll' is first 'iterated' or nested inside the consequent of another. Notice that the iteration that yields the nested 'scrolls' has the effect of lining up the antecedent of the inner scroll with the consequent of the outer scroll. 'Lines of identity' are then similarly 'iterated' and extended inwards to join the shared term. Composition or *cut* (now emphasized and with the usual meaning from the sequent calculus) is then performed to eliminate the middle term. In Pierce's words this involves the shared middle term (i.e. 'A') being 'deiterated' and then 'erased'. The result is a new 'scroll' with the requisite antecedents and consequents.

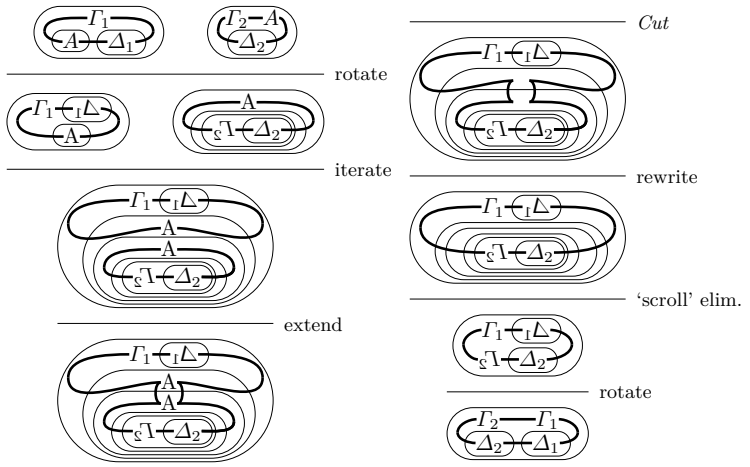


Fig. 38: Derivation of comp_4.

We could also perform the *cut* or composition step by employing linear distributivity followed by a elimination using the linear negation law. Similarly, while Peirce often nests the graphs vertically, as we have shown, we could use a 'one-sided' presentation where we reorient the 'scrolls' as following from the unit (i.e $\mathbb{I}$) and again perform linear distributivity and linear negation.

Four significant variations of *cut* or composition of 'scrolls' are given in Figure 39. These are the rules for planar polycategorical composition (see [23, p.14 & 16]). In terms of the general proof theory, these operations ensure that the respective (planar) ordering of the composed terms is preserved. All the conditions, however, are captured by using the bilinear operations and *cut*. We add that the four variants of planar composition in Figure 39 fall under Pierce's rule that any graph, a 'scroll' included, can be iterated within any 'positive' area. This includes in general the multiple locations allowed by consequents in the par'd context. This is a feature Peirce's Existential Graphs share with *deep inference*.[25]

Cockett et al. refer broadly to this relational setting as the *logic of generalized relations*. The Γ is a placeholder for a list of tensored (or, in our case, relationally composed terms), while the Δ is a placeholder for a list of par'd terms (in the dual presentation). Both are straightforwardly captured by lists in EGs, such as in Figure 40. Given the preceding discussion in this and the previous section, we would argue the Peirce understood this setting.

[25] See discussions of the comparison in [48,44] and in [14]. A further worthwhile direction between deep inference and cyclic linear logic is found in [28].

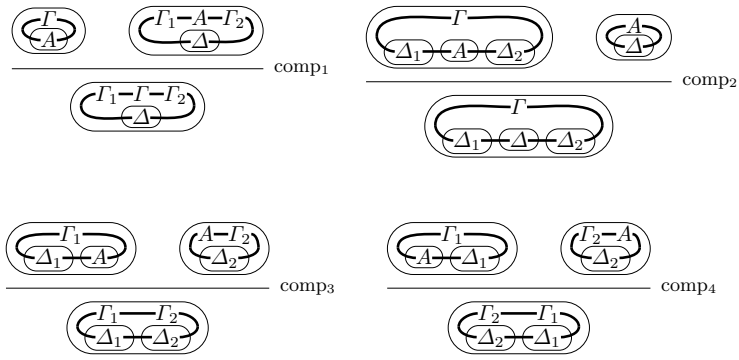


Fig. 39: 'Scroll' (i.e. planar) composition rules.

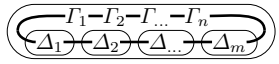


Fig. 40: A Generalized Entailment Relation

An advantage of circuit diagrams, like any wiring diagram, is the ease of expressing composition. A further question is whether nesting 'scrolls' can serve the same function. We believe it can, and that doing so has several advantages, but we are not yet convinced of its efficiency in this particular purpose.

The circuit diagrams from Cockett et al. were created following two major developments: (i) the circuits allowed for treating the units in terms of what are called 'thinning links' that include a variety of coherence conditions and rewrites, and (ii) the scope boxes, again with corresponding coherences and rewrites, were needed to keep track of the bounds of the currying operations in the context of derivations. As hinted at above, Peirce's 'scroll' serves the same function as the scope box. Further, in regards to the use of thinning links to keep track of the units, we note that Peirce's 'lines of identity' not only keep track of the position of the (possible) units but the cyclic presentation meets the exact coherence conditions given for treating the thinning links.

We point out a further advantage the 'scroll' has for linking connectives to their corresponding inference rules. When expressing tautologies for the logical connectives, the resulting introduction and elimination rules are captured (topologically) by these bilinear moves. As an example, we show how the tautology from the residual/linear implication (i.e. $A\backslash B \sqsubseteq A\backslash B$, which we write suggestively as $id_{A\backslash B}$) yields a corresponding modus ponens rule for its elimination in Figure 41. A further example comes from a basic tautology $A \sqsubseteq A$, which leads to the linear negation laws in Figure 42. The requisite inference rules for

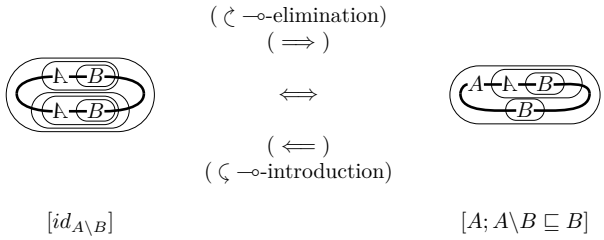


Fig. 41: Corresponding elimination rule from tautology.

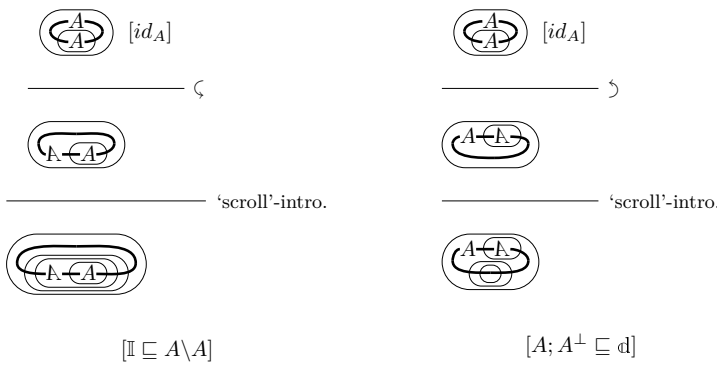
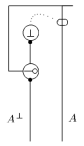


Fig. 42: (Topological) Derivations of Linear Negation Laws from $A \sqsubseteq A$

the bilinear case are all fashioned in this way: arising, as they do, out of the residuation operations from tautologies on the connectives.

We note these (again) correspond to the required rewrites in Cockett et al.'s circuit diagrams. Figure 43 shows the circuits corresponding to the bottom



$[\mathbb{I} \implies A^\perp \oplus A]$



$[A \otimes A^\perp \implies \mathbb{d}]$

Fig. 43: A comparison of Cocket et al's circuit diagrams for linear negation circuits.

graphs in 42. Peirce would have been aware of the manipulations that went into these circuit diagrams. Like a dial, the 'scroll' allows us to see the 'inner workings' of the operations that turn wires around in the more general setting.

As mentioned in Section 2, 'scrolls' can be added varyingly around the par'd contexts to reassociate (i.e. nest) par'd terms, add a par'd unit, or (as seen above) reorder an inclusion to yield the linear negation laws. 'Scrolls' can also be added to reassociate implicational terms. We give the following example in Figure 44, which expresses the key morphism for what Cockett et al. call a Bilinear Category [26, p. 103]. The addition of the 'scroll' in each respective location is (again) a



$[(A \multimap B) \dagger C]$ $[A \multimap B \dagger C]$ $[A \multimap (B \dagger C)]$

Fig. 44: Key morphisms for a Bilinear Category

simple way of expressing re-associativity. Cockett et. al recognize the morphisms as such, but we add that this is built into the rules of Peirce's original EGs.

We add a final example for posterity. In the bilinear case the left and right residuals and corresponding duals give rise to four different linear negations [42]. These are expressed, like a sequential variation of the intuitionistic case in Figure 23, by adding a 'scroll' on the requisite side to introduce a par unit. One can imagine, for example, adding these units to the derivations in Figure 42. While the connectives as presented in Figure 45 have the same meaning with the involution, they are distinguishable in the larger cyclic context and proof

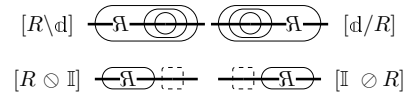


Fig. 45: Bilinear negations.

theory by how they behave, i.e. by keeping track of locations for corresponding expansion and elimination rules.[26] Again, and similar to the question above about whether nesting 'scrolls' can serve the same purpose as proof nets, this is a further direction worth making precise.

Peirce's 'scroll' simultaneously serves as a graph of inclusion and an involution. In this section we have demonstrated a number of further effects from adding a 'scroll' at various locations inside the inclusion. Adding 'scrolls' out of the unit wire or around a unit wire yields the variety of linear negation rules. Adding a 'scroll' can also reassociate in the implicational context, as in Figure 44, yielding the distinguishing morphism of Cockett et al.'s Bilinear Categories. In this section we have shown further that the bilinear operations can take the form of topological moves inside the 'scroll'. We continue demonstrating the graphical efficiencies of Peirce's 'scroll' in the next section, and show how the 'scroll' also keeps track of *refutations*.

5 Proofs and Refutations

While the multiplicative fragment of linear logic is characterized by *-autonomy, and in particular by linear distributivity and linear negation, another common characterization of the 'linear-ity' of linear logic is in terms of separate proof and refutation clauses for the linear connectives. While intuitionistic logic is said to be about proofs, linear logic can be thought of as a dialogical back and forth between a 'prover' and 'refuter'. This can be found, for example, as far back as [11], and also more recently in the anti-thesis interpretation by Shulman [71]. One of the first completeness proofs for linear logic is based on such a prover vs refuter approach in [1].

The intuitive idea behind this approach is that in a derivation the linear negations laws require every atomic expressions to have a linear dual. An appropriate proof then, such as those given in a valid a proof net, has the right number of 'pairs' of atomic expressions and in the right locations to ensure that the linear rules were followed.

Peirce often emphasized the importance of dialogical reasoning — both in thought and in signs [37,58,60].[27] Consider the following passage as one example:

[26] Further discussion on negation in the bilinear context and some further directions can be found in [47].

[27] Burch discusses one of the few instances of game semantics in EGs in [20]. See also [65].

> The answer I am reporting now goes on to show, what will hardly be disputed, that all deliberative meditation, or thinking proper, takes the form of a dialogue. The person divides himself into two parties which endeavour to persuade each other. [56, p. 180]

We note in particular that Hintikka's developments in game-theoretic semantics are based in part on the early ideas found in Peirce [37,38,58]. While the emphasis on proofs and refutations was not around at Peirce's time, he was aware of the refutation-clauses of the connectives, including the linear connectives. In fact Peirce's early derivations of linear distributivity were motivated from considerations of the refutation clauses of the connectives [53, see 'On the Logic of Relatives']. We also show, for the first time, how linear contraposition captures the well known linear equivalences in the graphs.

The refutation clauses of the Boolean connectives are well known. What we take to be interesting about the examples in Peirce's graphs is (i) that they show how the 'scroll' is an effective means of moving to the refutation context, and (ii) that the same principle works for the sequential/compositional terms, i.e. for the multiplicatives. We mention again that Peirce clearly employs intuitionistic reasoning about negation in these examples.

As a simple preliminary example, we show a more subtle presentation of the law of contraposition in Figure 46, stating "$A \sqsubseteq B \Leftrightarrow \bar{B} \sqsubseteq \bar{A}$". This is a



$A \sqsubseteq B$ $\qquad\qquad$ $\bar{A} \sqsupseteq \bar{B}$

Fig. 46: The Law of Contraposition

straightforward transformation in the graphs that follows from adding a 'double cut' around A. We reverse the direction of implication, $\sqsupseteq$, to signal reading the inclusion from the reverse direction and from bottom to top.

In the examples discussed below, Peirce adds more meaning to contraposition by recognizing that the implication $A \sqsubseteq B$ means that if B were to 'vanish', as he says, then A must also vanish. The vanish here is to go to False or absurdity, as in $B \Rightarrow \bot$ (and graphically: ). Such a meaning can be captured in the graphs, too, and involves making explicit an extra 'double cut' as seen below. We liken this, as in Section 2, to using the 'scroll' to add a par'd (or disjunctive) unit in the context of the graphs — the 'scroll' simply making explicit what is, or *could* be there — but now as a refutation. As in Figure 46, Figure 47 makes this explicit.

A further generalization is needed for the Beta case with sequential composition. We again, though, simply reason in the same manner about what would happen if a consequent were false. We now move to the main example from Peirce

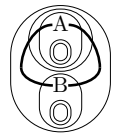


Fig. 47: $(A \Rightarrow \bot) \sqsupseteq (B \Rightarrow \bot)$

[53, p. 338-9] and start with the graph of "every lover of a servant is a benefactor of". Using $-l-$ for the 'loves' relation, $-s-$ for 'serves', and $-b-$ for 'benefactor', this graph is given in the top of Figure 48. Following the discussion above,

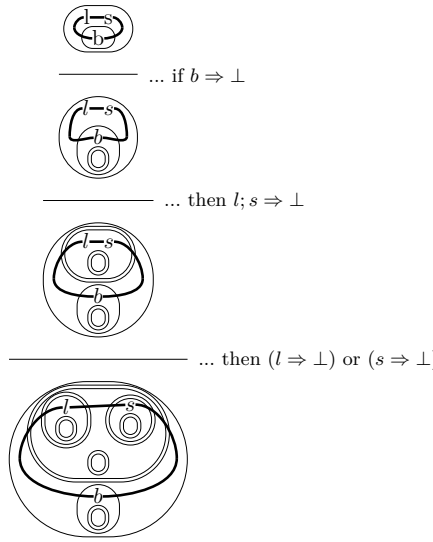


Fig. 48: Establishing the Refutation Clauses

Peirce notes that the meaning of this graph includes that if the consequent b were to vanish, then the antecedent must also vanish. Graphically, this sequence begins in the second derived line in Figure 48, where $b \Rightarrow \bot$ is represented by introducing ◎ within the consequent. We next capture the vanishing antecedent by adding further 'scrolls'. The larger 'double cut' around $l; s$ is needed because the orientation of the implication is reversed. The outermost 'cut' signals that

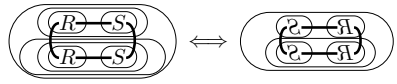
we 'peer inside' the first 'cut' to see what the consequent would be, and the consequent now reads (like the b before it) that $l; s \Rightarrow \bot$. Finally, notice that all we have done (and even all we will continue to do) is to add double cuts in or around various subgraphs. This is always appropriate, though the meaning of the graph only changes subtly.

The first important step is done. The next involves thinking about what it means for relational composition to vanish. The composition fails when either of the composed relations vanishes. Again, all we need to express the condition 'Either $l \Rightarrow \bot$ or $s \Rightarrow \bot$' is to add 'double cuts' in or around subgraphs, as in the third line in Figure 48.

At this point we have successfully traced the consequences of the refutation, i.e. what the result would be if the consequent were to vanish and the antecedent were likewise forced to vanish as well. '$l; s \sqsubseteq b$' implies that 'if $b \Rightarrow \bot$ then either $(l \Rightarrow \bot)$ or $(s \Rightarrow \bot)$'. And all this is shown in the graphs by making the series of additional 'double cuts' explicit.

The example shows that the 'scroll', standing as it does for an involution, is an effective means of tracking the move to the refutation context. The same principle also extends to relational composition, and its dual, and to the other linear connectives. Peirce goes on to use the same style of reasoning to derive the associativity of the dual of composition, to derive linear distributivity, and to derive the residuation equivalences. For example, when listing the Schröder equivalences Peirce points out that their refutation clauses are all the same and so concludes that these are equivalent expressions (or in our case, equivalent graphs).[28]

As a further example, we show how linear equivalences are captured in the graphs as instances of linear contraposition (as given in the derivation in Figure 42). In the linear case, the contrapositive also induces a converse as in the

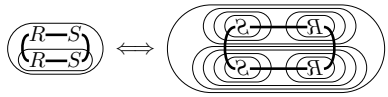


$[id_{R \dagger S}]$ $\quad$ $[id_{(R \dagger S)^\perp}]$ $[= id_{(S^\perp; R^\perp)}]$

Fig. 49: $R \dagger S \cong S^\perp; R^\perp$

[28] We mention again that this is for the binary/dyadic relations. The generalization to triadic relations requires that teridentity — the identification of three terms being the same — to be given a refutation where at least one is different. Peirce calls this the *triad of diversity*. This triad involves adding 'scrolls' as well. We also mention that falsity can be replaced with $\perp\!\!\!\perp$, which like $\top$ can be thought of as keeping track of the type of the wire. These are employed in the recent neo-Peircean calculus of relations given in [14].

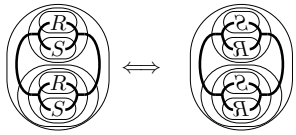
example shown in Figure 27(a). Note that the graph on the left in Figure 49 is the result of transforming the graph on the right by first rotating and inducing the converse, and then by adding a double cut to redirect the order of the inclusion. Following the modified converse in Figure 33, one could similarly turn the page upside down. We give relational composition — the other multiplicative — as another example in Figure 50. Finally, we show one of the additives as well



$[id_{R;S}]$ $\quad$ $[id_{(R;S)^\perp}]$ $[= id_{(S^\perp \dagger R^\perp)}]$

Fig. 50: $R; S \cong S^\perp \dagger R^\perp$

in Figure 51. Parallel operations are commutative, and so the final equivalence



$[id_{R \sqcup S}]$ $\quad$ $[id_{(R \sqcup S)^\perp}]$ $[= id_{(R^\perp \sqcap S^\perp)}]$

Fig. 51: $R \sqcup S \cong (R^\perp \sqcap S^\perp)$

follows from an application of commutativity. This is a key difference from the sequential case, where the involution *anti*-distributes rather than distributes. Regardless, the topological features of the scroll captures the difference. The key takeaway is that, given the residuation equations, the 'scroll' displays these topologically as the same graphs.

We list other well-known linear equivalences below in Figure 52. All can be captured in a similar manner topologically in EGs. Note, again, how the cyclic presentation naturally keeps track of the order and any necessary converses. We will come back to the listed exponentials in the next section.

The novelty here is: (i) that these have not been shown in the graphs, (ii) the examples show how the 'scroll' mediates between the proof and refutation contexts, and (iii) that the linear equivalences can be topologically captured by the same graph. The key takeaway is that the De Morgan duality commonly

$$(R \dagger S)^\perp \cong S^\perp ; R^\perp \quad (R \sqcap S)^\perp \cong R^\perp \sqcup S^\perp$$
$$(R ; S)^\perp \cong S^\perp \dagger R^\perp \quad (R \sqcup S)^\perp \cong R^\perp \sqcap S^\perp$$

$$(R^\perp)^\perp \cong R$$
$$\mathbb{T}^\perp \cong \perp\!\!\!\perp$$
$$\mathbb{I}^\perp \cong \mathbb{d}$$

$$(B \oslash A)^\perp \cong A \multimap B \cong A^\perp \circ\!\!- B^\perp$$
$$(A \oslash B)^\perp \cong B \circ\!\!- A \cong B^\perp \multimap A^\perp$$

$$(!R)^\perp \cong ?(R^\perp)$$
$$(?R)^\perp \cong !(R^\perp)$$

Fig. 52: Various linear equivalences.

recognized as captured by EGs also extends to the multiplicative (and so linear) case.

We note that these equivalences also follow from the line of identity following the laws of a special Frobenius algebra, and can be derived from those laws. The version of the modern graphical syntax presented in [36,14], based on Peirce's account, develops this view. Of interest here is the addition that the 'scroll' provides; namely, that with residuation the topological features of the scroll *already capture* these equivalences.

We have shown how the 'scroll' reassociates the par'd context and linear implication, the role it plays in capturing the linear negation rules, and we have now shown how the 'scroll' captures the contrapositive and linear contrapositive case. Indeed the only location in the inclusion we have not discussed is the location corresponding to the Boolean contrapositive and the single negation or complement. In [14] it is shown how negation as a unary connective arises out of requisite linear adjunctions as the converse of linear negation.

In this and the previous sections we have greatly generalized the function and purpose of the 'scroll', including its relation to the linear operations and connectives. In the penultimate section we turn towards resource sensitivity.

6 Resource Sensitivity

Linear logic is also described as the logic of resources. The linear negation laws allow for only strict resource production and annihilation, so that specific resources cannot be freely copied or duplicated. As we have made clear, Peirce was well aware of these rules as he arrived at and stated these very laws. We use this last section to mention how Peirce took these notions further and recognized that a notion of *quantity* follows from them. Finally, linear exponentials can be used to add selective copying or duplicating back into the logic. We end with a few remarks on what the linear exponentials look like in the graphs.

Peirce was aware that the linear negation laws, and in particular, the difference or diversity relation, allowed for a notion of quantity. We give an early

example from Peirce's algebraic work. Peirce states that from "Some A is B" and "Some A is not-B", we can conclude that "There are at least two A's" [CP 4:88]. This follows from the linear negation laws, where a B composed with a not-B asserts the difference of the remaining terms. The key point Peirce is emphasizing is that some A not being the same as some (other) A means that there are *at least two* A's.

As seen in the quotation about negation in Section 3 and Footnote 21, Peirce often emphasized that logic is a *linear* series of inferences. In the passage with the example given above, Peirce goes on to state that this derivation relies on the principle of contradiction, which he writes as "the non-identity of A and not-A". We note that this appears explicitly to be the multiplicative case, relying as it does on the diversity relation, and Peirce is reading the principle of contradiction specifically in terms of sequential or linear order. Peirce goes on to discuss the transitivity of linear implication, the role it plays in comparative relations, and eventually discusses multitudes, which is Peirce's term for expressing the size of a collection. We remind the reader of the earlier passage where Peirce states that linear implication "form[s] the basis of all quantitative thought". Indeed each of these developments arises from Peirce's study of the linear negation laws (and in particular the diversity relation) and the notions of quantity that he interprets to follow from them.

In the relational model, the linear exponentials are given by the relations in Figure 53.[29] Discussion of the exponentials, some of their properties, and their



$!R$ $\quad$ $?R$

$[= \mathbb{I} \sqcap R]$ $\quad$ $[= \mathbb{d} \sqcup R]$

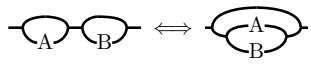
Fig. 53: Exponentials

representation in the relational model can be found in Section 14 in [42]. We know of no place where Peirce calls out these relations and emphasizes their significance as we might, at least with respect to the intuitionistic fragment, motivate them today. One noteworthy passage, however, is worth discussing below.

Perhaps the key feature of the !-exponential is that it converts a multiplicative into an additive. Equationally, this is expressed by the following: $!A \otimes !B \vdash !(A \wedge B)$. We show this graphically in Figure 54. Surprisingly, Peirce makes the same observation in MS 430 and reprinted in part below in Figure 55.[30] Peirce goes on to finish the passage by noting that the last two graphs, equivalent to those in Figure 54, are the same. Peirce is identifying, via topological features, the key

[29] We note again that $\mathbb{d} = \mathbb{I}^{\perp}$ and have left out the converse.

[30] We thank Jukka Nikulainen again for the digitization and presentation in the figure.



$!A; !B$ $!(A \sqcap B)$

Fig. 54: Key property of !-exponential.

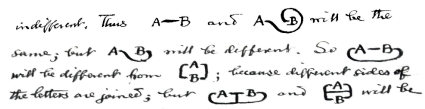


Fig. 55: Peirce's identification of a key feature of the !-exponential.

property of the !-exponential. This is a remarkable example of where a concern for graphical syntax and topological features can lead.

A list of well known linear equivalences were given in the previous section in Figure 52. One can show the equivalences for the linear exponentials using the 'scroll' in the same topological manner. We claim that one can express various further properties of exponentials described by Lambek in [42] using the graphs and Pierce's rules.[31] Finally, a key feature of the !-exponential is that it allows one to express intuitionistic implication [42, p. 233]. It is interesting to note how the !-exponential combines with residuation in this case. The additional identity relation has the effect of topologically turning a sequential, linear implication into a graph closer to the vertical (i.e. parallel) inclusion. These directions and their corresponding graphical outcroppings are all worth further study.

7 Conclusion

C.S. Peirce's early work on the logic of the relations has been absorbed into the relation algebraic tradition (following Tarski) and into the development of first-order logic (following his contribution to the discovery of quantifiers, for example). Much of this follows from an awareness of Peirce's earlier algebraic work. At the same time much of the work on Peirce's later Existential Graphs has remained outside the larger logic tradition and has remained, following Robert's initial systematic treatment of the subject, in the confines of first-order logic. This is slowly changing as alternative methods of diagrammatic reasoning — most notably the advent of string diagrams in category theory, but also the focus on graph rewriting more generally — have brought awareness back to Pierce's earlier work in diagrammatic reasoning.

This paper continues in this changing direction and calls attention to Peirce's early developments in linear logic. Peirce's presentation of the calculus of relations from 1883 is noteworthy for emphasizing the dual to relational composition,

[31] Such as, for example, using the rules and syntax found in [14].

linear distributivity, and the linear negation laws. Around the same time Peirce also carried out a systematic study of residuation and stated a rule for how composed terms can be rotated to the dual context on the other side of the inclusion. These are the key rules for bilinear logic and we conclude that Peirce understood these aspects of the relational model of bilinear logic.

This work on the calculus of relations occurred before Peirce's later development of the Existential Graphs. We further showed that these concepts play key roles in his later studies in the graphs. This includes derivations involving linear implications, studies of the par'd context involving the dual of relational composition, and more complicated expressions involving multiple residuals. We have shown some of these graphs, many of which have not been written about in the context of EGs before, and do so for the first time within the context of these bilinear operations.

We reiterate that this is the beginning of such work. Further directions that arose during the exposition include comparisons to the Lambek calculus and categorial grammar, potential generalizations towards proof nets and proof circuits, and new directions for intuitionistic variants of EGs. All of these we believe are worth further study.

We end with a takeaway comment about Peirce scholarship. Peirce emphasized the importance of the 'scroll' both as a key graphical feature of EGs and as a sign of the most primitive logical connective. This work restores the 'scroll' to a place of primary importance in the study of the graphs. The 'scroll' serves at one and the same time *as a sign of involution* and *as a sign of inclusion*. Much of this emphasis on the 'scroll' appears to be vindicated in the *-autonomous and bilinear settings, which emphasize these features.

In the introduction, we wrote of the hope that this work returns interest back to Peirce's early studies and contributions. We add here the hope that this work returns interest to what a concern for diagrammatic reasoning and corresponding topological features is capable of. We suspect Peirce's contribution along these lines to have a bright future.

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