THE EFFECTS OF CONCATENATION

Jean Cavaillès and Mathematical Philosophy

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To Elio. The tesi *is finished!*

Abstract

This dissertation proposes an interpretation of the positive doctrine in Jean Cavaillès's (1903-1944) essay *On Logic and the Theory of Science* [*LTS*] (1947). It is structured by a joint focus on Cavaillès's work as a historian of mathematics and on the received 'rationalist' history of philosophy with which the essay is in polemic. The first two chapters of the dissertation are genealogical, and give an account of how the concerns that will become central to *LTS* emerged through Cavaillès's epistemological research. In Chapter I, I offer an extended analysis of the way in which Cavaillès engaged with the debate around 'effective' mathematics as it ran from the period of the French analysts (Borel, Lebesgue) through to the formation of modern mathematical logic and proof theory. In Chapter II, I proceed to a reconstruction of how Cavaillès attempted to theorise the programmatic stakes of this earlier period of epistemological practice through the outlines of a philosophy of 'mathematical experience'. In Chapter III, I offer an analysis of *LTS* orientated around Cavaillès's use of the term 'concatenation [*enchaînement*]' to describe the movement specific to the history of mathematics, and from this his assertion that the 'nature of the intelligible' is that it 'cannot be all at once [*d'un seul coup*]'. The main objective of the dissertation is to show how this doctrine, which is explicitly stated in the vocabulary of Cartesian and post-Cartesian treatises on method, emerges from the ambition to develop what I call a theory of *mathematical time*, and to situate this programme in a broader history of theories of the necessarily successive character of thought.

Cavaillès is a difficult writer, which does not mean obscure. With him there is no eloquence, and incipient development is constantly held at bay. As a matter of fact, we are dealing with concatenations rather than developments, concatenations according to a very particular technique. The intermediate link is not necessarily where you'd expect it to be, between two successive propositions; rather, it is either implied or subtended by them. You won't understand this thought if you follow it step by step. […] Cavaillès knows his question so well in advance that he expresses himself much like he cites the authors that he has read: without being a slave to the strict linear order of exposition, by ranging over the whole spectrum of what is to be exposited. Thought can never just be contemporaneous with the phrase if it wants to hear it.

> *Georges Canguilhem, 'Jean Cavaillès (1903-1944)',* Mémorial des années 1939-1945, *Publications de la Faculté des Lettres de Strasbourg, 1947*

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Abbreviations used throughout for the works of Jean Cavaillès

OC = *Œuvres complètes de Philosophie des sciences* (1994)

TAE = *Remarques sur la formation de la théorie abstraite des ensembles* (1938), *OC* 221-374

MAF = *Méthode axiomatique et formalisme* (1938), *OC* 1-202

'Mathematical Thought' = 'La pensée mathématique' (1939/1946), *OC* 593-630

TC = *Transfini et continu* (1947, written 1941–1942), *OC* 451-472

LTS = *Sur la logique et la théorie de la science* (1947, written 1942–1943), *OC* 473–560

INTRODUCTION: TWO THESES, THREE SOURCES

> [A] bibliography of general philosophy is a paradox: nothing can replace familiarity with the great authors, and there is no point in dividing up their *œuvres* according to the divisions of a program. […] Students would be well advised not to get caught up in the fashions of the day: the best introduction to philosophy is still the careful and complete reading of a great work that serves as a unified point of reference for all fundamental problems (Plato or Descartes, Spinoza, Kant). In epistemology, on the contrary, reflection is inseparable from the actual thought at work in the sciences [*la pensée en acte dans les sciences*].
>
> Jean Cavaillès, 'Bibliographie sommaire pour le certificat de Philosophie générale et logique', *Bulletin de la Faculté des Lettres de Strasbourg*, 1941[1]

Anyone who knows the name 'Jean Cavaillès' likely associates it with the polemical conclusion of his posthumously published prison manuscript, *On Logic and the Theory of Science* [*LTS*] (written 1942–1943, published 1947), which circulates as a slogan of—depending on the generosity of the interpreter—between one and three lines. So widespread is recourse to this polemic in defining Cavaillès's work that it is in effect required to cite it when situating any analysis of his broader philosophical project. Let's take the most capacious version:

> Progress is material or between singular essences, its motor the demand that each of them must be surpassed. It is not a philosophy of consciousness but a philosophy of the concept that can yield a doctrine of science. The generative necessity is not that of an activity, but of a dialectic.[2]

These words have been made to serve a wide array of different, and sometimes incompatible, rhetorical functions. Thus one finds the 'Cavaillèsian' programme invoked alternately as a vituperative criticism of philosophies of experience and of the subject, as an internal corrective to Husserlian phenomenology, as the expression, within the discipline of the history of mathematics, of the methodological precepts of the French school of 'historical epistemology' (in which case Cavaillès will be mentioned alongside a litany of other founding figures such as Albert Lautman, Gaston Bachelard, Georges Canguilhem and Alexandre Koyré), or even as a forgotten precursor to the emerging paradigm of the 'philosophy of mathematical practice' with its ambition to focus philosophical reflection on mathematics as it 'actually done'.

1 Cavaillès 1941: 85, republished in Canguilhem 2015: 1151–1159.

2 *LTS* 78.

This dissertation attempts to put aside the rather tangled question of Cavaillès's posterity, and provide, in as far as such a thing is possible, an internal reconstruction of Cavaillès's particular form of 'mathematical philosophy'. The approach that I take in order to do so is to focus throughout on two closely interrelated aspects of Cavaillès's work:

> 1) The attempt, starting from Cavaillès's two doctoral studies on the history of set theory (i.e. *Remarques sur la formation de la théorie abstraite des ensembles*, 1938 [*TAE*]) and the Hilbert programme (i.e. *Méthode axiomatique et formalisme,* 1938 [*MAF*]), to develop a historiographical research programme in mathematical philosophy—i.e. in 'epistemology'—focused on the inter-theoretical character of the mathematical object as it is observed in 'effective mathematical work', and on producing a definition of mathematics in terms of the modalities of its history.
>
> 2) The received 'rationalist' history of philosophy to which Cavaillès constantly relates, both on the level of explicit doctrine and presumed vocabulary, principally shaped by the synthesis of Platonic, Cartesian and Kantian sources that circulated in Cavaillès's milieu in interwar France, and in particular by the commentaries of Cavaillès's teacher Léon Brunschvicg, whose *Les Étapes de la philosophie mathématique* (1912) is perhaps the central—if contested—methodological reference for Cavaillès.

The importance of this double focus is not only a matter of putting in order Cavaillès's sources, but corresponds to what I take to be the fundamental philosophical interest of this project. For the signature mark of Cavaillès's work is how a proposal concerning how to read the history of mathematics is always tied to a wide-reaching, even programmatic, philosophical standpoint. In a sense, what one finds in Cavaillès is an immanence of philosophy to mathematics and vice versa, but with the proscription that philosophical concepts be conditioned by reflection on the specific historical development of mathematics. What we are dealing with here is thus the history—or the historicity—of mathematics as a philosophical problem.

The first two chapters of this dissertation are genealogical, and give an account of how the concerns that will become central to *LTS* emerged through Cavaillès's first-order epistemological research. Chapter I offers an extended analysis of the way in which Cavaillès 'worked' a particular concept through a study of its history, namely that of the debate around 'effective' mathematics as it ran from the period of the French analysts (Borel, Lebesgue) at the turn of century through to the period of the formation of modern mathematical logic and proof the-

ory. In Chapter II, I then proceed to a reconstruction of how Cavaillès attempted to theorise the programmatic stakes of this earlier period of epistemological practice through the outlines of a philosophy of 'mathematical experience', a perspective which I argue is if not abandoned then at least fundamentally transformed during the composition of *LTS*.

I then proceed to a direct reading of *LTS*. I am in part concerned with giving the outlines of what I take to be at stake in the famous closing polemic concerning 'the philosophy of the concept', and indeed I give some interpretation of almost all the terms invoked in the passage cited above. But the gesture that underwrites my reading is to suggest that the core of the 'doctrine' found in *LTS* would be better condensed in two other central passages from the work, which I respectively call the Thesis on Evidence and the Thesis on the Intelligible. Both of these theses should be understood as transformations of broadly Cartesian vocabulary, and turn on the reactivation of the problem of 'concatenation [*enchaînement*]' as it is found first in Descartes's *Regulae ad directionem ingenii*, then in later rationalist treatises on method, now treated in light of then-contemporary results in mathematical logic. Paraphrasing, these two theses are as follows:

> The Thesis on Evidence (*LTS* 23–25). Modern mathematics replaces evidence with proof. This means that that the underlying conceptual problems posed by the project for a 'theory of science' rejoin 'Descartes's difficulty' concerning concatenation from the *Regulae*, that is, concerning the opposition between the immediacy of evidence and the successiveness of proof. But, on Cavaillès conception, concatenation assumes a historical modality, i.e. as *inter-theoretic* concatenation.
>
> The Thesis on the Intelligible (*LTS* 35–36). Concatenation has two essential characteristics: it is intrinsically progressive (i.e. it must move), and this progression (or movement) is conditioned at each step by the effective. In terms of the mathematical-logical conjuncture, this should be understood as the assertion that then-contemporary results concerning the limitations of formalisms (Gödel, Herband, Skölem, Church-Turing, etc.) had demonstrated something about the nature of the intelligible: that it 'cannot be all at once [*d'un seul coup*]'.

Conjoining these two theses, I suggest that *LTS* can be read as having advanced a positive doctrine concerning mathematical concatenation, which, through a transformation of Cartesian (cum Spinozist) categories, attempts to provide a theory of *mathematical time*, or of an in-

ternal logic of 'the before and the after', which Cavaillès understood to be necessitated by the 'not all at once' nature of 'intelligible concatenations'. Chapter III is concerned with showing the lineaments of this account, which, I argue, should be seen as having conjoined a strategic reactivation of a Platonic position with respect to the 'production of the intelligible' in order to arrive at a form of *rationalist nominalism*, fundamentally indebted to Spinoza. It is in light of this doctrine that we should should situate the famous critique of the 'philosophy of consciousness' announced at the end of *LTS*, and thus my interpretation puts less emphasis on Cavaillès as a critic of Husserlian phenomenology than other studies of Cavaillès. I suggest, however, the grounds on which a confrontation between Cavaillès's late doctrine and phenomenology ought to be staged: that of the nascent development of a concept of the act without a subject in Cavaillès's final essay. For, as Cavaillès states, the 'formal concatenation' through which he defines the progress of mathematics is a 'concatenation of acts'.[3]

The heuristic point to emphasise here is how an opposition between the simultaneous and the successive becomes central to the 'theory of reason' that Cavaillès wished to develop, and leads to the development of a concept of *intrinsic rational movement*.[4] The aim of this dissertation is thus both to specify how this doctrine emerges from a particular way of reading the history of 'effective mathematics', with an attendant definition of mathematics in terms of the specific form of its historicity, and to situate it in the *longue durée* of a philosophical concept: that of the intrinsic heterogeneity or variegation of 'concatenation', i.e. a definition of the essence of thought in terms of its successive and necessarily progressive character.

Before embarking on this elaboration, the role of this introduction is to give a first exposition of the two theses introduced above, and the problems and traditions that I understand them to invoke. This involves relating them to three major historical sources that I take to have been the basis for Cavaillès's conjunctural intervention in having developed a theory of mathematical or intelligible concatenation: Descartes's *Regulae*, Brunschvicg's *Les Étapes,* and Richard Dedekind's 1854 Habilitation address 'On the Introduction of New Functions in Mathematics'. Amongst these sources, the Cartesian text has quite a different character, for while Brunschvicg and Dedekind are explicitly (and biographically) important references for Cavaillès, Cartesian sources are something more like an assumed lexical and conceptual back-

3 *OC* 472.

4 For Cavaillès's announcement of the ambition to outline a 'theory of reason', see *MAF* 21/*OC* 29. In the table of contents for that work, this section has the sub-heading 'Necessity for a theory of reason' (*OC* 3), and the section gives an account of the 'antecedents' of that theory through the trajectory Descartes–Leibniz–Kant, i.e. exactly the trajectory we shall address here.

ground, the comprehension of which is a condition of legibility for reading Cavaillès. Indeed, for all that, as Jean-Toussaint Desanti summarises, the orientating gesture of the tradition of 'mathematical philosophy' initiated by Brunschvicg was to have made '[t]he slow emergence of the forms of mathematical intelligibility provide the *grille de lecture* through which the history of philosophies can be interpreted', we could fairly say that the implicit double of this programme was to have made Cartesian methodological categories the frame through which the very notion of 'mathematical intelligibility' was specified.[5] For this reason, I address the Cartesian context first. Following this, I give a reconstruction of how these Cartesianism themes were mediated for Cavaillès by Brunschvicg's 'intellectualist rationalism', both in terms of a methodology for philosophical reflection on the history of mathematics, and with respect to a polemic construction of the history of philosophy that connected categories from Cartesian treatises on method to a historicised post-Kantianism, in which the two strands condensed around the attempt to theorise the creative act or activity of the intelligence. Finally, I indicate how in a sense Cavaillès's programme commenced from the attempt to confront Brunshcvicg's philosophy with a reading of the specificity of modern mathematics in terms of a theme that Cavaillès extracted from Dedekind, in which the 'necessary generation of new concepts' was to be theorised in terms of a dialectical 'chain' of reciprocal constitution between objects and operations (or 'acts'). What emerges in this conjunction is a theory of the productivity of deduction in which succession is taken to be a characteristic of reason itself, rather than only being a consequence of the limitations of our faculties.

A. On Proof and Time: Two 'Cartesian' Theses

As just indicated, the two theses through which I read *On Logic and the Theory of Science* are both stated in a Cartesian vocabulary. In the case of the Thesis on the Intelligible, which I in a sense take to be the culmination of Cavaillès's whole late 'doctrine', this reference is implicit.

A.1. The Thesis on the Intelligible, or the Implicit Cartesian Citation

In the midst of Section II of *LTS*, Cavaillès turns to an extended elaboration on the relationship between syntax and semantics, at the centre of which he critiques Carnap's project for a 'general syntax' of science in the following terms:

5 For Desanti's comment, see his preface to the 1973 re-edition of *Les Étapes*, i.e. Brunschvicg 1972: vii.

> [T]here is a serious failing here, a misunderstanding of what is essential in formal concatenation [*l'enchaînement formel*]: its progression, which is both necessary and conditioned at every step by the effective. […] The fact that everything cannot be all at once [*d'un seul coup*] has nothing to do with history, but is the characteristic of the intelligible […].[6]

We shall return on many occasions to this thesis, and place it in the broader context in which it is developed in *LTS*, so for now I only cite the passage in reduced form. For the purposes of this introduction, I want to focus on two points.

First, we should place this thesis concerning the non-simultaneity of the intelligible within a broader problem field around the concept of *logical* or *mathematical time*. Heuristically, I use this concept to index the (perhaps paradoxical) attempt to think a concept of time without reference to life or to consciousness, i.e. concepts of time that are not derived from an experience of time. This perspective is well expressed in another central assertion from *LTS*, stated in the course of Cavaillès's exposition of Bernard Bolzano's 'theory of science':

> [S]cience moves outside of time—if by time we mean reference to the lived experience of a consciousness.[7]

Taking this as a positive assertion on Cavaillès's part, we can understand *LTS* to be concerned with trying to think a concept of *time without temporality.* I should immediately clarify that I am not concerned here with physical time, that is, the time of events, of chronology, which of course could be seen as another register of time without temporality.[8] Rather, I intend to position Cavaillès's work within a broader thought of intrinsic or internal time, which corresponds with the attempt to think the concept of concept as movement. A major question—condensed in the line just cited—is whether 'time' is even the appropriate concept to invoke here, as opposed to other terms—recalling other theoretical lineages—such as 'becoming', 'change', or, most directly, 'movement'. However, Cavaillès's own term for the concept at stake here was 'mathematical time', and in general that is the terminology I shall adopt.[9] Heuristically, much of

6 *LTS* 35–36.

7 *LTS* 22.

8 To recall, for example, Quentin Meillassoux's invocation of the mathematical dating—within the discipline of mathematical physics—of a material world that in principle precedes the genesis of consciousness. See Meillassoux 2006, Chapter I.

9 As cited by Granger 1996: 570–571; see my discussion at the end of Chapter II. Given the polemical weight attached to the difference between mathematics and logic in Cavaillès's context, it is wise not to too quickly assimilate 'mathematical time' to 'logical time'. However, in terms of showing how Cavaillès's work emerged from a dense field of reflections amongst his generation concerning the non-simultaneity of reason, it is notable that in 'Logical Time and the Assertion of Anticipated Certainty' (1945), Jacques Lacan proposed the syntagm *temps logique* more or less concurrently to Cavaillès's theory of mathematical time, and moreover

this thesis will be concerned with showing how Cavaillès attempted to extract such a theory of mathematical time from a methodology for reading the history of mathematics.

Second, in what sense is this a 'Cartesian' or an 'anti-Cartesian' thesis? Here, my point is in the first order lexical. Although Cavaillès does not mention Descartes anywhere in his formulation of the Thesis on the Intelligible, the central assertion that the intelligible is to be understood as a concatenation [*enchaînement*] that cannot be established all at once [*d'un seul coup*] is without doubt a reference to a particular doctrine from the *Regulae* concerning the *non tota simul* characteristic of an extended chain of reasoning (*catena*). This would have been obvious enough to Cavaillès's readership to make citation irrelevant, but it likely is not to the contemporary English-language reader, so it requires a brief exposition, both on the level of vocabulary and doctrine.

The Cartesian problem at stake here, to put it in the simplest terms, concerns the relation of proof to time. In the *Regulae*, this turns on the relation between two ways or modes of knowing: *intuition* and *deduction*, which are respectively associated with the 'simultaneous' and 'successive' poles of proof. Putting aside a great deal of doctrinal complexity, the essential point here is that *intuition* (or *evidence*) is to be associated with what can be *seen* 'all at once', whereas the other ways or modes of knowledge that Descartes will respectively refer to as 'deduction' or 'enumeration' can not be reduced to the survey enacted by a single gaze.

I refer here to a celebrated sequence, formed primarily by Rules III, VII and XI, in which Descartes first elaborates his notion of proof. Initially, in Rule III, this is posed as a rather brute opposition: proof can take two forms, intuition [*intuitione*] or deduction [*deductione*]. Intuition has a clear priority, in that it is defined as the 'conceptual act of a pure and attentive mind', and is effectively equated with evidence and certainty as such; that which is known

did so using exactly the same Cartesian figure in order to explicate it. Having proposed his celebrated 'sophism' intended to demonstrate a logical problem that required time for its resolution, Lacan writes that 'it is precisely because my sophism will not tolerate a spatialized conception that it presents itself as an aporia for the forms of classical logic, whose "eternal" prestige reflects an infirmity which is, nonetheless, recognized as their own—namely, that these forms never give us anything which cannot already be *seen all at once* [*être vu d'un seul coup*].' Lacan 2006: 166. Lacan cannot have read *LTS* when making this formulation, but Bachelard's subsequent (uncited) appropriation of the term can be read as an implicit synthesis of Lacan and Cavaillès: 'Epistemology […] places us in a logical time, with well-placed reasons and consequences, that is in a logical time that no longer has the slowness of real chronology.' Bachelard 1949: 96. For an application of this concept to the historiography of philosophy, see Goldschmidt 1953. Although—as I shall now outline—all of these formulations would have been well-understood to be implicit citations of Descartes, the only instance of which I am aware in which the term *temps logique* has actually been applied to the *Regulae* is Luporini 2020.

through intuition will be known immediately as evident and certain, and vice versa.[10] Deduction, on the other hand, can appear to be a secondary phenomenon, insofar as it only comes into play when the chain of reasoning necessary to know something is so extended that it escapes the immediate evidence of intuition. Thus the initial terms on which the opposition between intuition and deduction is introduced is as a strong bifurcation in the nature of knowing: intuition is certain knowledge insofar as it is simultaneous knowledge; deduction is knowledge insofar as it is successive, and its certainty is derivative of intuition.

Put otherwise, the disjunction intuition/deduction is strongly associated in the *Regulae* with the disjunction static/mobile: intuition does not move, while deduction moves, whence what Jean-Luc Marion refers to as Descartes's 'kinematics of thought'.[11] Proof qua deduction is a movement of proving, 'a continuous and uninterrupted movement of thought which perspicuously intuits individuals', which is required whenever the length of a chain of reasoning exceeds the limits of 'present evidence [*praesens evidentia*]', that is, when its complexity is such that is no longer possible to survey all of the links of the chain in 'one and the same ocular intuition [*uno eodemque oculorum intuitu*]'.[12] The continuous 'movement or succession [*moto sive successio*]' of thought at stake here is movement along a chain of inferences, and thus it is the very presence of a chain—or rather the characteristic of concatenation—that marks the difference between intuition and deduction.

Or so it seems. For as Descartes continues his exposition through Rules VII and XI of the *Regulae,* he complicates this blunt posing of the opposition between the 'stasis' of intuition and the 'movement' of deduction, and clarifies that 'intuition' is still a motion, that is a movement of inferring from one thing to another, but it is simply the smallest possible motion—the movement of immediate inference.[13] Whether a concatenation is to be considered as 'intuitive' or 'deductive' then ultimately comes to depend on the rapidity with which it can be traversed by the intellect. In an intuitive concatenation, the number of inferential steps is small enough that we can *see the whole movement at once*, and thus practically experience it as a 'single

10 AT X 368. Translation from Descartes & Heffernan 1998: 79.

11 Descartes et al. 1977: 242.

12 AT X 370. Descartes & Heffernan 1998: 83.

13 The degree to which the intellect is ultimately a *motion* is a classic interpretative problem in the reception of Descartes. In Cavaillès's context (broadly speaking), opposing answers were given by Jean Wahl—whose *Du rôle de l'idée de l'instant dans la philosophie de Descartes* states firmly that 'Descartes's idealism is an actualism', which tries to produce a theory of the indivisible instant (Wahl 1994: 60)—and Jean Laporte, who argued for a highly 'mobile' reading of Descartes, as most systematically expressed in Laporte 1945. Aside from the *Regulae*, the other *locus classicus* for this question is Descartes's *Conversation with Burmann.*

step'; another way of putting this is that if we can run through the chain fast enough that it is experienced as simultaneous, then it becomes intuition. On the contrary, if a concatenation is too long or too complicated to be so condensed, then the inferential motion found in it is not practically eliminated, so it is experienced as successive, and thus is deduction. Thus certainty is a matter of speed, and at the base of this account is a relativity of the intuition/deduction disjunction to the capacities of the reasoning agent. There is then no intrinsic (in principle) distinction between intuition and deduction; it is, if one likes, an external designation of the relation between a reasoner and given chain of reasoning.[14]

This account produces a whole host of difficulties, as Descartes is at pains to address throughout the *Regulae*. With respect to the relation between proof and time, chief amongst these is that insofar as deduction involves a time of reasoning (because it is required when reasoning exceeds 'present' evidence or intuition), it also involves memory, which is, in turn, a source of possible error. It is for this reason that Descartes introduces another 'mode of knowing' across Rules VII and XI: 'enumeration' or 'induction'. It is here that the great theme of the *continuity* or *completeness* of the inferential chain also comes into play. In the Rules in question, Descartes is responding to a practical problem introduced by his doctrine: if deduction must be reduced to intuition in order to be certain, how is this reduction to be so effectuated? The operation of enumeration is then introduced to assure us that in cases of chains of reasoning that are too extended (or too complex) to be apprehended all at once, and for which a temporally extended deductive apprehension is thus necessary (hence involving a correlative reliance on the 'weak' faculty of memory), no error can slip into the process because the chain *as a whole* (i.e. each and every constitutive inference that makes up the chain) can be systematically checked.[15] A major methodological concern of the *Regulae* is to produce a method

14 To anticipate an argument in Chapter I, Cavaillès will understand proof-theoretical debates around the nature of finitary reasoning and in particular around the formalisation of the concept of effective or computable procedure to bear on the same question, such that the transition from the 'Cartesian' period to 'modern' mathematics involves a development from an extrinsic and relative conception of the distinction between intuition and deduction to an intrinsic and absolute definition of the same.

15 It is worth noting that there are two ways this can be done, either in advance of or posterior to a deduction. Both operations are named by Descartes as enumeration, so we have, if one likes, two tenses of enumeration: the one is enumeration as 'comprehensive review' of *already* elaborated deductive links, the other is the 'complete enumeration' of all the possible options for thought *prior* to an investigation. It is because enumeration has this double aspect that it is also—in the case where its operation is *prior* to deduction—characterised as *induction*. With respect to an enumeration that is effectuated posterior to deduction, L. J. Beck emphasises well the temporal dynamics at play: 'In so far as enumeration is to be considered as a "comprehensive review", it may be properly described as a verification of the logical steps already traversed in the deductive process. […] In a sense, it is an attempt to exclude the extra-logical factor of duration by reducing the linear time-series of deductions to the essential unity of a *praesens evidentia*.' Beck 1952.

whereby successive apprehension can be eliminated through the elevation of intuitive certainty to longer deductive chains, but this is not always possible. Thus the truce at which Descartes arrives at is that *either* we can reduce extended concatenation to intuition by traversing it with sufficient speed, *or*, in cases where this is not possible owing to the length or complexity of the concatenation, we can utilise enumeration to ensure that the chain is at least 'complete'. This is how Descartes puts the matter in Rule XI:

> [W]e require two things for mental intuition, namely, first, that the proposition be clearly and distinctly understood, and, then, that it as a whole be understood simultaneously, and not successively [*tota simul et non successive intelligatur*]. Yet deduction, if we think of it as a process, as in the third rule, does not seem to be accomplished all at once [*non tota simul fieri videtur*], but rather does involve a certain movement of our natural intelligence, a certain movement of inferring one thing from another. [...] We are supposing, therefore, that the deduction is seen through intuition when it is simple and perspicuous, but not when it is multifaceted and complex. To the latter, we have given the name 'enumeration' or 'induction', because then it cannot be comprehended all at once by the intellect [*non tota simul ab intellectu potest comprehendi*], but rather does its certainty depend, in a certain manner, on the memory, which must retain the judgments about the individually enumerated parts in order that we may gather one thing from them all.[16]

Here, we can note an essential instability that has been introduced into the notion of deduction: in effect, deduction only exists as an independent mode of knowing when either it is a (successive) motion that has not been practically reduced to (simultaneous) intuition (as in Rule III), or when it has not been yet decided that it is in principal not so reducible, and thus that it will have to be known through enumeration (as in Rule VII). In the latter case, however, we can observe that the standpoint from which enumeration is guaranteed still involves what we could call simultaneity of the second degree: enumeration takes place from the standpoint of the 'completion of movement' from which a 'multifaceted and complex' reasoning can be retroactively surveyed in its constituent parts. Thus, in sum, deduction, which has been defined as the 'continuous movement of thought', ultimately refers to the *middle of reasoning*, i.e. to an effervescent existence of the successive in a process of reasoning which, insofar as it is certain, is simultaneous at both its outset and its conclusion.[17]

Thus, Descartes's methodological elaboration of the intuition/deduction dyad, which initially appears as a hard split between two independent operations of knowledge, the one

16 AT X 407-408. Descartes & Heffernan 1998: 134–137.

17 In the Thesis on Evidence, Cavaillès connects this status of deduction as *mediate* knowing to Plato's theory of *dianoetic* knowledge.

simultaneous and the other successive, in fact bottoms out in the view that only simultaneous apprehension can be knowledge proper (where we understand 'simultaneous' to mean something that is still a motion but which is a small enough motion to be practically experienced 'all at once [*tota simul*]'). The curious thing about the Cartesian problem of proof as presented in the *Regulae* is thus that proving essentially involves a temporal aspect—in that thought is a movement—but proof is only certain when time no longer plays any substantial role in the understanding of the proof. The elaboration of 'method' thus, in a certain sense, necessarily involves the attempt to eliminate time within time.

The notions invoked here form a background for Cavaillès's reflections that is lexical as much as conceptual, for French translators of and commentators on Descartes consistently rendered the whole complex of notions here by the vocabulary of simultaneous vision. Thus we find in the translations by Cousin (1824–1826), Sirven (1932) and Le Roy (1933) that both the notions of the *tota simul* and the *uno intuito* are translated by variations on the *seul coup*, the *même coup d'oeil,* or the demand that intuition be able to *embrasser* a whole concatenation *d'un seul regard*. This is of course correct insofar as Descartes's notion of evidence is, paradigmatically, that of veridical seeing.[18] In short, it is legitimate to interpret Cavaillès's various invocations of this terminology of instantaneous sight—which, as we shall see in Chapter I, are essential to his presentation of the programme of Hilbertian 'finitism'—as a reference to the Cartesian problem of the fact that the intelligence is a motion that only possesses certainty insofar as this motion is practically eliminated.

The *Regulae*, however, is not the only canonical referent for invocations of the *total simul* nature of proof. With respect to the immediate posterity of this notion in Cartesianism, reference must also be made to Leibniz's transformation of the vocabulary, which sets up a crucial heuristic opposition in the reception of the classical problem of method. For, contrary to Descartes, Leibniz argued that '[w]e don't usually grasp the entire nature of a thing all at once [*non totam simul naturam rei intuemur*], especially in a more lengthy analysis, but in place of the things themselves we make use of signs, whose explicit explanation we usually omit for the sake of brevity, knowing or believing that we have the ability to produce it at will.'[19] Whence Leibniz's famous (or infamous) assertion that in the extended arguments that

18 Indeed, Marion makes an argument for directly translating *intuitus*/*intueri* by *regard*/*regarder* throughout the *Regulae*. See Annexe I to his translation, Descartes et al. 1977: 295–302.

19 From 'Meditations on Knowledge, Truth, and Ideas' (1684), translation from Leibniz 1989: 24. In general, the notion of *totam simul* apprehension here is again usually rendered in French editions by *même coup d'oeil* or *seul coup*, for example in Prenant's translation: « Mais souvent, surtout dans une analyse un peu

compose the fabric of mathematics, we should 'usually call such thinking, which is found in both algebra and in arithmetic and, indeed, almost everywhere, *blind* or *symbolic*.'[20] Broadly, this sets the stage for a perennial—some would say reductive—opposition between 'Cartesian' and 'Leibnizian' conceptions of proof: are proofs evidential insights or mechanical deductions?[21] In terms of the dialectic between the simultaneous and the successive that I have been emphasising here, the 'Leibnizian' retort to Descartes is thus that successive syntactical manipulation ought to replace the ideal of simultaneous evidence, and that the guarantee of the justness of reasoning is not located in the veridical seeing of the truth of an argument by a reasoning agent, but is rather located in the external arbiter of a universal mathematics or formalism, a formalism that is moreover fundamentally combinatoric in character.

This is perhaps a contestable presentation of Leibniz's own views, but is the assumed interpretation of the 'Leibnizian' standpoint on mathematics that would have been a background to Cavaillès's work. Of particular significance here is the assimilation of the project for a 'universal mathematics' with the ideal of a purely formal combinatory logic, i.e. the assimilation of *mathesis universalis* to *ars combinatoria*. This is likely the mark of the significant influence of Louis Couturat on Cavaillès's milieu, for it was this assimilation which allowed Couturat to position Leibniz as the progenitor of Russell's and Peano's logics, which served to set up a rhetorical association of Leibnizianism with both then-contemporary logicism and with the project of universal or ideal language, both for mathematics and in a wider sense.[22] Even if

longue, nous ne saisissons pas l'objet de la pensée, d'un seul coup, dans toute sa nature, mais à sa place, nous utilisons des signes…. », Leibniz 1940: 79.

20 Leibniz 1989: 25.

21 For more recent statements of this opposition as orientating for the history of the concept of proof, see Verley 2014 and Hacking 2014. For a critique of this oft-cited division between 'evidential' or 'intuitive' and 'algorithmic' or 'calculatory' conceptions of proof, see Chemla 2012.

22 See Rabouin 2012: 192–194. Alongside being the principal expositor of early logicism in France and a major editor of Leibniz, Couturat was perhaps the most prominent philosophical advocate of a universal artificial language for science (see Couturat & Leau 1903). Here is a how Couturat states the 'doctrine' at stake in *La logique de Leibniz*: 'Indeed, given the limited capacity of the mind, which can neither encompass more than a small number of ideas at any one time, nor effectuate all at once [*d'un seul coup*] a deduction which is not immediate and simple, it is in danger of getting ensnared in the maze of complex notions, and getting lost in lengthy chains of reasoning. […] Leibniz envisions a careful and mechanical process that could guide and support discursive thought, eliminate its uncertainties and its tendency to progress by trial and error, and make its lapses and errors impossible. […] This process consists in representing ideas by signs, and the combinations by combinations of signs, such that the logical analysis of concepts is replaced by the material analysis of characters.' Couturat 1901: 90–91. For Cavaillès, the influence of Couturat was mediated by his friend Georges Friedmann, whose dissertation on Leibniz and Spinoza, written at the same time as Cavaillès's doctoral studies, would on publication be dedicated to Cavaillès and Marc Bloch, both of whom had by then been executed by the Nazi occupying forces (see Friedmann 1946: 6). In a letter of 27th Novem-

we can now challenge this as an interpretation of the Leibnizian corpus, it is how Cavaillès would have received matters. It is in this sense that Cavaillès will repeatedly associate a critique of the 'Leibnizian' dream of a universal combinatory, that 'old dream of panlogism', with the purported reactivation of this 'dream' by both Carnap and Husserl.[23] This kind of dramaturgy, in which various projects for the foundations of mathematics or science were seen as reactivating fundamental schisms in the reception of Cartesian method—and indeed something like an eternal conflict between 'intuition' and 'formalism'—was far from unique to Cavaillès in this moment. Exemplary in this regard is the assessment of the field of foundational research by Paul Schrecker in the special issue of the *Revue Philosophique de la France et de l'Étranger* produced to mark the *Congrès Descartes* of 1937:

> The new mathematical discipline of set theory, which is, in its scope, comparable to those created by Descartes and by Leibniz, was birthed in concourse with metaphysical thoughts and has developed itself more against the logistical current [i.e. 'logicism'] than through it. And it is logistics that, today, could assume the role of general science, which Descartes had reserved for algebra. There is however, elsewhere a pessimism about ever arriving, by virtue of logical form alone, at an Archimedean point on which one could found mathematics. This reaction against metamathematical panformalism has taken the form of intuitionism, which, in a certain sense, signifies a return to Cartesianism.[24]

This excursion in Cartesian doctrine shows something of the rhetorical backdrop to all of Cavaillès's writings, and will be assumed throughout this dissertation. Returning to the Thesis on the Intelligible, it should now be clearer why it is in one sense an inversion of 'Cartesian' terminology: intelligible concatenation cannot be seen all at once, thus the paradigm of certainty proper to it is not vision, that is, it is not intuition. To place emphasis on the necessity of proof in science is thus to know things *successively*, not *simultaneously*. In itself, however, this does not strictly mean that the Thesis on the Intelligible is anti-Cartesian: to define concatenation as what 'cannot be all at once' is also precisely a restatement of the *non tota simul* character of the extended chain of reasoning as Descartes characterised it, and which, as we have just seen, is the point that Leibniz—according to a certain reading—will displace onto the ideal of a formal system. What *is*, however, strictly not possible to square with either Descartes's or

ber 1929, whilst preparing classes on Leibniz as part of his duties as *caïman* at the ENS, Cavaillès thanks Friedmann for having pointed out sources from Couturat, Baruzi and Stein, signals his distaste for Leibniz's 'mania for the analysis and the quasi-grammatical classification of concepts', and then states that 'what I really find interesting is your [i.e. Friedmann's] hypothesis that Leibniz's interest in Spinoza stems principally from his interest in universal characteristics.' Ferrières 2003: 78/Ferrières 2000: 44.

23 *OC* 659.

24 Schrecker 1937: 346.

Leibniz's version of the doctrine is the claim that this very *non tota simul* character defines 'the intelligible' itself. Both doctrines are nonetheless the background to what we shall now discuss as the second—this time explicitly—Cartesian citation in *LTS*, which explains more concretely how Cavaillès understood the actuality of this debate to the field of research in the foundations of mathematics.

A.2. The Thesis on Evidence, or the Explicit Cartesian Citation

The simplest possible expression of Cavaillès's late philosophy might be to state that evidence is produced by a concatenation, but in the unique modality that he will give to that term, that is, as inter-theoretic concatenation, traversing the history of science. This is a thesis that emerges most clearly in Section I of *LTS*, in the course of Cavaillès's comments on Bolzano, a dense elaboration that is framed by the Cartesian problem of evidence, in the literal sense that references to Descartes both open and close Cavaillès's exposition. At the beginning of Cavaillès's presentation, we find the following:

> Placing the emphasis on the necessary character of science is precisely the predominant concern of Bolzano, for whom science is first and foremost proved theory. In his *Purely Analytical Proof*, Bolzano challenged an overhasty recourse to evidence, an evidence which Descartes invoked only for simple natures but which Kant saw as governing all geometric reasoning. The growing pains of analysis were taken to call for a change in the type of this evidence—as [Brunschvicg's] *The Stages of Mathematical Philosophy* has also shown. For Bolzano, this took the form of a radical transformation of observation into proof.[25]

The connections that Cavaillès is drawing here require two comments.

First, it should be noted that nowhere in Bolzano's *Purely Analytic Proof*, nor in the other works of Bolzano's on which Cavaillès plausibly draws, do we find mention of Descartes or the notion of evidence. What is at stake for Bolzano is, rather, the Kantian doctrine according to which mathematics is the discipline of the construction of concepts from pure or *a priori* intuitions.[26] Thus Cavaillès's comment here assumes that the Cartesian notion of evidence

25 *LTS* 19–20. I have translated Cavaillès's uses of *démonstration* (and variants) by 'proof' throughout, partly in order to emphasise the discursive continuity with the advent of 'proof theory' [*théorie de la démonstration*] in the (post-)Hilbertian sense.

26 A critique which is theoretically elaborated in Bolzano's *Contributions* (i.e. Bolzano 1817, translated at Bolzano & Russ 2004: 82–137) and only, if one likes, practised in the *Purely Analytic Proof*. It is indeed the *Contributions*—analysed extensively in Cavaillès's doctoral thesis on the Hilbert programme—that could be seen as the real referent for Cavaillès's methodological comments on Bolzano in *LTS*. We have to recall throughout that, given its conditions of composition, *LTS* is written without access to most of the sources cited therein.

(or *intuitus* in the sense given to it in the *Regulae*) is in some sense homologous with the Kantian doctrine concerning synthetic *a priori* intuitions in mathematics. That Cavaillès makes such an assimilation—which, as we will see, is closely related to another, viz. an analogy between the 'all at once' nature of Cartesian evidence and the 'homogeneity' of synthesis in Kant—is crucial for understanding his broader analysis of the 'theory of science', and itself in part rests on the polemical history of philosophy that was so central to Brunschvicg's project.

Second, when Cavaillès speaks here of the transformation of observation into proof, he is referring to Bolzano's methodological statement in the *Purely Analytic Proof* to the effect that '[…] scientific proofs should not merely be confirmations [*Gewißmachungen*], but rather groundings [*Begründungen*], i.e. presentations of the objective reason for the truth to be proved […].'[27] In context, Bolzano is explaining why it is that he is providing an extensive analytic foundation for a geometrical proposition that is 'extremely obvious', and why he takes it to be the case that while it 'therefore needs no proof in the sense of confirmation, it none the less does need a grounding.'[28] The essential theme here is a kind of epistemic inversion whereby instead of proofs being seen as proceeding from what is intuitively certain to other, derived, truths, what is experienced as evident is resituated as the product of a long chain of reasoning from which intuitive evidence has been subtracted. This is the orientating lesson that Cavaillès extracts from the transition to modern mathematics, most importantly from the work of Dedekind. It corresponds to the general historical assessment announced by Cavaillès in 1939, according to which the defining characteristic of 19th century mathematics was that:

> [O]wing to the development of different branches of mathematics and the necessity of abandoning the intuitive evidence upon which one had previously relied, one was led to place a greater emphasis on the notion of proof. Evidence gave way to provability. Whence the idea, widespread among almost all mathematicians, and which is found in researchers as different as Frege and Dedekind, that mathematics is a part of logic. What guarantees mathematical results is, in effect, the rigorous nature of the concatenations of inferences [*des enchaînements de raisonnements*] by which they are established.[29]

27 Bolzano 1817, translation from Bolzano & Russ 2004: 254.

28 Bolzano & Russ 2004: 254.

29 *OC* 597. The comment is taken from Cavaillès's oral presentation to the Société française de philosophie on 4th February 1939, which I discuss at length in §II.2. I have usually translated *raisonnement(s)* by 'inference(s)' in order to avoid the problem of English's lack of a plural 'reasonings'. Exceptions have been made in cases when the notion of reasoning *tout court* is most clearly at stake (e.g. *mode de raisonnement*/'mode of reasoning').

In *LTS*, Bolzano is presented as the first great theorist of this methodological inversion, which would so profoundly affect mathematical practice in the long 19th century.[30] And he is taken to have done so through a Leibnizian critique of the purportedly *finite* (and thus 'certain') evidence of number. Thus Bolzano's insistence that the 'theory of science' needed to elaborate a hierarchical organisation of logical dependencies, in which all proved terms were brought back to a simple enumeration of 'unproved' principles, and in which even elementary arithmetical reasoning is no longer accorded the status of intuitive certainty, is interpreted by Cavaillès as 'Leibnizian' in the following sense:

> Through a revolutionary reversal, number is banished from the domain of perfect rationality, and the infinite is let in. But it is no longer a question of a human intellect encompassing a complete concatenation in a single glance [*d'embrasser d'un seul regard un enchaînement complet*], even if it is only the system of the terms of a series. The necessary link that is perceived as holding between two terms guarantees the whole: the power of extension by iteration or combination turns out to be both the source of novelty and the guarantor of intelligibility. In the divine understanding the rational homogeneity of the finite and the infinite is thus expressed in the universality of logic: what is necessary is what can be proved.[31]

In brief, the challenge posed by Bolzano and his contemporaries and successors to Descartes's ideal of evidence (which, as noted, Cavaillès assimilates to the Kantian notion of 'finite' intuition) is 'Leibnizian' because it displaces the standpoint from which epistemic *simultaneity* is to be established: evidence, which has the characteristic of grasping all of its terms 'at once', is removed from the position of the (finite) human intellect and instead ascribed to the (infinite) intellect of God. The force of Cavaillès's analogy is to imply that this same 'Leibnizian' reversal is in some sense at stake in the transition from intuitive or informal notions of proof to the ideal of a formal deductive system as it was elaborated in a whole historical trajectory running in a bifurcated manner from Bolzano to the logicist (Russelian-Carnapian) and formalist (Hilbertian) programmes respectively. This analogy has metaphysical as much as methodological stakes, in that we can assume Cavaillès's comment here to partly recapitulate what he said about Leibniz in *Axiomatic Method and Formalism*, whilst addressing the Leibnizian theory of extension and the correlative 'solution' to the opposition between an assertion of the discrete plurality of substances and the phenomenal continuity of their spatio-temporal relations:

> Thus the difficulty splits: if primitive intuition is no more than a 'well-founded phenomenon', mathematics remains at its level; on the one hand the problem of founda-

30 See Gray 2008 for an excellent account of the emergence of this 'mathematical modernism'.

31 *LTS* 20-21.

> tions will be solved by a virtual reduction of mathematics to logic; on the other hand the relations between continuous quantity and discrete systems are transported from the mathematical plane to the metaphysical plane. Mathematics is simply the science of ideal relationships: in the divine will, everything is affirmed all at once [*d'un seul coup*]; there is no number of the totality of monads, and moreover an infinite number would be contradictory. […] To the extent that mathematics is affirmed, it will be nothing but the explication of axioms and of definitions: everything is reduced to a foundation of indefinitely varied combinations of a primitive system of simple notions.[32]

On this polemical presentation, Leibniz's 'mechanical' or 'combinatoric' view of proof is still 'Cartesian' in its conceptual architecture; it is because of *our* finite standpoint that we operate with locally successive syntactical chains in place of the full intuitive apprehension of the whole that this finitude disbars us from; in the infinite mind of God, however, the entire deductive chain is apprehended all at one, which is to say that deduction is reduced to intuition once more. If one likes, on this reading, it is not so much that Leibniz breaks from the elaboration of the problem of proof as articulated in the *Regulae* as that he radicalised the standpoint of 'complete enumeration' beyond the confines of a review of inferential steps by any particular reasoning agent. Proof thus still asserts its force through evidence qua simultaneity, but, perhaps paradoxically, this evidence is not evidence *for us*. The disagreement between Descartes and Leibniz on this matter is thus not so much centred on the *character of certainty* as it is on the *place of certainty*, on both a practical and a metaphysical level. In his closing comments on Bolzano, however, Cavaillès asserts that Bolzano's 'Leibnizian' turn towards the elaboration of a universal formal system rests on another fundamental Cartesian problem—that of the movement of concatenation.

> Not only is the structure of science proof, it is indistinguishable from proof. The essential traits of science are indeed found in it: unity, necessary and indefinite progression, and finally closure upon itself. The internal rule that governs proof posits each of its stages, and is contained in all of them, but is impossible to grasp *ne varietur* in any one of them: we are reminded of Descartes's difficulty concerning the evidences—or simple natures—in their concatenation [*la difficulté cartésienne des evidences—ou natures simples—d'enchaînement*]. We can multiply the hooks; if the demonstrative link consisted in these, we would have to indefinitely superimpose more and more of them. In reality there is no essential distinction between the hardened rings that seem to mark the terms and the movement that runs through them. But this movement does not halt: proof, by the simple fact that it posits its goal, extends and ramifies the domain it has created by means of the new combinations which it establishes as soon as they become possible. Ultimately, proof cannot be linked back to some unproved term: we come back to Plato's reservations with respect to the *dianoia* that borrows its hypotheses from the

32 *MAF* 25/*OC* 33.

> visible world, an arrangement without justification which can never yield any necessity. […] The heterogenous alliance of a pure concrete observable and a mode of rational concatenation is merely an image without thought. If there is science, then it is proof through and through, which is to say, it is logic.[33]

Here we see why Cavaillès claims that Bolzano names the historical point at which the project for a theory of the foundations of mathematics was brought back to consideration of 'Descartes's difficulty'. In sum, for all that it starts from the 'Leibnizian' premise that the change in the 'type of evidence' demanded by modern mathematics meant displacing intuition by recourse to a formal system, Bolzano's intention to construct the system of science as a *linear demonstrative chain*, in which all 'simple irreducible notions' (or judgements) are arranged in a progressive series of ground and consequence, brings him back to the Cartesian question of how to square the movement of thought with the demand that the elaboration of a system of knowledge be epistemically simultaneous. For the Bolzanian theory of science, as Cavaillès impressionistically reconstructs it, it is the whole 'system of science' (or 'structure of science') that is brought into play in the very attempt to give a unitary chain of 'groundings' for the individual propositions of science. It is for this reason that the core of the Thesis on Evidence is a discourse on proof as continuous motion, which draws on the metaphorical architecture of the *Regulae*. Proof is a productive movement that cannot be halted; nor can it be identified within any particular step or link in its internally connected progression; rather, the motion that proof is perpetually reorganises what can appear to be its isolable elements; this is what it means to posit, as Cavaillès understands Bolzano to have done, science as an 'object *sui generis*'.[34]

Yet if a background in Cartesian philosophy to some extent explains how it is that Cavaillès comes to theorise the movement of concatenation, what it does not explain is the equally essential presence of a wholly anti-Cartesian theme in all of his work: history. For in a crucial sense, the movement that Cavaillès sees in Bolzano's reactivation of the Cartesian problem—a movement that 'moves outside of time'—is only visible in the progression of actually produced mathematical works. Thus Cavaillès comes to extract from Bolzano a problem, which he thinks Bolzano himself could not resolve, but which sets the agenda for the rest of *LTS*. On Cavaillès's view, the great merit of Bolzono—a name which can be read here a synedoche for the whole project to found the *autonomy* of modern mathematics—is to have

33 *LTS* 24-25.

34 *LTS* 21.

avoided a double reduction: science, grasped in its true form *as* the movement of thought, is *neither* subordinated to already produced theories (*un existant historique*) *nor* to 'the absolute of consciousness'.[35] This movement is only itself. But if it is itself, how can it have any relation to effectively realised science? It cannot. As such, the theory of science is left with an insuperable division:

> Construction of a pure theory of rational concatenations on the one hand, on the other a relation to science as it has developed [*la science dévelopée*].[36]

A central topic for the remainder of this dissertation is how Cavaillès tries to theorise this twofold position by transforming 'concatenation' into an inter-theoretic category, i.e. by imbuing it with a historical modality. Thus, the Cavaillèsian 'programme', if we allow the term, will identify 'rational concatenation' and 'effective' scientific work. In order to understand what is at stake here, we need to refer the other two sources that I take to be the essential background to the two 'theses' that I extract from *LTS*: Brunschvicg, and Dedekind, and with them—or rather in their conjunction, which Cavaillès effected—a curious transformation of 'Kantian' categories.

B. Concatenation as Synthesis, or, the Productivity of Deduction

The whole of Cavaillès's theoretical endeavour was, in one sense, an internal critique of the epistemological programme he inherited from his dissertation supervisor Léon Brunschvicg, one which seeks to rectify that project in light of contemporary results in mathematical logic. Indeed, it is the peculiar synthesis of the history of philosophy and the history of mathematics that Brunschvicg practised that should be seen as the background to everything we have just discussed. What was at stake for Cavaillès is the way in which modern mathematics in an important manner both validated a crucial aspect of the Brunshcvicgian project and led to its necessary surpassing.

The fundamental background to be retained from Brunschvicg throughout this dissertation is twofold. First, an assertion concerning in what sense, both metaphysically and practically, mathematics and philosophy were to be seen as participating in a common history: in brief, that the 'object' of mathematical philosophy was the history of 'the "royal road" that the creative intelligence has traced.'[37] Second, the way in which this very synthesis is founded by

35 *LTS* 26.
36 *LTS* 26.
37 From Brunschvicg 1911: 176, an article that is then incorporated into Brunschvicg 1912: 459.

Brunschvicg on the concept of the 'act of the mind' (or of intelligence), drawn from a triple reference: Plato, Cartesian philosophy, and Kant. Cavaillès will ultimately extract a radically different concept of the 'act' (or operation) from his studies of modern mathematics, insofar as the concept of the act that is retained in Cavaillès's late work is no longer the act of the mind, and is radically severed from transcendental constitution, to the point where it approaches what I will later call a theory of the act without activity. This late conception is a theorisation of a much broader concern that runs through Cavaillès's writings concerning how to produce a concept that can function both as the unit of mathematical reasoning in the abstract—i.e. as a concept of the *operation*—and as a reference to the concrete work of the mathematician .

Thus one way of understanding the basic problematic of Cavaillès's work is as that of trying to affirm the creative dynamism of mathematics as Brunschvicg had sought to delineate it, but now without reference to the posit of the intelligence as the productive medium of its unity. In this respect, an essential impetus for Cavaillès's project was also Richard Dedekind's work on the foundations of mathematics, from which he extracted the methodological schema of the 'duality' between acts (or operations) and objects. It is against the backdrop of this conjunction of Brunschvicg and Dedekind that we can, at least in part, situate Cavaillès's later attempt to outline a non-psychological theory of creation in mathematics. The rest of this dissertation shall in large part be focussed on the development of this theory. For now, I conclude this introduction by outlining this conjunction.

B.1. Brunschvicg's Grand Narrative

What, then, was 'Brunschvicgian' epistemology?[38] This is not the place to give a general account of Brunschvicg's work, which was one of the defining intellectual projects of the turn of century in France, and which has suffered a fate similar to his contemporaries in Britain (British Idealism or Hegelianism) and Germany (Neo-Kantianism in the extended sense), which is to have been remembered only as the polemical Other of the various schools that would establish '20th century philosophy' as conventionally narrated, be it phenomenology or analytic philosophy.[39] What matters to us here is Brunschvicg's *own* polemic, which concerned a grand

38 Some parts of this and the subsequent sub-section draw from Hare 2022b. I thank Peter Osborne for his editorial comments on that article.

39 For a broader analysis of Brunschvicg's work and its context in the Neo-Kantian milieu of turn of the century France, see Terzi 2022. For a detailed analysis of how Cavaillès adopts certain Brunschvicgian categories, especially with respect to the problem of the modality of mathematical history, see Michel 2020; for a biographical sketch of their relation, Peden 2014: 32–38.

narrative of the history of philosophy, and advanced a series of normative bifurcations in the philosophical field. In Brunchvicg's writings, these bifurcations are always polemical or strategic, but what is at at stake is the construction of a univocal correct tradition in the history of philosophy, with the proviso that this correct line is never unwaveringly held by any single thinker.

It is here that, for Brunschvicg, mathematics—and the discipline of the history of mathematics—intervened. According to Brunschvicg's 'mathematical philosophy', it was the progress of mathematics which rectified philosophical errors, and never the inverse. Thus it was the internal movement of mathematics that vindicated the 'dynamic' Leibniz of the infinitesimal calculus *against* the Leibniz of the *characteristica universalis*, it was the development of non-Euclidean geometries that showed the superiority of the Kant of the Transcendental Aesthetic over the Kant of the Transcendental Analytic, and in his own time it was the development of the concept of intuition by mathematicians such a Poincaré that refuted a project such as Russell's. What mattered then, for Brunschvicg, was always the way in which mathematical history allowed a *reindividuation* of the history of philosophy, in order to bring forth an opposition: judgement opposed to concept, critical as opposed to psychological idealism, the logic of relations against the logic of classes, and so on. Perhaps the most constant of these divisions is what he called the division between *conceptualism* and *intellectualism* as 'two forms of rationalism'. Here is how he presents the split near the climax of *Les Étapes*:

> According to the first, the idea is a *concept* in the Aristotelian and scholastic sense; the essential role of the mind is to grasp the most general terms of discourse, even to the point of exhausting itself in the effort of enclosing them in an initial definition. The second is the intellectualist doctrine of the Platonists and the Cartesians, according to which the idea is an 'action of the mind [*action de l'esprit*]', which we could render as the connection that is operated by judgement and which expresses the very fact of understanding [*de comprendre*], *τό intelligere*. It must be admitted that these two forms of rationalism, the latent antagonism between which seems to us to organise the history of philosophy, have rarely been distinguished with sufficient clarity. Intellectualism, as we have repeatedly observed whilst tracing the course of the doctrine from the Platonic school through to the arithmeticism of the nineteenth century, has not always been able to resist the obsession with purely logical deduction: it has allowed itself to partake in the ambition of those doctrines which, proceeding from the general to the particular, inherently seek to envelop the universe of discourse in the consequences of some initial givens. But the critical examination of the logicist movement, as well as the positive results provided by the analysis of arithmetical and geometrical knowledge, make it pos-

> sible to re-establish and draw out the rhythm proper to the intelligence, as it is found in the production and composition of ideas, in the unification of scientific knowledge.[40]

In a sense, the essence of Brunschvicg's approach is contained here, at least with respect to what Cavaillès will take from it, both in the methodological priority of mathematics over philosophy, and in the unfamiliar scheme by which Brunschvicg organised the history of philosophy itself. On the one side, the path of the 'concept', which unites Aristotle, the 'scholastics', Leibniz's formalising ambition, the Kant of the table of the categories and of the ambition for a critical 'system', and ultimately issues in modern logic and the 'ideal languages' of Russell and Couturat. On this view—the *logical* path of rationalism—the 'idea' is a 'concept' in the sense that is *classificatory*, it follows a species-genera logic and proceeds by the subsumption of generals under particulars. On the other side, we have the path that runs from Plato, through the part of Cartesianism that was best able to emancipate itself from the scholastics, into the Kant of the synthetic judgement and which, of course, Brunschvicg viewed himself as extending. This was the path of properly *mathematical* rationalism, according to which the 'idea' is an *act,* which is always specific.[41] And the place where this act of the intelligence was given to philosophy was in the peculiar *productivity* of mathematical deduction. It was the unity of deduction and activity in mathematics that, for Brunschvicg, indicated the 'royal road' back to Platonism which, following the detour of scholasticism, had been rediscovered by the Cartesians, and specifically by the *Regulae*. Thus, in a passage that is plausibly the primary intertext for Cavaillès's Thesis on Evidence, Brunschvicg writes:

> It is one of the most original conceptions in the *Regulae*, that intuition is defined as an *act of the mind*, as the immediate understanding of a relation. […] There is therefore no difference in nature between intuition and deduction; […] [T]he rings [of the chain of deduction], which are successively forged by that thought whose proper office is the singular intuition, can be gathered together in a total conception in which the chain is *traversed* [*parcourue*] completely and with sufficiently rapid movement that the function of memory can be considered as having been eliminated. Practically at least, deductive synthesis ends up being equivalent to the simplicity of the intuitive act.[42]

40 Brunschvicg 1912: 537. The reference to an *action de l'esprit* in this passage has a footnote to Poincaré's *Science et méthode*; *τό intelligere* is a citation of Spinoza, *Ethics* II, Def. III, *Explicatio.* Brunschvicg then cites the phrase *conceptus actionem mentis exprimere videtur*: 'conception seems to express an activity of the mind'.

41 Elsewhere, Brunschvicg will repeatedly narrate this opposition as that between the 'logic of extension' and the 'logic of comprehension', another category from the post-Cartesian discourse on 'method', this time drawing on the Port-Royal Logic. See Brunschvicg 1904: 759 for a clear statement of this scheme, which draws on Part I, Chapter 6 of *Logic, or the Art of Thinking*, i.e. Arnauld & Nicole 1996: 39–40.

42 Brunschvicg 1912: 141. Emphasis mine. I already underline the notion of 'traversing' [*parcourir*] the chain of deduction, given the importance this concept will assume in Chapters II and III.

By the same token, for such a philosophy of the act of the 'creative' intelligence, which Spinoza would then develop to its fullest expression, 'the mathematical' is a *laudative* concept; it names what is living and dynamic in thought, with respect to which logical analysis and classification can be tools of investigation, but never grasp its essence.[43] To condition thought by this norm of productive *mathesis* is what it is to be Platonist in philosophy, which in fact puts 'intellectualist' rationalism at the antipodes from any philosophy that would assert the 'reality' of abstract, transhistorical forms.[44] In this sense, for all of the detailed work by Brunschvicg on the history of philosophy and of mathematics, both activities are downwind of his being *a* historian of the intellect, of its 'development' or 'unfolding'. Thus everything that appears as the *unity* of philosophy and mathematics is a consequence of the *unifying* power of the intelligence. Mathematics is only the highest expression of this intelligence; it is in a sense how we come to know it.

And indeed, this brings us to the last essential piece of Brunschvicg's construction, which is the way in which—and herein lies his 'Neo-Kantianism'—the virtue of the critical philosophy was to have identified the 'deductive synthesis' of mathematical activity with the real development of the history of science. This is in a sense the last crucial piece of Brunshcvicg's narrative. On the one hand, Kant rectified Cartesian philosophy by placing emphasis on the productivity of synthesis in the effective progress of science, and thus breaking with the tendency of 'classical' accounts of method to imagine the elaboration of a finite combinatory of ideas from which all future science would unfold. On the other, it was a 'Cartesian' (or rather Spinozist) emphasis on the impersonal power of the creative intellect that allowed the psychologism of Kant to be rectified, that is, which allowed the great errors of Kantianism—the doctrines of categories and of the faculties—to be jettisoned. What is at stake in both movements is a return to the fundamental insight of intellectualist rationalism: that the activity of the intelligence is prior to and productive of all the temporary 'forms' through which we then seek to apprehend the intelligence. This leads to an identification of the movement of the intellect

43 To draw on a metaphor from Imre Lakatos—whose discourse is surely the closest analogue to Brunschvicg's *philosophie mathématique* amongst the more celebrated 'options' in 20th century philosophy of science—the importance that logic has for the history of mathematics is thus like that which dissection has for the study of the living animal: it is useful in *some* cases. See Lakatos 1976: 3.

44 To anticipate an argument from Chapter III, this assertion leads to one of the most curious aspects of Brunschvicg's historical synthesis, which is that insofar as Platonically derived intellectualist 'mathematism' is to be pitted against realism about classificatory concepts, Plato is in a sense the true source of nominalism, so long as it is understood that for rationalist philosophy, 'nominalism' is a claim about the form of the concept (that it is specific, rather than subsumptive), not about its 'existence'.

with the movement of science as such. Whence a synthesis of Spinoza and Kant in which the internal activity of the 'intellectual system' is equated with the (historical) genesis of the 'system of science'.[45] What a post-Kantian approach allowed us to see was how it was the history of science, in its 'positivity', that we could apprehend the 'absolute unity of thought' qua intellectual *activity*. For on Brunschvicg's construction, the fundamental structure of the act at stake in the history of science is always that it is an act of unification.

It is this 'Kantian' identification of the synthetic power of thought with the real movement of mathematical history which explains an essential programmatic announcement by Cavaillès, which otherwise might give a mistaken impression of his relationship to Kantian philosophy. Responding to critical comments made by the mathematician Maurice Fréchet following the presentation of his doctoral works in 1939, Cavaillès states that:

> I do not seek to define mathematics, but, by way of mathematics, to know what it means to know, to think; this is basically, very modestly reprised, the question that Kant posed. Mathematical knowledge is central for understanding what knowledge is.[46]

Cavaillès's statement here is an axiom that prescribes an order of investigation: we ought to investigate mathematics in order to understand what thought is, and not the other way around; in other words, there is no *a priori* domain in which questions as to the essence of thought and knowledge can be posed *in advance* of (or conditioning) the progress of mathematics. We must accept this dogmatic aspect if we are to approach this philosophy in good faith. However, Cavaillès's placing of his project in Kant's lineage might be surprising, on two levels. First, because Kant is hardly generally received as having held that mathematics is the model of all thought, and second because Cavaillès is elsewhere firmly critical of the 'Kantian theory of mathematics' (see §I.3.), and *LTS* is in part structured by an extended critique of Kant and post-Kantian philosophies precisely on the grounds that Kantianism is a philosophy which attempts to delineate *a priori* conditions for thought prior to its effective development. So if Cavaillès's project shares a 'Kantian' aim, in what sense? Which 'Kant'? I suggest that it is not the historical 'Kant' who is at stake here, but Cavaillès tactically aligning himself with some-

45 For one of Brunschvicg's clearest statements of this quixotic synthesis, see the summary at the end of 'La place du Spinozisme dans l'histoire' (1906), republished as Brunschvicg 1923: 295–307. In one of his characteristically strident assertions, Brunschvicg concludes this piece by stating that 'it was thus not despite Kant, but through Kant himself, that there arose in the final years of the 18th century a rejuvenation and a renewal of the rationalism of the *Ethics*.' We should note that it is Brunschvicg's terminology of the 'system of science' which Cavaillès invokes with respect to Bolzano in the Thesis on Evidence.

46 *OC* 625. See Bertin 2025 for an excellent commentary on the programmatic side of this statement.

thing of the project advanced by Brunschvicg, who was in the audience when Cavaillès made this pronouncement.

Thus the last piece of Brunschvicgian 'doctrine' we need in order to situate Cavaillès's project relates to Brunschvicg's direct treatment of Kant in *Les Étapes*. On Brunschvicg's reading, 'mathematical philosophy' is 'the cornerstone of the *Critique of Pure Reason'*, and the core of the Kantian programme is to have posed the problem of synthesis in such a way that, as Kant puts it in the introduction to the B edition of the *CPR*, the questions 'How is pure mathematics possible?' and 'How is pure natural science possible?' are identified under the 'formula of a single problem': 'How are synthetic judgements *a priori* possible?'[47] As Brunschvicg summarises, 'Kant realised that the solution of the problem with respect to the science of nature is the corollary of an analogous problem that, rather than solely concerning physics—that is, the application of mathematics to experience—is internal to mathematics itself.'[48]

According to Brunschvicg, then, the novelty of Kant thus lay in his having attempted to resolve the problems inherent in the application of thought to reality by positing a strict analogy with, so to speak, *the application of mathematics to mathematics*, such that, to use the classic example, the subsumption of the concepts 'five' and 'seven' under the concept 'twelve' exhibits synthesis in its pure form, with respect to which empirical cases of synthesis (say, subsuming the manifold of sensory data that is experienced when looking at the fingers of a normal human hand under the concept 'five') is only a special case of this pure activity.[49] It is in this sense that the *CPR* can first and foremost be read as a 'mathematical philosophy', with respect to which so-called 'sensory experience' is only a derived form. In turn, Brunschvicg locates Kant's central innovations with respect to the problem of pure mathematics as a response to the problem of concatenation that d'Alembert takes up from Descartes: why is there—or indeed, is there?—more in a 'chain' of deduction (2 + 2 = 4) than in (immediate) intuition (2 + 2 = 2 + 2)? Why was the whole content of a mathematical proof not there from the start? D'Alembert's answer—drawing on Cartesian terminology—is that on the level of the rational contents themselves, there is no novelty in this process, but merely the progressive unfolding of an initial definition:

> What indeed are most of those axioms of which Geometry is so proud, if not the expression of a single simple idea by means of two different signs or words? Does he who says *two and two equals four* have more knowledge than the person who would be contented

47 Brunschvicg 1912: 257; Kant, *Critique of Pure Reason* (1998), B19-20.

48 Brunschvicg 1912: 256.

49 I am borrowing the phrase 'application of mathematics to mathematics' from Hacking 2014.

> to say *two and two equals two and two*? […] If one examines a succession of geometrical propositions […] it will be observed that they are all only the first proposition which is successively and gradually reshaped, so to speak, as it passes from one consequence to the next, but which, nevertheless, has not really been multiplied by this concatenation [*enchaînement*]*;* it has merely received different forms. […] Thus, the concatenation of several geometrical truths can be regarded as more or less different and more or less complicated translations of the same proposition and often of the same hypothesis.[50]

On this picture, what appears as progress for consciousness is only the gradual recognition of a rational arrangement that was latent in the relevant concepts. For Brunschvicg, the Kantian revolution lies in his having taken the opposite stance: there is more in the conclusion of a demonstration than was present at the outset, and this supplement is added by *a priori* synthesis, such that in the progressive steps of a demonstrative chain we glimpse the activity of the mind or intelligence in its pure form (in its *productivity*), which will be seen to be identical with the activity (or progress) of science itself. Kant thus answers the problem of concatenation or deduction by transforming it into the problem of synthesis:

> The place of *a priori* synthesis does not lie in the connection between the terms of a judgement, or in the demonstration of such and such particular 'numerical formula': it lies in the general process from which every particular number is derived, that is, *in the creation of the notions themselves*.[51]

As I will discuss momentarily, on this interpretation, Kant strongly prefigures Dedekind's analysis of mathematics as the 'necessary generation' of concepts. And this doubly so given that for Kant the primary moment of synthesis will be found in the basic arithmetical operations, interpreted as *acts*, such that, on Brunschvicg's reading, it will be 'in order to do justice to the sign +' that Kant will require the doctrine of the schematism, i.e. that it is the synthetic unity of the understanding that underwrites the entire synthetic process.[52] It is this ineliminable role given to the constructive activity of the mathematician which allows a reading of Kant as a philosopher of mathematical creativity, as Cavaillès notes in *MAF* (without yet taking the crucial step—which will define his later work—of rejecting Kant's theory for its very subjectivism): 'It is the synthetic activity of the *I think* that justifies the two characteristics of mathematical work: unpredictable becoming and absolute value. Absolute value because synthesis is

50 D'Alembert, 'Discours Préliminaire des Éditeurs' to the *Encyclopedie*, 1751, pp. ix., translation from d'Alembert 1995. Translation modified. For Brunschvicg's citation of this passage, see Brunschvicg 1912: 270.

51 Brunschvicg 1912: 270. Emphasis mine.

52 Brunschvicg 1912: 271. We would have to add here 'qua "inner sense"', but Brunshcvicg does not make this specification.

required by the unity of apperception, unpredictable becoming because there is an effective constructive activity.'[53]

It should be clear here why, on this reading, it made sense to ascribe to Kant the question that Cavaillès attributes to him: how does mathematics tell us what it is to think, what it is to know? But in all of this, we must note an essential assumption which Brunschvicg took to underwrite the Kantian approach, which is the assumption of the *homogeneity of mathematics*, indeed that the absolute unity of mathematics is what made it paradigmatic of the unity of thought. Methodologically, this was reflected in Kant's assumption that the activity at stake in elementary mathematical examples is sufficient to give a philosophical basis for the treatment of higher mathematics, such that developments at a higher level of technical complexity will merely appear as special cases, and cannot be expected to threaten the theoretical edifice that has been developed with reference to simple cases: the *form* of synthesis needed to assert that 7 + 5 = 12 could not be any different from that required by Galois theory or modern topology. Thus, in one sense, Kant made an essential contribution to 'mathematical philosophy' by producing an immanent philosophy of science:

> For the first time […] with Kant's doctrine concerning mathematics, the theory of science is, in relation to science itself, placed neither *above* science (as with Cartesian or Leibnizian metaphysics, which subordinate the principles of reason to theology), nor *below* science (as with English empiricism, which does not see mathematical notions as anything more than approximations of experience); the Kantian theory of science is exactly at the level of science itself.[54]

Yet, this immanence is achieved by insisting, by fiat, on the *operative homogeneity* of science, such that the concepts at work at its *avant-garde* will not be different in kind from those at stake at its most fundamental levels.[55] It is for this reason that, despite their internal divergences, the various attempts at the turn of the 20th century to delimit a 'secure' domain of mathematical activity—from Kronecker to Brouwer and Hilbert—by referring all constructions back to a finite intuitive basis, all index themselves as belonging to Kant's lineage. As far as Brunschvicg was concerned, writing in 1912, it was not necessary to take a final stance on this debate: one could maintain a division which recognised Kant's essential contribution to foundational questions whilst leaving other avenues open when it came to developments at the forefront of modern mathematics. The Cavaillèsian research programme essentially begins

53 *OC* 34–35.
54 Brunschvicg 1912: 271.
55 Brunschvicg 1912: 271.

once this *pax romana* is broken, that is, once it is no longer acceptable to separate the domains of 'technical' and 'foundational' mathematical work, which is precisely what he saw as the necessary result of the then-contemporary developments he approached in his epistemological studies. Answering 'the question that Kant posed', in the context of the foundational debates of the 20s and 30s, ultimately required a rejection of the Kantian programme at a quite fundamental level. The ground on which Cavaillès will articulate this break is the refusal of the thesis of the homogeneity of operations in mathematics, which equally entails a break with the thesis that mathematics deals with any particular or secure domain of 'objects'.

B.2. Dedekind and the 'Necessary Generation of New Concepts'

Whence Dedekind. For one way to read Dedekind bearing in mind all that has just been said is as having attempted to theorise Brunschvicg's theory of the synthetic act qua 'the creation of the notions themselves' directly on the level of mathematical practice. This is how Cavaillès received Dedekind's 1854 *Habilitation* address 'On the Introduction of New Functions in Mathematics'.[56] As indicated above, I take Cavaillès to have extracted two essential themes from this text, and from Dedekind's project more broadly. First, a model of the necessarily *successive* character of formalisation in mathematics. Second, an approach to the mathematical object (and its dual, the operation) which Cavaillès will return to in many guises over the course of his writings. Both themes are visible in the opening of Dedekind's address:

> This lecture is not about the introduction of a determinate class of new functions into mathematics (although one could perhaps interpret the title in that way); rather, it is about the general manner in which, in the progressive development of this science, new functions, or, as one can equally well say, new *operations* [*Operationen*], are added to the chain [*Kette*] of previous ones.[57]

We find here a fundamental problem that will inform all of Cavaillès's work: the conjunction of *necessity* and *creation*. The problem that Dedekind considers in this lecture is the movement involved in the introduction of 'ideal' objects and operations in the development of mathematics. On Dedekind's view, whilst every science develops through the gradual introduction of

56 This lecture was not published in Dedekind's lifetime, and was first widely circulated in 1932 through Emmy Noether's edition of Dedekind's complete works, i.e. in Dedekind 1932: 428–443. This means that Cavaillès encountered the text as a 'new' discovery during the period of his doctoral studies. He would go on to edit the Cantor-Dedekind correspondence in collaboration with Noether, which was published alongside his dissertations, i.e. Cavaillès & Noether 1937. All translations of Dedekind's writings here are taken from Ewald 1996a.

57 Dedekind 1996: 755.

new notions, the signal feature of mathematics is that such a process of introduction is necessary, i.e. the extension of the domain of objects and operations emerges from the kernel of the initial definitions in a regulated manner:

> [I]n this mathematics is distinguished from other sciences—these extensions of definitions no longer allow scope for arbitrariness but follow with an absolute necessity from the earlier primitive definitions, provided one applies the principle that the laws which flow from the initial definitions and which are characteristic for the concepts that they introduced have *universal validity* [*sind allgemeingültig*]. Then these laws conversely become the source of the generalized definitions if one asks: How must the general definition be conceived in order that the discovered characteristic laws be always satisfied?[58]

In technical context, Dedekind's initial focus is on the reciprocal extension of the field of objects (i.e. the *Zahlgebeit,* the number domain) and operations in the development of arithmetic and algebra. There is a double movement at stake here: the 'generation' of objects from operations, and the redefinition of operations in order to incorporate the objects that have been generated by the unrestricted application of the initial operations. Taken from one side, the extended application of the basic arithmetical operations of addition, multiplication and exponentiation and their inverse operations immediately necessitates that we 'create the entire existing domain of numbers anew': starting with an object domain restricted to the positive whole numbers, the rigorous filling out of the space implied by these basic operations and objects leads us to 'the negative, rational, irrational, and finally also the so-called imaginary numbers.'[59] It is in this sense that we have *created* the negative numbers, etc. Taken from the other side, a correlative modification of the domain of operations is now necessary, as they were not initially well-defined for all of the objects (i.e. number classes) that have been occasioned by their application: for example, exponentiation initially has no meaning for the case of negative numbers or fractions; it must thus be redefined in a more general setting by giving the general theorem for the addition of exponents, but this is to situate the operation of exponentiation on a higher plane, to give it a new meaning. Hence, we generate new definitions, such the law of exponentiation '$x^{a+b} = x^a \cdot x^b$', which is created by replacing all concrete numbers by abstract variables.[60] As Cavaillès summarises, in the dialectic at stake here:

58 Dedekind 1996: 756–757. Translation modified in line with Cavaillès's interpretation of the passage at *MAF* 54/*OC* 61.

59 Dedekind 1996: 757.

60 This is the form of abstraction that Cavaillès will call 'idealisation' or 'generalisation' in *MAF*, and later resituate in *LTS* as the concept of 'the paradigm'. See my discussion in Chapter III.

> Necessity intervenes in a double movement: on the one hand, the posited restricted operation demands, either in itself or in relation to already existing operations, an extension of the domain of individuals over which it ranges; on the other hand, in this new domain, the posited relations provoke the substitution of a new definition for the initial definition. […] The choice is not arbitrary: every posited definition immediately generates connecting threads with the previously existing system, but it is the whole bundle of them that is, in reality, to be understood as the new definition, which only condenses them to the highest degree.[61]

Dedekind has here sketched a research programme into the introduction of new operations or functions in mathematics. However, it contains a basic tension between two concepts that would traditionally be seen as opposed: necessity and creation (or 'generation' [*Erzeugung*]). Naïvely: if the 'new functions' introduced to account for the expanded application of operations were necessary, why were they not there already? Indeed, from a Kantian perspective the conjunction is nonsensical: given that, from the standpoint of the *Critique of Pure Reason* at least, *universality* and *necessity* are functionally identified, it makes little sense to speak of the necessary emergence of 'new functions', as *per definition* necessary and universal concepts should be *a priori*.[62] Cavaillès himself notes that Dedekind does little to resolve the tensions his account leaves concerning 'this necessary generation of new concepts'.[63]

Schematically, what Cavaillès will extract from this is a model of the dialectical co-constitution of operations and objects, i.e. what Cavaillès's student Gilles-Gaston Granger will call, in his various elaborations on the Cavaillèsian project, the perspective of *operation-object duality*.[64] In effect, this involves giving a primacy to the operation that is highly unusual in the history of philosophy: insofar as the domain of objects is seen as being *produced* by the development of operations, 'objects' no longer have any *a priori* status. Cavaillès's own language for this was that of the duality between *act* and *object*. Thus as he summarises in the conclusion to his doctoral thesis on the axiomatic method, returning to the earlier exposition on Dedekind:

> The duality formed by the object and the procedure acting on it is the mask of the dialectical surpassing of one method by another, the objects posited independently by the latter being the correlates of the former.[65]

61 *MAF* 53–54/*OC* 61–62.

62 For a profound elaboration on this 'hidden principle' of Kantian and post-Kantian philosophy, see Halimi 2013, in particular the second chapter.

63 *MAF* 54/*OC* 62. Note that Cavaillès has turned Dedekind's assertion about the introduction of 'functions' and 'operations' into one about the generation of *concepts*.

64 The elaboration of this perspective is a central theme in Granger's work, but see in particular Granger 1988: 67–92.

65 *MAF* 178/*OC* 186.

In the background of this statement is Dedekind's own vocabulary, for throughout his work he uses the terminology of 'act [*Akte*]' and 'operation' interchangeably. Thus, for Dedekind, in the process of the production of new definitions, which as Cavaillès summarised 'condenses' all of its connections with the previous mathematical situation, what this condensation produces is a new act. Now, according to Dedekind, this production of the new act relies on certain elementary 'natural' acts. This is a red thread running through Dedekind's work, such that the methodological perspective announced in the 1854 *Habilitation* address according to which mathematics was to be reconstructed as a chain of operations is the background to the more sophisticated formalisations he would later produce of the foundations of analysis and arithmetic, which Cavaillès will repeatedly present as crucial to the emergence of modern mathematics, both in terms of the development of set theory and the methodological self-consciousness of the axiomatic method. This is perhaps best expressed in a passage resuming the perspective of the 1854 *Habilitation* address in the 1872 essay *Continuity and Irrational Numbers*, i.e. the work in which Dedekind develops his celebrated definition of the real numbers in terms of so-called 'Dedekind Cuts':

> I regard the whole of arithmetic as a necessary, or at least natural, consequence of the *simplest arithmetical act*, that of counting, and counting itself is nothing other than the successive creation of the infinite series of positive integers in which each individual is defined by the one immediately preceding; the simplest act is to pass from an already-created individual to its successor that is to be created. The chain of these numbers already forms in itself an exceedingly useful instrument for the human mind; it presents an inexhaustible wealth of remarkable laws which one obtains by introducing the four fundamental operations of arithmetic. Addition is the result of bringing together into a single act an arbitrary number of repetitions of the simplest act; multiplication arises from it in a similar way.[66]

Over the course of his work, Dedekind will increasingly radicalise this perspective, such that he will attempt in the 1888 work *What Are Numbers and What They Should Be?* to subtract even the initial evidence of arithmetic, and reconstruct the notion of natural number on an even more primitive pair of acts, that of what he calls gathering things together into a 'system' (i.e. what will later be called a 'set'), and that of producing a 'mapping' or 'representation' between systems (*Abbildung*). Along with Cantor's work, this can be seen as the genesis of the 'set-theoretical' perspective in mathematics. Thus the 'act + 1', which would be a paradigm of evidence for Descartes and of *a priori* intuition for Kant, is now conceived as the 'mapping' $\mathbb{N} \to \mathbb{N}$, i.e. a map that goes from $\mathbb{N}$ to itself. It is for this reason that, more than any other figure,

66 Ewald 1996a: 768. Emphasis mine.

Cavaillès takes Dedekind's theoretical development—speaking of the whole *œuvre*, not only of the *Habilitation* address—to exemplify the transformation in the stakes of the 'problem of evidence' in the 19th century. Evidence is no longer the foundation of a chain of reasoning, but rather produced by a formalisation; not even the natural numbers are 'simple ideas'.[67] From the standpoint of Dedekind's late work, which on Cavaillès's analysis will lead more or less directly to the Hilbert programme and the development of proof theory, it is not the case that we 'know' what the sequence of natural numbers *is*, and from there produce an axiomatisation of this 'pre-formal' knowledge; on the contrary, it is in producing a formalisation that we understand what numbers are. Whence the tense character of Dedekind's treatment of arithmetical operations or acts: on the one hand, arithmetic and the associated number concept is seen as the most 'momentous' creation of thought, 'inseparably bound up with the mind's most inward nature'; on the other, the concept of number itself is not *primitively given* but rather *deduced*.[68] Thus the 'deduction' or 'creation' that produces the foundations or arithmetic corresponds to a general methodological perspective according to which 'numbers are free creations of the human mind'.[69] What is at stake here is thus not only a standpoint on the foundations of mathematics, but an entire ethos of epistemological practice according to which the seemingly intuitive must be viewed as the product of a long 'chain' of conceptual development:

> [M]ost people think that everything which has become familiar to them, unconsciously and mechanically, through lifelong practice, is also simple; they do not ponder at all how long the chain of thoughts [*Gedanken-Kette*] frequently is, that thereby enters into play.[70]

It is for this reason that Dedekind can be read as having most fundamentally thematised the methodological perspective proper to modern mathematics that Cavaillès expresses in the Thesis on Evidence with respect to Bolzano: evidence is produced by a concatenation, or, in Dedekind's own terminology, by a 'chain of operations'. Cavaillès's epistemological project can fruitfully be framed as having tried to apply this model to a philosophy of mathematical history, such that the chain of operations or 'concatenation' at stake here is transformed into a

67 As Ferreiròs summarises, this is a major development from the perspective of the 1854 address: 'Previously, natural numbers had been the foundation on which all of arithmetic was developed [...]. Now it seemed possible to define the natural numbers themselves on the basis of sets and maps.' Ferreirós 2002: 224.

68 The citation concerning 'mind's most inward nature' is from the *Nachlass,* Cod. Ms. Dedekind III, 2, translated in Ewald 1996: 837.

69 Dedekind 1888/Ewald 1996a: 791.

70 Dedekind, letter to Frobenius of February 1895, cited from Dugac & Dedekind 1976: 283. Translation from Ferreirós 2002: 242.

discourse on the historical production of theories. Thus the Dedekindian scheme of the 'necessary production of new concepts' becomes a discourse on the form of history that, in a sense, defines the mathematical enterprise. This of course involves a displacement, in which we can see something of Brunschvicg's project in the background. For in Dedekind, the successive movement of formalisation as presented under the banner of the chain of operations is the ideal model of mathematical construction, and not of real mathematical history. It is this post-Brunschvicgian transformation—or even appropriation—of the Dedekindian scheme that sets the stage for Cavaillès's reflections.

However, in the repeated references in Dedekind's discourse to the 'activity of the mind' or of the 'understanding'—which indeed are what make a sympathy with Brunschvicgian epistemology seem all the more plausible—we can find the point with respect to which Cavaillès will ultimately break from the 'Dedekindian' perspective. On Dedekind's view, the *successive* reconstruction of mathematics that he envisioned was required by the limitations of our faculties, i.e. by what he repeatedly referred to as 'our step-by-step understanding', our *Treppenverstand*.[71] It because *we* are limited, because *we* cannot 'see things all at once', that the methodological standpoint of the generation of concepts in required. This methodological standpoint was present from the 1854 *Habilitation* onwards:

> Science itself, which represents the course of human knowledge up to these results, is infinitely manifold, is capable of infinitely different representations [*Darstellungen*]. This is because, as the work of man, science is subject to his arbitrariness and to all the imperfections of his mental powers. There would essentially be no more science for a man gifted with an unbounded understanding [*Verstande*]—a man for whom the final conclusions, which we attain through a long chain of inferences [*Kette von Schlüssen*], would be immediately evident truths [*evidente Warheiten*]; and this would be so even if he stood in exactly the same relation to the objects of science as we do.[72]

With respect to the Thesis on the Intelligible, we can thus also read Cavaillès's 'inversion' of Cartesian terminology as a more or less precise radicalisation of the Dedekindian perspective on mathematics. In a sense, Cavaillès's dissatisfaction with Dedekind will be that in the re-

71 Most famously in the introduction to Dedekind 1888: 'This memoir can be understood by any one possessing what is usually called good common sense; no technical philosophical, or mathematical, knowledge is in the least degree required. But I know that, in the shadowy forms which I bring before him, many a reader will scarcely recognize his numbers which all his life long have accompanied him as faithful and familiar friends; *he will be frightened by the long series of simple inferences corresponding to our step-by-step understanding* [*Treppenverstand*], by the matter-of-fact dissection of the chains of reasoning on which the laws of numbers depend, and he will become impatient at being compelled to follow out proofs of truths which to his supposed inner consciousness seem at once evident and certain.' Ewald 1996a: 791. Emphasis mine.

72 Dedekind 1932/Dedekind 1996: 755–756.

course to the idea of a pre-constituted form of the understanding, with it's simple intuitions of 'system' and 'map', Dedekind had failed to critically apply his own procedure. In his concluding comments on Dedekind from *MAF*, Cavaillès in a sense closes the circle I have been drawing throughout this introduction by phrasing both the significance and the failures of the Dedekindian project in Cartesian terms:

> Thanks to them [i.e. the notions of system and map], the work [i.e. Dedekind 1888] gives the effective development of the first elements of number theory. It's not a reconstruction—such as Kronecker's arithmetisation—it is the theory itself, in which our 'step-by step understanding (*Treppenverstand*) doesn't want to skip any stage.' '*In Anfang war die Abbildung*'.

> The [i.e. Dedekind's] recourse to the activity of the mind, if it enables us to understand the unpredictable development of mathematics, at the same time confers upon it a perfect rigour and intelligibility, provided we accept *as clear and distinct* the correlative notions of system and application. We know how Russell's paradox in 1903 showed that this was an illusion [...].[73]

The failure of the Dedekindian enterprise in the face of the set-theoretical paradoxes was thus ascribed to a certain 'Cartesian' naïvety inherent to the project. In presuming that 'map' and 'set' (i.e. 'system')' were 'clear and distinct' notions, adequate to the nature of our understanding, Dedekind falls into the trap of positing simple natures (or 'intuitions') *prior to concatenation*, whereas his whole project ought to suggest (methodologically) the ruin of such an enterprise. Lying behind this claim is the truly radical nature of Cavaillès's project as stated in the Thesis on the Intelligible, which is that, as was in a certain sense ratified by then-contemporary results in mathematical logic, any posit of a pre-constituted structure to mathematical reasoning was doomed to be overcome by the very movement of mathematics. If we want to understand the line that runs from this assertion to the critique of the 'philosophy of consciousness', it can in a sense also be put as a transformation of Dedekindian (or rather Kantian) terminology: the successiveness or motion of science is not due to the limitations of our 'finite' faculties, but to the nature of reason itself. From *Treppenverstand* to *Treppenvernunft.* This dissertation will outline how Cavaillès arrived at this perspective.

73 *MAF* 57-58/*OC* 65-66. Emphasis mine.

I. THE EFFECTIVE AS THE ACTUAL AND AS THE CALCULABLE

> Personally, I am very reluctant to posit anything that would dominate the effective thought of the mathematician. I see an exigency in the problems themselves.
>
> Cavaillès, responding to Lautman, 4th February 1939[1]

> There are two things in this word 'sufficient grace'. There is the sound, which is nothing but the wind, and the thing itself, which is real and effective.
>
> Pascal, *Provincial Letters*, II, 29th June 1656

If Cavaillès's work—incomplete, more a promise than a theory—continues to assert a fascination for philosophers and historians of mathematics it is because he outlined a research programme that would aim to progressively reveal what he designated as the 'objectivity, mathematically founded, of mathematical becoming.'[2] Cavaillès's claim for an identity between the objectivity and the historicity of mathematics is what grounds his reputation as a central figure in the tradition of historical epistemology of mathematics. This same claim harbours an implicit definition of mathematics, that is, of its specificity: mathematics is the sole domain of thought whose rational character lies in its transforming its own objective (symbolic, conceptual, practical) conditions in a regulated manner, whence Cavaillès's characterisation of mathematics as an experimental activity. In turn, the singularity of mathematical concepts is to be sought not in their generality, but in the specificities of such transformations, i.e. in the way in which mathematical concepts change in history, in their mode of passage.

The novelty of Cavaillès's project thus consists in his having practised a new norm of reading the history of mathematics: to always read the body of already produced mathematical works under the image of *movement*—that is, as indicated above, as a becoming—rather than as a completed foundation. Central to the elaboration of this norm in Cavaillès's truncated *œuvre* is his definition of mathematics as *effective thought* [*la pensée effective*] or *effect-*

1 *OC* 628. An earlier version of this chapter, without the material on axiomatic set theory and the Hilbert programme, was published as Hare 2022a. It was developed out of a presentation given at FPMW 13, hosted at l'Université Côte d'Azur, 7-9 October 2021. I thank the organisers and respondents at that event, whose engagement has improved this work substantially.

2 *OC* 226.

ive work [*travail effectif*].[3] This designation intermingles philosophical reflection on the effective (or the efficacious) as the *actual* with a precise mathematical or technical history, namely the development of the concept of *effective calculability.* This can broadly be delineated into the three somewhat distinct phases of, first, the discourse of the 'French analysts' or 'French empiricists' (principally Émile Borel and Henri Lebesgue), second, the development of mathematical logic and proof theory in the context of the Hilbert programme, and, finally, the formation of the notion of (general) recursive functions in the Herbrand-Gödel formalisation and of the modern notion of computability as it is ultimately formalised in the work of Alonzo Church, Stephen Kleene and Alan Turing.

A focus on the concept of the effective highlights how, for all that Cavaillès has a reputation as a philosopher of *necessity*, he can equally be read as having advanced a philosophy of *actuality* in light of modern mathematics. In this sense, Hourya Benis Sinaceur is correct that attention to the different modalities of the concept of effectiveness should be a guiding thread for reading Cavaillès. However, I would like to suspend the gesture which grounds Sinaceur's analysis of the role of effectiveness in Cavaillès's 'theory of reason', which is to view Cavaillès's use of the concept as a direct extension of the Hegelian (and post-Hegelian) discourse on *Wirklichkeit*.[4] I do this not due to any ultimate incompatibility between the Cavaillèsian and Hegelian theories of actuality, but because it is necessary—at least initially—to preserve the autonomy of the referent in Cavaillès's philosophical lexicon if we are to analyse Cavaillès's contribution to mathematical philosophy. In any case, with respect to the philosophical referents of the term, it is important to note that within Cavaillès's immediate milieu, the term *effectif* (in its various noun and verb forms) was not straightforwardly analogous to the German *wirklich/Wirklichkeit*, even if it would later often assume that connotation.[5]

3 See *OC* 85, *OC* 100, *OC* 106, *OC* 180, *OC* 187, *OC* 540, *OC* 628.

4 See Sinaceur 2013: 86. 'L'effectivité hégélienne [...] constitue un fil conducteur essentiel pour lire Cavaillès [...].'

5 Exemplary in this regard is the entry *RÉEL* from Lalande et. al.'s *Vocabulaire critique et technique de la philosophie* (1902-23). There we find a sharp opposition between two senses of the 'real' in philosophy:

> Sense 1. The 'real' as that which is *given* [*donné*] or immediate, i.e. as that which is contemporary with thought (in the sense of *actuel*), and evident for it, 'present';
>
> Sense 2. The 'real' as what produces an effect, i.e. as that which 'acts effectively [*agit effectivement*]', or which has 'the character of efficaciousness [*efficacité*]'.

The authors of the *Vocabulaire* strictly identify Sense 1 (i.e. the real as the given) with 'the German terms *wirklich*, *Wirklichkeit*' (Lalande 1980: 901–902), and suggest that this is the sense at stake in the Kantian use of the concept. A general philological hypothesis could be that in this period, French philosophers chose *réel* or *donné* as a correlate to *wirklich* when they wished to emphasise Sense 1 and *effectif* (and corresponding forms) when they wished to emphasise Sense 2. But this latter modality of the term has its roots in

With respect to the mathematical referents, my interest here is not so much in giving a technical exposition of the problem of 'effectuability' in the mathematics of the first half of the 20th century, as in exhibiting Cavaillès's treatment of it, i.e. something of his epistemological practice as it appears in his doctoral dissertations. This is significant for the general economy of the our argument in this thesis, for two closely related reasons. The first concerns a periodisation of Cavaillès work. The second concerns how Cavaillès ultimately comes to formulate a critique of what he calls the 'Kantian theory of mathematics' in terms of its conflation of *effective process* [*procès effectif*] and *effectuated process* [*procès effectué*], a distinction which we shall argue gives the formal structure to Cavaillès's later conceptualisation of the problems posed by the 'theory of science'.

With respect to the periodisation of Cavaillès's philosophical development, my argument will be that his final essay *On Logic and the Theory of Science* should be understood as taking up philosophical problems posed by his earlier epistemico-historical studies of modern mathematics, but from a different standpoint, that of an abstract theorisation of the form of the production of mathematical concepts. Hence, Chapters I and II are primarily concerned with Cavaillès's epistemological methodology in the period prior to the composition of *LTS*, whereas Chapter III will be concerned with the conceptual categories developed in *LTS* for theorising what he comes to call 'mathematical concatenation'. In a word, if the period of the doctoral studies was devoted to writing a history of modern mathematical logic, in the final works Cavaillès aims to produce a *theory* of that history, or rather, a theory of what is at stake in the very historicity of mathematics. Heuristically, these two periods correspond to Cavaillès's respective projects for a philosophy of 'mathematical experience' and for the 'philosophy of the concept' advertised in the final lines of *LTS*. As we shall see, the question of the continuity between these two projects is rather fraught.

For what concerns Cavaillès's doctoral studies, they represent the culmination of a period of work that ran from 1928 to 1938 on the foundations of modern mathematics and logic. Cavaillès had completed his *licence* at the Lycée Louis-le-Grand in 1923, and entered the

post-Comtean discourse—i.e. in the (normative) opposition between 'positive', 'effective', or 'concrete' science (that is, science as it is 'actually practiced', or in the development of its 'effective history') and merely 'abstract' science—as much as it does in the German post-Kantian tradition. In the post-war, the translation of the specifically Hegelian conception of *Wirklichkeit* by *la réalité effective* would gain increasing currency, especially following Jean Hyppolite's convention in his translation of and commentary on Hegel's *Phenomenology of Spirit*, leading to a situation where nowadays one regularly finds *Wirklichkeit* rendered directly by *l'effectif* (e.g. in the title of Kervégan 2007), but this was not the case for Cavaillès's generation (e.g. Koyré 1931, where the term is still rendered simply by *la réalité*).

École Normale Supérieure in the class of 1924 along with his future collaborator Georges Canguilhem. His admission was owed to a special track allowing him to enter without Greek, based on his competence in mathematics, which had fascinated him from a very young age.[6] Having passed the aggregation in 1927, he commenced his doctoral studies under the supervision of Brunschvicg. Initially envisioned as an investigation into the emergence of set theory in the 19th century, this work would ultimately produce three major documents on the panorama of research into the foundations of mathematics up until the mid-1930s: his *thèse principale*, *Méthode axiomatique et formalisme* (1938), his *thèse complémentaire*, *Remarques sur la formation de la théorie abstraite des ensembles* (1938), and the publication of the *Briefwechsel Cantor-Dedekind* (1937), edited in collaboration with Emmy Noether.

If one theme predominates in Cavaillès's understanding of his work in this period, it is perhaps the ambition to produce a theory of creation in 'mathematical thought'.[7] These are the terms in which he presented his doctoral project in a letter to Abraham Fraenkel on 9th December 1930, while inquiring as to the status of Cantor's archive:

> I am preparing, under the supervision of Pr. Brunschvicg of Paris, a thesis on the origin and formation of set theory, up until around 1890; this will be an attempt at a philosophical presentation of mathematical creation. But first, insofar as such a thing is possible, I need to follow down to the finest details, from a psychological point of view, the development of fundamental ideas in creative mathematicians [*les créateurs*] and, in particular, in Cantor.[8]

A significant gulf separates this presentation of his project from his later theoretical commitments. By the time of Cavaillès's dissertation defence on 22nd January 1938, Brunschvicg would reproach him for not having adequately addressed the psychology of mathematics, and Cavaillès would respond, 'But sir, that is in no way the problem I have chosen to examine and it is of no interest for that problem.'[9] Yet if by the late 1930s Cavaillès was arrayed decisively against the 'psychological point of view', the focus on the necessary creation (or generation) of concepts in mathematics remained central, with Dedekind's foundational studies being a fun-

6 Racine 2002: 251.

7 In a letter of 22nd April 1928, after meeting with Brunschvicg to set his dissertation topic, Cavaillès writes to his sister Gabrielle Ferrières that 'the point of view that interests me' is to 'study mathematical thought, in its development, the mechanisms of its creations, and, perhaps—a bit utopian, but it's what I'm after—the hidden conditions that necessarily govern its unfolding.' See Ferrières 2003: 69/Ferrières 2000: 36.

8 Ferrières 2003: 92/Ferrières 2000: 59. Translation modified.

9 The remark is taken from a letter of 24th January 1938 from Cavaillès to his father, who had been been too ill to attend the defence. See Ferrières 2003: 139–141/Ferrières 2000: 105–107.

damental inspiration in this regard.[10] Indeed, the desire for a rigorously non-psychological theory of creation can be seen as a central aspect of the theoretical project later pursued in *LTS*.

Approaching the doctoral studies on their own terms, one is struck by what Pascal Bertin has described as an 'essentially methodological astonishment'.[11] Aside from some brief theoretical considerations in the introduction and conclusion to *TAE* and the long conclusion to *MAF*, Cavaillès's doctoral works are on the surface direct works of mathematical commentary. They are in some sense 'histories of mathematics', but they are histories in the making. Cavaillès is reading and commenting on—attempting to make intelligible—mathematics in the very act of its creation. He treats a wide range of historical mathematical material, but always in light of its most recent developments, of what was being produced by working mathematicians often only months before his own writing. Cavaillès's philosophical categories are thus always determined by a history he was living. This tends to make these very categories highly—conceptually, lexically—overdetermined. In the first order, this poses difficulties on the level of raw comprehension: in order to understand the concepts that Cavaillès deploys, one has to reconstruct something of their (often implicit) mathematical referents, but this reconstruction cannot rest on appeal to any external formalisation of the mathematical content of the concepts at stake. It is always a matter of trying to reconstruct how such and such concept was, often partially, grasped at a particular conjuncture. The difficulties here are increased by the idiosyncratic displacements made in Cavaillès's texts, often operated by means of translation. The philosophical interest of Cavaillès's work cannot be separated from 'this method of multilateral conditioning', as Canguilhem refers to it in the obituary he wrote for Cavaillès.[12]

At stake here are not only difficulties of comprehension, but also a problem that is at the centre of much contemporary contestation as to the global interpretation of Cavaillès's theoretical production: is the designation 'mathematical thought' in effect to be identified with 'thought' *simpliciter*? This question is tightly bound up with that of how to periodise Cavaillès's work, and the problem therein of the autonomy of philosophical discourse: is *On Logic and the Theory of Science* to be understood as a work of 'philosophical' logic that methodologically breaks with an equation between 'thought' and 'mathematical thought' that is im-

10 See my comments in the Introduction, B.2.

11 Bertin 2025 (forthcoming).

12 Canguilhem 1947: 151.

plicit in the period of Cavaillès's doctoral works?[13] The underlying question here—What is thought?—is in a sense the central problem which *LTS* will attempt answer.

It is a question that already implicitly orientates Cavaillès's doctoral studies, where it condenses around the concept of effective mathematical work. What should be underlined here is the intimate relation between Cavaillès's mature anti-psychologism and his rejection of the *a priori*. This might seem to result in an aporia: if thought is neither psychological nor *a priori*, what can it be? The answer that Cavaillès will give is to identify thought with objective historical production (i.e. with produced works). Understanding this point is crucial if we are to ward off a misinterpretation that suggests itself in formulations such as those cited in the epigram to this chapter, where we might expect Cavaillès's avowed interest in studying what he designates as 'the effective thought of the mathematician' to attach to the 'intuitive' thought of the working mathematician, in a psychological sense of intuition. On this reading, the force of Cavaillès's critical project would consist in asserting that the creative psychology of the mathematician necessarily exceeds 'merely' logical formalisation.[14] Yet Cavaillès's claim is the opposite: the category of 'effective thought' comes to be *opposed* to the immediacy of psychological thought. Thought is to be found in the mathematical work, not the consciousness of the mathematician. This is why we must investigate mathematics in order to know what thought is.[15] Reduced to a formula, we could say that Cavaillès progressively arrives at the position that the mark of a true mathematical creation is that the work that has effectively been done exceeds its anticipation by or recuperation in consciousness. The erasure of the figure of the individual 'creative' mathematician that is implied here is another sign of how far Cavaillès eventually travels from his initial presentation of his doctoral research in the above-cited letter to Fraenkel.

By the same token, the category of 'effective work' as Cavaillès comes to understand it is fundamentally positioned against the 'Kantian' theory of *a priori* intuitions in mathematics. Yet, Cavaillès's relationship to Kantianism in the broad sense, and in particular to the concept of intuition, was not consistent throughout his work. Examining how the concept of the effect-

13 This is, broadly, the consequence of Élisabeth Schwartz's explicitly Hegelian interpretation of *LTS*, as advanced in Schwartz 2020.

14 One certainly finds polemics to this effect in Cavaillès's milieu, for example in Poincaré's arguments against Russell and 'logicism', or at least a certain contested but influential reception of them. For opposed readings of what Poincaré really intended in the opposition between 'psychology' and 'logic' in mathematics, see Goldfarb 1988 and Detlefsen 1992.

15 A norm which, as we saw in the Introduction, Cavaillès places under the sign of Kant, but a rather curious Kant as read through Brunschvicg.

ive is modified along with shifts in its mathematical referents—and, specifically, the way in which Cavaillès temporalises the notion of the effectiveness of mathematics—is essential to understanding Cavaillès's changing views on this matter. On the level of epistemological practice, Cavaillès's rejection of the *a priori* here rejoins a central methodological tenet of his work and the relation it seeks to exhibit between mathematics and philosophy: the relationship between mathematical and philosophical concepts must not be that the former stand to the latter as *examples*—i.e. as particulars to universals—such that mathematics gives specific content to more general philosophical concepts. Rather, the relation between mathematics and philosophy is conceived according to singular points of contact.

For this reason, my attempt to show how Cavaillès followed the twists and turns of a specific concept in its historical development does not mean treating the effective as an example of some 'general' conception of mathematics. The concept of the effective is nonetheless *exemplary* for assessing Cavaillès's work in that it is a mathematical referent with which he was intimately familiar and which underwent a profound reformalisation in the course of his working life. This reformalisation forms the background for Cavaillès's extended confrontation with new work by Gödel, Church and Kleene around effectiveness and recursion in *Transfini et continu* (written in 1940–1941 and published posthumously in 1947, hereafter *TC*). Situated between the doctoral studies and *LTS*, I shall argue that this text is the bridge to the latter project.

It is a test of the rigour of Cavaillès's enterprise that the concept of the effective ought not to remain the same after the intervention of the new formalisation of the concept of effective procedure by Church et al., and this indeed is what we find. For it is through his analysis of this novel formalisation—work which Cavaillès was almost certainly the first to address in the French language, by date of composition if not publication—that Cavaillès formulates a general distinction between *the effectuating* and *the effectuated* in mathematical work, a schematic division that will then come to bear on all of the major doctrinal concepts advanced in *LTS*. Heuristically, this is what brings the 'ontology' advanced in that work into alignment with the gesture that underwrites many post-Kantian returns to Spinoza: the splitting of epistemological notions into their form as *producing* and their form as *produced*.[16]

This will be a major theme in Part II. In particular, I shall endeavour to show there how Cavaillès's stratification of the notion of mathematical concatenation into an intrinsic distinction between prior and posterior production, and his resulting assertion that 'the intelligible'

16 I thank Louise Morelle for helping me to arrive at this formulation.

is defined by its internal non-simultaneity, should be read as resuming the themes of his doctoral studies from the standpoint of a theory of mathematical time. As I laid out in the Introduction, this gives rise to the Thesis on the Intelligible (LTS 35-36), which I take as the orientating doctrinal claim of *LTS*. For now we need only emphasise that if the concept of the effective in part names the identification between the objectivity and the historicity of 'mathematical becoming' with which we began, it is the schematic division between the effectuating and the effectuated that provides what Cavaillès can be read as searching for throughout the doctoral dissertations: a theory of the form of mathematical history. The aim of this chapter is to show how he arrives at this division, in the plurality of its conceptual referents.

§I.1. The French Analysts in Cavaillès's Doctoral Dissertations

> [T]he analysts, Baire, Borel, and Lebesgue, militant in the concrete theory of sets, which they developed to the point of being new creators, are all empiricists. For them, it is a question of their own work: detached from any speculation, they want to determine which objects and methods they will be allowed to deploy without the risk of running into a contradiction.[17]
>
> *MA*F 6/*OC* 14.

Although Cavaillès's two doctoral dissertations are concerned with more contemporary (and primarily German) developments, respectively at what was then the forefront of set theory and the Hilbert programme, the earlier work of the French analysts played an important role both in framing the epistemological histories that Cavaillès wrote and in delineating the broader conceptual problematics that he extracted from these histories. Two points should be emphasised.

First, the stakes of the 'abstract' set theory dealt with in *TAE* are to a significant extent defined in opposition to the informal commitment to 'concrete' set theory as espoused by Borel, Lebesgue, et al. Cavaillès had in mind the following epistemological situation: on one hand, the French analysts had made crucial progress in the study of real functions, the theory of measure and the theory of integration by using modern set theoretical methods, and even extended set theoretical research in important directions; on the other hand, they expressed explicit methodological scepticism with respect to the autonomy of set theory as a domain of

17 *OC* 14.

mathematical research, a scepticism which they attempted to ground by reference to a preference for 'concrete' or 'applicable' mathematics.[18] It is this nascent commitment to the ideal of a concrete set theory—initially articulated in terms of an inchoate practical attitude—that Borel and Lebesgue will attempt to specify in the period following the debate of the famous 'Five Letters' (exchanged between René Baire, Borel, Jacques Hadamard and Lebesgue in 1905 over the reception of Ernst Zermelo's Axiom of Choice) by developing (somewhat opposed) quasi-formal notions of the effective.[19] The contested distinction between concrete and abstract set theory was initially understood in the French context in terms of a debate around what constituted an admissible definition in mathematics, and an attempt was made to delimit this in terms of *effective definability*. As we shall see, this notion of the effectively definable was not adequately formalised in the work of Borel and Lebesgue, but they did set a conceptual problematic that is the background for Cavaillès's own notion of effective work, insofar as they attempted to specify one modality of the effective (as 'actual' or practically efficacious mathematics) by reference to another (the effective as the calculable or as what could be concretely 'effectuated' in an unambiguous manner).

Second, Cavaillès's interest in the concept of the effective is connected to the necessity of developing 'a definition of mathematical work in general'.[20] This demand, which is found in both the final sentence of *TAE* and at the end of the opening of *MAF* (following the treatment of Lebesgue), can be read as the conceptual bridge between the two dissertations: the problems posed by the history of abstract set theory cannot be resolved without an interrogation of the concept of mathematical (or effective) work; in turn, it is the history of the debate around formalism and the axiomatic method that will be the essential material for this latter investigation.[21] In order to understand why these problematics are connected, it should be emphasised that the attempt to delimit 'concrete' mathematics in terms of an informal notion of effective definability was unsuccessful on the terms in which the French analysts posed it, insofar as, despite the expressed scruples of Borel, Lebesgue et. al., the theories that they had de-

18 For an overview of some of the vagaries of this nascent 'concrete' set theory, see Murata 1958, Murata 1959a, Murata 1959b, Murata 1960, Gispert 1995, and Bertin 2019: 87–91.

19 The 'Five Letters' are initially published as Hadamard et al. 1905 and republished as an appendix to the second and subsequent editions of Borel 1914: 150–160. Translation in Moore 1982: 311–320.

20 See *OC* 362 and *OC* 29.

21 I take this as one reason why Cavaillès published *MAF* as the sequel to *TAE*. While *MAF* is Cavaillès's *thèse principale* and *TAE* is the *thèse complémentaire*, and *MAF* was completed prior to *TAE*, *TAE* appeared as volumes 606 and 607 of the collection *Actualités scientifiques et techniques* published by Hermann, and *MAF* as volumes 608–610. Cavaillès states that *MAF* is the 'continuation' of *TAE* (*OC* 15). See Cassou-Noguès 2001: 19.

veloped in order to advance mathematical analysis were shown to depend on the very aspects of set theory which they contested.[22] In the introduction to *TAE*, Cavaillès frames the 'paradox' posed by set theory in terms of 'the ever more widespread recourse to a contested theory within incontestable mathematics', and expands on this by stating that '[t]he theoretical crisis which commenced in 1905 has only embarrassed restrictions and scruples concerning set theoretical modes of reasoning without arresting their internal power of expansion.'[23] We can understand this 'paradox' to be that the practical efficacy of set theory had proved it to be an essential matter for mathematical research even in domains and amongst mathematicians who had wished to delimit its content in purely instrumental terms. This is a paradigmatic example for what Cavaillès will call in the introduction to *TAE* the 'problem of explaining the unequal fecundity of methods.'[24] On Cavaillès's reading, the French analysts had posed this problem for themselves without having the adequate formal means to resolve it. He saw Hilbertian proof theory, and in particular the broad history of the expanded notion of recursive or effective definitions as it had been developed following Dedekind's work, as being essentially continuous with this unresolved problematic in Borel and Lebesgue. As such, interrogation of these two concepts of *the effective* and of *mathematical work* should be seen to inform each other throughout Cavaillès's œuvre.

Two innovations are important in Cavaillès's reading of this epistemological conjuncture. In the first place, Cavaillès progressively imbues the concept of the effective with a *temporal* connotation, viz. as a problematic of the relation between *already completed* (or effectuated) mathematics and future mathematical production. This means that Cavaillès's wish to do justice to the open-ended *effectiveness* of mathematical work is inseparable from his ultimate rejection of any *a priori* theory of mathematics, and his partial sympathy with an 'empiricist' project (in the sense of the French analysts).[25] Second, the dialectic of effectiveness and effectuation at play here must be indexed to Cavaillès's changing views on the idea of an intuitive basis for mathematical construction and hence to a shifting relation both to Brunschvicg's quixotic Cartesian-Kantian synthesis and to the Kantian underpinnings of finitist theories in mathematics.

22 For an analysis of the history of the relevant results here, c.f. Moore 1983.

23 *OC* 224. For a translation of the Introduction and Conclusion to *TAE*, on which I draw throughout this chapter, see Appendix II.2.

24 *OC* 227.

25 The relevant distinction between 'empiricism' and 'idealism' at stake also has roots in Emil du Bois-Reymond. See the commentary in Brunschvicg 1912: 527–533.

Turning to the 'Five Letters', the debate there, which sets the terms for the notions of the effective subsequently developed by Borel and Lebesgue, concerned what constitutes an admissible definition in mathematics. In this respect, a shared reference for the French analysts is a prior distinction drawn by Jules Tannery in his 1897 review of Couturat's 1896 *De l'infini mathématique*, in which Tannery affirmed the validity of mathematical correspondences that could be *determined* but which could not be *described* (i.e. what is broadly now referred to as the problem of 'impredicative' definitions).[26] Tannery's position on such correspondences is framed as an explicitly Cartesian commitment to the 'idea of determination', i.e. that we can intuit mathematical determinations (or laws determining correspondences) independently of the possible realisation (or description) of such correspondences:

> [I]t is important to observe that the idea of determination is independent of the possibility of formulating what this determination consists of. It is not necessary, for a set to be determined, that one can effectively identify with respect to [*effectivement reconnaître sur*] a given number whether it [effectively] belongs or does not belong to that set, it is sufficient that one knows that it [either] does or does not belong to it, and any reasoning based on the fact that one has the choice between these two suppositions will be valid.[27]

We can thus gloss this distinction as that between *abstract*—or *postulated—correspondences* (which can only be 'determined' but not 'described'), and *effective correspondences* (correspondences that can be concretely 'described'). In the 'Five Letters', Hadamard, Borel, Baire and Lebesgue each invoke this distinction, but in opposed terms. Indeed, the shifting lexical terrain around this referent in the discussion can be read as the sign of a contested concept.[28] For Hadamard (following Tannery), abstract or postulated *determination* is sufficient for mathematical *definition*; for Borel and Lebesgue *definition* or *determination* must equally entail (*effective*) *description*. In this sense, the later moves by Borel and Lebesgue to transform the 'effective' into a quasi-formal notion can be understood as differing attempts to give a descriptive definition of definition in mathematics.[29] The different proposals made by Borel and Lebesgue respectively push in constructive and purely descriptive directions. The two definitions they advance are as follows:

26 Tannery 1897: 132. Review of Couturat 1896.

27 Ibid., p. 134.

28 For an analysis of the same debate as an exemplary case for what Desanti calls an 'epistemological problem' in mathematics, see Desanti 1965, reprinted in Desanti 1975: 110–132.

29 It is in this sense that we are also discussing the genesis of descriptive set theory.

> Borel's definition of the *effectivement énumérable* (1908): A set is '*effectively enumerable*', if it is possible to 'state, by means of a finite number of words, a definite procedure for unambiguously attributing a determined rank to each of its elements.'[30]
>
> Lebesgue's definition of the *nommable* or *effectif* (1905): 'An object is defined or given when a finite number of words applying to this object and to this object alone have been uttered; that is, when a characteristic property of the object has been *named*.'[31]

In Borel's case, it is clear that what he has in mind is the individuation of mathematical objects. He will often speak of certain individual 'incommensurable numbers' such as π for which we can nevertheless determine the decimal representation up to any desired degree of approximation (these will be 'effectively enumerable' in a way that, e.g., 'the set of all the incommensurable numbers' is not). Borel intends this stricture to formally characterise the epistemological notion of a mathematical object being agreed upon in an unambiguous manner by working mathematicians. He thus wishes to characterise mathematical objectivity by reference to intersubjectivity—i.e. that any given mathematician will be able to determine the 'same' object to the same degree—and in turn to specify this notion of the intersubjective availability of the object by reference to calculability. This is clear when Borel later develops an even more restrictive notion of 'calculable numbers' or 'calculable functions': 'We state that number α is calculable when, given an arbitrary whole number n, we can obtain a rational number which differs from α by less that $1/n$.'[32] Again, the privileged referent for Borel in specifying his notion of effectiveness or calculability is the possibility of decimal expansion. In a footnote to this definition of calculable numbers, Borel then develops what I will call Borel's Thesis, i.e. that effectively calculable mathematics coincides with (or exhausts) applicable mathematics:

> Borel's Thesis: 'I am intentionally leaving aside questions concerning the greater or lesser practical length of the operations; what is essential is that each of these operations is executable in a finite time, by a certain and unambiguous method. For example, a decimal number β, such that its n^{th} decimal is equal to the decimal of π of rank n! must be considered as definite, even though its calculation, with the use of ten exact characters, may require, in the state of Analysis as it stands [*l'état actuel*], a time far exceeding the length of human life. *From the practical point of view, it may be said that the numbers*

30 Borel 1914: 65. Reprint of Borel 1908, wherein the constraint is initially articulated negatively in response to Richard's paradox. Cited by Cavaillès at *OC* 17.

31 Lebesgue 1905: 205. Cited by Cavaillès at *OC* 23.

32 Appendix to Borel 1914: 219, which is a revised version of Borel 1912. The definition of 'calculable function' given immediately afterwards follows the same schema.

> *effectively needed* [*dont on a effectivement besoin*] *can, in general, be effectively calculated* [*effectivement calculés*] *to the desired degree of approximation* [...].'[33]

I call this Borel's Thesis because there is an identification at stake, one which crosses our two modalities of the effective, the effective as the actual and as the calculable. The thesis is that all of the mathematical objects that practising mathematicians are going to need for what Borel will think of as *actual* mathematical work (i.e. which are of relevance to 'concrete' or 'applicable' mathematics) are those which can be *effectively calculated* (i.e. given in a computational manner). The identification at stake is thus between two different senses of the same term, as neatly condensed in the above citation: Borel is attempting to give a univocal interpretation to the word *effectivement* by using a formal notion (*effectivement calculés*) to specify an informal or conceptual one (those mathematics *dont on a effectivement besoin* 'from a practical point of view').

It is this identification that Cavaillès will oppose by giving a temporal reading of the restrictions at stake: for Cavaillès, Borel's definitions are constructive—and the Kroneckerian legacy is clear here, in that we are discussing constraints placed on mathematical reasoning—insofar as they are prescriptive with respect to future mathematical work. By contrast, Cavaillès takes Lebesgue's definition of effective mathematics (above) to be more open (or less restrictive), in that definitions are to be introduced according to the immanent demands of specific theories, rather than as a propaedeutic to their development. As Cavaillès summarises Lebesgue's position:

> [I]t is the notion of definition that has changed here: far from demanding [qua Borel] a possible construction solely based on a type of object posed once and for all, the effective [*l'effectif*, i.e. Lebesgue's notion] on the contrary demands that only that which is important for the present reasoning [*le raissonnement présent*] appears in the characterisation of the object.[34]

Thus for Lebesgue, all we need to do to define an object that we wish to describe is to *have named* its characteristic properties (and the past tense is crucial here), but there is no particular type of (practically effectuable) object which is being prescribed (nor proscribed) by this definition of the effective, and thus there is no prescription with respect to future mathematical work. The characteristic example here is Lebesgue's definition of measure, viz. that to

33 Borel 1914: 219, emphasis mine.

34 *OC* 23-24. Cavaillès is selective in his treatment of Lebesgue here. Lebesgue uses both descriptive and constructive definitions, and sometimes positions his work on measure as orientated towards producing constructive definitions of the central concepts of the theory of integration that would be *equivalent* to previously given descriptive ones. C.f. Lebesgue 1904: 100 and the commentary in Michel 1992: 85.

every set of points *E* we can associate a non-negative number *m* (*E*) that satisfies the following three conditions:

> (I) Any two equal sets have the same measure;
> (II) A set that is the sum of a finite or of a countable infinity of sets, without any points in common between them, has as its measure the sum of the measures of these sets;
> (III) The measure of the set of all the points (0,1) is 1.[35]

This definition simply defines the properties that we wish the concept of measure to have, but does not prescribe anything in particular about 'calculation': the stated stricture tells us exactly the conditions that the concept of measure is to fulfil, and *nothing more*, i.e. it does not extend to delimiting the modes of effectuation involved. Cavaillès thus expresses his sympathy for Lebesgue's project (at least at this point) in terms of the attempt to honour the immanence of modes of definition in mathematics to what Cavaillès will call 'effective work'. Having cited the above definition of measure and then referenced the general advances made as a result in the theory of integration, Cavaillès writes:

> They [these advances] stem from the simplifications achieved thanks to the descriptive character of the definition: all that matters is what is named there, whereas a construction that has nothing to do with the problem posed [again, i.e. Borel's strictures] could only be a burden or an arbitrary restriction. In a way, what is affirmed here is the homogeneity of the materials of an enterprise, the simultaneity of mathematics with the work before it: this remains empiricism, since it is only a question of describing effective work [*le travail effectif*], but it is an empiricism of actual thought [*la pensée en acte*], with no other reference than to the unpredictable becoming of mathematics. The modes of definition are left up to the variations and demands of its movement: with each new acquisition, new possibilities appear. The enrichment of the nameable coincides with the enrichment of science itself.[36]

Once again, in praising this more capacious definition of effective work (or actual thought), Cavaillès has progressively imbued the notion of effectiveness with a temporal connotation. This informs Cavaillès's qualified aligning of himself with an 'empiricist' project for the description of 'mathematical experience', and against the possibility of an *a priori* theory of science.[37] Indeed, this conviction constitutes the core of the Cavaillèsian programme: the effectiveness of mathematics is defined in terms of the *unpredictability* of its development; the rejection of the *a priori* is a rejection of *prediction*.

35 Lebesgue 1904: 103.

36 *OC* 25-26.

37 C.f. Cavaillès's 1938 letter to Paul Labérenne, where he credits 'this same empiricism' as having played a role in his break with 'Brunschvicgian idealism'. See Appendix I, pp. 201-202 for a translation of this letter.

§I.2. The Efficacy of Axiomatisation in Abstract Set Theory and the Hilbert Programme

We can note that this way of conceiving the effectiveness of descriptive definitions brings the work of the French analysts into close contact with contemporaneous developments around the axiomatic method in the Hilbert school. This was certainly a common conception in Cavaillès's immediate milieu. Thus, when, in his 1935 paper 'Variations of Mathematical Style', Cavaillès's friend Claude Chevalley introduces the 'axiomatic style' as exemplified by Hilbert's axiomatisation of geometry in the *Grundlagen der Geometrie*, in contrast to the 'constructive style' of 19th century analysis as exemplified by Weierstrass, he does so by noting the descriptive nature of Hilbert's approach.[38]

> This theory [i.e. Hilbert's axiomatisation of geometry] starts from a point of view diametrically opposed to the constructive point of view mentioned above; in effect, instead of defining points, straight lines, etc., on the basis of other notions, in order to then deduce their properties, it leaves the nature of these objects completely undetermined, and contents itself with giving a *descriptive* definition that consists in the statement of a certain number of their fundamental properties, called axioms.[39]

According this presentation, Lebesgue and Hilbert, despite important differences, belonged to a common effort to liberate mathematical inquiry from restrictions based on the 'nature of entities' to be described; on the contrary, one would now proceed from the axioms to the entities.

> Hilbert's axiomatic example would not remain isolated. Within the multitude of mathematical objects defined on the basis of numbers, a certain crystallisation was taking place; theorems were grouped together, using analogous methods; in these groupings were the seeds of autonomous theories that were soon to break away from construction on the basis of real numbers. One example is Lebesgue's descriptive definition of the integral, which, although it did not yet take the exact axiomatic form of Hilbertian geometry, nevertheless followed its fundamental approach, since it first of all presents the integral in terms of its fundamental properties, and only subsequently to this demon-

38 I take this paper by Chevalley to have been significant for Cavaillès's formulation of his methodological programme in the doctoral works, in that the notion of mathematical 'styles' developed by Chevalley therein was very much conceptually adjacent to the conception of the unity of theories that concerned Cavaillès in the doctoral works (see §I.2.i., below). Cavaillès would later include the paper in a syllabus for students in logic and general philosophy (i.e. Cavaillès 1941) while teaching at the university Strasbourg in 1941, by then displaced to Cleremont-Ferrand by the occupation. For an analysis of the relevance of Chevalley's approach to contemporary debates in the historiography of mathematics, see Rabouin 2017.

39 Chevalley 1935: 380.

> strates that the conditions so imposed are compatible with the object to be constructed.[40]

Thus for Chevalley at least, the move towards descriptive definitions was akin the nascent structuralist approach to mathematics, which Chevalley and others would soon radically develop under the name of Nicholas Bourbaki.[41] On such a conception, contemporary mathematics was seen as replacing 'the old question: is such or such a statement true? (i.e. can it be deduced from the properties of real numbers)' with a new question, 'in which theory (i.e. according to which hypotheses) is the statement true, and in which theories is it false?'[42] Lebesgue and Hilbert could then both be perceived as exemplifying this theoretical tendency. It is notable that Lebesgue himself did not see things exactly this way. For example, in the 2nd edition of the *Leçons sur l'intégration*, published in 1928 (the 1st edition was 1904), he adds a footnote distancing his approach from Hilbert's, stating that the *Grundlagen der Geometrie* exemplifies the axiomatic method insofar as it provides an enumeration of the axioms of a theory as something which is 'sufficient unto itself and forms a completed whole', prior to the concrete elaboration of the theory.

> In contrast, descriptive definitions laid down in the course of the development of a theory, such as the [i.e. Lebesgue's own] definition of the integral, do not claim to enumerate all the axioms on which they are based; they do not form a complete whole and cannot be isolated from the rest of the theory.[43]

What to make of these rather opposed presentations of the methodological front? Two points should be emphasised here. First, the very fact that as late as the mid-1930s it was ambiguous whether a 'descriptive' approach to mathematical definitions should be received as in the lineage of the so-called 'constructive' or 'axiomatic' styles in contemporary mathematics is of course exemplary of the kind of difficulties faced by Lebesgue and Borel in their respective definitions of the effective, and indeed of the polysemy of that term, which depending on the standpoint adopted seemed to allow the demand for effective definitions in mathematics to attach both to the strictures of step-wise construction and the liberty of axiomatic definition. Second, it should be underlined that this very ambiguity as much as its presumed overlap with methodological problems proper to the Hilbertian approach to mathematics was central to

40 Chevalley 1935: 380–381.

41 Indeed, Schappacher goes so far as to present Chevalley's 'Variations' essay as 'the first manifesto of Bourbaki'. Schappacher 2022: 166.

42 Chevalley 1935: 383.

43 Lebesgue 1928: 107.

how Cavaillès understood matters, and this conjunction is significant for the structure of the doctoral works, where, as we shall see, the development of the category of effective mathematical work is intimately tied to reflection on the nature of axiomatisation.

Running through the wide array of mathematical material covered in those works are two central questions:

A) What is the unity of a mathematical theory?

B) What is the efficacy of axiomatisation?

Grosso modo, these two questions are articulated by Cavaillès around what he refers to as the 'double unity' of theories and the 'double given' of axiomatisation. At stake in both cases is a particular conception of the relation between the intuitive content of mathematical reasoning and a given stage of technical formalisation. It is by focussing on this relation that Cavaillès sketches, at least in outline, a theory of mathematical work that is at the same time an account of the historicity proper to mathematics.

As mentioned above, it is the development of this definition that is the bridge between the two doctoral works. For Cavaillès this was also a historical development, dictated by the conjuncture: the state-of-the-art of research in the axiomatisation of set theory (subject of *TAE*) had in effect demonstrated that a more profound reflection on the 'definition of mathematical work in general' was necessary (subject of *MAF*). It was the Hilbertian school that had undertaken this task, in two phases: the 'first Hilbert' of the mutually elaborated axiomatisation of geometry and arithmetic (associated with the *Foundations of Geometry* [1899], 'On the Concept of Number' [1900] and the celebrated lectures on 'Mathematical Problems' [1900] and, on the threshold between the two periods, 'Axiomatic Thought' [1918]), and the 'second Hilbert' of the Hilbert Programme proper and the development of metamathematics and proof-theory. In his analysis of the latter, Cavaillès concludes that post-Hilbertain proof-theoretical investigations—in particular those by Gödel, Herbrand and Gentzen—required a reconceptualisation of the relation between intuition and mathematical work. Perhaps more than any other reference, it is Herbrand's mutually elaborated definitions of finitary reasoning and of the concept of recursive function that played a determining role here—one which, in the French context of the 1930s, was also naturally read as an evolution of the problem of effective definability as it had been posed by the preceding generation.

It is thus necessary to take leave of the French analysts, and consider this later theoretical trajectory. I will approach this in three sub-sections: first, through an outline of the prob-

lems that Cavaillès took to be posed by abstract set theory; second, through his analysis of the conceptual stakes of axiomatisation in general; and finally through an outline of Cavaillès's reading of the methodological development of mathematical finitism. With respect to the third sub-section, we shall see that part of the originality of Cavaillès's analysis was to have often implicitly translated problems in proof-theory directly into Cartesian language. In terms of the broader economy of our reading of Cavaillès's philosophical project, this will lead to outlining a direct proof-theoretical index for the Thesis on the Intelligible from *LTS*, to compliment the Cartesian index outlined in the Introduction. Thus we shall see that the precise formulation there—viz. that the nature of the intelligible is that it cannot be established 'all at once'—has a direct correlate in Cavaillès's understanding of finitism.

§I.2.i. The Double Unity of Theories in *Remarks on the Formation of Abstract Set Theory*

We first need to consider in some more detail the broad epistemological research programme outlined in *TAE*. Crucial here is Cavaillès's attempt to produce an account of what he called in that period the 'central intuition' or 'central operation' unifying theories. Recall what was said above concerning the 'paradox' of the practical efficacy of set theory with respect to Borel and Lebesgue's generation. In effect, Cavaillès understood this paradox to touch on a broader theoretical problem that he considers throughout the doctoral works: that of the individuation or 'existence' of mathematical theories. Broadly speaking, Cavaillès—in this period at least, for we shall see that he came to modify this view—considered mathematical theories as being individuated by two possible 'unities': first, the unity of technical practice at a given historical moment in the development of mathematics; second, by the unity of an intuitive process of reasoning, i.e. what he shall also refer to as an intuition or 'gesture'.

Here, it is important to take account of a general methodological tenet of Cavaillès's approach to 'creative mathematics': the appearance of a new mathematical theory is marked by the appearance of a new mode of reasoning, which is to say a new norm.[44] Thus, for the case of 'The Cantorian Creation' (the title of the first volume of TAE as published by Hermann), Cavaillès's claim is that one does not find set theory proper in the period of Cantor's work from 1873–1883—e.g. in the papers on trigonometric series—because insofar as one is still considering sets of points or numbers, said work is still 'characterised by its intimate subor-

44 We shall later consider this in terms of Cavaillès's *rationalist nominalism* and its Spinozist sources (see Chapter III).

dination to analysis.'[45] Set theory only arrives as an 'autonomous' mathematical work once, from 1883 onwards, one transitions to consideration of abstract sets and moreover to the new methods demanded by their employment.

> There is no set theory until there is the appearance, not so much of notions, as of an original mode of reasoning; such as that which Cantor used for the first time in generalising Heine's theorem. An essential part of the future edifice depends on this—which we will later find again under the name of transfinite iteration.[46]

On Cavaillès's analysis, what is really crucial here—i.e. what marks a new mode of reasoning or, in the later language of *LTS*, a new mode of concatenation—is not so much the appearance of a new object domain, but rather of a new mode of operating that is required by said objects. In other words, the novelty of a theory is the novelty of a 'method'. The problem—as the history of mathematics has amply demonstrated—is that the technical elaboration of a method is no simple work, and frequently (we might say constitutively) produces results that were unpredictable from an 'intuitive' standpoint. As Cavaillès states in the introduction to *TAE*, which functions as something like a methodological manifesto for the doctoral works: '[A] method is not something that can be arbitrarily isolated according to the demands of a problem.'[47] This obviously poses a problem for an epistemological project that would seek to isolate the 'necessary emergence' of new theories, for such a project has to maintain that there are meaningful criteria by which particular theories can be isolated from the contingent fabric of mathematical practice—at least for the purposes of analysis. It is this concern that brings Cavaillès into proximity with the kind of descriptive endeavour that Chevalley had sketched under the category of 'mathematical styles'. Cavaillès's aims, however, are not only descriptive; he positions the project of *TAE* is that of divining necessary and autonomous theories from those that are merely a 'plurality encased in a contingent system'.[48] The task of the epistemologist, faced with the complexity of mathematical history, is thus positioned as a kind of purification, which must treat genuine mathematical theories in their integral novelty. Thus Cavaillès proposes two criteria of individuation for theories so understood:

45 *OC* 270.

46 *TAE* 60/OC 258.

47 *OC* 225. What follows here is a reconstruction of Cavaillès's argument in this very dense passage; a translation is included in the Appendices.

48 *OC* 227.

> Unity 1: Technical Method: The definition, through the investigation of the procedures of a theory, of the objects that compose the field of that theory (i.e. axiomatisation or formalisation).
>
> Unity 2. The Central Intuition: The primitive operative gestures which are the basis for the 'organic complex' (or 'state of mind') that the procedures of Unity 1 compose.

This second unity (or so Cavaillès claims) condenses mathematical work around the development of a theory *even in* conditions in which a formal definition of the theory in question is not available, and in which this coincidence is not visible to the mathematicians that are developing it, at least initially.

> Whence the profound kinship that unites—without their having any full awareness of it, as an implicit agreement—the researchers of a single discipline, at least in a given period, *vis a tergo*, an objective momentum [*élan objectif*] of the research itself. Does set theory possess this double unity, which, for a theory, is the justification of its mathematical existence? Researching the first [unity] is technical work: we have seen the difficulties that it encounters. But, perhaps, if only in order to evaluate its chances of overcoming them, it is possible to try and detect the second [unity], there where it exists, beneath the juxtapositions—perhaps factitious—of an arbitrary construction. It is here that history intervenes.[49]

It is in these terms that Cavaillès situates the conceptual problem of axiomatisation, which is at the same time a problem of the historicity of mathematical work. On Cavaillès's view, the elaboration of a mathematical theory is both fundamentally historical in character (it is 'locally' determined by the historical status of existing mathematical production) and defines itself according to a new plane of intelligibility that breaks from the historical situation, whence the unique theoretical problems posed by the study of it: 'Mathematical history seems, of all histories, the least connected to that of which it is the medium.'[50] In the same manner, the 'militant mathematician' is both located in history and tasked with the refusal of a historically sedimented arrangement of technical practice.

> [W]ith what means do they operate? The history-negating work is brought about in history. Double relation: with the problems posed and studied at a time (choice of rebellion), and with already existing methods, material with which to forge the new instrument.[51]

49 *OC* 225.
50 *OC* 225-226.
51 *OC* 226.

The immediate conceptual difficulty posed by this account is that of the 'identity' (if we allow the term) between the notions of prior and posterior theory; in the economy of *TAE*, the notion of the double unity of theories cited above is in part intended to account for this.

> The unquestionably objective elements are those knots that constitute theories: here, in the intertwining of notions and methods, causal links disappear and are replaced by relations of intelligibility. Systematic exposition—axiomatisation, for example—makes this process visible but is not indispensable to it. It is possible to perceive its unity in the central intuition according to which its concatenations proceed, but also to methodically detect it solely by the description of the phenomenon. To follow the genesis of notions, to clarify above all their effective links with problems and isolate those general procedures whose inner gesture comes closest to the central intuition which is impossible to describe, such is the work—at once subject to history and a critique of it in the name of its results—that has some chance of yielding an objective result.[52]

Perhaps no passage in Cavaillès's early writings better expresses his working method in the doctoral studies, nor the (often-implicit) methodological assumptions that will be if not abandoned at least fundamentally displaced from the standpoint of *LTS*. Nevertheless at this point he clearly intended it as a more general project, with respect to which 'what is at stake here is only a limited application of this procedure to the particular question posed by set theory.'[53]

It is worth considering then how Cavaillès poses this question of the unity or 'existence' of theories with respect to the specific status of 'abstract' or 'axiomatic' set theory circa 1935 (roughly), i.e. the period discussed in *TAE*.[54] Heuristically, Cavaillès presents this history as follows. Set theory emerges from a diverse set of mathematical practices and associated problems, with the most pertinent being those in analysis (defining the uniqueness of representation by trigonometric series, attempts to develop a general notion of function) and in arithmetic (outlines of a general theory of numerical operations). Starting from these respective domains, Cantor and Dedekind each attempt to unfold what we now call set theory, under the rubric of founding mathematics on the systematic elaboration of primitive operations: in the case of Cantor, the two 'principles of generation' of numbers (as outlined in the *Grundlagen einer allgemeinen Mannigfaltigkeitslehre* of 1883); in the case of Dedekind the two simple 'acts

52 *OC* 226-227.

53 *OC* 227.

54 As noted, *TAE* and *MAF* are both published by Hermann in 1938. However, they are written over the course of 1936 and the first part of 1937, and Cavaillès does not seem to have done substantial new research for the dissertations after 1935, at least judging by the fact both works almost exclusively cite texts from 1935 or earlier. In the case of *TAE*, there are two citations from 1937 (Cavaillès's own publication of the Cantor-Dedekind correspondence and Fraenkel 1937) and nothing from 1936; in the case of *MAF* there are only two citations from 1936 (Heyting 1936 and Gentzen 1936).

of the mind' of collecting things together into a *system* and of putting said systems into correspondence (i.e. *mapping*).[55] Remarkably, the elaboration of these respective operations, developing out of quite distinct fields of mathematics and problems proper to them, come to coincide over the course of the 19th century in the formation of set theory in its first 'naive' phase.

The intrinsic drama of this evidently fascinated Cavaillès from early on in the composition of the doctoral works; it both gives the scheme of *TAE* (which is divided into a chapter on Cantor and Dedekind respectively, preceded by one on the 'prehistory' of set theory in 19th century mathematics) and motivated Cavaillès's editorial work on the Cantor-Dedekind correspondence. Thus already in his 1932 review of Cantor's *œuvres*, Cavaillès presents the interest of said correspondence as being that it shows the difference of 'two modes of thought', each of which is 'generative of distinct methods, such as transfinite recursion on the one hand, and reasoning by chains on the other.'[56] Yet, for all of their specificity, the effect of systematisation lead to the 'miraculous' coincidence of these trajectories around certain technical problems. 'Here, as elsewhere, it is moreover remarkable that the particularities of these creations have rapidly dissipated under the pressure exerted by the demands of problems.'[57] The tension here—between the individuality of methods and the unity of problems—was orientating for Cavaillès's early epistemological project, and accounted for the central importance of set theory. Thus, as he writes in introducing the final book of *TAE*, on Dedekind and the subsequent axiomatisations of set theory: 'Such is the decisive singularity in the history of set theory, that its notions and several of its essential results were rediscovered, as if involuntarily, via a detour through research into the foundations of arithmetic.'[58]

This was the situation for the first—pre-axiomatic—stage of set theoretical research. Whence the significance of the discovery of the paradoxes, which—according to this presentation—were devastating for 'naive' set theory insofar as they result from the deployment of the most primitive or 'intuitive' operations of the theory. Thus for both the case of the Burali-Forti paradox and Russell's paradox, respectively arrived at through Cantor's 2nd principle for generating the ordinals and through diagonalisation,

> it is an essential procedure of the theory that, defined abstractly, results in a contradiction […]. The situation is the same for the two cases: a theory is constituted in order to

55 As Cavaillès notes *en passant* in *MAF*, the latter gesture is Cartesian in spirit, in that *system* and *map* are accorded the rank of 'clear and distinct' ideas. See *OC* 66.

56 Cavaillès 1932: 443.

57 Cavaillès 1932: 443.

58 *TAE* 121/*OC* 319.

> respond to problems effectively posed by existing mathematics; the novel notions and procedures thus forged allow results to be obtained that are independent of the particular structure of the mathematical objects from which one began. […] But the set of all the ordinals is inconsistent; but the set of all is sets is inconsistent. The fault lies not with the procedures—the simple executive power of notions—but rather it is the basic notions themselves which disappear. What, now, is a set? And if we change its definition, what remains of the naive theory?[59]

From this, the necessity for an axiomatic reconstruction of set theory, as carried out by Zermelo in a quite Dedekindian spirit of 'logical criticism', an act that at the same time gave birth to the new endeavour of abstract set theory (precisely, that is, the 'abstract' theory which the French analysts questioned the intelligibility of, and to which they opposed their 'concrete' theory) and its various adventures: debates around the axiom of separation and the axiom of choice, competing axiomatic formalisations offered by Zermelo-Fraenkel and Von Neumann, etc. Cavaillès places especial emphasis—concluding the historical presentation in *TAE*—on the latter's generalisation in 1925 of results from Skolem into a conclusion concerning the non-categoricity and thus the essential 'relativity' of set theoretical concepts, both with respect to the definition of the theory's basic 'naive' concepts and with respect to the attempt to reconstitute the theory axiomatically, i.e. an 'objection' which 'does not concern the consistency but the univocality [*Eindeutigkeit*] (categoricity) of set theory'.[60] Thus an ultimate 'divorce' between the initial intuitive theory and its axiomatisation: 'the immediate conjunction between the naive movement of mathematical creations and axiomatised systems is impossible.'[61]

The broad strokes of this historical analysis are what inform Cavaillès's above-outlined conception of the 'double unity' of theories and its specific relevance to the case of set theory. As stated, for the case of set theory, the state-of-the-art concerning Unity 1 is the primary object of *TAE*. In the conclusion we find that the judgement concerning this unity is, if not definitively negative, at least agnostic; in particular, results concerning the non-categoricity of the theory are taken to have shown that at least on the technical level there is no single (univocal) system of objects and procedures called 'set theory'; there are only 'set theories'. On the other hand, Unity 2 is taken to be in some sense incontestable owing to the applications of the the-

59 *OC* 340.

60 Von Neumann 1925: 240, translation from Heijenoort 1967: 413. As Cavaillès summarises in (implicitly) Platonic terms, this leads to '[r]elativity as much at the base (finitude and well-ordering) as at the summit'. *OC* 356. We shall return to the profound importance of Skolem and the non-categoricity of theories for Cavaillès in Chapter II.

61 *OC* 356.

ory; the 'reality' or 'existence' of set theory—the 'objective momentum of the research itself'—formed a part of the historical *a priori* of mathematics in Cavaillès's moment. This posed a problem that was descriptive as much as theoretical: the mathematical unity of the theory could not be established in formal terms, and yet clearly it 'existed' as a commonly recognised—and indispensable—mode of reasoning amongst working mathematicians. This paradox was all the sharper because of set theory's intended purity, i.e. its aim to characterise the mathematical object as such, as a propaedeutic to the investigation (and systematisation) of individual theories and their specified object domains.

> The privileged place of set theory, and the difficulties it has encountered, give its history its philosophical importance: the ambition for a development prior to and as a precondition for the rest of mathematics ought to allow the notion of a mathematical theory to be clarified, isolated as *in vitro*. After the paradoxes, the demand was to exhaustively enumerate the hypotheses, to clarify the notion of proof itself. If the results attained only produce an immediate transformation of the axioms, can we still speak of a theory? The reality of a reasoning finds itself in some sense measured by its admixture of mathematical material, by the unification that it obtains between already mathematically existing objects or procedures. Whence the difficulty of making divisions of jurisdiction within a living tissue, with respect to which their claims would only be superstructures of these procedures in which the unpredictable deviation of a gesture at once obtains novelties and preserves connections. Rejecting or founding a theory is not definitive work, it is not devoid of degrees; moreover, as with all characterisations of unity, it cannot be carried out by simple logical investigation.[62]

It was this situation that required an investigation of the concept of 'mathematical work in general'.

§I.2.ii. The Double Given of Axiomatisation in *Axiomatic Method and Formalism*

> It is only a realist prejudice that leads us to preoccupy ourselves with objects, whereas [with Hilbert] in fact all that matters with respect to the succession of our assertions is what governs this succession, namely effective intellectual work.
>
> *MAF* 77/*OC* 85

It might seem that we have strayed somewhat from the concept of the effective. Yet this detour through Cavaillès's reading of the history of axiomatic set theory was necessary to situate the philosophical stakes of his attempt to develop a conception of mathematical work qua 'effect-

62 *OC* 36–362.

ive thought', which initially appear quite distinct from the whole problem field around effective calculability. For the other referent which the term 'effective' assumes in Cavaillès's lexicon is to mathematics as it is actually practised, not reconstructed in an *a priori* theory, i.e. the 'already mathematically existing objects and procedures' of the conclusion to *TAE*. This is the sense of the effective as the actual, as opposed to the effective as the calculable (qua—if we wish to emphasise a combinatorial register—the 'enumerable'). In turn, the category of 'effective work' refers to the relation to be thought between said mathematical actuality (at a given moment) and the creation of novel theory. Here we touch on a certain epistemological experience that in some sense Cavaillès's work was devoted to developing on: the experience of reading the history of mathematics.[63] From this the need to produce a concept of the historicity proper to mathematics. The material from which this concept was to be worked up was the concrete study of the dialectical relations that obtain between various formalisations and the intuitive content of informal mathematical theory. Thus, in his 1937 presentation to the Congrès Descartes, summarising the results of *MAF* and in particular his critique of Brouwerian intuitionism and what he takes Brouwer to get wrong in his 'characterisation of mathematical work', Cavaillès states that the intuitionistic scheme of mathematics as being progressively constructed from the dyad,

> too quickly exhausts the original character of intuition: if it only is insofar as it is effective, its intention is that what it attains should be independent of the event through which it attains it. Dialectic of mathematical history: the liberation from the contingent by the effective, yet the effective in its effectuation is contingent.[64]

We see here the highly overdetermined modal character of Cavaillès's conception of mathematical work: it is necessary, unpredictable and actual. The concept of the effective emerges from Cavaillès's attempt to think through a series of characteristics that he wishes to attribute to mathematical work and its historical development, characteristics which appear as strongly in tension with one another.

A) Mathematical work is *objective*—it proceeds according to effective constraints that are independent of the individual psychology of the 'working mathematician'. The ob-

63 On this topic, see Desanti's comments on the tradition of 'philosopher-readers' of mathematics in France in his Preface to the 1972 reprint of Brunschvicg's *Les Étapes*.

64 *OC* 578. I use the translation 'dyad' for Cavaillès's references to to the Brouwerian *dyade* [*twee-eenigheid*], as opposed to the conventional English translation as 'two-ity', because I take the continuity with the Platonic terminology to have been significant for Cavaillès's understanding of intuitionism. For a summary of Cavaillès's reception of intuitionist sources, c.f. van Atten & van Gemert (forthcoming). For a translation of Cavaillès's presentation (i.e. Cavaillès 1937), see Appendix II.1.

jectivity in question is historical. But—and this is methodologically crucial—the intended sense of historical objectivity here is strictly intra-mathematical; it concerns the presence of certain 'instruments' and 'problems', not external sociological factors.[65]

B) The progress between mathematical theories is (at least locally) *necessary*—given a certain effective arrangement of mathematical practices, specific developments have to occur: they are demanded by the problems as much as the instruments present in a historical period.

C) There are genuinely *novel* mathematical creations—a new theory cannot be predicted by any *a priori* theory of science.

The conjunction between characteristics B) and C) is the most mysterious; as we saw in the Introduction, it is condensed by Cavaillès into the (Dedekindian) slogan of the 'necessary generation of new concepts'.[66] How can it be that certain theoretical developments in mathematical history are at once absolutely necessary and strictly unpredictable? The category of the effective in its various determinations names this conjunction in Cavaillès's work.[67] As mentioned above, in the general elaboration of *MAF* this conception of the effective as bearing on the history of mathematics is developed primarily through an analysis of the Hilbert programme, in two phases: the first Hilbert of the axiomatisation of geometry and arithmetic, and the second Hilbert of mathematical finitism and proof theory, i.e. the Hilbert programme proper as it was carried out by both Hilbert himself and those who followed in his wake.

With respect to the first, 'axiomatic' Hilbert, the basic problem that interests Cavaillès is that of the efficacy of axiomatisation. Thus in the sub-section of Chapter I of *MAF* labelled *Rôle effectif des axiomes* (in which, by the by, the word 'effective' does not appear), the advance made by Hilbert's 'axiomatic criticism' of 19th-century geometry partly lies in his elaborating more rigorously than any of his forebears (Pasch et. al)on the methodological perspective according to which '[w]hat is important in an axiom is not what is descriptively stated in it, but

65 The division between the 'internal' and the 'external' here if of course hard to make in principle—'it is difficult to make a division between social or psychological concatenations and mathematical necessity' (*OC* 226) —, but the presumption that there *is* such a division is in a sense primitive for all of Cavaillès's work.

66 *MAF* 54/*OC* 62.

67 To anticipate a later argument: a minimal way of resolving this tension is to say that for Cavaillès what is 'necessary' is *that* mathematics moves, but not any *particular* movement, i.e. that there *be* the generation of new concepts, but not that any *given* concept is generated. This is certainly the more defensible version of the Cavaillèsian research programme. It is not, however, entirely consistent with certain of his comments on Cantor and Dedekind in the doctoral works, which seem more specifically necessitarian. On my reading, something like this analysis only emerges clearly around the time of *LTS*; I shall discuss its formulation and the rather quixotic metaphysical ('mobilist') underpinnings it requires in Chapter III.

rather the efficacy [*l'efficacité*] that it possesses relative to a certain result.'[68] This is more or less a restatement in terms of efficacy of Chevalley's analysis of the 'axiomatic style', as well as an aspect of Cavaillès's general historical analysis as we discussed it in the Introduction, according to which the development in 19th-century mathematics had served to replace evidence with proof. For the local case of geometry, the argument here is that the 'meaning' or 'content' of a geometrical axiom is longer to be found in certainty derived from intuitive representation (I represent two points on a plane in my mind and 'see' that a straight line can be drawn between them), but rather from the consequences that can be derived starting from a group of axioms, where here 'consequence' should be understood as naming something like a hypothetical delimitation of operational space (these axioms prescribe that such and such objects can be manipulated in such and such ways, but given a different set of axioms they could be manipulated differently). Thus, 'what is important' in an axiom is not its initial 'evidential' content, but rather what it eventually allows us to 'effectively' (in the sense of actually) prove. This would be the essential difference between the old 'Euclidean' ideal of the axiomatic method and the new perspective of Hilbert and his successors, for whom 'the axiomatic method alone could found and extend mathematical work, because the axiomatic method expresses its essence'.[69]

Yet this standpoint produces, if not a paradox, at least a tension, which concerns the productivity of axiomatisation, i.e. a different sense of its 'effectiveness'. Axiomatisation comes after informally elaborated theoretical practice, and seeks to purify it.[70] On the other hand, axiomatisation—i.e. the choice of axioms—at the same time creates a new theory, the content of which is the previously given operation(s), but now unfettered. Purification is also liberation. Yet what assures us that the new operation(s) in the axiomatised theory in some sense line up with 'the same operation(s)' in its/their earlier, restricted form?[71] The history of the *décalage* between 'naive' and 'abstract' set theory was exemplary of these theoretical problems; the advantage of Hilbert was to have attempted to investigate them directly in mathematical terms.

68 *MAF* 72–73/*OC* 80–81.

69 *MAF* 75/*OC* 83.

70 C.f. Cavaillès comments on Desargues, *MAF* 69/*OC* 77.

71 To change frames, I understand Cavaillès's overriding problem to have been something like what Rabouin has called (drawing on Putnam) 'the problem of transtheoretical identity' with respect to mathematical objects, i.e. the question of how to resolve the tension between 'referential stability' and 'variability of contexts' in mathematical practice, given that it seems as if this question must be posed—however problematically—in strictly 'internalist' terms. C.f. Rabouin 2023.

Now, it is in light of the interpretation of the effective as the (historically) actual that we should understand Cavaillès's reading of this question of efficacy, for one of the peculiarities of *MAF* (with respect to other approaches to the philosophy of mathematics) is the fact that the question of the nature of axiomatisation is first and foremost understood in terms of the relation between historically prior and posterior theory. Cavaillès's attempt to produce a series of theoretical categories for thinking through the connection between pre-axiomatised theory and the axiomatisation of that same theory is concomitant with a more general epistemological project of how to understand the internal dialectic of mathematical history. This is what leads to Cavaillès's analysis of the 'double given' of axiomatisation. As Cavaillès characterises matters, the standpoint of the first Hilbert, in which the axiomatic method was proposed as an autonomous foundation for mathematical work, faced two primary obstacles:

> 1st [obstacle]: When considered as elements of an implicit definition of one or more words [in ordinary mathematics], the axioms must make appeal for the rest of their statement to previously introduced notions [...]. Thus, axiomatisation only isolates something from an already constituted conceptual system; axiomatisation cannot be represented in a vacuum.
> 2nd [obstacle]: This isolation is not arbitrary: we have seen that axioms, either in systems or in sub-groups, each time characterise the unity of an original intellectual process. Axiomatization generally takes place after the theory has been constituted, and merely strips away the adventitious to make that theory purely what it is. In cases, on the other hand, where the choice of axioms creates a new theory (such as with algebra or set theory), the content of such new theories is never anything other than the description of a certain operation— already defined in a wider theory—but now freed, for example, from restrictions: thus we find in topology that we can define neighbourhoods as arbitrary open sets, and open sets as invariant sets with respect to finite intersection and infinite sum. Or the axiomatic definition of the Lebesgue integral is intended to ensure complete additivity.[72]

Here one of course recognises the same conceptual issues that were at play in Cavaillès's reading of the history of abstract set theory. Approaching them again through the problem of axiomatisation allows them to be posed at a higher level of generality. It is important to note that both in terms of the implicit concepts at stake and the specific examples, this passage also anticipates the analysis of the two modes of mathematical abstraction, the paradigm and the thematic, that we shall have to consider at length in the positive doctrinal elaboration of *LTS* (see Chapter III); there, the Lebesgue integral shall be given as a central case of 'thematisation' in mathematics. In the context of *MAF*, it is clear that the recourse here to 'the unity of an ori-

72 *MAF* 88/*OC* 96.

ginal intellectual process' resumes the theme of the central intuition from *TAE*, and more broadly that the double relation sketched in this passage between intuitive operations and their formal elaboration mirror the earlier scheme of the double unity of theories, which is now stated as a double given of mathematical history. Thus, as Cavaillès immediately summarises following his stating of the above 'obstacles':

> In any case, axiomatisation makes a double reference to a given: externally, to the given of the system from which it borrows its concepts; internally, to the given of an operative unity that it does no more than characterise.[73]

These two 'givens' both bear on the effective: the given of historical technical practice is the actuality of mathematical work, the given of an 'operative unity' is the foundation of its efficacy, which exceeds any particular formalisation. It is in preserving the independence of the latter 'operative' pole that Cavaillès locates the specific contribution of Hilbert's approach. What the adventure of axiomatisation had ultimately shown was that far from an axiomatised theory reducing mathematical work to mechanical deduction, axiomatisation serves to exhibit the profound sense in which 'reasoning consists in operative concatenations guided by the very intuition of these operations.'[74] Attending to these primitive 'operative concatenations' required a deepening of the notion of proof, and thus a transition from axiomatisation (programme of the first Hilbert) to proof theory and 'formalism' (programme of the second Hilbert).

§I.2.iii. Finitism and the Proof-theoretical Stakes of the Effective

> Long before the appearance of the paradoxes, Hilbert had insisted on the importance of intuitive concatenations in true mathematical work.
>
> *MAF* 92/*OC* 100

The fact that Cavaillès reads the proof-theoretical programme inaugurated by Hilbert and his collaborators as fundamentally being concerned with the autonomy of 'intuitive concatenations' in mathematical work might be surprising, given that the Hilbert programme was frequently received in Cavaillès's time (and since then) as a quasi-totalitarian reduction of mathematics to a mere 'formula game'. Yet this is precisely how Cavaillès understood matters: the programme of the 'second Hilbert' consisted not in the suppression of intuitive mathematical work in favour of a 'mechanical' formalisation (an accusation which Cavaillès will, however,

73 *MAF* 88/*OC* 96.

74 *MAF* 89/*OC* 97.

level at logicism), but rather an attempt to preserve the essential autonomy of both intuitive reasoning and rigorous formalisation by producing a mathematical discourse on the line between the two. Thus an essential task of proof theory was to mathematically demarcate the domain of intuitive or finitary reasoning, i.e. to 'specify the zone of effective thought'.[75] In this sense, the project of proof-theory was to produce a mathematical discourse on the autonomy of mathematics. Cavailles's reading on this point concerning mathematical autonomy brings his interpretation into line with Curtis Frank's presentation of the programme, according to which the fundamental ambition of the Hilbertian analysis was that mathematising the notion of proof itself would serve to liberate mathematical work from any extra-mathematical (i.e. philosophical, logical, or epistemological) determination.[76]

In Cavaillès's terms, an affirmation of this anti-foundationalist view on the autonomy of mathematics coincided with an analysis of its effectiveness, such as Hilbert tried to carry out. The Hilbertian standpoint views mathematics as a register of pure efficacy: insofar as it is what it does, there can in principle be no (externally articulated) 'epistemology *of* mathematics', because it is of the very essence of mathematics to be indifferent to questions of justification. Otherwise stated, mathematics *works*, and the crucial question is 'why and how does it work?', but this is only a mathematical question.[77] Proof-theory is the intra-mathematical discourse on this functioning, its forms and its limits. Further, the sense of 'works' intended here is similarly intra-mathematical; it concerns the fact that mathematics answers its own questions, not to any external criteria of adequacy (such as say its 'application' in physics). There is more at stake here than simply an analysis of Hilbert; for Cavaillès this standpoint characterised the whole project of specifically contemporary mathematics, as it had crystallised following the period of the 'foundations crisis' in the 1920s and 1930s. Thus, as summarised by Gilles-Gaston Granger in a very Cavaillèsian spirit, the transition to contemporary mathematics and the birth of mathematical logic was characterised by the 'demand for effectiveness', such that even mathematicians as opposed as Hilbert and Brouwer were involved in a common project, i.e. a particular 'manifestation of rigour, which consisted in substituting the oper-

75 *MAF* 98/*OC* 106.

76 'Hilbert's innovative stance was this: Mathematics is a way of knowing that cannot and need not be justified on any *a priori* grounds. […] Nevertheless mathematics can be the subject of foundational insight, through a self-evaluation the outcome of which is that questions about how and why mathematical techniques work the way they do can be given purely mathematical answers.' Franks 2009: 61–62.

77 In some ways, Cavaillès converges here with Mic Detlefsen's 'instrumentalist' interpretation of Hilbert. See Szczeciniarz 2022.

ative intuition of effective constructions for the passive intuition of "evidences"'.[78] This is an apt summary of the overriding conceptual insight behind Cavaillès's designation of mathematics as 'effective thought'.

Such at least was the ideal version of the Hilbert programme that was so significant for Cavaillès, and which he partly endorses under the name of 'modified formalism' in the conclusion to *MAF*, even if he had many critiques of Hilbertianism in its concrete elaboration.[79] It was the development of such a discourse on efficacy that required the specification of the notion of 'finitary' reasoning, i.e. a *mathematical* definition of the finite (where we understand 'mathematical definition' to mean 'not based on external psychological, epistemological or philosophical categories'). This is the other essential aspect of the metamathematical standpoint that Cavaillès emphasises in his reading of the whole proof-theoretical trajectory that he analyses in *MAF*, running, broadly, from Hilbert and Bernay's first writings on the metamathematical programme in the early 1920s to the work of Herbrand and Gentzen in the 1930s.

There are two essential points to emphasise in Cavaillès's presentation of this history. The first is that Cavaillès implicitly translates the conceptual categories of Hilbertian finitism into Cartesian terms. The second is that, as Cavaillès presents matters, the later developments of the proof theoretical programme in the course of various (ultimately frustrated) attempts to provide a consistency proof for the arithmetic axioms (and the attendant *Entscheidungsproblem*, i.e. the task of defining a general decision procedure for the validity or 'satisfiability' of expressions in first-order logic) had served to repose the problem of *effectuability* at a metamathematical level. The work of Herbrand and the expansion of the concept of recursive function served to make this connection most explicit. These two points are intimately related, for if we schematise Cavaillès's rather terse expositions on these questions, the general picture that emerges is that proof-theory represents a dialectical development on the opposition between the simultaneous (*tota simul*) and the successive (step-by-step) in the characterisation of reasoning that we discussed with respect to Descartes and Dedekind in the Introduction. Thus these points together form what we called above the proof-theoretical index of the Thesis on the Intelligible in *LTS*. Let's deal with them in turn.

With respect to the first point, a Cartesian interpolation is present from the outset of Cavaillès's engagement with proof theory. Thus, in his 1935 review of the first volume of Hil-

78 Granger 1988: 85. C.f. the section of the third chapter of *Pour la connaissance philosophique* (1988) entitled '*L'exigence d'effectivité*', pp. 85–92.

79 *MAF* 179/*OC* 187.

bert and Bernay's *Grundlagen der Mathematik* for *Recherches philosophiques*, Cavaillès characterises the methodological contribution of the project as follows:

> Its [i.e. the Hilbert/Bernays programme] originality lies in the fact that it has raised the finitist method one stage higher: its intuitive material will no longer be those drawings that are the characters, but those other drawings, the proofs, realities that are just as concrete and easy to take in at a glance [*embrasser du regard*].[80]

On this presentation, the proof-theoretical programme serves a very Cartesian purpose: by presenting proofs as finitistically graspable proof arrays, it allows long chains of reasoning to be condensed to the point where they can be apprehended in a single (ocular) intuition. In making this reference to the Cartesian scheme (which as outlined in the Introduction, would have been understood by his audience as having a precise doctrinal referent), Cavaillès touches on the sense in which the basic categories of finitism can reasonably be said to take up the problems that attached to the definition of proof in the *Regulae*, and the attendant distinction between intuition and deduction. The degree to which these problems return in the Hilbertian setting is surely remarkable, for a fundamental conceptual problem in the determination of a concept of 'finitary' reasoning is that it is supposed to encompass both the notion of finite reasoning as representationally simple (i.e. bearing on finitely and thus immediately intuitable signs) and as terminating procedure (we could call these the *objectual* and the *operational* poles of the finitist project). As Wilfred Sieg summarises, the full expression of the concept in the Hilbertian setting ultimately identifies these two senses.

> The word finitist is intended to convey the idea that a consideration, a claim or a definition represents (i) that *objects are presentable*, in principle, and (ii) that *processes are executable*, in principle.[81]

We shall return to this double sense of the finite below, with respect to Herbrand. In the initial formulations of the finitist project, Hillbert's language for the first criteria is that the objects of finitary metamathematics must be *surveyable* [*überblickbar*], a notion that would have an important afterlife in the epistemology of the computable.[82] When Cavaillès discusses this demand for surveyability, he again translates Hilbert into the language of Descartes. Thus in presenting Hilbert's argument from his first systematic paper on the proof-theoretical founda-

80 Cavaillès 1935: 425. The concept of *dessin-démonstration* [*Beweisfigur*] indexed by Cavaillès here is often rendered by 'proof array' in English-language literature on proof theory.

81 Sieg 2019: 138.

82 For a recent survey of the semantics of this concept and its significance for both Wittgenstein and Turing, c.f. Floyd 2023.

tions of mathematics—i.e. the 'New Groundings of Mathematics: First Report' article of 1922—that 'contentual' mathematical reasoning should replace the logicist foundational programme attempted by Frege and Dedekind, Cavaillès summarises the polemic as follows:

> The idea is indeed Kantian: mathematics is more than just logic, insofar as it is effective thought, and that all effective thought presupposes the application of abstract thought to an intuition. […] But it is not the sterility of logic that is targeted here, so much as—and this is the extra-Kantian aspect—its insecurity: 'if logical deduction is to be secure, it must be concerned with objects that can be encompassed by the gaze from all sides [*embrassables du regard sous toutes les faces*], such that their distinctive signs and their reciprocal relations are intuitively given alongside them, as something that cannot be reduced to anything else and has no need of such a reduction.'[83]

To underline the point: in citing Hilbert, Cavaillès here directly renders *überblicken* in the vocabulary of the *Regulae*.[84] It is insofar as metamathematics founds mathematical work on such primitive (i.e. irreducible, *tota simul*) apprehension of simple objects (the sign-complexes) that it can be sure of proceeding on a secure basis. By this same gesture, the psychological notion of intuition is in effect replaced (at least in principle) by reference to the material manipulation of signs, subject to finitary constraints. Thus:

83 *MAF* 92/*OC* 100. I have here rendered as directly as possible Cavaillès's translation of the passage from Hilbert 1922: 163.
Cavaillès's French rendering reads: « si la déduction logique veut être assurée, elle doit porter sur des objets embrassables du regard sous toutes les faces, tels que leurs signes distinctifs, leurs relations réciproques soient intuitivement donnés avec eux, comme quelque chose qui ne se laisse réduire à rien d'autre et n'en a pas besoin .»
The original German reads: „Soll das logische Schließen sicher sein, so müssen sich diese Objekte vollkommen in allen Teilen überblicken lassen und ihre Aufweisung, ihre Unterscheidung, ihr Aufeinanderfolgen oder Nebeneinandergereihtsein ist mit den Objekten zugleich unmittelbar anschaulich gegeben als etwas, das sich nicht noch auf etwas anderes reduzieren läßt oder einer Reduktion bedarf." Hilbert 1922: 163.
In the translation by William Ewald from Mancosu 1998: 202, we find: 'If logical inference is to be certain, then these objects must be capable of being completely surveyed in all their parts, and their presentation, their difference, their succession (like the objects themselves) must exist for us immediately, intuitively, as something that cannot be reduced to something else.'

84 Cavaillès repeats this gesture by translating Gentzen's observation that the problems posed by transfinite induction for the finitist project is that of how to understand that inductions can be 'reasonably perspicacious [*einigermassen übersehbar*]' in a '*precise finitist sense*' as the demand they ought to be in *une certaine mesure être embrassé d'un coup d'œil*. C.f. Gentzen 1936: 559, Gentzen & Szabo 1969: 196, *OC* 171.

> Metamathematics, or proof theory, then becomes the true science: its objects will be the assemblies of signs or formulas, and their organisation into unities of dependence or theories. It is in the grouping of these—i.e. the addition of axioms, the testing of their relative fruitfulness—that real work, capable of procuring truth, consists. Moreover, thought is always certain of itself, since full consciousness accompanies each of its steps [*la pleine conscience accompagne chacune de ses démarches*], which are of finite number: intuitionist requirements are thus rigorously satisfied here.[85]

Such was Cavaillès's fundamentally Cartesian understanding of the simultaneous pole of finitism; it must be complemented by the successive pole, which is clarified by the development of a robust metamathematical definition of recursion.

Here we turn to our second point of emphasis, viz. how the development of a general metamathematical definition of recursion in the context of attempts to prove the consistency of arithmetic leads to a resituating of the stakes of both the effective and the Cartesian concept of proof. Cavaillès treats these issues at length in Chapter IV of *MAF*, but here we will restrict ourselves to his comments on Herbrand. Roughly speaking, it is the attempt to generalise the concept of a consistency proof that leads to the central role of the *Entscheidungsproblem* in the development of proof theory, in that the import of the latter problem is that it became necessary to define a general procedure that would allow a consistency proof for the arithmetical axioms, where the relevant sense of 'consistent' in effect becomes that it is impossible to deduce a contradiction in a finite number of steps.[86] It is not at all self-evident that this notion of the finite as terminating in 'finite' time should coincide with the epistemological demand that mathematical reasoning be securely grounded on finitely apprehendable signs or marks. As noted above, post-Hilbertian proof-theoretical developments often turn on the question of grounding this coincidence.

It is in this context that Cavaillès's treats the work of Herbrand, in particular the method of treating consistency proofs in terms of their possible 'satisfaction in a domain', which Cavaillès paraphrases as the general method of 'effectuation'.[87] This method crystallises

85 *MAF* 98-99/*OC* 106-107.

86 C.f. Mancosu 1998: 151 for a commentary on this sense of 'consistency' for Hilbert and his collaborators, a use which predates the more general posing of the definition of a decision procedure for first-order logic. The canonical statement of the *Entscheidungsproblem* is Hilbert & Ackermann 1928: 77, where it is referred to as *das Hauptproblem der mathematischen Logik*, a designation that Herbrand mirrors on many occasions. It should be underlined that owing to the sources Cavaillès had available to him, the *Entscheidungsproblem* was still treated as open in *MAF*. The negative answer that would be secured by Church and Turing will only register for Cavaillès in *TC*.

87 Cavaillès also refers here to the method of 'elimination' as outlined in Bernays 1932, but notes that as only 'brief indications' have been given as to that method in that text, he has replaced its exposition by treatment of 'the method which derives from Herbrand's Theorem.' See *OC* 133, *OC* 143.

the aforementioned identification between our two senses of the finite, in that Herbrand proceeds by the elimination of quantifiers through the *reduction* of a theory to its statement in terms of 'individualised' quantifier-free propositions (i.e. by putting it in *prenex form*), which can then be treated uniquely in a recursive, step-by-step manner, in which 'mathematical functions […] are replaced by a particular individual for each particular value of their argument.'[88] As Herbrand summarises in the introduction to his 1930 dissertation *Recherches sur la théorie de la démonstration*—a startlingly Cartesian text—this guarantees the security of effective mathematical reasoning by giving a formalisation of the 'method of recursion', in a manner that, in context, could be read as replacing the (Cartesian) ideal of intuition as simultaneous evidence. The intuitive (i.e. the finite) is now schematised as that which 'stops in the finite', and (the method of) recursion is what allows us to properly demarcate these halting points. This is why 'method' must correspond to a 'handbook' of effectively constructible operations. Thus, for a given proof and a given property A:

> This recursion which 'stops in the finite' [*« s'arrête dans le fini »*] has such an intuitive certainty that the most rigorous mind could not refuse to acquiesce to it; for an argument which relies on it is basically only a description of the operations which are to be executed [*à effectuer*] in each particular case; it is a handbook [*un manuel opératoire*] which permits us to verify property *A* for the conclusion of any proof translated into signs that we will be able to effectively construct [*que nous pourrons effectivement fabriquer*].[89]

Herbrand is then able to establish an identification between such effectively constructible operations and certain number-theoretic functions, such that the *Entscheidungsproblem* is 'arithmetised' (which does not mean it is solved).[90] In other words, a meta-theoretic problem (how to procure a general procedure for determining whether a given statement is a theorem of a given theory, i.e. of a set of axioms) is shown to be equivalent to a specific class of arithmetical problems, i.e. the 'the effective solution of diophantine equations.'[91] It is in this sense that arithmetisation realises a central ambition of the Hilbert programme, which we discussed

88 *OC* 147.

89 Herbrand 1968: 37–38/ Herbrand 1971: 50. Translation modified.

90 As Walter Dean underlines, this conception of the 'method of arithmetisation' was developed in part by Hilbert and Bernays, as presented in Bernays 1930, which Cavaillès had read. This is germane to our analysis in that it highlights how Hilbert and Bernay's work in mathematical logic is naturally developed by computability theory (c.f. Dean 2021: 365). This is entirely consistent with Cavaillès's interpretation.

91 Herbrand 1968: 213/Herbrand 1971: 276.

above under the heading of the autonomy of mathematics: questions *about* mathematics are to be considered rigorously as mathematical questions.[92]

In terms of our central theme in this chapter, it is remarkable that this achievement is brought about through a reformalisation of the notion of the effective, by a kind of double movement: depending on the perspective taken, one could say that the problem of effectiveness had been raised from the calculatory level to the metamathematical level, or that metamathematical problems had been concretised by rendering them in arithmetical terms. Either way, this movement, by which the general question of what it means to understand theorems as consequences of axioms became identified with a particular class of number-theoretic problems, was a coming together of our two modalities of the effective, as indexed by the terms effective (or 'actual') mathematical work and effective calculability.

Attendant to this reformalisation of the concept of the effectuable, the methodological prioritisation of recursion, and with it a precise formulation of finitary reasoning, has the surprising result that the paradigm of secure reasoning is successive, not simultaneous. This is of course an inversion of 'Cartesian' epistemology. Now, it should be noted that Herbrand does not explicitly mention Descartes, but, as outlined in the Introduction, it is was quite natural in the historical context to read this refoundation of the concept of finitary reasoning on a rigorously 'step-by-step' basis as an affirmation of the successive pole of the Cartesian picture of proof. This is clear in Albert Lautman's presentation of Herbrand's results in his 1935 address to the International Congress of the Philosophy of Science:

> For the Hilbertians, metamathematics cannot be part of formalised mathematics, and this is explained by Herbrand as follows: a metamathematical reasoning must be discursive and for the same reason must be based upon a recursion in the finite, the validity of which comes from the fact that, as Descartes required, said reasoning can be revised so rapidly that it becomes intuitive. This metamathematical recursion is thus the necessary condition for the proof of the consistency of reasoning by ordinary recursion in arithmetic.[93]

Nowhere in *Recherches sur la théorie de la démonstration* does Herbrand himself suggest that his 'method of recursion' resumes the Cartesian theme of the rapidity or condensation of reas-

92 As Franks emphasises, it is for this reason that on Herbrand's view the importance of arithmetisation was not necessarily (or even probably) that it would help to resolve the *Entscheidungsproblem,* or even that it would make it *easier*, but rather that it showed the mathematical nature of the profound *difficulty* of its resolution. '[F]or Herbrand the value of arithmetization is evident when a metatheoretic problem is shown to be too hard to solve just as it is when the reduction leads to a solution: In each case a metatheoretic problem has been cast into the precise terrain of mathematics.' Franks 2009: 97.

93 Lautman 2006: 55. Lautman adds a footnote to this comment referencing Herbrand 1930.

oning, but taken alongside Cavaillès's own Cartesian interpolations of proof-theoretic terminology, Lautman's presentation demonstrates a shared epistemological background for a 'Cartesianised' reception of the Hilbert programme. It is notable that on Cavaillès's reading it is also the focus on the notion of (effectuable) recursive procedure that allowed the proof-theoretical programme to achieve its sought after depsychologisation of the notion of a finite or 'intuitive' basis for mathematical work in a manner that was more rigorous than Hilbert's earlier informal presentations of the ideal of 'surveyability', i.e. that with Herbrand and his successors the notion of the finite no longer rested on its (implicitly subjective) 'evidential' character. This at least is what Cavaillès expresses in a letter to Lautman on 13th June 1936, written whilst revising the section of *MAF* treating Herbrand and Gödel: '[T]he great merit of Herbrand—which I haven't emphasised enough—is to strip his approach of all intuitive signification (i.e. interpretation of quantitative signs).'[94] This splitting of the idea of the finite from the ideal of evidence was methodologically crucial insofar as it allowed properly finitary methods to be applied to the successive construction of mathematical objects (i.e. certain forms of transfinite induction) that in principle exceeded any simultaneous apprehension. It is in this sense that Cavaillès observes that by making the notion of finitary reasoning more precise, Herbrand had also 'enlarged' it.[95] Here Cavaillès cites Herbrand's definition of intuitionistic or finitary reasoning from 'Sur la non-contradiction de l'Arithmétique' (1931), i.e. a crucial point in the genesis of what will later be called the Herbrand-Gödel formalisation of general recursive functions:

> By an intuitionistic argument we understand an argument satisfying the following conditions: in it we never consider anything but a given finite number of objects and of functions; these functions are well-defined, their definition allowing the computation of their value in a *univocal* way; we never state that an object exists without giving the means of constructing it; we never consider the totality of all the objects *x* of an infinite collection; and when we say that an argument (or a theorem) is true for all these *x*, we mean that, for each *x* taken by itself, it is possible to repeat the general argument in question, which should be considered to be merely the prototype of these particular arguments.[96]

94 Sinaceur 1987: 120. This comment is all the more remarkable for having been written prior to Cavaillès's engagement with the Church-Turing formalisation.

95 C.f. *OC* 161.

96 Definition from Footnote 3 in Herbrand 1931a: 3, translation from Herbrand 1971: 288–289, emphasis mine. Herbrand's term 'intuitionistic argument [*raisonnement intuitionniste*]' should be understood as referring to the demands of *finitary* reasoning in the sense of the Hilbert programme, rather than to *intuitionistic arguments* in the sense of Brouwer, hence why Cavaillès introduces the cited passage at *OC* 161 as an enlargement of *raisonnement fini* (on this point, see van Heijenoort's introduction to Herbrand's 1931 paper in Heijenoort 1967: 618–620, reproduced at Herbrand 1971: 282–284).

Herbrand then goes on to define the set of recursively defined functions with the demand that —considered 'intuitionistically' in the sense just cited, i.e. in a 'finitary' manner—such functions must 'must make the effective computation [*permettent de faire effectivement le calcul*]' of values possible 'for every particular set of numbers.'[97] With respect to these definitions, Cavaillès states:

> Indeterminacy now only remains with respect to the construction: it can be understood as a progression of stages, each of which is finite in the sense mentioned above. There is no longer a way to encompass all of these steps in a single glance [*Il n'y a plus moyen d'embrasser du regard toutes les démarches*]: if consciousness ceases to be coextensive with their succession, it is at least present in each of them.[98]

This formulation is remarkable for the extent to which it condenses a series of themes which we have sought to highlight in Cavaillès's considerations on the notions of the effective and of proof. The essential point to underline is again one of conceptual translation, for Cavaillès's interpretation of Herbrand intercalates in very precise terms the legacy of Dedekind's *Treppenverstand* ('*notre entendement à marches*', as Cavaillès translates the term in *MAF*) with the Cartesian problem of the simultaneity or the successiveness of proof.[99] Cavaillès's formulation here anticipates his definition of formal concatenation in the Thesis on the Intelligible from *LTS*: the characteristic of the intelligible is that not everything can be established at once [*d'un seul coup*], i.e. that no single consciousness (for which we can read: no single formalism, no single theory) can simultaneously survey the elements of mathematical work. There is thus a precise proof-theoretical sense in which Cavaillès later assertion of the non-simultaneity of the intelligible is a *consequence* of the fact that formal concatenation is 'conditioned by the effective'.[100] The effectiveness of mathematical work and its *non tota simul* character coincide.

97 Herbrand 1931a: 5, translation from Herbrand 1971: 291, translation modified to replace Goldfarb's 'actual' with 'effective'. See Sieg 1994 and Sieg 2005 for commentary on the conceptual developments between Herbrand and Gödel's respective definitions of recursion.

98 *OC* 161.

99 *OC* 66.

100 Indeed, we could go further and hypothesise that both the above-cited summary of Herbrand's results and the later Thesis on the Intelligible from *LTS* are implicit citations (conscious or unconscious) of Herbrand on Cavaillès's part. Immediately following the definitions just cited from Herbrand 1931a, Herbrand notes in an appendix on Gödel's recently announced incompleteness theorems that if one defines 'Group C' as the set of effectively calculable functions (in the sense just outlined, i.e. recursively definable in Herbrand's sense), then Gödel's use of the method of diagonalisation proves that said set of recursively definable functions cannot itself be enumerated by a recursive function. This has the consequence that, as van Heijenoort summarises, 'the metamathematical description of the system cannot be projected into the system.' Herbrand 1971: 284. With respect to the (thus defeated) ideal for a complete metamathematical characterisations of 'all procedures for the construction of functions', Herbrand writes: 'One could object that it is perhaps possible

This at least is the position at which Cavaillès will arrive; it is only anticipated in *MAF*. Here we must again attend to historical context. The metamathematical programme as it had developed up to the moment surveyed in *MAF* had produced a redefinition of the problem of effectuability in mathematics. This had served to conjoin problems that would scarcely have seemed connected to the generation of Borel and Lebesgue. In particular, it was surely not clear at the turn of the century that the sense of 'succession' at stake in the deployment of the natural numbers (arithmetic)—i.e. in 'effective calculability'—and that at play in Dedekind's ideal for a successive reconstruction of mathematics that would correspond to the limitations of our 'step-by-step understanding' would somehow come to turn around a common formalisation. Yet this is what the adventure of investigations into proof theory and first-order logic eventually produced. The technical problem of determining the bounds of the *calculable* and the epistemological problem of characterising the stepwise nature of proof in mathematics (as much the immediacy of its 'evidence') had condensed around a common set of procedures; in turn, a question of the highest generality concerning the limits of the *solvable* had been identified with the resolution of a particular class of arithmetical problems. This had served to repose problems around the boundaries of 'concrete' or 'definable' mathematics as Borel and Lebesgue had considered them, but on a radically different terrain. Cavaillès does not directly draw this connection in the text of *MAF*, but it is implicit in his arrangement of the historical material. Throughout that work, one has the impression Cavaillès is searching for a precise formulation of his own conception of the effective, as exemplified by the paradoxical situation he outlines in the conclusion, which was that contemporary developments had shown mathematics to be 'effective work' which in principle could not be restricted to 'any system of effectively constructed objects.'[101] Such is the background against which Cavaillès will interpret the closely related developments of a new formulation of the notion of effective procedure and a negative answer to the *Entscheidungsproblem* achieved by Church et al.

to describe at once [*d'un seul coup*] all the schemata that are included in C for the construction of functions. This would mean that it is possible to describe outright [*d'un seul coup*] all intuitionistic procedures for the construction of number-theoretic functions. But this is impossible.' Herbrand 1931a: 8/Herbrand 1971: 296. Cavaillès never directly refers to these lines; they are, however, the most plausible index for Lautman's above-cited interpretation of Herbrand in terms of the *Regulae*.

101 *OC* 184.

§I.3. Effective Process and Effectuated Process in *Transfinite and Continuum*

Written in 1940–1941 and published posthumously in 1947, *Transfini et continu* marks a new moment, both in Cavaillès's thinking concerning the effective and in the broader mathematical history of the concept.[102] The essay is predominantly concerned with Cavaillès's technical exposition of two new mathematical developments: first, the formalisation of the Church-Kleene notion of effective calculability (or computability) in the papers they authored from 1935 to 1938, i.e. the formation of the so-called Church-Turing thesis and the attendant negative solution to the *Entscheidungsproblem*; second, Gödel's 1938/1940 results on the consistency of the continuum hypothesis.[103] Cavaillès then concludes with a dense few pages in which he develops a new critique of Kantian projects in the foundations of mathematics. The point I wish to focus on is in what way Cavaillès's use of the concept of the effective with respect to the earlier debate between the French analysts is transformed by his engagement with the Herbrand-Gödel notion of recursive function and the Church-Kleene formalisation of effective calculability in this new moment. For our purposes here, it is significant that Cavaillès directly

102 The essay was intended for *Revue philosophique* and submitted in September 1941, but the journal was abolished by the Nazi occupying forces before it could be published. A typescript, annotated by Cavaillès, was preserved by Suzanne Lautman (the widow of Albert), and corrected by Henri Cartan and Chevalley for publication as Cavaillès 1947. See Appendix I for more information on its conditions of composition. See Appendix II.3. for a translation of the final 4 pages of the publication, which form a continuous exposition on the 'Kantian theory of mathematics'.

103 A note on Cavaillès's sources: with respect to Church and Kleene, the papers cited by Cavaillès in the original publication of *TC* are Kleene 1935a, Kleene 1935b and Church 1936b (see Cavaillès 1947: 15). The editors of Cavaillès 1962 (i.e. the version of *TC* reproduced in the *Œuvres complètes*) insert references to Church & Kleene 1937, Kleene 1938, and Church 1938, without signalling that these were not necessarily read by Cavaillès (c.f. *OC* 463). With respect to Gödel's results, the only paper cited directly by Cavaillès in *TC* is Gödel 1939, which does not directly mention the new definition of general recursive functions (again, the editors of the 1962 *Philosophie Mathématique* collection insert additional citations; compare Cavaillès 1947: 17 with *OC* 467). Cavaillès does however appear to directly quote Gödel's 1938 abstract announcing the results, albeit in paraphrased form (compare Gödel 1938: 556 with *OC* 467); again, Gödel makes no direct mention of general recursion there. Cavaillès does not seem to have had access to the canonical formulation of the notion of general recursive functions in Gödel's Princeton lectures as transcribed by Kleene and J. B. Rosser and published as K. Gödel, 'On undecidable propositions of formal mathematical systems' (reprinted with revisions in Gödel 1986: 346–371). However, Cavaillès's exposition in *TC* draws on Paul Bernays's Supplement II to the second volume of Hilbert and Bernays' *Grundlagen der Mathematik* (Hilbert & Bernays 1939: 392–421, miscited by Cavaillès as being published in 1940), which recapitulates Gödel's innovations (as does Church 1936b, which, as noted, was read by Cavaillès). This Supplement—entitled 'A precise explication of the concept of calculable function and Church's Theorem on the decision problem [*Eine Präzisierung des Begriffs der berechenbaren Funktion und der Satz von CHURCH über das Entscheidungsproblem*]'—likely influenced the specific manner in which Cavaillès interrelates the new results by Church, Gödel and Kleene.

relates Borel and Lebesgue's notions of the effective to Church's recasting of the concept of 'effective procedure (or process)' by introducing a split between what Cavaillès will call 'effective calculation' and 'effectuated calculation', which then informs his critique of the 'Kantian theory of mathematics' on the grounds that it conflates 'effective process' and 'effectuated process'. This formal disjunction between the effective (or the 'effectuating') and the effectuated is the major conceptual advance in *TC*, and will play a crucial schematic role throughout *LTS*.

The fact that Cavaillès understands Church and Kleene to be formalising the same notion as Borel and Lebesgue merits comment, given that neither Church's own papers nor prominent commentaries on the Church-Turing hypothesis tend to mention this connection.[104] Church, Kleene and Turing's formalisations are instead usually indexed to the general pre-history of the notion of recursive definition and the search for a resolution to the 'decision problem', whose major points include the introduction of the chain rule in Dedekind's *Was sind und was sollen die Zahlen?* (1888), Hilbert's use of transfinite recursion and the epsilon calculus in the attempted proof of the continuum hypothesis in the 'Über das Unendliche' lecture of 1925, work by Hilbert, Ackermann and Bernays on the *Entscheidungsproblem*, and the Herbrand-Gödel notion of general recursive function. These are doubtless the major references explicitly made by Church.[105] It is important to note, however, that Church repeatedly refers to his and Turing's definitions as providing a formalisation of an already widespread 'ordinary (not ex-

104 Exceptions include Avigad & Brattka 2014 and Dean & Walsh 2017, as well as some Russian histories, probably owing to the influence of Nikolai Luzin (e.g. Semenov & Uspensky 1981, in particular pp. 105–106 and pp. 176–181). Whilst Church doesn't directly index the notion of effective procedure to Borel et. al., he does cite Borel's notion of the effectively enumerable in Church 1936a: 157, along with other works by the French analysts.

105 It remains a matter of lexical curiosity from where exactly Church derives the terminology of the effective. He appears to have coined its general use as a technical notion in the proof-theoretical literature, as evinced by the fact that Turing does not use it in his contemporaneously written Turing 1937, and Hilbert and Bernays only begin using the term *effektive* in the second volume of the *Grundlagen*, i.e. after having read Church. One might expect that the source is Herbrand, except that in a letter on 29th November 1935 Church writes to Kleene that Herbrand's definition of recursiveness 'has no regards to effective calculability' (c.f. Sieg 1997: 160). Plausibly Church came across the terminology whilst researching his 1934 paper on Richard's paradox, which would have involved reading French sources, and in which he speaks in passing of (effective) enumerability, i.e. Borel's terminology (c.f. Church 1934: 359). In any case, this again highlights that Cavaillès's immediate perception of a trajectory running from Borel and Lebesgue to Church-Kleene was not so widely shared, and likely shaped by his location in France. Amongst Cavaillès's rough contemporaries, Gustav Choquet also saw this as a natural connection: in a 1945 address to the Colloque de Logique et Philosophie entitled 'Épistémologie du transfini', he observes that amongst the previous generation of mathematicians 'many were in agreement that mathematics should be limited to the study of definable and clearly conceived entities; except, they were no longer in agreement as to what to put under these terms', and presents Church's work on 'effective constructibility' as more adequately formalising the notions at stake. Choquet 1980: 7.

plicitly defined)' or 'vague' notion of effective calculability (indeed, the status of the Church-Turing thesis as a 'thesis' is that it suggests such an identification between a formal and an informal notion, and thus is not 'provable' in formal terms).[106] I take Cavaillès to have seen this informal notion to be precisely what was at stake in the debates over the notion of effective definability among the French analysts. In having seen this connection, Cavaillès in *TC* is to an extent anticipating the history of effective descriptive set theory.[107]

When Cavaillès introduces the notion of effective calculability in *TC*—and the immediate reference here is to Gödel, but the middle section of *TC* will be concerned with the Church-Kleene formalisation in a way that shows that Cavaillès is connecting the two developments—he does so by recalling the same point which framed the introduction to *TAE*, i.e. the discussion between Borel and Lebesgue concerning 'the notions of the nameable and of the effective.'[108] He then cites Lebesgue's 1905 definition of the *nommable* or *effectif* (see above, §I.1.), before remarking:

> The vagueness of this requirement—which was intended to limit the domain of authentic mathematics—is due to the indeterminacy of [the notion of] *words*. If these include 'all objects [such] that... there is an object that...', where is the finitude? The development of formalisms, in particular thanks to Gödel's methods, makes it possible to avoid this pitfall. By obliging a calculation to be effective [*être effectif*], Lebesgue was led to transform it into an effectuated calculation [*calcul effectué*], [at least] if he wanted to chase away the indeterminacy of 'a calculation'. But this would have made all progress impossible. The correlative notions of the rule-governed calculus [*calcul régulier*] and of quasi-recursive functions introduced thanks to Gödel's work escape this dilemma.[109]

What is going on here? The point concerning the 'indeterminacy of words' is clear: if we say that all we need to do in order to have defined a mathematical object is to name a characteristic property as the *definiens* of the *definiendum*, without having placed any constraints on the acceptable modes of definition involved, then nothing prevents us from naming precisely

106 C.f. Church's review of Turing's 'On computable numbers, with an application to the Entscheidungsproblem' (i.e. Church 1937) and Church 1938: 226–227.

107 I thank Brice Halimi and Walter Dean for clarifying conversations on this point.

108 *OC* 461–462.

109 *OC* 462–463. The (non-standard) terminology of *calcul régulier,* as well as that of 'quasi-recursive functions' are both drawn from Bernays' Supplement II to the second volume of the *Grundlagen der Mathematik*, i.e. I understand *calcul régulier* to be Cavaillès's translation of the calculus of *regelrecht auswertbaren* functions introduced by Hilbert and Bernays as a generalisation of Church, Gödel and Kleene's definitions (c.f. Hilbert & Bernays 1939: 394–395). Wilfrid Sieg suggests 'reckonable functions' as a translation for this notion (c.f. Sieg 1997: 166); I adopt the plainer 'rule-governed' from Copeland 2024. The later French translation by F. Gaillard, E. Guillaume and M. Guillaume renders it by *fonctions à évaluation réglée* (c.f. Hilbert & Bernays 2001: 466–467).

those 'paradoxical' properties that the notion of effectiveness was intended to prohibit, i.e. the problems of impredicativity return in full force. The only practical means advanced for avoiding this problem at Lebesgue's time was to (implicitly or explicitly) refer back to the kind of constraints on calculability that characterised Borel's notion of the effectively enumerable. This explains the ambiguity of Lebesgue's position that we discussed in §I.1., wherein Lebesgue both seems to propose a wider notion of the effective than Borel and states that he is attempting to express the same constraint.[110] From Cavaillès's perspective this means that *in practice* Lebesgue's openness to the demands of effective work collapses into constraining mathematics solely to the domain of *effectuated* calculations, i.e. that Lebesgue's quasi-formal notion of effectiveness is only clarified by reference to a particular class of (already) effectuated objects. Lebesgue's position thus collapses back into Borel's in the course of its elaboration. Cavaillès's claim is that first Gödel and then Church-Kleene, in formalising the modern notion of effective computability, allow us to avoid this pitfall:

> By means of this numbering [i.e. the Gödel numbering] and these recursive relations, there is a permanent control of the effectiveness of the calculation [*l'effectivité du calcul*]: each of these steps falls under our grasp since we are sure to reach it by means of a canonical sequence of steps. The determination lies in the process of recursion. The truth is that this is not fixed once and for all, which is precisely what prevents [future] mathematics from being limited to [already completed] contemporary mathematics [*A la vérité celui-ci n'est pas fixé une fois pour toutes, ce qui évite justement de borner les mathématiques aux mathématiques actuelles*]. But a function, for example, will be said to be *calculable in a rule-governed manner* [régulièrement calculable] if it is possible to construct a special formalism for it that meets the conditions laid down.[111]

The development here is dialectically subtle. As Cavaillès notes, it is Gödel's work that inspired Kleene and Church to develop what Church calls an *absolute* definition of constructibility or effectiveness, where 'absolute' is understood to signify that the notion is 'independent of reference to a given formal system'.[112] This absoluteness or independence of the new formal defini-

110 E.g. Lebesgue frames his commitment to descriptive definitions as being distinct from Borel's constructive definitions; on the other hand, immediately preceding his definition of the effective Lebesgue writes: 'I shall try never to speak of a function without effectively defining it; I shall thus adopt a point of view very similar to that taken by M. Borel in his *Leçons sur la théorie des fonctions.*' Lebesgue 1905: 205.

111 *OC* 462. Again, Cavaillès is drawing on Hilbert and Bernays's exposition of *regelrecht auswertbaren* functions.

112 *OC* 463. The editors of *Cavaill*ès 1962 add a footnote at this point referencing Church 1938, which Cavaillès does appear to be paraphrasing: 'Much of the interest of the proposed definition lies [...] in its absoluteness, and would be lost if it could be shown that it was in any essential sense relative to a particular scheme of notation or a particular formal system of logic.' (224). As noted above, Cavaillès does not cite this paper in the original version of *TC*. The question of whether Cavaillès read Church's 1938 paper is significant, given that it would have been a source available to Cavaillès which mentions Turing's contribution to the so-called

tion of effectiveness is what makes it adequate as a proposed formalisation of the informal notion. Yet the inverse of this movement of absolutisation is that the notion of an effectuable object is now purely *relative* to what Cavaillès will call 'a given conceptual system' (i.e. to a given formalism).[113] In terms of the temporalised reading that we have been discussing, Cavaillès is asserting that it is precisely because the notion of effectiveness in the Church setting is solely a notion of *procedure* or *process* that it doesn't attempt to limit mathematics to any particular class of effectively or intuitively graspable objects, and thus is more supple with respect to the demands of mathematical theories that have *not yet* been developed.

Thus the novel point here—which depends 'on the definition of effective process [*processus effectif*]'[114]—is that the same movement in the history of the concept of the effective can at once be read as an *absolutisation* of the notion of constructibility and a *relativisation* of the notion of intuition. The notion of intuition thus no longer refers to any given class of 'evident' objects upon which construction could be founded, but rather, in Cavaillès's terms, intuition is internal to concatenation. It is this inversion which grounds Cavaillès's new critique of Kant, or, more precisely, of the Kantian underpinnings of competing projects of mathematical finitism:

> An irreducible intuition would only be a halt, devoid of thought. [*Une intuition irréductible n'est qu'un arrêt sans pensée*]. The Kantian theory of mathematics [...] conflates the dialectical moment of the positing of the concept and the transcendental moment of its schematisation. [...] Moreover, the misunderstanding appears which takes the construction itself to be representable; this is the confusion already noted between effective process and effectuated process [*procès effectif et procès effectué*]. [...] The fundamental ambiguity of transcendental philosophy is that it posits the conditions of knowledge as relative to a synthetic activity which is defined and directly apprehended in a knowledge that itself escapes from critique: the condition of representation is conflated with the representations of this condition.[115]

To summarise, the problem with Borel's position was that his proto-constructivist notion of the effectively enumerable sought to condition mathematics by something extrinsic to mathematics, i.e. the notion of a finite number of words or characters. This is connected to the notion of an intuitive basis on which (future) mathematical work could be grounded. In turn, Lebesgue's more liberal 'descriptive' notion of the effective was tied to the same strictures. We

'Church-Turing Thesis', and Turing is never mentioned by Cavaillès. Turing's result is however mentioned in passing in Gentzen 1938, which Cavaillès had read.

113 *OC* 472.

114 *OC* 464.

115 *OC* 469–470.

can say that Borel and Lebesgue share a certain 'Kantianism' that we also find in Kronecker, Hilbert, Brouwer and other finitist projects (albeit in very different ways) insofar as they make a double attempt, at once descriptive and restrictive: the first is to describe the constructive (or intuitive) activity of mathematical understanding as it actually (or effectively) proceeds; the second is that they aim to ground this constructive activity in a certain well-defined or intuitively graspable class of effectuated objects (or intuitions), that is to say well-defined (or individuated) with respect to *our consciousness* and its limitations.

Hilbert's methodological comments in the 1927 Hamburg address on 'The Foundations of Mathematics' are exemplary in this regard. On the one hand, he positions the project of proof theory in descriptive terms: 'The fundamental idea of my proof theory is none other than to describe the activity [*Tätigkeit*] of our understanding, to make a protocol of the rules according to which our thinking actually [*tatsätlich*] proceeds.'[116] On the other, this ambition to describe construction is prefaced by the assertion that 'as a condition for the use of logical inferences and the performance of logical operations, something must already be given to us in our faculty of representation [*in der Vorstellung*], certain extralogical concrete objects that are intuitively [*anschaulich*] present as immediate experience prior to all thought.'[117] What Cavaillès argues, to the contrary, is that by delinking effectiveness and effectuation, Gödel and Church allow mathematical philosophy to maintain its descriptive ambition without its restrictive supplement. This is founded in the recognition that the meanings given to any type of object that we might wish to serve as a foundation or as an intuitive basis for mathematical reasoning (such as points, marks, whole numbers, 'the continuum') are continually altered and rewritten by the formalisms (or concatenations) that specify them. There is nothing prior to thought: this is the consequence of Cavaillès's characteristic conjunction ‘mathematical experience’.

What is at stake in the critique of the Kantian theory of mathematics in the conclusion to *TC* is Cavaillès's confrontation with various projects of finitism and constructivism spanning the preceding fifty years, one that is remarkable because it displaces certain technical problematics from the history of mathematical finitism into Cavaillès's own philosophy of mathematical history. The new theory of intuition that is outlined here is thus Cavaillès's contribution to a relativisation or historicisation of the transcendental:

116 Hilbert 1928, translation from Heijenoort 1967: 475, emphasis mine.

117 *Ibid*., 464. Hilbert’s text of 1928 repeats a portion of the text from Hilbert 1922; in both texts the lines just cited immediately precede the sentence formulating the notion of the ‘surveyable [*überblickbar*]’ which we discussed above in §I.2.iii.

> The mark of history is the submission of the transcendental to its stages […]. Necessity appears after the fact [*après coup*]. No analysis of the consciousness of acts permits prediction, any more than it can procure an arbitrary permanence. *Intuitive is a synonym for effective (or effectuating) consciousness, transcendental for constitutive (or constituting) consciousness*, relative to a given conceptual system. No more than there is a closed system is there an absolute transcendental: formal unity is the continuity of *a concatenation of acts*, the concrete is the novelty of unifying a multiplicity that is still conceptually recognised as such.[118]

Mapping this passage onto the one cited just above, the effectuating pole is placed on the side of 'effective process', while the transcendental or 'constituting' pole is placed on the side of 'effectuated process'. For Cavaillès, the (temporal) disjunction here between the effective (now identified with the effectuating) and the effectuated poles is not something that is contingent or *extrinsic* to the mathematical concept, but is something that is *constitutive* of the concept itself. The new concept of concept presented here is continuous with Cavaillès's redefinition of the notion of the effective, which now names the whole array of symbolic and conceptual material (objectively) available to a working mathematician at a given moment. In other words, the effective becomes the site of what Petitot has called Cavaillès's theory of the '*continuous actualisation* of intuition [actualisation continue *de l'intuition*]', an actualisation which is rather a perpetual reactualisation that is each time mediated by the specificities of mathematical work, and not through any reference to the putatively *a priori* structure of consciousness, nor through any pre-individuated object domain.[119] The conclusion to *TC* should then be read as the transition to the object of Cavaillès's mature work in *LTS*, namely that of providing a theory of the logic of the passages between the positing of objects and their transformations in new formal settings, which is to say a theory of mathematical time that is no longer to be indexed to the time of consciousness, as with the Kantian and Husserlian theories.

The formulation of the effectuating/effectuated disjunction produces a hard-won formula for Cavaillès's conception of mathematical work and its internal historicity: it is a dia-

118 *OC* 471-472, emphasis mine.

119 Petitot 2014, an address delivered at a commemoration for Cavaillès held at the ENS, 17th February 2014. On Petitot's reading, Cavaillès's gesture in *TC* is in fact to simultaneously historicise the two Kantian regimes of intuition and the concept: on the one hand, 'intuition (i.e. the transcendental aesthetic) is historicised through reactualised intuitive superposition, [that is] intuition is connected to effective practices [*pratiques effectives*] and no longer to an original donation as with Husserl'; on the other, 'categoricity (i.e. the transcendental analytic) is historicised through the dialectic of the concept.' (*ibid*.) In other words, what interests Petitot in *TC* is that in the same movement by which Cavaillès abolishes the idea of an external donation which is to act as the basis for mathematical synthesis, he reworks the doctrine of the schematism as internal to mathematical work.

lectic between the actual and the effectuable, but these two poles must be maintained as a necessary formal distinction. The fact that transcendental constitution has been placed on the side of the effectuated is the core of his critique of Kantian philosophy, and in a sense the whole of *LTS* will be organised around this disjunction. In the context of *TC*, it allows Cavaillès to place the project of the doctoral works on a new theoretical ground. Above, we outlined how Cavaillès's general project in the doctoral studies was to develop an account of the internal dialectic of mathematical history and the attendant problem of theory invariance across transformations between formal settings. Cavaillès's epistemological work was orientated towards an account of how mathematical theories emerge, and the development of conceptual categories specific to thinking the relationship between prior and posterior theory, a relationship that must be able to account for both the genuine novelty of new theories and the sense in which they meaningfully transform specific notions in prior theory. In the conclusion to *MAF* he summarises that dialectic as follows, with respect to what he calls the three roles of generalisation or idealisation in Hilbertian formalism:

> In each case [i.e. in each of the three forms of generalisation], the fecundity of effective work is obtained by these ruptures in the mathematical fabric, by this dialectical passage from a theory that carries its boundaries within itself to another superior theory which can disregard it despite of and because of issuing from it. Such striking through of lacunae**,** such intentionally operated reversals—nothing could be more opposed to the progressive construction of intuitionism, which links everything together through processes that are generated by each other and takes the dyad as its unique starting point.[120]

In *TC*, this analysis is rearticulated in terms of concatenation (which we can now read in the expanded, substantive sense of the term that will predominate in *LTS*). In specifying his critique of the 'Kantian theory of mathematics', everything comes to turn on the assertion that, for Cavaillès, the philosophical import of the discovery of 'new notions' in mathematics should not lie in the attempt to render these notions legible within the armature of pre-existing philosophical consciousness, but rather in trying to think how transcendental structuration is transformed subsequent to their production.

> Each time, the solution does not lie [...] in the discovery of a translation of these new notions into an immutable spatial intuition [...]. The solution rather lies in a transformation of the intuitive zone, which is to say in a transformation of the rules that lay out the use of thought in it, i.e. the schematic system. The most striking example is the numerical continuum in the definition given by Dedekind. Otherwise put, *intuition in its quiddity progresses parallel to the dialectical concatenation of concepts*. It [i.e. intuition]

120 *MAF* 172.

> is nothing other than the manifestation for empirical consciousness of a relative independence of methods and theories, which permit those autonomous elaborations the results of which occasion encounters and reversals. The transcendental schema represents the motive power of a new synthesis, subsequent to such reversals: the internal force of a method, the underlying unity of a theory. To the extent that it schematises an intuition, it fixes the theory in a rigid aspect, acquiring for it a kind of uniformity the very intelligibility of which makes it seem definitive.[121]

In a word, the transcendental itself is produced by a concatenation, and the multiplicity of different 'intuitions' in mathematics is an effect of this production. This represents a break with the notion of the 'central intuition' as the medium of the unity of theories as it had orientated the doctoral dissertations, for in *Transfini et continu* 'theory' is in no sense the emanation of an invariant intuition. The patina of finality that seems to attach to a mathematical theory (at a given 'stage' in its development) is rather the consequence of the transformations occasioned by the 'dialectical concatenation of concepts' (or 'concatenation of acts'), which results in new intuitions through reschematisation. If, as Cavaillès states elsewhere, mathematics possess a 'necessary rhythm', like a musical composition, then the relation therein between intuition and concept is contrapuntal. All of this is consummate with Cavaillès's mature nominalism, and his correlative critique of formal sufficiency: a refusal in principle of the elaboration of an *a priori* meta-theory that could act as a universal (and trans-historical) plane of comparison for mathematical theories; from the perspective elaborated in *TC*, the presumption of an intuitive basis for mathematical work falls to the same critique. Yet the consequences of this interpretation of contemporary mathematics—the framework that would justify it—remained to be elaborated. 'The link between this intuitive superimposition and the dialectic of the concept remains the fundamental problem of the philosophy of mathematics.'[122]

121 *OC* 470, emphasis mine. Sinaceur glosses this passage as follows: 'An intuition is the mark of the constraining presence of the object, but this presence is not a donation, whether the source [of that donation] be internal or external. Objectivity lies not in the adjustment or the coincidence between subject and objects; it lies in the autonomy of viable formations. Moreover, there are not objects in the proper sense, there are only fulcrums for concatenations.' Sinaceur 1994: 91.

122 *OC* 471. As we shall discuss in Chapter III, 'superimposition' is a term of art for Cavaillès.

II. FROM EXPERIENCE TO CONCATENATION

> One does not pass without a break from a practice to its concept, from [an] experience to its concept.
>
> Louis Althusser, letter to Jacques Lacan, 4th December 1963.[1]

On Logic and the Theory of Science sets out from the point where *Transfinite and Continuum* ends, that is, with the task of providing an account of 'the dialectical concatenation of concepts', or what will be called in *LTS* 'the effective concatenation of intelligible contents'.[2] The distinction between the effectuating and the effectuated, the technical context of which occupied us throughout Chapter I, will be presented there as governing the internal structure of concatenation, which will in turn be identified with the essence of 'thought' or 'the intelligible' itself. This is the positive interpretation of *LTS* that I shall outline in Chapter III. In order to situate the theoretical terms at stake there, however, it is first important to give an account of how Cavaillès arrived at this new perspective. My aim here is thus partly genetic. The preceding chapter was focussed on Cavaillès's work as an epistemologist and a historian in the doctoral studies and afterwards, and his methodological orientation around the study of effective mathematical work. I will now consider how Cavaillès was lead to certain inventions through his attempt to theorise this earlier period of epistemological practice.

To this end, we will now traverse for a second time the period up to the composition of *LTS*, but focus on Cavaillès's more general philosophical reflections on the concept of 'mathematical experience'. This latter term is often taken be the central concept in Cavaillès's work, such that *LTS* is read as a final and partial attempt to provide a theory *of* mathematical experience.[3] This is not the interpretative path we will follow here. On the contrary, I wish to emphasise that in Cavaillès's late writings the concept of experience is displaced by that of concatenation. This does not mean, of course, that concatenation assumes all of the theoretical content of the concept of 'experience', for in that case there would be no force to the transition;

1 Althusser 1996: 154–155.

2 *OC* 470; *LTS* 18/*OC* 500.

3 This perspective was especially common amongst Cavaillès's students, such as in Desanti's preface to the 1994 publication of the *Œuvres complètes*: 'Cavaillès wished to understand "mathematical experience" in its necessary movement of constitution, and [to locate] in it, consubstantial with this necessity, logic itself.' *OC* 7. It is also the assumption behind commentaries such as Cortois's 'The Structure of Mathematical Experience According to Jean Cavaillès' (1996) and Cassou-Nogues's *De l'expérience mathematique* (2001). See also Sinaceur 1994: 83: 'This final text [i.e. *LTS*] was for the most part a prelude to Cavaillès's projected book on mathematical experience.'

'concatenation' is importantly much more restricted in its scope, which is why it represents a theoretical advance.

At stake here is, in part, a question concerning a double unity: first, the internal unity of Cavaillès work (of his writings), and, second, the unity between that work and his life. The assertion that we find an expansive concept of 'experience' in Cavaillès's work has served to ground the claim of such a double unity, such that one can claim to trace a straight line from the rigours of Cavaillès's epistemological research to his later 'heroic' stature as a member of the Resistance. Exemplary here are the various obituaries and commentaries by Canguilhem. In one of the last of his many eulogies for Cavaillès, 'A Life, A Work: 1903–1944, Jean Cavaillès, Philosopher and Resistant', given as a radio address to France-Culture on 27th April 1989, Canguilhem begins with the following assertion:

> At the heart of Jean Cavaillès's philosophy is a question, the statement of which in traditional terms has led some otherwise competent and good faith commentators to hesitate to comply with it. This question concerns the nature, function and importance of experience in mathematics.[4]

For Canguilhem, the confusion that can arise from interpreting Cavaillès's concept of 'experience' in 'traditional terms' lies in mixing up the notion of lived experience with the project for a philosophy of mathematical experience announced by Cavaillès, in which, rather than referring to something immediate and pre-conceptual, 'experience' would in effect be another name for the concept of 'mathematical work', i.e. the 'resolution of a problem' according to a rule-governed series of gestures. Properly understood, '[t]here can be no confusion between *this* 'experience' and the 'experience' celebrated by empiricist philosophers.'[5] Canguilhem thus took the development of such a 'non-empiricist' concept of experience to be both the central ambition of Cavaillès's work that was to realised in his final manuscript, and the medium of continuity between his philosophical and resistance activity.

> Experience, gesture, work; these agents, which Cavaillès deemed to be responsible for the progress made by mathematics in its search for its foundations, were also, for him, immediately the ways and means of his conduct in the Resistance: chosen, planned, and pursued as a perilous enterprise.[6]

This kind of eulogy is central to what we might call the folkloric reception of Cavaillès's life and work in the French philosophical field, in short to his reputation as a *figure*: the 'resistant

4 Canguilhem 1989, *OC* 683.
5 Canguilhem 1989, *OC* 683.
6 Canguilhem 1989, *OC* 684.

according to logic', the 'terrorist philosopher'.[7] It creates a synthetic schema, sometimes explicit, sometimes implicit: Cavaillès, the rationalist philosopher of the internal necessity of mathematical experience was lead to embrace a different necessity, that of practical action, through another formative experience, that of the resistance. However, a reader that approaches *LTS* with such a narration in mind will likely be surprised do discover that in that manuscript, composed in a prison cell in the midst of Cavaillès's involvement in the resistance, the basic theoretical terms invoked by Canguilhem either do not appear or do so only in a marginal, elliptical manner. Indeed, one discovers an enigmatic 78 pages of writing, posthumously entitled *On Logic and the Theory of Science,* which supposedly develops a 'philosophy of mathematical experience', in which the phrase 'mathematical experience' is not present, and which can only in a rather quixotic sense be said to make any positive propositions with respect to either 'logic' or the 'theory of science'.

At the very least, what needs to be explained is a twofold observation with respect to the text of *LTS*. First observation: although the word 'experience' appears many times there, it no longer responds to the positive theoretical ambition that Cavaillès had seemingly harboured for the coinage 'mathematical experience' or, at times, for the correlated expression 'mathematical activity'. 'Experience' is now only a term used to report on and analyse the other positions which Cavaillès treats in the essay; it is no longer used in his own voice. We are thus not dealing here so much with a question of attestation as one of absence. Whence the importance of a second observation: what we *do* find in *LTS* is a series of moments in which doctrines that Cavaillès had previously been developing with respect to the notion of experience are directly rearticulated in the language of concatenation, with the term appearing in various forms: *enchaînement rationnel*, *l'enchaînement intelligible*, *l'enchaînement démonstratif*, *l'enchaînement formel*, and, yes, *l'enchaînement mathématique*. The substantive use that predominates here—*l'enchaînement*—is moreover mostly new to the vocabulary of Cavaillès's writings, with the most relevant exception being the use which emerges at the end of *Transfinite and Continuum* and which I have just cited above: *l'enchaînement dialectique des concepts*. In this restatement of Cavaillès's theoretical project, 'mathematical experience' is not the only term that disappears: the concept of the 'gesture', which Cavaillès had earlier advanced as the fundamental unit of mathematical experience, is substituted by the concept of 'act' (or 'operation').

7 Canguilhem 1969, *OC* 677, 678.

Some interpretation of this transition is an essential prelude to any analysis of *LTS*, and the vexed question of how to the relate the programmatic announcement there for a 'philosophy of the concept' to Cavaillès's other writings. The methodological question here is perhaps the most essential: can one cite writings from across Cavaillès's small corpus with the assumption of a 'doctrine' that is shared by the earlier epistemological works and *LTS*? This has been the norm in most writing on Cavaillès—although not all.[8] I think matters are more complicated than this, for reasons of periodisation that shall concern us below. In brief, what this chapter aims to establish for the purposes of interpretation is that I read *LTS* as expressing a quite distinct project from that which Cavaillès was pursuing in his doctoral studies and the period immediately after. To underline the point: I say 'for the purposes of interpretation' because I make no claim here as to Cavaillès's intention; nowhere will you find a record of him inveighing against the concepts of 'experience' or 'gesture'. I am interested in the internal logic of the writings, in what they contain and what they do not. With this in mind, I want to foreshadow that, as I read it, the manuscript of *LTS* contains at least the outlines of both a positive and a negative project, both of which can best be understood in the relief cast by Cavaillès's prior attempt to develop a 'philosophy of mathematical experience'.

With respect to the negative project, *LTS* is organised as a dismantling of the possibility of a 'theory of science' insofar as, in broadly post-Kantian terms, it would aspire to be a theory of the *unity* of science and—or rather *qua*—the unity of experience. For all that *LTS* is in part a reckoning with the long arm of *Wissenschaftslehre*, we misread that work if we imagine it as a conjunctural rectification of a field of post-Kantian 'epistemology' defined by its then-contemporary transformation into the respective camps of logical empiricism and phenomenology.[9] It is closer to the order of a refutation. The form taken by this refutation—expressed as

8 Indeed, we might say that the central 'contemporary' question internal to the small world of Cavaillès scholarship has become how to interpret the 'break' between Cavaillès's early and late works, if it exists. But, tellingly, there is little agreement as to what constitutes this difference. Schwartz 2020 and Cortois 2024 are exemplary in this regard, respectively making the transition to the 'philosophy of the concept' turn on an encounter with Hegel's *Logic* and Cavaillès's engagement with post-logical empiricist debates in the philosophy of physics (readings which of course do not strictly preclude each other, but suggest rather different theoretical horizons).

9 Indeed, in a sense the major editorial intervention from Canguilhem and Ehresmann is having given the work the title *On Logic and the Theory of Science*, which had the function of bestowing a degree of theoretical generality on a rather conjunctural and specific text. Moreover, for all that they attempted in their introduction to qualify matters by insisting that 'the title we have proposed is a label of convenience rather than a programmatic statement, and makes no reference to the spirit of the doctrine to which it applies' (*LTS* xii/*OC* 482), it has surely given the impression that Cavaillès's essay outlines a positive contribution to the 'theory of science'. It is at least as viable to argue that the opposite is the case.

Cavaillès's polemic against the 'philosophy of consciousness'—relies essentially on an argument against the notion of the field of experience as both the unifying ground and object of scientific knowledge, that is, in an argument against any philosophy of transcendental 'constitution'. The fatal problem with the concept of experience is that it *is* a concept of unity, or at least of unification. Recall that we are here interpreting an absence: that in Cavaillès's final writings, 'experience' no longer appears as a central positive term in his reflections on mathematics. The reading I will develop is not that Cavaillès ceases to be concerned with 'experiences', but that in the conjuncture in which he was writing, invoking a 'philosophy of experience'—however much one might specify the notion of 'mathematical experience'—could not but be read as advancing some principle of unity, i.e. of the homogeneity of a constituting field. This is in large part what is at stake in the polemic against the univocity of consciousness that will close *LTS*, and signals his most profound break with phenomenology.[10] Thus for all that Cavaillès's late philosophy could from another angle justly be called a philosophy of the intrinsic heterogeneity of experience, the suppression of reference to experience in his final text served to clarify the radicality of the thesis which marks the insuperable gap between the 'philosophy of the concept' and post-Kantian thought: 'experience' has no common form, and insofar as there is no (*a priori*) form of experience in general, it is useless to conceive philosophy as an investigation into the 'conditions of possible experience'—or *a fortiori* into the conditions of scientific knowledge in general. Minimally, it is important to consider the scope that Cavaillès had attached to the concept of 'experience' in order to understand what is at stake in this 'negative' conclusion, a conclusion which for all of its appearance of being a sceptical or even destructive endpoint for the philosophy of science, is in fact at the core of Cavaillès's rationalism.

Such considerations are all the more crucial if we are to situate the positive doctrine in *LTS*: Cavaillès's development of an account of mathematical or 'intelligible' concatenation. For the pages of *LTS* in which Cavaillès primarily develops this account, that is, in his outline of the concepts of *the paradigm* and *the thematic*, are undeniably in some sense a restatement of conclusions that Cavaillès had previously presented under the banner of a project for 'mathematical experience', wherein those concepts—or rather their earlier correlates—bore the

10 *LTS* 78. 'The term 'consciousness' does not entail univocity of application any more than 'thing' does an isolable unity.' To ward off potential confusion, the sense of 'univocity' at stake here is not that of the so-called 'univocity of being', but concerns the capacity of consciousness to 'uniquely' define its objects. See footnote 62, below.

names *idealisation* (or, alternatively, 'generalisation') and *thematisation*.[11] Thus at the very heart of *LTS* we find what appears to all the world like Cavaillès finally writing down his doctrine of mathematical experience—without once invoking the term 'experience' in the course of this condensed exposition.

In this new positive statement of things, the role previously played by 'experience' is now taken up by the notion of concatenation, but in a novel sense, which I propose to schematise as *inter-theoretical concatenation*, that is as the concatenation between different historical theories, or what Cavaillès will call in his lectures in the early 1940s—in a crucial moment for my narrative of this transition—the 'concatenation of systems'.[12] Here, the reader might well ask why emphasise the *novelty* of Cavaillès's deployment of the concept of 'concatenation' in *LTS*, given that the term is often present in Cavaillès's doctoral studies, and indeed I invoked it more than once in the previous chapter. My claim is that in that earlier period of Cavaillès's work, 'concatenation' was used for the most part to refer to the admissible chains of inference internal to an already constituted rational domain, that is, to what we can call *intra-theoretical concatenation*. This is the 'classical' sense of the term, more or less as it is found in Descartes.[13] Thus concatenation in effect named an established 'mode of reasoning', or the working through of a proof, internal to a given theory. This was counterposed to the wider field of historical mathematical 'experience' or 'activity', out of which which these theories could be said to be constituted. Such an opposition is what we do not find in *LTS*. On the contrary, for the 'mature' (or at least, for factually the last) Cavaillès, inter-theoretical concatenation becomes the name of the history within which intra-theoretical concatenations are produced. Thus a distinction between experience and concatenation is replaced by an internal stratification of the notion of concatenation. The 'dialectic' of mathematical history that Cavaillès will then evoke is to be understood as the dialectic between these different kinds of concatenations, and the forms of movement proper to them.

This chapter attempts to explain something of how this transition in standpoint took place. This involves reckoning with questions both as to what Cavaillès intended by the concept of experience, and what is at stake in his final turn towards interest in a 'logic' of the theory of science. These questions are not independent of a rather vexed historiographical

11 I.e. *LTS* 26–33, 'the most precious pages' of the work, as Paul Cortois refers to them (Cortois 2024: 98), with which we shall be concerned at length in Chapter III.

12 The quote is taken from Cavaillès's 1941 lectures at the Sorbonne, which we shall discuss in §II.2.

13 See my comments on this in the Introduction.

question concerning what projected work, exactly, Cavaillès's final essay is to be read as the partial realisation of. On the one hand, in the period we shall now discuss, Cavaillès made reference in a letter to Brunschvicg to having wanted to write a book entitled *L'Expérience mathématique.*[14] On the other, just prior to his arrest and imprisonment, and the attendant writing of the manuscript we receive *LTS*, Cavaillès taught a course on Logic at the Sorbonne over the course of 1941 and 1942, which Bachelard had commissioned him to write up for publication. It is at least partly in this context that Cavaillès made references to wanting to write a *Treatise on Logic*, which is the name by which he refers to the *LTS* manuscript when handing it to his sister Gabrielle Ferrières.[15] It is in effect impossible to adjudicate finally whether *LTS* is best understood as a partial draft of one work, the other, or neither. Broadly speaking, however, the interpretation that I will take here is that the surviving manuscript that Canguilhem and Ehresmann publish as *Sur la logique et la théorie de la science* should first and foremost be read as a reworking of the material Cavaillès had just presented in his Logic course at the Sorbonne, but that in the act of writing Cavaillès went beyond the aims of those lectures, and folded in certain modifications of the doctrine of mathematical experience as he had been developing it the period that we will now discuss.[16]

All of this leaves open the very general question: what is the project of logic, as Cavaillès came to conceive it? And how is logic to be related to the 'theory of science'? One way of rephrasing this question is: *is there or ought there to be 'a logic of science'* (if this is taken to mean the same thing as 'a theory of science') or perhaps '*a logic of the theory of science*' (if it is not)? And—to resume the question Cavaillès will, in *LTS*, take Bolzano to have posed most sharply—what discipline would constitute it? It is important here to emphasise the radicalism of Cavaillès's (implicit) negative answer to these questions: the orientation ultimately adopted by *LTS* is that no such *externally* constituted discourse for the articulation of an autonomous logic of science is to be accepted. This is intrinsically bound up with the question at the core of *LTS*: what is thought? Or so I will argue. This leaves a difficult methodological question for the interpreter: what relation to posit between the history of logic and Cavaillès's own 'positive' doctrines in *LTS*? This is partly also the question: what relation to posit between *LTS* and

14 Ferrières 2003: 182.

15 See Ferrières 2003: 201.

16 For some of my rationale on this point, see the history of the composition of the *LTS* manuscript that I include as Appendix I. Recounting such a history inevitably involves a partial account of Cavaillès's resistance activities.

Cavaillès earlier epistemological or historical studies? These are questions that shall occupy us for the remainder of this chapter.

In what follows, I will first attempt to delineate as narrowly as possible what Cavaillès's seems to have intended by the project for a philosophy of 'mathematical experience' circa 1938–1939, as well as detailing some of the theoretical difficulties encountered by that project. Of crucial importance here will be the largely implicit anthropologism of the concept of mathematical experience—i.e. its essential reference to 'human activity'—which I will argue is in large part what Cavaillès breaks with in his final work, and, in doing so, signals a renewed confrontation with the most intimate problems of both phenomenology and 'Brunschvicgian' epistemology. Second, I will indicate what I take to be Cavaillès's announcement of a new problem in his lectures at the Sorbonne immediately before writing *LTS*, i.e. what he will call the question of how to think the 'difficulty of the traversal of problems [*parcourir les problèmes*]', or of providing a 'logic' of prior and posterior production in mathematics. The turn to such a logic of production is indicative of another question that will be central to *LTS*: that as to the unity or diversity of 'mathematical becoming', and, moreover, of 'thought'.

*

With respect to the texts written between Cavaillès's doctoral works and the composition of *LTS*, the most important sources here, in chronological order:

1) The presentation with Albert Lautman to the Société française de Philosophie, published as 'Mathematical Thought' (February 1939);
2) *Transfinite and Continuum* (written 1940–1941);
3) The surviving transcripts of the 1941–1942 Sorbonne courses on logic (spring 1941 through summer 1942);

In Chapter I, we already considered *TC* and the schematisation therein of a formal distinction between the effectuating and effectuated internal to the concept of mathematical work. This development will thus be assumed here, and we shall limit ourselves to consideration of the first and third of these sources, which respectively represent the most developed documents we possess of Cavaillès's projects for a work on *Mathematical Experience* and for a *Treatise on Logic*. Comparing these two texts serves to both delineate a fairly consistent project that emerged for Cavaillès in this period, and to set up the shift in standpoint that is already rep-

resented in *TC* and which will be orientating for *LTS*. Reading between the lines, what we find here is Cavaillès arriving for the first time at a more systematic statement of the problem he intended to address—that is to say, an account of the aim of his work as a philosopher-historian of mathematics—whilst at the same time seeming to hesitate as to the philosophical foundations of this project. Thus, in the text sent by Cavaillès to the Société, we find the only time in which he directly describes himself as an 'epistemologist' of mathematics:

> This becoming [i.e. of mathematics] seems to be autonomous; it seems possible for the epistemologist to recover a necessary concatenation beneath the accidents of history; the notions that were introduced were necessitated by the solution to a problem and, solely by virtue of their presence amongst the notions that already existed, they pose, in their turn, new problems.[17]

Cavaillès resumes the same point in his oral exposition:

> This becoming is autonomous, which means that while it is impossible to place ourselves outside of it, we can, by studying the contingent historical development of mathematics such as it is presented to us, discern necessities beneath the concatenation of notions and procedures. Evidently, the word 'necessity' here cannot be made precise in any other way. We make note of problems, and we realise that these problems demanded the appearance of a new notion; that's all that we can do, and it is certain that such a use of the word 'demand' comes all too easily to us, since we are standing on the other side of things, since we can see the successes. What we can say is that the notions that have in fact appeared have really provided a solution to the problems that were effectively posed.
>
> I believe that it is possible, beneath the picturesque contingency of the concatenation of theories, to carry out such work. For my part, I have tried to do this for the case of set theory [...].[18]

Here we have Cavaillès's first proper invocation of *inter-theoretical concatenation*. But, perhaps surprisingly, it is used to refer both to the ideal or necessary 'concatenation' of problems and to the factual or historical 'concatenation' of theories (*l'enchaînement des notions et des procédés*). This enabled a clear delineation of the task of the epistemologist: to recover the necessary concatenation of problems, as it were, *ex negativo* (and only retroactively) through the study of the empirical history of mathematical production, that is, through (or beneath) historical concatenation. 'Experience' was in a sense conceived here as the common medium of these two forms of concatenation. As Cavaillès saw matters, such an excavation beneath the 'picturesque contingency' of the archive of mathematical texts is what he had made an attempt

17 *OC* 594.
18 *OC* 600.

at in his *thèse complementaire* on set theory (i.e. *TAE*), and he was now attempting to formulate the precepts of such an approach as, if we accept the terminology, a historiographical research programme. I do not believe Cavaillès ever renounced this ambition. But what is striking in these texts—and which, as we shall discuss at length below, one finds little trace of in *LTS*—is that in the very moment in which Cavaillès formulates the notion of 'necessary concatenation' as the object of epistemology or of the philosophy of mathematical experience, he can only do so by invoking its requisite doubling—as material—with contingent, that is, *human* mathematical activity. Thus the concept of 'necessary concatenation' was first proposed in a rather untheorised manner as a concept of 'history', wavering between reference to an ideal history of problems grasped in the sequence of their necessary emergence and an essentially *anthropological* history of what Cavaillès called, in a letter to Paul Labérenne, 'the effective work of the mind—that is, of human technique—in experience.'[19] In the first instance, this posed problems of ontology: what can we say of the nature of the mathematical object, such that it can be in some sense the common referent for these two 'concatenations'? That is, how to account for its transtheoretical character?[20] But perhaps more acutely, it posed a problem of discourse: having at once asserted that epistemology ought to be concerned with necessary as opposed to contingent concatenations *and* methodologically limited his work to the study of effectively realised mathematics, as opposed to any *a priori* theory, from what position could the line between the necessary and the merely historical be demarcated?

§II.1. Three Theses Concerning Mathematical Experience in 'La pensée mathématique' (1939)

In the case of the 'Mathematical Thought' transcript, the summary provided by Cavaillès to the Société is very instructive, for it both gives an account of the results of the doctoral works from a rather different perspective—that of the positive elaboration of a concept of mathematical experience—and points towards later developments. It begins by giving the rationale for the two conceptions of mathematics that Cavaillès takes to have been disbarred by contemporary

19 Mougin 1945: 79. No precise date is given, but Mougin states that the letter is from the end 1938. Significantly for the development I am outline here, in the same letter Cavaillès asserts 'my complete break with idealism, even of the Brunschvicgian variety […] I have nothing in common with university optimism, the philosophy of the Third Republic, that *ersatz* of religion […].' For a translation of the theoretical part of the letter, see Appendix I.

20 I am appropriating the terminology of the 'transtheoretical' character of the mathematical object from Rabouin 2023. See §I.2.ii., footnote 71.

technical results. On the one hand, logicism must be rejected for two reasons, both relating to what has been revealed by 'the effective formalisation of mathematics.'[21] First, it has been shown that the formalisation proposed by logicism does not solely rely on 'purely logical notions', but rather requires reference to properly mathematical theories such as combinatorics. Second, Gödel's results have proved that 'it is impossible […] to insert [the whole of] mathematics into a unique formal system.'[22] On the other hand, the hypothetico-deductive conception of mathematical proof, which Cavaillès takes to have been given the maximum degree of formal precision by Von Neumann, must be considered invalid because the idea of a *sui generis* axiomatic starting point has been shown to be a fiction, owing to the dependence of deductive notions on previously elaborated informal mathematical theory (with, as we shall discuss below, Skolem's 'paradox' as the presumed background to this latter point). In short: 'there is an essential solidarity between the various parts of mathematics, such that it is impossible to envision a regression back to an absolute beginning.'[23] These considerations Cavaillès presents as the 'incontestable' part of his analysis (because technical), but also as being only 'negative' in character.[24] Having stated this conjunctural analysis, Cavaillès outlines three philosophical theses, which are a condensed expression of the broader project on which he intended to embark at this point, and which by contrast we can take as his attempt at a 'positive' (but by that dint also 'contestable') contribution to the question of the 'essence of mathematical thought'.[25] These theses are admirable for their clarity—they are doubtless amongst the clearest 'doctrinal' expressions that Cavaillès ever produced—but each of them also entails theoretical difficulties which in part anticipate the programmatic change in standpoint that is found in *LTS*. In a reduced form, these theses are as follows.

21 *OC* 593. The text of 'La Pensée Mathématique' is composed of three parts: 1) respective summaries that were communicated by Cavaillès and Lautman to the Société in advance; 2) a main oral presentation by each of them expanding on the same points in greater detail; 3) a transcript of the ensuing discussion. Parts 1) and 2) thus involve a significant degree of repetition, and here I move between them quite freely in order to condense the essential points. For all citations from the 'Mathematical Thought' transcript, I have drawn on Robin Mackay's translation (i.e. Cavaillès & Lautman 2015), but modified the language for consistency with my general exposition.

22 *OC* 593–594.

23 *OC* 594.

24 *OC* 596.

25 *OC* 596.

§II.1.i. First Thesis on Mathematical Experience

> Mathematics is a becoming which cannot be defined. Qua becoming, it is necessary, autonomous, and unpredictable.[26]

As a becoming, mathematics has two primary characteristics. First, it is autonomous, which is to say necessary. The equation here is, implicitly, a kind of Spinozist Kantianism: to say that something is autonomous is to say that it is not determined by anything outside of itself, but this is also to say that it fully determines itself, without contingency, such that the two notions mutually imply each other: 'Autonomy, thus necessity.'[27] Second, qua a 'veritable becoming', it must be authentically or originally 'unpredictable [*imprévisible*]', i.e. there can be no *a priori* structure that determines its development in advance. Rather, this development is shaped each time by the precise contours of its effectuation. Taking these two points together, Cavaillès's thesis is that in mathematics (and *only* in mathematics) the three Kantian modal categories coincide: in the becoming of mathematics, actuality (effectiveness) = necessity = possibility (unpredictability); this tripartite equation determines the relevant concept of autonomy here.[28] From all of this Cavaillès makes a corollary assertion: mathematics cannot be defined (where 'definition' is understood to signify recourse to an external standpoint on said becoming); whence a rejection of the whole problem of 'foundations'. Specifically, the rejection of 'definition' here turns on the ineliminable historicity of mathematical thought, i.e. on the assertion that qua structure of indefinite effectuation, mathematics will repeatedly produce new elucidations of its essence, but these will precisely be partial apprehensions internal to its continuous unfolding.

> Not only is it impossible to reduce mathematics to something other than itself, but every definition, within a given epoch, is relative to that epoch—that is to say, to the history of which this definition is the culmination. There is no eternal definition. To speak of mathematics is always to remake mathematics.[29]

However, for all the stridency which which Cavaillès makes this rejection of definition qua external measure, we could say that he does implicitly advance a minimal definition of mathematics that shall come to the fore in *LTS*: mathematics is defined solely by its mode of passage,

26 *OC* 600–601. For my statements of each of Cavaillès's central 'theses' presented to the Société, I am making a synthesis of the major programmatic points advanced by Cavaillès across the session.

27 *OC* 601.

28 We can note that Cavaillès is thematising in terms of 'necessary becoming' the same modal characteristics we discussed with respect to the concept of 'effective mathematical work' in §I.2.ii.

29 *OC* 594.

by the way in which it moves. Implicit here is a definition of mathematics as rational or intrinsic movement, as it will be elaborated on throughout Cavaillès's final essay. In sum, Cavaillès's comments to the Société concerning mathematical becoming express the position that he will soon state succinctly in the unpublished fragment 'Mathematics and Formalism'.

> The image of the gesture must not deceive us: however free the invention of a method may seem, the development of the whole of mathematics follows a necessary rhythm: there is a reciprocal conditioning of notions and of the expansions provoked by their mandatory application in neighbouring domains. Specifying the modalities of this conditioning by closer examination of this history, which is not a history, may help us to understand, in any case not to define, if definition signifies reduction. The failure of the simplistic endeavour whereby all of mathematics was pruned back to the poorest of its branches, to combinatory calculation, is a sufficient example. It is useless to try it on a more ambitious level, such as Cantor ultimately wanted to do with his theory of sets. Mathematical activity is an object of analysis and possesses an essence: but like a smell or a sound, it is itself.[30]

Cavaillès's stated ambition here to produce what one could call a purely descriptive theory of mathematical becoming leaves some rather fundamental theoretical tensions. His polemic against the adequacy of 'logicist' definitions of mathematics lead him to abjure any direct recourse to a formalised theory of the essence of mathematics or to a 'combinatoric' account of the symbolisation of forms, i.e. to any putative reduction of the open history of mathematical operativity to the framework of mathematical logic in a narrow sense. It is by this token that—at the very least in the period of the doctoral works, which is in a sense summarised in the presentation to the Société—Cavaillès's project was exclusively epistemological, and aimed to constrain mathematical philosophy to an *a posteriori* reflection on the real movement of mathematics, a movement that would in principle exceed any particular formal theory. This is one of the most recognizably Brunschvicgian aspects of Cavaillès's work. At the same time, Cavaillès's ambition was increasingly to produce an anti-psychologistic theory of mathematics that would consider mathematical becoming as an autonomous process in which 'the necessary generation of new concepts' was in no sense to be reduced either to the creative consciousness of the individual mathematician, or the 'external' standpoint of consciousness, or 'intelligence' in general, with this latter refusal signalling a break with Brunschvicg's idealism.[31] The problem here is that if the progress of mathematics can be neither accounted for

30 *OC* 664. As I will discuss below, given that the final lines of *LTS* will reject the notion of activity, the closing formula here is one more reason to place the text as part of an earlier theoretical moment, in which the concepts of 'activity', 'gesture' and 'experience' form a certain consistency.

31 See my introduction to Chapter I.

through a psychological theory of creation, nor through a general theory of the intelligence, nor through a logical reflection on form, then there seems to be no place left for philosophical reflection on mathematics besides an effervescent and passive apprehension of a generative activity the motor of which, in the last instance, escapes positive conceptualisation. This is what Brice Halimi has recently emphasised as a fundamental lacuna in Cavaillès's theoretical project:

> Cavaillès leaves the rich logicist reflections on the symbolisation of forms unexploited. Yet, is not a theory of operations and of concatenations that is deprived of any logical theory of their symbolisation condemned to remain on the side of a philosophy of the act, of experience, of consciousness?[32]

Taken as an address to the project presented in 'Mathematical Thought', this critique is very apt. What is patently required in some sense or another—even if negatively defined—is an account of the 'ontology' of mathematics, and a corresponding confrontation with philosophical logic. Otherwise, by invoking ontological notions such as 'becoming' while refusing the logical resources necessary for their theoretical articulation, one is forced back to an ineffable philosophy of contemplation.[33] It is not clear that Cavaillès ever fully escapes from this 'contemplative' error, but minimally we can state that around the time of 'Mathematical Thought' the theoretical question as to the position of philosophical reflection on mathematics was profoundly unresolved. It is if one likes the problem of theory itself. It is in this framework that we should evaluate Cavaillès's proposal of the concept of 'mathematical experience' and the related conceptual architecture around the notion of the 'gesture', which were at this juncture his attempt, however minimal, to positively theorise mathematical becoming. Whence Cavaillès's second thesis.

§II.1.ii. Second Thesis on Mathematical Experience

> Mathematics is an experimental activity. Mathematical experience is to be understood as a 'regulated', recursively defined chain of gestures on gestures that, in the last instance, bear upon 'concrete' or 'sensible' objects.[34]

32 Halimi 2020: 160–161.

33 This contemplative stance is what Halimi sees as a consequence of Cavaillès's refusal of the logicist tradition: 'Forms are not simply "immanent", in a sense left undefined, to mathematical concatenations. The formal can and must be the object of a theory, even if this theory cannot be a science of forms: this theory is philosophical logic as conceived in the logicist tradition.' Halimi 2020: 161.

34 *OC* 601–602.

Here, we have to start by accepting a certain constitutive polyvalence in Cavaillès's use of the term *expérience*, which carries both the connotation of 'experience' and 'experiment'. With respect to the projected work that would have borne the title *L'Expérience mathématique*, it is clear that the modality to emphasise is that of Cavaillès's ambition to produce a concept of 'mathematical experience' in general, that is, not the 'lived' experience of the mathematician, but mathematical experience as a name for the particular structure of object constitution proper to mathematical work. The concept was thus supposed to name something close to what was intended by the slogan of Brunschvicg's which Cavaillès cites in the section of the conclusion to *MAF* devoted to dialectical experience [*l'expérience dialectique*] as the necessary correlate of the 'existence' of the mathematical object: '[m]athematical analysis is a suggestion made by experience for the extension of experience itself.'[35] With respect to particular statements in 'Mathematical Thought' and elsewhere in which the concept of experience is deployed, however, the word often appears in the course of an elaboration in which restricting the translation exclusively to 'experiment' is equally—or more—valid. Exemplary here is the fact that the only direct definition that Cavaillès gives of the phrase *l'expérience mathématique*, taken from his main oral presentation to the Société, comes during the course of an assertion as to the 'experimental' nature of the mathematician's activity, such that 'the mathematical experiment' is in many senses the proper translation. While elaborating on the notion of mathematical becoming, Cavaillès observes that the 'fundamental dialectic of mathematics' is that the generation of new notions appears to be locally necessary (owing to the specific exigencies of posed problems) but that at the same time these new notions or 'procedures' appear with 'complete novelty'. He then proceeds:

> I would characterise this novelty by way of the second point of my conclusion: namely, that the activity of mathematicians is an experimental activity [*une activité expérimentale*].
>
> By experiment [*expérience*], I understand a system of gestures, rule-governed and subject to conditions independent of these gestures. I'm aware that this definition may seem vague, but I believe it is impossible to remedy this wholly without taking up effective examples. I mean by this that each mathematical procedure is defined in relation to a foregoing mathematical situation upon which it partly depends, but from which it also boasts an independence, such that the result of this gesture is to be observed in its fulfilment. This, I believe, is how the mathematical experiment [or experience] [*l'expérience mathématique*] might be defined.[36]

35 *MAF* 176/*OC* 184. The citation is from Brunschvicg 1912: 570.

36 *OC* 601.

Cavaillès's statement of the concept here must partly be understood in terms of the opposition to physical experiments (or experience). Cavaillès wants to draw out both a proximity between experiments in physics and the creation of new notions in mathematics and an absolute distinction between the two modes of thought. On the one hand, mathematics like physics is a material activity that is intrinsically embodied in the 'sensible world'. On the other hand, mathematics and physics are to be distinguished by the modes of passage between prior and posterior theory in the two disciplines. The resolution of a mathematical problem is 'experimental' in that it is 'a construction subject to possible failure, but carried out in conformity with a rule […] and which, ultimately, unfolds within the sensible.'[37] But unlike in physics, where the criterion of the success or failure of an experiment is indexed to the external world to be described, in mathematics the norm of experimental validation is purely internal, i.e. the content and consequences of the positing of a new mathematical concept are fully and solely determined by the relation between the body of prior mathematics and the novel modes of reasoning instituted by said new mathematical theory, and not by any external (i.e. extra-mathematical) criteria of adequacy. Thus in mathematics:

> Operations and rules only have meaning relative to an anterior mathematical system: every representation that is effectively thought (as opposed to a pure lived experience [*distincte du pur vécu*]) is a mathematical system in so far as it is thought—that is to say, regulated organisation of the sensible (in virtue of the continuity between mathematical gestures, from the most elementary on up).[38]

The position at stake here is quite original. Mathematical thought and physical thought both unfold continuously as the 'systematic coordination of efficacious gestures' acting on and out of the sensible world, but do so in radically different senses.[39] This statement of the mathematics/physics differend is hence tightly bound up with what we could call Cavaillès's modal theory of abstraction, according to which mathematical thought is not separate from the world, but rather is a qualitatively distinct manner through which the world is lived. Mathematical experience is experience *insofar as* it is necessary; to think the world mathematically is to think the same world as that of practical life—i.e. the world as it is perceived in 'lived experience'—but to think it under the aspect of necessity. Thus Cavaillès develops a critique of an entire tradition in which abstraction is conceived as a progressive process of idealisation that would go from unconceptualised (particular) sensory experience to the development of more

37 *OC* 601.

38 *OC* 594.

39 The phrase *coordination systématique de gestes efficaces* is taken from 'Du collectif au pari' (1939), *OC* 650.

general 'mathematical' notions. On the contrary, mathematical notions are mathematical from the outset, and bear on sensory experience primarily negatively. In a sense, then, Cavaillès is sceptical of the very notion of abstraction, as clearly outlined in 'From the Collective to the Wager', written around the same time as the presentation to the Société:

> [G]eometry does not idealise anything. What is there to think in roundness if not already the circle? It is a naïve epistemology that would see mathematical objects as being born through abstraction from the real. In fact, there is an autonomous development of operations which, from the start, are mathematical: practical applications come from the success of the operations codified in the real. If there was abstraction, it would be, not from the real (where there is nothing to abstract) to the operations, but from the operations to the real (the disregarding, in the latter, of certain mathematical properties) […].[40]

On this view, the application of mathematics to physics thus does not come down to generalising empirical experience, but rather to using mathematics to suppress or delete the contingent in experience. Schematically, there is a definition of sorts at stake here: mathematics is the real viewed under the mode of necessity—recalling that, qua necessary, it must move—whereas physics is the real as it is coordinated in particular moments of action. There is thus no 'ontological' distinction between the 'world' as it is thought in mathematics or in physics, but there is an absolute disjunction between these two ways of thinking the same world. A distinctive mark of Cavaillès's programme is that this disjunction is to be thought methodologically through the difference in the respective forms of historicity of physical and mathematical theory, which leads to a quite radical assertion of the non-unity of science. This is what came to the forefront in a debate between Cavaillès and Gonseth, only a few months prior to the presentation to the Société, in which Cavaillès's asserted an 'essential polymorphy' to the concept of truth, as opposed to the unifying 'general idea of truth' proposed by Gonseth.[41]

40 *OC* 642, translation from Cavaillès 2014: 85–86.

41 Gonseth 1939: 41. Cavaillès's use of the term *polymorphisme essentiel* should be underlined, for it anticipates the stance against the possible homogeneity of scientific knowledge that will be one of the central theses of *LTS*, and which seems to have been arrived at in part by Cavaillès's sharpening his arguments against Gonseth. In brief, if in the debate with Gonseth, Cavaillès invokes an *external* polymorphy between the notions of truth proper to mathematics and physics, which implies that there is no common form of experience between the two disciplines (a judgement which *a fortiori* would then imply a correlative polymorphism of the notion of experience proper to other less 'formal' disciplines such as biology or psychology), then in *LTS* this position is radicalised into the assertion that *internal* polymorphy is an essential characteristic of the intelligible itself, i.e. of all forms of 'rational concatenation' (*LTS* 28). We will be concerned with this development at length in §III.3.

> I do not believe it is possible to unify mathematical and physical experience under the same concept. There is an autonomous mathematical knowledge that is sufficient unto itself, and that therefore requires an idea of truth that is unrelated to physical truth.[42]

As indicated, the difference that Cavaillès seeks to articulate here between mathematics and physics concerns their mode of passage, i.e. the relationship between anterior and posterior theory in the respective disciplines or modes of thought. At least as Cavaillès was considering this relation circa 1938–1939, the difference at stake is that while in physics new theories *replace* old theories, in mathematics new theories dialectically transform old theories while preserving their results in a new framework.

> Between the Euclidean geometry of Euclid and the Euclidean geometry of contemporary mathematicians there is not a passage comparable to that from classical mechanics to relativistic mechanics. In the latter case, there is the substitution of one theory for another in order to give an indication towards the world; in the former case, there is a deepening of reflection [*approfondissement de la réflexion*], in which precision is given to the axioms but the results are preserved.[43]

What we see emerging here is a formal definition of mathematical history in terms of a logic of the prior and the posterior that will predominate through *LTS*, and which will lead to Cavaillès developing a concept of the 'superimposition' of mathematical theories proper to his own Spinozist nominalism. But there it will be reconstituted through the effectuating/effectuated disjunction (as discussed in Chapter I with respect to *Transfinite and Continuum*),s and thus assume a certain 'ontological' force. We should also note that it is the logic of prior and posterior theory in mathematics sketched here that will be restated in reduced terms as a central problem of the 'doctrine of science' in the conclusion to *LTS*—that 'progress cannot be a mere increase in volume by juxtaposition, the prior subsisting with the new, but must be a perpetual revision of contents by way of deepening and erasure [*approfondissement et rature*]'—but that the formulation there is stated from a different theoretical ground.[44] We shall return to all of this later. What needs to be emphasised for now is only that Cavaillès's position as articulated in the debate with Gonseth—congruent with the definition of mathematical experience presented to the Société—involves such an extreme division between mathematics and physics that problems pertaining to their relation are hardly even questions as to the nature of knowledge.

42 Cavaillès to Gonseth in Gonseth 1939: 41.
43 Cavaillès to Gonseth in Gonseth 1939: 42.
44 *LTS* 78.

> There is an essential difference here which gives the mathematical experiment an advantage over the physical experiment: the latter is *negative*, and can only undermine a [prior] theory as a whole, without having anything to say concerning its content; in contrast, the former is *positive*, it is the very fulfilment of the [prior] theory. Whilst both experiences stem from the same intuitive sensory activity, they thus each represent the culmination of two diametrically opposed evolutions.
>
> The description of these evolutions, and the study of the relations between them, seem to me to belong more to general anthropology than to epistemology.[45]

If we take Cavaillès's reference to anthropology seriously, then on his account the relation between mathematics and physics poses an essential problem for philosophy, but it is not, as traditional epistemology would have it, the problem of the rational ordering of the world or of the unity of scientific practice, but that of how to think the fact that in the contingent history of human societies we find points of contact between two incommensurable regimes: that of the production of rational knowledge and that of the organization of practical activity.[46] Yet

45 Cavaillès to Gonseth in Gonseth 1939: 43.

46 It should be noted that Paul Cortois's recent and highly compelling article 'Jean Cavaillès's struggle with the problem of coordination' (2024) marshals almost exactly the same series of citations as I lay out here, in order to give a diametrically opposed interpretation of their role in the development of Cavaillès's ultimate position. For Cortois, far from having arrived at theoretical formulations that he will radicalise in *LTS*, Cavaillès's comments to Gonseth represent a moment of rhetorical overreach, forged in the heat of polemical discussion, which served to show that Cavaillès's 'mathematism' was an untenable stance in the epistemology of science. On this reading, Cavaillès momentarily arrived at a sort of high theoreticist error which would need to be corrected in *LTS*: 'Cavaillès, by following this path, could find himself confronted with a prospect that would present itself as a failure: the prospect of having to conclude that physics is impossible as a science […].' Cortois 2024: 92. For Cortois, then, having been 'lead to posit conditions of impossibility rather than conditions of possibility for theoretical physics' (ibid.) is a view that Cavaillès was forced to abandon under the pressure of its absurdity, and indeed this accounts for what Cortois sees as Cavaillès's renewed attempt to develop a positive 'theory' or 'doctrine' of science in his final essay, i.e. a position within the extended field of *Wissenschaftslehre* that could square his radical commitment to the autonomy of mathematics with the practice of mathematised physics, that is, with the application of mathematics to the world, which must involve the 'coordination' of mathematical theory (or theories) with the specificity of the experimental situation. There is no space here to systematically engage with Cortois's interpretation; I find his reading of Cavaillès's 'doctrine' concerning physics convincing whilst taking a very different stance on how this fits within the general theoretical edifice of *LTS*. I will limit myself to two further comments. First, it is clear that Cavaillès would never have ascribed to the 'impossibility' of theoretical physics as a scientific discipline in its own right. This would not only be factually absurd, but moreover inconsistent with Cavaillès's rather extensive interest in the problems of physical theory, a strain of his thinking that seems to have only intensified in the final period of his work, as evinced by 'From the Collective to the Wager' (1939), his lectures at the Sorbonne the subject of causality, and the important section of *LTS* on the difficulties posed for logicism by the 'fundamental problem of physical epistemology' (i.e. *LTS* 40–43), which as Cortois notes are usually 'passed over in silence' in existing commentaries on Cavaillès (Cortois 2024: 97). But to have a theory of mathematics *and* a theory of physics does not have to mean that one must absorb them under the same concept. Second, it seems to me that the theoretical problem at stake here does not so much concern the conditions of possibility for science, as of the discourse on science (that is:

Cavaillès's dismissive invocation of 'anthropology' here has a certain irony to it, insofar as his own definition of mathematical experience as a unified concatenation of gestures bearing on primitive practical operations with sensible objects is, itself, a kind of transcendental anthropology, according to which human experience—embedded in history—leads by a 'necessary concatenation' to results that lie outside of all experience. Thus as he states in his summary comments following the debate with the Société (primarily responding critically to remarks from Maurice Fréchet and Paul Lévy):

> The historical, contingent mathematician may stop, may be tired, but the demands of a problem impose the gesture that will resolve it.
>
> This, if you like, is what I pointed to, calling it the reality of knowledge: that which, from the very point of view of an anthropology or a philosophy of the human constitution, is the extraordinary miracle of human destiny; independently of life in the lived world, it presents problems that necessitate solutions and lead outside of that which is, by way of a necessary concatenation.[47]

There is no reason to believe that Cavaillès ever really abandoned this view. But the fraught question—which will not have the same status in 1942 as it did in 1938/1939—is the relation of this 'necessary concatenation' to the empirical or historical concatenation of human constitution and of material mathematical practice. It is here that questions around the theoretical status of the concept of the gesture return in full force. This is because it is the concept of the continuity of the chain of gestures that underwrites this whole picture, in which mathematics and physics are modally differentiated but nevertheless can equally be construed as material practices that unfold throughout the history of 'human activity'. It is relevant in this respect that within the framework of developing a general theory of mathematical experience as he presented it to the Société, Cavaillès presents the distinction between idealisation and themat-

'epistemology' as it would have been conceived in Cavaillès's context). On my reading, in his comments to Gonseth, Cavaillès is already on his way to the position he will assert more forcefully in *LTS,* which is that there is no general form of knowledge that could deserve the name of *a* 'theory of science' (in the singular), in the first order because the forms of historicity (i.e. of movement) proper to different scientific discipline have no common measure, and more fundamentally because mathematics, that classical paragon of theoretical 'generality', in fact *in principle* has no unified 'form'. Thus to do justice to the thought at work in mathematics as much as in the 'effective' historical sciences, philosophical rationalism required a nominalism that would break with the unifying assumptions behind Neo-Kantian epistemology, including those of his 'master' Brunschvicg. Science has no common aim, least of all 'to know the world', a programme which would 'represent the abandonment of complete creative autonomy.' (*LTS* 19) This is part and parcel with Cavaillès's implicit break with the concept of 'experience' (again, in the singular), such as we are tracing it here. But this is a claim about the foundations of the 'theory of knowledge', not about the efficacy of particular sciences.

47 OC 604.

isation as it had already been schematised in the conclusion to *MAF*—i.e. what will later become the distinction between the paradigm and the thematic in *LTS*—directly in terms of the concept of the gesture: *idealisation* occurs when 'an operation which was, in an accidental manner, limited to certain conditions that are extrinsic to its fulfilment, is freed from this extrinsic limitation'; in *thematisation*, by contrast:

> [G]estures that were carried out with respect to a [specific] model or domain of individuals can, in their turn, be considered as individuals on which the mathematician then works, taking them as a new domain. For example: the topology of topological transformations; we could find many other examples. This procedure allows mathematical reflections to be superimposed upon one another, and it also has the interest of showing us that there is an unbroken connection between the concrete activity of the mathematician from the first moments of their development—placing two symmetrical objects side-by-side, swapping them around—and the most abstract operations; for each time this connection is found in the fact that the system of objects under consideration is a system of operations which, themselves, are operations upon other operations which, ultimately, are founded upon concrete objects.[48]

What is asserted here is a univocal concept of the gesture (or of the operation). I pick up two objects and move them around—picking them up, moving them, these are both gestures—and in seeing certain properties that pertain in this shifting juxtaposition I form the concept of 'symmetry', and this is itself a gesture. Or to take another example, I place three objects in a line (a gesture) and then conceptualise matters such that the object on the left is the 'first object', the object in the middle is the 'second object', the object on the right is the 'third object'. I then extend the notion: if another object were placed to the right of the third, it would be the 'fourth object', and so on indefinitely, whence the idea of 'ordinal counting', i.e. the 'central intuition' of arithmetic. Alternatively I could (for a certain—limited—number of objects, in our case three) apprehend these objects all at once, and count them as 'three objects'; thematised, this gives us the concept of 'cardinal counting'. Then, at higher levels of abstraction we can begin to consider all of the intricate mathematical or metamathematical properties of these thematic concepts: ordinal counting can be characterised, if one likes, in terms of the Peano axioms, and so on, up to the development of a fully fledged mathematical framework such as set theory, or rather various competing 'set theories'. Each step in this process is a gesture, and as a chain of gestures on gestures the body of theory that we call mathematics is so constituted. Thus the gesture is mathematical from the beginning, but it is still a 'sensory' activity, i.e. 'for all this, we never leave the sensible world'.[49] The consistency of this view rests on the

48 *OC* 602.
49 *OC* 626.

modal theory of abstraction outlined above. It is at once intended as a description of the practice of the working mathematician at a given moment and as a way of theorising the progressive development of mathematics in the history of the human species.

The gesture as it is thought in these texts is thus a descriptive category intended to solder an abstract theorisation of the act of creative thought to an empirical characterisation of the mathematician's activity, in particular their work with sensory sign complexes. This was what was at stake in Cavaillès's earlier uses of the concept of the gesture in the course of his reading of Hilbertian metamathematics as 'experiments made with signs [*expérience sur les signes*]' in the conclusion to *MAF*. In this way, a theory of 'mathematical work' derived from a particular reading of Hilbert could function to create continuity across the different scales of 'human activity' in its most primitive form—'the child faced with their abacus is already a mathematician'—and the 'dialectical' creation of concepts as conceived through Cavaillès's reading of post-Dedekendian mathematics.[50] As we saw in Chapter I, an analogous role was played by the notion of the 'central intuition' throughout the doctoral works.

In sum, what we find in this second thesis from 'Mathematical Thought' is a very condensed expression of the same project at stake in the above-cited letter to Labérenne, and which was in a sense the orientating notion for the whole period of Cavaillès's doctoral studies and immediately thereafter: the elaboration of a quasi-anthropological account of mathematical work according to which the corporate body of mathematics would be understood as the necessary and progressive unfolding of 'gestures on gestures, …, on gestures on the primitive sensible world.'[51] This was the ontology underwriting Cavaillès's ambition to produce a descriptive epistemology of mathematical creativity that would bear a legible index both to the figure of the individual working mathematician—i.e. their 'experimental activity'—and to their relationship to (and transformation of) historically constituted mathematical material, which is in turn available to the 'epistemologist' as an archive (of mathematical writings). In the economy of this project, the theory of the gesture functioned as a kind of naturalisation of Dedekind's 'chain of operations', one that could give transcendental anthropological ground to this descriptive intent.[52] This enterprise was both recognisably Brunschvicgian in its theoretical lineage and could be seen as adjacent to a form of descriptive 'phenomenology' of mathematics. It was, in short, the project to develop a theory of mathematical experience, for which

50 *MAF* 172/*OC* 180.

51 *MAF* 178/*OC* 186.

52 See Introduction, B.2. for an outline of what Cavaillès is appropriating from Dedekind.

the gesture served as perhaps the central methodological concept. The division between idealising and thematising gestures was then advanced as one proposal to understand the structure of the recursively defined natural chain of operativity that Cavaillès at this juncture understood to be the essence of thought.

We should ask: is the unity of strata assumed here between the human activity of the mathematician and the dialectical history of mathematical concepts—moreover, a progressive unity in which the history of mathematical activity is one of *unification*—operative in the 'ontology' presented in *LTS*, i.e. once this theoretical elaboration is resumed from the standpoint of 'the intelligible' itself? For, in marked contrast to the doctoral works, the standpoint of both *Transfinite and Continuum* and *On Logic and the Theory of Science* is that of an ontological analysis of mathematical concatenation or the concatenation of rational contents. By the same token, in the assertion from *TC* that mathematics is to be understood as a 'concatenation of acts' we should understand the relevant concept of 'act' to be of quite different theoretical standing to Cavaillès's earlier invocations of the notion of 'act' in the sense of 'act of the mind'. It is notable in this regard that having advanced a tentative definition of mathematical experience as a 'system of gestures' around 1938–1939, Cavaillès seemingly—depending on where one dates the late fragment 'Mathématiques et formalisme'—never again uses either the terms 'mathematical experience' or 'gesture' in texts bound for publication.[53] What, then, to make of this evaporation of the concept of gesture from Cavaillès's later theoretical lexicon?

My argument is that Cavaillès in effect comes to replace the quasi-anthropological concept of the gesture with the elaboration of a theory of the act without a subject, or, to emphasise the tensions inherent to this notion, of *acts without activity*. This concept of the act is the correlate of Cavaillès's attempt to develop a theory of *time without temporality* as the proper concept of the internal movement of science. This is part of a shift whereby the identification of mathematics with 'experimental activity' is implicitly problematised in *LTS*, which is a central stake in the famous refusal of a (Brunschvicgian-Kantian) notion of (synthetic) 'activity' in the final lines of that work. All of this of course leaves the relevant concept of the 'act' completely underdetermined. It must be thought in relation to its proper dual, which is at stake in Cavaillès's third thesis as presented to the Société and which will equally come to the forefront in *LTS*—the problem of the object.

53 See Appendix I, footnote 19 for considerations on where to place 'Mathématiques et formalisme' (i.e. Cavaillès 1949) in the chain of Cavaillès's writings.

§II.1.iii. Third Thesis on Mathematical Experience

> The question of the existence of mathematical objects is the question of the existence of the objects of thought. What modern mathematics—and Skolem's paradox in particular—shows us is that the existence of an object is relative to the actualisation of a method. Thus 'object' = the reality of the act of knowing; the mathematical object does not exist 'in itself', but is the correlate of a human activity.[54]

The elaboration of this third thesis from 'Mathematical Thought' is the antechamber to the project of *LTS*. In effect, it announces a central problem of mathematical philosophy which Cavaillès will later find Husserlian phenomenology to have posed acutely—and to have floundered upon: that of the 'reciprocal independence' of act and of object.[55] In *LTS* this bears on the thesis that Cavaillès will advance as to, first, the *non-homogeneity* of mathematics (or concatenation) and then, by homology, the *non-univocity* of consciousness, which is central to the rejection in that work of any philosophy of transcendental constitution. In the presentation given to the Société, Cavaillès elaborates extensively on the first part of this argument—intrinsic heterogeneity of the object domain of mathematics—without asserting the inference to the correlative heterogeneity of constituting activity. Thus, in reading the 'Mathematical Thought' transcript alongside *LTS*, we find a central tension, which must be explained by any global interpretation of Cavaillès's work.

> – In 'Mathematical Thought', despite Cavaillès's recognition of the heterogeneity of the object domain of mathematics, and even (or moreover) of the mode of generation of mathematical objects, he still holds the thesis that mathematical objects are to be considered as 'the correlate of an activity'.[56] This activity is then qualified as specifically 'human activity'.[57]
>
> – In the final line of *LTS*, we find to the contrary a rejection of the idea that 'activity' is the motor of the necessarily progressive character of science.[58]

54 *OC* 603–604.
55 C.f. *LTS* 44–45, i.e. the transition from Section II to Section III of *LTS*.
56 *OC* 604.
57 *OC* 604.
58 *LTS* 78.

This latter criticism of the notion of activity, which has received less attention than Cavaillès's correlated polemic against 'the philosophy of consciousness', can initially seem frankly confusing, given that 'mathematical activity' is a central positive theoretical term in Cavaillès's earlier writings: *MAF* begins by outlining the task of 'situating mathematical activity [...] with respect to the other activities of consciousness', a task which in the economy of Cavaillès's doctoral theses would be equivalent to producing the sought-after definition of mathematical work, and, as discussed above, the late fragment 'Mathematics and Formalism' closes with an assertion as to the 'essence' of 'mathematical activity'.[59]

What, then, accounts for Cavaillès's coming to take the opposite view, ultimately opposing the notion of activity in such strident terms? Proper elaboration on this point requires getting into the exposition of the text of *LTS*, so here I will restrict myself to three comments concerning, respectively, the periodisation of Cavaillès's sources, the mathematical background to Cavaillès's comments on the non-homogeneity of mathematical objects, and the anthropological assumptions behind the notion of 'activity' at stake here.

First comment. With respect to the sources of this 'thesis', it is striking that Cavaillès's formulation that the mathematical object is 'the correlate of an activity' was seemingly developed prior to his returning to serious engagement with Husserl, given that it is surely one of the pronouncements that brings him closest to a kind of phenomenology of mathematics. It is only around 1941, in the context of his teaching at the Sorbonne, that Cavaillès appears to have started reading Husserl again—following an interest in the early 1930s—and in so doing resumed what he refers to in a letter to Brunschvicg as his 'old quarrel with transcendental logic'.[60] In other words, on my interpretation, Cavaillès's most 'Husserlian' moment came prior to his seriously rereading *Formal and Transcendental Logic*. My interpretation of the general development at stake here is that the major contribution of this engagement with Husserl was to clarify that Cavaillès's project would have to break with some of the assumptions that had seemed to make it adjacent to phenomenology.

This theoretical proximity went deeper than the language of 'correlation', 'activity', and 'experience'. It concerned a shared attempt to take what we could call a middle position on the 'reality' of the mathematical object, i.e. that the objects with which mathematics was concerned were not an 'in itself' hovering outside of temporal experience (contra 'Platonism' in the sense of 'Mathematical Platonism'), nor were they to be found in empirical experience (a

59 *MAF* 5/*OC* 13; *OC* 664.

60 Ferrières 2003: 182. For a translation of this letter, see Appendix I.

mathematical object is not a 'concrete' object of reference like a chair or a tree), but that they were nonetheless *objective* and non-psychological. The reality of the mathematical object was to be located in the objectivity of its mode of generation, not through the transcendence of its 'properties'. For Cavaillès, the way to the articulation of such a middle position was through exacting study of the relationship between different formal systems and the 'objects' with which they were concerned. Thus, as we saw in Chapter I, a major part of Cavaillès's work from 1928 to around 1940 was orientated by the attempt to answer one question, 'How do new concepts arise in mathematics?', by investigating another, more technical one: 'What is the relation between the axiomatic (or formal) determination of concepts and their "intuitive" content?' Implicit here is an analogy between the relation 'thought—object of thought', and the relation 'mathematical theory—object of that theory', such that conclusions with respect to the latter allow us to state things about the former.

Now, this analogy is the assumption behind Cavaillès's general assertion to the Société that 'the notion of the existence of mathematical objects interests us [...] because it poses the problem of the existence of the objects of thought.'[61] But it is also an analogy that Cavaillès will later come to locate in Husserl, in the restricted form 'set of axioms—object generated by the axioms'. He takes it to ground both a shared theoretical horizon and set of problems concerning the definition of objects common to the project of formal phenomenology as elaborated in Husserl's *Formal and Transcendental Logic* and the 'Hilbertian' perspective on axiomatisation (that is, principally, the standpoint of the 'first Hilbert' as discussed in Chapter I). This will be the condition of the argument in *LTS*—often implicit, but theoretically central—that results stemming from modern mathematical logic (and principally from post-Hilbertian proof theory) are ruinous to phenomenology just as much (and for the same reasons) as they are to the 'naïve' standpoint on axiomatisation. Whence a central equation: the non-categoricity of 'real mathematics' = the non-univocity of consciousness.[62] In other words, the position that

61 *OC* 604.

62 Here a word of terminological caution with respect to the following exposition: when Cavaillès speaks, as he often does, of the problem of determining mathematical objects *de façon univoque* (*OC* 223), and equivalent formulations, he is referring to what would in contemporary mathematical English more often be called the difficulties of giving a 'unique' or 'one-to-one' characterisation of objects; it is in this sense that there is a natural connection between the 'failure' of categoricity for axiomatic systems and the failure of 'uniqueness' for the definition of their objects, as was already touched on in §I.2.i. with respect to Von Neumann's comments on the threats to the 'univocality [*Eindeutigkeit*] (categoricity) of set theory' in Von Neumann 1925, to which Cavaillès attached much importance. One cannot—indeed should not—assume that every use of this term by Cavaillès is intended to invoke 'univocity' in the philosophical sense, and this could be seen to weigh in favour of translating his technical uses of the term by 'uniqueness', so as to not overcode mathem-

Cavaillès will arrive at is that studying the history of mathematics reveals to us the internal heterogeneity of thought, indeed a *definition* of 'thought' in terms of this internal heterogeneity, or what I shall later term 'variegation'. But this is a position proper to *LTS*, not the project for a philosophy of mathematical experience. As indicated, at the moment we are describing here, Cavaillès still held something like the 'phenomenological' thesis that underwriting the diversity of theoretical objects ('idealities', in phenomenological jargon) was an essentially unitary manner in which said objects were constituted. This was the 'mathematical activity' which, at this juncture, Cavaillès held to be his essential 'object of analysis'.[63] The path to such an analysis was through consideration of the essential dependence of the mathematical object on the system of axioms or 'operations' that generated it.

Second comment. In the context of the exposition to the Société, all of these themes are filtered through Cavaillès's comments on 'Skolem's paradox'. Here is how Cavaillès introduces the reflections that I summarise as his 'thesis' above, i.e. his third 'positive' conclusion concerning his research at that time:

> The existence of objects is correlated with the actualisation of a method and, as such, is non-categorical, but is always dependent upon the fundamental experience of an effective thought. That the possibility of exhaustive description (or *ex nihilo* generation) [of objects] by the axioms is an illusion, foiled by Skolem's paradox, can be explained by the necessary non-coincidence [*décalage*] between exposition and authentic thought. […] They [i.e. mathematical objects] are thus neither an in-itself, nor are they within the world of lived experience—they are the very reality of the act of knowing.[64]

We of course recognise here the 'middle position' on the existence of mathematical objects just indicated. Bypassing the technical details, it is significant that Cavaillès accorded a central role to Skolem's results in order to ground this position. What Cavaillès is invoking here is what is now usually referred to as the Löwenheim-Skolem theorem, i.e. that, to restrict ourselves to Cavaillès's statement of the theorem, '[i]f we have a model which we suppose to satisfy a system of axioms, it is always possible to construct a countable model satisfying this same system of axioms.'[65] It is perhaps not immediately obvious why this constitutes a 'paradox'. It does so

atical referents by philosophical jargon. However, in the other direction, by the time of *LTS*, Cavaillès's criticisms of the *univocité d'application* (*LTS* 78) of consciousness to its objects in the phenomenological analysis are directly related to the mathematical scope of the terminology. Thus I have chosen 'univocal' throughout in order to preserve what I take to be the intended resonance of the mathematical content in Cavaillès's philosophical uses of the word.

63 To invoke once more the final line of 'Mathematics and Formalism', *OC* 664.

64 *OC* 595.

65 *OC* 603.

because it produces the surprising situation that a system of axioms that explicitly characterises an uncountably infinite object domain can be fully and adequately modelled by a countable system of objects. This is immediately problematic for the view that what an axiom system *does* is to characterise all of the possible systems of 'objects' that fall under it isomorphically, i.e. the definition of the well-definedness of an axiom system in terms of its *categoricity*.

I will not enter into contestation about the interpretation of this 'paradox', as we are only interested here in what Cavaillès extracted from it. It is useful to consider the project negatively: what would the absence of Skolem's result allow us to assert? In other words, what is disbarred, on Cavaillès's view, by Löwenheim-Skolem? Fundamentally, much of this turns on what we understand by an *axiomatisation* of a theory, i.e., what Cavaillès characterises as a conception of axiomatisation in terms of the *ex nihilo* generation of objects from axioms. What is meant by this?

At base, what is at stake here is what is sometimes called an 'algebraic' (or broadly 'model-theoretic') conception of axioms, along the lines of that espoused by the 'first Hilbert' in the *Grundlagen der Geometrie*, according to which the content of the concepts (or 'objects') characterised by an axiomatic theory are fully and adequately determined by axioms. To take the example often attributed to Hilbert: in a properly axiomatised theory, the content of the concepts 'plane', 'line' and 'point' will be so fully determined by the axioms that if we replace those words with arbitrary others ('exercise bike', 'podcast', 'kombucha', etc.) we don't lose any of that content, because all that 'content' *is* is the system of relations determined by the axioms. In other words, on this view there is no 'pre-axiomatic' (intuitively given) object which the axioms attempt to *approximate*; a 'circle' just is a particular set of axiomatically determined relations (at least for the purposes, that is, of the theory in question). This view, according to which the objects of a theory are completely determined by the axioms is what Cavaillès refers to as the view that the objects are 'generated' *ex nihilo* by the axioms.

This should make the counter-case clear. In brief, if Skolem's result *did not* hold, then it would be possible to consider 'the object'—or rather the field of objects determined by a set of axioms—as being determined univocally. Thus one could proceed from a 'well-defined' set of axioms which posit an (uncountably) infinite domain to a univocal (that is to say categorical) determination of the correlative object field. To take the example of set theory, this would mean that having set up something like Zermelo's axioms would be to concretely define (or uniformly generate) a single object domain, the universe of sets, which said axioms adequately

characterise. There would thus be a pleasing transition from one univocity (the unique axiomatically-defined foundation of the 'theory of sets') to another (all of the sets defined by these axioms). This is the view that Cavaillès describes in both *MAF* and 'Mathematics and Formalism' as 'radical formalism', which he viewed as having been decisively undermined. And it is Skolem and associated generalisations concerning categoricity results that Cavaillès understood (largely following Von Neumann) to have been central here. It is this situation on the foundational front that Cavaillès is summarising for the Société. And here we should underline the very general terms in which Cavaillès understood Skolem's 'paradox'.

> This paradox, upon which Skolem himself and many others (this summer, Gentzen) have long reflected, amounts to saying that it turns out to be impossible to exhaustively characterise a model satisfying a system of axioms. If we suppose the axioms—that is to say, the enumerations of the properties we need for the objects—to be posited, we cannot expect these axioms at the same time to generate the objects. Rather, we are obliged to suppose the existence of a field of objects, and then, from the properties of these objects in this field, we can deduce other properties. What we cannot say is that our field of objects can be characterised in a *uniform* fashion by our system of axioms.[66]

Thus, Cavaillès understood Skolem's 'paradox' to imply two essential things: first, the *heterogeneity* of the object domain of mathematics, and second what he will call, also taking up language directly from Skolem, its internal *relativity*. Now, especially once we take into account that Cavaillès will understand Husserl's 'formal logic' to embody the 'naive' perspective on axiomatisation that Skolem disbars, it is very plausibly reflection on *this* point—on the 'non-uniformity', qua 'non-categoricity' of the 'generation' of objects from axioms—that should be read as motivating Cavaillès's rejection of the unity of constituting activity in *LTS.* Or so it seems. Yet what is especially curious, on such a reading, is that in the 'Mathematical Thought' transcript, Cavaillès seems for a moment to have taken the opposite stance. For, elaborating on the consequence of this interpretation of Skolem's results at the close of his exposition, Cavaillès states:

> What is it for an object to exist? Here we find ourselves in the presence of the fact that the very type of certain, rigorous knowledge—which is precisely what mathematical knowledge is—prevents us from positing objects as existing independently of the system that is fulfilled by these objects, or even as existing independently of a necessary concatenation from the very beginning of human activity.

66 OC 603–604. Emphasis mine. The article from 'this summer' to which Cavaillès refers is Gentzen 1938. This is quite possibly where Cavaillès learned of Church's results.

> Which means that we can neither posit the objects [as existing] in-themselves, nor exactly say: here is the world—a world that we describe. In each case, we are obliged to say: these are the correlates of a certain activity. All that we think in themselves are the rules of mathematical reasoning that are demanded by the problems that pose themselves, and that there is even an overflowing, a demand for excess made by unresolved problems, which obliges us to posit yet other objects, or to transform the definition of objects that were posited at the start.[67]

And here I believe we are really dealing with a point of real theoretical tension: almost in the same voice, Cavaillès has asserted that the object must be in excess of any (given) axiomatisation, *and* that it is the correlate of an activity, not a reality in itself.[68] The only conditions under which this position was legible was to conceive of 'activity'—for which the model was, of course, synthesis in the Kantian sense—as the unified operational pole of the entire field of mathematical objects, i.e. as a kind of general form attending the diversity of object constitutions in mathematics. With the assertion that such a conception leads us to consider the 'existence of the objects of thought', Cavaillès also implies that, from a broader philosophical stand-

67 *OC* 604.

68 It is worth emphasising the continuity of this 'problem' throughout Cavaillès's work. In *MAF*, Cavaillès had attacked the 'prejudice' that underwrote 'non-critical ontology', namely that 'the object should be defined prior to the operation, whereas in fact it is inseparable from it.' (*MAF* 178/*OC* 186). We recognise here, of course, the central methodological notion of 'object-operation duality' that Cavaillès wished to extract from the (post-) Dedekindian perspective on mathematics, and which was very much in line with a 'formalist' approach to mathematics (whence Cavaillès's tentative ascription of his own position, at least at some point, as 'modified formalism'). It is very plausibly having taken such a stance that the reality of the object was inseperable from that of the operation which served to make Skolem so impactful for Cavaillès, because on his very general understanding of the Löwenheim-Skolem theorem, what this theorem showed was that *any* system of operations of sufficient complexity (i.e. invoking uncountable infinities) would not be able to 'univocally' (that is, categorically) define its objects. Whence a curious situation: the mathematical 'object' had no reality except as the dual (or the 'correlate') of an operation (or system of operations), and yet it could not be totally defined by any particular axiom system (it exhibited an intrinsic 'excess' or 'transcendence' to formalisation, if we allow the terms). Cavaillès's interpretation was to make this curious situation *constitutive* of the intrinsically 'transtheoretical' character of the mathematical object: it was not an in-itself, but nor was it defined adequately by any given theory; rather its 'existence' was as the 'fulcrum' (to use Sinaceur's terminology) for the different historically located 'mathematical concatenations' that bore upon it. To take an example: the inter-theoretical 'reality' of a concept such as 'function' would be its *commonality* to intra-theoretical concatenations as diverse as those at work in Eulerian analysis, Fourier series, Dedekind's set-theoretic notion of 'mapping', Borel and Lebesgue's discourse on 'real functions', and so forth. It is *this* kind of 'ontology' of the object that Cavaillès will unfold in *LTS*, as we will describe under the banner of his 'rationalist nominalism'. We can say Cavaillès's position as expressed in the 'Mathematical Thought' transcript was not contradictory, but it was unstable. He seemed to wish to accord a greater 'reality' to the object than he had in his doctoral theses, which required a confrontation with 'ontology' that he had previously wished to avoid. A momentary settlement, which was not to hold, was to locate the 'reality' of the object in its role as the locus of the 'activity' of different historical mathematicians. Here I only wish to underline that an interpretation of Skolem in terms of the necessary relativity of the object to any particular 'system of axioms' was central to his posing of this problem.

point, this scheme can be applied to the general problem of object constitution. But that such a thesis is viable and desirable will be precisely the assumption underlying the 'philosophy of consciousness' that Cavaillès will inveigh against in *LTS*.

Third comment. Worse: the activity that the mathematical object has been supposed to be a correlate of is, specifically, *human* activity. As already discussed, for the Cavaillès of the 'Mathematical Thought' transcript, the 'necessary concatenation' that mathematical history is was identified with the unfolding of 'human activity' in the sensible world. Indeed, this was a primary theoretical function of the whole 'mathematical experience' concept: in the 'modal' terms laid out above, mathematical experience was the name of human activity viewed under the aspect of necessary (as opposed to contingent, historical) concatenation. *Enchaînement* and *activité* were thus seen as two faces of the same experience. And not just any 'experience'. The experience at stake here was not the abstract field of experience in general, but the historically situated experience of a particular collective entity: the community of human mathematicians.

In this respect, the project for a philosophy of mathematical experience owed a profound debt to Brunschvicg, for the construction in *Les Étapes de la philosophie mathématique* is precisely a transcendental anthropology of this kind, in which observations drawn from ethnography, pedagogy, psychology and history are all marshalled in favour of a developmental picture of 'intellectual activity' as the unifying *telos* of the human species.[69] Whence a crucial turning point in that work, in which, having defended the centrality of 'the experience of history' to what he calls 'the problem of mathematical philosophy', Brunschvicg inserts a discourse on the primitive operation of exchange as one of the transcendental grounds for the development of mathematical abstraction, insofar as it allows science to grasp a 'true relation': in this case, the primitive operation of equivalence. This precedes a passage that one cannot but read as intertext for Cavaillès's closing polemic in *LTS.*

> This leads us to a philosophical conception capable, to borrow an expression from Harald Höffding, of preserving the essential value of both *truth* and *activity*. *The philosophy of the concept* and *the philosophy of intuition* were in agreement when they assumed the incompatibility of these two values, even if this meant that they gave two contradictory interpretations of this incompatibility, and neutralised each other in an endless controversy, one which only served to highlight their common inadequacy. But this circle is

69 Broadly speaking, one can say the same of Brunschvicg's subsequent major work on physics, *L'expérience humaine et la causalité physique* (1921). Cavaillès would surely have been aware of the echo of this work in proposing a book bearing the title *L'expérience mathématique*, and presumably in part intended this as an explicit tribute to Brunschvicg.

> broken if we have succeeded in proving that the initial operation of science is the determination of a true relation.[70]

The 'philosophy of the concept' invoked here is Brunschvicg's old bogeymen of Aristotelian 'conceptualism', which he had critiqued in these terms as far back as *La modalité du jugement*.[71] This is evidently not what Cavaillès means to affirm by using the term in the conclusion to *LTS*. But this hardly undermines the resonance between the two passages, nor of Cavaillès's having chosen *this* term to oppose Brunshcvicg with. In Brunschvicg's discourse, it was in order to resolve the contradiction between 'truth' and 'activity' (that is, in order to think both at once) that philosophy must commence from the specificity of the mathematical operation as a human activity that already contained its reflective element. In the mathematical operation, truth and activity were one. Whence a methodological assertion: 'The primary material for philosophical reflection is men that act.'[72] This sounds a lot like pragmatism, as Brunshcvicg was aware. Yet, while travelling part way along that road, Brunshcvicg demurred, for:

> Contemporary pragmatism, less enamoured of real action than of oratorical generalities, makes the error of stopping where real analysis ought to begin; it has failed to recognise the specific distinction exhibited by *the different forms of human activity,* and to reserve a special place for what makes humanity an exceptional species amongst the species of animal.[73]

A special place, that is, for mathematical—which is to say operative—activity. Placing emphasis on the specificity of the mathematical operation was thus essential to preserve another truth, indeed a definition: 'Man' is, essentially, the mathematising animal.

Although Cavaillès never directly repeated this story, his work up until around 1940 remained profoundly indebted to its anthropological horizon. The assumption underlying the project for a 'philosophy of mathematical experience' was that one could make the embodied field of human activity in the 'sensible world' occupy the 'operative' pole in a theory of object constitution. In other words, what Cavaillès shared in this period with Brunschvicgian epistemology was the idea of human activity as the unified constituting field from which all object constitutions sprang, even if their 'mode of generation' had to be heterogeneous (qua Skolem). Activity so conceived was the 'common soil of all rational activities', which mathem-

70 Brunschvicg 1912: 470. Emphasis mine.
71 Brunschvicg 1897.
72 Brunschvicg 1912: 470.
73 Brunschvicg 1912: 470. Emphasis mine.

atical philosophy had to make its object.[74] Indeed, there is the shape of an argument here, even if hard to perceive: according to Cavaillès's discourse at the time, the very heterogeneity of the object domain of mathematics—as had been demonstrated by the various results on the limitations of formalisms emerging from then-contemporary mathematical logic—demanded that mathematics be conceived as the product of a unitary activity, in order that the unity of mathematics be preserved. This was the essentially Brunshcvicgian aspect of the project of mathematical experience that his final theoretical struggle would allow him to break from.

In sum, with his later rejection of 'activity' and his correlated shift away from the concept of 'experience', Cavaillès is taking leave precisely of what could be seen as a cluster of commitments common to both Brunshvicg and Husserl, which (with modifications to be sure) Cavaillès himself had held at the time of developing his project for a concept of mathematical experience in the late 1930s. The common name for this set of assumptions was: the philosophy of consciousness, i.e. a philosophy of constituting experience that had as its necessary double a transcendental anthropology. Cavaillès's ambition to study mathematical experience could appear as a kind of restriction of this general transcendental anthropological ambition to the empirical community of *mathematicians*, through the important medium of the 'case study' (set theory from the 19th century through to the 1930s, and so on). In *LTS*, a profoundly different discursive operation will be at stake. Cavaillès will ask: can one think mathematical becoming no longer from the side of the common community of mathematical *agents*, but rather from the side of the 'concatenation itself', i.e. from the side of the *acts* that compose it? Cavaillès will thus move towards the view that while in some sense the mathematical object is always the 'correlate' of an act, this act is not to be theorised as the product of a genetic activity, least of all the activity of 'mind' or consciousness. Thus, if *LTS* can be said to retain from the project for a philosophy of mathematical experience something of the thesis that 'the activity of mathematicians is an experimental activity', this thesis is no longer, strange as it may seem, asserted in terms of an analysis of *activity*, nor of *experience*. *LTS* does not analyse the standpoint of 'the mathematician'; it analyses the standpoint of the 'act' itself.

Assertions made by Cavaillès in the late 1930s such as those we have discussed here concerning the mathematical object as a 'correlate of activity' or the 'reality of the act of knowing' have not ceased to contribute to a misunderstanding of this project. For example, in their recent introduction to their new translation of the 'Mathematical Thought' transcript (which they entitle 'Mathematical Thinking'), Gabriella Crocco and Frédéric Jaëck assert that, in con-

74 *MAF* 21/*OC* 29.

trast to the 'stronger, objective sense' given by Lautman to the 'existence' of mathematical objects:

> For Cavaillès, the existence of mathematical objects can only be asserted in a weak sense: they are merely the correlate, in actual and sensible mathematical experience, of the structures that the mind generates.[75]

This interpretation has a polemical bearing on contemporary developments in the philosophy of mathematics, which is to assimilate Cavaillès's notion of 'experience' as a precursor of sorts to the emerging paradigm of a 'philosophy of mathematical practice', with Crocco and Jaëck going so far as to directly identify 'experience' and 'practice'.[76] I do not want to enter here into the question of the relation of either Cavaillès's or Lautman's stances to the contemporary methodological debate around mathematical 'practices', to which indeed it does seem that they could make a rather fundamental contribution. What I do want to point out is that nothing could be farther from the position that Cavaillès will take in *LTS* than to hold that the mathematical object is in the last instance only 'generated' by 'the mind' (albeit as the 'correlate of a structure'). And herein lies the rub. In the transcript of his presentation to the Société, and in particular in the ensuing discussion, Cavaillès is evidently trying to articulate a rather unstable position. On the one hand, a 'Platonic' assertion of an 'in itself' of the mathematical object was to be stridently rejected 'for the quite simple reason that it is totally useless'; useless, that is, with respect to explaining effective mathematical work and its development.[77] On the other, he was already resolutely opposed to a psychological explanation of mathematical creation. In the attempt to wind his way between these two errors, a proto-phenomenological position that would understand mathematical entities to be the 'objective correlates' of a constituting activity had surely seemed an appealing *detente*, one that would make the project for a philosophy of mathematical experience legible within the broad field of other post- and neo-Kantian critiques of 'psychologism', including Brunschvicg's. It would take actually returning to work out Husserl's position to rid him of this illusion.

75 Cavaillès & Lautman 2024: 16. Crocco and Jaëck's translation was published too late in the process of revisions of this thesis to integrate it into this chapter. Here, Crocco and Jaëck more or less repeat Lautman's own initial response to Cavaillès's presentation. 'It seems to me that in what he calls mathematical experience, M. Cavaillès attributes a considerable role to an activity of the mind, which determines, in time, the object of its experience. Thus, according to him, there are no general characteristics that are constitutive of mathematical reality; on the contrary, this reality is affirmed at every moment of the history of mathematics, whereas Platonism is identified with a theory of the "in-itself" existence of mathematics.' *OC* 605.

76 Cavaillès & Lautman 2024: 14.

77 *OC* 603.

§II.2. The Traversal of Problems in the 1941–1942 Sorbonne Lectures on Logic

We now turn to Cavaillès's lectures at the Sorbonne from early 1941 through to summer 1942. It appears that Cavaillès taught four courses in this period: two on logic, the first in April–May 1941 and the second running from October 1941 to April 1942; one on physics and the problem of causality, from April to June 1941; and one on the problem of space and time, from October 1941 to May 1942. I understand Cavaillès's teaching in the logic courses to have been a first exposition of the considerations that would eventually produce the manuscript of *LTS* (see some considerations on this point in Appendix I). As such, despite the often elliptical nature of the record we have of this teaching—these are student notes, and must be treated with the appropriate hermeneutical caution—it is essential to an understanding of the intervention that Cavaillès intended to make with his 'treatise on logic'.[78] In terms of the periodisa-

78 Some words are necessary concerning the status of the texts I will now discuss. As mentioned above, Cavaillès's lectures on logic were attended by a number of future luminaries of the French philosophical scene (see Appendix I). There are, or rather were, two known student transcripts of Cavaillès's first course on logic, respectively made by Geneviève Rodis-Lewis and Marie-Louise Gouhier-Dufour. The former set of notes were handed to Santiago Ramírez by Rodis-Lewis whilst Ramírez was completing his doctoral thesis on Cavaillès. We do not know exactly what these notes covered, but from the single (and very hard to decipher) page photocopied in an appendix to Ramírez's thesis (Ramírez 1989: 671) it is clear they concerned the first logic course, at least in part. From Ramírez's description, we can moreover assume they were quite substantial: 'Mme. Rodis-Lewis's notes consist of two large notebooks. One of them has come unbound and consists of thirteen pages of small handwriting full of abbreviations. The second notebook has been properly preserved. It consists of 66 pages and is entitled "*cours de mardi*"' (Ramírez 1989: 350). Unfortunately, as is confirmed by a letter of 10 July 1984 addressed by Rodis-Lewis to Ramírez, she sent Ramírez « *les originaux* » on the expectation of their eventual return (letter from the private papers of Ramírez). It seems that in the event Ramírez brought the notebooks back to Mexico with him, and he stated to Carlos Alvarez, with whom he collaborated on the Spanish translation of *MAF* (i.e. Cavaillès 1992), that he intended to publish them (personal communication with Alvarez). However, Ramírez passed away without doing so, and the just mentioned letter from Rodis-Lewis was all that Alvarez was able to find in Ramírez's remaining papers, upon investigation in October of 2024 into the boxes pertaining to his work on Cavaillès in the attic of Ramírez's daughter in Mexico City. The sad conclusion must be that most probably, appearing to be a random collection of scribblings, and not even in Ramírez's hand, they were discarded by the family on his death. As a consequence, the only surviving course notes are those produced by Gouhier-Dufour, a copy of which is found at CAPHÈS at the ENS (GC.32.3). These cover only the first lecture of Cavaillès's logic course, and then the entirety of the course on physics. In the late 1980s, Paul Cortois worked with Gouhier-Dufour to produce a transcript of these notes and clarify certain points in the handwriting. A full publication of this revised transcript is in preparation, but at present a first typewritten version of the surviving logic lecture can be found at https://cavailles.hypotheses.org/files/2024/11/Cavailles-Cours_Logique-notes_de_Marie-Louise_Dufour-Gouhier.pdf. In what follows I will cite the pagination from this transcript as Cavaillès et al. 1941. The genre of this text poses a double difficulty. First, being student notes, one has to take them more as an indication of what Cavaillès was covering in the course than as a verbatim transcript. Second, they are syntactically elliptical, which often requires a higher degree of interpretation on the translator's part. As such, I also include all passages discussed here in the French, minus my interpolations. Another quite sub-

tion I have given, the notes are one of three documents (along with *TC* and the published version of *LTS* put together by Canguilhem and Ehresmann) that we possess of Cavaillès's mature theoretical project, as defined by the transition from an epistemology of 'mathematical experience' to a theory of 'concatenation' which was to be articulated from a new methodological standpoint, that of the project for a 'philosophy of the concept'. And, indeed, it is in the Sorbonne transcript specifically that we find what I take to be the statement of this transition, in Cavaillès's reorientation of the problems of mathematical philosophy around an account of what he will call the 'traversal of problems' or the 'concatenation of systems', i.e. what we propose to name 'inter-theoretic concatenation'. Given the nature of the text we are dealing with here, I only draw attention to some key points that can be established with relative certainty.

1. Central to this reframing of Cavaillès's project was a renewed attempt to pose the problem of the nature of 'logic' at a certain degree of generality, encompassing both 'philosophical' and 'mathematical' logic. What is perhaps most striking here is the highly synthetic conception of 'logic' assumed here; Cavaillès returns to problems posed by Descartes, Kant, Bolzano and Husserl with the idea that they must be rethought *after* modern mathematical logic. There is no strict division of 'disciplines' here; rather a direct line is drawn from the problems of 'method' in Cartesianism, to the relationship that consciousness bears to its *a priori* structure according to Kant, through the whole tradition of post-Kantian *Wissenschaftslehre*, and on to the Husserlian project of transcendental logic, all in light of and in direct contact with the most recent technical results from post-Hilbertian proof theory. Thus we see something like what Brunshcvicg wished for in the project of 'mathematical philosophy', but Cavaillès's approach involves a similar 'immanence' to mathematical *logic*, which was totally alien to Brunschvicg. There is a systematic ambition here that is new to Cavaillès's work, but what remains from the period of the doctoral studies is the conditioning of philosophical reflection by effective mathematical work in its most contemporary form, with mathematical logic being understood as a proper part of this. Both in terms of the references invoked and the broad narration of their trajectory, it is beyond doubt that what we have here is a first version of the analysis that will be recapitulated in *LTS*.

stantial albeit telegraphic set of notes by Gilles-Gaston Granger has just recently been unearthed, which allows the above breakdown of the four courses which Cavaillès taught from 1941–1942. As I only discovered the existence of these notes in March 2025, I have not been able to take proper account of them in the present exposition. I thank Ties van Gemert and Marjolein Holvoet for following a lead and transcribing some of their content.

2. How, then, is 'the problem of logic' posed? In the conception of 'logic' implicit in the Sorbonne lectures, it is the historical becoming of thought that is at stake, as a common matter (if one allows the term) for 'philosophical' and 'mathematical' reflection. And what we can say about this history is that with respect to the question of *method*, it has been riven by a fundamental opposition between the *absolute* and the *relative*. Thus:

> [H]istorically [speaking]: philosophy oscillates between the positing of an absolute and the necessity of an authentic development.[79]

In very broad terms, the problem that Cavaillès seems to be trying to pose in the lectures is the following: given that 'thought' (or science), for which mathematics is the essential case, has a history, philosophy is faced with the seemingly insuperable difficulty of how to at once account for the absolute validity of knowledge and the fact of its development. Logic must not oppose the absoluteness of knowledge to development; it must think the absolute *in* this development, which is to say in the development of effectively realised mathematics. In other words, our 'problem' is how we are to think the movement of thought as something necessary. Thought must have some rationally graspable structure, but any theory of this structure must be able to account for its motion, or else it would betray thought. 'Logic' is, in a sense, the discourse on this problem, which is to say that the object of study for logic is 'the problem of the possibility of the structure of thought', so long as we understand that this structure must also account for the fact of the historical development of thought (by which we here, of course, are to understand 'mathematical thought').[80]

3. It seems that we have arrived, indeed, at a rather Kantian problem: 'logic' as the discourse on the conditions of possibility of knowledge, insofar as this problem is thought formally, that is, in terms of 'the form of thought, of contents, of structures'.[81] But Kant is presented as only one, insufficient, response within the field of this problem. For as Cavaillès outlines matters here, the Kantian solution leaves an essential question unresolved: is 'form' something that is governing *prior* to the development of thought, or is it something *immanent* to the development of thought? This question will be reposed in opening section of *LTS*, where it serves as the prism through which Cavaillès will split the options for the 'theory of science' after

79 Cavaillès et al. 1941: 2. « Historiquement : la philosophie oscille entre position d'absolu et nécessité d'un développement authentique. »

80 Cavaillès et al. 1941: 11. « Mais problème de la possibilité de structure de pensée (c'est à dire de la logique). »

81 Cavaillès et al. 1941: 11. « Logique: classiquement: Forme de la pensée, des contenus, des structures »

Kant. In the lecture we possess from spring 1941, it is already posed in more or less these terms:

> Either: [1] knowledge can be prescribed *a priori*; in which case logic = organon;
> Or [2] knowledge is a development which makes itself at each moment; in which case logic is contemporaneous [with knowledge] (and thus we do not know what the form demanded by the content [=] *x* will be).[82]

In a critical manner we shall later discuss at length, Cavaillès is already on the way to aligning himself with the second of these options. Methodologically, the problem at stake here is one of autonomy: how not *predetermine* the analysis of science? In any case, the dilemma has already been stated.

4. As we have stated, the general picture that Cavaillès draws here is exactly what will be repeated in *LTS,* both in terms of the argumentative throughline and the litany of names at stake: Leibniz, Kant, Bolzano, Husserl… But there is one crucial difference in the historical presentation: in this introductory Sorbonne lecture, a great emphasis is placed on Descartes and the continuity of post-Kantian debates about logic with the classical problem of 'method', whereas in *LTS* this trajectory is mostly assumed, even if it is very present on the level of vocabulary. This, I believe, helps to explain something curious about the *genre* of *LTS*. What does it mean to state that *LTS* is a 'treatise on logic'? For what it is surely not, in any sense, is a textbook, nor an explicit survey of the state-of-the-art of mathematical logic (though we can expect some more indications of this type to have been present in the introduction had planned to include with the text). What the Sorbonne lectures suggest is that *LTS* can very much be read in the long tradition of rationalist 'treatises on method' exemplified by Descartes's *Regulae* and Spinoza's *Emendatione* (and arguably running back to Proclus's 'Prologue' to the *Commentary on the First Book of Euclid's Elements*).[83] To be sure, it resumes these problematics *after* modern (post-Hilbertian) mathematical logic. But it is not primarily a positive work on the 'logic of science' in the vein of Bolzano or Carnap, and is fundamentally orientated against elaborating a 'formal logic' of foundations such as Russell and Whitehead's *Principia*. It is of course reasonable to ask if Husserl's *Formal and Transcendental Logic* should be placed in the same line. The reference to Descartes is moreover essential for two pieces of

82 Cavaillès et al. 1941: 12. « – Ou bien connaissance peut être prescrite a priori = alors logique = organon – ou bien connaissance est un développement qui se fait à tout instant = alors logique est contemporaine (nous ne savons [pas] ce qui sera la forme exigée par contenu x) »

83 See Maronne & Rabouin 2022.

vocabulary: *enchaînement*, of course, but also the following passage in which Cavaillès addresses the problem that the 'development of knowledge' posed for Cartesianism:

> 2nd difficulty [of Cartesianism]: [the] absolute character of the development of knowledge. This is resolved for Descartes by [recourse to the notion of] order: necessary passage from simple intuitions to complicated systems of thought. The validity of what results is guaranteed by the necessity of the development [of knowledge]: it is viewed *uno intuito*, so there must be an internal affinity between its elements [i.e. the elements of knowledge], such that they can be encompassed simultaneously [by the gaze].[84]

Regardless of any infelicities that may exist in the transcription, the reference here is undeniable: Cavaillès is invoking the *Regulae*, in terms which show that he was thinking again about the *tota simul* character of Cartesian knowledge immediately prior to composing *LTS*.[85] Further, in the economy of the lecture, it is clear Cavaillès is arguing that Skolem and other results on the limitations of formalisms can be seen to bear on this 'Cartesian' problem: the discovery of non-categoricity shows that there is no 'internal affinity' to the elements of knowledge, and thus the 'development' of knowledge cannot be viewed *uno intuito*, i.e. insofar as it is an 'authentic development', knowledge cannot be 'encompassed simultaneously'.

5. As just mentioned, Skolem's paradox is once again crucial to how Cavaillès presents these matters in the logic course. Emphasising this point is necessary to understand the mathematical subtext of what I take to be the central passage in which Cavaillès states his 'new' problem. First, Cavaillès introduces the essential notion of 'categoricity':

> Given a system of axioms, we can define an object such that it belongs to that system— this is the *categoricity* of the system — [= the possibility of] Translation of one theory in another, so long as they obey the same system of axioms.[86]

Skolem's paradox is then introduced in general terms similar to those in the 'Mathematical Thought' transcript, but with much greater technical detail on the problem of the 'relativity' of countable infinite models to axioms which claimed to posit properly uncountably infinite sets, from which the 'paradox' derives its force: for any uncountably infinite set, categoricity seem-

84 Cavaillès et al. 1941: 2–3. « 2e difficulté : caractère absolu du développement de la connaissance : résolue pour Descartes par l'ordre : passage nécessaire d'intuitions simples à systèmes de pensées compliqués — Validité du résultat garantie par nécessité du développement : regardé *uno intuito* donc possède une affinité interne entre ses éléments, pour qu'on puisse les embrasser simultanément. »

85 Indeed, Cavaillès here makes reference to the exact passages of the *Regulae* that should be read as the intertext for the Thesis on the Intelligible and the Thesis on Evidence in *LTS*. See my philological comments on this point in the Introduction.

86 Cavaillès et al. 1941: 6. « Étant donné tel système d'axiomes : on peut définir un objet qui y est compris — c'est la catégoricité du système — Traduction d'un théorie dans l'autre, quand obéissent au même système d'axiomes. » (Here I correct the Cortois transcript, which has 'Traduction dans théorie dans l'autre'.)

ingly cannot be neatly defined in the way just indicated. Following this, Cavaillès summarises the consequences of the paradox. Gouhier-Dufour's notes are not one hundred percent clear here, so I make my best interpolation of the technical point:

> Whenever a system of objects is fixed by a set of axioms, this system of objects and their properties appears as *relative* to this set. — [In order to avoid this relativity] It would be necessary to possess in advance an absolute, a set from which to choose. [But we cannot 'posit' such an absolute.][87]

Cavaillès is drawing directly here on Skolem's terminology. For example, in the presentation given as part of the *Entretiens de Zurich* hosted by Gonseth, Skolem states the consequence of his generalisations of the Löwenheim-Skolem theorem as follows:

> All set-theoretical concepts and, as a consequence, all the concepts of the whole of mathematics are thus found to be relativised. The meaning of these concepts is thus not absolute; it is referred back to the axiomatic domain that serves as their basis. It is perhaps relevant to note explicitly that the concept 'finite' is relativised in the same way.[88]

From what was said above with respect to the emphasis that Cavaillès placed on Skolem in his presentation to the Société, it should be clear what the philosophical stakes are here: it is always necessary to relate the object to the way in which it has been generated; but, various non-categoricity results stemming from the Löwenheim-Skolem theorem show that there is no 'absolute' mode of generation; thus mathematics as a whole is 'relativised'.[89] And herein lies what I take to be a central moment in the transition of Cavaillès's project. Cavaillès seems to have taken returning to the problem of object constitution *after* the non-homogeneity results of Skolem et al. to require the transition to a different problem, i.e. what he will now call the theory of the 'traversal of problems' or the 'concatenation of systems'. In other words, Cavaillès's claim will be that the process through which the mathematical object is generated is not (or not only) a process of 'subjective' object constitution, insofar as this could be mod-

87 Cavaillès et al. 1941: 8. « Chaque fois que système d'objets fixés par ensemble d'axiomes, apparaissent comme relatifs à cet ensemble ainsi que leurs propriétés — Il faudrait posséder d'avance un absolu, ensemble au sein duquel on choisirait. »

88 Gonseth 1941: 37–38. Cavaillès will repeatedly ask after this volume in his letters to the Albert and Suzanne Lautmans whilst revising the *LTS* manuscript, see Appendix I.

89 It worth nothing that today there is some disagreement as to whether the order of reasons being imputed to Skolem here is correct, viz. that non-categoricity *implies* 'relativity'. See, for example, the argument in Jané 2001 to the effect that Skolem in fact understood 'relativity' as a basic fact, with 'non-categoricity' being simply an unsurprising confirmation of this. But in any case it is reasonable to say that on the basis of the material available to him, Cavaillès did understand things this way: non-categoricity, *thus* relativity. In particular, set theory (or indeed any mathematical theory that deals with uncountable infinities) is not univocal, and thus not 'absolute' (with respect to the generation of 'its objects').

elled on the relation 'axiom-system—object', it is *mathematical history*, understood in its rational form as 'concatenation'. Here is how he puts matters in the central quote from the lectures, which immediately follows his restatement of the consequences of Skolem's results:

> Necessity of grasping the process through which the mathematical object is generated. The necessity of mathematics negatively obliges the mathematician to distinguish, at the heart of the solution which has been posited as absolute, something arbitrary [in that solution]. — From which stems the necessity to always surpass [the initial] positing.[90]
>
> *Difficulty of the traversal of problems* = each problem does not analytically contain the following [problem] ([rather,] it contains the means of production of the following [problem]).[91]
>
> Is *the concatenation of systems* necessary? Is a problem posed by the appearance of a solution to the preceding problem? — *Relative* to a problem, there is only one way to posit the instruments that will serve to solve it. The solution thus appears in the same necessary positing of these instruments. Thus *an absolutely necessary internal relativity.*[92]
>
> [Is there, then, a] movement of mathematical thought? And thus a validity to what results? *That which comes after, contains that which came before, but with a new meaning.* That which has been [newly] obtained preserves what came before by changing its meaning […].[93]

The synthesis that Cavaillès draws here is truly remarkable. In a word, Skolem's proof of the 'relativity' of mathematical objects shows that mathematical philosophy must 'grasp the process through which the mathematical object is generated'. But this cannot be 'grasped' through any abstract 'axiomatic' standpoint. In other words: the 'absoluteness' of mathematics is not formalisable, because mathematics exhibits, in a striking formulation the technical subtext of

90 Cavaillès et al. 1941: 8. « Nécessité de prendre conscience du processus par lequel l'objet mathématique s'engendre. Nécessité des mathématiques oblige négativement le mathématicien à distinguer au sein de la solution posée comme absolue, un arbitraire — D'où nécessité de dépasser toujours une position. » The Cortois transcript has *une position*, while Ramirez gives *première position*. The Gouhier-Dufour ms. simply has *1. position*.

91 Cavaillès et al. 1941: 8. Emphasis mine. « Difficulté de *parcourir* les problèmes = chaque problème ne contient pas analytiquement le suivant (contient moyen de production de suivant). » Emphasis mine.

92 Cavaillès et al. 1941: 8. Emphasis mine. « Enchaînement des systèmes est-il nécessaire ? Problème posé par apparition de la solution du précédent ? — Relativement à une problème il n'y a qu'une façon de poser les instruments servant à le résoudre. Solution apparaît dans position même nécessaire de ces instruments — Donc relativité interne absolument nécessaire. »

93 Cavaillès et al. 1941: 8. Emphasis mine. « Mouvement de pensée mathématique ? — alors validité du résultat ? Ce qui vient après, contient ce vient avant, mais avec une sens nouveau. Ce qui avait été obtenu va se conserver en changeant de sens […]. »

which (i.e. Skolem's 'paradox') would be easy to miss, 'an absolutely necessary internal relativity'. Nevertheless, as Cavaillès began this lecture, we must think the absoluteness of mathematics. From whence comes Cavaillès's solution: this 'absoluteness' is identified with the *internally heterogeneous movement* of mathematics—i.e. its necessity of producing new problems. The object of mathematical philosophy thus becomes that of thinking the 'traversal of problems', which is to say, the way in which new problems are produced, and in being so produced, transform the 'meaning' of prior problems. And what form does this 'traversal' take? *Concatenation*, which is no longer thought only internal to a constituted mathematical or logical theory, but as the name for the movement between theories, which is to say the name of the production of theories.

From this remarkable synthesis, a whole series of questions arises, which will more or less define the field into which *LTS* will intervene. In particular, the problem of 'logic' has been reconstituted as that of how to think the form of the 'before' and the 'after' which is intrinsic to the movement of inter-theoretic concatenation ('the concatenation of systems'). Thus the object of logic is not the *a priori* conditions of thought, but the relation between prior and posterior production as it can be observed in 'effective' thought. And indeed, *LTS* will abound with theoretical formulations on this point, some of which directly mirror the language here, most notably the conclusion to the essay, in which immediately preceding the famous statement concerning the 'philosophy of the concept', Cavaillès asserts that 'one of the essential problems of the doctrine of science' is how to account for the fact that in its dialectical development:

> What comes after is more than what came before, not because it contains it or even because it extends it, but because it exits from it necessarily and bears within its content the singular mark, each time, of its superiority.[94]

What Cavaillès also seems to have arrived at in the logic course is the crucial thesis of the *local necessity* of mathematical movement, which is methodologically key for his late perspective. It is 'relative' to a (particular) problem that there is a 'necessary' positing of the instruments that will resolve it; this does not mean that one ascribes to a 'global' necessity—i.e. teleology—of mathematical history. Quite the opposite. But how to think the rapport between this 'local' necessity and the movement of the 'whole chain' of mathematical theory, i.e between mathematics in the singular and the diversity of actually realised mathematics? This will become a motivating question in *LTS*.

94 *LTS* 78.

This problem—that as to the 'unity' or 'diversity' of mathematical becoming—is of course a very classical problem: that of the One and the Many. So it is appropriate that Cavaillès's lexical formulation here—*Difficulté de parcourir les problèmes* = problem of the necessity of the *enchaînement des systèmes*—is entirely consistent with the Cartesian-Neoplatonist synthesis that I will argue in Chapter III is the proper historical background through which to understand what Cavaillès will theorise in *LTS* under the name of 'thought'.[95]

6. Thus Cavaillès was lead to the problem of his late philosophy: to think the form of the progress of mathematical theories as 'necessary concatenation', where this was understood to specifically mean 'inter-theoretic' concatenation. It is also in arriving at the formulation in its precision that he fully begins to make the displacement we have been discussing here: formulas that were previously developed with respect to the concept of experience are now recapitulated in the language of concatenation. A crucial moment in this respect is when Cavaillès returns to the exact point he had begun to draw out in the discussion with Gonseth concerning the different forms of historicity proper to mathematical and physical experience. This is now directly restated in terms of the opposition between two forms of concatenation.

> What is physical—in opposition to mathematical—is the effective action of the physicist. Physical experience is situated in history whilst mathematical experience is not. […] Mathematical thought and physical thought mutually exclude each other (necessary concatenation on the one hand and purely historical concatenation on the other).[96]

This is the hinge to Cavaillès's restatement of the relation at stake here directly in terms of 'mathematical concatenation' and 'physical concatenation', such as we find it in *LTS*:

> But physical concatenation has no absolute beginning, no more than mathematical concatenation does. On the one hand, insofar as they are mathematical relations, the statements of physical concatenation only acquire their meaning as elements in a system that is already posited, and that already possesses a more or less precise experimental meaning. On the other hand, experience itself, as a system of acts, is internally organised in such a way that it is impossible to interrupt its unfolding, except by way of a superficial abstraction: experimental acts generate yet more experimental acts by way of a *sui generis* concatenation which, at least insofar as it is such, is independent—because it is of another essence—from mathematical concatenation. At the intersection of these

95 Indeed, the notion of the 'traversal' of a concatenation—i.e. the specific term *parcourir* (and variants)—is the word which Brunschvicg uses in *Les Étapes* to describe the movement of the intellect through a proof in Descartes (see my Introduction) and, in a different register, a quite Plotinian choice of terminology (see §III.3.).

96 Cavaillès et al. 1941: 10. « Ce qui est physique—par opposition au mathématique—c'est l'action effective du physicien—Exp. physique se situe dans l'histoire (alors que mathématique non). […] Pensée mathématique et du physique s'excluent mutuellement (enchaînement nécessaire d'une part et enchaînement historique pur d'autre part. »

> two processes the physical relation appears; from their more or less complete coordination there emerges theoretical physics, which it falls to the doctrine of science to situate and to make sense of. But this escapes *a fortiori* all formalisation.[97]

Thus we arrive at the promised displacement: the doctrine that Cavaillès had been developing with respect to 'mathematical experience', and which in the passage from the Sorbonne lectures Cavaillès associates with 'necessary concatenation', is synthesised in *LTS* into a statement about the nature of 'mathematical concatenation'. At the same time, reading across the two passages, we can see that 'physical concatenation'—which is still properly associated with the term 'experience'—is in some sense 'purely historical concatenation'. I hope to have made it clear that this is more than a mere lexical point, but it is worth recapitulating the development we see here from the moment in which Cavaillès made much the same arguments to Gonseth in 1938. As discussed above, in the division between mathematical and physical experience that Cavaillès had made at an earlier moment, what was at stake was one field of (human) experience that was modally differentiated by its two forms of historicity. In *LTS* on the other hand, this modal disjunction is partly preserved, but it is no longer differentiating the field of experience. Instead, the relationship at stake is between two forms of concatenation: physical (i.e. historical) concatenation, and mathematical (i.e. necessary) concatenation. And in this new formulation, the only form of relation that can be established here is between *singular or specific points* in the respective concatenations. The problem of the physics-mathematics relation becomes that of 'coordinating' these points, and this is asserted to be a core problem for the 'doctrine of science'. Interpolating the passage from the Sorbonne lectures just cited, it is also clear that this problem stands metonymically for the relation between reason and history. But in the same breath, Cavaillès states that this problem—this relation—must 'escape all formalisation'.

At stake here, I believe, is a genuine theoretical innovation in Cavaillès's late work: the form of movement proper to these two concatenations is fundamentally different. Physical or historical concatenation has an *external* form of movement; the reason for its alteration is not intrinsic to its conceptual motion; rather it comes from an encounter with some obstacle outside of itself (e.g. in the observation of an experimental failure). Mathematical or necessary concatenation, on the contrary, is a concatenation the motion of which is *internal*. In the terms we have been laying out, this means that in physics there is no 'traversal of problems', not in the sense in which 'logic' as the discourse on necessary concatenation is interested in them.

97 *LTS* 40.

We might say that physics only has difficulties, not 'problems': physics encounters specific technical difficulties, which may or may not be resolved, but in physics the problem qua problem cannot provide the motor for its own surpassing by another problem. Thus Cavaillès arrives at a basic methodological assertion for the 'philosophy of the concept': the difference between physics and mathematics, as theoretical disciplines, is that physics *can stop* whereas mathematics *cannot stop*.[98] This 'doctrine' has a rather stark consequence, which is that not only physics, but all application of knowledge to the world is no longer the proper object of 'logic'. In a sense, the problem with physics, and *mutandis mutatis* the problem with life, as with all merely historical experience—that is, with historical experience insofar as it loses itself in the vanishing singularity of the moment—is that it does not exhibit the pure movement of the concept which is the true object of philosophical logic. Only the concept moves.

7. I have begun to parody Cavaillès's position in these last lines, but only a touch, mostly by placing some emphasis where Cavaillès would not care to. This was in order to make a rhetorical point. For I have passed over the figure with which Cavaillès begins his lectures on logic, whom the reader cannot but have found conspicuous in his absence from the preceding exposition, and who now seems truly unavoidable: Hegel. For what could be more Hegelian than to have first identified the absoluteness of logic with the movement of the concept, and then attempted to derive a doctrine of science through stratifying the different forms of movement proper to different domains of the concept? So what to make of the fact that Cavaillès's first port of call in the logic lectures is to signal the 'insufficiency of the Hegelian solution' to the problem of logic?[99] Now, Cavaillès does not reference the *Science of Logic* here but rather Hegel's critique of mathematics's pretension to grasp the absolute from the Preface to the *Phenomenology of Spirit* (i.e., I presume, paragraphs §42–46). At a crucial moment in this exposition, Hegel asserts the following judgement:

> The actual is not something spatial as it is taken to be in mathematics; neither concrete sensuous intuition nor philosophy wastes any time with the kinds of non-actualities which are the things of mathematics. In such non-actual elements, there are then only

98 In passing, we can note that this formulation brings to a higher level of philosophical expression something that had concerned Cavaillès as far back as *MAF*, in the conclusion to which he asserts that 'although there is nothing to think in physics besides the mathematics that is found there, nevertheless the technical intention—in the sociological sense, i.e. the affirmation of human life in the world, bringing man as an individual being to figure among things—intervenes in order to halt the normal dialectical process.' (*MAF* 179–180/*OC* 187-188) This observation that the demands of physical or technical application 'halts' the dialectal development of mathematical theory has now been transformed into a definition of the two disciplines, qua their respective forms of concatenation.

99 Cavaillès et al. 1941: 1.

> non-actual truths, which is to say, fixed, dead propositions; one can call a halt to any of them, but the next begins anew on its own account without the first itself having moved on to another and without any necessary connection arising out of the nature of the thing at issue.[100]

I take this to be the claim that Cavaillès repudiates directly: if we follow Hegel, there is not a 'true movement' in mathematics but rather only a 'formal movement'. For this very reason, Cavaillès glosses Hegel as having asserted that:

> In order to posit the absolute validity of its knowledge, mathematics is obliged to kill the internal movement of concepts.[101]

This, of course, is a position that Cavaillès could never accept. His construction of Hegel's critique of 'mathematicism' thus was that, on Hegel's view, science must seek to attain the absolute; but mathematics merely *posits* the absolute, i.e. as something completed, *already* attained; thus mathematics cannot think 'true movement': the movement of the Concept. Only philosophy can do this, and it is in this sense that philosophy is the path to science, not mathematics. For Cavaillès, it is on this point that the 'Hegelian solution' must be rejected; on the contrary, for Cavaillès mathematics not only exhibits 'true movement', but is the *only* domain of intrinsic rational movement. Nonetheless, as we have begun to reconstruct it, his underlying argument is that the way out of philosophy's 'oscillation' between the 'positing of an absolute' (Cartesianism) and the 'necessity of an authentic development' (Post-Kantianism) is to identify the absolute *with* necessary development. Is this not, precisely, a 'Hegelian solution'?

It can indeed seem as if the disagreement between Hegel and Cavaillès is rather small, that it concerns disciplinary distinction (the *where* of the concept) rather than conceptual form (the *how* of the concept). On this reading, Cavaillès would simply have attempted to elaborate Hegelianism *internal to mathematics*, a kind of mirror world Hegelianism. This would be how Cavaillès ends up (implicitly) asserting on mathematical terrain what do, indeed sound like the eminently Hegelian conclusions just listed above concerning the movement of the concept. In the history of the reception of Cavaillès's project, this issue has divided commentators from the start, and I will not enter into this discussion now in a systematic way.

To close, however, I would like to suggest that the 'question of Hegel' with respect to Cavaillès is very intimately connected to understanding what Cavaillès meant by logic, and

100 Hegel, *Phenomenology of Spirit*, §45. Translation from Hegel 2018: 27.

101 Cavaillès et al. 1941: 2. « [...] pas de mouvement véritable. Mouvement formel seulement. Mathématique pour poser validité absolu de sa connaissance, est obligée de tuer mouvement interne des concepts. »

thus how to read *LTS*, but not primarily because of the question of influence. Rather, it is because one of the most tender points concerning Cavaillès's late 'philosophy of the concept' is whether he believed there was a logic of the concept that could be articulated in philosophical discourse, independent of effective mathematical history. In other terms, was there any possible *positivity* to the discourse of a specifically philosophical logic? My argument is that Cavaillès refuses this, and that in this sense he remained 'anti-Hegelian' even if he was, it seems, increasingly attentive to Hegellian conceptual categories. Thus one can say that for Cavaillès, the concept also does not move in philosophy; there is no 'traversal' of the philosophical problem.[102] This is the aspect of his work that has great resonance with later 'structuralist' interventions. Put in the terms we've been developing, this is to say that philosophical *discourse*, insofar as it attempts to erect a 'theory of science' on *a priori* grounds, cannot but 'kill' the movement internal to the properly scientific (which here only really means: mathematical) concept. This is if one likes the shape of the critique that Cavaillès will systematically elaborate in *LTS* against proposals for a post-Kantian 'theory of science' in his conjuncture. It means, also, that the most difficult problem that faces the interpreter of *LTS* is that of reckoning with the degree to which that text enacts a performative contradiction: qua discourse 'on' concatenation (that is, on the form of production), is it not self-defeating?

This is the problem of Cavaillès's *writing*, as we shall return to it in Chapter III. But before we do, it seems worth noting that in terms of the long history of the question in the French philosophical field concerning 'Hegel or Spinoza?', it is this very question of the autonomy of philosophical discourse which Cavaillès brings to the forefront. At the very least, we can suggest that the 'Cavaillèsian' option was to insist on restricting himself to the mathematical example in order to avoid falling into a rapture of the self-sufficiency of philosophical

102 This puts me at odds with the most 'Hegelian' reading of Cavaillès. For Elizabeth Schwartz, an embrace of philosophical as opposed to only mathematical logic is the core of the 'break' between the Cavaillès of the doctoral studies and the Cavaillès of *LTS*, and Hegel is the key figure in having motivated this change. On Schwartz's interpretation, this turns on a basic question that I also posed in the introduction to Chapter I: for Cavaillès, does thought = mathematical thought? According to Schwartz, the doctoral works, in effect, do identify thought with mathematical thought, whereas *LTS* does not, *because LTS* makes a transition to philosophical logic—on broadly Hegelian grounds—as reflection on the dialectical history of Logic, which is in no sense to be identified with the effective history of mathematics. Thus the logic of thought, if one likes, is articulated on the basis of an entirely different ontological ground than the epistemology of mathematics, and philosophy returns as the proper site for elaborating said ontology. On this reading, Cavaillès's closing polemic in favour of the concept and contra consciousness is in fact a methodologically self-reflective Hegelian riposte to his own earlier philosophy of mathematical experience: 'In brief, what is at stake in the polemic against the philosophy of consciousness is first and foremost the debate between Cavaillès the dialectician of mathematical experience and the Cavaillès of the logic of the concept.' Schwartz 2020: 187.

discourse, and that this, putting aside any claim as to his Hegelianism or otherwise, was always a central stake in his attachment to Spinoza. For if under a certain reading, the profound proximity between Hegel and Spinoza in the history of philosophy is that they produce the two great modern rationalist discourses on the movement of thought, it is the distinctively Spinozist path to make *mathesis* the locus of that movement, insofar as it is thinkable. It appears that Cavaillès, to the end, agreed, for as Granger notes,

> I recall a lecture on the subject of time given on 20th May 1942 as a response to one of our presentations for the aggregation at the ENS, where Cavaillès insisted on the specificity and irreducibility of a 'mathematical time', of 'a before and an after' that cannot be eliminated from the operative process [*du processus opératoire*], the recognition of which is 'one way to think eternity in Spinoza's sense.'[103]

This was, it seems, the last lecture that Cavaillès gave before his arrest, and the writing of *LTS*.

103 Granger 1996: 570.

III. VARIEGATION AND SPECIFICITY

(The Positive Doctrine in *On Logic and the Theory of Science*)

> Mathematics is the science of acts without things—and by that token the science of things that one can define though acts. *To make*, *to be able*, are its essential words.
>
> Paul Valery, Cahiers 2, 811 (= 1935, XVIII, 202)[1]

In a 1975 survey of the main stances taken on the relationship between philosophy and the sciences, Desanti aptly summarises, in passing, the way in which *On Logic and the Theory of the Science* has been received. 'At the end of a very profound essay […] Jean Cavaillès concluded with some propositions of a programmatic nature. He declared that only a philosophy of the concept can produce a theory of science. Interpreted negatively, the formula is true.'[2] To be sure, this reputation rests on certain signal features of the text. It is an essay that concludes with a 'programmatic' statement that it is '*not* a philosophy of consciousness, *but* a philosophy of the concept that can yield a doctrine of science.' Yet this programmatic announcement only seems to have a negative content: it is clear (or relatively so) what is being refused, but not what is being affirmed. The subsequent reception of the essay has in general rested on its critical aspect. Amongst those for whom the book—at least notionally—was to serve as a north star in the struggle against both 'positivism' and 'the philosophy of the subject' from the 1960s onwards, it was understood as having been a broadside on Cavaillès's part against both the then-emerging paradigm of logical empiricism in the philosophy of science and the post-Husserlian project of phenomenology—one that was developed prior to the dominance of either, but went sadly unheeded. But it was rarely read with the intention to extract the commitments latent in the work. In turn, even in subsequent affirmations of the project for a 'philosophy of the concept', the notion of 'the concept' primarily occupies a negative place: it is not the subject, not consciousness, not some mythically concrete empirical experience, not merely 'tautologous' logical form… but as to what it *is*, matters are much less clear.

1 Cited in Szczeciniarz 2021: 89.

2 Desanti 1975: 62–63.

We shall begin from the other side of things, and examine the doctrinal content of the 'philosophy of the concept', independent of its polemical position against the 'philosophy of consciousness'. It is only on that basis that we can return to the critical import of this project. In this respect, the most crucial pages of *LTS* are the dense elaboration therein of the disjunction between *the paradigm* and *the thematic*, running from *LTS* 26–33. This chapter shall in effect be an extended commentary on this elaboration. As signalled in Chapter II, these pages of *LTS* are in part a restatement of certain doctrines that were previously developed under the banner of a project for a philosophy of 'mathematical experience', but which are now articulated in terms of the idea of concatenation, understood in the sense of the dialectic between *intra-theoretic* and *inter-theoretic concatenation* as it had emerged around the time of Cavaillès's Sorbonne logic lectures of 1941–1942, that is, in terms an account of prior and posterior production in mathematics, or a 'logic' of 'the before and the after' intrinsic to the movement of mathematical history (that is, internal to 'mathematical time'). It is in terms of such a logic that we should understand the disjunction in question: reflection on the nature of proof, and in particular on ways in which 'form' is separated from individual mathematical results, leads us to a necessary difference internal to the movement of mathematics:

> The process of separation is twofold: longitudinal, i.e. coextensive with demonstrative concatenation, and vertical, i.e. inaugurating a new system of connections [*système de liaison*] which uses the old one as its basis, and no longer takes the old system as a stage that a movement has passed through, but rather makes the manner in which it is currently moving its object of reflection. In each case what manifests is a constitutive property of the essence of thought—or of intelligible concatenations: the *paradigm* and the *thematic*.[3]

In—for now, extremely—abstract terms, the doctrine that the text then outlines with respect to theses two properties of 'the essence of thought' is that the paradigm is characteristic of *effectuated concatenation* and the thematic is characteristic of *effectuating concatenation*. In turn, these will be associated with the stratification I have outlined in Chapter II with respect to the historicity of mathematics: the effectuated is associated with the 'intra-theoretic' level of concatenation (it is 'longitudinal'), and the effectuating is associated with the 'inter-theoretic' level (it is 'vertical'). The crucial question then becomes: what is the relation between these two forms of 'intelligible concatenations'? And herein lies a central theoretical intervention of this text: while we might reasonably expect the effectuating to be conceived as necessarily prior to the effectuated—that is, that producing is necessarily prior to what is produced

3 *LTS* 27.

—what we find in *LTS* is that, on the contrary, *at least methodologically speaking* the effectuated is prior to the effectuating, i.e. effectuating concatenation takes effectuated concatenation as its 'basis'. This is a central pillar of the positive doctrinal contribution made by *LTS*, and, as we shall see, it has radical consequences in both an epistemological and an ontological register. With respect to the latter, I will suggest a translation of the doctrine just expressed: another way understanding the claim that effectuating concatenation takes effectuated concatenation as its basis is as the assertion that *specificity* is founded on *variegation*. In terms of the philosophical referents that are invoked in the text of LTS, this comes down to using what I will call a strategic Neoplatonism to ground a methodology of rationalist nominalism, fundamentally indebted to Spinoza.[4] What is perhaps most difficult to grasp here is how, through this account, a position on the nature of the intelligible, indeed what we could cautiously call an 'ontology', is wedded to a historiographical research programme in the philosophy of mathematics. The task of this chapter is to define these terms.

§III.1. Locating the Positive Doctrine in *LTS*

Before, however, diving into the subtleties of this account, some preliminary comments, bearing on how the pages we will focus on here are situated in the structure of *LTS*, my methodology for interpreting them, and the tradition I take this 'doctrine' to have been situated in.

§III.1.i. Position of the Analysis of the Paradigm and the Thematic in *LTS*

With respect to the organisation of *LTS*, an attentive reader may be surprised to note that I have just now signalled out as the core 'doctrinal' part of the work a section (*LTS* 26–33) which contains neither of the two 'theses' that I previously presented as the core of the essay: the Thesis on Evidence (*LTS* 23-25) and the Thesis on the Intelligible (*LTS* 35-36). However, the section on the paradigm and the thematic must be understood as the essential bridge between these two theses. Here it is important to have in mind the structure of the essay as a whole, which is broadly as follows:

> *LTS* 1–26: Section I. Critique of the Kantian theory of science and the opposing options for its development in the conjuncture, as defined by whether an emphasis is put on

4 I thank Silvia Mollicchi for suggesting the phrase 'rationalist nominalism' as a way of describing Spinozism.

'demonstrative system' (i.e. 'formalism') or on 'mathematical organon' (i.e. 'epistemologies of immanence').[5] This section is thus subdivided into:

LTS 1–14: Exposition of the Kantian theory.

LTS 15–19: Problems with epistemologies of immanence, represented by Brouwer and Brunschvicg.

LTS 19–26: Analysis of the 'formalist' view, represented by Bolzano.

LTS 26–43: Section II. Analysis of the pure theory of rational concatenations, or the problem of proof. This section contains extended critiques of what are taken to be two insuperable difficulties facing the logical empiricist account of the logic of science, i.e. first, the aporias faced by the attempt to develop a 'pure syntax', and, second the problem of attempting to suture this syntactical account to 'physical epistemology'. As such, the section is subdivided into:

LTS 26–33: Outline of the theory of the paradigm and the thematic.

LTS 33–40: The problem of syntax and semantics, containing an exposition on Tarski, and, therein, a critique of Carnap.

LTS 40–44: The problem of physics, and the incapacity of Carnap and Wittgenstein (of the *Tractatus*) to deal with the same.

LTS 45–78: Section III. Analysis of Husserl's *Formal and Transcendental Logic*, and a reckoning therein with the phenomenological attempt to found theory of science in an 'ontology' of constituting consciousness, i.e. with the project of transcendental logic.

Considering this organisation of the work, the most puzzling thing about the elaboration on the paradigm and the thematic is, at least initially, what it is doing here at all? For what we are presented with in these 78 pages is a polemical survey of the field of the epistemology of science after Kant, reckoning in turn with the major options for developing a theory of science within the horizon opened up by the critical project, which contains at its centre a sort of paroxysm in which Cavaillès apparently elaborates his 'own' theory of the movement of science, without signalling it as such, or directly connecting it back to the criticisms he is directing at other figures.

5 *LTS* 14.

I believe this impression is mistaken, or at least has to be read in light of a positive argument that runs through text, which turns on the connection between the Thesis on Evidence and the Thesis on the Intelligible, through which the whole scheme of *LTS* appears in miniature. As we saw in Chapter II, in his Logic courses at the Sorbonne in 1941-1942, Cavaillès was already assimilating a Cartesian category (the *tota simul* or *uno intuito* nature of evidence) to a post-Kantian problem field (the unity of science qua the homogeneity of synthetic activity). As indicated in my Introduction, I take this to be crucial to the organisation of *LTS*, in something like the following form. On Cavaillès's reading, a central development of mathematics in the 19th and 20th centuries had been to replace 'evidence' with 'proof'. This is taken to imply that it is the movement of proving, understood as the historical production of the intelligible, which produces evidence or intuition. In terms of (now curiously inverted) Cartesian doctrinal terminology, this means that evidence is the product of a concatenation, not its foundation (Thesis on Evidence). Yet what modern mathematical logic (Gödel, Skolem, etc.) has also shown is that the character of this concatenation is internally heterogeneous, i.e. it cannot be established all at once. Thus, insofar as we hew to the maxim of rationalism (that we must study mathematics in order to know what thought is), we arrive at the positive claim that the nature of the intelligible lies in its intrinsically *non tota simul* character (Thesis on the Intelligible). Thus: neither 'science' nor 'knowledge' (if there be any difference between the two) can be 'univocal', at least not if it is to be adequate to the intelligible.

It is based on this argument that we find the celebrated formula against the philosophy of 'consciousness' or 'activity' in the final lines of the essay. The terms of this rejection have been suggested in Chapter II: the attempt to found the theory of science in constituting activity is fatally flawed because it assumes a univocity of constitution which, in principle, makes it inadequate to the heterogeneity of the modes of generation of the objects of thought. This is a more general way of stating the epistemological thesis that the 'theory of science' must not be erected as prior to and governing for the development of 'effectively realised' science. Thus, in the last instance, what is denied at the climax of the work is the theory of science *qua* theory of the unity of science, as it would have been understood in the broadly Neo-Kantian discourse in which Cavaillès was intervening, and which Husserl is presented as having purified through a transcendental analysis of consciousness. This does not mean—in a sense we shall consider below—that the problem of the 'unity' of science disappears for Cavaillès; but what we can say is that it must not be thought as the unity of a *constituting field*.

Now, what is needed in order to articulate this account is a discourse on the form of the heterogeneous movement of science, i.e. some positive account of the *production of concepts* that would not be a theory of the *constitution of concepts*. And, broadly speaking, this is what the notions of the paradigm and the thematic are supposed to provide. Thus, if one wishes to locate a positive account of the movement of science in the text of *LTS*, it in effect begins in rather peremptory fashion in the middle of the treatment of Bolzano, i.e. with the statement at *LTS* 23—paraphrasing a little—that *the unity of science is movement*, and that this movement 'moves outside of time'. The medium of this non-temporal movement is concatenation, in a sense that should make us 'recall Descartes's difficulty', i.e. the problem in the *Regulae* of how to square the immediacy of intuition with the successive nature of deduction (*LTS* 25). This I take to be a statement of the stance adopted in *LTS*, not simply a gloss on Bolzano. The logical structure of this 'movement' (that is, of the relation between effectuated and effectuating concatenation) is then outlined from *LTS* 26–33 under the heading of the paradigm and the thematic. Finally, in the course of the treatment of the problem of syntax, it is made clear in *LTS* 35–36 why the discourse just developed requires a radically different stance on 'the intelligible' than that assumed by logicism and more broadly by a whole tradition of thinking *logic* in terms of *intrinsic simultaneity* that is taken to run from Descartes and Leibniz to Carnap and Husserl. To think, on the contrary, the non-simultaneous and thus intrinsically successive nature of the intelligible, what is required is a discourse, however minimal, on mathematical time, and that is what the elaboration of the paradigm and the thematic is; it thus stands in contraposition to the various positions that are addressed in the rest of the work.[6] This is the arc we shall be concerned with in this chapter.

§III.1.ii. *LTS* as a Text

In what follows, I take a rather different approach than in the preceding chapters. There, the account was in some sense genetic: we considered how it was that Cavaillès, the figure, developed the vocabulary and conceptual instruments that would be wielded in *LTS*. Here we will be interested in the text. For another innovation that I take to be at stake in *LTS* concerns the status of writing, and its relation to theory. Thus one last comment on the development of Cavaillès's views prior to composition of *LTS*, and the question of the place of philosophical discourse as it emerged at the end of Chapter II.

6 The background to these 'two theses' is given in the Introduction.

As discussed earlier, the Sorbonne lectures contained the outlines of a new philosophical project, but posed a stark methodological problem. At base, this concerned the status of philosophical discourse on the development of mathematics, as localised in the problem of the relation between philosophical and mathematical logic. The project which Cavaillès had envisioned was to give an account of the intrinsic movement of mathematical concepts which would resume themes from the whole tradition of reflection on problems of 'method', conceived as running from, at the very least, Descartes to Husserl, but now conditioned by then-contemporary mathematical logic, in particular by the results that had emerged in the 1920s and 1930s concerning the 'internal limitations of formalisms' (Löwenheim-Skolem, Gödel, Herbrand, Tarski, Church-Kleene, etc.).[7] What was proscribed, from this perspective, was an external philosophical discourse on mathematical becoming, i.e. any *a priori* theory or purely philosophical logic of science.

This poses a difficulty, which could be described in general terms as *the autonomy problem*, and which emerges from the attempt to avoid two errors, which I will call the error of the *a priori* and the error of contemplation. Let's agree that we want to give an account of the 'autonomous' movement of mathematical thought without 'reducing' it to anything else, i.e. the polemic that Cavaillès repeatedly advanced against providing a 'definition' of mathematics (that is, an 'eternal definition').[8] Given this *desideratum*, any abstract philosophical discourse that would condition the movement of mathematics in advance is to be rejected, for it would lead us to lose the object that, as epistemologists, we are supposed to be studying: the 'effective' becoming of mathematics. Thus what in particular must be avoided is the erecting of a system of abstract logical categories which effective mathematical work would only serve to give content to. This in a word is the criticism of the Kantian project of General Logic with which *LTS* begins, and which gives the scheme to his treatment of most of the figures therein; it is further why specific mathematical theories must not be conceived as 'examples' of general philosophical concepts. In fact, this prohibition does not only bear only on general *a priori* categorisation. It is also prohibits a 'psychologistic' theory of the faculties of the 'working mathematician', and, perhaps more radically, it means that mathematical philosophy also cannot follow a project of logical 'foundations' in the style of, say, Russell and Whitehead, or even Carnap, a project which is likewise to be rejected because it would erase the specificity of the

7 To adopt the title of Ladrière 1957—a highly 'Cavaillèsian' treatise—as a synthetic term for this constellation.

8 *OC* 594. See §II.1.i. for my summary of this polemic.

movement of mathematics (for if mathematics were in the last instance reduced to logic, then the specificity of mathematical history would obviously be irrelevant with respect to philosophical reflection on 'form', which would be essential transhistorical). This whole constellation of views—the partisans of which would doubtless object to their assimilation—is *the error of the* a priori.

So far, so Brunschvicgian: mathematical philosophy, it seems, must limit itself to 'reflective analysis' on the becoming of mathematics, and thus condition itself as a descriptive enterprise that embarrasses the pretensions of 'first philosophy'. It is also the 'positivist' side of the project, in the sense that term would have held in France at the time, i.e. post-Comtean positivism, not 'logical positivism'.[9] Putting aside many complications, it is fair to say that this was the method at stake in the period of Cavaillès's doctoral works, which had been supposed to issue in a work on 'mathematical experience'.[10] However, the very stridency of this opposition to the *a priori*, that is the very embedding of mathematical philosophy in immanent reflection on mathematical becoming, threatens a second error, with which *LTS* is no less concerned, and which is addressed there under the banner of the 'epistemologies of immanence', even if the formulations we find on this point are more reserved, perhaps owing to Cavaillès's desire not to appear too vituperative in his denouncement of his 'master'.

This is the *error of contemplation*. It stems from the essentially different historical faces ascribed to the figures of the 'mathematician' and the 'philosopher' (or the 'epistemologist'), that is, the different temporalities they inhabit. Only the mathematician actually participates in becoming, even if they are not its agent. The central theoretical equation of necessity and unpredictability thus holds firm for them: the mathematician faces forward in history, and by dint of this alone can they claim to be a 'revealer of necessities', necessities which they reveal through their 'history-negating work'.[11] Correlatively, the philosopher-epistemologist faces backwards towards a necessity that only 'appears after the fact [*après coup*]', and attempts to

9 See Chapter XIII of Brunschvicg 1912, in which for all of his disagreements with positivism, Brunschvicg affirms that Comte, rather than Fichte, Hegel or Schopenhauer, is the true 'inheritor' of Kant, because he had retained the essential methodological posit that mathematics was a 'constituted discipline' with respect to which philosophy must disavow the ambition of either 'transformation or alteration', for 'mathematics furnished philosophy with the definitive type of knowledge'; the central positivist innovation, which Brunschvicg partly aligned himself with, was then to displace the Kantian question as to the *justification* [*le droit*] of mathematical knowledge by reflection on the *fact* [*le fait*] of this knowledge. Brunschvicg 1912: 283.

10 Indeed, in a letter of April 22nd 1928 to Ferrières, Cavaillès excitedly describes having agreed on the topic of his doctoral studies with Brunshcvicg—i.e. what would eventually become *TAE*—as an application of 'the method on which we both agreed' to the history of set theory. See Ferrières 2003: 69.

11 *TAE* 28/*OC* 226.

honour the 'unexpected inflexions of mathematical becoming' that they behold in history (or alongside it).[12] But this latter position is fundamentally ambivalent: it is a laudable role only insofar as the philosopher manages to dwell within the infinity of the forward movement that they seek to apprehend, without prematurely inserting a halting point. In practice, the philosopher is more easily given the role of Orpheus, who in casting their vision back a little too suddenly makes the fatal mistake of defining, from a single vantage, the progression that advances behind them, and thus petrifies its animating principle. In brief, the error of contemplation is that in taking philosophical expression as merely epiphenomenal to the real movement of mathematics, one treats mathematical history as an *indefinite* becoming, and the very definiteness of conceptual expression then becomes a barrier to the experience of this rapturous unfolding of mathematical necessity.

In a sense, the methodological problem here concerns the immanence of philosophy to mathematics (and vice versa). The discourse on the movement of the mathematical concept, for all that it is rooted in the history of mathematics, cannot itself *be* the history of mathematics. Indeed, the project of giving an account of the 'logic' of 'mathematical time' qua inter-theoretic concatenation is, we might say, intrinsically not 'mathematical', for it is precisely on the intra-theoretic level that the mathematician carries out their work. On the other hand, the project of describing the logic of mathematical concatenation cannot stand outside of mathematics, pronouncing on what it wishes mathematics to be. It can appear that this dilemma is insuperable: the sought after discourse is neither 'mathematics' nor 'theory', in the sense of the 'theory of science' (or the 'philosophy *of* mathematics'), so what can it be?[13]

This dilemma is internal to the project of Brunschvicgian epistemology, at least as it is presented in *LTS.*

> *[R]ational immanence means that no external purchase on the idea is possible*. But what place does this leave for a philosophy that would consist in more than the mere explication of the scientist's intentions? Might we even say that the history of progress is no longer external, a cathartic useful to the scientist for staving off the temptations of the social, and that *only the effective concatenation of intelligible contents attains to the level of spirit*? Even this is not satisfactory: the terms 'spiritual' and 'immanence' presuppose

12 *TC* 274/*OC* 472; *TAE* 29/*OC* 227.

13 Here we return to worries discussed in §II.1.i., and I again direct the reader to Halimi 2020 for an excellent exposition of this problem establishing the 'right distance' between mathematics and philosophy. However, I think that it is harder than Halimi believes to reconcile the Cavaillèsian 'programme' on this point with 'logicist' reflection on the symbolisation of forms.

> the possibility of an ascesis or a deepening of consciousness that would be distinct from scientific comprehension.[14]

As I read it, the central assertion here that 'rational immanence' implies that no 'external purchase' on the idea is possible is not only a summary of Brunschvicg's view. This is what Brunschvicg held, of course, and Desanti recounts a motto of sorts that Brunschvicg used to share with his students, stating that he 'always preferred the immanence of the intelligence to the transcendence of the intelligible.'[15] It also, however, captures something very intimate about *LTS* as a whole. A signal characteristic of the text is to adopt a version of the (basically Spinozist) thesis that the idea cannot be grasped externally, and thus must be considered internally, that is, only in light of its own norm, whilst inverting the terms of Brunschvicg's motto: we must choose the immanence of the intelligible over the transcendence of any intelligence or consciousness that would putatively govern its movement. This at least is what I read as being at stake in the following continuation of the commentary on Brunschvicgian epistemology, i.e. that even though this is presented as a choice without preference, the logic of *LTS* makes it clear that there is an implicit affirmation here. I rearrange the text a little, to make the 'choice' at clear:

> What is needed is either[:]
> – [1] an absolute intelligibility that legitimates the Spinozist superimposition of the idea of the idea,
> – or [2] a reference to a generative consciousness characterised by an ability to grasp itself immediately in its authentic acts.
> In both cases, an ontological analysis seems necessary while, on the other hand, no answer is given to properly epistemological questions.[16]

We should recognise here that the second option refers to the Husserlian 'ontology' that will be addressed in Section III of *LTS*, which is one possible way of grounding Brunschvicgian 'immanence', but it is also presented here as very much in line with Brunschvicg's own invocation of the active intelligence in its impersonal progress through rational history. The first option, on the other hand, is the alternative that will be opposed to the Husserlian path; that is, the 'positive doctrine' that *LTS* develops. The strange task that *LTS* then sets itself is to *write* from the standpoint of 'absolute intelligibility', and not from the standpoint of the progressive un-

14 *LTS* 18-19. Emphasis mine.

15 From Desanti's preface to the re-edition of *Les Étapes*, i.e. Brunschvicg 1972: *v*. In opening a special session of the Société française de philosophie of 2nd June 1962, commemorating the 50th anniversary of *Les Étapes*, Jean Hyppolite relays the same anecdote in a more commanding tone: « *Il ne faut jamais substituer la transcendance de l'intelligible à l'immanence de l'intelligence.* » Dubarle et al. 1963.

16 *LTS* 19.

folding of 'the intelligence'. It is only by undertaking this effort of positive—that is, determinate—elaboration that the error of contemplation can be avoided.

Is this a viable path? Can one even claim that the idea of 'writing the intelligible' has a meaning? Here I will only say that, as I understand it, *LTS* is by its very form properly inscribed within a tradition of thought that intrinsically links metaphysics to writing.[17] This is not something that Cavaillès ever states, but I find it fundamental to the singularity of *LTS* as a text, that is, as an essay. An evocative comment on this point was made by Jean-Claude Milner in response to a question from Knox Peden about the importance that the tradition of 'French Rationalism' held for the structuralist milieu of the late 1960s:

> Yes, I think that there is a continuity in which Descartes indeed plays a part. But I would not put it in terms of rationalism. I would say that there is in the French tradition a sort of prose of thought, but we desire or need these writings to encounter questions along the way that make prose explode. I think of Descartes's *Meditations* and Pascal's *Pensées*; I could add Cavaillès's last text. This is not German systematicity, which is architectural; this is of the order of the concatenation of reasons [*de l'ordre de l'enchaînement de raisons*], which is linear. […] In the continuity that I am thinking of, the effect of concatenation remains, regardless of the length of the chain; it remains even if the chain at one point has an encounter with an element that breaks it.[18]

The construction of such a tradition seems to be entirely apt for situating the specificity of *LTS*. To hew closer to Cavaillès's vocabulary, it is a consequence of the characteristics of the intelligible itself (i.e. of its *non tota simul* nature, and thus its internal demand for 'progress') that it take the form of effective work. It is legitimate to paraphrase this as the assertion that *thinking* is essentially *prose-like*; the value of thinking is that it is successive and that it contains an explicit and determinate content, which is not transparent to its 'author'; this is finally why 'writing' in the empirical sense, that is as effective, material inscription, tells us more about the nature of 'thought' than any demand for introspective reflection.[19] We can recall here, also, the preface that Cavaillès, Aron and Lautman had given to their series for Hermann.

17 What does the 'autonomy' of thought mean here? It means that reason and writing obey the same law, which is their own.

18 Milner & Peden 2012: 243–244. Translation modified. Original French accessible at: http://cahiers.kingston.ac.uk/interviews/milner.html.

19 We are speaking here about Cavaillès's philosophical practice. But it is worth noting that this aspect of his work sits in a long tradition in French thought—in which alongside Cavaillès and Lautman, we could include the names of other 'philosophers' such as Alain Badiou, Gilles Châtelet, Jacques Derrida, Quentin Meillassoux, but also 'artists' such as the Comte de Lautréamont, Stéphan Mallarmé, Paul Valéry, the OULIPO collective—that has reflected deeply on the centrality of inscription to the nature of mathematics, and which has often taken mathematics to be purest form of writing. For surely mathematics is the exemplar of a practice in which a whole 'rational concatenation' can depend on or be shifted by a single inscription, by an index or a bracket?

> For us, reflection begins at the level of specific difficulties, arising from effective research, and subject to the sanctions of failure.[20]

On my understanding, from the standpoint of *LTS,* this can be reread as a claim about the intelligible as such, not just about epistemological practice. What is retained from the earlier focus on 'experience' qua 'experiment' is that, in order to be *specific*, all rational work must be 'subject to a possible failure', and in this sense on the level of writing (that is, of philosophical discourse), the 'genre' proper to the intelligible is the *essai*.[21] This is coherent with the nominalist thesis that we shall consider below: to attempt a general theory of science is to abandon rationalism, for reason, in its necessary development, takes the form of specificities, not generalities. Thus the way out—partial, *tentative*—of the dilemma of trying to avoid an *a priori* theory of science without falling into a rapture of the extra-discursive is to hold that writing is the inverse of theory.

This will be our guide for what follows. In attempting to limn the 'positive doctrine' in *LTS*, my aim is not to hypothesise some unwritten doctrine in the 'Cavaillèsian' corpus, but to give an account of the written doctrine. I will thus take leave of Cavaillès the figure, and give some indications of how I believe *LTS*, the text, can be read. This does not mean that *LTS* is the only text we shall be concerned with here. It should be acknowledged that one of the central innovations in *LTS* concerns the invention of a paratactic style. The work's signature lack of rhetoric, the absence of passages intended to convince or even orientate the reader, is perhaps why it holds such a fascination. Indeed, the Cavaillès of *LTS* seems to have decided to submit his writing to a norm of *ellipsis*. This is not just a matter of concision, even if on a first reading one has the impression that Cavaillès has used two words wherever three would have sufficed. It also bears on the repeated erasure of the subject of clauses, in a manner that can at times seem deliberately ambivalent, but which I take to be constitutive of the text's 'ethics' (*ellipsis as ethics*, this is a fine slogan for the late Cavaillès).[22] And a part of this ellipsis—now in a more extended sense—concerns the text's *referents*, which often take the form of implicit citations.

20 Lautman 2006: 236/Lautman 2011: 196. Translation modified.

21 That is, as both literary genre and as 'attempt'.

22 To give a pertinent example which will be discussed at greater length below, the Thesis on the Intelligible (*LTS* 35-36) concludes with a stark assertion:

> *Abstraire de la sorte n'est pas fixer l'essence mais arrêter.*
> To abstract in this way is not to fix the essence, but to halt.

All norms of philosophical grammar ask that a subject be added to this sentence, or else we ask: what is it that is 'halting'? The essence? But Cavaillès does not write « l'arrêter », nor « s'arrêter ». To abstract in the way that is being critiqued is simply to… halt. *LTS* abounds with cases such as this.

Whether this latter form of ellipsis is conscious or simply an artefact of the text's very particular conditions of composition is by the by; this is how the writing functions. To treat *LTS* as a text thus also means to reckon with the history of words. Which brings me to my third point.

§III.1.iii. On the Tradition in Which to Read *LTS*

As Bachelard states in his preface to the 1960 re-edition of *LTS*:

> This book, which throughout assumes acquaintance with an entire mathematical and logical culture, is nevertheless a book that lies very close to the preoccupations of the philosopher.[23]

What Bachelard, however, could not so easily have seen is the extent to which *LTS* assumes a rather peculiar *philosophical culture*, to which Bachelard himself also belonged. This was the culture of post-Brunschvicgian 'rationalism', with its highly particular admixture of Platonic, Cartesian and Kantian sources. And what gave the unity to this 'culture'? We might say it was the conjunction of a *norm* with a received history of philosophy, each grounding the other. Here, it is useful to refer to the critical category of 'mobilism' advanced by Julien Benda—a figure from the generation previous to Cavaillès—to characterise the French philosophical field of 1945, i.e. more or less simultaneous with the composition of *LTS*.[24] Benda identifies an axiom of French philosophy, established some time in the late 19th century, at the threshold between the 'old' and the 'new' rationalism, as something like the following normative evaluation: that which moves is superior to that which does not. This 'mobilism' was in turn grounded in a characteristic set of philosophical references, which varied in their weight for the different authors in this milieu, but almost inevitably involved some index to a synthesis of Cartesian and Kantian themes, orientated around an account of the 'productivity' of thought. We shall have occasion to consider the appropriateness of Benda's 'mobilist' label to *LTS* in more detail below (see §III.4.ii), but for now I only wish to note that in Cavaillès's inversion of Brunschvicg's doctrine—that is, in the move from the immanence of the intelligence to the immanence of the intelligible—there is a common problem at play for this conjuncture in French philosophy, summarised well by Dr. Pol Le Cœur's comments on the relationship of his brother and fellow resistance fighter Charles Le Cœur's unwritten work to Cavaillès's own writings:

23 *LTS* iv/*OC* 476.

24 See Benda 1945, republished in Benda 1950.

> Both of them go beyond the problem of knowledge, in order to reach the problem of the creation of the intelligible.[25]

It is with respect to the thinking of this problem of the 'creation' or 'production' of the intelligible that *LTS* fundamentally intervenes, and this must be assumed in order to understand the play of citations it implicitly and explicitly invokes. In particular, the confrontation one finds there with a field of 'post-Kantian' epistemology bifurcated into 'logical positivist' and 'phenomenological' options can give the reader the impression that Cavaillès is attempting to take a position within that enumeration, unless they understand from the start that this confrontation is being articulated from the standpoint of problems that come from the 'mobilist' context of turn of the century French philosophy, not 'from' logical positivism or phenomenology.

The essay is thus extremely difficult to understand without taking into account this tradition and its signature problems. Some of this context has just been given with respect to what I called the 'problem of contemplation', as well as in the Introduction (§B.1.). But nowhere is this implicit context more striking than in the philosophical stakes attached to the concept of 'thought' in Cavaillès's exposition, that is, in the assertion that the paradigm and the thematic are characteristics of the 'essence of thought'. It is a condition for analysis of the discourse *LTS* will develop on this point that we understand from the beginning that the reference to 'thought' here is already at the antipodes from 'consciousness'; that, to cite a formulation made by Emile Bréhier with respect to Plotinus, which we shall return to put in context later,

> Thought itself does not think.[26]

A fortiori, 'thought' as it is intended here has nothing to do with empirical cognition. Rather, what is at stake for this philosophical culture is a robustly metaphysical and idealist conception of thought as the name for the movement that produces all that is thinkable; this thought cannot itself be grasped, but can only be known through its products, which must, precisely, be conceived *as* products. This equation between 'thought' and 'production' is something like an absolute assumption of the subset of French philosophy to which Cavaillès belonged, and had already been present at least as far back as Brunschvicg's doctoral dissertation, *La modalité du jugement* (1897), in which he presents the central question posed by Cartesian epistemology as follows:

25 Ferrières 2003: 60.

26 Bréhier 1928: 179/Bréhier 1958: 189.

> Descartes does not distinguish between the spontaneous development of thought in the mind of the mathematician and the systematic exposition of science in the writings of mathematicians. This exposition [...] consists in laying out the products of thought for the outside world, and it therefore presupposes that thought has already applied itself in a very particular manner to grasping these products and becoming conscious of them. *Consciousness thus pertains to the products of thought, not to production, which is thought itself.* [...] If [drawing on Leibniz] it is true that the progress of knowledge is due to a synthetic activity, we may well ask whether it is not the case that this activity is unconscious, not by accident, but by essence, i.e. whether this activity is not such that reflection can never immediately reach it in its intimate reality, but can only interpret it after the fact and in a roundabout way, *through analysis of its externalised products.*[27]

According to this received history of philosophy, then, there is first, a relation of identity between 'thought itself' and 'production itself', and, second, a methodologically prescribed formal difference: in order to 'think thought', one has to take the disjunction producing/produced as primitive. Now, as can perhaps be intimated from the conjunction here of Bréhier's Plotinus with Brunshcvicg's 'Cartesian' (cum Leibnizian) account of the 'unconscious' progress of the impersonal intellect, what is at play here is something like a negative theology of science. In making these citations the background to my interpretation of *LTS*, I do not intend to imply that it should straightforwardly be read as adopting such a negative theological method, nor—and this will be a crucial point of contestation—an idealist 'metaphysics'. But I do want to insist that in the equation of 'thought' with 'intelligible concatenations' which *LTS* announces, and in the stratification therein of thought so-conceived into its effectuating and effectuating poles, it would be understood by a contemporary reader of the text that *this* is the presumed theory of thought that is being reckoned with.

Why do I insist on this so firmly? Because it is only in light of these assumed references that we can understand one of the crucial nodes of the section on the paradigm and the thematic, which is that Cavaillès first indexes the notion of the paradigm to a specific citation of Plato (that is, to the 'second hypothesis' from the *Parmenides*), and then indexes the thematic, albeit only implicitly, to a citation of Spinoza's *Treatise on the Emendation of the Intellect.* These are very much treasured texts by Bréhier and Brunschvicg. I note this not to bring us back to a genealogical mode, but only to signal how we shall now proceed. The difficult situation that any interpretation of *LTS* should address is that we have an extremely condensed text, written for the most part without references (for extrinsic reasons), which expresses its 'own' doctrine concerning *the form of production as such* in the rigours of its writing, but

27 Brunschvicg 1897: 101–102. Emphasis mine.

which is nonetheless constantly doubled by references to a highly polemical received history of philosophy, which is not explicitly addressed, but is rather present in the form of a lexicon that is only implicitly indexed to its sources.

Whence the strategy that I shall pursue in the remainder of this chapter, in three rather distinct sections. First, I shall give a general account of what I understand the 'doctrine' of the paradigm and the thematic be, through a commentary on the key passages in that elaboration. Then, we shall approach the same material from another angle, that is, from the side of its intended indexes: from Plato to Spinoza.

§III.2. The Paradigm and the Thematic — Statement of the Doctrine, or, the *allure* of Science

To state matters, then, in very general terms, what *work* are the concepts of the paradigm and the thematic doing in *LTS*? We do not have space here for an *ad litteram* commentary on the pages in which the notions are advanced, so I restrict my comments to four central points.

§III.2.i. Relation of the Paradigm and the Thematic to the Earlier Doctrine of Idealisation and Thematisation

The notions that we find under the name of 'the paradigm' and 'the thematic' in *LTS* are in the first order a proposition concerning how to read the history of mathematics. They are a restatement of the opposition that had previously been developed between 'idealisation' (or 'generalisation') and 'thematisation' as two modalities of mathematical abstraction, which was present as a distinction in both the doctoral studies (i.e. in the conclusion to *MAF*) and as a core part of the doctrine of mathematical experience. In that context, the disjunction concerned specific working practices in 19th and 20th century mathematics, principally indexed to Dedekind (that is, the concept of the 'chain of operations' as an account of progressive generalisation of object-operation domains) and Hilbert. What is new in *LTS* is that the concepts are being posed in much more general terms, now tied to an account of 'the essence of thought', and are placed in a longer philosophical lineage.[28] It thus worth recalling at the outset the 'effective mathematics' to which these notions are indexed.

With respect to 'idealisation'—that is to say what will become the characteristic of 'the paradigm'—the orientating notion at stake is arriving at a form by replacing objects with vari-

28 I am influenced on this point by Sébastien Maronne (overheard conversation).

ables. To make recourse to the cliché, the moment of transition from a concrete formula (7 + 5 = 12) to a general statement of that formula in terms of any object whatsoever ($a + b = c$) is thus an 'idealising' operation. More broadly, this kind of abstraction is indexed by Cavaillès to any invocation of 'ideal elements', such as in ideal number theory in the style of Kummer and Hilbert. These are the terms in which the notion was presented in *MAF*:

> We have already seen the triple role of generalisation, under the different aspects of descriptive definitions that it generates (such as the addition of ideals, or complete or partial axiomatisations): [1] the liberation of operations from conditions that are extrinsic to their accomplishment, [2] the dissociation or identification of accidentally united or separated procedures [*processus*], and, finally, [3] the positing of new objects as the correlates of operations now recognised as autonomous.[29]

Heuristically, then, 'idealisation' involves the production of a form through the autonomisation of operations, that is by rendering them more 'general' with respect the space of objects on which they bear. The fundamental form of abstraction at stake here is precisely the reciprocal 'generation' of operations from objects and objects from operations that Cavaillès had extracted from his analyses of Dedekind's *Habilitation* address, i.e. the schema of 'object-operation duality'. In *LTS*, few new mathematical examples are given, so we must assume the same practices to have been intended. In terms of philosophical referents, this kind of abstraction is now indexed to a particular 'paradigmatist' reading of Plato, and to Leibniz's *ars combinatoria*.

Thematisation, by contrast, involves the production of what we could call 'meta-theoretic' notions, that is, a 'transversal' approach to operations, which allows them to be classified as 'types' of operation. This is why Cavaillès in *LTS* refers to the thematic as a *liaison-type*, as opposed to the paradigm, which is a *liaison-acte*. An elementary example here is the notions of associativity and commutativity (and thus the possible classification of 'systems' into the associative and the non-associative, etc.). In *MAF*, Cavaillès had described thematisation as 'the transformation of an operation into an element of a higher operative field, for example in the topology of topological transformations (essential in a general sense to group theory).'[30] In the section of *LTS* on the thematic, exactly the same example is given, along with a few others, such

29 *MAF* 172/*OC* 178. To this, Cavaillès appends a footnote directing the reader to his respective analyses, earlier in *MAF*, of Dedekind (principally Dedekind 1888) and Hilbert (specifically the use of ideals in Hilbert 1926), and with respect to the latter he explicitly draws a connection to the method in Dedekind 1854, and cites exactly the same example (i.e. the abstract law of exponentiation) that was used in his analysis of that text earlier in *MAF*: 'The addition of ideals substitutes, in the place of primitive objects, a system of symbols that are both the starting point and result of operations that are now defined uniquely by their formal properties ($a^x \cdot a^y = a^{x+y}$ for raising of powers, for example). These are the successive generalisations studied by Dedekind in his Habilitation address.' *MAF* 97/*OC* 105. See Introduction §B.2. for an elaboration on this point.

as the general theory of operations as it developed from Hankel and Grassmann through to the logical calculuses schematised by Boole, Schröder and Frege, and the descriptive definition of the Lebesgue integral (we will return to the latter).[31] In terms of philosophical referents, even though the terminology of 'thematisation' was initially drawn from Husserl, the primary index is now to Spinoza, specifically to the 'idea of the idea' from the *Emendatione.*

In sum, what is 'novel' in the notions as we find them in *LTS* is not so much the mathematical examples given, as, first, the broader conceptual history in which they are now placed, and, second, the fact that they are now related to each other in an explicitly temporal logic: the thematic *operates on* or is *posterior to* the paradigm. The first point will occupy us in §III.3. and §III.4.; we shall now be concerned with the second.

§III.2.ii. Why 'Effectuated' and 'Effectuating' Concatenation?

My reading of the paradigm and the thematic doctrine emphasises that the two concepts are respectively assigned to the positions of the 'effectuated' and the 'effectuating'. This is a fine point of vocabulary, as this identification is not stated as such in *LTS*. The terminology of the effectuated and the effectuating is rather a general scheme that I extract from the disjunction elaborated in *Transfinite and Continuum* between *procès effectué* and *procès effectif* (with the latter term, as I endeavoured to show in §I.3., being equivalent to *procès effectuant*).[32] With respect to *LTS*, what is at stake is the conceptual function of a lexical distinction, for throughout the text the grammatical difference between the active and passive form of concepts (that is, their form as producing and their form as produced) structures the exposition given by Cavaillès. Thus if in *TC* we find the idea that the disjunction between *effectuated process* and *effective (or effectuating) process* is necessary to the internal structure of 'effective mathematics', *LTS* puts this scheme to work.

It is in this sense that I take the effectuated and the effectuating as the general name for a series of disjunctions that suffuse *LTS*. In the first order, this concerns the concepts now un-

30 *MAF* 177/*OC* 185. Indeed, example of group theory is clarifying for the concept of the thematic: the way in which the group theoretical '+' (or '*')—as defined by its properties of i) associativity, ii) presence of an identity element, and iii) presence of an inverse element—characterises both, say, the symmetrical rotations of a square in a plane *and* the '+' of arithmetic (under certain conditions), is an elementary case of the kind of 'thematisation' of the operation about which I understand Cavaillès to be speaking. For an application of the concept, along similar lines, to the definition of 'morphism' in category theory, see Szczeciniarz 2021: 78–81.

31 *LTS* 31-32.

32 See Chapter I for an extended elaboration on this point.

der the discussion, the paradigm and the thematic, which are respectively identified with *posited meaning* [*sens posé*] and *positing meaning* [*sens posant*], and then with *the operation as operated* [*opération en tant qu'elle est operée*] and *the operation as operating* [*opérante*].[33] They are also associated—in a move that has great significance given that, as I am arguing, what is at stake here is a proposal concerning the 'creation of the intelligible'—with the 'created term' and 'the way in which the term was created'.[34] The disjunction is equally central to the critiques developed in the text, where various figures will be taken to task for failing to observe it. Thus a central problem with the Kantian theory of science concerns the fact that it tries to 'insert' *condition* and *conditioned* into a single 'system' (a point that is repeated almost verbatim with respect to Leibniz), and the concluding critique of Husserl in part turns on the way in which phenomenology is forced to posit an absolute coincidence between *constituting constitution* [*la constitution constituante*] and *constituted constitutions* [*les constitutions constituées*].[35] Suffice to say that at least at the level of philosophical grammar, the disjunction at stake here is essential to the positive account developed in *LTS*. Insisting on said disjunction would further have been an immediately recognisable gesture in Cavaillès's milieu, as the same point is central to many other polemics in that 'mobilist' context, and would of course broadly have been understood as having roots in the Spinozist division between *natura naturata* and *natura naturans*.[36]

33 *LTS* 31–32. One of course wonders whether Cavaillès, who was deeply familiar with Lutheran theology, intended the latter coinage to evoke the distinction between *ex opere operato* and *ex opere operantis*.

34 *LTS* 30.

35 *LTS* 6; *LTS* 30; *LTS* 65. Granger assimilates this critique of Husserl to a comment made by Cavaillès in an exposition of 22nd April 1942 on the subject of 'Time and Tragedy', just prior to the composition of *LTS*, in which he criticised the error of confusing 'representing history [*histoire représentante*]' with 'represented history [*histoire représentée*]'. See Granger 1996: 572.

36 There are numerous examples of the degree to which this lexical scheme was common currency in the milieu from which *LTS* emerges. Perhaps the most pertinent cases concern a running polemic in this period about the nature of reason, that is, whether it should be seen as something 'constituted' or 'constituting' (the correct response of course being: both). André Lalande extensively developed an opposition between *raison constituante* and *raison constituée* in his teaching from 1909 onwards, ultimately schematising the distinction in Lalande 1948, and in the *Vocabulaire* entry on *raison* we find a debate between him and F. Mentré over whether that terminology is preferable to the opposition between *raison raisonnante* and *raison raisonnée* (see Lalande 1980: 883–886). It is in the same discursive context that Brunschvicg inveighs against 'reasoned' reason as merely '*reasonable* reason': 'The existence of a plurality of geometric systems serves to prove that over and beyond *reasonable* reason [l*a raison raisonnable*], which develops in accordance with experience, there is room in the field of mathematics for a *reasoning* reason [*une raison raisonnante*], which is capable of initiative and fecundity.' Brunschvicg 1912: 518. Polemics of this type circulate throughout the French philosophical field long after the 1940s—consider Jacques Alain Miller's celebrated intervention concerning *la structure structurée* and *la structure structurante* (Miller 1968: 95)—and indeed a preference for the *-ant(e)* over the *-é(e)* could fairly be said to remain an axiom of French philosophy

For the sake of concision I shall throughout refer to the general disjunction here as that between effectuated and effectuating, i.e. between any concept insofar as it is grasped in its form as produced and that concept insofar as it is grasped in its form as producing.[37] Again, schematically, if we accept the effectuating/effectuated disjunction as the general name for this split in *LTS*, the essential thing to note is that the disjunction between the paradigm and the thematic tracks it: the paradigm will consistently be associated with the effectuated pole (qua 'posited meaning' or as 'the operation insofar as it is operated') and the thematic with the effectuating pole (qua 'positing meaning' or as 'the operation insofar as it is operating'). Thus if the whole section on the paradigm and the thematic is in an important sense a direct theorisation of the dialectics of effective mathematical work, the fact that the concepts are introduced in order to characterise the relation between *producing* and *what is produced* as constitutive of 'the essence of thought' is one sense in which they are posed on different grounds to the older idealisation/thematisation pair, even if the concepts bear on related historical mathematical 'examples'.

§III.2.iii. In What Sense is the Paradigm Prior to the Thematic?

Having established this as the general scheme of the opposition between the paradigm and the thematic, we must now ask after their *relation*. That is, in general terms, what is the relation between the effectuated and the effectuating? And here, in a sense, lies the central turn in the work. Recall for a moment how we got here: as I understand it, the fundamental ambition of the positive doctrinal part of *LTS* is to outline a logic of the 'before and the after' that is intrinsic to 'mathematical time', i.e. to 'eternity in Spinoza's sense'.[38] One would expect that having set out to provide such an account of prior and posterior production, and having then formulated the concepts of the effectuating and the effectuated as, schematically, giving the internal structure of this 'production' (that is, the internal structure of concatenation), the effectuating would be positioned as conceptually prior to the effectuated, the producing as prior to what is produced. Yet in an important sense the opposite is the case. Here we should consider at greater length the paragraph cited above in which the notions are introduced, which I di-

today.

37 I appropriate this sense of 'disjunction' as referring to conceptual differences that are *modal* insofar as they bear on one thing expressed in two different ways or modes from Caygill 2022, and thus, at a great distance, from Duns Scotus.

38 To recall the formulations from Cavaillès's lecture of 20th May 1942, with which I closed Chapter II.

vide up for the sake of legibility. The exposition is introduced with the following assertion, which picks up immediately from the exposition on Bolzano which concludes Section I of *LTS*:

> This is what a faithful theory of science leads us to acknowledge: the demonstrable has not been given its proper due. But unity does not mean uniformity: beneath the identity of [its] essence, the demonstrative [*le démonstratif*] may take on many aspects.[39]

It is thus these different 'aspects' of demonstrative science (that is, of proof, or 'the pure theory of rational concatenations') that we shall now attempt to distinguish. The first is what will be called the paradigm, or *liaison-acte*:

> [A] zone of singularities wherein what is proved adheres to the proof to the point of being indistinguishable from it, characterising *a unique moment of science*.[40]

And the second is what will be called the thematic, or *liaison-type*:

> *[T]he set of concatenations proper*, which, although they only impose themselves to the extent that they effectively prove something, and thus cannot be renewed in their entirety, nonetheless *display commonalities of type* from group to group, which is the mark of the unity of movements and which can be manifested in the abstract.[41]

Finally, the relation between these two 'aspects' of the movement of proof is stated, i.e. the initial definition with which I began this chapter:

> Thus there appears the notion of logical form, a notion indispensable to a theory of proof. The process of separation is twofold: longitudinal [that is, paradigmatic], i.e. coextensive with demonstrative concatenation, and vertical [that is, thematic], i.e. inaugurating a new system of connections which uses the old one as its *basis*, and no longer takes the old system as a *stage* that a movement has passed through, but rather makes *the manner in which it is currently moving* [*son allure actuelle*] its object of reflection. In each case what manifests is a constitutive property of the essence of thought—or of intelligible concatenations: the *paradigm* and the *thematic*.[42]

Here we can note something rather confusing in the initial posing of the opposition: immediately subsequent to the designation of the thematic (i.e. the 'vertical', effectuating movement) as 'the set of concatenations proper', the paradigm (i.e. the 'longitudinal', effectuated movement) is designated as 'demonstrative concatenation'. This leads one to ask, is demonstrative concatenation, then, not 'concatenation proper'? This would, in terms of the lineage of the term, be quite a startling claim on Cavaillès's part! For in the canonical, Cartesian, sense, con-

39 *LTS* 26. NB that I have endeavoured to consistently render all forms of the term *démonstration* by 'proof' (and variants), but being forced to coin *provative* seemed a bridge too far.

40 *LTS* 26. Emphasis mine.

41 *LTS* 26. Emphasis mine.

42 *LTS* 26-27. Emphasis mine, except for the words 'paradigm' and 'thematic', which are italicised by Cavaillès.

catenation is intrinsically bound to proof, that is, a chain of reasoning. It is the distinction on which I have been insisting between inter- and intra-theoretical concatenation that explains this puzzle. Inter-theoretical concatenations are the set of 'concatenations proper' that use intra-theoretical 'demonstrative concatenations' as their 'basis'. It is thus tempting to chalk the reference to 'concatenations proper' up to a certain infelicity of terminology. Indeed, soon after this, the more substantive elaboration on the thematic will be introduced as follows:

> *Thematisation takes the concatenation as its starting point*, but this time grasped in mid-flight, on a trajectory that moves toward meaning. Thought is no longer directed towards the created term, but sets out from the way in which it is created, so as to discover its principle *by way of an abstraction that is of the same nature* [as the paradigm], but this time directed transversally.[43]

This is close to unintelligible if the 'concatenation' which the thematic takes as its starting point (i.e. the paradigm) is not a 'proper concatenation'. However, what must be underlined is that in stating that 'the set of concatenations proper' refers to the *inter-theoretical level*, Cavaillès can also be read as affirming precisely the methodological standpoint that is orientating for *LTS*: concatenation 'proper' concerns mathematical time, that is to say, the movement between different historically constituted mathematical theories; what was referred to in the Sorbonne lectures as the 'concatenation of systems' qua the 'traversal of problems'. This also explains in an elementary way why 'concatenation proper' takes 'demonstrative concatenation' as its 'basis': the trans-theoretical dimension of concatenation can obviously only be asserted on the basis of certain already existing theories, established at different historical moments. We also begin to see something here of the methodological sense in which 'effectuating' concatenations can only be grasped on the basis of already 'effectuated' concatenations: it is obvious that if the role of thematic concatenation is, in part, to grasp 'commonalities of type', it can only do so starting out from at least two different prior concatenations, or else there would be no difference across which common characteristics could be established. Finally, however, it should be noted that although there is a fundamental distinction at stake here between the different 'levels' (intra- vs. intra-theoretical) or 'directions' (longitudinal vs. transversal) of concatenation, both paradigmatic and thematic concatenations are 'abstractions of the same nature'.

How are we to understand this claim that the paradigm and the thematic are in some sense 'of the same nature'? Well, first and foremost that they are both concatenations. But they

43 *LTS* 30. Emphasis mine.

are also both *movements*, whence the use of processual language to describe each side of the opposition. This is in fact central to the doctrine being expressed, which is that the opposition between the effectuated and the effectuating is not *simply* that between the static and the moving; it is more 'dialectical' than that. The paradigm is a 'unique moment' or 'stage' of the movement of science. The thematic, by comparison, is reflection on the way in which that movement is moving. This at least is the—perhaps contentious—translation that I propose for the formulation cited above that the thematic grasps concatenation *dans son allure actuelle*. One could of course opt for something plainer in translating the phrase, such as 'current shape' or—as Mackay and Peden have it—'current configuration'. Such formulations, however, miss the modal nature of the disjunction at stake: the paradigm is characteristic of the movement or production of mathematics *insofar as* it is grasped as a moment of movement; the thematic is that same movement or production *insofar as* what is apprehended is *how* that movement is moving right now; but what is at stake here is a one movement, for, as we will find later in the essay: 'it is the entire body of mathematics that develops in a single movement across stages and under diverse forms.'[44]

It is crucial to emphasise that paradigmatic or effectuated concatenation and thematic or effectuating concatenation are internal to a 'single movement', because what is at play here is not a polemic against the paradigmatic, but an argument that it is insufficient *if* taken to be the only modality of abstraction. Indeed, as we shall see below, this is what I take to be the motivation for the choice of the Platonic terminology of 'the paradigm' to name this form of abstraction, that is, in the choice of this name for the production of abstractions qua the production of general 'forms'. Insofar as the paradigm—or the 'moment of the variable'—characterises the capacity for mathematics to put concrete objects into variation, that is, to create forms into which arbitrary objects can be *substituted*, there is no mathematics without the paradigmatic 'moment'.[45] The error, however, is that restricting ourselves to the kind of 'generalising' abstraction that the notion of the paradigm is intended to name (an error which is, broadly, taken to be characteristic of 'logicism'), would be to ignore the sense in which:

> The previous analysis was incomplete: in the moment when meaning takes shape, a duality appears between *positing meaning* [*sens posant*] and *posited meaning* [*sens posé*], between the signification of an operation in so far as it is *operated* [*opérée*]—i.e. abstracted from the problem whose exigency created its active singularity, and divested of all that was accidental in its starting point through the possibility of substitution—and

44 *LTS* 72.

45 *LTS* 29.

> its signification as *operating* [*opérante*], that is, no longer [only] the meaning of the passage [as such] but that of the *quomodo* of this [same] passage, wherein the result is no longer considered in its essence, however pure it may have become, but in its simple position as a result that results, and hence as an indication of the modes [of operation] that generated it, and on which interest is then focussed.[46]

The reference here to the paradigm as characteristic of the 'passage [as such]', as opposed to the thematic as the '*quomodo* of passage', of course recapitulates the opposition we have been discussing between the 'unique moment' of movement (that is, a 'stage' of movement) and the way in which that movement is moving (that is, in its current *allure*). To repeat: the effectuated aspect of concatenation corresponds to the passage as such, while the effectuating aspect of concatenation corresponds to the *quomodo* of that passage. In associating this scheme with the opposition between the operation as operated (that is, as having produced a form by abstraction from a concrete case) and the operation as operating (that is, bearing on the way in which the operation was able to be operated), Cavaillès has also arrived at a final transformation of the Dedekindian scheme of object-operation duality into something like an internal logic of mathematical history. Time (in the sense of mathematical time) has been inserted into the very concept of the operation. Now what is central to the philosophical point being made here is that the '*quomodo* of passage'—the 'operating', the 'how', the 'in what way'—in which thematic abstraction is interested is something *specific*, indeed its rational value comes from the fact that it is specific in the way that the passage as such is not. This nominalist thesis is central to the structure that *LTS* tries to exhibit with respect to the nature of production, and which will constitute the 'Spinozist' moment of Cavaillès's exposition. In effect, this concludes the statement of the doctrine. In order to explain the force I attach to this point, however, we must first pass through a different, Platonic, hypothesis.

§III.3. 'The Polymorphism of Rational Concatenation' (Platonic Index of the Paradigm)

The three paragraphs devoted to a direct explication of the paradigm (i.e. *LTS* 27–30) are surely a candidate for the most dialectically fraught in the essay. The initial opacity of these pages can be put down to at least two fundamental difficulties.

First, although the tone of these pages is that of a continuous elaboration, the argument here is in fact divided into two distinct parts. In a first pair of paragraphs (*LTS* 27–29), we are

46 *LTS* 31.

presented with an argument concerning the essential contribution of the paradigmatic moment to the movement of science, one which is bound up with a polemic against Kant. This concerns the *putting into variation of the sensible*, and, or moreover, the internal ‘multiplicity’ or ‘polymorphism’ of thought. Then, in the third and final paragraph of the exposition (*LTS* 29–30), we turn to a critique of an exclusive focus on the paradigmatic mode, and the way this leads to an erasure of movement, with the exemplary figure here being ‘Leibniz’, i.e. a Leibniz who implicitly stands for the style of abstraction proper to ‘logicism’ (broadly conceived). These moves within the exposition are not sharply distinguished, and it is easy for the reader to be left rather uncertain whether this is an affirmation of the paradigm, or a polemic against it. It is thus essential to keep these two strands of the elaboration apart, and below we shall treat them in turn.

Second, this entire elaboration is made to turn on a pair of citations from Plato’s *Parmenides*, specifically *Parmenides* 142d–143a, that is, the ‘second hypothesis’ concerning the One. It is thus without doubt that the choice of the name ‘the paradigm’ (in the singular) for what was previously called idealisation or generalisation is intended to invoke Platonic doctrine. However, *what* exactly is being evoked is difficult to discern, for the citations made by Cavaillès do not explicitly mention *paradiegma* at all. Now, the theory of the ‘forms’ as ‘paradigms’ *is* mentioned in the first half of the *Parmenides* (i.e. at 132d), but that has a rather lateral and contested relation to the second half of the dialogue, which enumerates the various hypotheses on the One, and from which Cavaillès draws his citations. Further, Cavaillès is very clearly against a ‘naive’ reading of ‘Platonism’ qua an affirmation of mathematical forms as ideal ‘abstracta’ (i.e. in the sense of so-called ‘Mathematical Platonism’).[47] So what is the function of this Platonic reference?

I propose a—perhaps rather blunt—instrument to hack through this thicket of difficulties. It is that citing *this* hypothesis invokes particular points of Neoplatonic doctrine, that is, tied to the association of the second hypothesis from the *Parmenides* with the position of the Intelligence (or the ‘second hypostasis’).[48] In a sense, this part of my argument is a ‘likely

47 That is, ‘Mathematical Platonism’ in the sense outlined in Bernays 1935, which is perhaps the *locus classicus* for analytic philosophical literature on the topic, especially following its republication as Bernays 1984. Cavaillès certainly knew this article, and I take the sense of ‘Mathematical Platonism’ which Bernays describes to be the object of Cavaillès’s criticism of the ‘recourse to Plato’ in ‘non-critical ontology’ his concluding comments to *MAF*. See *OC* 176.

48 In placing so much emphasis on the choice of this particular citation from Plato, I am influenced by Regnault 1968, which to my knowledge is the only extended elaboration of the importance of the *Parmenides* in *LTS*. Regnault’s strategy, however, diverges from mine: starting from the same Cavaillèsian clue concerning the

story'; its value is not that it explains Cavaillès's 'intentions' (in which case it would be rather doubtful), but that if we read the reference to the paradigm in Neoplatonic terms—in particular, through Plotinus as interpreted by Bréhier—it helps to clarify a broader conceptual logic at work in *LTS*. First, it provides a way to understand the identification of *unity, movement,* and *diversity* which determines much of the conceptual architecture of the essay, whilst shifting the problem of the 'unity of science' away from the Kantian stakes of that problem, at least as Cavaillès constructs them. Second, it explains a different series of identifications that we have already signalled as being crucial to the critical project of the work, i.e. the identification of non-simultaneity with non-univocity or heterogeneity, which is what underwrites Cavaillès's conclusion that the intrinsic variegation of science both founds its 'necessarily progressive' character and means that it cannot be encompassed by any *a priori* theory of science. Further, the fact that variation is the condition for this variegation is another sense in which the paradigm, as discussed §III.2., 'enables' the thematic.[49] Finally, the hypothesis of a Plotinian background to the *critique* of the paradigm (that is, of exclusive focus on the paradigm) places Cavaillès within a line of a strategic Neoplatonism, the central 'doctrine' of which could be said to stem from Plotinus's argument that 'the intelligibles' (that is, the Ideas or the Forms) are not prior to but rather produced by 'the intelligence', a position which is articulated in terms of a polemic against a 'paradigmaticist' (or 'exemplarist') reading of Plato.

Given the importance I assign to this structurally 'Plotinian' interpretation of *LTS*, I approach the text as follows. First, I will discuss the initial statement of the Platonic hypothesis in *LTS*, i.e. the *affirmative use* of 'the paradigm'. Second, I will give a general exposition on how I take this to be connected, in an implicit manner, to a particular Plotinian 'doctrine'. Third, I will discuss how related 'Neoplatonic' themes are present throughout *LTS*, albeit implicitly, most importantly for our purposes in Cavaillès's comments on Bolzano—and thus the Thesis on Evidence. Finally, we will return to see how this interpretation helps us to understand the *critical use* of the paradigm, and with it something essential about the polemical stance that *LTS* is taking in its conjuncture, as expressed in the Thesis on the Intelligible.

citation of the second hypothesis in the context of elaborating a theory of science, Regnault proceeds to produce a full enumeration of the eight (or nine) hypotheses of the *Parmenides* as a 'matrix' of all possible relations between 'science' and the 'theory of science'. Regnault does not however assign one or the other of the 'hypotheses' as Cavaillès's 'own' position. For a brilliant recent use of Regnault's 'matrix' in the field of psychoanalysis, see Mäki 2024: 167–210.

49 As I will briefly discuss below, this can also partly be understood as an assertion as to the *dianoetic* character of mathematics, to adopt another piece of Platonic *argot*.

§III.3.i. The Affirmative use of the Paradigm. Why Cite *Parmenides* 142d–143a?

The exposition on the paradigm begins with a stark assertion: '[t]he *paradigm* is characteristic of actualisation [*l'actualisation*].'[50] Given the contours of my interpretation of *LTS*, I take 'actualisation' here to correspond to *effectuation*, specifically in its modality as *effectuated*. Now, the 'actualisation' being invoked here is the effectuation of a form 'from' the sensible, that is, the passage from a concrete (particular) instance to a general schema through the suppression of what was specific in the initial situation; it is the transition from an instance to a generalisation of that instance, which can then stand for any other arbitrary fulfilment of the form; thus the sense of actualisation intended here is the opposite of 'participation', insofar as we might understand that to refer to the 'realisation' of a pre-existent form through particularisation.[51] What is thus being 'actualised' or 'effectuated' here is the form itself, and the means of production of this form is, as Granger glosses the paragraph we are discussing, 'the mobilisation of a relation through a schema that has been detached from its terms.'[52] Thus, as Cavaillès expands on his comment:

> The *paradigm* is characteristic of actualisation. Not an actualisation *hic et nunc* in lived experience [*dans le vécu*] from which the extrinsic has been eliminated in thought's simple taking possession of itself, but an actualisation necessitated by the meaning of that which is posited, namely a relation which, as such, is only affirmed in the singularity of the realisation of the concatenation, but which treats this singularity as something arbitrary [*quelconque*], and thus eliminates it even as it posits it, and thereby reveals an internal principle of variation. The same liberating decoupling [*dissociation*] of meaning is operative from arithmetical reasoning concerning whole finite numbers to the most abstract concatenations.[53]

50 *LTS* 27.

51 We can note a comparison with Victor Goldschmidt 1946 study *Le paradigme dans le dialectique platonicienne*, which is orientated by a central ambiguity in the word 'paradigm' as it is deployed throughout the economy of Plato's *Dialogues*: 1) On some occasions, we find the Forms defined as Paradigms with respect to sensible things (i.e. Forms as the 'models' of things); 2) On others, we find sensible things defined as Paradigms of the Forms (i.e. as examples, or particular instantiations). Goldschmidt is explicitly concerned with the second sense of paradigm, whereas, as just noted, Cavaillès is concerned with the first, i.e. the paradigmatic relation in *LTS* goes from things to Forms, not from Forms to things. Despite this divergence, both Cavaillès and Goldschmidt can be said to have returned to Plato's theory of the paradigm at roughly the same moment in order to ask the question: *what is the role of examples in science* (i.e. in 'method')? See Goldschmidt 1951 for an extension of this analysis to Descartes.

52 Granger 1988: 73. Granger's exposition of the paradigm/thematic disjunction from pp. 70-81 of *La connaissance philosophique* is perhaps the richest yet produced, and my interpretation is firmly indebted to it.

53 *LTS* 27.

This is the ineliminable function of the paradigm for mathematical abstraction: it is actualisation or effectuation qua the revelation of an 'internal principle of variation', that is, a liberating 'decoupling' from the immobility of the particular instance. In a sense, the picture of abstraction here is quite classical; it is the conception of abstraction as despecification. For all that *LTS* will criticise the sufficiency of this kind of abstraction, it is nonetheless clearly a necessary condition for mathematical work. It is immediately following these lines that we find the first citation of the Platonic hypothesis:

> Since 'it is' is said of this *one* which is, and 'one' is said of this *being* which is one, and since 'being' and 'one' are not the same thing, but both belong to the same thing [...] it follows necessarily that 'the one which is' is a whole, and 'one' and 'being' are its parts. (142d)[54]

Then, having noted that the independence so achieved is the 'source of plurality', Cavaillès inserts the second half of the citation, i.e. the conclusion of the second hypothesis from the *Parmenides*:

> In this way, then, 'the one which is' must be unlimited in multitude. (143a)[55]

Cavaillès then proceed to a striking conclusion, in which he alligns this use of the Platonic hypothesis with a critique of the Kantian doctrine of synthesis:

> It is this alone that makes polymorphism within one and the same rational concatenation possible [*la polymorphie dans l'unique enchainement rationnel*]: these successive fractures of independence, each of which detaches from what came before the imperious outline of what will necessarily come after in order to surpass it. The synthesis that Kant detects in thought does not depend upon some manifold supplied to it from elsewhere or which is different from it, but only upon thought itself, which, in its moments and its progression [*son progrès*], is a multiplicity: what is unified is not preliminarily given as a manifold [...] in fact, it is the very unfolding of acts themselves [...]. In this way, synthesis is coextensive with the engendering of the synthesised.[56]

Now, why cite the second hypothesis of the *Parmenides* in order to stage a confrontation with the doctrine of synthesis? We should first recall the general sense in which Cavaillès critically takes up an argument from Brunschvicg by identifying concatenation with synthesis, i.e. by understanding the Kantian innovation as, first, the transformation of the Cartesian problem of the demonstrative chain into that of the synthetic act, and, second, the location of this synthetic act in the historical *progress* of science.[57] It is because we make this identification that,

54 *Parmenides* 142d, cited at *LTS* 27-28. I have cited the Plato passage from Cornford 1939: 137, but modified it slightly in line with the text of *LTS*, for consistency of terminology below.

55 *LTS* 28. Cornford 1939: 139.

56 *LTS* 28.

57 See my comments to this effect in the Introduction, §B.1.

as a corollary, the central 'problem' of the Kantian framework as exposed by Cavaillès in the opening pages of *LTS* becomes that of the relation between *condition* and *conditioned*, which can be thought as the relation between *demonstration* and what is *demonstrated*. Cavaillès presents this at *LTS* 4–5 as a kind of fork. The first option is that the concatenation between condition and conditioned is given in advance in a system, which has the effect of erecting an absolute logic (in Kantian terms: 'General Logic'), prior to synthesis, which consciousness then simply manifests. The alternative is that the concatenation between condition and conditioned is 'realised in a singular fashion each time', but then the only progress at stake in a demonstration is that of the 'concatenation itself' (i.e. of synthesis), and we must abandon both the absolute *a prioricity* of consciousness and the posit of a radically 'external' matter to be synthesised (i.e. the two poles of Kantianism on Cavaillès's understanding).[58]

In light of the central assertion in *LTS* that the effectuated/effectuating disjunction is internal (i.e. immanent) to concatenation (i.e. to its 'progression'), it should be clear that this statement is an implicit endorsement of the second side of the 'fork' just outlined. This is the argument that is resumed at the outset of the elaboration on the paradigm, and we can see how Cavaillès's central claim here—viz. that '[s]ynthesis is coextensive with the engendering of the synthesised'—becomes the thesis that it is not that 'thought' is 'one', and that movement is extrinsic to it, driven by the need to unify a diverse manifold given 'from the outside', but rather that thought itself is already ('in its moments and its progression') a multiplicity, and thus that its movement is intrinsic. This is the first sense we can give to the assertion that has just been made concerning to 'polymorphy of rational concatenation': concatenation must move, and it possesses no single unifying internal form; indeed these two characteristics imply each other. This 'polymorphy' is what is revealed in the 'abolition of singularity' which is characteristic of the paradigm.[59] Cavaillès views these conclusions as a radicalisation of the doctrine of synthesis beyond strictly Kantian bounds.

Insofar as any of this is clear as a criticism of 'Kantian' doctrine (a topic I put aside), why is it a 'Platonic' argument? I suggest that these very conclusions concerning 'the polymorphy of rational concatenation' can also be read as a structurally Neoplatonic corrective to Kant: the claim Cavaillès is making is that the doctrine of the synthetic unity of consciousness is not adequate for conceptualising the internally variegated nature of science, that is, its in-

58 *LTS* 4-5. We can note in passing that this is precisely the fork that was outlined with respect to the Kantian theory of logic in the Sorbonne lectures of 1941–1942.

59 *LTS* 28.

trinsic diversity, which is then identified with unity at a different level, a unity that is in motion. Thus, insofar as Kant can conceptualise the internal movement of science, he betrays it by subordinating it to the pre-constituted formal unity of consciousness. My proposal is that the three characteristics of paradigmatic concatenation found in the paragraphs just discussed—in sum, that the paradigm is characteristic of actualisation, in the sense of the production of forms; that this actualisation creates internal variation or polymorphy; and, finally, that the multiplicity that is revealed through this actualisation does not stem from the extrinsic intervention of some world 'outside' of thought, but from the intrinsic productivity of thought itself—can be be mapped onto a particular reception of Plato and Plotinus. Clarifying this point will require a minor excursus.

§III.3.ii. Short Interlude on 'Platonic' Doctrine

Now to our likely story: could it be that the function of citing *this* particular hypothesis is to recall a twofold argument taken from Plotinus, broadly as it would have been understood through Bréhier's commentaries?[60] I will put aside some niceties here, as my point is not genealogical—although certainly, Cavaillès the figure was familiar with these sources—but rather concerns a structural logic that is at play throughout *LTS.*

Here, we need to bear in mind two pieces of Platonic, or rather, Neoplatonic doctrine. The first is the sense in which the opposition between the first and the second hypothesis in the *Parmenides* is made to stand for that between absolute unity and variegated motion. The second is the way that analysis of the latter side of that opposition—identified with the position of the Intelligence as an infinitely productive movement—leads to a critique of an 'exemplarist'—or 'paradigmaticist'—interpretation of Plato.

α) The 'Second Hypothesis' as the Position of Internal Movement

First, let us recall the context in which the 'second hypothesis' is stated. The *Parmenides* is divided into two parts: a first half which contains a prolonged discourse on the forms, and the second, which applies the method of dialectic to Parmenides's 'own' hypothesis, namely 'that

60 Bréhier arguably defined the study of Plotinus for a generation of French scholars. Cavaillès was Bréhier's student as early as 1924, and even though Cavaillès sided with Brunschvicg in an argument over their opposing views (see the letter of Cavaillès to his sister of February 3rd 1924, Ferrières 2003: 51), he remained an influence, and Cavaillès wrote an extensive review of Bréhier's *La philosophie de Plotin* in 1928 (i.e. Cavaillès 1928). For an extended consideration of Plotinian themes in Cavaillès's early theological writings, see Cortois 2020.

"the One is"'. This gives rise to a labyrinth which depends on the different possible interpretations of the copula, and which ultimately yields eight possible 'hypotheses' concerning the One. Here we need only be concerned with the first two 'hypotheses', the relationship between which in any case broadly gives the scheme of the rest:

1) The One Is [or 'the One that is One'], i.e. the 'absolute' One;
2) The One has Being [or 'the One that Is'], i.e. the 'relative' (or participated) One.

As just stated, the basic conceptual opposition here turns on the interpretation of the copula: either we interpret the hypothesis that 'the One is' as pleonastic—i.e. as asserting that the nature of the One is that it is 'one'—or we interpret it as asserting a *relation* between two terms, 'oneness' and 'isness'—i.e. that the One *participates* in Being, that the one 'exists'. The dialogue then draws out the consequences of these respective hypotheses, in diametrically opposed manners. On the first, the One is found to be *deprived of all predicable properties* (it is *neither* in motion nor at rest, it is *neither* in space nor outside of space, etc.). On the second, the One is found *to bear every possible predicate* (it is *both* at motion and at rest, *both* in space and not in space, etc.).

It is here that the 'Neoplatonic' interpretation comes in, at least according to Bréhier. Bréhier identifies the core doctrine advanced by Plotinus as arising from, first, making *The Parmenides* the central text in the Platonic corpus, and then proceeding to give a substantive interpretation of the various 'hypotheses' by transforming them into the 'hypostases'. The unfolding of the dialogue is then read as an account of the logical ontogenesis of reality *from* the One as it descends into its derived forms. In particular, the section of the text of the *Parmenides* on the 'first hypothesis' is to be read as enumerating the properties (or, rather, the lack of properties) of 'the first hypostasis', that is, The One, and the section on the 'second hypothesis' is read as enumerating (this time positively) the properties of the 'second hypostasis', that is, of the Intelligence. Thus, the crucial difference between the One and the Intelligence is that while the One is subtracted from Being and thus itself completely without either motion or properties, the Intelligence, progressing 'out of' the one, is a principle of pure movement, and generates an infinite plurality of properties. It is in this sense that the descriptions given of the competing hypotheses in the dialogue of the *Parmenides* are taken to provide the pure logical

schema of the hypostases.[61] Thus, as Bréhier's summarises this point in his introductory note to *Enneade* V. 3. 10:

> [T]hought is not a rest, but a progression [*la pensée n'est pas un repos, mais un progrès*]: where there is rest, there is not thought at all; therefore, if the intelligence thinks itself, there must be an internal multiplicity in it […]. Plotinus here calls the *Parmenides* to his aid: it is there that he finds that the 'One that is one' can only be stated by denying itself a multiplicity of predicates (*Parmenides,* 137c–142a; cf. 32–34), whereas the 'One that is' must say of itself: I am this or that, thus admitting in itself otherness, number, and 'many other things', i.e. all the properties Plato demonstrates of 'the One that is' (142a–151e).[62] In this short passage ([V. 3. 10.] x, 32–39), we see how Plotinus makes the first two hypotheses of the *Parmenides* (either the One is one, or the One is) correspond to his first two Hypostases, the One and the Intelligence.[63]

Schematically, we thus have the following opposition:

> 1) The First Hypothesis = The One Is [or 'the One that is One'], i.e. the 'absolute' one = The One = The First Hypostasis
>
> *Characteristics: a) static; b) unique/simple; c) resistant to all predication*.
>
> 2) The Second Hypothesis = The One has Being [or 'the One that Is'], i.e. the 'relative' (or participated) one = The Intelligence = The Second Hypostasis.
>
> *Characteristics: a) pure motion; b) infinitely (internally) multiple; c) possesses (successively) all possible predicates.*

This gives the first essential piece of context for the reading that I propose: that the citation of the second hypothesis of the *Parmenides* in *LTS* is an invocation of the 'second hypostasis' qua the *position* of pure motion and internal plurality accorded to the Intelligence, and that this is the reason why the affirmative use of the paradigm is associated with the 'polymorphy' of 'intelligible concatenation', that is, of 'thought' qua production.[64]

61 For a comment to the effect that Bréhier was responsible for this widespread reading of Neoplatonism, see Emilsson 2007: 3. This interpretation is frequently taken to have been established contemporaneously with Bréhier's reading by Dodds 1928. However, for a critique of an identification between the hypotheses and the hypostases as a frame for reading Plotinus, see Gerson 2016.

62 I.e. the hypothesis cited by Cavaillès.

63 Plotin & Bréhier 1931.

64 I emphasise the term 'position', insofar as we should treat the role of the 'Intelligence' at stake here solely as a formal position characterised by the above-enumerated characteristics, and not *prima facie* as a reference to 'mind' in any more substantive (anthropomorphic) sense.

β) The Critique of 'Exemplarism'

Here, we should cleave a little longer to Bréhier's analysis, for it leads to a striking analogy with aspects of Cavaillès's exposition. Bréhier characterised the Plotinian notion of the intellect as responding to an internal tension in the interpretation of Platonic doctrine concerning the Ideas (or 'the intelligibles'), i.e. as emerging from Plotinus's criticism of an *exemplarist* or *paradigmaticist* reading of Plato. According to Bréhier's reconstruction in *La Philosophie de Plotin*, the Platonic tradition prior to Plotinus left open a twofold question:

> Question 1) Are Ideas external to the Intelligence that contemplates them?
> Question 2) Are Ideas like external archetypes of the things which imitate them? (i.e. the problem of the theory of the paradigm)

On Bréhier's reading, 'traditional' or mid-Platonism—commencing from a too-literal interpretation of the *Timaeus*—answers both of these questions in the affirmative, with the consequence that Ideas are taken to pre-exist the act of the Intelligence.[65] The theory of participation correlatively becomes central to the interpretation of Plato. Contrary to this, on Bréhier's construction, Plotinus reverses traditional or mid-Platonism by answering both questions negatively. The consequence is that with respect the first question, the intelligible(s) are not separate from the Intelligence, but exist '*in* Intelligence', and the knowledge that the Intelligence has of the intelligibles (i.e. of Ideas) is self-knowledge, rather than operating by analogy with sensory knowledge (i.e. it is the knowledge of a subject, and not of objects); hence, 'the intelligible is fused with Intelligence itself.'[66] With respect to the second question, Bréhier reads Plotinus's epistemology as a rejection of what he calls the 'exemplarism' of the naïve mid-Platonic interpretation of the theory of participation.[67] For if the Ideas were really 'external archetypes' then, having reached then, the Intelligence would stop; this would mean the Ideas are 'limit' to the Intelligence.

Taken in concert, these two theses mean that the Plotinian doctrine concerning the Intelligence can be understood as 'the true forerunner of the idealistic doctrines which pose spirit as concrete and substantial reality, affirming itself through itself and independently of things', i.e. those of 'St. Augustine, Descartes, and Hegel'.[68] In other words, it is through an ana-

65 See Bréhier 1958: 98–100, i.e. the English translation of Bréhier 1928.

66 Bréhier 1958: 100; 101. Emphasis mine.

67 Bréhier 1958: 101. NB that the English translation simply has 'Plato's theory of Ideas' for Bréhier's 'l'exemplarisme chez Platon'. See Bréhier 1928: 101.

68 Bréhier 1958: 103. Brunschvicg would of course also be placed in the line of this enumeration.

lysis of the Intelligence which is critically arrayed against 'exemplarism' that Plotinus extracts from Plato the principle of 'constructive dialectic' according to which the potency of thought lies in its intrinsic (and infinite) self-differentiation, a self-differentiation which 'produces' the forms.

> The fact is that there is something else in the affirmation of thought through itself than the affirmation of an empty identity in which every difference disappears. It signifies also that Intelligence is a dynamic principle which cannot be fixed in any concrete and halted form [*forme concrète et arrêtée*]. […] [I]f we desire to look for the motivating force of this dialectic in Plotinus, we shall discover it in the fact that it is impossible for thought to stop at a definite limit whatever it may be. To fix attention upon a specific object of contemplation is to stop thinking. 'If it [thinking] does not progress toward a different state, it will come to a halt, and once halted it will cease to think [*il s'arrêtera, et, une fois arrêté, il ne pensera pas*].' (V. 3.10) As a consequence, total thought, thought thinking itself, is the end of the movement which produces successively the thought of all things.[69]

According to Bréhier's reconstruction, then, the affirmative and critical parts of Plotinus's doctrine concerning the Intelligence coincide. The generative dialectic between the One and the Intelligence is characterised as that between 'total' or 'pure' *thought* (which, as we have seen, 'does not think') and *thinking*, which can only think *insofar as it moves*.[70] In order for this movement to be possible, the intelligible must be 'internally variegated, a conclusion which is inseparable from the argument that the intelligibles (the ideas) are internal to or produced by the Intelligence. The classic statement of this argument is *Enneade* VI. 7. 13., 'How the Multiplicity of Ideas Came to Exist, and on the Good', 31–42. I present the passage in Gerson's English with Bréhier's French interpolated at the crucial point:

> Intellect is everywhere itself. It, then, has a constant wandering. Its wandering is on 'the plane of truth', which it does not leave. It has taken it all into its possession, and has made it in a way a place for its motion; the place is identical with that of which it is the place. This plain is variegated, so that it may traverse it. *Were it not in every respect and always variegated, it would come to a standstill insofar as it is not variegated. If it comes to a standstill, it does not think* [*Cette plaine est variée pour offrir une carrière à* ***parcourir*** *; si elle n'était pas sans cesse et partout variée, l'intelligence s'arrêterait ; et si elle s'arrêterait, elle ne penserait plus*].[71]

69 Bréhier 1958: 103–104. Emphasis mine. Translation modified.

70 I.e., the context for my citation of Bréhier's thesis in §III.1.iii: 'The one is no longer thought directed towards itself, but it is, as Plotinus says quite simply, "thought" […]. But thought is that which causes beings to think […]. Thought itself does not think.' Bréhier 1928: 179/Bréhier 1958: 189.

71 Plotinus & Gerson 2018: 819./Plotin & Bréhier 1938: 84–85. Emphasis mine. I again thank Louise Morelle for drawing my attention to this passage.

Note the order of reasons here. As 'active thought' (i.e. as think*ing*), the Intelligence *must* move; if it came to a halt, it would cease to think. But in order for the Intelligence to maintain this intrinsic and necessary movement, the 'objects' that it contemplates (which are in the last instance internal to itself) must exhibit variegation [*poikilos*]. As a corollary, if the intelligible were smooth (homogeneous), the Intelligence would not find the traction necessary to propel its movement, and thus would not be able to think. Thus the necessary variegation of the intelligible is a consequence of the intrinsic movement of the Intelligence, but according to a very singular order of reasons: the space of the intelligible (the 'plane of truth') *must* be internally variegated *so that* the Intelligence can traverse it, and it is for the this reason that the plane *is* so variegated, rather than the inverse thesis, i.e. that the Intelligence would be characterised by movement *because* 'the intelligible' is variegated 'prior' to this movement. This order is central to the Plotinian account of genesis: the multiplicity of the Ideas ('the intelligibles') is produced by the (self-reflective) movement of the Intelligence, rather than vice-versa. And what is the term which Bréhier uses for the 'traversal' that is enabled by this variegation? *Parcourir*, that is, exactly the term which Cavaillès uses to assign the central object of study for mathematical philosophy in his Sorbonne lectures on Logic: to think the 'traversal of problems [*parcourir les problèmes*]'.[72]

§III.3.iii. Of Movement and Halts

Let us now pursue the idea that it is *this* structural logic that is at stake in the citation of the 'second hypothesis' in *LTS*, in both its 'affirmative' and 'critical' aspects, i.e. as indexing both the necessity of a progressive self-differentiating movement, founded on variegation, and in the refusal of an exemplarist reading of the Forms. What can be seen as a heuristically 'Plotinian' (*après* Bréhier) logic in *LTS* is thus the following set of claims: First, what is at stake in the 'actualisation' of a form is the production of a form, not a fulfilment of the form. Second, this production of forms is the condition of the movement of science, but qua 'products' the forms remain internal to this movement. Third, the very 'liberation' through putting into variation that enables the production of forms has as its double the 'paradigmatist' error, which is to treat 'produced' forms as if they were uncreated, that is, established prior to a movement of thought that would then be cast only as the retroactive apprehension of these forms. This is the error of abolishing the necessary distinction between the effectuated and effectuated, and

72 See my discussion in §II.2.

treating a moment of effectuated science as the eternal form of science. In terms of the 'doctrine of science' (i.e. a theory of scientific epistemology), this corresponds to the fantasy of taking a particular moment in the development of science and erecting it as transcendental and governing for both the past and future development of science. In a word, the becoming of science structurally produces forms which in their very certainty—that is, through their *evidence*—provoke the conditions for the forgetting of the becoming which occasioned them.

I will return to this third 'critical' use of the concept of the paradigm in a moment, but want to focus for a moment on what is affirmed in the first two theses. The notion of the productive movement of science at stake here must be related to another theme—importantly, no less central to Plato's 'second hypothesis'—which is that of the one and the many, or of unity and diversity. In some sense, *LTS* still proposes discourse on the unity of science, but it is not the post-Kantian notion of 'unity' that is at stake here. It is rather a problem of how to think unity-in-diversity that, as I have just attempted to outline, follows a structurally Neoplatonic logic. In a sense, the relation between the *oneness* and the *heterogeneity* of mathematics (and thus of science) can be seen to orientate Cavaillès's whole late discourse. The task of mathematical philosophy is to think mathematics as a whole as an intelligible 'concatenation of systems' which, as discussed §III.2., 'develops in a single movement'.[73] At the same time, as Cavaillès had written in the late fragment 'Mathematics and Formalism', '[e]ach independent part of mathematics possesses its own modes of concatenation, which characterise it.'[74] If, as I have have been arguing *LTS* is concerned with trying to discern a logic of production—i.e. of the before and the after—in mathematical history, it is through treatment of the problem of the one and the many, as found in Plato, that the conceptual categories required for thinking this logic are found.

Nowhere is this clearer than in the long treatment of Bolzano which immediately preceeds the statement of the paradigm/thematic doctrine, and which is the context for Cavaillès's statement of the Thesis on Evidence. Let's recall the dilemma which, as I argued in the Introduction, Cavaillès extracts from Bolzano's theory of science, and which *LTS* is in large part an elaboration on. Having posited science as an 'object *sui generis*', requiring a 'pure theory of rational concatenations', it seems that it is impossible to connect this theory back to 'developed science'.[75] In light of the exposition just given on the Platonic categories at play in *LTS*,

73 *LTS* 74.

74 *OC* 663.

75 *LTS* 21; *LTS* 26. See my commentary in the Introduction.

it is possible to see that in Cavaillès's posing of this dilemma, what is at stake is—bluntly paraphrased—precisely the relation of 'participation' to the problem of the many and the one. As Cavaillès presents matters, Bolzano's claim is that science is unparticipated, i.e. that its unity is not affected by its diverse realisations. But if science is independent and one, such that the 'theory of science' could be—and indeed must be—elaborated prior to examination of the historical development of the sciences, then why does science change, why does it move? The whole dialectic at stake here thus turns on the epistemological status of this claim to the independence of the theory of science. For Bolzano, Cavaillès writes, science is in part defined by its opposition to 'the universe of cultural objects'.[76] While the latter are intrinsically characterised by their participation in sociological phenomena,

> [science's] mode of actualisation is, on the one hand, extrinsic, [...] while on the other hand it demands unity, which is to say that it cannot accommodate an effective multiplicity of singular realisations. There are not diverse sciences [*diverses sciences*], nor diverse moments of a single science [*divers moments d'une science*], and neither is there an immanence of one unique science to a variety of disciplines. Yet these disciplines condition one another in such a way that both the results and the meaning of any one of them, in so far as it is a science, requires the use of others, or their collective insertion into a system. A theory of science can only be a theory of the unity of science.[77]

The resonance here with the characteristics of the 'first hypothesis' from the *Parmenides* as outlined above should be obvious: we move from the assertion that science is unitary to the assertion that, being unitary, it resists all predication. This explains what can seem like the paradoxically negative enumeration of the possible predicates that science could bear here: if there can be 'no different sciences', nor 'different moments of [one] science', nor the 'immanence of one unique science' to a diversity of sciences, have we not claimed that 'science' is devoid of all possible content? What is remarkable here is that Cavaillès proceeds to offer a resolution to this problem by identifying the unity of science with its internal and necessary movement, a movement which produces the diverse modalities of science through 'the spontaneous generation of intelligible elements':

> And this unity is movement [...]. As it is only autonomous progress, a self-enclosed dynamism with neither absolute beginning nor end, science moves outside of time—if by time we mean reference to the lived experience of a consciousness. In science, change means an increase in volume through the spontaneous generation of intelligible elements. […] [A]ny concept or system of concepts, by the very fact of its being posited, at once excludes and requires the Other [*l'Autre*]: in this sense, the representation of an

76 *LTS* 21–22.

77 *LTS* 22.

> absolutely simple infinity of all knowledge is an image that bears no relation to the reality of militant science except that it pushes to the limit a certain property of its movement—the absorption of the anterior by a posterior that both ratifies it [*le justifie*] and in a certain sense eliminates it [*le supprime*]. […] [T]he true meaning of a theory lies not in an aspect that the scientist themselves understands to be essentially provisional, but in a conceptual becoming that cannot halt [*qui ne peut s'arrêter*].[78]

I read this passage as Cavaillès directly puppeteering Bolzano to speak to his own problem. In terms of the Platonic intertext for *LTS*, the point to emphasise is that we have moved from statements that resemble the first hypothesis to an analysis that is redolent of the second hypothesis. Instead of claiming that the unity of science demands that the theory of science be independent of the differential realisation of science in the sciences, and thus 'static', the schema is flipped on its head, and unity is now identified with movement, from which it follows that science will progressively inhabit every possible determination or predicate. Whence the invocation of the characteristically Neoplatonic notion of 'movement outside of time', that is, of a movement that is not characterised by its imbrication with (lived) temporality, but by an internal logical order of the before and the after, an order which precedes and (if we follow the whole Neoplatonic ontogenesis) generates what we perceive as time.[79] Formally, then, it seems that *LTS* in some sense places science in the position of the second hypostasis or of the Intelligence.

As stated at the outset, my analysis here follows the logic of the likely story: if we hypothesise a Plotinian intertext to *LTS*, how would we read that work? In the passage just cited,

78 *LTS* 22-23. I leave aside analysis of the other great Platonic theme in this passage, that of the Other.

79 Indeed, we might say that the very questions which immediately occur with respect to this assertion by Cavaillès—'What can "movement outside of time" possibly mean, if it is not metaphorical?" "What is the medium of this movement?"—are questions proper to the Neoplatonic tradition. As has been most forcefully underlined by Eyjlófur Kjalar Emilsson, the tension between the concept of movement (both internal to the Intellect and in the generation of the Intellect from the One) and the refusal of 'time' is a central problematic in the interpretation of Plotinus: 'What "happens" at the levels of the One and Intellect is outside time so that there can be no question of temporal process here. Nevertheless, the language Plotinus uses to describe the genesis of Intellect from the One is unmistakably the language of events and it is not clear how we are to think about this at all without the terms suggesting a process in time. […] [I]t should be confessed that the last question about the application of temporal notions to the atemporal remains unaddressed and unresolved.' Emilsson 2007: 24. The core of this problem is arguably Aristotelian rather than Platonic, having its roots in Aristotle's assertion in *Metaphysics* Theta 8 that *energeia* [actuality] has 'priority [*πρότερος*]' over *dunamis* [potentiality]. A classic problem here is that of interpreting what 'priority' is supposed to signify if it is not 'temporal priority', i.e. that of clarifying the relationship between priority of consequence and priority of creation; we could say that in Plotinus this becomes the problem of how to think 'genesis' if it is not to be thought as a 'temporal genesis'. For an analysis of this claim of 'priority' in Aristotle, c.f. Beere 2009. I again thank Silvia Mollicchi for insisting on the continued importance of these problems, which, it seems to me, are central to the interpretation of Cavaillès's work.

following this reading highlights another distinctively Neoplatonic theme in Cavaillès's analysis: the problem of the unity and/or diversity of science plays out in a conceptual dialectic between *movement* and the *halting of movement*. In this sense, the doctrine presented in *LTS* fullfils a programme that was latent in Cavaillès's critique of the 'Kantian theory of mathematics' in *Transfinite and Continuum*: '[a]n irreducible intuition would only be a halt, devoid of thought [*un arrêt sans pensée*].'

§III.3.iv. The Critical Use of the Paradigm

This dialectic between movement and the halting of movement is the core of the *critical* use of the concept of the paradigm. To return to the central passage we are discussing here (*LTS* 27–30), having elaborated his analysis of the doctrine of synthesis—the heuristically 'Neoplatonic' context of which should now be clear—Cavaillès turns to a critique of what we can heuristically call the 'logicist' error, which comes from the exclusive use of the paradigm. Recall first the affirmative use: the paradigm has the underlying characteristic of revealing or enabling 'variation' (i.e. polymorphy), narrowly construed here as the capacity for one (general) formula to range over an infinite number of (particular) substitutions (if one likes, the moment of the paradigm allows the transition from *examples* to *forms*). The critical use turns on the same analysis, since, Cavaillès claims, a dangerous illusion arises from taking this paradigmatic mobilisation of form as the sole modality of mathematical abstraction:

> Whence the distinction at each stage between matter—the original singularity—and form—the current meaning. A distinction which *if* it is projected into the absolute would seem to be applicable from one end of the concatenation to the other, like a continuous fault line running through the nodes of local distinctions. […] Yet it is illegitimate because it misundestands what it generalises, i.e. that which drives the necessarily progressive passage from the act to its meaning: there is no meaning without an act, and no new act without the meaning that engenders it.[80]

What Cavaillès argues here is that exclusive focus on the paradigm yields a kind of ossified concatenation, insofar as it takes the (particular) product of a process of abstraction—realised by deleting a particular 'matter' (or content) from a situation—and then erects it as a general form, which is then both projected and retrojected throughout the past and future chain of mathematical development. This corresponds to the illusion that the axiomatic presentation of a mathematical theory comes before its systematic development, whereas in fact it is only possible after a theory has been developed. Here we should recall the terminology developed

80 *LTS* 29. Emphasis mine.

in §III.2. For what we see here is also critical version of the claim that the paradigm is effectuated concatenation. The error that Cavaillès is describing here consists of taking a particular (intra-theoretical) effectuated concatenation and treating it as if it were adequate for the conceptualisation of the whole (inter-theoretical) effectuating movement of mathematics. It is in short the attempt abolish the disjunction between effectuated and effectuating through recourse to the notion form, and thus abolish movement. Once again, the foil is Leibniz:

> Of course, for a given length of concatenation, we can, by placing ourselves at its endpoint, then view its meaning as one unique form that runs through the whole succession of acts. This is how we proceed in the systematic exposition of a mathematical theory, beginning by giving an inventory of procedures, an inventory that is the result of analysing the most recent meaning of the concatenations [*du sens dernier des enchaînements*] and artificially considering them as having anticipated and led up to [that meaning] without any deviations along the way or any subsequent twists and turns. This is the *moment of the variable*: by replacing the determinations of acts with an empty place where a potential substitution can be made, one progressively raises oneself to a degree of abstraction that yields the illusion of an irreducible formalism [*d'un formel irréductible*]. This is what Leibniz tried to do by passing to the absolute via the mirage of an infinity whose simplicity renders the conditions and the conditioned simultaneous. […] Here the spatial image of juxtaposition and the employment of an elementary combinatory, which retains the simple characteristics of the finite in the infinite as it reveals itself to itself, are both bound up with the origin of the enterprise, and are the cause of its failure.[81]

In other words, the failure of 'logicism' (in both its Leibnizian and contemporary guises) stems from it giving a spatial (simultaneous) rather than a temporal (successive) image of formalism. The downfall of purely formal theories of mathematics is thus that they believe themselves to have abolished the distinction between the act (or succession of acts) and the product of that (or those) acts by rendering 'condition and conditioned simultaneous', by imaging them to be 'juxtaposed' in single system.[82] With respect to my proposed Neoplatonic reading of Cavaillès, this has strong resonance with Bréhier's reconstruction of Plontinus' critique of the exemplarism (i.e. paradigmaticism) resulting from an excessive focus on the theory of participation: logicism tries to abolish the logic of the before and the after that must be retained as a formal characteristic of the production of the intelligible. Another way of putting this is to say that purely formal theories of science, enamoured with the 'moment of the vari-

81 *LTS* 29–30. It is obvious here how Cavaillès is assimilating critiques of logicism, Carnap and 'Leibnizianism'. As noted earlier, Couturat's particular 'combinatoric' interpretation of Leibniz is partly in the background here. See Introduction, footnote 22.

82 Note the exact repetition of the argument concerning 'condition' and 'conditioned' that was marshalled against Kant at *LTS* 4–5, as discussed above.

able', fail to see that the real value of the putting into variation of the sensible which is enabled by mathematical abstraction is that it produces variegation, i.e. that 'plane of truth' that is produced by formalisation is a variegated plane. This is the core of Cavaillès's claim concerning 'the polymorphy internal to a single rational concatenation': that variation always produces variegation, that the formal is never smooth, never homogeneous.

Mapping this onto an argument concerning how to read the history of science (i.e. that we must read the products of historically produced science in order to reveal the 'productive' core of science, without presuming to *posit* the essence of scientificity) serves to give a characterisation of Cavaillès approach to writing the history of mathematics as, heuristically, a kind of critical or strategic Neoplatonism. This should be understood against the background I indicated above (§III.1.iii), viz. that in a heuristic sense the so-called 'French Epistemological' approach stemming from Brunschvicg's work is, methodologically speaking, negative theology, but abstracted from theophany and applied to the history of science. It also allows us to fully situate the Thesis on the Intelligible, which should be read as an application of the critique of the paradigmaticist error just outlined (inherent in the 'freezing' of forms that the 'moment of the variable' tends to produce) to an analysis of Carnap's project for a 'general syntax'. Here, Cavaillès fully synthesises the affirmative and critical appropriations of the Platonic scheme that we have been discussing here:

> More than just an error of foresight, there is a serious failing here [i.e. in Carnap], a misunderstanding of what is essential in formal concatenation: its progression [*progrès*], which is both necessary and conditioned at every step by the effective. A simultaneous positing of all possibilities [...] contradicts the notion of a formal system, the meaning of which implies that it be generated by successive movements of fragmentation and surpassing. The fact that everything cannot be all at once [*d'un seul coup*] has nothing to do with history, but is the characteristic of the intelligible—to misunderstand this is to take leave of the security of its immediate presence in favour of a projection, not only the internal emptiness of which, but also its collapse into [*déchéance vers*] the historical, is revealed in its being bound up with the actualisations of a certain time and the contingencies in which they are clothed. There is no attainment of the absolute here, only a hypostasis [*hypostasie*] of systems and procedures whose status is but transitory. To abstract in this way is not to fix the essence, but to halt [*arrêter*].[83]

83 *LTS* 35–36. Cavaillès will go on to strictly recapitulate this argument with respect to Husserl's ideal of 'formal mathesis': 'Indeed this is the idea of a universal syntax, as Carnap sought to describe it in a single stroke [*d'un seul coup*]. And we know what came of it: mathematical concatenation has an internal cohesiveness that does not allow itself to be rushed. Progressiveness is is essential to it, and decisions that neglect this are lost in the void.' (*LTS* 70)

In Chapter I, we saw how this thesis could be read as Cavaillès's inversion of 'Cartesian' epistemology in light of then-contemporary technical developments in the theory of 'effective calculability'. There, I argued that the central claim that the characteristic of the intelligible is that everything cannot be all at once is in one sense Cavaillès's most condensed philosophical formula concerning the consequences of the various results around the 'inexhaustibility' of formalisms discovered by Gödel, Herbrand, Church and Kleene. However, returning to the same passage in light of the preceding discussion of a Neoplatonic context for Cavaillès work, it is remarkable that this exact same thesis concerning the inexhaustibility of mathematics can be read in Plotinian terms. The nature of formal concatenation— i.e. of 'thought'—is found in its *progression* [*progrès*]; this has the consequence that the intelligible elements through which concatenation passes must be successive and not simultaneous, i.e. they must contain an intrinsic logical order or 'motion'; the 'succession' at stake here is moreover not smooth, but defined by internal 'fragmentation'; this is then expressed as a thesis concerning the notion of the intelligible itself (i.e. concerning its variegation). Indeed, synthesising the two mathematical and philosophical senses that ought to be attributed to this passage, Cavaillès's claim should be understood as being that developments in-contemporary mathematical logic had in a sense ratified this Neoplatonic scheme, in that they had demonstrated in an 'absolute' manner the internal variegation of every 'formal concatenation'.[84] And how is this claim staked out? In terms which, allowing a different intertext here than the obvious use of Cartesian terminology, can also be read as the language of the second hypothesis from the *Parmenides*: that the 'the one which is' is not 'all at once'.

Here it is germane to make one final comment on the connection between the Thesis on the Intelligible and the Thesis on Evidence. Recall the context of the citation we are discussing here: the whole elaboration that runs from the discussion of Bolzano, through the paragraphs on the paradigm and thematic, to the passage just cited, is around fifteen pages, and can be read as a continuous argument, which I have attempted to outline throughout this dissertation. But one aspect of the Thesis on the Intelligible which I have not emphasised the fact that Cavaillès immediately connects his assertion that the Bolzanian theory of science redis-

84 This conjunction of a reading of the incompleteness results from Gödel et al. that takes them as evidence of an inexhaustible potency of mathematical creation and a Neoplatonistically inclined spiritualisation of these results is developed extensively by Jean Ladrière, who seems to me to have been engaged in a profound *implicit* elaboration of the aspect of Cavaillès's project that I have presented here. See Ladrière 1966. Yet I also think that in an important sense, this spiritualising extension of a 'Cavaillèsian' perspective on modern mathematics is an error, because on Cavaillès's reading, 'production' is not a *potency.*

covers a 'Cartesian' problem concerning the movement of concatenation to the Platonic category of *dianoia*. Speculatively, I think we can read a line that runs from this point to the Thesis on the Intelligible. The problem with Bolzano's attempt to think science as an 'object *sui generis*' is that it would require that the 'principles' of science (the 'unproved' terms) be *posited* 'all at once', i.e. outside of discursive thought. But then science—or proof—could have no possible connection with effectively realised science. Science would be the One of the first hypothesis, absolutely outside of Being. But on the contrary, science must be thought as proof in movement, which is to say that it always has a *dianoetic* character, it is discursive. Now, in Plotinian terms, the distinction between *noesis* and *dianoia* is precisely that the former operates all at once, as a kind of mystical 'seeing' beyond (empirical) sight, whereas the latter is successive, it is not all at once.[85]

If there is a 'theologisation' of mathematics at stake here, it would have to be understood as follows. Research into the foundations of mathematics—the various results concerning the limitations of formalisms had—shown that it was impossible to insert mathematics into a system. This was, as Cavaillès will state when opposing Gödel to Husserl, 'the revenge of technique, which overthrows constructions effectuated in an abstraction that sought to surpass it.'[86] What this implied was the necessarily successive (and progressive) character of mathematics: the could never be an absolute system of proof, which would stand outside of mathematics and put an end to mathematical work. All that 'mathematical philosophy' can then think is the diversity of effectively realised proofs, which means to think 'dianoetically'. In LTS, this is transformed into the claim that the intelligible itself (and not only the intelligence that traverses it) is in some sense 'discursive'. This is a final way to read the manner in which the critique of Carnapian 'logicism' is, in the Thesis on the Intelligible, rhetorically connected to the Leibnizian theory of the divine intellect, which intuits everything *d'un seul coup.* On the contrary, for the Cavaillès of *LTS*—who, handing the manuscript to his sister, included a note stating 'In numbers, I find God again'—the conclusion towards which modern mathematics draws us is that not even God can think everything all at once.[87]

85 As Emilsson summarises, 'what non-discursive thought grasps in a single blow, discursive thought has divided into steps, premisses, or reasons, for the fact to be ascertained.' Emilsson 2007: 179–180.

86 *LTS* 71.

87 It should be noted that *this* is not exactly a Plotinian doctrine. Nonetheless, insistence on the *dianoetic* character of mathematics is a hallmark of Neoplatonic accounts, particularly following Proclus. See Maronne & Rabouin 2022. The outline of how a history of the *non tota simul* as a theological—and specifically angelological —problem can be traced from Thomism to Cartesianism can be found in Dubouclez 2014, Dubouclez 2017. For the context of Cavaillès note « Dans les nombres, je retrouve Dieu », see Appendix I.

§III.4. 'The Progress Between Singular Essences' (Spinozist Index of the Thematic)

> Je hâtai les pas. Je ne pouvais mettre un nom à ces essences singulières.
>
> Jules Verne, *Voyage au centre de la Terre*, 1864, Chapter 30[88]

Can we halt here? Is *LTS* only a 'mobilist' manifesto that is all the more forceful because it is articulated, rigorously, on mathematical terrain? It seems this would expose the doctrine to another error, the lineaments of which were given above. I began by outlining how, on my reading, the logical substructure of *LTS* is an argument concerning the metholodogical priority of the effectuated with respect to the effectuating. In reading a Neoplatonic critique of 'exemplarism' into Cavaillès's use of the concept of the paradigm (qua effectuated concatenation), have I not returned Cavaillès to a position of lionising the infinite potency of mathematical creation, overflowing the bounds of any of the temporary forms that would enclose it? That is, to 'contemplation' of a boundless effectuating capacity of the intellect, superior to any of its effectuations? To something like Brunschvicg's view, in which thought, which finds its highest expression in *mathesis*, is a 'creative activity escaping all norms'?[89] Here it is necessary to return to the 'nominalist' thesis announced above concerning the specificity of the effectuating (§III.3.iv.). Which means we can return to the end.

> Progress [*progrès*] is material or between singular essences [*essences singulières*], its motor the demand that each of them must be surpassed. It is not a philosophy of consciousness but a philosophy of the concept that can yield a doctrine of science. The generative necessity is not that of an activity, but of a dialectic.[90]

In closing *LTS* this way, what Cavaillès signals is that his critique of both the 'philosophy of consciousness' (Kant, Husserl) and of 'activity' (Brunschvicg) is fundamentally a consequence of his Spinozist nominalism. In other words, it is a consequence of the doctrine of the thematic. Here, to close my exposition of the 'positive doctrine', must distinguish Cavaillès's programme from his polemic.

88 'I hastened forward. I could not give any name to these singular creations.' Verne 1877.

89 *LTS* 17/*OC* 499.

90 *LTS* 78. The syntax of Cavaillès's first sentence here makes the translation 'Progress' preferable to 'Progression', but it should be underlined that it is the same term at stake here as in the Thesis on the Intelligible: one essential characteristic of formal concatenation is its *progrès*.

§III.4.i. The Programme

It first must be underlined why this invocation of the Spinozist concept of the singular essences at the close of the essay returns to the argument at stake in the earlier elaboration on the paradigmatic/thematic disjunction. What Cavaillès states in the section on the thematic is that thematic abstraction, in contradistinction to paradigmatic abstraction, is an increase in determinacy. The view that is being opposed is one that would see abstraction only as despecification. *This* is what limiting the theory of abstraction to paradigmatic abstraction would yield. Whence a remarkable thesis, which is a consequence of the ordering that is proposed in *LTS* between the effectuated and the effectuating: knowledge of the effectuating is achieved (insofar as it can be) in knowledge of specificity; to know 'how something was made' (or how it is moving) requires a different kind of abstraction than producing a 'form' by abstracting away from particular details; it requires specifying rather than generalising abstraction; but specifying abstraction must *depart* from a particular generalised—that is, effectuated—abstraction. This is articulated precisely at the moment in *LTS* when Cavaillès makes his other explicit (to his audience) citation of Spinoza, when, at *LTS* 32-33, in summarising the 'double interplay' of the paradigm and the thematic, he makes a final statement concerning their relation:

> On the one hand, not only do both issue from the same emergence of meaning, but the abstraction of the former [i.e. the paradigm] *enables* the latter [i.e. the thematic] […]. On the other hand, positing meaning [i.e. thematisation], like all meaning, is an act whose development is longitudinal [i.e. it also has a 'paradigmatic' character], which is to say that all positing meaning is simultaneously the posited meaning of another act (something that may call to mind the traditional principle that every idea has a formal reality). Thus, once initiated, the deepening movement [i.e. thematisation] sets off a new concatenation of the first type [i.e. a paradigm]: *the idea of the idea* manifests its generative power on the plane that it defines, without discounting *the possibility of unlimited superimposition*.[91]

Both the reference made here to the 'idea of the idea' and the closing use of the 'singular essences' are citations on Cavaillès's part of a continuous argument from Spinoza's *Treatise on the Emendation of the Intellect*, i.e. the nominalist side of Spinozism. Indeed, the paragraph is more or less a direct gloss (from memory) on paragraphs 32-34 of the *Emendatione*, where Spinoza famously argues that 'the idea of a circle' is different from the circle itself, but nevertheless intelligible 'through itself'. Thus, Spinoza argues, the relationship between what he

91 *LTS* 32. Emphasis mine.

calls the 'formal' and 'objective' essences, is one in which every formal essence can be taken as the object of another objective essence, but the objective essence will also in its own right be something 'real and intelligible'. It is on this basis that Spinoza asserts what Cavallès summarises here: 'the idea of the idea […] again has its own essence, which can also be the object of another idea, and so on without end.'[92] This is precisely the logic that has just been assigned to the 'double interplay' between the paradigm and thematic (between produced and producing). It is also this relation between the 'formal' and the 'objective' essence that I understand Cavaillès to invoke by stating that the progression between the singular essences is 'material'. Within the economy of the *Emendatione*, this logic, in which the essences relate to each as particulars to particulars, with each being both intelligible for itself and through the other, is the core of what could be called Spinoza's rationalist nominalism, which is precisely what Cavaillès seeks to recall in the polemic concerning the opposition between the philosophy of the concept and the philosophy of consciousness, i.e. *Emendatione* paragraphs 93–94:

> [A]s long as we are engaged in an enquiry into real things, *it will never be permissible for us to draw a conclusion from what is abstract*, and we shall take great care not to mix the things that are merely in the intellect with those things that are in reality. The most secure conclusion is to be drawn from *some particular affirmative essence* [*essentia aliqua particulari affirmativa*], i.e., from a true and legitimate definition. For, starting from universal axioms alone, the intellect cannot descend to particulars [*ad singularia*], since axioms are of infinite extension and do not determine the intellect to contemplate one particular thing [*ad aliud singulare*] rather than another. So the correct path to discovery is to develop our thinking from the basis of some given definition, and progress will be more successful and easier as a thing is better defined.[93]

I take this to be the methodological core, in a sense, of Cavaillès's late doctrine: passage through the paradigmatic moment of abstraction is necessary but not sufficient, for to really know a thing—to define it—is to know it more specifically. This is both a prescription for rationalist philosophy and a practical programme for how to read the history mathematics. The 'progress' between mathematical theories is not one of increasing *generalisation*, with the aim of arriving at a 'higher genera'; on the contrary, the aim of reading the progress of mathematics is to better distinguish things that had previously been mixed up, and this can only be achieved by knowing prior theories through other, posterior, theories. But there is no 'theory of theories', all there is the 'unlimited superimposition' that Cavaillès presents as an interpret-

92 Spinoza & Shirley 2002: 10.

93 Spinoza & Shirley 2002: 25. Emphasis mine.

ation of the doctrine of the 'idea of the idea', or as Cavaillès states immediately afterwards, 'an unlimited intelligible complication'.[94]

Here it helps to consider one of Cavaillès's cited cases of 'the thematic', i.e. the Lebesgue integral. In the 'progress' of 19th century analysis, in the classical narrative that goes from Cauchy to Riemann to Lebesgue, one finds both a splitting of concepts that were previously confused, and the production of more general concepts, attendant to the development of the concept of function.[95] Thus on the side of the splitting of the concept, Cauchy made the assumption that all integrable functions are continuous, but this was shown to be false, and the advantage of the Riemann integral is that it divides the notions of 'continuous' and 'integrable' functions. It is then Lebesgue that allows a further differentiation, by using set theoretical tools to divide the concept of integrability and differentiability, and so on into all the complications of the 20th century theory of functions. Thus the concept of 'function' is better and better distinguished depending on the domain of its application. On the other side, directly accompanying and indeed enabling this development, is an increasing generalisation of the concept of 'function': whereas at the outset of the 19th century, the concepts of 'analytic function', 'Fourier series', and so on might well have appeared to be entirely distinct, in a development that starts from Direchlet's conception of the 'arbitrary function' and runs all the way to the foundations of set theory, by the dawn of the 20th century, they are all abstractly to be described as forms of the 'function in general'.[96]

Now, what is the proper way to conceptualise this progress? One natural way to do so —the standpoint of the textbook, let's say—is to say that what was arrived at by the early 20th century is a better classificatory scheme, in which all functions are now presented as *predications* of the general concept 'function': we have 'analytic functions', 'pointwise discontinuous functions', 'integrable functions', and so forth. The narrative of 19th century analysis would thus be that of an increasing triumph of *generalisation*, according to the logic of species and genera.[97] But an alternative way to understand this 'history', is to say that from a given posterior 'stage' in the development of the mathematics—i.e. that which is synthesised in Lebesgue's standpoint—the prior stages—i.e. those represented by Cauchy and Riemann—are now 'comprehended' (as in, 'understood') with a higher degree of specificity. We come to un-

94 LTS 32.

95 My presentation here draws on Michel 1992, a work that is itself very much indebted to Cavaillès.

96 See Chorlay 2016 for an account of the epistemological issues posed by this development in mathematical practice.

97 This is what Brunschvicg will call the 'logic of extension'.

derstand the *singular essence* of the Cauchy integral through the other singular essence (or idea) of the Lebesgue integral. But this 'progress' (if one holds onto the term) does not abolish the content of the earlier theory: qua 'ideas', each link (each theory) in this sequence will remain 'intelligible through itself' as well as being intelligible 'through another idea' (or theory). Thus one can still hold that the practicing mathematician of the early 19th century well understood the objects with which they operated, according to their own 'norms'. What changes when one idea is conceived through another is a shift in relations of intelligibility.[98] If one likes, we can 'make differences' (distinctions) that were previously obscure, distinguish what was previously conflated. But this does not abolish the internal specificity of prior stages of mathematical development; it simply allow this specificity to be grasped in a way that might have been invisible to practitioners immersed in the whole network of methods that composed that theory. This alternative narration of the historical progress between mathematical theories can thus be read in terms of the other slogan of Spinoza's nominalism, from paragraph 55 of the *Emmendatione*: 'the more generally existence is conceived, the more confusedly also it is conceived […] the more particularly it is conceived, the more clearly it is understood.'[99]

As I read the doctrine of the thematic, it is this conception of the history of theories—of the 'logic' of 'the before and the after' that is being affirmed. And it is easy, on this account, to see how it is that the paradigmatic moment enables thematic specification. For, in our particular example, it is the initial detaching of a form through the generalising abstraction of set theory that 'enabled' the specifying notion of the Lebesgue integral. There is no 'descriptive' concept of integration without the initial 'moment of the variable' by which the set theoretisation of the notion of space renders structure indifferent to specific content, and allows the formation of the theory of functions at a different level.[100] But this cannot be where abstraction stops. It is in this sense that the doctrine that the thematic is posterior to the paradigm is, in one sense, a proposal for how to read mathematical history.

98 This is what Brunschvicg will call the 'logic of comprehension'.

99 Spinoza & Shirley 2002: 15.

100 Refer here, for a closely related example, to Lebesgue's 'descriptive' definition of measure as given at the close of §I.1. It is clear that this kind of definition both defines a much more specific class of objects than 'set' *simpliciter*, but still relies on the initial 'generalising' abstraction of 'set' for such a specifying abstraction to be stated. On this point, see Granger's discussion Granger 1988: 72–73.

§III.4.ii. The Polemic

But, as noted, it is also a polemic. And I believe we can say it is a polemic in a double sense, both against the 'empiricist' abstraction practiced by Carnap, and against what was called above 'mobilism' (towards which our Neoplatonic hypothesis brought us). Here, it is important to note the remarkable way in which, with his statement against 'activity', Cavaillès chooses to close *LTS* with an argument against Brunschvicg that is precisely articulated in Brunschvicg's own concepts. Indeed the very notion of rationalist nominalism is a profoundly Brunschvicgian programme, as Brunschvicg himself expressed in a debate with Étienne Gilson in 1931: 'Descartes, Malebranche, Spinoza considered that the first condition in order to be a rationalist is to be a nominalist.'[101] For Brunschvicg too, this was expressed as polemic against an exclusively generalising notion of abstraction:

> For Spinoza, ontology and empiricism are two opposed errors of the imagination, like the two poles of its perpetual and impotent oscillation. To make the specific givens of a problem vanish into the ever poorer, ever more futile shadow of their logical similarities is to turn one's back on science […]. The role of science is to make thought circulate not over and above individuals, but through individuals. […] In the logic of extension, the unity of the class is opposed to the specificity of beings; in the logic of comprehension, the unity of the system is the intelligible foundation of specificity.[102]

It is in this context that Brunschvicg repeatedly mobilised the notion of the 'singular essences'.[103] In a sense, Cavaillès's philosophy of the concept only adds—but this is no small

101 Gilson et al. 1931: 78.

102 Brunschvicg 1923: 299–300.

103 Indeed, one can plausibly claim that Brunschvicg properly invented the concept of the *essences singulières*, by translating Spinoza's use of *essentia particulari affirmativa* into an assertion about the nature of ideas in a more metaphysical register. In other translations of Par. 93 of the *Emendatione* from this period, Saisset, Appuhn, Callois and Koyré all chose (appropriately) some variation on *essence particulière.* Now, in the *Ethics* one finds references to the 'essences *of* singular things', but never to the 'singular essences'. Brunshcvicg's gesture was, however contestable it may be, to identify Spinoza's *essentia particulari affirmativa* with the ideas in the Platonic sense, such that he could assert in 1904 that it was engagement with Neoplatonic sources that 'led Spinoza to reconstitute the Platonic ideas in the notion of the "singular affirmative essences" [*essences affirmatives singulières*]' Brunschvicg 1923: 797. By the time of *Les Étapes*, this is condensed into the syntagm *essences singulières*. Thus during his polemical construction of what he calls the 'reaction against mathematicism' in 19th century French spiritualism, which he attributes in part to a profound misunderstanding of Spinoza exemplified by the Aristotelian spiritualisms of Ravaisson and Taine, we find the following assertion: 'In this respect, let us reread the short preface that Taine placed at the head of his *Essai sur Tite-Live* (1856): there, the intention to express "a talent... by a formula", to discover in man "a master faculty, whose uniform action communicates itself differently to our different cogs, and imbues our machine with a necessary system of already predicted movements," sis placed under the patronage of Spinoza, that is to say, of the philosopher who saw most clearly the vanity of all classification into faculties as well as of all general ideas, who most insisted on *the indefinite complexity of singular essences* [*la complexité indéfinie des essences singulières*].' Brunschvicg 1912: 563. Clearly, Cavaillès's assertion as to the 'unlim-

thing—that to think this 'circulation' through the category of the intelligence—that is, through 'activity'—is another way of making 'the specific givens of a problem' vanish in an indefinite becoming. One can indeed fairly make this critique of Brunschvicg's writings, where for all the subtleties of his historical analyses, they ultimately serve to exhibit the same triumph of the same intelligence. Counterposed to this, the discourse of the specificity of modes of production such as it is outlined in *LTS* is orientated towards refusing such a dissolution of all products into the indifferent generality of some underlying productive unity. The means by which this is achieved is by insisting that different 'producings', which are always enabled by prior productions, are still singular and differentiated. For, as Cavaillès already stated in 1937, inveighing against intuitionism: 'the demand for possible arithmetisation (Kronecker-Brouwer) is a misunderstanding of what is specifically mathematical: the unlimited procession of original intuitive modes.'[104]

I believe this insistence on specificity can be read as a profound reckoning with the entire intellectual context in which *LTS* was produced, that is with the 'mobilism' common to Brunschvicg, Bréhier, Bergson, Valéry, and others besides. Part of the originality of Cavaillès's work lies in the fact that he attempts to develop a theory of the pure movement of thought without recourse to a metaphorics of life or the organic. The strangeness and also the interest of *LTS* is that it is traversed by an argument for the mobile over the static, but this opposition between what moves and what does not is not in any explicit way rooted in a more fundamental opposition between some dynamic force and the exhaustion of that force. It is essential that the movement of mathematics is never characterised as being expressive of some other 'real' locus of dynamism: Nature, God, *élan vitale*. What is finally at stake in Cavaillès's theory of mathematical time is thus an attempt to outline a purely rational theory of movement that is not vitalist. This is the other reason why the Spinozist corrective to the Neoplatonic theory of the movement of the intelligence is necessary.

Here we can return to the strident terms in which Benda advanced the category of mobilism, as introduced in §III.1.iii. On Benda's analysis, the normative insistence on the superiority of movement was a shared background for almost all of then-contemporary philosophy in France, such that theoretical divergences were be rooted in different accounts of the locus of movement, rather than in disagreement as to the valorisation of movement as such. Benda

ited superimposition' of the idea of the idea take up this line of interpretation. I thank Étienne Balibar and Jean-Pierre Moreau for a clarifying conversation on this point.

104 OC 579.

positioned this in terms of a new 'Spinozist' rationalism of perpetual becoming and an older 'Cartesian' current which, for Benda, places the highest rational value on determinate ideas:

> In sum, there are two conceptions of thought: according to the first, thought does not know any kind of halt [*aucune espèce d'arrêt*], however brief it may be, but only an incessant surpassing of itself, not solely in the field of its competence, but in the nature of its activity; according to the second, thought always exerts itself on a determined object, which can be a mobility, and interprets its experiences according to principles that are invariable, at least in their essences; according to the first, to adopt the expressions of Brunschvicg, thought consists in a perpetual *giving* [*donnant*]; according to the second, thought knows only what is *given* [*donné*]; according to the first, thought seeks to be indefinitely 'open'; according to the second, thought accepts the evocation (Hamelin) of a definitive knowledge. The first is the metaphysical—i.e. poetic, lavish—conception of thought; it is that of Spinoza, of Hegel (minus the belief in the transformation of the nature of thought with time), of the 'new rationalism'; the second is the scientific conception; it is that of Descartes.[105]

Doubtless, Benda's critique makes some conflations that we would wish to avoid, not least because the primary enemy aimed at throughout the argument is Bergson, with Benda identifying figures as diverse as Émile Boutroux, Louis Rougier, Édouard Le Roy, Poincaré, Bachelard and Brunschvicg as fellow travellers. This is surely not how Brunschvicg and most of all Bachelard would have viewed things. However, for all that it is, historiographically, overdetermined by vociferous opposition to 'Bergsonism' in philosophy, Benda's critique nevertheless evinces the acuity proper to the conservative: to see points of commonality that would not appear as such to protagonists on the progressive front. What is all the more striking, from the perspective of the analysis I have been developing here, is that he finds this common point in the lexical opposition between the conception of thought as something unhalting [*non-arrêtée*] and as halted [*arrêtée*], which he locates in a register that is aesthetic (Valery: 'To halt on an idea is to halt on an inclined plane') as much as philosophical (Bergson: 'With halts, however numerous they may be, we shall never make mobility').[106] Benda's primary polemical target in drawing these references together is what he sees as a fallacy underwriting this shared insistence on the mobility of thought, that in order to grasp the mobility of the real, thought itself must be mobile, i.e. that because the real is unhalting, thought must also not halt. There is of course of Spinozist inspiration to this argument: just as the idea of a circle is not round, the idea of a river does not flow, and the same for more ideal forms of motion. Why

105 Benda 1950: 1950.

106 Valéry 1942: 10; Bergson 1934: 213. Cavaillès was certainly familiar with the latter, and assigned it his bibliography students in logic and general philosophy whilst teaching at Strasbourg (i.e. Cavaillès 1941). See Canguilhem 2015: 1157.

should the idea of movement have to move? Contrary to this fallacy, Benda's argument was that although science is, indeed, characterised by an incessant movement, its essential scientificity consists in its each time formulating new concepts as 'clear and distinct, which is to say halted [*arrêtée*], ideas':

> A scientific thought is a fixed position of the mind [...] and the mobility of science consists in the passage from a fixed state A to another fixed state B, not in the absence of all fixity. The 'spiritual modification' that is given to us as presiding over science, and singularly over contemporary science, is a movement that passes [...] from one halt of the mind to another halt [*d'un arrêt de l'esprit à un autre arrêt*], it is a modification by *leaps and bounds*, by *pulsations*, not by a continuity of movement, exempt from every halt.[107]

It is on this last point that Cavaillès's 'Spinozist' doctrine is articulated precisely in agreement with Benda, and against the prevailing currents in his milieu. The affirmation of the 'progress between singular essences' that closes *LTS* is an affirmation that the movement of mathematics that he wishes to theorise is a movement from singular point to singular point; thus, the thought proper to science is not continuous, but internally discontinuous, striated. Indeed, the heuristic argument that I have laid out above that the 'master argument', so to speak, of *LTS* is to found specificity on variegation, in a sense culminates this point: the stratification or heterogeneity of the intelligible—that it is not smooth—is what allows it to be thought, rather than its continuous unfolding. Indeed with respect to Cavaillès's assertion that 'science moves outside of time—if time refers to the lived experience of consciousness', this could precisely be read as asserting that it is the very *continuity* of lived experience that means that it cannot exhibit mathematical time. Cavaillès's project thus—qua Benda's (quite just) argument above—is profoundly directed at a 'Bergsonian' conflation of movement and continuity.[108] This is why the movement that the philosophy of the concept tries to think is *definitionally* outside of lived experience.

This, I think, is the final sense we can give to the polemic concerning generalisation and specifying abstraction that runs through the exposition on the paradigm and the thematic. *LTS* in a way concludes by stating that specifying abstraction is needed to rectify the errors of both 'mobilism' and 'formalism'. This is a Brunschvicgian argument, but one that is no longer grounded in a Brunschvicgian metaphysics of activity. The notion that abstraction is despecification is, Cavaillès thinks, common to Husserl, to Bergsonian mobilism, and to all traditions of 'em-

107 Benda 1950: 19. The reference to the 'spiritual modification' of contemporary science is to Bachelard 1934: 142.G. Bachelard, *Le Nouvel Esprit Scientifique*, p. 142.

108 For a brief comparison of Benda and Cavaillès views, see Engel 2024.

piricism', from Hume to Russell and Carnap. The only difference is the normative weight they assign to this characterisation.

It is finally in these terms that we should understand the 'logic' of *LTS*, that is the insistence therein that the aim of the philosophy of the concept is to think movement, but to think it specifically. It is in this sense that the logic of the disjunction between the paradigm and the thematic allows us to understand what a 'concept' is for Cavaillès: it is the point where abstraction allows matters to be thought more specifically than they can be either in experience or through the moment of generalising abstraction. Here we can perhaps admit a cinematic analogy. Like with Muybridge's *The Horse in Motion* (1878), the object of enquiry for the philosophy of the concept is 'How is it moving?' The concept is like film in that it allows the *allure* of movement to be seen. Now, the invention of film makes a way of moving—what Serge Daney calls a new type of pose—visible by multiplying the instantaneous (that is, the photographic moment) in a determinate manner.[109] In the same way, the gait of the concept is only seen in the concatenation of specific theories, and this is what a philosophy informed by the history of mathematics tries to make its object. Cavaillès's opposition to Bergson's mobilist credo is very precise: it is *only* with 'halts' that one can 'make' movement. This is how I understand what can appear paradoxical in Cavaillès's perspective: that the epistemology of mathematics, precisely by focussing on the specificity of how theories were produced in history, allows us to think 'eternity in Spinoza's sense', understanding of course that for Cavaillès and others of his generation, this meant an eternity that was in movement. This at least, to step back for a moment into the genetic mode, is what Cavaillès claimed in a letter to Étienne Borne as early as 1930:

> I even wonder to what extent it is possible to attain the true naivety of the Saint without a prior submission to this necessity which manifests the approach of God, immanent in mathematics, transcendent in love. And it is here that I locate my grievance against [Gabriel] Marcel, his ignoring of the absolute value of the intelligible, of the rational: there is something divine even in the concept, *at least in the passage from one concept to*

109 'What did photography do compared to painting? It made the instantaneous possible. It became sensitive to the "poses" that reality keeps on taking but that we struggled to perceive because there were no means to "capture" them. It has invented the "ordinary movement [*mouvement quelconque*]" that Deleuze talks about. And what did cinema do compared to photography? It multiplies the photograph twenty-four times to obtain a second of movement. It increases the number of ordinary moments, of interstices, thus creating a type of "pose" (and a type of beauty) which is not quite the same as that of painting or photography. The "beauty" of a shot is of a new type.' Daney 1993: 29, translation from: https://kinoslang.blogspot.com/2024/12/the-exercise-was-beneficial-sir-serge.html [accessed 10.04.25].

> *another*. And it is here that we have the true Spinozist ontology, incomplete, but definitive in what it asserts.[110]

There is a line that runs directly from this intuition to all of the twists and turns of the text of *LTS*, some of which I have just tried to trace. We can at least say this much: the 'concept' as the philosophy of the concept tries to think it is only grasped in the passage from one concept to another. The role of reflection is to understand the *how* of this passage, and in order to conceptually grasp this 'how', the only available method is to produce more concepts, which bear on each other in the 'unlimited superimposition' of the idea of the idea. Another way of framing this final point, and thus of taking up one last time the sense in which, qua modalities of mathematical concatenation, the paradigm and the thematic are 'of the same nature', is to state that what is at stake in the retroactive superimposition of mathematical theories—which is to say in the bearing of one group of intelligible concatenations on other, prior intelligible concatenations—is always specific acts acting on other specific acts. For in this 'dialectic', which produces mathematical history:

> The new operations that intervene bring the evidence of their completion to bear upon previous operations—or acts.[111]

And what, then, is an act? This is the question which *LTS* manages, at least, to pose. The section of *LTS* we have been commenting on in this chapter and, in effect, throughout this whole dissertation ends with the statement that the theory of proof—or of rational concatenation—'turns out to require an ontology' (*LTS* 43). And if *LTS* presents such an ontology, it is an ontology of the act. Yet the act implied here it is not an act of consciousness, nor, as we have just seen, can it even be said to be the act of any kind of activity.

The whole confrontation with Husserl that runs throughout the remainder of *LTS* concerns this concept. We might even say that the theory of the duality between act and object that Cavaillès had extracted from Dedekind, and which in *LTS* had finally been reconceptualised as a logic of mathematical time, required an engagement with Husserl through its formal resemblance to the post-Brentanian theory of the noetic-noematic correlation. Section III of *LTS* is thus concerned with an intricate internal critique of the phenomenological project as expressed in *Formal and Transcendental Logic*, one that arrives the point of refutation just sig-

110 Cavaillès, letter to Borne, 7th October 1930, from Cavaillès 2010: 27–28. Emphasis mine.
111 *LTS* 37.

nalled: mathematical history is a chain of acts acting on acts, but these are not the acts of consciousness.[112]

I do not claim here to give any analysis of Cavaillès's arguments on this point, but would like to close by indicating that future 'Cavaillèsian' research would have to proceed by developing this concept of the subjectless or agentless act in mathematical practice. It is hard to overstate the radicality of Cavaillès's position here, for ultimately what is developed in the final section of *LTS* is an argument against the very idea of transcendental philosophy—which for Cavaillès would include Husserlian phenomenology as a proper part—on the grounds that philosophy must break with the very idea of 'object constitution' which Husserl had so brilliantly purified out of the Kantian project. As Cavaillès writes, from a Husserlian perspective,

> [Kant] posed the critical problem in a manner that was too narrow, and set out at too high a level. It [that is: the critical problem properly posed] consists first of all in the discovery of logical entities as [1] independent objects possessing their own reality and yet [2] engendered by the consciousness that must undertake transcendental analysis.[113]

Here we can see something of the proximity that Cavaillès evidently saw between his position and Husserl's concerning what I called in Chapter II an 'intermediate' position on the reality of the mathematical object. Cavaillès and Husserl are both working through an attempt to hold onto two seemingly incompatible theoretical *desiderata*, i.e. that the object is both 'independent' and 'generated'. Now, as Cavaillès's understands matter, in Husser these objects are ultimately engendered by consciousness. Thus, as Cavaillès summarises, Husserl's liberation of transcendental philosophy from the assertion of any 'pre-constituted' nature of the subject (a reading that is, of course, profoundly indebted to Brunschvicg) involves a correlative transformation in the whole problem of logic, such that from the phenomenological standpoint:

> [T]he meaning and being of all objects can only be conferred upon them by a subjectivity that is their source and ground: the problem posed by logic is transformed into a problem of the transcendental constitution of objective entities.[114]

Now, it might seem that the problem of 'the transcendental constitution of objective entities' is, in a certain sense, precisely Cavaillès's own problem, as he would have read it through the Dedekendian project of thinking 'the necessary generation of new concepts' in mathematics.

112 For a thorough reconstruction of the fundamental issues that unite and divide Cavaillès and Husserl on this point, see Pradelle 2013, which I partly draw on here. Pradelle 2020 is, in a sense, a sophisticated attempt to produce a positive synthesis of the two figures, via the mediation of Desanti.

113 *LTS* 59.

114 *LTS* 59

But the entire conclusion in *LTS* is structured around an intricate argument that is the very perspective of 'transcendental constitution' that must be abandoned. The discourse on the form of production such as it developed throughout the essay is ultimately counterposed to any discourse of constitution. Thus, as Cavaillès states near the end of his exposition on *Formal and Transcendental Logic*, what Husserlian phenomenology cannot think, insofar as it is wedded to theory of object constitution *by* consciousness, is the fact that:

> [N]o consciousness is witness to the production of its contents by an act (LTS 75)

What concept of the act is at stake here, such that it can be said to 'produce' the contents of consciousness? Any future development of a Cavaillèsian project would have to start from this point. Mathematics is composed of acts, but they are not acts of subjects, not of agents. This is what it meant to rigorously develop a non-psychological theory of creation in mathematics. This of course poses a new question: are these acts themselves 'constituted'? The whole point of Cavaillès's doctrine of the act without activity seems to be to answer this question in the negative. Acts simply *are*, they can be related ('superimposed'), but they are not the product of some prior capacity. This rejoins, of course, a much older problem:

> The only Cartesian contribution to the emergence of the subject in its modern, 'Cartesian,' sense—that is, the first-person relationship between thought and existence, not to say personal identity—is to accept the general axiom that *we cannot conceive an act without a subject* […].[115]

Cavaillès, on the other hand, states that this is what mathematics has been busy 'conceiving' all along.

115 Libera 2008: 200.

APPENDICES

APPENDIX I – COMPOSITION AND PUBLICATION HISTORY OF *ON LOGIC AND THE THEORY OF SCIENCE*

The role of this appendix is to give, as best as I can reconstruct it, a chronology of Cavaillès's work in the period that runs from the completion of his doctoral studies to the composition of LTS*. Although fragments of such an account exist in other work on Cavaillès, principally in his sister Gabrielle Ferrières's biography—on which I draw extensively here—there is no account that relates the rather startling outlines of Cavaillès's life in this period to two important questions for the reading of Cavaillès's final essay:*

1) The chronology of what record we have of Cavaillès's work in this period. This covers:

A) The relatively small number of letters of philosophical and scientific content that survive from the period of 1938 up until his execution in 1944, a situation that can be attributed both to the general chaos of the wartime situation and to the clandestine nature of Cavaillès's activity following 1939.

B): The posthumous publications that were put together by Canguilhem and collaborators after the war, principally Transfini et Continu*, the late fragment 'Mathématiques et formalisme', and, of course,* LTS *itself.*

C) The surviving notes of Cavaillès's teaching at the Sorbonne from 1941-1942.

2) The question of the intended sources of LTS*. As we receive the work, the original manuscript of which seems to have been lost, it contains a very scant collection of citations, many of which were added by Canguilhem and Ehresmann. However, Cavaillès's letters demonstrate repeated attempts to acquire a wide range of material on mathematical logic in order to revise the text, and to write an introduction in order to situate it in technical context.*

This Appendix is a first attempt at an account of these points, which is inevitably partial. It also includes a number of translations from Cavaillès's letters, or at least the most relevant parts of them for the question of an établissement du texte*. I hope it can be of use to future travellers.*

*

First, a summary of the facts, which are essential to the status of *LTS* as a sort of myth, but a true myth. Having completed his doctoral works and published them in 1938, Cavaillès began a period of research towards a work or works that he would variously describe by the proposed name *L'Éxperience mathématique* or as his 'Treatise on Logic'. As Bachelard recounts,

> Cavaillès's theses were ready. As he used to say, now that his theses were completed, he would be able to *get to work*. We planned projects: to collectively defend rational thought, to remind philosophy of the demands of proof.[1]

As it happened, this period of work, which would be rather short, was structured by two external factors: teaching responsibilities and the outbreak of the war, with the latter leading to Cavaillès's role in the founding of the French resistance. Following his thesis defence in January of 1938 Cavaillès transferred to a teaching position at the University of Strasbourg and for a while was busy with administrative responsibilities. In September he travelled to Amersfoort for a meeting that had been organised by Ferdinand Gonseth with a view to coalescing a movement in the philosophy of science to counter the Vienna Circle.[2] While there he met 'the Pole Tarski' and engaged in a lively debate with Gonseth on the relation between mathematical and physical experience.[3] Finally the conference was cut short, for Cavaillès at least, when the announcement of a partial mobilisation of French troops lead to him and others returning early to France.

In the end the war was not to commence quite yet, and the academic year of 1938-1939 was spent between teaching, editorial work, and some writing, as well as embedding himself more publicly in debates in mathematics and philosophy in France. On the mathematical side this was the period in which he was most associated with the nascent Bourbaki group, with whom he was sociologically close if not necessarily doctrinally aligned; André Weil was also teaching at Strasbourg at this time, and between February and April of 1939 Cavaillès corresponded with Jean Dieudonné concerning an early draft of what would become the first volume of Bourbaki's 'Elements of Mathematics' series, on set theory.[4] On the philosophical side, he initiated, in collaboration with Raymond Aron and Albert Lautman, a collection of essays to be published by Hermann. After a period of debate in which said collection was almost called

1 Bachelard 1950, in Ferrières 2003: 237.

2 As noted in Cortois 2024: 93, this movement never coalesced in any systematic sense, but it did produce the journal *Dialectica*, founded after the war by Gonseth, Bachelard and Paul Bernays in 1947. C.f. Lauener 1985.

3 Ferrières 2003: 143/Ferrières 2000: 109. See §II.1.ii. for an analysis of Cavaillès's discussion with Gonseth.

4 Part of this correspondence is published as Cavaillès & Dieudonné 2020.

Clarté and then *Élements* (the latter a suggestion from Weil, who presumably then took it himself for the Bourbaki series), it ultimately bore the title *Essais philosophiques, publiés par Jean Cavaillès*.[5] In the introduction to the first volume of that series—namely Lautman's short essay *New Research on the Dialectical Structure of Mathematics*—, signed by Cavaillès and Aron, but seemingly also written by Lautman, it is announced as a series on 'dogmatic philosophy without restriction of any kind', with 'philosophy' being defined as 'any requirement for clarification which is not satisfied by the instruments of scientific technique in their current status or in the course of their normal development.'[6] As it turned out Lautman's essay and Sartre's *The Emotions: Outline of a Theory* (both 1939) would be the only volumes published in Cavaillès's lifetime.[7] He and Lautman gave a joint presentation to the Société française de Philosophie on the 4th of February 1939, the transcript of which would be published as 'Mathematical Thought'.[8] In August of 1939 he completed work on 'From the Collective to the Wager', an essay on the mathematics of probabilities that puts in order certain material on which Cavaillès had worked on from the beginning of his doctoral studies.[9] This would be his last piece of writing before the outset of his resistance activities.

The lineaments of Cavaillès's attempts in this period to develop a philosophy of mathematical experience are visible in a letter written to the mathematician and Marxist Paul Labérenne at the end of 1938. It is worth citing the theoretical part in full, for it demonstrates Cavaillès's growing distance from Brunschvicgian epistemology.

> As for set theory, I have wished to be very cautious, because it seems to me to be impossible to limit, except after the fact, the domain of effective works: even if works such as those by Fréchet are sterile, from time to time results appear from other quarters, for example with Lusin. On the other hand, this same empiricism allows me to adopt a more definite attitude with respect to the problem of the foundation of mathematics.[10] Even more than I have emphasised in my thesis, in taking these things up again, I believe that all recourse to any *a priori* whatsoever is dishonest: thus my complete break with idealism, even of the Brunschvicgian variety, in defining logic as a primary natural technique which, being the simplest one, is then used by all the others (as arithmetic is

5 C.f. letters to Lautman of the 23rd of November and the 6th of December 1938, Sinaceur 1987: 124–126.

6 This introduction is reprinted in Lautman 2006: 235–236/Lautman 2011: 195–196. In a letter of 6th December 1938 Cavaillès discusses 'merging' his own ideas with a draft that Lautman had already sent him. C.f. Sinaceur 1987: 125–126.

7 Lautman's *Symmetry and Dissymmetry in Mathematics and Physics* (1946) and Cavaillès's own *Transfinite and Continuum* (1947) would complete the series, posthumous publications for both authors.

8 Initially published as Cavaillès & Lautman 1946, and then reprinted at *OC* 593-630.

9 Cavaillès 1940, reprinted at *OC* 631-652.

10 Here I take Cavaillès to be referring to the empiricism of Fréchet and Lusin (deriving from the school of Borel and Lebesgue), rather than to his own 'empiricism'.

> used by other mathematical theories). In short, logic is completely subordinated to an experience, which is undoubtedly not a historical experience, since it allows us to obtain results the validity of which is outside of time, but which are born of historical experience. This is too brief and I'm just in the process of trying to get it all clear, and perhaps, moreover, it is not the subject of your discussions. In any case, I wanted to emphasise, better than I did in my book [i.e., presumably, *MAF*], because there I was too close to the material I was working with, that I have nothing in common with university optimism, the philosophy of the Third Republic, that *ersatz* of religion—and that the history of consistency proofs shows on this point, with great clarity, that there is no kind of way to pre-emptively dominate or decide the effective work of the mind—that is, of human technique—in experience.[11]

The quasi-anthropological conception sketched here according to which 'mathematical experience' is to be defined as the historical unfolding of progressively more complicated operations carried out on and transforming prior 'natural' or 'human' techniques is consistent with the definition of the concept of experience in terms of a 'system of gestures' that Cavaillès will advance in this period, and ultimately in some sense abandon, or at least modify. But we are getting ahead of ourselves; this was in any case the state of the concept at the dawn of 1939.

Then war.[12] In September of 1939 Cavaillès is mobilised into the French army and stationed in Forbach on the French-German border. Throughout the winter he engages in skir-

11 This letter is reproduced in Mougin 1945: 79, with no information as to its provenance or exact date. It appears to be a response to an invitation from Labérenne to speak on the importance of recent results in set theory. At the time, Labérenne was involved in disseminating Ernst Kol'man's 'Marxist' critiques of the 'idealism' of the French Analysts, which were developed in the context of a violent domestic polemic against Lebesgue's disciple Nikolai Luzin in Russia. C.f. Graham & Kantor 2009: 148. It is likely against this background that Labérenne connected with Cavaillès's work, for both *TAE* and *MAF* deal with Luzin at some length; whence Cavaillès's qualified defence of 'empiricism' here. It would also explain why the letter begins with one of Cavaillès's rare explicit addresses to political questions: 'Although in philosophy I am not orientated by dialectical materialism (I distinguish the former from politics, in which choice according to personal decision plays a different role), I have already told you that I find myself led to results which are perhaps not so excluded by your attitude.' Laberenne's use of Kol'man in Labérenne 1936 was highly polemical, and oriented around two arguments. The first concerned a Marxist critique of Brunschvicgian epistemology: Labérene asserts a common ground between Brunschvicg and Kol'man in recognising that mathematics like any science is a historically given activity, but critiques Brunschvicg's overly 'Hegelian' (not exactly an accurate claim) or 'idealist' account of this activity when counterposed to Kol'man's 'Marxist' or 'materialist' approach to the history of science (pp. 21-23). The second argument involved mobilising Kol'mans critique of Lebesgue and Luzin's 'abstract' use of set theory, which Labérenne connects with Borel's critique of Lebesgue on the same point, in order to militate in favour of Borel's 'concrete' set theory, whence a materialist valence given to this older epistemological debate amongst the French analysts (pp. 30-31). It is hard to imagine Cavaillès having been especially drawn by this polemic favour of the 'concreteness' of Borel's mathematics pitched against the 'bourgeois ideology' of Lebesgue et. al., and most likely he is being polite in his comment about the possible reconciliation of his and Labérenne's 'attitudes'. Cavaillès's 'materialism', if we accept the term, is not in any articulated sense Marxist materialism. Cavaillès's friend Georges Friedmann was a collaborator of Labérenne's, so it is plausibly through Friedmann that they were put in touch.

12 The outlines of the biographical account here are drawn from Ferrières 2003 and Aglan & Azéma 2002.

mishes across the Warndt forest region, before being commanded back to Paris in January 1940 to join the cryptography division. Throughout the spring he complains to his superior officers of not being on the front, but does not succeed in securing another transfer. This dissatisfaction culminates in a chaotic period in which he appears to have taken it on his own initiative to abandon his posting and depart for the front from the Gare du Nord on the 10th of May, on the eve of the German invasion. He becomes tangled up in the retreat of French troops around the 8th of June, goes missing during a reconnaissance operation, and is falsely reported as having been killed in action.

In fact, Cavaillès had been captured on the 11th of June just south of Saint-Remyen-l'Eau and interred in the temporary prisoners of war camp that had been erected in the Citadell in Cambrai. He remains there for more than a month, contracting dysentery in the process, until the decision was made to deport all French officers to Germany on a transport passing through Belgium and departing on the 25th of July 1940. He is transported through Tournai, then Renaix, then Gand, before the transport pauses at Lokeren, near Antwerp. There Cavaillès effects his first escape, scaling down from the second floor of a mill where the prisoners were briefly being held and into the garden of a convent, where he is assisted by a nun in changing into civilian clothes. From there he returns first to Paris and then in the autumn to the free zone in the South where he resumes his teaching post at Clermont-Ferrand, the faculty from Strasbourg having been forced to decamp there. All of his possessions are abandoned in his old Strasbourg apartment.

It is there that in early 1941 Cavaillès forms Libération-Sud along with Emmanuel d'Astier de la Vigerie and Raymond and Lucie Aubrac (the latter having been an acquaintance back in his teaching days at Amiens). At this point the organisation is primarily conceived as a propaganda operation, with its major organ being the clandestine newspaper *Libération*. In early March 1941 Cavaillès drafts the opening tract of this publication in collaboration with d'Astier and Jean Rochon: 'This newspaper will tell you some truths that our government, on the orders of the Germans, is obliged to hide from you. This newspaper will indicate some tasks that fall to the French who have not GIVEN UP. As such, this newspaper will not be a piece of paper, but an act.'[13]

13 Opening page of *Liberation 1*, July 1941. Ferrières doesn't mention Rochon in her account of the authorship of this text at Ferrières 2003: 178/Ferrières 2000: 144, but the three names are listed by Alya Aglan in Aglan & Azéma 2002: 85.

On the 18th March 1941, Cavaillès is requested to return to Paris to teach at the Sorbonne, temporarily replacing René Poirier in the chair of methodology and logic of the sciences.[14] That spring onwards he gives two lecture courses: the first on logic and the philosophy of mathematics, the second on physics and general philosophy of science and bearing the title 'Causalité, nécessité, probabilité'.[15] In the autumn he continues to teach on logic and teaches a second general philosophy course on 'Space and Time'. These lectures are attended by, amongst others, Suzanne Bachelard, Jules Vuillemin, Gilles-Gaston Granger, Francis Courtès, Jean Gosset, Geneviève Rodis-Lewis and Marie-Louise Gouhier-Dufour.[16] It is owing to the latter that we have a partial transcript of the first course on logic and a full transcript of the course on physics; some notes of Granger's also survive.[17] These courses cover material that will eventually be directly recapitulated in the *LTS* manuscript. Alongside these teaching activities Cavaillès works under a number of pseudonyms to establish links between Libération-Sud and the nascent Libération-Nord. The fact that his name is publicly advertised in association with the courses he is teaching causes him to worry that he will be arrested after each lecture; the rest of his official life is carried out under the assumed name Jacques Carpentier, who is renting an apartment on the rue du Docteur-Roux.[18] Some time in the midst of all of this, that is between his escape from the prisoner transport in July 1940 and the autumn of 1941, *Transfinite and Continuum* gets written. Canguilhem also places the posthumously published fragment 'Mathématique et formalisme' in this period.[19] In the summer of 1941

14 Aglan & Azéma 2002: 85.

15 C.f. Cortois 2019: 51.

16 C.f. Schwartz 2020: 174, Sinaceur 1987: 119, Cortois 2019: 51.

17 Granger's notes can be found at the BNF, fonds Gilles-Gaston-Granger, cote NAF 28690 (81). It is not entirely clear whether Cavaillès taught one extended course on logic, or two successive courses. At the limits, we can say that Cavaillès was teaching at the Sorbonne from March 1941 through to the end of May 1942. See my discussion in §II.2.

18 C.f. Ferrières 2003: 183/Ferrières 2000: 150 and Aglan & Azéma 2002: 87.

19 C.f. Canguilhem's editorial note to the original publication, Cavaillès 1949: 158. This note is not reproduced in the *OC* version, i.e. *OC* 659–664. To be precise, Canguilhem argues that the piece can be definitively dated as being written posterior to the period of the doctoral works—i.e. from later than 1937—and prior to the composition of *LTS* in late 1942. He takes the high quality of the handwriting and Cavaillès's having placed his signature at the end to indicate that it was destined for publication. He then goes on to suggest that the metaphor therein of the mathematician's gesture closing what was left open in a problem 'like a pounce after the hunt [*comme un saut après l'aguet*]' by its military resonance suggests that the piece was written after the commencement of Cavaillès's combat activities, i.e. later than September 1939. This latter hypothesis seems to me less certain. Indeed, certain aspects of this text—not least the concept of gesture itself—bring it on the contrary more in continuity with the period immediately following the doctoral works. Unfortunately Canguilhem gives no indications as to where amongst 'Cavaillès's papers' the manuscript was found—a fact that is itself surprising, given that said papers were lost several times in the course of the war

Cavaillès writes to Brunschvicg—who being of Jewish heritage is by this point a refugee in Aix-en-Provence—concerning the status of his publications in preparation, and gives the only reference we have to *L'Expérience mathématique* as the title for a projected work, presumably having previously discussed it in person.

> [Tran Duc] Thao has written me an excellent dissertation on Husserl, a Husserl that has been a bit Hegelianised—or even Finkised.[20] Moreover since the 15,000 pages of Husserl's papers have been made available to him and since he is bursting with energy I have advised him to take a look at them over the vacation. The results could be published… if we had the paper. You know that both journals are out of action.[21] I had written a rather long article for *Revue philosophique* on recent work by Kleene and Church and above all on Gödel's admirable theorem which uses transfinite induction to confirm the consistency of the axiom of choice and the continuum hypothesis with the other axioms of set theory: an elevation for intuition.[22] Except that there is no paper. *L'Expérience mathématique* is slumbering; I'm only a bit grumpy about that. Before writing it I should like to have resumed an old quarrel with transcendental logic, and especially with Husserl's logic which Thao's thesis has given me the occasion to reflect on again.[23] In the *Krisis* there is a somewhat exorbitant use of the *cogito*—and the superimposition of descriptions of acts is deceiving and worrying.[24]

Such is the position of Cavaillès's work one year before the period of composition of *LTS*. He has completed a couple of epistemological studies posterior to his doctoral works, given some synthetic lectures on modern logic, and, in the course of doing so, found the intimations of a new philosophical standpoint. He understands that articulating this will involve an explicit confrontation with both Brunschvicg and Husserl, and the planning of a new work developing on the concept of mathematical experience. Around this time he also spoke several times of wishing to write a 'Treatise on Logic', and Bachelard invited him to write his course at the Sor-

—and it is not found amongst the otherwise well-organised documents relating to Cavaillès in the Canguilhem archive at CAPHÉS. As such, questions as to the precise dating of the text must be left up to hermeneutical intuition.

20 This was an early version of Thao's later *Phénoménologie et matérialisme dialectique,* published in 1951. Sadly the original thesis that Cavaillès would have read is lost. C.f. Benoist & Espagne 2013: 92.

21 Presumably a reference to *Revue de métaphysique et de moral* and *Revue philosophique de la France et de l'étrangers*.

22 The article in question is of course what is later published as *Transfini et continu*.

23 It should be emphasised that—on my reading of events—this is probably the occasion for Cavaillès's first serious re-engagement with Husserl. Cavaillès had read 'Husserl's *Logic*' in the early 1930s and at some point seems to have wanted to gain a greater familiarity with the phenomenological movement, making efforts to see both Husserl and Heidegger during his travels in Germany around that time, but aside from adopting the vocabulary of 'thematisation' from Husserl in *MAF*, he ultimately did not pursue this engagement in any programmatic sense until writing *LTS.* It is significant for my interpretation in §II.1.iii that I assume Cavaillès did not read Husserl again, at least seriously, until *after* the 'Mathematical Thought' presentation with Lautman to the Société française de Philosophie in February 1939.

24 Ferrières 2003: 182.

bonne up for publication with PUF.[25] It is viable to either interpret things such that there is a single projected work at stake here or that Cavaillès's intention had been to write up his survey of the logical field and develop his criticisms of Husserlian phenomenology *before* embarking on a separate positive doctrinal work on mathematical experience. In any case less than one whole work would ultimately be completed. In that text the term *l'expérience mathématique* will not appear.

The first half of 1942 was, according to Ferrières, marked by increasing dissatisfaction with propaganda work and a commitment to direct action.[26] Nevertheless, Cavaillès ends up with joint responsibility (along with his old Normalien friend Jean Texcier) for the production and dissemination of a second clandestine newspaper, *Libération-édition de zone Nord*, between March and September of 1942. The newspaper does not seem to have flourished under his editorship and Cavaillès himself assigned little importance to it.[27] The priority however has become re-establishing contact with the resistance leadership in London. To this end a mission is organised, with Cavaillès and others to leave by submarine from a beach in Narbonne in early September 1942. The submarine launch is noticed by customs officials; some of the *réseau* manage to swim aboard, but not Cavaillès or his collaborator Christian Pineau. They attempt to flee but are caught and arrested on the outskirts of town, and Cavaillès is interred in the prison in Montpellier.

His time in Montpellier is split between philosophical work and an escape attempt involving sedative (for the guard) and a hacksaw smuggled in by Lucie Aubrac. The plan doesn't go off and Cavaillès appears to have resigned himself for a moment to writing up his logic course. Owing to the leniency of the commander of the military forces in the Montpellier region, General Jean de Lattre de Tassigny, Cavaillès is allowed visits from friends who bring him some necessary books.[28] On 4th November 1942 he writes to Lautman requesting some of these.

> Your offer of books comes at the right time. I am trying to write up my lectures on Logic, cut off from my notes as well as the books I have in Paris, which no one here has heard of. I refer, in particular, to Hilbert [and] Ackermann (2nd ed.), Lewis and Langford, Tarski,

25 As Canguilhem and Ehresmann will later note in their introduction to *LTS*, he would wryly comment after stating this ambition that '"I'll only have time to write it in prison." Which is exactly what happened.' *LTS* ix/*OC* 479.

26 C.f. Ferrières 2003: 183/Ferrières 2000: 150.

27 Aglan & Azéma 2002: 87–89.

28 At least, many sources state versions of this anecdote, citing books in the plural. It is unclear whether any except Husserl's *Formal and Transcendental Logic* in fact arrived.

> Gödel, and the *Entretiens de Zurich*, which would prove most helpful. Also if you could find in Toulouse Husserl's *Formale u. transzendentale Logik*: it is with reference to him and a bit against him that I try to define my own position… It would be precious to be able to talk to you about all of that. (Do you possess a copy of our old enemy the *Logische Syntax der Sprache*?)[29]

Some time between the 4th and the 11th of November Lautman comes to visit from Toulouse and, as is evident from the presence of multiple references in *LTS*, is indeed able to bring Cavaillès a copy of *Formal and Transcedental Logic*.[30] It does not seem as if Lautman was able to locate the other works that Cavaillès had requested. During this period Cavaillès sends a postcard to the pastor at the prison, Jean Cadier, stating that he 'desired the presence of the church around him.' On his first visit, Cadier asks Cavaillès if he can get any books sent to him. Cavaillès responds 'The New Testament and Hegel's *Logic*, if possible the Berlin edition.' We know that a Bible was provided because Cavaillès thanks the priest for having found a copy of the New Testament with the psalms as '[t]hat way, the Israelites here in prison with me will be able to read it.' Given the almost complete absence of citations of Hegel in *LTS*—there is one, seemingly transcribed from memory—we can assume that Cavaillès was not so lucky when it came to the *Wissenschaft der Logik*.[31]

This more luxurious portion of Cavaillès's imprisonment comes to an end on 11th November 1942 when the German forces invade the free zone in the south of France. General

29 Ferrières 2003: 188/Ferrières 2000: 154. The reference to the *Entretiens de Zurich* is to Gonseth 1941.

30 C.f. Canguilhem and Ehresmann's comment to this effect, *LTS x*/*OC* 480.

31 The one quote from Hegel in Cavaillès's manuscript—which is the only reference to Hegel in his published writings—is a paraphrase from the introduction to the 'Subjective Logic', which Cavaillès mobilises against what he sees as Kant's impossible invocation of a purely extra-conceptual 'matter': 'This is what led Hegel to observe: "A matter (or content) without its concept is something extra-conceptual, thus without essence."' *LTS* 5. As Canguilhem and Ehresman note, Cavaillès did not include a page reference, and they had to consult with André Kaan to divine what exactly from Hegel Cavaillès was drawing on. I take this as strong evidence—if not definitive proof—that Cavaillès did not have the *Science of Logic* to hand while he was writing. Paul Cortois (personal communication) surmises that Cavaillès's request for the 'Berlin edition' of Hegel shows that his familiarity was already more than passing at this point. What is certain is that Cavaillès at some moment engaged extensively with the work, owing to the recent discovery of a collection of undated, handwritten notes of Cavaillès's containing transcribed passages from the *Logik* (42 A5 handwritten pages, mostly sections from Book I and Volume II), as well as from Plato (28 pages, mostly passages from the *Parmenides* and some from the *Republic*), and some difficult to parse notes on Gödel (4 pages). Given the content of these notes, we can presume them to have been made ether for the purposes of Cavaillès teaching at the Sorbonne in 1941-1942 or while revising the manuscript of *LTS* in early 1943; it would of course be interesting to establish if they are from before or after the main period of composition of *LTS*. The notes are accessible online here: https://cavailles.hypotheses.org/vie-de-cavailles/manuscrits-correspondance-et-cours-de-jean-cavailles. The account of Cavaillès's request for sources in Montpellier is taken from a letter from Cadier written to the journal *Le Christianisme au XXieme siecle* and published in the issue of 16th November 1950, pp. 492-493. It is catalogued as Fonds Cavaillès, box 2, item 79.

de Lattre refuses the order from the Vichy government to stand down the troops under his command and he is arrested. On 12th November Cavaillès and several other important prisoners are transported to the internment camp Saint-Paul-d'Eyjaux outside of Limoges. The resistance attempts a breakout en route and Pineau escapes but Cavaillès does not. Conditions at the camp are very harsh. It is cold and Cavaillès contracts both impetigo and *carotène*, the latter an effect of a diet consisting of carrot soup produced from animal feed. Cavaillès continues to write *LTS* while communicating with Libération-Sud and making escape plans. On 25th November he gets a package via Marcel Ferrières, the husband of Gabrielle. On the 29th he gives a lecture on Descartes to the other prisoners.

> This afternoon, there was a lecture in the yard by professor Cavaillès, holder of the chair of philosophy at the University of Strasbourg, connected to the chair of logic and comparative philosophy at the Sorbonne, interned here for the past fifteen days: now that is what one calls a restoration of values! A remarkable afternoon is every way, owing both to the high level of thinking and the effort towards vulgarisation made by the speaker. The subject treated was 'Descartes and the Discourse on Method'. A certain number of comrades were disappointed however, as they had expected to learn new tricks for playing *coinchée* or *belote*.[32]

So things go until late December. A draft of *LTS* is completed. Cavaillès has been informed of another escape attempt involving Marcel Ferrières, and decides to pre-empt this so as not to risk the family's involvement. On the night of the 29th of December, assisted by a Communist worker whom he has befriended and who has been allowed to maintain a workshop on the camp perimeter so that he can complete certain repairs, Cavaillès walks to said workshop carrying his toolbox and the manuscript of *LTS*. He escapes into the snow. Weakened by hunger and by the maladies he has contracted in the camp, and evidently an escaped prisoner, he almost doesn't make it very far, but on the road to Limoges a truck stops and allows him to stow away in the back.

From there the resistance network is able to help. He travels to Toulouse intending to rendezvous with Lautman, but has mixed up rue Théophile-Gauthier with rue Théodore-de-Banville, where Lautman in fact resided.[33] He continues to Lyon, reconnecting with Libération-Sud, and then on to Paris in the early days of January 1943. On his return he resigns from information work in order to focus directly on organising paramilitary actions.[34] Things get

32 Georges Rougeron, *Journal Inedit*, 1942, cited in Douzou 1998: 149. At the cost of explaining a joke, *coinchée* and *belote* are traditional French card games—*jeux « de cartes »*.

33 Sinaceur 1987: 127.

34 Aglan & Azéma 2002: 118–120.

politically complicated, as this is a matter of some controversy within Libération-Nord: on one hand, the socialist and syndicalist members of the network want to focus more on political propaganda and the development of a mass line, and are consequently less committed to tactics of outright combat; on the other it bears on a growing power struggle between Cavaillès and Pineau over control of the *réseau* and its relation to the command in London. On Aglan's interpretation of events, this becomes a major factor in Cavaillès's urgency in organising a second attempt to travel to London, because it will allow him to resolve this domestic dispute by invoking the high command.[35] While this struggle is going on, Cavaillès continues to work on *LTS*, at least enough to produce a typescript and attempt to put the references in order. On 14th February 1943 he writes to Lautman:

> You will have learned by now of how the confusion between Banville and Gauthier deprived me of the joy of seeing you in person. Too bad, as our good master used to say, what you expect never comes to pass. What doesn't arrive either when you put a false address is parcels. I had my clothes sent from Limoges by rail (a 'Canadian', a ski suit, loafers, the only good ones I had), and above all I'm afraid [some] booklets by you, my dear friend, and a few of my books. If you can find them, you'll take a big weight off my mind.[36]

To which Cavaillès appends a list of the items that had been sent to Toulouse, including works by Herbrand, Gentzen, Hilbert and, primarily, multiple articles by Tarski.[37] Ultimately there are no citations of any of these works in the surviving typescript of *LTS*, and amongst the authors the only ones mentioned by name are Hilbert and Tarski. It is also unclear whether the requested works were needed for revision of the existing text of *LTS* or for the historical and technical introduction that Cavaillès planned to write in order to situate it. Either way, before his departure for London at the end of February, Cavaillès gives a copy of the text as it stands to his sister: '"If I don't come back," he said, "Publish this. It's my philosophical testament. I am, I'm afraid, unhappy about the [lack of an] introduction without which, however, the book will prove too difficult and demanding for my readers."'[38] To this version of the manuscript he appends a note which, had it been included at the head of the published version, would likely

35 Aglan & Azéma 2002: 120–122.

36 Sinaceur 1987: 127.

37 To be precise, the works listed by Cavaillès are: Lewis & Langford 1932, Herbrand 1931a, Herbrand 1931b, Gentzen 1935a, Gentzen 1935b, Gentzen 1938, Hilbert & Ackermann 1928, Tarski 1933, Tarski 1935a, Tarski 1935b, Tarski 1935c, Tarski 1936 and Tarski 1938.

38 Ferrières 2003: 201. This is the same passage in which Ferrières refers to *LTS* as Cavaillès's *Traité de logique*.

have given a rather different impression of the work than that suggested by its present title: 'In numbers, I find God again.'[39]

On 24th February 1943 Cavaillès departs successfully for London on the second attempt, again via submarine. He spends 6 weeks there, meets with de Gaulle, and refuses an offer to remain in England in order to be involved in the reconstruction after the war. While there he also meets with Aron, the source of the oft-cited anecdote on Cavaillès's 'Spinozism', taken from Aron's obituary for Cavaillès: 'I'm a Spinozist; I believe we submit to the necessary everywhere. The concatenations of the mathematicians are necessary, even the stages of mathematical science are necessary. This struggle that we carry out is necessary as well.'[40] On 15th April 1943 he returns to France via parachute.

What follows is an even more intense period of direct action, it seems to the increasing displeasure of some other members of the Cohors. In effect a rupture was opened up between those members of Libération-Nord who were focussed on the political future after the war and the increasingly violent and hazardous activities being carried out by Cavaillès and his collaborators. Any narrative on these points is contested; in any case the rupture became quite severe and on 15th May 1943 Cavaillès ceased communication with the Bureau central de renseignements et d'action (BCRA) over security concerns.[41] Soon Cavaillès was organising actions completely on his own, and a message delivered to London on 20th August 1943 reads rather dryly '[a]s you must have learned already, I have given up my duties as a regional delegate [for Libération-Sud] and as a member of the directing committee of Libération Zone Nord in order to devote myself exclusively to military tasks.'[42] All of this was of course attracting a lot of attention to him from the occupying authorities, and that same day a report delivered by Pierre Thiébaut to the resistance command back in London explicitly requested for Cavaillès to be brought back over the channel for his own protection.[43] On 28th August 1943 he was arrested for the final time on the Boulevard Saint-Michel, between Port-Royal and Luxembourg stations; the same night his sister and her husband would be apprehended at their apartment.

39 « Dans les nombres, je retrouve Dieu » . Allocution de Maurice Leenhardt, item 55, fond D, Fonds Cavaillès. C.f. Peden 2014: 48.

40 Cited by Canguilhem in his 1967 inaugural address for the amphithéâtre Jean-Cavaillès, reprinted at *OC* 674. Translation from Peden's introduction to Cavaillès 2021: 19.

41 That is, with the information bureau of the organised resistance.

42 Aglan & Azéma 2002: 129.

43 Aglan & Azéma 2002: 133. 'I would also like to specifically urge you to issue directions for the withdrawal of our friend Marty [Cavaillès's pseudonym in resistance communications], for whom a prolonged absence is truly essential. I know that he is actively sought after and every move he makes puts him in real danger.'

Throughout this period revisions on *LTS* had not ceased, at least not entirely, and on 16th August 1943 Cavaillès had managed to send a last letter off to Lautman's wife Suzanne, which contains what can in context only be read as a slightly obsessive enquiry about the same works at stake in the letter to Albert of 14th February 1943.

> I've started working again at logic—on the one hand I should like to know if you finally managed to recover those precious booklets, on the other also I should like to have the *Entretiens de Zurich,* which are decisive for many-valued logics. Since all my notes have been lost and I must reconstitute it all, dear Łukasiewicz's article would be a precious aid in this task that requires patience. Send all that to my sister also.[44]

From the three letters to the Lautmans and the comments recounted by Ferrières, it is clear that Cavaillès was continuously working on revising *LTS* right up until his arrest. Most probably he was attempting to write new material for the missing introduction. This at least is the best explanation for his request for the article from Łukasiewicz in the last letter, given that nothing even really adjacent to many-valued logics is mentioned in the extant text of *LTS*. What is less clear is what state of redaction the surviving copy of the manuscript represents. For it is not the typescript given to Ferrèires by Cavaillès before his departure to London at the end of February 1943 that ends up being edited for publication by Canguilhem and Ehresmann. On the contrary, Ferrèires's copy was seemingly lost in the arrest of her and her husband and the attendant sacking of their apartment. The copy that survived was rather a typescript sent at some point by Cavaillès to his friend Mme. Foulquier, and which would be handed to Canguilhem and Ehresmann (via Ferrières) after the war.[45] It is thus not possible to determine from exactly what point in the process of revisions this typescript came, but we can say with confidence that it was not the most up-to-date version. Most probably it came from the same period of initial revisions to the manuscript in January-February 1943 as the copy handed to Ferrières and then lost, but possibly it was from the second period of April-August 1943.

Either way, Canguilhem and Ehresmann make by their own account the decision to exercise a deliberately light touch on the editorial side, effectively restricting themselves to fixing typographical errors and supplying some missing references. In particular, and in con-

44 Ferrières 2003: 215/Ferrières 2000: 180. The article to which Cavaillès refers is Łukasiewicz 1941. Plausibly there is some error of transcription here around the reference to 'cher Łukasiewicz', with whom Cavaillès does not seem to have had personal relations.

45 Ferrières 2003: 201/Ferrières 2000: 167. In their introduction to *LTS,* Canguilhem and Ehresmann describe the typescript as having preserved by Ferrières, but it is Foulquier's copy which is at stake. C.f. *LTS xi*/*OC* 481. The Foulquier typescript is in neither the Fonds Cavaillès at the ENS nor the Canguilhem archive at CAPHÉS, and must be presumed lost.

sultation with Ferrières, they decline proposals that either Gonseth or Dominique Dubarle provide an introduction in lieu of the one that Cavaillès was not able to complete. The case of Gonseth here is intriguing. In a letter written to Ferrières on May 7th 1946 he states that Cavaillès had in fact written a draft introduction and then destroyed it. 'I was informed that Cavaillès had torn up and burnt the introduction he had written. The work lacks, in effect, an introduction that would situate it, and which would show how it intervenes in the most current questions, how it brings an invaluable judgement on the most modern currents. I am ready to write such an introduction.'[46] Gonseth also proposes to publish the manuscript in his collection *La science dialectique*. Ferrières forwarded this proposition to Canguilhem, and in a draft of an angry letter preserved at CAPHÈS, Canguilhem refers to this notion as *indécente*, citing the agreement of Ehresmann, both on the grounds that he would expect such an introduction to simply enlist Cavaillès's work into Gonseth's own philosophical project and that it would be inappropriate for a Swiss philosopher to present the legacy of such an important text from an executed French philosopher-resistant.[47] In the case of Dubarle, he was with Ferrières's agreement approached in early 1947 by Aron, who was president of the newly formed Société des Amis des Jean Cavaillès, to write a technical introduction to the work, which Dubarle accepted with the proviso that he would only be able to do so after the summer holidays. In the end it was decided that the urgency to publish the text meant leaving it without an introduction and *LTS* is published in 1947.[48]

What, then, to make of all this textual history? Aside from the intrinsic dramatisation of the order and concatenation of things and of ideas therein, the importance of the preceding accounting is a follows. A first cluster of points concerns its references, both present and absent. A second concerns the aim of the text and its genre.

In sum, *LTS* is composed between early September and the final days of December of 1942. It was revised after that but we cannot assume that the version we have substantially reflects those revisions. With respect to the references, we know that Cavaillès had a copy of

46 CAPHÉS Dossier G.C.32.3.2. Ferrières for her part never spoke of the intentional destruction of a draft introduction, but only of manuscripts as having been lost in the chaos of her and Cavaillès's arrest (personal communication with Paul Cortois). This is thus probably a misunderstanding on Gonseth's part.

47 CAPHÉS Dossier G.C.32.3.2. Gonseth would ultimately write up his interpretation of *LTS* in Gonseth 1948: 324–327.

48 See the account in Schwartz 2020: 179. It is also possible that there was disagreement about Dubarle's highly Hegelian interpretation of the text, which would eventually be published in two parts in the *RMM* the following year as Dubarle 1948a and Dubarle 1948b. For a history of the Société des Amis des Jean Cavaillès see Baptiste Mélès's account in Mélès & Szczeciniarz 2019: 127–140.

Husserl's *Formal and Transcendental Logic* with him whilst writing, but not that he had much else, despite references in the secondary literature to his having been supplied with 'works' in the plural while in prison. This is relevant for giving one possible account of the prominence that *FTL* assumes in the text. It is clear that Cavaillès had intended for some time to have 'resumed an old quarrel with transcendental logic'. But the outsised role of the section interfacing with *FTL* in *LTS* (there is no other remotely comparable textual engagement with another philosophical work in Cavaillès's writings) must also at least in part be chalked up to the fact that this was the work he had as a companion. Alongside this, the Kant citations in Section I of the text are also substantial enough to presume he had a chance to consult the Jäsche *Logik*, either in Montpellier or (more likely) whilst doing revisions in Paris in early 1943. He did however make repeated efforts to acquire material specifically bearing on the state of the art of the Hilbert programme and on model theory in the Tarski/Łukasiewicz line. We can presume this to have been for the purposes both of writing the introduction and of bringing Section II of *LTS* up to the same standard as Section III so far as references are concerned. A reasonable presumption—impossible to confirm—is that a version incorporating some of these references was being worked on, posterior to the typescript sent to Foulquier, and that the intended version of Section II of the text would have revised some of the material on Tarski and semantics (i.e. *LTS* 33-40), but it is possible that Cavaillès considered the existing text to be close to finished, and that the sources he was pursuing were intended solely for the separate introduction, be it unwritten, unfinished, or simply lost.

These points about Cavaillès's intended sources also bear on the genre of the text. What *LTS is* is a text on logic. The most certain way to read it is not as a draft of a book called *L'expérience mathématique*, but rather as a writing up of the 1941-1942 Sorbonne lecture courses. Here, we minimally have to disagree with Sinaceur's judgement that the manuscript we receive as *LTS* was a draft of an original work (either a 'treatise on logic' or on 'mathematical experience' or both), separate from Cavaillès's writing up of the logic course demanded by Bachelard, which she takes as a separate project.[49]

More generally, *LTS* ought to be read as Cavaillès's mature accounting with the project of philosophical logic, so to speak, 'after' the advent of modern mathematical logic. Whence a trajectory that goes from Descartes through Kant and then to Bolzano, Carnap, Tarski, and Husserl. It is significant that this material on the philosophical tradition of logic was expected

49 Sinaceur 1987: 119.

to be supplemented by an introduction that can be expected to have related the conceptual development in the main body of the text to precise technical issues in modern logic and proof theory. In particular, we can presume from the texts that Cavaillès was requesting in his correspondence (Hilbert, Herbrand, Gentzen) that the puzzling absence of the Hilbert programme (or indeed much concrete detail on modern mathematics and logic at all) from *LTS* is an artefact of its truncated composition. In any case, we can hazard that the projected work, including the introduction, would have contained a quite different dialectic between effective mathematics and philosophical conceptualisation than we find in the more narrowly 'philosophical' extant text.

APPENDIX II – TWO THESES FROM *ON LOGIC AND THE THEORY OF SCIENCE*

Cavaillès was a writer for whom the basic unit of thought was the paragraph. This poses a problem for the expositor, given that the length of said paragraphs makes it unwieldy to cite them whole cloth. As outlined in the Introduction, this dissertation is structured as a commentary on two such units from On Logic and the Theory of Science, *both of which (explicitly and implicitly) refer to the 'Cartesian' doctrine of concatenation, and which I take as expressing central 'theses', insofar as it is legitimate to extract such a thing from that work. Characteristically, these theses are announced only laterally, in the course of Cavaillès's treatment of other writer's problems (principally Bolzano and Carnap respectively, but also Descartes, Plato, Kant, Tarski…), and thus not presented as doctrines. Nonetheless, I understand them to be points where Cavaillès implicitly affirms something of his own standpoint on the 'theory of science'.*

I present here in full the paragraphs in which these 'theses' are advanced, as they will be referred to passim. In the case of the Thesis on Evidence, *it is necessary to include the end of the preceding paragraph for legibility.*

This is also perhaps the appropriate place to mention the relationship of all the translations from LTS *in this dissertation to Robin Mackay and Knox Peden's invaluable 2021 English edition of that book. This was published roughly half-way through my primary research, when I had already been working with my own renderings of many of the central passages. I was also fortunate to have been able to give some minor input on a late draft of the published translation (be it for good or for ill, a consequence of this is that I have inserted my translation of* enchaînement *by 'concatenation' into the Anglophone reception of Cavaillès's work). Having access to Mackay and Peden's rendering of this very difficult text was of course an essential aid in producing my own interpretation. However, in the palimpsest that a doctoral study inevitably becomes, the line has eroded between my uptake of passages from Mackay and Peden's translation, modifications of the same, and original renderings on my part. As such—and given that Mackay and Peden retain a marginal index to the original pagination of* LTS*—I have opted not to independently reference their version, but each time only to cite the* LTS *numbers. All translated citations (and the errors therein) are to be treated as my 'own', but build on the work done by previous translators, both Mackay and Peden (2021) and the earlier version by Theodore J. Kisiel (1970).*

The Thesis on Evidence, or the Explicit Cartesian Reference – *LTS* 23-25

[…] The autonomy of science is simultaneously expansion and closure: a negative closure through its refusal to borrow anything from, or end up arriving back at, what is outside of it. If the idea of total knowledge is meaningless—with absolute consciousness there is a hiatus just as real as that with opinion, such that there can be no question of either preparing for it or of leading up to it—the idea of the radically extra-scientific fares no better. In order for it to be posited as outside [of science], it would have to be placed on the same plane, and thus it would already be an experience. Science is a Riemannian volume which can be at once closed and not have anything outside of it. No method that issues from the doctrine [of science] can recover Descartes's decision: '*sub scienti[am] non cadit*'.[1]

The difficulty encountered here not only concerns how to justify these characteristics and how to specify them, but how to situate the discipline that posits them. The doctrine of science makes claims both to validity and to intelligibility; it seeks to be the science of science, and thus a part of itself. It follows that its statements must not be constitutive of any particular development, but rather must appear immediately in a self-illumination of the movement of science, while being distinct from this movement owing to their perpetual emergence. Such is the role of structure. By defining a structure of science which is nothing other than a manifestation to itself of what it is, one specifies and justifies the preceding characteristics, not through an explication that would mark out its own place and which could then become, in its turn, an object of reflection, but rather through a revelation that is not distinct from what is revealed, present in its movement, principle of its own necessity. Structure speaks of itself. Yet this is only a way to impose itself through an authority which borrows nothing from the outside; there is only a mode of unconditional affirmation, that is, proof. Not only is the structure of science proof, it is indistinguishable from proof. The essential traits of science are indeed found in it: unity, necessary and indefinite progression, and finally closure upon itself. The internal rule that governs proof posits each of its stages, and is contained in all of them, but is impossible to grasp *ne varietur*[2] in any one of them: we are reminded of Descartes's difficulty concerning the evidences—or simple natures—in their concatenation. We can multiply the

1 {'it does not fall within the scope of science', Descartes, letter to Mersenne, 13 November 1629 (*AT* I 73)}

2 {unaltered, without varying}

hooks; if the demonstrative link consisted in these, we would have to indefinitely superimpose more and more of them. In reality there is no essential distinction between the hardened rings that seem to mark the terms and the movement that runs through them. But this movement does not halt: proof, by the simple fact that it posits its goal, extends and ramifies the domain it has created by means of the new combinations which it establishes as soon as they become possible. Ultimately, proof cannot be linked back to some unproved term: we come back to Plato's reservations with respect to the *dianoia* that borrows its hypotheses from the visible world, an arrangement without justification which can never yield any necessity. True science never takes leave of what is proved. The modern conception of hypothetical-deductive systems is subject to the same critique. How could a principle or group of principles which in their content and arrangement are not themselves intelligible, serve as the point of departure for an intelligible unfolding? The heterogenous alliance of a pure concrete observable and a mode of rational concatenation is merely an image without thought. If there is science, then it is proof through and through, which is to say, it is logic.

The Thesis on the Intelligible, or the Implicit Cartesian Reference – *LTS* 35-36

The difficulty lies elsewhere: it is a question of knowing how the system that will be the basis of all syntaxes is to be constructed. What Carnap calls general syntax is only a set of abstract rules, and moreover everything precise about it is borrowed from effectively realised mathematics and its syntax: the syntactical imagination seems to lose itself in the emptiness of a radical abstraction. By way of the several details he provides, one can even show that the paths predicted in advance, paths that were supposed to have been taken by later science, have remained barren. More than just an error of foresight, there is a serious failing here, a misunderstanding of what is essential in formal concatenation: its progression, which is both necessary and conditioned at every step by the effective. A simultaneous positing of all possibilities, even if it is taken to be unrepresentable and as such simply admitted as a basis, contradicts the notion of a formal system, the meaning of which implies that it be generated by successive movements of fragmentation and surpassing. The fact that everything cannot be all at once has nothing to do with history, but is the characteristic of the intelligible—to misunderstand this is to take leave of the security of its immediate presence in favour of a projection, not only the internal emptiness of which, but also its collapse into the historical, is revealed in its being bound up with the actualisations of a certain time and the contingencies in which they are clothed. There is no attainment of the absolute here, only a hypostasis of systems and procedures whose status is but transitory. To abstract in this way is not to fix the essence, but to halt.

APPENDIX III – TRANSLATIONS

Reflections on the Foundations of Mathematics (1937)[1]

Summary: The current state of the problem of the foundations of mathematics is examined. We are obliged to reject the formalist solution. The intuitionist solution is criticised. Analysis of the process of mathematical abstraction demands a position of immanence.

The failure of the formalist school in the face of the technical problem on which it had made the foundations of mathematics depend has had the advantage of clarifying both its own philosophical position and those of opposing theories. In effect:

1) Gödel's theorem has shown the illegitimacy of the Hilbertian attempt to ally the gains of finitism with the body of classical results, given that the metamathematical domination of a formalism is impossible with less powerful means.

2) This same theorem restores the correlative notions of language and of mathematical work to their true relationship [*rapport*], which formalism had confused in its image of a game of meaningless symbols. A symbol is only what it is owing to an intention that is directed towards intuitive operations. Mathematical language is only an indication that points towards effective acts in the mind of the mathematician.

Ought we, then, adopt the Brouwerian solution, with, at the same time, its definition of the specificity of the intuitive activity that gives rise to mathematics, and the transformations of technical development that it then deduces from this definition? It does not seem so, once we analyse the reasons through which Brouwer justifies it. There is, in effect, an incoherence:

1) In the general attitude of intuitive empiricism which, in its being empiricist, rejects two incontestable facts:

a) The existence of a non-intuitionistic mathematics or the agreement of the consciousness of mathematicians [*des consciences mathématiciennes*] concerning the results discovered by means of classical methods; the Brouwerian can

1 {Translation of Cavaillès 1937, reproduced at *OC* 577–580. This was a brief summary of the argument of *MAF*, as presented by Cavaillès to the 9th International Congress of Philosophy (*Congrès Descartes*). In general, footnotes throughout these translations are Cavaillès's own; additional notes by myself are in braces.}

> characterise these as inadequate consciousness [*consciences inadequate*], but if the methods employed are empty of meaning and they nevertheless produce certitude [*la conviction*], it must be a simple accident with respect to these results that are external to those of true mathematics; the permanence of these accidental encounters then becomes unintelligible. (This, in other words, is the argument given by Hilbert: how is that classical analysis, every part of it having been ploughed over and over, never gave rise to impossibilities?)
>
> b) The fecundity of the axiomatic method, which only has a heuristic value for the intuitionist, given that the system of axiomatised objects is not given by a construction (i.e., given that there is no arithmetical interpretation of the axiomatic). And this heuristic value cannot, itself, have any justification in as deliberately imprecise a theory of the process of mathematical creation as that provided by Brouwer.

2) In the characterisation of mathematical work. In effect, Brouwer misunderstands the internal dialectic of intuitive activity: for all that he defines the dyad (origin and type of mathematical work) as the process of consciousness in order escape from time, he nevertheless so completely subordinates mathematical creations to time that history then intervenes in the definition of objects (for example: *fleeing* properties, which are only such in relation to mathematical history). This too quickly exhausts the original character of intuition: if it only is insofar as it is effective, its intention is that what it attains should be independent of the event through which it attains it. Dialectic of mathematical history: the liberation from the contingent by the effective, yet the effective in its effectuation is contingent.

From these critiques, it seems that we can conclude:

1) In agreement with Brouwerianism: it is impossible to characterise mathematical activity by reducing it to something else; it is not only an intuition, but an original historical becoming. Its initial objects are neither whole numbers, nor elements of some special intuition, but rather result from the concrete steps taken in order to organise the action of consciousness in the world. The mathematical given is, at each moment in history, the system of objects with which contemporary mathematics is concerned.

2) Classical mathematics is legitimated thanks to the specification of the process of abstraction that is essential to mathematical development. Two aspects:

a) Idealisation, as analysed by Dedekind and Hilbert: an operation is posited as being possible to effectuate in an unconditioned manner (so long as the restrictions placed on this effectuation are extrinsic to the intuitively perceived reality of the operation). The addition of ideals serves as an example: the ideal objects added to one another are as much concrete objects as any other, insofar as they are possible results of the posited operation.

b) Thematisation, through which an operation becomes in turn the point of application of a superior operation (for example, the topology of topological transformations). In this case, the new domain is an intuitive field given through coordination with the previously available intuitive domain: its operations are complications of the operations of the primitive domain; yet, when one considers only what pertains to the particular properties in question, they engender new intuitive concrete acts.

Thus, the demand for possible arithmetisation (Kronecker-Brouwer) is a misunderstanding of what is specifically mathematical: the unlimited procession of original intuitive modes. Mathematics never leaves the sensible world—whence the partial truth of the Kantian theory and its modification in Hilbert's theory of the sign—but is analysis without end of the core of sensible gestures. It is only through ontological prejudice or psychologistic error that one could presume to to define them or the thought that is immanent in them.

Consequences for the problem of foundations:

1) Suppression of the unity of *theory*: the process of axiomatisation is an aspect of the double process of idealisation and thematisation; it is not the only one—and in any case it only has meaning relative to an already existing mathematical concrete. The characteristic of being consistent—which only concerns verbal expression, and which is in fact never directly studied (it always being a matter of interpretation or construction of a model)—is only the mask of an effectuation of certain intuitive operations, which is always recognised as being possible.

2) Introduction of the infinite through the notion of the arbitrary, the result of a particular process of abstraction: in a reasoning, the length of a well-ordered sequence does not intervene (potential infinity); in the reasonings of analysis, the study of a convergent sequence of rational numbers effectively only uses these two properties (being convergent and being composed of rational numbers), thus, all convergent sequences of rational numbers are ob-

jects of reasoning (Cantor's axiom). This definition of the continuum is analogous with that given by the intuitionists, but here, by abstracting from the particularity of the substituted numbers, it is always possible to establish the order (even if it is not in fact established). In the same manner, impredicative definitions can be admitted to the extent that they introduce operations the effectuation of which only requires the intervention of properties of the defined set.[2]

3) The same can be said of set theory, which is a collection of operations on any arbitrary mathematical object whatsoever, yet which already exist mathematically. The radical abstraction of the set seems to mark a rupture in the dialectical development of mathematics (set, origin of mathematical objects).[3] This abandonment of the unity of theory also makes it possible that processes of a similar degree of abstraction, the cardinal and ordinal operations, are not related to each other: the example of the problem of the continuum, which is probably unsolvable because it is insufficiently determined.

2 Thus, equally, the law of the excluded middle does not posit an object, but permits reasoning about an objects solely by taking into account its negations (the “whole” is defined by the dependence between two operations).

3 Whence the fact that the paradoxes only arise in abstract set theory.

Introduction and Conclusion to *Remarks on the Formation of Abstract Set Theory* [*TAE*] (1938)[1]

Introduction to *TAE* (*TAE* 25-29/*OC* 223-227)

There are few histories as winding as that of set theory. After a difficult beginning, in the face of mistrust justified by the failure of prior speculations on the actual infinite[2], its construction, consecrated to a mathematical universe, was hardly completed when the discovery of antinomies arrived to shake its foundations, in a disturbance which threatens to extend to analysis, and against which no remedy has so far been found to be efficacious. In 1925, Von Neumann put an end to a renewed attempt at axiomatisation by demonstrating the irreducibly non-categorical[3] character of every system of axioms that would preserve what is essential in the theory: 'We can do more than note that there is here a new objection against set theory and that for the time being no means of rehabilitating this theory is in view.'[4] These developments equally have the result of eliciting a juxtaposition of more or less different theories of the basic Cantorian *oeuvre*: there is no longer a single set theory, but that of Zermelo-Frankel, that of Von Neumann, that of Brouwer. Yet, the applications of set theory, at least in classical mathematics, demand its unity. There appears another fact here, which is not the least paradox of this history, namely the ever more widespread recourse to a contested theory within incontestable mathematics. In 1873, in a letter to du Bois-Reymond, Weierstrass made use of the countability of the algebraic numbers—which Cantor had just demonstrated—in order to advance a theory concerning the functions of real variables.[5] From 1884 with Poincaré's memoires on the structure of the domain of existence of automorphic functions, to Mittag-Leffler on the possibility of the construction of functions with given singularities, a series of applications develop which today have the highest importance in analysis.[6] Contrary to Cantor's expectation, which was directed towards analytic functions, it is overall with respect to the study

1 {Translation of introduction and conclusion to Cavaillès 1938.}

2 For example Fontenelle's *Geometry of the Infinite*, a tissue of incorrect reasonings. [General question for the editor: in Cavaillès's French footnotes are placed mid sentence, on particular notions. Preserve this or move them to the end of the sentence in accordance with the English norms?]

3 That is to say that the system of axioms considered as an implicit definition does not determine its object in a univocal manner. We consider this notion in J. Cavaillès, *Méthode axiomatique et formalisme*. Paris, Hermann, 1937, ch. II.

4 Von Neumann 1925: 240. {English translation in Heijenoort 1967: 413. Translation modified in line with Cavaillès's rendering.}

5 Weierstrass 1923: 206. {Cavaillès mistakenly cites the volume of 1924, which does not exist.}

of functions of real variables that the theory has made the most considerable progress. The works of Baire, Borel, Lebesgue have provided new modes of generating or of putting into correspondence in Cartesian space and transformed the theory of integration. The theoretical crisis which commences in 1905 has only embarrassed restrictions and scruples concerning set theoretical modes of reasoning without arresting their internal power of expansion. Today these restrictions and scruples pass to a second level, since the problem of foundations seems further than ever from a solution, all while three disciplines manifest a direct and essential dependence on set theory: first, functional analysis or general topology founded in 1909 by Fréchet, which itself finds new fields of application each day; second, the theory of measure and integration developed by Borel, Lebesgue and Denjoy; finally, combinatorial topology. Without a doubt it is a question of certain applications [*utilisations*] that each time can be precisely delimited, and certain mathematicians even take care to separate off the results obtained thanks to the more contestable parts of the theory, such as the different more or less strong statements of the axiom of choice. Yet, a method is not something that can be arbitrarily isolated according to the demands of a problem. The very demand for security requires that a method be stripped of its accidental surface, found in a particular case, in order to clarify the necessary and sufficient conditions of its application [*application*]. If it succeeds for certain sets, this can only be in virtue of properties that are definable in them: there will be rigorous mathematics only once we have, through the investigation of the procedures themselves, defined the field of objects with which they agree.[7] On the other hand, these procedures are concatenated together in an organic complex: it is a state of mind that constitutes the secret basis from which they burst forth with an apparent unpredictability, and to which it is necessary to return if one wishes to understand it and not merely to exhibit it but to take it further. Whence the profound kinship that unites—without their having any full awareness of it[8], as an implicit agreement—the researchers of a single discipline, at least in a given period, *vis a tergo*[9], an objective momentum [*élan objectif*] of the research itself.[10] Does set theory possess this double unity, which, for a theory, is the justification of its mathematical existence? Researching the first [unity] is technical work: we have seen the difficulties that it encounters.

6 C.f. Denjoy 1937: 12. 'The greatest revolution that mathematics has known since the invention of differential and integral calculus has just commenced.'

7 {i.e. Unity 1.}

8 [Remove the 'of it'?]

9 {Propelling them from behind.}

10 {i.e. Unity 2.}

But, perhaps, if only in order to evaluate its chances of overcoming them, it is possible to try and detect the second [unity], there where it exists, beneath the juxtapositions—perhaps factitious [*factices*]—of an arbitrary construction.[11] It is here that history intervenes.

Mathematical history seems, of all histories, the least connected [*liée*] to that of which it is the medium; if there is a connection [*lien*], it is *a parte post*, of use for curiosity, but not for understanding [*l'intelligence*] the result: what comes after explains what comes before. The mathematician has no need to know the past, because it is their vocation to refuse it: to the extent that they do not submit to what seems incontestable by dint of the fact that it is, to the extent that they reject the authority of tradition, disregard an intellectual climate, to that extent alone they are a mathematician, which is to say a revealer of necessities. Yet, with what means do they operate? The history-negating work is brought about in history. Double relation [*liaison*]: with the problems posed and studied at a time (choice of rebellion), and with already existing methods, material with which to forge the new instrument. In neither case does the individual will or the style of an environment suffice for an explanation: even if one conceives of mathematics as a system in itself, the sinuous parts of the processes of revelation would be in relation with the structure of the already revealed parts. Said otherwise, there is an objectivity, mathematically founded, of mathematical becoming; it is the exigency of a problem that makes it necessary to strip a method of those accidents that no reflection could fail to perceive as useless, it is the internal vigour of a method that surpasses its primitive field of application and poses new problems. There is a connection [*connexion*] of a mathematical system to a time, through which the reciprocity of actions excludes equally interior lacunas and the imaginary of an exterior void in which the fixity of a model could be situated.

Within this indetermination, it is difficult to make a division between social or psychological concatenations and mathematical necessity—one sees this with respect to simultaneous discoveries, which could equally be explained by the presence of an instrument or the acuity of a question, or alternatively by the attention that was sociologically focussed on them. The unquestionably objective elements are those knots that constitute theories: here, in the intertwining of notions and methods, causal links [*liens*] disappear and are replaced by intelligible relations. Systematic exposition—axiomatisation, for example—makes this process visible but is not indispensable to it. It is possible to perceive its unity in the central intuition ac-

11 {By using the term 'factices', Cavaillès perhaps intends merely to signal here that the existing 'juxtaposition' is *artificial* or *fabricated* in a common language sense, i.e. it is something created; or perhaps he intends to invoke Descartes's doctrine of *idées factices* [*factae*], usually rendered by 'facticious ideas', as opposed to 'innate ideas' and 'adventitious ideas'.}

cording to which its concatenations proceed, but also to methodically detect it solely by the description of the phenomenon. To follow the genesis of notions, to clarify above all their effective links with problems and isolate those general procedures whose inner gesture comes closest to the central intuition which is impossible to describe, such is the work—at once subject to history and a critique of it in the name of its results—that has some chance of yielding an objective result. In the face of such an endeavour to remain situated within the mathematical system, while at the same time characterising an autonomous structure, factitious architectures should disappear, and perhaps the sterility of certain questions should even be discerned. There is an endlessly renewed source of surprise in the unexpected inflexions of mathematical becoming, in its ironic abandonment of the paths that an attempt at systematic planning laid out before it. A sufficiently detailed critical study of neighbouring theories ought to be able to throw some light on this problem of explaining the unequal fecundity of methods. However, what is at stake here is only a limited application of this procedure to the particular question posed by set theory. Necessary emergence or—and to what extent—historical fantasy? Autonomous structure or plurality encased in a contingent system?

Conclusion to *TAE* (*TAE* 161-164/*OC* 359-362)

It is possible to give the process of axiomatisation a less absolute meaning: a kind of condensation of the active material of a theory around a few intuitively graspable procedures, for which the question of approbation, if not justification, is easier to resolve.[12] From this point of view, the research of Zermelo, Fraenkel and Von Neumann is as useful in the development of the theory as it is in drawing the outline of its history. Thanks to them the scope of the system and its true connections [*liaisons véritables*] seem to be established—at least insofar as this is possible at the present moment.

The theory of equivalence was born in the heart of analysis, with methods borrowed from analysis.[13] If Bolzano had already presented the notions of the union and product of sets, Weierstrass, in the context of discussing the complex numbers, gave them precise definition, which Cantor only had to then transpose.[14] With its simple operations and poverty of results,

12 {Less absolute, that is, than the forced choice, following Von Neumman's extension of Skolem's results, between a comprehensive project of formalist reconstruction and the intuitionistic reform of mathematics proposed by Brouwer, with which Cavaillès has just closed the final chapter of *TAS*.}

13 {By 'the theory of equivalence', Cavaillès refers to Cantor's definition of the comparisons of the 'powers' of sets in terms of one-to-one correspondence between elements, i.e. what will later be called 'cardinality'.}

14 Cantor 1872. See above, p. 18. {I take this to be a typo, with Cavaillès in fact referring to the definition of the notion of equivalence at p. 78 of *TAS*, i.e. *OC* 276.}

the whole interest of set theory lies in the manipulations that it makes more simple, in the relations [*les mises en rapport*] that it establishes between sets that had already been developed in analysis. However much one wishes it to be a precondition for analysis, set theory only exists to the extent to which its connection [*liaison*] with analysis becomes one of continuity rather than mere contiguity, lest it vanish into the logical void. Number and function, these intuitive acts of the mind are instruments that forge themselves, for nothing exists before them. That conceiving mapping as an operation logically anterior to the function was a deception is revealed in the flowering of the paradoxes: after Zermelo, it is making the operation of separation[15] precise, through the construction of functions by recursion, which allows Fraenkel to once again raise Dedekind's edifice. Coming after the functions of union, set of subsets, etc., the non-predicative moment enables an unlimited generation that opens up the perspectives of the theory. Can it go further with only these resources (that is, without the axiom of substitution)? It is difficult to decide: aside from the definition of order and well-ordering, it has hardly achieved results since the theorems of Bernstein and Koenig. In order to bear fruit, it seems to need the objects—sets or functions—defined by classical mathematics: additive function, measure in abstract integration, the notion of neighbourhood for topological spaces. It is then we see a proliferation of results and security won thanks to the internal mathematical solidarity of acts that they call for, to the effective control to which their more concrete character subjects them. There is thus no longer the singular development of an autonomous theory. The problem of universal comparability thus reveals the fantasy of an in-itself of sets, which underlies Cantor's work. How to affirm the existence of a mapping function if the two sets in questions are not defined by anything outside of their existence? It seems as if the operations of equivalence do not allow this aim to be attained. Tied up with this problem there is doubtless a new axiom, which Hartogs[16] has shown to be equivalent to the affirmation of well-ordering (and by extension the axiom of choice); its independence with respect to the other axioms thus follows from Fraenkel's work on the axiom of choice.

With the theory of ordinals, a radically new movement of thought appears—characterised by the axiom of substitution—more appropriate, it seems, for procuring the development of a true theory. We see that its subordination to considerations of equivalence was the result of an historical accident. But it is still necessary to note that it is not well-ordering which is a

15 Skolem had already employed it in Skolem 1922**.** {English translation in Heijenoort 1967: 293–301.}

16 It is evident that comparability follows from well-ordering (since all well-ordered sets are comparable). Hartogs has proved the inverse. C.f. Hartogs 1915.

precondition for numeration [*nombrement*], but numeration which is a precondition for well-ordering.[17] No set in effective mathematics, outside of the series of the integers, serves as an example of well-ordering; even if it is definable with the axioms of equivalence, no interior moment of that theory would suggest well-ordering if were not already provided by the creation of the ordinals. It seems as if—whatever its advantages might be in other respects— axiomatic presentation only takes up again, slightly disguised, naïve concatenations: the validity of the former, in the absence of any proof of non-contradiction, can only be based on the reality of the latter; that is, a return to Cantor's principles of generation. What is remarkable, moreover, is that an axiomatisation such as Von Neumann's on the one hand eliminates a whole series of reasonings concerning the theory of equivalence (Berstein's proof, Zermelo's proof of well-ordering), whilst on the other hand throwing no light on the continuum problem: whilst it is possible to prove that the cardinality of a set M is less than that that of the set of its subsets $\mathfrak{P}M$, we have no way to compare $2^{\aleph_0}$ and $\aleph_1$. The existence of certain entities, posited by the axiom *N*—or by Frankel's axiom of substitution—as well as those given by other axioms, such as the set of parts of the union set, seem impossible to put in relation [*mettre en rapport*] with each other except via a new axiom. This also serves to give plausibility to a second continuum hypothesis, taken up again by Lusin,[18] which, up until now, has not given rise to any contradiction, save that it excludes the first hypothesis: $2^{\aleph_0} = 2^{\aleph_1}$.

The privileged place of set theory, and the difficulties it has encountered, give its history its philosophical importance: the ambition that it be developed as a precondition for the rest of mathematics ought to allow the notion of a mathematical theory to be clarified, isolated as *in vitro*; after the paradoxes, the necessity of exhaustively enumerating the hypotheses clarifies the notion of proof itself: if the results attained only produce an immediate transformation of the axioms, can we still speak of a theory? The reality of a reasoning finds itself in some sense measured by its admixture of mathematical material, by the unification that it obtains between already mathematically existing objects or procedures. Whence the difficulty of making divisions of jurisdiction within a living tissue, with respect to which their claims

17 On the other hand, we know no method for ordering a set other than by well-ordering, such as we know from simply ordered sets. {The relevant notion of 'numeration' here refers to von Neumann's work on transfinite induction; see 'On the introduction if transfinite numbers' (1923), translated in **Heijenoort 1967: 346**–354**.** Cavaillès uses the terms *nombrement* and *nombrabilité* to translate von Neumann's *zählung* and *Zählbarkeit*, i.e. the possibility of ordinal numbering, as opposed to *dénombrabilité* for *Abzählbarkeit*, i.e. 'countability'. C.f. *TAS 152*-153/*OC* 350-351 and *OC* 624.}

18 Lusin 1935.

would only be superstructures of these procedures in which the unpredictable deviation of a gesture obtains at once novelties and preserves connections [*les liaisons*]. Rejecting or founding a theory is not definitive work, it is not devoid of degrees; moreover, as with all characterisations of unity, it cannot be carried out by simple logical investigation. For all that axiomatics seem to disassociate two theories—equivalence and ordinal numbers (and perhaps a third, which would be the study of the continuum)—the security of which then decrease, it is the pragmatic considerations of the militant mathematician that have the final word. It is in the whole analysis that the problem of existence and consolidation is posed, with respect to which a definition of mathematical work in general is necessary.

Concluding Comments to *Transfinite and Continuum* (*TC*) (1947)[1]

There thus appears a kind of reciprocity between transfinite superimposition and the impredicative moment: just as in projective geometry the intervention of a new dimension of space renders the axioms of congruence irrelevant (by passing from the line to the plane, from the plane to three dimensions), in the same manner the elevation to a sufficiently high transfinite order replaces impredicative definitions and justifies them by a predicative process. It is this same idea which guides Gentzen in his effort to demonstrate the consistency of classical analysis. The originality of Gödel is to have advanced to the class ω_ω, and at the same tome to have consolidated what was believed since the failure of meta-mathematics to be outside the possibilities of any [further] consolidation.

Is it reasonable to demand or only to hope for more? The conditions of strict observance of mathematical existence posed by the finitists evince a misunderstanding of what the very reality of the object consists of. There is not any absolute starting point: what to say of the point, of the continuum, of the whole number, if not that they are elements and origins of concatenations through which they gain their meaning and which surpass them? An irreducible intuition would only be a halt, devoid of thought. The Kantian theory of mathematics is still an obsession of logicians and mathematicians, but this is only because it has a superficial aspect which lends itself to being illegitimately interpreted in psychological terms. Moreover, even when it is authentically understood, the Kantian theory remains inadequate in that it conflates the dialectical moment of the positing of the concept and the transcendental moment of its schematisation. The notion of construction in intuition must reconcile the requirement of domination by the concept with the insertion of its object in a multiplicity through which it acquires a kind of independence with respect to all present [*actuel*] content of thought.[2] There is in the singular mathematical object something like an affinity with what is not it, by which it escapes its definition and justifies progress beyond itself. But how, in the Kantian theory of the concept, can one posit in intellectual terms a relation [*liaison*] which

1 {Translation of the final four pages of Cavaillès 1947, reproduced at *OC* 469-471.}

2 {The word *multiplicité* poses classic problem for the translator of French mathematical philosophy, as it can properly refer either to 'multiplicity' as the other of 'unity', or to the concept of the 'manifold' in a technical sense, i.e. both in the mathematical context (recall that Cantor's initial name for 'set theory' was *Mannigfaltigkeitslehre*, and thus in French translations *la théorie des multiplicités*) and in Kantian jargon (where 'manifold' in the sense of 'the synthetic unity of the manifold' is sometimes rendered by *multiplicité*/*multiple*, sometimes by *divers*). This capacious meaning should be borne in mind throughout this text.}

seems to contain an irreducible element of exteriority? Whence the idea of an intuitive knowledge in which multiplicity and exteriority subsist in essence, and of an intuitive construction which submits this exteriority to unity without abolishing it as such, since it remains both interior and exterior to the concept. Its virtual rationality, the links it has with the concepts of a finite intellect also pose problems: one could ask what bearing the concept of *quantum* has as an intelligible content independent of its numerical schema. Moreover, the misunderstanding appears which takes the construction itself to be representable; this is the confusion already noted between effective process and effectuated process [*procès effectif et procès effectué*]. Since the time of construction is transcendental time, it ought to escape from all representation, and consequently from anything that would condition the fulfilment of its synthesis. But would this still be a time? And could one still speak of fulfilment? The fundamental ambiguity of transcendental philosophy is that it posits the conditions of knowledge as relative to a synthetic activity which is defined and directly apprehended in a knowledge that itself escapes from critique: the condition of representation is conflated with the representations of this condition. Whence the exclusion of infinite number: how to construct in time that which requires an infinite number of stages? Yet it is only because these stages are disassociated from their unique function as *elements* and turned into *events* that these objects are then taken to require a singular representation and exist only through that representation. Hence, mathematics is impossible and Eleatism is correct.

If one seeks to already give a role to the *arbitrary something* [*quelconque*] in the definition of a synthetic process—both as *an arbitrary something* appearing in the discernment of the stages, and as that which is opposed to the singularity of the latter and to the final result—then the process itself must be defined as a rule, that is as the rational unity of a certain multiplicity, such that the reference to intuition supposes a relativity of this intuition to other schemas, to other regulated concatenations. At each stage of the progress of mathematics, new notions are refused their citizenship, on the grounds that they cannot procure a representation in the pre-existent intuitive system. This was the case for negative numbers, for the imaginary numbers, and for the infinitely small and the infinitely large at the time of Kant. Each time, the solution does not lie—as is superficially believed—in the discovery of a translation of these new notions into an immutable spatial intuition (representation of the negative numbers on a line, of the imaginary numbers in a plane), even if we were able to give a meaning to the confused notion of translation. The solution rather lies in a transformation of the intuitive

zone, which is to say in a transformation of the rules that lay out the use of thought in it, or the schematic system. The most striking example is the numerical continuum in the definition given by Dedekind. Otherwise put, intuition in its quiddity progresses parallel to the dialectical concatenation of concepts. It is nothing other than the manifestation for empirical consciousness of a relative independence of methods and theories, which permit those autonomous elaborations the results of which occasion encounters and reversals. The transcendental schema represents the motive power of a new synthesis, subsequent to such reversals: the internal force of a method, the underlying unity of a theory. To the extent that it schematises an intuition, it fixes the theory in a rigid aspect, acquiring for it a kind of uniformity the very intelligibility of which makes it seem definitive. This is the case with the arithmetisation of analysis, and with the intuition of pure number spoken of by Poincaré, which in reality is the schema of complete induction.

But the demands of actualisation [*actualisation*] are each time relative to the intuitive stage, which is to say to the multiplicity inherent in a mathematical conceptual system: if the number series is an intuition, it is just as impossible to represent it by a particular number (as in imagining a chiliagon) as it is to decide the properties of the numbers (the unresolved problems of arithmetic).[3] The works by Church and Kleene on the constructibility of functions and transfinite numbers represent a further step: by utilising quasi-recurrent functions, traversing a segment of Class II, they require a more complicated intuition to which the schematism of quasi-recurrent functions is then applied. It is the same with unrestrained transfinite induction in Gödel's theorem: the schema is precise because of what it allows us to prove, the intuitive zone is constituted by the system of schemas (at an inferior stage) of the logico-mathematical formalism of set theory. Cantor said "I represent a set as an abyss."[4] Progress by intuitive superposition has clarified this intuition through the multiplicity of modes of use given to the

3 {The *locus classicus* for 'imagining a chiliagon' is Descartes's 6th *Meditation*, but also see Kant's critique of Ebhard in his 1790 'On a discovery whereby any new critique of pure reason is to be made superfluous by an older one', Kant 2002: 303–304. With respect to the *problèmes non résolus de l'arithmétique* here, Cavaillès seeming means to refer to 'unsolvable problems' in the sense of Church 1936b.}

4 {A reference to the verbatim account given by Felix Bernstein, having acted as an intermediary between Cantor and Dedekind while visiting the latter in Harzburg in 1897. The account was published in Noether's edition of Dedekind's works (Dedekind 1932: 449): 'The following episode should be of special interest. Dedekind said, with respect to the concept of set, that he imagined a set as a closed sack that contains completely determinate things—but things which one does not see, and of which one knows nothing except that they exist and are determinate. Somewhat later, Cantor gave his own conception of a set. He drew his colossal figure upright, made a magnificent gesture with his raised arm, and said, staring into the indeterminate [*ins Unbestimmte*], "A set I imagine as an abyss."' (Translation from Ewald 1996b: 836.)}

sign-set in the systems of Zermelo-Frankel or Von Neumann; here again, it is a regulated mobility, a relation [*relation*] to the indeterminacy of unsolved problems.

The link between this intuitive superposition and the dialectic of the concept remains the fundamental problem of mathematical philosophy. The description is well-known of those historical reversals in which a result shatters a method and the whole system from which it originated: the procedures required for the solution of a problem provoke, in the very actualisation that gives them meaning, so complete a change of aspect that it is already necessary to abandon the notions that form their structure. But the intellectual relations [*liaisons*] surpass empirical history: it is their dialectical development which at the same time ensures their movement and—according to their own standards [*par elles-mêmes*]—the permanence of their validity. The mark of history is the submission of the transcendental to its stages: the need for transition [*l'obligation du passage*] is recognised in a failure, the necessity for progress in the indeterminacy of a discovery. Necessity appears after the fact [*après coup*]. No analysis of the consciousness of acts permits prediction, any more than it can procure an arbitrary permanence. Intuitive is a synonym for effective (or effectuating) consciousness, transcendental for constitutive (or constituting) consciousness, relative to a given conceptual system. No more than there is a closed system is there an absolute transcendental: formal unity is the continuity of a concatenation of acts, the concrete is the novelty of unifying a multiplicity that is still conceptually recognised as such. But if there is multiplicity only at the outset, every attempt at unconditioned regularisation is rendered null and void: equally for the case of Kronecker's finite arithmetism, for the Weierstrassian theory of the infinitesimal calculus, and for Hilbertian finitism.

Man always arithmetises[5]

If we wish to retain Dedekind's motto, we must extend the notion of numeration further than that of *mapping*, further than any determined transfinite, further even than any arbitrary numeration [*nombrement quelconque*], in order to mark the effective solidarity of a regulated development the intelligible justification of which at the same time lies outside of the human condition—in order to be complete—and yet which—at least as an idea—is perhaps one of its essential characteristics.

5 {A reference to the title page of Dedekind 1888.}

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6 Cavaillès to Gabrielle Ferrières, letter of 5th July 1934.