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Author(s): Orna Harari
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The Concept of Existence and the Role of Constructions in Euclid's Elements

ORNA HARARI

Communicated by A. JONES

Dedicated to the memory of Yonathan Begin

Summary of the argument

This paper examines the widely accepted contention that geometrical constructions serve in Greek mathematics as proofs of the existence of the constructed figures. In particular, I consider the following two questions: first, whether the evidence taken from Aristotle's philosophy does support the modern existential interpretation of geometrical constructions; and second, whether Euclid's *Elements* presupposes Aristotle's concept of being. With regard to the first question, I argue that Aristotle's ontology cannot serve as evidence to support the existential interpretation, since Aristotle's ontological discussions address the question of the relation between the whole and its parts, while the modern discussions of mathematical existence consider the question of the validity of a concept. In considering the second question, I analyze two syllogistic reformulations of Euclidean proofs. This analysis leads to two conclusions: first, it discloses the discrepancy between Aristotle's view of mathematical objects and Euclid's practice, whereby it will cast doubt on the historical and theoretical adequacy of the existential interpretation. Second, it sets the conceptual background for an alternative interpretation of geometrical constructions. I argue, on the basis of this analysis that geometrical constructions do not serve in the *Elements* as a means of ascertaining the existence of geometrical objects, but rather as a means of exhibiting spatial relations between geometrical figures.

1. The existential interpretation of geometrical constructions

A survey of Greek geometrical texts reveals a distinction between two types of proofs: proofs that establish the correctness of certain constructions and proofs that establish the truth of certain assertions. Since late antiquity, these two types of proofs are called "problems" and "theorems" respectively. In Euclid's *Elements* a problem is formed as an infinitive expression, proposing the construction of one figure in relation to a given figure;

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for instance, "to construct an equilateral triangle on a given line" (*Elements* I.1).[1] By contrast, a theorem is stated as either a conditional proposition or a universal proposition, relating a property to a given geometrical figure; for instance, "if in a triangle two angles be equal to one another, the sides which subtend the equal angles will also be equal to one another". (*Elements* I.6) Further, Euclid marks the distinction between these two types of proofs by ending theorems with the expression "which was required to be proved" and problems with the expression "which was required to be done".[2] There corresponds to the distinction between problems and theorems an analogous distinction within the first principles of the first book of Euclid's *Elements*. Euclid's list of first principles includes basic assertions, called "common notions" and basic construction tasks, which are listed among the postulates. Euclid's geometry, then is based upon two types of principles: (1) principles that assert quantitative relations, such as "if equals are added to equals the wholes are equal" (common notion 2) and (2) principles that license certain construction tasks, such as "to draw a line from any point to any point" (postulate 1).

The correspondence between the two types of first principles and the two types of derived propositions implies a distinction between two relations that subsist between first principles and derived propositions. In the derivation of theorems from common notions, the common notions serve as premisses and the theorems serve as conclusions; that is to say the relation between common notions and theorems is a deductive relation, by which one truth-valued assertion is logically derived from another truth-valued assertion. In contrast, the relation between postulates and problems cannot be analyzed in deductive terms. The construction step: "from point A to point B let the straight line AB be drawn" cannot be regarded as an inference from the postulate: "to draw a straight line from any point to any point"; as the former sentence has a perfect passive imperative form and the latter is formed as an infinitive expression, these sentences have no truth-value.[3]

Typically, a Euclidean proof proceeds from a preparation stage (κατασκευή), in which a certain figure is constructed, to a deductive stage (ἀπόδειξις), in which certain quantitative relations are deduced. Hence, in spite of the different logical significance of constructions and proofs, the constructive and the deductive aspects of Euclidean proofs are interwoven. That is to say, apart from procedures leading either from basic construction steps to other construction steps or from basic assertions to *derived* assertions, Euclid's proofs are based also upon a relation between construction steps and assertions. Given this relation between the deductive and the constructive aspects of Euclid's proofs, there arises the question of the role of constructions within the deductive structure of the *Elements*.[4]

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[1] All English translations of Euclid's *Elements* are quoted from: Heath, 1956.

[2] The expressions "which was required to be proved" and "which was required to be done" might be interpolations, added to the text in late antiquity. The present analysis of the *Elements* is based upon Heiberg's and Stamatis' edition of the text. Hence, in referring to Euclid's practice, I refer in fact to his practice as it is reflected in the edited text.

[3] Ian Mueller points to the difference between these two types of derivations. He accounts for the relation between postulates and constructions steps in terms of the distinction between rules of inference and deductions. Mueller, 1974, p. 39.

[4] Problems of constructions were introduced in Greek mathematics not only as part of a deductive system. In most cases these constructions, such as squaring the circle or the trisection

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The traditional answer to this question states that Euclid's postulates and problems are logically equivalent to existential propositions of the form "there is an x". This interpretation is associated with the mathematician H.G. Zeuthen, who based his view mainly upon the first book of the *Elements*.[5] In this book, problems of constructions are introduced as preliminaries to other problems or theorems in which their results are used. For instance, problem I.10, "to bisect a given finite straight line" is introduced before theorem I.16, which proves the quantitative relations between an exterior angle of a triangle and either of its interior and opposite angles. Accounting for the order of these propositions, Zeuthen maintains that Euclid "does not dare to use the midpoint of a segment in proof I.16 before he has demonstrated the existence of this point through its construction in I.10.[6] Hence, in Zeuthen's view, the order of the propositions in Euclid's *Elements* indicates that the employment of constructions is aimed at securing the existence of the defined geometrical terms.

Prima facie, Zeuthen's interpretation seems to capture adequately the *logical* relations that hold between problems and theorems. From the point of view of modern formal logic, a theorem will be formed by means of the universal quantifier (i.e., a proposition governed by the expression "for all x"), while a problem will be formed by means of the existential quantifier (i.e., a proposition governed by the expression "there exists an x such that. . ").[7] However, the logical equivalence between constructions and existential propositions does not necessarily amount to theoretical equivalence. The introduction of existential propositions into a logical system carries with it theoretical presuppositions that constructive formulations do not necessarily presuppose. The following analysis of Zeuthen's interpretation discloses these presuppositions.

In accounting for the theoretical significance of constructions, Zeuthen views the awakening of existential questions in Greek mathematics as a consequence of the Pythagorean discovery of incommensurable magnitudes. That is to say, in Zeuthen's view, the discovery that there exist no ratios of integers that can express the ratio between certain geometrical magnitudes led to the understanding that mere assumptions cannot secure the existence of mathematical terms. Therefore, explicit existence proofs were introduced as a response to the theoretical problems that arose with the discovery of incommensurable magnitudes. In relating the allegedly existential import of constructions to the discovery of incommensurable magnitudes, Zeuthen's interpretation implies

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of an angle were introduced for the sake of discovering the manner of the construction of certain figures or curves. (See, Knorr, 1986). This paper deals exclusively with the role of constructions within a deductive system.

[5] Zeuthen, 1896.

[6] Ibid, p. 223.

[7] The view that the existential interpretation provides an adequate description of the logical structure of the *Elements* is held also by the opponents of the existential interpretation. For instance, Knorr argues that Zeuthen's thesis is a valid account of the *essence* of geometric problems but it is not a valid *historical* account of the geometers' view of their own technical efforts (Knorr, 1983, p. 127). Similarly, Mueller's criticism of the existential interpretation is confined to the claim that "Euclid's restriction of proofs of existence to constructions ... [is] not a result of the conscious adoption of a constructivist philosophy of mathematics" (Mueller, 1981, p. 150).

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that the correspondence between a defined term and the reality to which it refers cannot be taken for granted, but it rather should be established by means of proofs. This idea is one of the prominent characteristics of modern philosophy and it finds expression *inter alia* in Leibniz' distinction between nominal definitions and real definitions. An explicit application of this distinction to Euclid's *Elements* is found in Saccheri's *Logica Demonstrativa* (1697), where he lays down the distinction between *definitiones quid nominis* and *definitiones quid rei*.[8] Viewing Euclid's *Elements* from the perspective of this distinction, Saccheri claims that Euclid's construction of a square in *Elements* I.46 is aimed at turning the nominal definition of a square (def. I.22) into a real definition.[9] From a mathematical point of view, the application of the distinction between nominal definitions and real definitions to Euclid's *Elements* seems plausible in light of the development of mathematics in the twentieth century. This interpretation enables one to view the theoretical implications of the Pythagorean discovery of incommensurable magnitudes as analogous to the theoretical implications of the modern foundational crisis of mathematics.[10]

The modern foundational crisis of mathematics is a consequence of the discovery of the paradoxes of set theory. These paradoxes have various manifestations, but from a methodological point of view they can be formulated in terms of the following question: whether it is permissible to mark off a set of objects by mere indication of a conceptual characteristic and then infer that this class determines a valid mathematical object. The discovery of paradoxical sets entails that not every combination of characteristics suffices for determining a mathematical object and guaranteeing its existence. As a result, the paradoxes of set theory give rise to a sharp distinction between the conceptual realm, in which a set is conceived of and the objective realm, in which the members of a given set exist. The modern identification of constructions with existence proofs, associated with the intuitionist school is a response to the distinction between mathematical definitions and mathematical existence that arises from the discovery of paradoxical sets.[11] Given this distinction, the intuitionists require a justification of the validity of mathematical terms by means of constructions. Thus, existence proofs secure the validity of a certain concept by answering questions of the form "is there a particular instance that accords with a given concept?" In the assumption that the discovery of incommensurable magnitudes had similar consequences for Greek mathematics, the existential interpretation ascribes to geometrical constructions a similar role; namely the role of justifying the validity of a concept, by forming a particular instance that accords with it.[12]

Generally speaking, the formulation of problems by means of the existential quantifier carries with it the distinction between the universal and the particular. Hence, by

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[8] Saccheri, 1980, pp. 193–197.

[9] On the relation between the notion of existence proofs and modernity see: Lachterman, 1989, pp. 50–61.

[10] The analogy between the modern foundational crisis of mathematics and the discovery of incommensurable magnitudes has been drawn in: Hasse and Scholz, 1928.

[11] An explicit relation between the existential interpretation of constructions and intuitionist philosophy of mathematics can be found in Becker, 1927.

[12] In the following I use the expressions "modern thought" or "the modern concept of existence" in referring to this view.

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interpreting the employment of constructions in existential terms, the existential interpretation reads into Euclid's *Elements* a conception of mathematical objects, according to which geometrical figures are particular representations of a universal system. From a logical point of view, this interpretation implies that geometrical constructions are means of justification; i.e., they are conceived of as a logical procedure that aimed at establishing the truth-value of a given content. The following analysis of the ancient origin of the existential interpretation sets the historical background for examining the applicability of these assumptions to Euclid's *Elements*.

Explicit evidence for the existential motivation that allegedly lies behind Euclid's employment of constructions cannot be found in Euclid's *Elements*. Euclid does not explain the rationale lying behind his constructive propositions, nor does he employ existential terminology in formulating problems. On the contrary, certain formulations of construction problems in the *Elements* suggest that the question of existence is separated from the formulation of problems. For instance, in Book III of the *Elements*, Euclid establishes the existence of a tangent to a circle in proposition III.16 and yet in proposition III.17 he formulates a problem, showing how to draw a tangent to a circle from a given point outside this circle.[13] Nevertheless, it has been argued that the existential interpretation is indeed an adequate analysis of Greek mathematical practice, since it can be found in ancient philosophical discussions of mathematics.[14] The main evidence in supporting the existential interpretation is found in Proclus' commentary on the first book of Euclid's *Elements*, where he accounts for the sequential priority of problems over theorems in existential terms:

> For unless he had previously shown the existence of triangles and their mode of construction, how could he discourse about *properties that are accidental to them in themselves* (καθ'αὑτὸ συμβεβηκότων) and the equality of their angles and sides? (233.21-25)[15]

The historical credibility of the existential interpretation is reinforced by the Aristotelian terminology that Proclus employs in accounting for the priority of problems over theorems.[16] In the *Posterior Analytics* and in other contexts, Aristotle maintains that demonstration is an inquiry into properties that are related to a given genus in a certain predicative relation called, "accidental in itself" (καθ' αὑτὸ συμβεβηκότων). It is in this context that Aristotle puts forward the requirement that the existence of the genus ought to be assumed prior to the demonstration that establishes the predicative relation between the genus and its "accidental in itself" attributes.[17]

The Aristotelian origin of Proclus' existential interpretation contributes to the historical adequacy of the existential interpretation, as Aristotle's theory of demonstrative

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[13] Further evidence for the distinction between geometrical constructions and existence claims are introduced in Knorr, 1983, pp. 128–130.

[14] See for instance, Niebel, 1959.

[15] The English translation is based upon Morrow, 1970. I revised Morrow's translation as his translation conceals the Aristotelian terminology that Proclus employs. All references to Proclus' commentary on the first book of Euclid's *Elements* refer to pages and line numbers of Friedlein's edition.

[16] See for instance, Heath, 1956, p. 195 and Niebel, 1959, pp. 89–102.

[17] See for instance, *Posterior Analytics* I.7 75a37-b2.

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knowledge and Euclid's *Elements* seems to be resting on common theoretical foundations. This view is based mainly upon the parallelism between the axiomatic structure of the *Elements* and Aristotle's account of first principles. In the *Posterior Analytics* I.2, Aristotle introduces a distinction between two classes of first principles: (1) principles that are common to more than one branch of knowledge, called axioms; and (2) principles, which are unique to each of the branches of knowledge. This class is divided into two sub-classes of first principles, called definitions and hypotheses. Traditionally, Aristotle's definitions are interpreted as predicative propositions that specify the meaning of a term and hypotheses are interpreted as existence claims.[18]

The first book of Euclid's *Elements* differs from other Greek mathematical texts in classifying the first principles under three groups: common notions, postulates and definitions.[19] These three types of first principles seem to be parallel to Aristotle's threefold distinction between axioms, definitions and hypotheses. One of Aristotle's examples of an axiom is the proposition: "if equals are taken from equals the remainders are equals,"[20] which is one of Euclid's common notions. Further, the characterization of the axioms as common to more than one branch of knowledge accords with Euclid's list of common notions, as they can be extended beyond geometry. Similarly, Euclid's introduction of the distinction between definitions and postulates seems to correspond to Aristotle's distinction between definitions and existence claims. That is to say, the introduction of three types of first principles seems to stem from the attempt to accommodate the structure of the *Elements* with Aristotle's requirement, according to which the existence of the defined terms should be established.[21]

In the following, I examine the question of whether Aristotle's requirement that each branch of knowledge should establish the existence of its terms can serve as an evidence for the existential interpretation of geometrical constructions. This examination leads to the conclusion that Aristotle's concept of being and the modern concept of existence refer to two different objects; in Aristotle's philosophy ontological status is ascribed to universal concepts, whereas the modern concept of existence refers to particular instances. An analysis of the presuppositions that underlie Aristotle's concept of being will detach the Greek question of mathematical existence from the question of justification, thereby it will pave the way to an alternative interpretation of the role of geometrical constructions in the *Elements*.

2. Aristotle's concept of being

In *Metaphysics* VII.10, Aristotle introduces a mathematical example in considering the question of whether the material parts of an entity correspond to the parts of its

[18] This view is held by the majority of Aristotle's interpreters such as Barnes, 1993, pp.100–101; von Fritz, 1971, pp. 361–366; McKirahan, 1992, pp. 122–125.

[19] Euclid's classification of first principles into three groups is an exception in Greek mathematical texts; usually the list of first principles includes only definitions.

[20] *Posterior Analytics*, 76a42, 77a30 and *Metaphysics* 1061b9.

[21] Among the proponents of this view are: Heath, 1970, pp. 53–57; Hintikka, 1981, p. 142; von Fritz, 1971, p. 390.

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definition. This discussion discloses the main difference between Aristotle's concept of being and the modern concept of existence.

> A part may be a part either of the form (i.e., the essence), or of the compound of the form and matter, or of the matter itself. But only parts of the form are parts of the formula, and the formula is of the universal ... But when we come to the concrete thing, e.g., this circle, i.e., one of the individual circles, whether sensible or intelligible (I mean by intelligible circles the mathematical, and by sensible circles those of bronze and wood), of these there is no definition, but they are known by the aid of intuition or perception; and when they go out of our actual consciousness it is not clear whether they exist or not. But they are always stated and cognized by means of the universal formula. But matter is unknowable in itself. And some matter is sensible and some intelligible, sensible matter being for instance bronze and wood and all matter that is changeable, and intelligible matter being that which is present in the sensible things not qua sensible, i.e., the objects of mathematics. (1035b31-1036a12)[22]

In this passage, Aristotle draws an analogy between sensible objects, such as wooden circles and mathematical objects, which are considered to be intelligible objects. In drawing this analogy, Aristotle maintains that despite the distinction between sensible matter and intelligible matter, both circles are particular instances of a universal concept or formula. Hence, by introducing the term "intelligible matter", Aristotle views the idealized objects of mathematics as particular objects, even though their matter is not changeable. In Aristotle's view, then, mathematical objects have two facets: (1) a form, which is a universal concept common to all of the particular instances of a given species and (2) matter, which is the element that distinguishes each of the instances of a species from one another. By introducing the distinction between form and matter, Aristotle applies to mathematics the distinction between the universal and the particular. That is to say, he draws a distinction between universal concepts, such as triangularity and their corresponding particular instances, such as triangles.

Furthermore, according to this passage, form and matter do not differ from one another only in their extension; rather form and matter have different cognitive significances, as the former is an appropriate object of definitions, while the latter is characterized as "unknowable" (ἄγνωστος). The cognitive significance of the distinction between form and matter has implications on the cognitive value of the particular object. As the scope of application of definitions is restricted to universal concepts alone, a particular object, which is a compound of matter and form cannot be defined exhaustively; namely it cannot be known in the strict sense. However, Aristotle's attitude towards the particular object is not confined to the cognitive realm; rather Aristotle draws an ontological conclusion, claiming that it is impossible to ascertain the existence of a particular mathematical object. The inference that Aristotle draws from the cognitive aspect of the distinction between the universal and the particular to its ontological aspect indicates that in his view the existence of an entity is determined by means of definitions.[23]

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[22] In this paper, I have generally chosen to follow the translation offered in Barnes, 1984. In certain cases, however, I have changed the translation for the sake of clarity or precision.

[23] The idea that definitions determine being is not foreign to Aristotle's thought. For instance, throughout his *Physics*, questions such as whether void or time exist are answered by verbal

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The analysis of the passage quoted above discloses the main difference between Aristotle's concept of being and the modern concept of existence. Like Aristotle's concept of being, the modern concept of existence presupposes the distinction between the universal (i.e., the concept) and the particular (i.e., the object). In contrast to Aristotle's view, modern thought regards concepts and definitions as verbal accounts that determine merely the meaning of a term. In this view, the concept includes no indication whatsoever as to existence of the objects that satisfy its definition. The existence is determined by adding to the concept further characteristics, taken from experience, that indicate the conditions under which a concept can be realized. The dependence of existence upon experience confines the means of establishing existence to an examination of particular instances. In contrast to the modern view, Aristotle's concept of being is applied to universal concepts and it is ascertained by means of definitions.

In the *Posterior Analytics* II.7 Aristotle introduces the distinction between nominal definitions and real definitions, in accounting for the relation between definitions and being:

> For it is necessary for anyone who knows what a man or anything else is to know too that it is (for of that which is not, no one knows what it is – you may know what the account or *the name signifies* when I say goatstag, but it is impossible to know what a goatstag is. (92b4-8)

According to this passage, the distinction between real definitions (i.e., answers to the question "what it is"?) and nominal definitions (i.e., answers to the question "what a name signifies"?) corresponds to the distinction between real objects and empty terms, such as "goatstag". Terms that refer to non-existent objects are characterized as definable only in the nominal sense, whereas existent objects are marked as those objects to which an essence or a definition can be ascribed. In other words, the possibility of answering the question "what it is" serves as a criterion for distinguishing existent objects from non-existent objects.[24]

In his discussion of definitions in *Metaphysics* VII.12, Aristotle compares the relations subsisting between the components of definitions to those subsisting in non-definitional predications. He exemplifies definitions and non-definitional predications

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characterization of void and time (*physics* IV 7,10). Similarly, in the *Posterior Analytics* II.8 Aristotle maintains that the question "whether it is?" is answered by means of definitions, such as "a thunder is a sort of noise in the clouds, or "an eclipse is a sort of privation of light" (*Posterior Analytics* 93a21-24). Furthermore, in their analyses of the Greek concept of being, Owen and Kahn have argued, albeit on different grounds, that the notion of existence is almost absent from Greek thought. According to this analysis, the verb "to be" does not denote existence in the sense of "there is an *x*", but is rather employed in a predicative sense, i.e., "X is Y". In this sense "being" means "to be something" (Owen, 1965, Kahn, 1986).

[24] The relation between definitions and being is traditionally interpreted in terms of the modern distinction between concept and object. According to this view, definitions gain their existential import from an implicit reference to an empirical experience of the defined object, e.g., Bolton, 1976, pp. 530–533. According to another related view, definitions provide a reliable means of selecting genuine instances of investigated phenomena. See for instance, Demoss and Devereux, 1988, pp. 141–146.

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respectively by the following compound expressions: "a biped animal" and "a white man". Analyzing the difference between these compound expressions, Aristotle maintains that the compound expression "a biped animal" differs from a non-definitional compound expression in conveying one meaning, that is to say the meaning of "man". By contrast, the non-definitional expression "a white man" has no unitary meaning, but it rather expresses two separate meanings, namely those which are conveyed by the words "man" and "white" (1037b9-20).

The criterion of forming a unitary meaning can be applied to the distinction made in the *Posterior Analytics* II.7 between real definitions and nominal definitions. In this chapter, Aristotle introduces the terms "a goatstag" (τραγέλαφος) and the "*Iliad*" (92b32) in exemplifying entities, to which the question "what the name signifies?" applies. Both examples illustrate a case in which a single name does not convey one single meaning. The *Iliad* is a single name that denotes the Homeric epic, that is to say Homer's tales of the Trojan War constitute the meaning of the word "*Iliad*". However, the word "*Iliad*" expresses not just one single meaning, but a complex of meanings, which correspond to the various scenes and stories described in the *Iliad*.[25] Likewise, the word "goatstag" is a single word that denotes two different animals, i.e., a goat and a stag. It follows from this analysis that real definitions differ from nominal definitions, which are given to non-existent things, in constituting one unitary meaning.

Aristotle applies this criterion in defining substances, which are the primary ontological entities of his philosophy. In *Metaphysics* VII.4, Aristotle introduces the distinction between a definition and a name's signification in answering the question of what a substance is:

> But we have a definition not where we have a word and a formula identical in meaning (for in that case all formulae would be definitions; for there would be a name for formula whatever, so that even the *Iliad* would be a definition), but where there is a formula of something primary; and primary things are those which do not involve one thing being said of another. Nothing, then, which is not a species of a genus, will have an essence (1030a7-11)

Definition, then, unlike the signification of a name, determines being since it can be given only to primary things, that is to say to substances. Such a definition does not determine being in the existential sense; that is to say, it does not assert, "there is an X". The expression "there is an X" refers to a particular instance, while according to Aristotle, a particular instance cannot be defined (e.g., 1039b27-31). Rather, a definition distinguishes an object from other objects, thereby indicating that the defined object is a certain unitary entity (τόδε τί). In other words, the term "being" means for Aristotle being a definite entity or in Aristotle's terms being "a this".[26]

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[25] In the *Poetics* XX, Aristotle expresses this view explicitly, saying: "A sentence is said to be one in two ways, either as signifying one thing or as a union of several speeches made into one by conjunction. Thus the *Iliad* is one speech by conjunction of several; and the definition of man is one through its signifying one thing". (1457a27-30)

[26] Although in the *Categories* Aristotle sometimes mentions particular instances as examples for "thisness" (3b10–18), in *Metaphysics* VII and VIII, he employs the expression *tode ti* in reference to forms (1037b27) or to composites of matter and form (1033a31; 1033b19-26;

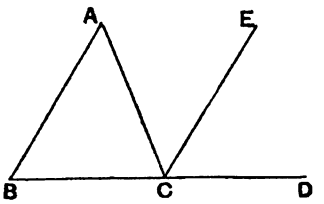
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Hence, Aristotle's requirement according to which each branch of knowledge should establish the existence of the terms that it employs cannot serve as evidence in supporting the existential interpretation of geometrical constructions. The existential interpretation ascribes to Euclid a notion of existence that first holds for particular instances and second provides a justification for the validity of geometrical terms. However these characteristics do not hold for Aristotle's concept of being. Unlike the modern concept of existence, Aristotle's concept of being applies to universal concepts and further it is not considered to be a means of justification, but rather a means of determining an entity, by specifying its essential attributes.

3. Aristotle's concept of being and the ontological significance of constructions

In considering the relationship between Aristotle's concept of being and Euclid's mathematical practice, I analyze two geometrical proofs that Aristotle reformulates in his writings. The first example that I consider here is Euclid's proof of the proposition that the sum of the angles in a triangle is equal to two right angles (*Elements* I.32).

In any triangle, if one of the sides be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of the triangle are equal to two right angles.



Let ABC be a triangle, and let one side of it BC be produced to D.
I say that the exterior angle ACD is equal to the two interior and opposite angles CAB, ABC, and the three interior angles of the triangle ABC, BCA, CAB are equal to two right angles.
For let CE be drawn through the point C parallel to the straight line AB.
Then, since AB is parallel to CE, and AC has fallen upon them, the alternate angles BAC, ACE are equal to one another.
Again since AB is parallel to CE, and the straight line BD has fallen upon them, the exterior angle ECD is equal to the interior and opposite angle ABC.
But the angle ACE was also proved equal to the angle BAC; therefore the whole angle ACD is equal to the two interior and opposite angles BAC, ABC.
Let the angle ACB be added to each; therefore the angles ACD, ACB are equal to the three angles ABC, BCA, CAB.

1037a1-2; 1038b5-6). For instance, in *Metaphysics* VII.3, Aristotle rejects the contention that matter is a substance, claiming that it is not a "this" (1029a28), while in *Metaphysics* VIII.1, he contends that matter is only potentially a "this" (1042a26-28). It seems, then, that "thisness" is determined by the form and that the composite of matter and form is a "this" due to its universal aspect, i.e., the form.

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> But the angles ACD, ACB are equal to two right angles;
> Therefore the angles ABC, BCA, CAB are also equal to two right angles.

In *Physics* II.9, Aristotle refers to this proof saying:[27]

> Necessity in mathematics is in a way similar to necessity in things, which come to be through the operation of nature. Since the straight line is what it is, it is necessary that the angles in a triangle should be equal to two right angles. (200a15-17)

In this passage, Aristotle considers the conclusion that the sum of the angles in a triangle is equal to two right angles to be derived from the definition, or the essence of a straight line. Aristotle assumes here that the term "line" and the term "triangle" are related to one another in a conceptual relation, that enables the explication of the triangle's properties from the definition of line. This obscure account of the Euclidean proof can be understood in light of Aristotle's characterization of the predicative relations, upon which demonstrative syllogisms are based. In the *Posterior Analytics* I.4, Aristotle says:

> One thing belongs to another in itself, if it belongs to it in what it is, for instance line to triangle and point to line (for their substance is from these, and they belong in the account which says what they are). (73a34-37)

According to this passage, the relation between a line and a triangle is not regarded as a relation between a figure and the constituents from which it is constructed. Rather, the term "line" is considered here to be a part of the definition of the term "triangle". In applying his theory of definition to geometrical objects, Aristotle views the term "line" as a genus, under which the term "triangle" is subsumed as one of its species. Indeed, in *Metaphysics* V.28, where the various meanings of the term "genus" are specified, Aristotle maintains that geometrical figures are species of certain genera, such as planes and solids. Hence, in formulating a geometrical proof, Aristotle substitutes the relation of a whole to its parts by the relation of subordination, that subsists between the genus "line" and the species "triangle".

By contrast, in Euclid's proof the terms "triangle" and "line" do not function as universal concepts. Rather, this proof involves breaking the triangle into parts and considering them separately. For instance, in the first stage of the proof, the triangle that is given at the setting out stage (ἔκθεσις) is not examined as a whole, but it is rather viewed as a part of the whole configuration that results from the construction step. Viewed from this perspective, the sides of the triangle (AB, AC) are no longer considered as parts of a triangle, but as two lines that have different properties. That is to say, side AB is considered as a line parallel to line EC, while side AC is regarded as a line falling between two parallel lines (ἡ εὐθεῖαεμπίπτουσα). Hence, the relation between the whole and its parts, which Aristotle's reduces to conceptual relation is an indispensable presupposition in Euclid's proof.

It follows from this analysis that Aristotle's application of conceptual relations to geometrical proofs does away with Euclid's geometrical reasoning. The term "equal to two right angles" is not included either in the definition of a line or in the definition of

[27] That Aristotle was familiar with the Euclidean version of this proof can be seen in *Metaphysics* IX.9, where Aristotle describes this proof as involving drawing a line parallel to the triangle's side.

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a triangle, therefore it cannot be explicated from these definitions. This consequence of the application of the conceptual approach can be seen in the *Prior Analytics* I.35, where Aristotle attempts to cast this proof into syllogistic form:

> Sometimes too a mistake will result from such a search, i.e., the belief that there is a deduction of immediate propositions. Let A stand for "two right angles", B for "triangle" and C for "isosceles". A then belongs to C because of B, but it belongs to B not because of anything else (for a triangle has its angles equal to two right angles in virtue of itself). (48a31-36)

Aristotle considers here the attempt to prove the relation between a triangle and the sum of its angles to be a fallacy (ἀπάτη), resulting from the failure to see this relation as immediate. Thus Aristotle's logic can accommodate only the conceptual relation between a universal concept like triangle and a special case, like isosceles, that falls under it. However, the non-conceptual relation between a triangle and the sum of its angles cannot be cast into syllogistic form and therefore it is assumed at the outset as an immediate or indemonstrable relation.

Similar difficulties can be found in the *Posterior Analytics* II.11, where Aristotle presents in syllogistic terms the following geometrical example:

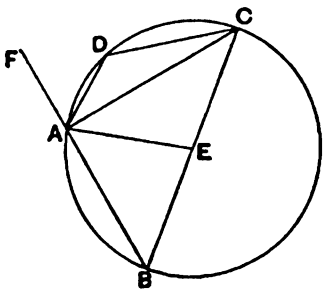
> Why is the angle in the semicircle right? It is right if what holds? Well let right angle be A; half of two right angles B; the angle in the semicircle C. Thus B is the cause that A, right angle, belongs to C the angle in the semicircle. For this is equal to A and C to B; for it is half of two rights. So if B half of two rights holds, then A belongs to C (that is the angle in the semicircle is right. (94a28-31)

From this passage the following first figure syllogism can be formulated

(1) A right angle belongs to all half two right angles;
(2) Half of two right angles belongs to the angle in the semicircle;
(3) Therefore, a right angle belongs to the angle in the semicircle.

The Euclidean proof of this proposition (Euclid III.31) has three parts, of which only the first is relevant to this comparison; therefore only this part will be considered here:[28]

In a circle the angle in the semicircle is right.



[28] In accounting for above syllogistic formulation of the geometrical proof, Heath refers to another version of the proof, which is found interpolated into the Euclidean text (Heath, 1970, pp. 71–73). This proof may indeed correspond better to the *Posterior Analytics* II.11. My analysis of the discrepancy between Euclid's proofs and syllogistic inferences holds for this proof as well.

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(1) Let ABCD be a circle, let BC be its diameter and E its center . . . I say that the angle BAC in the semicircle is right.
Let AE be joined and let BA be carried through to F.
(2) Then, since BE is equal to EA, the angle ABC is equal to the angle BAE (Euclid I.5; base angles of an isosceles triangle are equal).
(3) Since CE is equal to EA, the angle ACE is also equal to angle CAE (Euclid I.5).
(4) Therefore, the whole angle BAC is equal to the two angles ABC and ACB.
(5) The angle FAC exterior to the angle ABC is also equal to angles ABC and ACB (Euclid I.32)
(6) Therefore the angle BAC is equal to angle FAC.
(7) Therefore each is right (Euclid def. I.10)
(8) Therefore the angle BAC in the semicircle is right.

An examination of Aristotle's proof in light of the Euclidean proof reveals conspicuous differences between them. In fact, in comparison to Euclid's proof, Aristotle's syllogistic inference does not appear to be a proof at all. In Aristotle's proof, it is assumed that the angle in the semicircle is equal to half of two right angles; this assumption is tantamount to the conclusion that the angle in the semicircle is right. On the other hand, in Euclid's proof the equality of the angle in the semicircle to a right angle is established by showing the equality between angle BAC and angle FAC. It might be argued that Aristotle's formulation does capture adequately the logical structure of Euclid's proof, as it can be made to correspond to steps 6-8 of the Euclidean proof. In both proofs, the conclusion, concerning the equality of the angle in the semicircle to a right angle, is deduced from the definition of a right angle.[29] Yet, in Aristotle's syllogistic inference, the applicability of this definition to the angle in the semicircle is not established. By contrast, in Euclid's proof, the applicability of the definition of a right angle to angle BAC is derived from the construction, that is carried out in step (1). That is to say, line BF is shown to be a straight line, since it is produced from line AB, according to the second postulate. Hence, the application of the distinction between the universal and the particular to Euclid's proofs results in a loss of the logical relations subsisting in the proof. In applying this distinction, Aristotle reduces the relation between a whole and its parts to a conceptual relation. As a result, his formulations of mathematical proofs either beg the question (as in his formulation of the theorem concerning the angle in the semicircle) or are logically deficient (as in his formulation of the theorem concerning the sum of a triangle's angles).

The discrepancy between Aristotle's logic and mathematical reasoning has consequences both for the historical credibility of the existential interpretation of geometrical constructions and for its theoretical adequacy. From an historical point of view, the comparison between Euclid's geometrical proofs and Aristotle's syllogistic formulations of these proofs indicates that Aristotle's philosophy and Euclid's practice are based upon

[29] Novak, 1978 bases his contention that syllogistic reasoning can accommodate mathematical proofs upon the fact that both Euclid's proof and Aristotle's depend on the definition of a right angle.

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different theoretical foundations. Aristotle's syllogistic inferences and Euclid's geometrical proofs presuppose different objects. In formulating syllogistic inferences, Aristotle confines the range of application of syllogistic terms to the intermediate realm of generic concepts that can serve both as subjects and as predicates in categorical propositions (43a41-42). Due to the nature of generic concepts, the restriction of syllogistic inferences to such concepts affects the logical relations that a syllogistic inference can establish. Generic concepts are formed from the sensible multitude, by selecting from the wealth of particular objects, those features that are common to several of them. The formation of generic concepts consists, then, in selecting from the plurality of objects only similar properties, while neglecting the rest. In this process the features that distinguish one particular instance from another particular instance are considered to be contingent and therefore they are excluded from the generic concept. As a result, syllogistic reasoning cannot accommodate non-predicative relations, such as spatial relations. By contrast, Euclid's proofs focuses exactly on the features that the formation of generic concepts excludes. In Euclid's proofs, geometrical figures are considered in their particularity; that is to say, the characteristics that the proof examines are not those that belong to a certain figure inasmuch as it is subsumed under a universal concept. Rather, geometrical proofs consider the characteristics that are determined by the spatial relations between geometrical objects. Therefore, a geometrical figure is not viewed from the perspective of its essential attributes, but from the perspective of the relations that it attains by being part of a certain spatial configuration. Hence, the difference between Aristotle's view of geometrical objects and the geometrical objects that underlie Euclid's proofs, casts doubt on the possibility of substantiating the existential interpretation of construction by appealing to textual evidence taken from Aristotle's philosophy.

Furthermore, the discrepancy between Aristotle's form of reasoning and Euclid's proofs undermines the adequacy of the existential interpretation on theoretical grounds. According to the existential interpretation, geometrical constructions serve as a means of representing a universal concept by introducing a particular instance that corresponds with it. That is to say, geometrical constructions are not regarded as a positive means of explicating content, but rather as a means of providing a concrete representation of universal concepts. In this view, then, geometrical constructions do not contribute content to the proof, but they serve rather as a means of ascertaining an already given content. This construal, however fails to account for the discrepancy between Aristotle's syllogistic inferences and Euclid's proofs. If constructions were only means of instantiating a universal concept, Euclid's proofs could have been affected by reference to the universal concepts alone. Nevertheless, this analysis of syllogistic formulations of Euclidean proofs shows that the substitution of particular instances by universal concepts does affect the content of the proof. Hence, it seems that the role of geometrical constructions in the *Elements* is not confined to justification.

4. An alternative account of the role of constructions

The above-discussed examples of Aristotle's formulation of mathematical proofs have a common feature; in both examples Aristotle renders construction steps in terms of definitions. In formulating the proof to the effect that the sum of the angles of a triangle

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is equal to two right angles, Aristotle transforms a proof based upon the construction of a line parallel to the triangle's side into a proof that is based upon the definition of the term "line". A similar attitude is found in Aristotle's syllogistic formulation of the proof to the effect that the angle in the semicircle is right. Although both Aristotle and Euclid base this proof upon the definition of a right angle, each of them refers to a different definition. In Aristotle's formulation the term "right angle" is defined as "a right angle is half of two right angles", while Euclid posits the following definition: "When a straight line *set up* on a straight line makes the adjacent angles equal to one another, each of the equal angles is right and the straight line standing on the other is called perpendicular to that on which it stands". Unlike Aristotle, Euclid does not define the term "right angle" in terms of a property that belongs to it, but rather in constructive terms. That is to say, Euclid's definition characterizes a right angle as a result of a certain construction (setting up) that generates it.

This comparison may suggest that Aristotle views geometrical constructions as a certain account of ontological questions. In reformulating geometrical proofs, Aristotle renders constructions in the ontological terms of his philosophy. That is to say, Aristotle attempts to determine the ontological status of mathematical objects by specifying their definitions. Furthermore, this comparison implies that Aristotle considers the mathematical way of handling ontological questions to be deficient, so that by reducing constructions to definitions he attempts to amend it. The following examination of the analogy between definitions and constructions paves the way towards an alternative interpretation of geometrical constructions.

While delineating his view of mathematical objects in *Metaphysics* XIII.6, Aristotle presents three accounts of numbers – two philosophical and one mathematical (ὁ μαθηματικός ἀριθμὸς). According to Aristotle, the philosophical accounts of numbers differ from the mathematical account of numbers in their view of the relations between the units that constitute each number. Both philosophical accounts of numbers assume that the units are non-comparable with one another; according to one view, any unit is non-comparable with any other unit, while in the second view numbers are collections of units that are internally comparable within each number but non-comparable with units in other numbers. By contrast, the mathematical account of numbers does not presuppose any distinctions between units, but views all units as comparable and undifferentiated.[30] (1080a16-37) Given that in Aristotle's philosophy the term "non-comparable" (ἀσύμβλητος) characterizes the coordinative relation between different genera,[31] it follows that the philosophical accounts of numbers differ from the mathematical account of numbers in drawing a generic distinction between units.

Aristotle's criticism of the view according to which units are internally comparable but externally non-comparable discloses the difference between the philosophical standpoint and the mathematical standpoint.

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[30] In presenting this classification of the different accounts of numbers, I follow Annas, 1976, pp. 163–164.

[31] *Metaphysics,* X.4 1055a6-7.

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> The cause of the mistake they fell into is that they conducted their inquiry at the same time from the standpoint of mathematics and from that of the universal formulae; so that from the former standpoint they treated unity, their first principle, as a point; for the unit is a point without position. They put things together (συνετίθεσαν) out of smaller parts, as some others have done. Therefore the unit becomes the matter of numbers and at the same time prior to two; and again posterior, two being treated as a whole, a unity and a form. But because their inquiry was universal they treated the unity, which can be predicated of a number, as in this sense also a part of the number. But these characteristics cannot belong at the same time to the same thing. (1084b23-32)

The mathematical standpoint and the conceptual standpoint are characterized here as two different accounts of the relation between the whole and its parts. In the mathematical standpoint, the primitive term is the part, which is introduced in accounting for the whole. Thus, from this point of view the unit is theoretically prior to the number two. Presupposing the priority of the part over the whole, the unity of numbers is explained in terms of constructions or arrangement of units that are put together in a certain order. The conceptual standpoint, by contrast presupposes the priority of the whole over its parts. By assuming the priority of the whole over its parts, the conceptual standpoint views the whole as a universal concept and the parts as its particular manifestation. This universal concept serves, in this view, as a unifying principle that turns a collection of ingredients into a definite whole. Hence, it follows from this analysis that Aristotle considers the role of definitions to be analogous to the role of constructions; for both are considered to be different accounts of the relation between the whole and its parts. Bearing in mind the relation between Aristotle's concept of being and the question of unity, it may be concluded that the ontological significance, which is ascribed to geometrical constructions lies in their being an account of the relation between the parts and the whole, and not in their being an answer to questions of the form: "is there an x?".

With regard to Euclid's arithmetic, Aristotle's characterization of the mathematical standpoint seems to accord with Euclid's practice. Book VII of the *Elements* posits the following definition of number: "a number is a multitude composed of units". This definition accounts for the whole (i.e., the number) in terms of its parts (i.e., the units), but it does not provide the conceptual means by which the relation between the whole and its parts can be mediated. This gap seems to be bridged by the definition of "unit": "a unit is that in virtue of which each of the existing things are called one" (VII def. 1). In this definition, as in Aristotle's characterization of mathematical numbers, the part, i.e., the unit is considered to be the account that forms a unified number out of the parts that constitute it. The role that the unit plays in accounting for the unity of numbers can be explicated by a comparison between Euclid's definition of number and other definitions that are found in Greek sources.

The term "unit" is a significant constituent of any Greek definition of number. Yet, although the idea of "a collection of units" underlies the Greek definitions of number, a distinction can be made, within these definitions, between two conceptions of the relation between the number and the units that constitute it. In addition to Euclid's definition of number that characterizes a number as an aggregation of units, one can find in Greek sources definitions that refer also to the relation between the units. In these

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definitions, whose source is probably Pythagorean, a number is defined as "a collection of units or a progression (προποδισμὸς) of multitudes beginning from the unit and a recession ceasing with the unit"[32] or "the whole realm of number is a progress from the unit to the infinite by means of the excess of one unit". Like Euclid's definition, these definitions ascribe to the unit the role of arranging the collections of units in a serial order. By introducing the idea of "progression", these definitions allude to the role that the unit has, in turning a collection of items into an ordered sequence. According to these definitions the collection of units turns into a whole, i.e., a number, as a result of a *successive* reproduction of the unit.

These definitions then differ from Euclid's definition in referring to the relation of succession that subsists between the units. The relation of succession that results from the act of counting off units or adding units is abstracted, in these definitions from concrete numbers and then it is incorporated into the unit as its attribute. By contrast, Euclid's definition does not include an explicit reference to the order of units; yet by ascribing to the unit the role of a unifying principle, it presupposes the act of counting off units. The comparison between Euclid's definition of number and the Pythagorean definitions discloses two facets of the concept of number: the quantitative aspect, based upon measurement and the qualitative aspect that finds expression in relations of order. Euclid's definition emphasizes the quantitative aspect of numbers i.e., the units, while providing a non-conceptual account for the qualitative facet of the concept of number, i.e., the act of counting. Hence, in Euclid's arithmetic operations serve as a unifying factor that provides a non-conceptual account for the relation of succession that subsists between the units. In light of this analysis, it might be asked whether geometrical constructions and counting have an analogous role in Euclid's mathematics; that is to say, whether geometrical constructions, like counting provide a non-conceptual account of qualitative relations.

On the face of it, Euclid's formulation of geometrical terms seems to be contrary to his formulation of arithmetical terms. In Euclid's arithmetic, the primitive term is the part, i.e., the unit, whereas the whole results from the parts by means of addition. By contrast, the definitions of the basic geometrical terms in the first book of the *Elements* proceed from the whole to the part. Definitions I.3, I.6 describe the lower dimension in terms of the higher dimension; that is to say, according to these definitions points are the limits of a line and lines are the limits of a surface. Likewise, in Book XI a surface is described as the limit of a solid. However, despite the difference between Euclid's arithmetical definitions and his geometrical definitions, these definitions rest on similar presuppositions.

In his commentary on the first book of Euclid's *Elements*, Proclus contrasts Euclid's definitions that assume the priority of the whole over its parts to the Pythagorean approach that ascribes priority to the parts. By presupposing the priority of the parts over the whole, the Pythagorean approach accounts for geometrical terms in arithmetical terms. In this view, the basic geometrical constituent is the point and geometrical objects are formed out of points by means of addition. According to this view, the point is regarded

[32] Theon of Smyrna, p. 18,3; cf. Iamblichus, pp. 10, 16. This definition may be attributed to the neo-Pythagorean Moderatus (first century A.D.). On these definitions see: Klein, 1968, pp. 52–53.

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as analogous to the unit, the one-dimensional figure, i.e., the line is identified with the number two; the two-dimensional figure, i.e., the surface is identified with the number three and the three-dimensional figure, i.e., the solid is identified with the number four.[33] Hence, geometrical objects are analogous to arithmetical objects, as both are viewed as a result of a successive addition of a basic constituent. However, the application of the arithmetical notion of addition to geometry is insufficient for accounting for geometrical figures. Whereas the successive addition of units does account for the formation of arithmetical objects, successive addition of points does not provide an exhaustive account for the addition of spatial dimensions. The generation of geometrical figures by means of successive addition lacks the notion of a determined position, which is indispensable for the transition from one dimension to another. As a result, the Pythagorean account of geometrical terms does not provide the means of determining the position of geometrical objects, but it rather presupposes the spatial position as a property of points. This notion finds expression in the Pythagorean definition of a point as "a unit having position".[34] Hence, in accounting for mathematical objects, the Pythagorean view assimilates mathematical relation with the mathematical objects. That is to say, the relation of succession that underlies arithmetical objects is conceived of as a property of the unit and spatial positions are conceived of as a property of the point. In adopting this view, the Pythagoreans consider mathematical relations to be given as part of the characterization of mathematical objects; that is to say, their view of mathematical objects presupposes the notions of succession and of geometrical space.

By contrast, in Euclid's definitions of geometrical terms spatial relations are not incorporated into the objects, but they rather presuppose these objects. In the first book of Euclid's *Elements* the basic geometrical figures, i.e., points, lines and planes are characterized twice; in definition I.1 a point is defined as "that which has no parts" and in definition I.3 is it said that "the extremities of a line are points". Likewise, in definition I.2 a line is defined as a "breadthless length" and in I.6 lines are characterized as the extremities of a surface. The same approach is found in Book XI, where Euclid posits the following two definitions: XI.1 "a solid is that which has length, breadth and depth" and XI.2 "an extremity of a solid is a surface". These definitions can be classified under two groups: (1) definitions that determine the quantitative aspect of geometrical objects (I.1, I.2, and XI.1) and (2) definitions that determine the qualitative aspect of geometrical objects (I.3, I.6, and XI.2). The definitions of the first group refer to the notion of divisibility as a means of determining the measurable aspect of the spatial object. That is to say, a point is characterized as a non-measurable entity, as it has no parts that can measure it. Similarly, the other definitions of this group discern the measurable aspect of the geometrical figure from its non-measurable aspect. A line is measurable with respect to its length, while it is non-measurable with respect to its breadth. A similar consideration can be applied to the two other quantitative definitions. By contrast, the definitions of the second group account for the aspect that the quantitative definitions left undetermined.

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[33] Proclus, 97,18.

[34] Ibid. 95, 25–96,1. A similar definition can be found in Aristotle's writings; see for instance, *Metaphysics* 1026b26, 31; 1084b26; 1069a12.

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That is to say, these definitions characterize the non-measurable aspects of the figures, such as the points in the case of lines, as limits of the quantitative object. By accounting for the non-measurable aspect of geometrical figures, Euclid draws a distinction between the material aspect of geometrical figures, i.e., the divisible or measurable aspect and the positional aspect, regarded as a limit of the measurable figure. Yet, in Euclid's geometry spatial position is not given in the abstract, rather the Euclidean notion of spatial position presupposes the measurable object as its underlying substrate. Hence, in this view, spatial position is conceived of as imposed on spatial magnitudes, by an act of limiting.

The priority that Euclid ascribes to spatial magnitude over spatial position has consequences on the adequacy of the existential interpretation of geometrical constructions. According to the existential interpretation, geometrical constructions are aimed at providing a concrete representation of a given universal system. However, the above analysis of Euclid's geometrical definitions indicates that Euclid's geometry does not presuppose the notion of geometrical space. As a result, constructed geometrical figures cannot be viewed as particular instantiations of a universal system of spatial relations. Euclid's space is rather a generative space, in which geometrical relations and therefore geometrical objects are generated through an act of limiting divisible (i.e., material) magnitudes. This conclusion is further supported by the fact that no *clear* distinction can be made between definitions and constructions; for in addition to descriptive definitions, such as the definition of a straight line, Euclid's list of definitions includes also generative definitions, such as the definition of a right angle (def. I.10) or the definition of a diameter (def. I.17).[35] In this latter type of definitions, Euclid characterizes a geometrical entity in terms of the procedure that leads to its construction. Hence, the distinction between nominal definitions and real definitions does not provide an adequate account, for instance, of the difference between the definition of a right angle and problem I.11, in which a right angle is constructed. Real definitions are considered to be different from nominal definitions, in indicating the existence of an entity by adding constructive characterizations to the content that has been explicated in the nominal definition, whereas the difference between definition I.10 and problem I.11 cannot be explained by reference to constructive terms.[36] In the absence of a *clear* distinction between nominal definitions and real definitions, geometrical constructions cannot be construed as a realization of a given content, but rather as a means of formulating a given content.

In considering the logical implications of the discussion of Euclid's geometrical definitions, I analyze Euclid's proof to the effect that the base angles of an isosceles triangle are equal to one another (I.5).

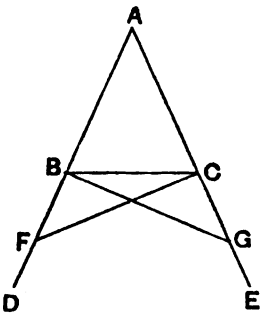
> *In isosceles triangles the angles at the base are equal to one another, and, if the equal straight lines be produced further, the angles under the base will be equal to one another.*

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[35] "A diameter of the circle is any straight line *drawn* through the center and terminated in both directions by the circumference of the circle, and such a straight line also bisect the circle".

[36] A definition that appeals to constructive terms presupposes the possibility of construction. Therefore, on the assumption that constructions secure the existence of the terms, actual constructions would become redundant.

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Let ABC be an isosceles triangle having the side AB equal to the side AC; and let the straight lines BD, CE be produced further in a straight line with AB, AC. (postulate 2). I say that the angle ABC is equal to the angle ACB, and the angle CBD to the angle BCE.
(1) Let a point F be taken at random on BD;
(2) From AE the greater let AG be cut off equal to AF the less (I.3).
(3) And let the straight lines FC, GB be joined (post. 1).
(I) Then since AF is equal to AG and AB to AC, the two sides FA, AC are equal to the two sides GA, AB respectively; and they contain a common angle, the angle FAG.
Therefore the base FC is equal to the base GB, and the triangle, and the triangle AFC is equal to the triangle AGB, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend, that is the angle ACF to the angle ABG and the AFC to the angle AGB (I.4).
(II) And since the whole AF is equal to the whole AG, and in these AB is equal to AC, the remainder BF is equal to the remainder CG.
(III) But FC was also proved equal to GB; therefore the two sides BF, FC are equal to the two sides respectively; and the angle BFC is equal to the angle CGB, while the base is common to them; therefore the triangle BFC is also equal to the triangle CGB, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend; therefore the angle FBC is equal to the angle GCB, and the angle BCF to the angle CBG.
(IV) Accordingly, since the whole angle ABG was proved equal to the angle ACF, and in these the angle CBG is equal to the angle BCF, the remaining angle ABC is equal to the remaining angle ACB; they are at the base of the of the triangle ABC. But the angle FBC was also proved equal to the angle GCB; and they are under the base.

The first inference upon which Euclid's proof is based leads from the conjunction (A) "line AF is equal to line AG and line AB is equal to AC" to the conjunction (B) "the angle ACF is equal to angle ABG and angle AFC is equal to angle AGB". These propositions rest on the construction steps that precede the proof, yet the role that the construction steps play in the tacit inferences that lead to these propositions is different. Proposition (A) is nothing but a reiteration of the content, which is given in the setting out stage (*ekethesis)* and the construction step (2). In putting forward the relation of equality between line AF and AG, the construction step (2) serves as a means of measurement; that is to say, it determines the quantity of line AG in comparison to the quantity of line AF.

By contrast, proposition (B) does not rest solely on the content, which is given in the setting out, the construction steps and the first congruence theorem. Rather, the derivation

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of proposition (B) requires a modification of the meaning of the ingredients that form the figure in question. That is to say, the application of the first congruence theorem requires an act of visualization, in which a configuration of lines is treated as a figure of a certain sort. For instance, prior to the proof, line AB is considered to be a side of a triangle, while line BF is regarded as a part of the additional line BD. The application of the first congruence theorem to this proof requires an act of visualization, in which line AB and line BF are treated *as* sides of a triangle. This act of visualization introduces the triangle AFC, thereby the other components of the configuration gain new meaning; line AC does not serve as the side of triangle ABC and line FC turns into a base of a triangle. Similarly, the meaning of components of the spatial configuration undergoes further modification at stage (III). In this stage, line BF is detached from the whole AF and it is treated *as* a side of the triangle FBC. The introduction of triangle FBC modifies the meaning of the other components of the configuration in question. That is to say, line FC that served in stage (I) as the base of triangle FAC turns into one of the sides of triangle BFC. As a result, line BC that served, at stage (I) as the base of the triangle ABC serves at stage (III) as the base of triangle BFC.

Applying similar considerations to other Euclidean proofs,[37] it may be concluded that the role of constructions in Euclid's geometrical proofs extends beyond instantiation. The analysis of the above-quoted proof shows that the application of problems to theorems contributes content to the proof in two ways: it serves either as a means of measurement by which quantitative relations are deduced or as a means of exhibiting qualitative relations, i.e., the order or the position of geometrical figures. The former employment of geometrical constructions can be characterized as deductive; that is to say it serves as a means of *explicating* the content that is already given in the setting out stage. By contrast, the latter employment of constructions serves as a means of *developing* content; that is to say, it enables one to go beyond the content, which is given in the setting out stage, by placing the elementary geometrical figures (i.e., lines) in different spatial relations.[38] This procedure results in a modification of the geometrical significance of the entities, as it endows geometrical entities with attributes, by changing the relations that subsist between them. Hence, Euclid's reasoning procedure and his geometrical definitions rest on the same presuppositions. In both cases the material aspect of geometrical space (i.e., its quantity) is regarded as given, while the relational aspect of geometrical space (i.e., it quality) is generated by certain operations like limiting and constructions. In other words, Euclid's *Elements*, unlike modern geometry does not presuppose the distinction between a set of objects (points, lines and planes) that serve as variables and a universal system

[37] For the sake of simplicity and clarity, I have chosen examples from the first book of the *Elements*. I believe, however, that my analysis holds for any application of construction procedures in the other geometrical books of the *Elements*.

[38] In his discussion of the ancient method of analysis, Hintikka and Remes ascribe to constructions heuristic value and yet they view them as existential instantiations (Hintikka and Remes, 1974, pp. 41–48). However, according to my interpretation, instantiation cannot have a heuristic value, as it does not add any content to the instantiated concept.

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of relations that serves as a function.[39] Rather, the spatial relations are generated by means of constructions and therefore they are not clearly distinguished from the objects that satisfy these relations. The existential interpretation of geometrical constructions reads into the *Elements* the distinction between relations (i.e., functions) and objects (i.e., variables) and therefore ignores the heuristic value of constructions.[40]

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[39] Hilbert's *Grundlagen der Geometrie* for instance, assumes three sets of objects: points, lines and planes and relations, such as "lie", "between" or "congruent", which are described in the axioms.

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Dibner Institute for the History of Science and Technology
MIT E56-100
38 Memorial Drive
Cambridge, MA 02139, USA
oharari-eshel@dibinst.mit.edu

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