HISTORIA MATHEMATICA 11 (1984), 442–470

The *Erlanger Programm* of Felix Klein: Reflections on Its Place in the History of Mathematics

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The commonly held view that Klein's *Erlanger Programm* was one of the most significant and influential works for the history of mathematics during the half-century following its publication in 1872 is questioned on the grounds that insufficient attention has been paid to the complex web of related mathematical activities of the period. By sketching some of these that are connected with Lie and his school, we present a first approximation to a more informed assessment of the place of the *Erlanger Programm* in the history of mathematics.
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Die allgemein verbreitete Auffassung, daß Kleins Erlanger Programm eines der bedeutendsten und einflußreichsten Werke für die Geschichte der Mathematik während des auf die Publikation von 1872 folgenden halben Jahrhunderts war, wird in Frage gestellt, weil bisher dem komplexen Netz verwandter mathematischer Aktivitäten dieser Periode nicht genügend Aufmerksamkeit geschenkt wurde. Eine Skizze einiger der mit Lie und seiner Schule verbundenen Aktivitäten wird als ein erster Entwurf zu einer angemesseneren Bewertung der Stellung des Erlanger Programms in der Geschichte der Mathematik vorgelegt.
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Il est généralement accepté que le *Programme d'Erlangen* de Klein fut l'une des oeuvres les plus significatives et les plus influentes de l'histoire des mathématiques au cours du demi-siècle ayant suivi sa publication en 1872. Nous mettons en doute cette assertion en prenant en considération l'attention insuffisante portée antérieurement à la complexité du réseau que forment les activités mathématiques de cette période. En esquissant quelques-unes de ces activités impliquant Lie et son école, nous proposons une première approximation d'une évaluation plus documentée de la place du *Programme d'Erlangen* dans l'histoire des mathématiques. © 1984 Academic Press, Inc.

Historians and mathematicians have been remarkably uniform in their emphasis upon the considerable historical significance of the *Erlanger Programm.* It is regarded as one of the most influential works in the history of geometry, and more generally mathematics, during the half-century after its publication in 1872. For example, according to Courant the *Erlanger Programm* "ist vielleicht die einflussreichste und meistgelesene mathematische Abhandlung der letzten 60 Jahre geworden" [1925, 200]. In a similar vein, J. L. Coolidge declared that it "probably influenced geometrical thinking more than any other work since the time of Euclid, with the exception of Gauss and Riemann" [1940, 293]. In her biography of Klein, Renate Tobies pointed to the manifold translations of the *Programm,* which commenced in the early 1890s, as evidence of the fact that its value was quickly recognized outside Germany [1981, 37]. For Tobies the *Erlanger Programm* "bildete . . . eine entscheidenende Zäsur für die Geometrie des 19. Jahrhunderts. Die Verwendung des Gruppenbegriffes durch Klein unterstützte die Ansätze strukturellen mathematischen Denkens, die sich gegen Ende

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des 19. Jahrhunderts herausgebildeten" [1981, 36–37]. As a final example we cite the recent note by Rowe, who described Klein's *Erlanger Programm* as "unquestionably" his "single most important mathematical accomplishment. . . . This work remained a *leitmotiv* not only for Klein but for much of the mathematical world throughout his lifetime" [1983, 448] [1].

Although the *Erlanger Programm* and its historical background have been analyzed in detail [2], I know of no comparable analysis of its influence. The historical basis of the above sort of claims is thus not entirely clear, and to me, at least, the validity, or even the meaning, of those claims is not obvious. My experiences investigating the history of the structure and representation of Lie algebras have led me to suspect that such blanket assertions as the above are misleading and need to be qualified, perhaps revised, and certainly elaborated on the basis of further research [3]. The problem of assessing the influence of a particular work on the course of history can be difficult. A historical event, such as the publication of a mathematical work, does not exist in isolation but as part of a "collage" of events linked together by the institutions through which mathematics is cultivated, communicated, and evaluated. The more we know about the entire collage the easier it is to appreciate the significance of one particular piece. In the case of the collage of late 19th- and early 20th-century mathematics to which the *Erlanger Programm* belongs, our knowledge is rather sketchy at best. My research on the history of Lie algebras has acquainted me with a few of the pieces which, taken together, have a direct bearing on the *Erlanger Programm* and the question of its influence.

In the hope of stimulating further research and discussion, and fully aware that out knowledge of this collage is still inadequate for definitive judgments, I offer the following tentative conclusions: (1) the *Erlanger Programm* remained largely unknown during the twenty years following its publication in 1872; (2) during this period several mathematicians—Poincaré, Wilhelm Killing, and Eduard Study—arrived independently at similar ideas, and their work had a definite impact upon subsequent developments; (3) it is impossible to extricate the influence of the *Erlanger Programm* from the influence of the school of mathematics that Lie founded in Leipzig with a branch, so to speak, in Paris. In the language of my metaphor: the many pieces of our collage represented by the work of Lie and his school are intertwined with the *Erlanger Programm* with sufficient complexity that no single piece stands out as dominant. It does seem, however, that Lie and his school were involved in a mathematically more substantial way than the *Erlanger Programm* in fostering the development of its ideas and in bringing about the trend toward structural thinking that has come to characterize 20th-century mathematics. In what follows I shall attempt to justify these conclusions. To that end it will be necessary to first briefly review the gist of the *Erlanger Programm* and Klein's mathematical activities in the years immediately following its publication.

The *Erlanger Programm* was composed in 1872 when Klein was only 23 years old although already appointed to a full professorship at the University of Erlangen. Bearing the title "Vergleichende Betrachtungen über neuere geometri-

sche Forschungen," it expounded the thesis, supported with numerous examples, that the diverse geometrical investigations that had been undertaken during the 19th century could be unified and classified by viewing geometry as the study of those properties of configurations in a "manifold" which are left invariant by an underlying group of transformations [4]. From this viewpoint Klein considered, for example, projective geometry, Plücker's line geometry, and Lie's sphere geometry. The new viewpoint also brought with it the idea that two geometrical theories that were seemingly different could turn out to be "equivalent" [*gleichbedeutend*] in the sense that, by a one-to-one correspondence between suitably chosen "space elements" in each geometry, an isomorphism of the associated groups would be established. The new viewpoint also suggested a general, albeit somewhat vague, geometrical research program since the study of geometry was conceived within the framework of the general problem: Given a manifold and a transformation group acting on it, one should investigate the properties of configurations which are unchanged by the transformations of the group [1872, 463]. From an algebraic standpoint this problem, Klein asserted, was tantamount to that of developing a theory of invariants relative to the given manifold and associated group. By viewing geometry as embedded within this more general framework, Klein was able in particular to present the methods and views underlying recent work by Lie on contact transformations and partial differential equations. With the concurrence of Lie, Klein concluded by calling for the development of a theory that would be analogous to the theory of finite groups presented in Jordan's *Traite des substitutions* (1870), a theory of groups of transformations which act upon "continuous manifolds" [1872, 489] [5]. He pointed to certain portions of his joint work with Lie on W curves [1870, 1871] as a contribution to such a theory. That is, they had posed the problem of determining the curves in the plane left invariant by some one-parameter family of linear (not necessarily homogeneous), commuting transformations; and in order to solve it they had to determine all such systems of transformations. Although Klein had noted at the beginning of the *Programm* that a group of transformations need not be continuous, all the examples presented in the text were in fact continuous [6].

Published as a "Programmschrift," the *Erlanger Programm* had a limited circulation and was not widely known (as we shall see) during the period 1872–1892. Klein himself did not develop its ideas into a workable research program nor did he encourage his students to do so. There seem to have been two reasons for this. One involved the unexpected death of Klein's mentor and supporter, Clebsch, in November of 1872 [7]. Clebsch's death prompted Klein to dedicate himself to continue the work of his revered mentor. This also involved continuing the direction of some of Clebsch's students, who came to Erlangen from Göttingen after his death. Many were older than Klein, and Klein tended to encourage them in the directions that already interested them. Also, Clebsch's interest in Riemann's theory of functions ultimately led Klein to study Riemann's work itself and to discover the "geometrico-physical viewpoint" underlying it but discounted by Clebsch [Klein 1923a, 477 f.]. Under Riemann's influence Klein developed his

conception of "geometrical function theory." Thus when he was appointed to the Chair of Geometry at the University of Leipzig in 1880, he inaugurated his tenure there with lectures on this subject [8]. Geometry in the sense of the *Erlanger Programm* was not cultivated during the Leipzig years.

This conclusion requires further discussion since Klein in his autobiography [1923b, 18] stated that the *Erlanger Programm* remained the greatest guiding principle [*Richtlinie*] for his subsequent research. Certainly this is the case in the sense that the conviction with which he presumably wrote the *Programm*, namely, that the group concept could bring order, unity, and new perspectives to diverse areas of mathematics, was a continuing source of inspiration to him in his later work on the theory of equations and complex function theory, work which did involve consideration of groups such as the finite groups of the regular solids and the discontinuous groups that arise in what was later called the theory of automorphic functions [9]. But this work of Klein's was in no specific sense a working out of the explicit concerns of the *Erlanger Programm*. As Klein himself explained in 1893, the *Erlanger Programm* was concerned with the application of the group concept to geometry in the traditional sense of that word [10]. By 1893 Klein was willing to describe in retrospect his work on the theory of equations and complex function theory as part of "higher geometry" since it involved consideration of "geometric groups," that is, groups of transformations acting on some "continuous" manifold.

Klein's liberal usage of the term "geometry" reflected his disdain for the compartmentalizing tendencies of specialized research [11]. The vision of the *Erlanger Programm* was thus eventually subsumed under Klein's broader vision of the group concept as a unifying principle for mathematics in general. But such a vision was neither explicit in the *Programm* nor exclusively Klein's [12]. It was shared by Lie and by Poincaré. All three were impressed by the work of Jordan, particularly his *Traité des substitutions* (1870), where groups were applied to geometry as well as to the theory of equations. Indeed if one were concerned to identify the most influential source of the general idea that the consideration of groups is important in mathematics, Jordan's *Traité* would be the obvious candidate. Therefore it is the viewpoint of this essay that in considering the influence of the *Erlanger Programm* it is necessary to restrict attention to its content as opposed to a more general attitude toward the applicability of the group concept that may have inspired it.

A second reason why Klein probably did not attempt to develop the ideas of the *Erlanger Programm* in the period 1872–1892 was that to do it in any substantial way required the development of a theory of continuous transformation groups, the type of groups which alone had been considered by way of example in the *Programm*. The creation of such a theory out of nothing was a herculean task that Klein was apparently not disposed to undertake, despite the concluding remarks of his *Programm*. Thus he made no effort to develop or promulgate the ideas of the *Erlanger Programm* until shortly after Lie commenced publication of systematic expositions of his theory of continuous transformation groups. Lie's theory

provided the appropriate mathematical context for the working out of the ideas of the *Erlanger Programm*. Klein himself implied this when, on the occasion of its republication in translation in the early 1890s, he linked the decision to give it a wider circulation to the appearance of Lie's *Theorie der Transformationsgruppen* [1888, 1890, 1893a] [13]. We shall return to this point further on.

It was thus Lie, not Klein, who devoted himself to the development of a theory of continuous groups. As indicated above, Lie was in close personal contact with Klein during the three years prior to the publication of the *Erlanger Programm* and shared Klein's belief in the value of the group concept for the study of geometry and other branches of mathematics. The belief was inspired in both by Jordan's *Traité des substitutions* [1870] and led to their joint work on W curves [1870, 1871]. It is a moot point whether the ideas of the *Erlanger Programm* were the joint property of Lie and Klein, as Lie subsequently claimed [1893a, xvi] or whether they are completely Klein's, as Klein later seemed to suggest [1921, 41]. In the *Erlanger Programm* itself, Klein explained that he was writing it in order to present certain methods and results contained in recent work by himself and Lie which, despite the diversity of subject matter, shared the tendency toward the general viewpoint that he now wished to expound at length [1872, 461]. On the basis of the early joint papers of Klein and Lie and Klein's own comments thereon [1921, 415], it seems reasonable to conclude that the idea of using continuous groups of transformations and associated infinitesimal transformations in geometrical studies was mostly due to Lie, but that the idea to use groups to systematically classify existing geometrical theories originated with Klein. Lie's interest in groups and geometry was from the outset informed by his interest in differential equations in the geometrical spirit of Monge and his vision of a Galois-type theory of differential equations. These differences proved vital to Lie's creation of his theory of transformation groups to which he devoted most of his energy in the 1870s and thereafter. The critical moment that apparently gave Lie the courage to undertake the task of creating such a theory occurred in the fall of 1873 when he discovered that he could determine all continuous transformation groups in one variable by utilizing the associated "Lie algebras" of infinitesimal transformations.

Lie worked in relative isolation in his homeland of Norway until 1886 when he accepted, with Klein's encouragement, the chair in geometry at Leipzig vacated by Klein's move to Göttingen. Lie came to Leipzig expressly to create there a school dedicated to the further development, application, and dissemination of his theory of transformation groups. In creating his theory, Lie was guided by the paradigm of the theory of finite groups of permutations, especially as epitomized by Jordan's *Traité*. Whereas the theory of finite groups was largely developed after its applicability to the study of algebraic equations had been revealed by Galois and his expositors, Lie hoped to develop his theory and its applications more or less simultaneously. Thus the two intertwined and fundamental goals Lie set for the development of his theory were (1) to cultivate those aspects of the pure theory that appeared essential for the (somewhat vaguely) envisioned appli-

cations; and (2) to demonstrate the fruitfulness of the pure theory by successfully applying it in other branches of mathematics.

Lie remained in Leipzig until 1898 when ill health forced his return to Norway. He died there the following year at the age of 57. Given the relative brevity of his tenure at Leipzig, the ill health that plagued him throughout the period, and his mercurial, often abrasive personality, it is remarkable how successful he was in achieving his objective. Three *Privatdozenten* at Leipzig, Friedrich Engel, Friedrich Schur, and Eduard Study, turned their research interests to aspects of Lie's theory. Engel, in fact, became Lie's right-hand man and devoted his entire career to the development and exposition of Lie's ideas. It was with Engel's invaluable assistance that Lie was able to express his theory in a systematic, literate form as the three-volume *Theorie der Transformationsgruppen* [1888, 1890, 1893a]. Another mathematician who should be mentioned along with Engel, Schur, and Study is Wilhelm Killing, a professor at the Lyceum Hosianum in Braunsberg, East Prussia. Killing's work on the foundations of geometry led him to the view that Lie's theory of transformation groups, elements of which he had discovered independently, was indispensable for such foundational studies [14]. Although Killing himself made only one brief visit to Leipzig and was never able to maintain a cordial relationship with Lie, he maintained a vital contact with Leipzig through his extensive correspondence with Engel; and he contributed in a fundamental way to Lie's theory. More will be said about Killing further on.

Lie also managed to attract a number of doctoral students at Leipzig, among whom we mention Ludwig Scheffers, who edited three volumes of Lie's lectures for publication, including the widely read *Vorlesungen ueber kontinuierliche Gruppen mit geometrischen und anderen Anwendungen* [1893b] and *Geometrie der Berührungstransformationen* [1896], and Gerhard Kowalewski, whom Lie described as one of his best "German students" [15]. The qualification was necessary because Lie attracted students from other countries as well. Most significant were the students from the prestigious École Normale Supérieure who, with the expressed encouragement of the leaders of the Paris community of mathematicians (Darboux, Picard, Poincaré) studied Lie's theory in order to apply it in areas of interest in Paris, especially to differential equations. Many *normaliens,* such as Ernest Vessiot, Wladimir de Tannenberg, and Arthur Tresee, spent a year in Leipzig to study with Lie, although some, including Élie Cartan, never actually visited Leipzig.

It is against the backdrop of Lie's school that the work of Killing, Poincaré, and Study should be discussed. I shall begin with Poincaré. Influenced, like Klein, by Jordan's work and by non-Euclidean geometry but unaware of the *Erlanger Programm,* Poincaré had likewise come to the conclusion that a geometry is "the study of the group of operations formed by displacements to which we can subject a figure without deforming it." Those words were written in 1880, the year after Poincaré had received his doctorate, in an essay containing the beginnings of his theory of Fuchsian functions [16]. Thus when Lie visited Paris in 1882, he found Poincaré already convinced of the importance of group theoretic ideas in geome-

try—and throughout mathematics. According to Lie's report to Klein in a letter from Paris, Poincaré had explained to Lie that all of mathematics was a tale about groups [*eine Gruppengeschichte*] [17]. Lie also indicated that Poincaré did not know the *Erlanger Programm,* and so he described it to him. There is no evidence, however, that Poincaré ever studied the *Programm* or was particularly influenced by it since he himself had already arrived at a group theoretic interpretation of geometry. By contrast, Poincaré was greatly impressed by Lie and his theory of groups, as was Émile Picard, another promising young mathematician two years younger than Poincaré. By Lie's own contemporaneous account, it was Poincaré and Picard who were quick to grasp his ideas when he presented them in Paris [18]. As a consequence, between 1883 and 1892 both Poincaré and Picard wrote a number of papers involving applications of Lie's theory to diverse areas of mathematics—complex function theory, algebraic geometry, hypercomplex number systems, differential equations, and foundations of geometry. The paper on foundations of geometry written by Poincaré [1887] applied Lie's theory (especially Lie algebras) to solve Helmholtz' "space problem" in two dimensions. Poincaré also took every occasion he could to popularize the idea of geometry as the study of groups.

Lie strengthened his ties to the mathematicians of Paris by a return visit paid in 1887, the year following his move to Leipzig. By then Poincaré and Picard had established themselves, along with Gaston Darboux, as the leaders of the mathematics community in Paris. Darboux had remained in close contact with Lie since they first met in 1870. It was shortly after this visit that Picard, Darboux, and Jules Tannery, the director of scientific studies at the École Normale, did something unprecedented. They decided to encourage graduates of the École Normale—the prime source of future French mathematicians—to study transformation groups, if possible with Lie in Leipzig, so as to write a thesis applying the theory to matters of interest in Paris. This shows just how highly Lie's theory was esteemed in Paris. As Picard put it in a letter to Lie in 1888, "Vous avez créé là une théorie d'une importance capitale, et qui comptera comme une des oeuvres mathématiques les plus remarquables de la seconde moitié de ce siècle . . ." [19]. Thus the leaders of the Paris community encouraged the development of a "branch" of Lie's school in Paris, populated by ambitious and bright *normaliens.* "Voilà Paris," Picard wrote to Lie in 1893, "devenir un centre de groupes: toute cela fermente dans ces jeunes cerveaux, et on aura un excellent vin quand les liqueurs seront un peu reposées" [20].

Next I turn to Killing. Since I have already analyzed his work and its background in considerable detail elsewhere [1980, 1982], it will suffice to recall here that, motivated by many of the same elements in 19th-century geometry that inspired the *Erlanger Programm* (e.g., the discoveries in non-Euclidean geometry and the foundational essays of Helmholtz and Riemann) and guided by the disciplinary ideals of the Berlin school of Weierstrass, Killing articulated (unwittingly) a group theoretic approach to the study of geometries that had much in common with the *Erlanger Programm,* which was unknown to Killing. There were of

course significant differences as well between the *Erlanger Programm* and Killing's own *"Braunsberger Programm"* [1884]. Whereas Klein focused upon the classification of known geometries, Killing emphasized the need for a systematic and exhaustive classification of all possible "space forms," which, on the infinitesimal level on which Killing operated, corresponded to Lie algebras of infinitesimal motions. The research program articulated by Killing involved, among other things, the problem of a complete classification of all Lie algebras over the complex field. That Klein did not automatically accept such abstract generalities as geometry in the sense of the *Erlanger Programm* is clear from his many critical remarks on "geometrical" work that transcended the bounds of intuition and experience [21].

It was through Killing's subsequent contact with Lie and his student Engel that he was encouraged to develop the mathematical techniques and relationships by which he hoped to implement his research program on space forms in a focused way. That is, he concentrated on a problem of importance to Lie's envisioned Galois theory of differential equations: the classification of simple structures. Killing's work [1888a,b, 1889, 1890] as perfected by Élie Cartan in his doctoral thesis [1894] added an unexpected and profound dimension to Lie's theory of the structure of transformation groups, a dimension which opened the way to the substantial geometrical applications made by Cartan, who was perhaps the most significant single contributor to the group theoretic approach to geometry during the first half of the 20th century [22]. In view of this fact, we shall return below to briefly consider the background and nature of Cartan's geometrical work and its relation to the *Erlanger Programm.*

I now turn to the case of Eduard Study. During the late 19th and early 20th centuries, Study was the foremost contributor to the study of geometry in the sense of the *Erlanger Programm,* but Study's inspiration for carrying out such a program came through his contact with Lie and his school rather than through Klein or his *Programm.* By the time Study came into contact with Lie, he had already formulated a general research program for his career as a mathematican. Under the influence of Grassmann's *Ausdehnungslehre,* which he had digested as a university undergraduate, Study set himself the task of carrying out research aimed at providing geometrical knowledge with a suitable conceptual and algebraic framework [23]. Initially he was of the opinion that the *Ausdehnungslehre* provided an adequate basis for geometry, but by the time he received his doctorate from Munich in 1884 he had become convinced that such a basis was to be found rather in the theory of invariants and particularly in the symbolical method employed by Clebsch and Gordan. The theory of invariants thus became, and remained, at the center of Study's master plan for geometrical research.

In 1885 Study completed his *Habilitationsschrift* [1886] under the supervision of Klein at Leipzig. But neither Klein nor the other mathematicians at Jena, Strassburg, Munich, and Leipzig whom he had encountered during his predoctoral studies exerted any significant influence upon him and his master plan for research [Engel 1931, 173]. In the case of Klein this may appear surprising, especially since

in the concluding remarks of the *Erlanger Programm* Klein had indicated [1872, 488] that in the group theoretic study of a manifold the theory of invariants was the formal means for expressing geometrical properties (in Klein's sense). But Study's relationship with Klein was strained and distant. When in 1892 Klein complained to Study that many ideas in his *Programm* were now being put forward by Study as his own, Study pointed out in a letter to Klein that when they were together in Leipzig in 1885 and Study had expressed some of the ideas of his master plan to Klein, Klein never referred him to the *Erlanger Programm* and, at least in Study's mind, discouraged him from developing his ideas [24]. As a consequence, Study never even thought to examine Klein's publications for ideas congenial to his own research program. Study's letter to Klein also captured the fierce independence, bordering on eccentricity, with which he pursued his personal research goals [25]. The deepest cause of the discord that had characterized their relationship, Study opined, resulted from Klein's desire to treat him like his other students: "Ich hatte aber damals schon den Weg vorgerechnet, den ich nacher inne gehalten habe, und, soweit es in meinen Kräften steht, auch weiter verfolgen gedenke." Despite Study's staunch individualism he quickly became a functioning member of Lie's school for personal and intellectual reasons I shall now indicate briefly.

Study's first exposure to the ideas of Lie's theory of groups probably came through contact with Engel, who had returned to Leipzig from Norway in July 1885 after spending ten months there with Lie in preparation for writing a *Habilitationsschrift* on transformation groups [26]. Study, who was also preparing his *Habilitationsschrift*, quickly became a close personal friend. During the four semesters in 1885–1888 that Study spent as *Privatdozent* in Leipzig, he was constantly in the company of Engel [Engel 1931, 137]. Thus when Lie arrived in Leipzig in April of 1886, Study was already favorably disposed toward Lie's theory as a result of his association with Engel. Lie, for his part, took a personal liking to Study [27]. Their relationship was much more amicable than the one Study had with Klein. One reason was that Lie did not attempt to do what Klein had done. Seeing that Study was a geometer with considerable knowledge of, and commitment to, the theory of invariants, a subject far too algebraic for Lie to deal with himself, Lie encouraged Study to continue work on his master plan and to develop the fruits of the connections of invariant theory with his theory of groups [28]. As a further inducement, Lie asked Study to lecture on the theory of invariants in a manner that would bring out the connections with his theory of groups. This was to be for the benefit of Lie's better students and was to occur during the Summer Semester of 1887 [29]. Perhaps this was one reason why Study decided to spend the month of January 1887 in Erlangen, intensively studying invariant theory under the personal tutelage of Gordan.

By the time of his visit to Gordan, Study had clearly become a believer in Lie's theory of groups. This is evident from his letters to Engel from Erlangen which recount his proselytizing efforts to convince Gordan of the importance of Lie's theory [30]. Study also voiced his belief in the importance of Lie's theory in his review [1889b] of the first volume of *Theorie der Transformationsgruppen* [1888].

Here he endorsed Lie's view that his theory is justified in the first place by virtue of the envisioned Galois theory of differential equations. The application to differential equations thus required a penetrating study of transformation groups themselves [pp. 171–172]. Study also felt that such research would benefit mathematics in general. Like Lie, he believed that "es wenige Gebiete der mathematischen Wissenschaft geben wird, die nicht durch Anneigung der Grundgedanken der neuen Disciplin eine wesentliche Förderung erfahren könnten. Vor allem aber werden den nächtsverwandten Gebieten der Differentialgleichungen und der Geometrie eine Fülle neuer Ansichten und Aufgaben zugeführt" [p. 189]. What Study had in mind regarding geometry he explained in his treatise [1889a] on the invariant theory of ternary forms:

> Leibniz und auch Grassmann glaubten, dass es *einen* solchen Calcul gäbe, welcher aller geometrischen Gedanken mit gleicher Einfachheit und Vollendung zum Ausdruck brächte. Heute können wir sagen, dass dies nicht der Fall sein kann, dass das verschiedenartige Interesse, oder, um den Gedanken schärfer auszudrücken, dass die Geometrie verschiedener Transformationsgruppen auch verschiedenartige Hilfsmittel erfordert. Die Aufgabe wird jetzt, zunächst für die wichtigsten dieser Gruppen ein geeignetes Algorithmensystem, oder eine Invariantentheorie zu schaffen. [p. 202, n.7]

One can see why passages such as this prompted Klein to accuse Study of plagarism. It expressed essentially the same vision of geometry as the *Erlanger Programm*. But Study arrived at it through the influence of Lie and his school and sought to carry it out with the tools he had learned there.

Thus between 1872 and 1890, while the *Erlanger Programm* remained relatively unknown and inaccessible, Lie developed his monumental theory of continuous groups; and the idea that geometry should be regarded in group theoretic terms was arrived at independently by Poincaré, Killing, and Study, all of whom were influenced in one way or another by Lie and his theory. In view of these developments, Klein realized that the time had come to give a wider circulation to his *Programm*. As he explained in 1892:

> My 1872 Programme, appearing as a separate publication . . . had but a limited circulation at first. With this I could be satisfied more easily, as the views developed in the Programme could not be expected at first to receive much attention. But now that the general development of mathematics has taken, in the meanwhile, the direction corresponding precisely to these views, and particularly since Lie has begun to publish in extended form his *Theorie der Transformationsgruppen* . . . it seems proper to give a wider circulation to the expositions in my Programme. [1893a, 215]

Translations of the *Programm* were quickly published in Italian [1890], French [1891] and English [1893a], and in 1893 Klein had the German original reprinted with some additional footnotes in the pages of *Mathematische Annalen*. During the academic year 1892–1893 Klein also presented lectures on higher geometry which provided him with the occasion to expound the contents of the *Erlanger Programm* in greater detail and in conjunction with the requisite geometrical background material. A handwritten copy of these lectures was published in lithograph form in 1893, although they were not given a wider circulation until

1926 when they were published, in part, by Blaschke [31]. Finally, it seems safe to assume that Klein was the prime mover behind the decision to include the article "Kontinuierliche geometrische Gruppen. Die Gruppentheorie als geometrisches Einteilungsprinzip" in the *Encyklopädie der mathematischen Wissenschaften* [Fano 1907].

Undoubtedly these efforts on Klein's part had some effect, but that was because the *Erlanger Programm* was read in the light of Lie's theory of transformation groups. Furthermore, Lie's influence upon Poincaré, Killing, and Study which was independent of any influence of the *Erlanger Programm* also had a lasting effect upon the subsequent development of the group theoretic approach to geometry, an effect that cannot be ignored when considering the impact of the ideas of the *Erlanger Programm*. It is consequently impossible to extricate the influence of the *Erlanger Programm* from the influence of Lie and his school, although it would seem that the role of Lie's work was more substantial. In order to illustrate these points I shall conclude by briefly considering the geometrical work of Fano in Italy, Study in Germany, and Cartan in France.

One of the relatively few mathematicians who were familiar with the *Erlanger Programm* before the 1890s was Corrado Segre. Segre had obtained his doctorate in 1883 from the University of Turin at the age of 20. By that time he was already in correspondence with Klein, through whom he had submitted a paper to *Mathematische Annalen;* and by 1885 he was familiar with the contents of Klein's *Programm* [32]. In 1888 Segre assumed the Chair of Geometry at the University of Turin. That was also the year that the first volume of Lie's *Theorie der Transformationsgruppen* appeared, and Segre decided that the time was ripe to encourage an Italian translation of the *Erlanger Programm,* especially since young Italian geometers did not seem to be acquainted with its contents. One of Segre's students at Turin, Gino Fano (1872–1952) undertook the task of a translation, which was published in 1890. During the next ten years, both Fano and another young Italian geometer, Federigo Enriques (1871–1946) published several papers, some jointly, relating Lie's theory of groups to geometry [33]. Many of these papers were devoted to the study of algebraic surfaces or hypersurfaces left invariant by some Lie subgroup of $SL(n,C)$. The rationale behind this line of research was explained by Enriques in one of his first papers [1893, 1590–1591]: Klein in his *Erlanger Programm* had introduced the idea of geometry as the study of the properties of figures left invariant by all the transformations of the *Hauptgruppe*—the group G defining the geometry. Thus the most important classes of configurations from Klein's viewpoint were those forming fields, viz., those mapped into themselves by G. However, in the concluding remarks of the *Programm,* in order to bring his joint work with Lie on W curves [1870, 1871] within its framework, Klein added:

> Bei der Behandlung einer Mannigfaltigkeit unter Zugrundelegung einer Gruppe fragen wir entsprechend zunächst nach den Körpern (§5), nach den Gebilde die durch alle Transformationen der Gruppe ungeändert bleiben. Aber es gibt Gebilde, welche nicht alle aber einige Transformationen der Gruppe zulassen, und diese sind dann im Sinne der auf die Gruppe gegründeten Behandlung besonders interessant. . . . [1872, 489]

Paraphrasing this remark, and noting the great advances that had been made by Lie on the subject of continuous groups in the intervening twenty years, Enriques proposed for the complex projective geometry of space ($G = SL(4,C)$) to determine algebraic surfaces left invariant by some finite dimensional Lie subgroup of $SL(4,C)$.

Thus Enriques, and also Fano, devoted a large part of their group theoretic geometrical research to a problem that was tangential to the main thrust of the *Erlanger Programm,* a problem that was just as much, and perhaps more, of Lie's devising than Klein's. In fact, unbeknownst to Enriques, Lie had already investigated it [34]. It was the lack of knowledge of continuous transformation groups that had discouraged immediate sequels to the papers on W curves, but Lie's theory of such groups now encouraged them. This point was made explicitly by Lie in his first paper studying surfaces left invariant by continuous transformation groups [1882, 179–180]. I shall consider one of the more interesting papers written by Fano on this theme in order to illustrate the point that these papers were inspired as much by the contributions of Lie and his school as they were by the *Erlanger Programm.*

After spending the academic year 1893–1894 in Göttingen with Klein, Fano returned to Italy and, following the lead of Enriques, commenced publishing on the application of Lie's theory to geometry. In two memoirs [1896a, b] he considered the problem of determining $(n - 1)$-dimensional algebraic varieties in n-dimensional space which admit a continuous group of projective transformations, with special attention accorded to the case $n = 4$. Fano rejected the methods that had been employed by Lie and Enriques in the case $n = 3$ as unsuitable for the extended problem because they involved extensive calculations lacking any conceptual coherence [1896a, 188]. At least for nonsolvable groups he was able to devise a conceptually simple yet general method based on a theorem that Study had discovered while studying the theory of invariants under the influence of Lie and his school [35]. Stated in modern terms, it is the complete reducibility theorem for $sl(2,C)$ [36].

Fano's method proceeded as follows. Let $L \subset sl(n + 1,C)$ be nonsolvable. Then by a theorem due to Engel [1893], L necessarily contains a subalgebra M with the structure of $sl(2,C)$. Since an algebraic variety left invariant by L is *a fortiori* left invariant by M as well, the sought-for varieties are contained among those left invariant by a projective group M with the structure of $sl(2,C)$. Fano therefore sought to determine this class of varieties. Study's complete reducibility theorem showed that $C_{n+1} = \Sigma V_i$, where each $(h_i + 1)$-dimensional vector space V_i is left invariant by M and the restriction M_i of M to V_i "leaves nothing planar invariant," that is, leaves no proper vector subspaces invariant. Fano realized that each M_i with $h_i \neq 0$ is the group which indicates how the coefficients of the general binary form of degree h_i are transformed when its variables are subjected to a linear transformation. (This is the irreducible representation of $sl(2,C)$ of highest weight h_i.) Thus Fano's problem reduced to the problem of determining the invariants of these forms. In the case $n = 4$ this problem is manageable since $\Sigma(h_i + 1) = 5$.

Fano stressed the noteworthiness of this connection between his purely geometrical problem and the fundamental problem of the theory of invariants.

Two points about Fano's work are particularly relevant to our discussion. The first is that the problem dealt with by Fano was not a central concern of the *Erlanger Programm* but was analogous to the problem considered by Lie and Klein in 1870 and 1871. Indeed, as noted above, Lie had himself dealt with its generalization to surfaces. The second is that Fano's work relied critically upon the concepts and results of Lie's school, such as the concept and properties of solvable Lie algebras (e.g., Lie's theorem on solvable algebras), Study's complete reducibility theorem, and Engel's characterization of nonsolvable Lie algebras. Without them his paper would not have been conceivable. Thus, even in the case of a mathematician such as Fano who studied the *Erlanger Programm* and had been in direct contact with Klein, it is impossible to extricate his achievements from the influence of the work of Lie and his school.

In addition to several papers on continuous groups and group theoretic aspects of geometry, Fano wrote the article "Kontinuierliche geometrische Gruppen. Die Gruppentheorie als Einteilungsprinzip" for the *Encyklopädie der mathematischen Wissenschaften* [1907]. As the title suggests, the article was devoted to those mathematical developments which could be regarded as falling within the purview of the *Erlanger Programm*. The article thus has a special significance for the present essay since it is here that we should find fairly complete information on how the group theoretic approach to geometry was developed, and by whom, during the thirty-five years since publication of the *Erlanger Programm* and, in particular, during the years since its republication and translation in the early 1890s. What we learn is that the principal contributors in this period to a group theoretic study of geometry were three Italians, Enriques, Fano, and Ugo Amaldi; an American, H. B. Newson; and Eduard Study.

We have already discussed a portion of the work of Enriques and Fano. The other major area of interest was the classification of finite-dimensional Lie groups of birational or Cremona transformations, a matter even more in the spirit of goal (1) of Lie's research program as discussed above [37]. The same can be said of the work of Amaldi [1901, 1905, 1908], who considered problems suggested directly by Lie's work. The work of Newson, a professor at the University of Kansas, was also inspired by Lie's research [38]. In his *Vorlesungen ueber continuierliche Transformationsgruppen* [1893b], Lie had determined all projective groups in one and two variables (i.e., all Lie subgroups of $SL(n,C)$, $n = 2,3$). Newson set as his goal to rederive Lie's results by purely synthetic methods on the grounds that "in the domain of Projective Geometry, analytic and synthetic methods are mutually supplementary" [1895–1896, 71]. Thus, although Newson stressed that his work was "in no sense to be regarded as a contribution to Lie's theory of continuous groups" [1902, 109], it was motivated by it rather than by the central concerns of the *Erlanger Programm*. Newson was, however, familiar with that work and in fact quoted from Klein's Note I ("Ueber den Gegensatz der synthetischen und analytischen Richtung in der neueren Geometrie") to justify redoing Lie's results

synthetically [39]. Newson's work was rather eccentric and inconsequential, but it further illustrates how inextricable are the influences of Klein's *Programm* and the work of Lie.

Fano's article [1907] left no doubt as to who the major contributor to geometry in the sense of the *Erlanger Programm* was. Nine sections of [1907] accounting for 22% of the text, are devoted to a description of Study's work. Whereas the results of the above-mentioned mathematicians are accorded a few sentences or perhaps merely a footnote, Study's geometry of elements of second order in the plane (§16), his geometries of dual and radial projectivities (§§17–19), his projective and pseudoconformal geometries of somas (§20) and his systematic group and invariant theoretic treatment of problems in elementary geometry (§§38–39) and the projective geometry of quadric surfaces (§§41–42) are accorded the attention they rightly deserved [40]. The same is not true of Study's paper [1905] on "Hermitean geometries," possibly because Fano did not have sufficient time to digest its contents. Klein had shown that the standard non-Euclidean geometries could be described by means of a metric defined in terms of a real nondegenerate quadratic form. Study observed that further real geometries can be created by defining a metric in terms of a nondegenerate Hermitean symmetric form. The *Hauptgruppe* of such a geometry consists of all projective transformations (i.e., members of $SL(n,C)$) which leave the form invariant. Because it corresponded by analogy to elliptic geometry, the geometry associated with the form $\Sigma z_k \bar{z}_k$ was accorded special attention by Study. In this case the group is $SU(n)$. Study realized that all the real groups associated with a Hermitean geometry were simple and that, as *complex* groups, they all had the structure of $SL(n,C)$ [1905, 365]. Study's work on Hermitean geometries was a source of motivation for Cartan's research into the structure and representation of real Lie algebras (discussed below) [41].

Study, more than anyone else we have discussed, pursued original, genuinely geometrical research in the spirit of the *Erlanger Programm*. The prominent place accorded his work in Fano's article [1907] undoubtedly proved stimulating to other mathematicians looking for research problems. One of these was Wilhelm Blaschke (1885–1962), whose doctoral thesis (1908) under the supervision of Wirtinger at the University of Vienna dealt with Study's geometry of "spears" [*Speere*] [42]. Blaschke then spent the academic year 1908–1909 studying with Study at the University of Bonn, where Study was Full Professor. After spending the winter semester 1909–1910 in Pisa (to study with Bianchi) and the summer semester 1910 in Göttingen (the chief attractions being Hilbert and Klein), Blaschke returned to Bonn, and Study, to compose his *Habilitationsschrift* [1910b], which was also on the geometry of spears. Blaschke left Bonn for Greifswald in 1911 because Engel, who was Full Professor there, had obtained a lectureship for him. Thus during the formative years of his career, Blaschke, who went on to become a leading exponent of geometry in the spirit of the *Erlanger Programm*, was in close contact with Lie's students—including Kowalewski, who was also at Bonn. The influence of Lie's mathematics is evident in Blaschke's papers. Blaschke was also deeply influenced by his contact with Study and his work [43].

Even Blaschke's pioneering work on differential geometry, which culminated in his *Vorlesungen ueber Differentialgeometrie* [1921, 1923, 1929] was inspired by the approach to geometry advocated and practiced by Study, to whom Blaschke dedicated the first volume of his *Vorlesungen* [44]. As he explained in the foreword:

> Als ich 1908 nach Bonn kamm, hat mich E. Study trotz seiner schweren Erkrankung seines persönlichen Unterrichts gewürdigt und mir die kritischen Untersuchungen zur Differentialgeometrie vorgetragen, an denen er damals arbeitete. In dankbaren Gedenken an die schönen Bonner Tage sei Herrn Study dieses Bändchen gewidmet. [1921, vii]

Study published some of his "critical investigations" on differential geometry in [1909]. They display the same group and invariant theoretic orientation characteristic of his other geometrical work and undoubtedly provided further encouragement to Blaschke to pursue differential geometry in the same spirit. Thus while Blaschke wrote in the foreword that Klein's *Erlanger Programm* would serve as his pole star [1921, vii] and, in the foreword to the second volume [1923], paid homage to Klein since "[v]on ihm stammt die auf dem Begriff der stetigen Transformationsgruppen beruhende geometrische Denkart, die allem Folgenden zugrunde liegt," it was in reality Study and his work that had caused him to appreciate the value of such a "geometrische Denkart." (Blaschke was apparently unaware that Study had arrived at such a "geometrische Denkart" on his own under the influence of Lie and Engel.)

Study had other students as well who, although not as distinguished and influential as Blaschke, contributed to the promulgation of geometry in the spirit of the *Erlanger Programm:* for example, J. L. Coolidge, whose books [1909, 1916, 1924] conveyed many of Study's ideas and results, and Hans Beck, whose *Koordinatengeometrie* [1919] was the first textbook that sought to consistently develop geometry in the sense of the *Erlanger Programm* [45]. Study's students understandably hailed the *Erlanger Programm* as the earliest statement and symbol of geometry as they practiced it. But their actual work along those lines was inspired by Study's influence and contributions. Beck summed up the situation well in the preface to his book:

> Für die Koordinatengeometrie wurde erste neue Gedanke nach langer Zeit 1872 von F. Klein in seinem Erlanger Programm ausgesprochen. . . . Klein erkannte wie sich die unübersehbare Fülle geometrischer Einzelerscheinungen in eine Reihe von Systemen einordnen lässt. Damit wurden die Invariantentheorie und die Lehre von den Transformationsgruppen Hilfswissenschaften der analytischen Geometrie. . . . In der Lehrbücher haben diese Gedanken Kleins ein Echo bisher nur gefunden im. . . [Heffter & Koehler 1905]. Auch auf das inhaltsreiche Werke von Clebsch-Lindemann [1906, 1910] sind ohne merklichen Einfluss geblieben und ebenso lange Zeit auf die Forschung, da Klein selbst nicht allzuviel zu Verbreitung seiner Ideen getan hat. Erst die letzten zwanzig Jahre etwa haben, dank der Arbeiten Studys und seiner Schüler, so viel aus der Sache herausgeholt, dass heute von einem gewissen Abschluss, wenigstens in der Gebiete der Geometrie der Ebene, gesprochen werden kann. [1919, iii–iv]

According to Beck, there was currently no textbook in any language that expounded the preceding half-century of group theoretic geometrical research. His

book was intended as a contribution toward filling that gap. Thus it was Study and his students who made the sort of vague ideas found in the *Erlanger Programm* into a viable, multifaceted research program and who presented the results of that program to the mathematical world in the form of textbooks and treatises.

While Study was perhaps the most significant contributor to geometry in the spirit of the *Erlanger Programm* at the turn of the century, Élie Cartan was the major architect of a group theoretic approach to geometry during the first half of the 20th century. We now consider the nature of his geometrical work and its relation to the *Erlanger Programm* and other developments discussed in the preceding pages.

When a French translation of the *Erlanger Programm* was published in the *Annales* of the École Normale in 1891, Klein expressed his delight at the translation into French since "actuellement, la théorie des groupes semble, plus que jamais, occuper en France l'attention, et que, par suite, le contenu de mon programme y excitera peut-être quelque intérêt" [1891, 87, n. 1]. Klein had in mind the activity in Paris, by Poincaré and Picard and many graduates of the École Normale, which focused on applications of Lie's theory. As can be seen from Fano's article [1907], however, the appearance of the *Erlanger Programm* did not generate any research activity in Paris. There interest on group theoretic matters tended to be focused for a long time on applications of Lie's theory to differential equations, in accordance with Lie's own emphasis at the time. Cartan's work during the period 1894–1910 was typical in this respect. His thesis work on Killing's classification of simple structures had, as he emphasized, a direct bearing on applications of Lie's theory to differential equations. His subsequent work on "infinite continuous transformation groups" was also motivated by applications to differential equations. It led to a new approach to structure and to Cartan's classification of infinite continuous simple groups. What distinguished Cartan's work from that of other *normaliens* was the emphasis accorded to structural considerations, which he sought to show were valuable in applications.

Perhaps another reason why the *Erlanger Programm* stirred up so little interest in France was that it did not seem to have any affinity with the type of differential geometry and kinematics cultivated by Darboux and his students, who dominated the teaching positions at the École Normale. Geometry was thus a vital ingredient in the mathematical education of Cartan and other *normaliens*, but it was a different sort of geometry. Thus it is not surprising that when Lie's theory was eventually brought to bear upon geometrical matters, the motivation came from geometry in the sense of Darboux rather than from the *Erlanger Programm*. The first *normalien* to do this was Émile Cotton (b. 1872). In his thesis [1900], which was dedicated to Darboux and Goursat, Cotton studied three-dimensional manifolds which are left invariant by some transformation group in the sense of Lie and obtained canonical forms for the line element ds^2. A few years later he made a more far reaching observation on the relation of Lie's theory to differential geometry. Darboux had introduced the method of moving frames of reference to study the infinitesimal properties of curves and surfaces in Euclidean space. In a note in

the *Comptes rendus* of the Paris Academy [1905] the Belgian mathematician A. Demoulin pointed out that a similar method could be introduced into "Géométrie cayleyenne"—the metric geometries introduced in projective space by means of a quadratic form (the absolute)—thereby making possible the extension of Darboux's method to the study of these geometries, such as hyperbolic and elliptic geometry. Demoulin's point of view was not group theoretic, but his remarks were viewed in this light by Cotton, who observed that the method of moving frames could be generalized by replacing the group of Euclidean displacements by any transitive finite-dimensional transformation group [Cotton 1905]. Cotton's paper thus made Lie's theory of groups geometrically appealing by relating it to the strong interest in infinitesimal geometry and kinematics within the Darboux school. In addition, Cotton utilized group theoretic work on differential equations by Tresse and Vessiot and provided what he regarded as a more intuitive presentation of some of Vessiot's results on differential equations.

"J'espére," Cotton wrote, "que le present travail facilitera l'utilisation des groups (autres que celui des mouvements) qui se présentent naturallement dans questions classiques. . ." [1905, 43]. His work did far more than he anticipated, for it provided the fillip for Cartan's grand synthesis of his newly developed approach to the structure of groups with the geometrical and kinematical perspective of the Darboux school. Having been educated in this school, Cartan evidently took great satisfaction in this opportunity to relate the theory of structure to the mathematics he had learned as a student:

> En présentent les choses d'un point de vue cinématique, on arrive à reconnaître la généralisation d'une théorie classique, celle de trièdre mobile due à M. Darboux. Dans le cas particulier ou le groupe est celui des déplacements de l'espace . . . [l]es formules de structure coïncident alors tout simplement avec les équations aux dérivées partielles du premiere ordre que M. Darboux a rendues classiques dans la théorie des déplacements à plusieures paramètres. [1910b, 251]

Cartan's papers [1910a, b] relating structure and moving frames inaugurated a new phase in his career as a mathematician. From now on geometrical considerations were to play a far more dominant role in his research. As in the papers of 1910, the interests of the Darboux school, particularly differential geometry, were central in this phase of his career, although his own originality and penchant for structural considerations continued to set his work apart from that of his colleagues. From his work on moving frames, Cartan went on to develop his theory of generalized spaces (1922–) in response to Einstein's general theory of relativity and then his theory of symmetric spaces (1926–). These investigations by Cartan led him to a group theoretic conception of geometry which embraced within its scope not only geometry in the sense of the *Erlanger Programm*—geometry in the sense of Poincaré and Klein, as Cartan sometimes referred to it [46]—but also Riemannian geometry and related theories, such as Levi-Civita's parallel transport, all of which were outside the purview of the *Erlanger Programm*. Cartan articulated a research program in geometry which effectively supplanted the *Erlanger Programm*; and his program involved the theory of Lie groups in a more

profound way than had geometry in the sense of Poincaré and Klein [47]. He thus had good reason to conclude, in an address, "Le rôle de la théorie de groupes de Lie dans l'évolution de la géométrie moderne," that

> rien ne montre mieux que ces rapproachments entre tant de théories [géométriques] en apparence si diverses la puissance d'unification et de création qu'apporte la théorie des groupes, due aux génie du grand géomètre norvégien, S. Lie. [1936, 103]

Although Cartan's geometrical work was largely within a framework of differential geometry, he also concerned himself with some aspects of geometry in the sense of Poincaré and Klein. He was of course familiar with Poincaré's views on the importance of groups for geometry, but early in the "geometrical" phase of his career he also digested the contents of Fano's *Encyklopädie* article [1907] in the course of preparing a revision [Cartan, 1915] for inclusion in the *Encyclopédie des sciences mathématiques* [48]. Part of the motivation for his papers on the representation of real and complex semisimple Lie algebras arose from this encounter with geometry in the sense of the *Erlanger Programm*. This aspect of Cartan's mathematics warrants further discussion here because it illustrates very well the tangled web of relationships linking the *Erlanger Programm* with developments emanating from Lie and his school.

It is evident that Cartan put considerable time and thought into the revision of Fano's article. Whereas Fano's article was 97 pages long, Cartan's ran to 135 pages; and much of the new material originated with him. The emphasis upon Lie groups was also even greater. Whereas Fano explained to his readers that Lie's theory would be continually used in his article [1907, 294], Cartan declared: "Le présent article n'est que l'exposé des applications à géométrie de la théorie des groupes continues de S. Lie" [1915, 4]. Among new applications that Cartan included were those to infinitesimal geometry facilitated by the group theoretic generalization of Darboux's moving frames and those involving infinite-dimensional groups. Cartan also greatly expanded Fano's treatment of projective groups. For example, he showed how Fano's rendition [1907, §§28–29] of Klein's Principle of Transfer (discussed below) could be used to construct new projective groups with a given structure from given ones, thereby generating geometries equivalent to the ones defined by the original groups [1915, 99–105]. These constructions are linked to Cartan's solution [1913] to the problem of determining all projective groups which "leave nothing planar invariant." Expressed in modern terms, this is the problem of determining all faithful irreducible representations of semisimple structures. Here it will suffice to discuss the geometrical motivation behind this problem which relates to Cartan's interpretation of Klein's notion of equivalent geometries and its implications.

Central to Klein's classification of geometries—in the *Erlanger Programm* as well as in Fano's exposition [1907]—was the idea that geometries which *prima facie* were distinct could actually turn out to be "equivalent" [*gleichbedeutend*]. Equivalence was established by means of a biunique mapping ϕ from one space M on which a group G acted (thereby defining a geometry in Klein's sense) to

another space M'. By means of ϕ, a group G' was made to act upon M', thereby defining another geometry. The original geometry was thereby "transferred" to the new setting by the mapping ϕ, and this procedure became known as Klein's Principle of Transfer [*Uebertragungsprinzip*]. In symbols: for any T in G the corresponding T' in G' is $\phi T\phi^{-1}$. In considering the geometry induced upon a space by a group, however, it was also important to realize that the spatial elements themselves need not be points. Plücker's work had inspired the idea that the spatial elements of a geometry could be lines or planes or ellipses or some more general type of configuration. The sole restriction on the choice of space element was that their totality form a "field" [*Körper*], a word Klein borrowed from Dedekind. Klein's own definition of a field is of interest in the light of Cartan's interpretation of equivalence:

> So wähle man ein einzelnes räumliches Gebilde . . . aus und wende auf dasselbe alle Transformationen der [Gruppe] an. Man erhält dann eine mehrfach unendliche Mannigfaltigkeit mit einer Anzahl von Dimensionen, die im allgemein gleich der Zahl der in der Gruppe enthaltenden willkürlichen Parameter ist, die in besonderen Fällen herabsinkt, wenn nämlich das ursprünglich gewählt Gebilde die Eigenschaften besitzt, durch unendlich viele Transformationen der Gruppe in sich übergeführt zu werden. Jede so erzeugte Mannigfaltigkeit heisse mit Bezug auf die erzeugende Gruppe ein Körper. [1872, 473]

A group which, say, originally acts upon points induces a group action as well upon other space elements comprising a field. According to Klein—in a passage essentially quoted by Fano [1907, 297]—the content of the geometry is not changed by a change in space element. That is,

> jeder Satz der bei *einer* Annahme des Raumelements sich ergab, ist auch ein Satz bei beliebiger anderer Annahme, nur die Anordnung und Verknüpfung der Sätze ist geändert. Das Wesentliche ist also die Transformationsgruppe, die Zahl der Dimensionen, die wir einer Mannigfaltigkeit beilegen wollen, erscheint als etwas Sekundäres. [1872, 471]

Cartan carried Klein's viewpoint one step further. He concluded that, in effect, equivalent geometries correspond precisely to groups with the same structure [1915, 99] [49]. Cartan referred to his paper on moving frames [1910b] as documenting this fact. There Cartan made the following observations [1910b, 254 ff.]. Suppose G is a continuous r parameter group of transformations T_a with equations $x_k' = f_k(x_1, \ldots, x_n; a_1, \ldots, a_r)$, $k = 1, \ldots, n$. According to Cartan, a configuration $\mathcal{F}$ always exists such that $T_a\mathcal{F} \neq \mathcal{F}$ for all $T_a \neq I$. If, e.g., G is projective, we may take for F an orthonormal basis (with order of elements of course counting as part of the definition of $\mathcal{F}$). This is the n-dimensional analog of Darboux's *trièdre mobile*. A field then arises by considering all $\mathcal{F}_a = T_a\mathcal{F}$, where T_a runs through G. (Cartan himself did not speak of fields in [1910b] but rather of reference frames. When he wrote [1910b] he may not yet have read [Fano 1907] or the *Erlanger Programm*.) Now the action defined by G on this field defines what Lie called the parameter group $H = \{P_b\}$ of G. P_b sends $\mathcal{F}_a$ into $\mathcal{F}_{a'}$ where $\mathcal{F}_{a'} = T_b\mathcal{F}_a = T_bT_a\mathcal{F} = T_{a'}\mathcal{F}$. The geometry of G acting on points and, in effect, its parameter group acting on the space elements $\mathcal{F}_a$ are essentially the same in the

Kleinian viewpoint. The same procedure may be applied to another group G^* and leads to its parameter group acting upon some field of configurations $\mathfrak{F}^*_{a^*}$, $a^* = (a^*_1, \ldots, a^*_s)$. Now Lie had proved [1888, 414] that two groups have the same structure if and only if their parameter groups can be made to coincide by a suitable transformation of parameters. This implied that the geometries associated with G and G^*, respectively, were equivalent if and only if these groups had the same structure. For Cartan, the essence of a geometry was determined by its structure, its diverse manifestations by the representations of that structure. In particular, to determine equivalent projective geometries was to determine all the (inequivalent) linear representations in $SL(n,C)$ (or $SL(n,R)$) of the underlying structure determined by these complex (or real) geometries. The problem of determining all projective geometries was deemed important by Cartan because every known geometry was equivalent to a projective geometry, and he conjectured this was true in general [50].

Thus, from Cartan's perspective, the theory and representation of real and complex Lie algebra structures was of considerable importance to the study of geometry in the sense of Poincaré and Klein, a point not usually made in discussions of Cartan's work on representation theory. But that is not the whole story behind the making of Cartan's trilogy of memoirs on the representation of complex semisimple structures [1913] and on the structure and representation of real semisimple structures [1914a, b]. While it is true that they—or at least [1914a] and [1914b]—were written with geometrical applications in mind [51], there were other sources of motivation as well, especially with regard to the memoir [1913], which represents the keystone of the triolgy. Both these sources and the principal tools employed by Cartan in the memoirs—the Killing–Cartan theory of structure and the theory of weights—originated with work done within Lie's school bearing no relation to the *Erlanger Programm*.

Killing's attempt to classify all Lie algebra structures as a part of his "Braunsberger Programm" had led him to study the properties of the secondary roots [*Nebenwurzeln*] of the characteristic polynomial associated with a Lie algebra. In this connection, he proposed a problem [1889, 116], the resolution of which, he claimed, would constitute the major step en route to a complete classification of structures. An exposition of Killing's theory of secondary roots and its connections with his problem will be given elsewhere [52]. For the purpose at hand it suffices to realize that Killing's secondary roots were essentially the weights of the s-module r, where $g = s + r$, s semisimple, $r = \mathrm{rad}(g)$, in accordance with the radical splitting theorem. In his doctoral thesis [1894], with an eye toward applications to differential equations, Cartan corrected and perfected Killing's theory of secondary roots. He also observed that Killing's problem was equivalent to the following: For each semisimple structure determine all Lie algebras of linear homogeneous transformations with the property that they leave nothing planar invariant. We shall also refer to Cartan's reformulation as Killing's problem.

In his thesis Cartan solved Killing's problem in a special case that was directly relevant to differential equations. For each simple structure, he determined the

linear group of the type described above in the smallest possible number of variables. Later his work on the classification of infinite Lie groups of simple structure, which was also motivated by the prospects of applications to differential equations, required him to return to secondary roots and Killing's problem and to solve it in certain special cases dictated by his classification problem [1909]. By this time he was also aware of some of Kowalewski's work on the determination of primitive Lie groups [1899] which led Kowalewski to formulate the problem of determining all projective groups which leave nothing planar invariant [53]. In [1909] Cartan had established a theorem which implied that the projective groups in Kowalewski's problem were necessarily semisimple, so that Kowalewski's problem was equivalent to Killing's.

Cartan was also aware of Study's work on projective Lie groups which grew out of Lie's encouragement of Study to combine the theory of invariants with his theory of transformation groups. Because Study never published his results on the grounds that they were incomplete, Lie, who sensed their potential, communicated them in the third volume of his *Theorie der Transormationsgruppen* [1893a, 785–788]. Study's work was inspired by the theory of invariants and particularly by the method of series expansions [*Reihenentwickelungen*] of Clebsch and Gordan [54]. By formulating their results in a geometrical and group theoretic manner, Study turned them into what we would now describe as complete reducibility theorems for certain representations of $SL(2,C)$ and $SL(3,C)$. Moreover, on that basis he conjectured privately to Lie that a complete reducibility theorem would probably hold for all representations of any semisimple structure. Although in print Study (or Lie) limited the conjecture to $SL(n,C)$, the possibility of an extension to any semisimple structure would have been an obvious consideration for anyone, such as Cartan, acquainted with the theory of structure; and in fact Cartan did formulate such a conjecture [55]. The general conjecture also brought with it another problem which Study also considered. That is, if a projective group with a given structure can be decomposed into projective groups which leave nothing planar invariant, then the obvious problem is to seek to determine all of this type of group. Expressed in modern terms, it is the problem of determining for a given structure all its faithful irreducible representations. Although Study's results on this problem were limited in scope, Cartan perceived its identity with the problems of Killing and Kowalewski.

Thus by 1910, when Cartan's mathematical interests turned more and more in the direction of geometry, he had many reasons for wishing to determine, in effect, all faithful representations of complex semisimple structures. And the theory of weights by means of which he accomplished this objective was an elaboration of the theory of secondary roots which had its origins in Killing's papers on the structure of Lie algebras, papers which emerged when Killing developed the mathematical ideas of his "Braunsberger Programm" along the lines encouraged by Lie's vision of a Galois theory of differential equations.

Having now described, albeit superfically, some of the pieces of that fascinating historical collage to which the *Erlanger Programm* belongs, we leave it to the reader to draw his own conclusions regarding its significance and influence. We

have attempted to show that the view that it can be meaningfully regarded as one of the most significant and influential documents in the history of mathematics circa 1872 to 1922 is overly simplistic and essentially unhistorical. Since mathematicians tend to respect priority, the *Erlanger Programm* was pinpointed as the source of the group theoretic study of geometry, but it should be clear from what has been presented here that such a view, if interpreted as a historical statement of influence, ignores the contributions of Lie and his school. That would be a serious mistake since, as we have seen, no significant development of the ideas of the *Erlanger Programm* occurred without the involvement of the ideas and results of Lie and his school. It would, of course, be equally simplistic and unhistorical to deny to the *Erlanger Programm* any significance or influence. Our point is that a more balanced and genuinely historical understanding of the *Erlanger Programm* and its place in the history of mathematics will emerge when it is viewed within the broader context of that collage. Along with such an understanding comes a deeper understanding of the collage itself, which has a far greater intrinsic value. Much still needs to be done along these lines. The geometrical work of Study is certainly worthy of further attention. The same can be said with even greater emphasis of the geometrical work of Cartan. And so on. This essay will have accomplished its objective if it has succeeded in conveying some sense of the fascinating wonders of this collage and if it has whetted the appetite of others to explore it more fully.

NOTES

1. See also [Caratheodory 1919, 48, 51; Kline 1972, 920; Russo 1968, 5–6; Veblen 1929, 181–182; Burau & Schoeneberg 1973, 397]. Klein himself, in his lectures on the development of mathematics in the 19th century, described the *Erlanger Programm* more modestly as belonging to that class of writings which "seek to inspire something new by bringing order to what is already at hand" [1926, 28].

2. For example, [Russo 1968; Wussing 1969, 123–143].

3. The results of this investigation are contained in [1980, 1982] and in an essay currently being readied for publication entitled "Élie Cartan and the Prehistory of the Representation Theory of Lie Algebras." I gratefully acknowledge the support of the National Science Foundation.

4. Here "manifold" is used in the 19th-century sense of *Mannigfaltigkeit*, meaning a mathematical object which could be assigned coordinates $(x_1, \ldots, x_n)$. The viewpoint was (unconsciously) strictly local with no concern for coordinate patches and how they fit together. On the history of the concept of a manifold in the 19th century, see [Scholz 1980].

5. In his lectures on higher geometry, after listing diverse applications of "geometrical groups" (groups defined for all points of some "continuous manifold"), Klein wrote: "In Anbetracht dessen verlangen wir hier und verlangten Lie und ich im Jahre 1872 in den Schlussbemerkungen meines Erlanger Programmes ausdrücklich eine Gruppentheorie, d.h. eine rationelle Aufzählung und Classification aller geometrischen Gruppen, die es überhaupt gibt" [1893c, 52].

6. "Die Transformationen einer Gruppe brauchen übrigens durchaus nicht, wie das bei den im Texte zu namenden Gruppen allerdings immer der Fall sein wird, in stetiger Aufeinanderfolge vorhanden zu sein. Eine Gruppe bildet z.B. auch die endliche Reihe von Bewegungen, die einem regelmässigen Körper mit sich selbst zur Deckung bringen, oder die unendliche, aber diskrete Reihe, welche eine Sinuslinie sich selber suponiert" [1872, 462, n. 10].

7. See Klein's remarks [1921, 412; 1923b, 17].

8. These lectures were published in lithograph form as [Klein 1892]. They contain no consideration of groups of any kind.

9. On Klein's work in these areas see [Gray 1982a, 1984]; [Wussing 1969].

10. See Klein's comments in his lectures on higher geometry [1893c, 27–28].

11. At the conclusion of his lectures on higher geometry Klein stated: "Wenn wir nun zurückblicken auf alles das, was wir in den beiden letzten Semestern zusammen durchlaufen haben, dann tritt gewiss hervor, dass wir uns um die Abgrenzung der einzelnen mathematischen Disciplin nicht gekümmert haben. Zahlentheorie u. Differentialgleichungen, Functionentheorie, projective Geometrie, Invarianten u.s.w., alle diese Gebiete sind in gleichen Weise herangezogen worden. Hierin ruht gerade das Charakteristische der gegenwärtigen Vorlesung, (und vielleicht meiner Vorlesungen überhaupt) dass diese einzelnen mathematischen Disciplinen in ihrer Wechselbeziehung zu einander zur Geltung gebracht werden. Demgemäss steht aber unser Weg, den wir eingeschlagen haben, im Gegensatze zu den von sehr vielen Mathematikern gegebenen Weisung, dass man sich concentriren müsse, um etwas leisten zu können" [1893c, 385–386].

12. The closest Klein came to stating such a vision was in the following remark by Klein which seems to have been inspired by Lie's work: "Bei der Auffassung der Geometrie, wie sie hier zugrunde gelegt ist, kann es gleichgültig sein, wenn diese Gebilde seither nicht sowohl als geometrische, sondern nur als analystische betrachtet wurden, die gelegentlich geometrische Anwendung fanden, und wenn man bei ihrer Untersuchung Prozesse anwandte . . . , die man erst in neuerer Zeit bewusst als geometrische Umformungen aufzufassen begonnen hat. Unter diese analytische Gebilde gehören vor allen die homogenen Differentialausdrücke, sodann auch die partiellen Differentialgleichungen" [1872, 482].

13. [Klein 1893a, 215]. Similar remarks are made by Klein in the Italian translation [1890, 307 n.], to my knowledge the earliest translation to appear in print.

14. See [Hawkins 1980, 1982].

15. Letter of introduction by Lie on behalf of Kowalewski (August 1, 1898). Quoted by Kowalewski in his autobiography [1950, 85–86].

16. This essay did not receive the prize and was not published. Recently it was rediscovered and analyzed by Jeremy Gray [1981, 1982b]. See also [Gray 1984].

17. Letter of October 1882 (Niedersächsische Staats- und Universitätsbibliothek Göttingen, Cod. Ms. F. Klein 10, Nr. 685).

18. Letter from Lie to Klein referred to in note 17.

19. Letter dated 25 June 1888 (Universitetsbiblioteket i Oslo, Brevsammlung Nr. 289).

20. Letter dated May 1893 and located in the collection referred to in note 19.

21. For a further discussion of this point, see pp. 316 ff. of my [1980].

22. The term "structure" originated with Lie's French students, Vessiot and de Tannenberg. See [Hawkins 1982, 161–162].

23. See [Engel 1931, 140; Weiss 1933, 109].

24. Letter dated 5 April 1892 (Göttingen, Cod. Ms. F. Klein 11. Nr. 1238). The relevant passage reads as follows: "Ich erkenne auch an, dass die Schlussbemerkungen [of the Erlanger Programm] thatsächlich die Gedanken enthalten, die ich später vertreten und theilweise ausgeführt habe. Ich bedaure, dass mir dieser Umstand früher entgangen ist. . . . Dagegen kann ich nach Lage der Sache *nicht* zugeben, dass Sie Recht haben, aus unserem Verhalten auch Tadel gegen Engel oder mich herzuleiten. Sie selbst haben ja s.Z. in Leipzig die Brauchbarkeit der Gedanken, auf deren Urheberschaft Sie *heute* Werth legen, auf das Entschiedenste bekampft. Sie haben im Gesprach einmal zu mir geäussert, ich sei zu denselben Ansichten gelangt, die *Clebsch* in den letzten Jahren seines Lebens vertreten habe; aber niemals haben Sie eine Andeutung davon gemacht, dass Sie selbst schon ähnliche

Gedanken veröffentlicht hätten. Ist es nun wohl zu verwundern, wenn ich unter solchen Umständen nur allzu sicher war, dass solche Dinge in Ihren Schriften gar nicht stehen *könnten*?

25. Hilbert, who spent some time with Study in Leipzig and in Paris, described Study's eccentricities in less favorable terms. See [Reid 1970, 20].

26. According to [Weiss 1933, 115]: "Zur Beschäftigung mit Transformationsgruppen kam Study durch eine Vorlesung von F. Engel. . . ." (Weiss was one of Study's students.)

27. This is reflected in Lie's letters to Klein (Göttingen, Cod. Ms. F. Klein 10, Nr. 735, 736, 741, 750, 751, 762).

28. See Lie's letters to Klein written ca. November 1886 and 8 December 1886 (Göttingen, Cod. Ms. F. Klein 10, Nr. 728, 730).

29. Letter of Lie to Klein ca. November 1886 (Göttingen, Cod. Ms. F. Klein 10, Nr. 728).

30. Letters of Study to Engel dated 22 January 1887 and 11 April 1887. These letters are located at the Mathematisches Institut, Justus-Liebig Universität Giessen (Nachlass Friedrich Engel). That Study failed in his efforts to convert Gordan is indicated by a letter from Engel to Study (26 February 1893) describing an evening Engel spent with Gordan during which the latter remarked disdainfully [*höhnisch*] "dass Lie und ich immer mit der Gruppentheorie renommirten, aber wenn man etwas von uns wissen wollt, nichts könnten."

31. [Klein 1926]. Klein's original lectures [1893b, c] were divided into three sections. Blaschke published the first two, which originally comprised the first volume [1893b]. The first two sections were essentially an exposition of the *Erlanger Programm* and the geometrical background required to appreciate it. The second volume [1893c] began with an introduction which surveyed the contents of the first volume in a comprehensive group theoretic manner. Together with the first volume, it represents the exposition of the *Erlanger Programm*. The second volume [1893c], by contrast, is devoted to Klein's more far reaching vision of the theory of geometrical groups as a means of bringing unity to otherwise disparate branches of mathematics. A substantial portion of the second volume is devoted to Lie's theory of continuous groups and its applications to the Helmholtz space problem (dealt with by Lie between 1886 and 1890, at Klein's suggestion), to differential invariants, and to differential equations (Picard–Vessiot theory) [1893c, 54–302]. This was followed by a brief discussion of discontinuous groups and their connection with elementary geometry (the regular solids), crystallography, complex function theory, and the theory of numbers [1893c, 302–385]. As Blaschke noted in the foreword to [Klein 1926], the material of the second volume is only loosely connected to that of the first (and hence to the *Erlanger Programm*). For this reason and because the material would require substantial revision to be publishable, Blaschke decided against publication. In the third section of [Klein 1926] Blaschke presented instead an exposition of some topics of his own choosing, one of which was Study's line geometry.

32. See [Segre 1885, 1–2, 11 ff.].

33. These papers, which span the period 1892–1899, were expounded and referenced in [Fano 1907]. See especially Sections 8, 21, and 22.

34. See [Lie 1882; 1893a, Chaps. 9–10; 1895].

35. Fano was unable to devise a correspondingly systematic and general method for dealing with the problem in the case of solvable groups, and he consequently presented just the highlights of his results in print [1896b].

36. Study never published the theorem himself, but Lie presented it in his *Theorie der Transformationsgruppen* [1893a, 785]. According to Lie, Study had devised a proof with some assistance from Engel. Fano was the first to attempt a proof in print [1896a, 192 ff.], although it is unclear whether it is in any sense completely viable. These matters are discussed more fully in "Cartan and . . . Prehistory" (note 3).

37. See, e.g., [Enriques & Fano 1897; Fano 1898a, b].

38. See [Newson 1895–1896, 1897, 1901, 1902]. A paper by Newson's student A. Emch [1896] is along the same lines.

39. The quoted passage reads as follows: "Es ist zwar immer an der Forderung festzuhalten, dass man einen mathematischen Gegenstand noch nicht als erledigt betrachten soll, so lange er nicht begrifflich evident geworden ist, und es ist das Vordringen an der Hand des Formalismus eben nur ein erster aber schon sehr wichtiger Schritt" [1872, 490–491].

40. References to Study's publications are given throughout [Fano 1907] and especially in the indicated sections. See also [Cartan 1915], which is more complete. Many of Study's results were incorporated into his book, *Geometrie der Dynamen* [1903].

41. Further details will be presented in "Cartan and . . . Prehistory" (note 3).

42. According to Strubecker [1982, 37], Blaschke's paper [1910a], written while he was with Study at Bonn, incorporated the results of his doctoral dissertation.

43. On Study's influence on Blaschke see Sperner's memorial address [1982, 112] and virtually every page of Stubecker's analysis of Blaschke's papers [1982, esp. p. 35].

44. The dedication reads: "Meinem verehrten Lehrer E. Study zum sechsigsten Geburtstag."

45. The doctoral dissertations of Coolidge [1904] and Beck [1905] already combine group and invariant theoretic perspectives in the manner characteristic of Study's work. Beck's *Elementargeometrie* [1929, 1930] was written in the same spirit. On Coolidge, see also [Struik 1955, esp. pp. 671–672, 674–675].

46. See [Cartan 1937, 1.88.12–13].

47. On Cartan's program, see his talk [1936], his encyclopedia article [1937], his paper [1926], and his booklet [1935].

48. Due to the turmoil of the World War I, apparently only the first portion of Cartan's article appeared in 1915. The entire article is contained in his *Oeuvres*, Pt. III, Vol. 1 [1955a].

49. Independently of Klein and Lie, Killing had arrived at essentially the same conclusion in his "Braunsberger Programm" [1884, 19].

50. [Cartan 1914c, 451–452; 1915, 92]. Compare [1908, 449, n. 249]. The theorem of Ado (1934) confirmed the veracity of Cartan's conjecture, at least on the local or Lie algebra level on which Cartan tended to operate during this period.

51. See, e.g., [1914a, 265–267; 1914b, 149–150].

52. Such an exposition is contained in the essay "Cartan and . . . Prehistory" referred to in note 3.

53. Kowalewski and his students worked on aspects of this problem between 1899 and 1913. Their work is discussed in "Cartan and . . . Prehistory" (note 3).

54. These matters are discussed at length in "Cartan and . . . Prehistory" (note 3).

55. In a letter to Hermann Weyl in 1925, Cartan wrote that for some time he had regarded a complete reducibility theorem for semisimple structures as very reasonable although he lacked the means to prove it. (Letter dated 12 March 1925 and located in ETH–Bibliothek Zürich, Wissenschaftschistorische Sammlungen, Hs 91:501.)

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