# Proclus' Account of Explanatory Demonstrations in Mathematics and its Context

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*Abstract*: I examine the question why in Proclus' view genetic processes provide demonstrative explanations, in light of the interpretation of Aristotle's theory of demonstration in late antiquity. I show that in this interpretation mathematics is not an explanatory science in the strict sense because its objects, being immaterial, do not admit *causal* explanation. Placing Proclus' account of demonstrative explanation in this context, I argue that this account is aimed at answering the question whether mathematical proofs provide causal explanation as opposed to grounds. I show further that Proclus can answer this question in the affirmative due to his realist view of mathematical objects and the priority he ascribes to causal relations over logical relations.

Proclus' commentary on Euclid's *Elements* contains a formulation and an attempt to answer the question whether mathematical proofs explain. In so doing, he seems to presuppose an Aristotelian conception of explanatory demonstrations, in which an explanatory proof should show that a subject has certain attributes in virtue of its being. However, understanding Proclus' account of mathematical explanation solely from the viewpoint of Aristotle's theory of demonstration fails to explain a central and unique feature of Proclus' explanatory reformulation of certain Euclidean proofs: the employment of kinematic constructions.[1] In the following, I answer the question why the explanatory factor in Proclus' explanatory reformulations of Euclid's proofs is a process that generates mathematical objects rather than their definitions. I argue that Proclus' account of mathematical explanation rests on the assumption that Aristotelian explanations, being grounded in essences or definitions, lack explanatory worth. Hence in his explanatory reformulations of certain Euclidean proofs, Proclus does not merely

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[1] I justify this interpretation of Proclus' account of mathematical explanation in the following section.

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consider whether mathematical proofs can establish essential relations; he also considers whether proofs based on definitions or essences can provide explanation. In substantiating this contention, I examine Proclus' conception of explanatory demonstration in light of the interpretation of Aristotle's theory of demonstration in late antiquity. I show first that the question of mathematical explanation presupposes a conceptual shift in the notion of cause in later antiquity due to Stoic influence. This conceptual shift results in a distinction between *causes*, which are factors that bring about certain effects by being active, and *explanations*, which specify the reasons or grounds of certain facts or conclusions. This distinction leads in turn to the contention that the mathematical sciences do not inquire into causes because mathematical objects, being conceptual entities, cannot be causally affected. Second, I argue, through an analysis of Philoponus' commentary on the *Posterior Analytics*, that both the distinction between cause and explanation and its consequence for the explanatory worth of mathematical demonstrations were incorporated into the interpretation of Aristotle's theory of demonstration in late antiquity. Finally, in light of these discussions, I show that Proclus can give an affirmative answer to the question whether mathematical proofs are causal (as opposed to explanatory) due to his ontology, in which mathematical objects are viewed as projections of discursive reason-principles (λόγοι) onto the imagination, and due to the priority he ascribes to causal relations over logical relations. I open my argument with Proclus' formulation of the question of mathematical explanation and his attempt to answer it.

## *The Problem of Mathematical Explanation in Proclus' Commentary on Euclid's Elements*

Proclus formulates the question of mathematical explanation in his comments on the first proof in the *Elements*, where he examines Euclid's proofs in light of Aristotle's notion of demonstration:

> We shall find sometimes that what is called "proof" has the properties of demonstration, in proving the sought by means of definitions as middle terms – and this is a perfect demonstration, but sometimes it attempts to prove from signs. This should not be overlooked. For although geometrical arguments are absolutely necessary on account of their underlying matter, they do not always draw their conclusions by demonstrative methods. For when it is proved that the interior angles of a triangle are equal to two right angles from the fact that the exterior angle of a triangle is equal to the two opposite interior angles, how can this dem-

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> onstration be from the cause? How can the middle term be other than a sign? For the interior angles are equal to two right angles even if there are no exterior angles, for there is a triangle even if its side is not extended.[2] (206.12–26 Friedlein)

In arguing that Euclid's proof of the equality of the sum of the angles of a triangle to two right angles (*Elements* I.32) is not explanatory, Proclus appeals to Aristotle's conception of demonstrative explanation, in which such demonstrations ground the object's properties in its essence or being. In Proclus' view, Euclid's proof of proposition I.32 is not explanatory because it is based on an auxiliary construction, in which the triangle's base is extended. This auxiliary construction does not provide an explanation, in Proclus' view, first because there is no necessary relation between the extension of the base and the attribute "having the sum of the angles equal to two right angles"; a triangle has this attribute regardless of its having an exterior angle as a result of the extension of the base. Second, the extension of the base has no bearing on the triangle's being; there is a triangle, Proclus says, even if its side is not extended. In Proclus' view, then, Euclid's auxiliary construction has no explanatory worth because triangles have the sum of their angles equal to two right angles by virtue of being triangles and not of having an external angle. This passage may suggest, then, that in questioning the explanatory worth of Euclid's proof of proposition I.32, Proclus addresses the question whether mathematical proofs can establish essential relations between subjects and attributes.

However, an examination of Proclus' attempt to render Euclid's proof of proposition I.32 and of two other related propositions (I.16 and I.17) explanatory indicates that although Proclus does address the above question, his explanatory reformulations of these proofs do not merely meet the requirement that proofs should establish essential predications. In his comments on propositions I.16 (the exterior angle of a triangle is greater than either of the interior and opposite angles), I.17 (any two angles of a triangle are less than two right angles), and the second part of I.32 (the sum of the angles of a triangle is equal to two right angles), Proclus attempts to derive the above propositions from a kinematic construction, in which a triangle is constructed out of two perpendiculars to a given straight line that move toward each other up to their intersection point, which is the triangle's vertex. Clearly, from his comments on propositions I.16 and I.17, Proclus introduces this kinematic construction to account for the explanatory worth of these proofs. This kinematic

[2] All translations are mine.

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construction is first introduced in Proclus' comments on proposition I.16 as a supplement to a corollary that he derives from proposition I.16. This corollary states that, if a straight line falling on two straight lines makes the exterior angle equal to the interior and opposite angle then the lines cannot form a triangle.[3] Although the proof of this corollary presupposes proposition I.16, Proclus goes on to show by means of the kinematic construction that if the two straight lines form a triangle by moving toward each other and intersecting, the exterior angle is greater than each of the interior and opposite angles. Proclus' treatment of this corollary raises the question of why he tries to establish proposition I.16 in proving a corollary that presupposes this proposition. The answer may be found in Proclus' discussion of the enunciation (πρότασις) of proposition I.16: "In any triangle, if one of the sides is produced, the exterior angle is greater than either of the interior and opposite angles". Proclus states that those who omitted from this enunciation the words "if one of the sides is produced" exposed Euclid to the criticism that this proposition does not hold for triangles as such, because triangles have no exterior angles by virtue of being triangles (305.25–6 Friedlein). Although Proclus argues that the inclusion of these words in the enunciation prevents this criticism, he does not altogether dismiss it; he uses, in fact, the same argument in the above quoted passage, where he claims that Euclid's proof of proposition I.32 is not explanatory because "there is a triangle even if its side is not extended" (206.25–6 Friedlein). Understood in light of the above passage, Proclus' defense of Euclid's enunciation seems to stress that this proof is not meant to hold for triangles as such. In insisting on the inclusion of the words "if one of the sides is produced", Proclus may be understood as claiming that Euclid's proof of proposition I.16 is not meant to be explanatory, i.e., it does not consider the attributes of triangles as such. The concluding paragraph of Proclus' comments on proposition I.16 suggests that his kinematic construction is aimed at rendering this proof explanatory:

> Therefore, if there will be a triangle and lines AB and CD will intersect, of necessity the exterior angle will be greater than the interior and opposite angle.[4] For the exterior angle increases while the interior remains the same, or the interior decreases while the exterior remains the same, or the interior contracts and the exterior expands farther. The motion of the lines is the cause of these [facts] […] and you can infer (ἔχεις συλλογίζεσθαι) from this how the generations of things bring to our view the true causes of the conclusion. (309.16–310.8 Friedlein)

[3] This corollary is equivalent to proposition I.28 in the *Elements*.

[4] AB and CD refer here to the perpendiculars that become the triangle's sides when intersecting each other.

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In Proclus' view, then, the kinematic construction establishes a necessary connection between the generation of a triangle and the attribute of having an exterior angle greater than its opposite interior angle.[5] Proclus does not explain this contention but the rationale behind it seems to be this: the inclination of the perpendiculars is not only the procedure through which the exterior angle becomes greater than its opposite and interior angle, it is a necessary condition for the production of a triangle; the inclination of the perpendiculars makes the two interior angles less than two right angles. Hence, by the parallel postulate, these inclined perpendiculars will necessarily intersect if extended indefinitely and thus will form a triangle.[6] Accordingly, the derivation of proposition I.16 from the kinematic construction renders this proof explanatory, since, unlike Euclid's proof, it shows that the attribute of having an exterior angle greater than the interior and opposite angle holds for triangles by virtue of being triangles and not by virtue of having an extended base.

This conclusion finds further support in Proclus' comments on proposition I.17, where he employs again the kinematic construction of a triangle, stating that one must look at the generation of triangles (γένεσις τῶν τριγώνων) to discover the cause of this proposition (310.19–311.1 Friedlein). Here he explicitly claims that proposition I.16, from which Euclid derives the present theorem, cannot serve as its cause. According to Proclus, the extension of the base, from which the exterior angle results, is contingent so it cannot explain the necessary fact that the sum of any two interior angles of a triangle is less than two right angles. The cause of this conclusion, he argues, is the inclination of the perpendiculars toward the base, which decreases the right angles (311.15–23 Friedlein). In his comments on this proposition, as in those on the previous one, Proclus considers explanatory a procedure in which the attributes of triangles are derived from their mode of generation. Proclus' preference of the kinematic construction over Euclid's construction comes out in the concluding paragraph of his discussion of proposition I.32, where he refers to the kinematic construction as evidence for the truth of this theorem:

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[5] Through this construction Proclus does not succeed in proving Euclid's theorem, which states that the exterior angle of a triangle is greater than *either* of the interior and opposite angles. He only proves that the exterior angle is greater than the base angle.

[6] Here Proclus does not consider all possible cases. He ignores the cases in which both exterior angles decrease or increase when forming a triangle.

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> It seems that the truth of this theorem is in line with our common notions. For if we think of a straight line and of lines standing in right angles at its extremities, then if they incline so that they generate a triangle, we would see that in proportion to their inclination, so they reduce the right angles that they made with the straight line; the same amount that they have subtracted from these [angles] is added through the inclination to the angle at the vertex, so of necessity they make the three angles equal to two right angles. (384.13–21 Friedlein)

The reasoning presented in the above passage is mathematically deficient. The kinematic construction does not entail the vertical angle of the triangle being equal to the amount subtracted from the base angles as a result of the rotation of the perpendiculars.[7] Proclus' reference to the kinematic construction, even though it cannot be used in proving proposition I.32, may be understood in light of his comments on propositions I.16 and I.17. So understood, Proclus' appeal to the kinematic construction in this context seems to be aimed at securing the explanatory worth of this proof. In this interpretation Proclus offers here an evocative explanatory procedure rather than a valid mathematical proof.

Proclus' comments on propositions I.16, I.17, and I.32 indicate that Aristotle's notion of explanatory demonstration underlies his formulation and answer to the question of mathematical explanation. Proclus' attempts to anchor these propositions to a kinematic construction seem to answer the question whether mathematical proofs can ground the proved attributes in their subject's being. Yet, this account does not provide an exhaustive explanation for the rationale behind Proclus' employment of kinematic constructions. In particular, it does not answer the question why the explanatory factor in Proclus' account is a process that brings about a certain result. If Proclus' sole objective had been to accommodate Euclid's proofs to Aristotle's theory of demonstration he would have shown that a triangle has its mathematical attributes by virtue of being a triangle through an attempt to derive these attributes from the definition of a triangle. Definitions, however, do not feature in Proclus' efforts to render Euclid's proofs explanatory. Instead Proclus derives the triangle's attributes from a process that generates triangles. Thus Proclus' explanatory reformulation of Euclid's proofs reflects

[7] The derivation of proposition I.32 from the kinematic construction requires further reasoning. If one divides the vertical angle by a perpendicular drawn from the triangle's vertex to its basis, one can show that each of the parts of the vertical angle is equal to the angles that the triangle's sides make with the original perpendiculars (i.e., the perpendiculars out of which the triangle was constructed), because these angles are alternate angles. Proclus, however, does not offer such reasoning in the above passage.

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only partial adherence to Aristotle's conception of demonstrative explanation. Proclus adheres to Aristotle's general requirement that proofs should ground the subject's attributes in the subject's being, yet in accommodating Euclid's proofs to this requirement he ascribes explanatory worth to active factors that generate mathematical objects, thereby deviating from Aristotle's theory of demonstration. In the following two sections I place Proclus' discussion of mathematical explanation in the context of the ancient interpretation of Aristotle's theory of demonstration. I thereby show that in his comments on propositions I.16, I.17 and I.32, Proclus in fact attempts to tackle a problem regarding the explanatory worth of Aristotelian demonstrations, in which essences or definitions, rather than active factors, are considered explanatory. I open my discussion with an analysis of the presuppositions that gave rise to the problem of mathematical explanation.

## *The Emergence of the Problem of Mathematical Explanation*

In his comments on the first proof in Euclid's *Elements*, where the questions considered by geometry are discussed, Proclus ascribes to Aristotle the view in which geometry is not an explanatory science:

> It seemed to many people that geometry does not study the cause and the question "Why"? – Amphinomus is of this opinion, while Aristotle originated it.[8] But Geminus says that one could find the investigation of this question in geometry too […]. For whose task would it be, if not the geometer's, to seek and find an answer to this question? (202.9–19 Friedlein)

Proclus' association of Aristotle with those who claim that geometry is not an explanatory science is difficult to understand, as Glenn Morrow pointed out.[9] On numerous occasions throughout the *Posterior Analytics*, Aristotle explicitly states that demonstrations should ground their conclusions in the cause, and he considers the question "Why?" (διὰ τί) one of the principal questions asked in demonstrative sciences. Moreover, nowhere throughout the *Posterior Analytics* does Aristotle qualify these contentions with regard to mathematics; on the contrary, in the *Posterior Analytics* I.14 he counts arithmetic, geometry, and optics among the sciences that inquire into the reason why (78b19–21). Although it is clear that Proclus' report on Aristotle's view regarding

[8] Amphinomus was probably a contemporary of Speusippus.
[9] Morrow 1970, 158 n. 7.

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the explanatory worth of geometry has no basis in any of Aristotle's extant writings, it is still worth considering why Proclus ascribes this view to Aristotle. Such a consideration proves Proclus' erroneous report on Aristotle's view to be more significant than it seems at first glance.

Proclus' commentary on Euclid's *Elements* is not the only ancient source, which traces to Aristotle the view in which mathematics is not an explanatory science. Simplicius makes a similar claim when he presents an excerpt from Geminus' epitome of Posidonius' *Meteorologica* in his commentary on Aristotle's *Physics* (291.21–292.31 Diels). Here Simplicius says that Posidonius' view, in which explanations can be found only in the natural sciences, whereas the mathematical sciences, such as astronomy, can at best provide plausible hypotheses, has its origins in Aristotle (τὰς ἀφορμὰς ἀπὸ Ἀριστοτέλους λαβοῦσαν) (291.22–23 and 292.31 Diels). An analysis of this excerpt provides us with the information, missing from Proclus' report, as to the reasons that led certain philosophers to exclude explanations from mathematics. It thereby brings to light the presuppositions that gave rise to the formulation of the question of mathematical explanation and may clarify why Proclus ascribes to Aristotle the view, in which mathematics is not an explanatory science. The relevant passage in Geminus' epitome is the following:

> Further, in many cases the natural philosopher will apprehend the cause by looking at the productive power (ποιητικὴν δύναμιν), whereas when the astronomer (ἀστρολόγος) proves from external predicates, he does not become a competent contemplator of the cause, for example when he explains (ἀποδίδωσιν) that the earth or the stars have spherical shape. In certain cases he does not even intend to grasp the cause, for example whenever he discusses an eclipse; but sometimes he finds out by hypothesis, thereby explaining in certain ways, whereby if certain assumptions are taken for granted the phenomena will be saved. For example, why do the sun, the moon, and the planets appear to move irregularly? If we may assume that their circuits are eccentric or that the stars revolve in epicycles, their apparent irregularity will be saved; and it will be necessary to examine accurately in how many ways these phenomena can be brought about, so that the study of the planets resembles aitiology according to a possible method. (292.9–20 Diels)[10]

[10] Posidonius, Fr. 18 (Edelstein/Kidd). This view is attributed to Posidonius by Seneca, *Epistulae*, 88.21–8 (Fr. 90 Edelstein/Kidd). In Seneca's report, Posidonius distinguishes on the same grounds geometry, rather than astronomy, from natural philosophy. Hence it seems that Posidonius' view of the explanatory worth of astronomy holds for geometry as well.

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In this passage Posidonius distinguishes natural philosophy from astronomy, stating that the latter does not study the cause. He argues at the beginning of this passage that the astronomical demonstration that the earth or the stars have a spherical shape, for instance, is not explanatory, because the astronomer bases this demonstration on external predicates (ἀπὸ τῶν ἔξωθεν συμβεβηκότων). The exact meaning of this contention may be inferred from the first part of Geminus' summary, where natural philosophy and astronomy are distinguished with respect to their subject matter. (291.23–31 Diels). According to this discussion, natural philosophy studies substance, power, quality, and generation and corruption, whereas astronomy studies the shapes, magnitudes and distances of the earth, sun, and moon. This distinction underlies Posidonius' account of the different methods employed in natural philosophy and astronomy, in which the natural philosopher draws his conclusions from substance, power, value (ἄμεινον οὕτως ἔχειν), or generation and change, whereas the astronomer bases his demonstrations on attributes of shapes (ἀπὸ τῶν συμβεβηκότων τοῖς σχήμασιν) (292.3–9). Thus, Posidonius' distinction between natural philosophy and astronomy is rooted in the distinction between substance and attribute. Natural philosophy studies celestial bodies as substances; that is, it studies the sun, the moon and the stars themselves. Astronomy, by contrast, studies celestial bodies as having certain attributes; in so doing it does not study celestial bodies themselves, but the attributes that hold for their shapes, magnitudes, and motions.[11]

Posidonius' contention that astronomy does not study substances accords with Aristotle's view that mathematical objects are not substances in the strict sense.[12] But Aristotle, unlike Posidonius, does not question the explanatory worth of mathematical demonstration on the basis of this ontological stance. Aristotle's theory of demonstration does not require that the objects of demonstrative knowledge be substances; it only requires that they can be treated *as* substances. In the *Posterior Analytics* I.13 Aristotle qualifies his contention that mathematical objects are not said of any underlying subject, saying that even

[11] I.G. Kidd interprets this distinction differently. In his view, the natural philosopher argues deductively from universal principles or axioms, whereas the astronomer proceeds from "the observation and calculations of the properties of the particular phenomena which form his field of study" (Kidd 1988, vol. II (i), 131–132). I see nothing in this passage that implies that Posidonius' distinction between astronomy and natural philosophy rests on the distinction between universals and particulars.

[12] *Metaphysics* XIII.2 1077b14–17.

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if geometrical objects are said of an underlying subject, they are not studied *as* (ᾗ) being said of a subject. The procedure of treating mathematical objects as substances (i.e., as self subsistent entities), which Aristotle elsewhere calls abstraction (ἀφαίρεσις), suffices, in Aristotle' view, to secure demonstrations in mathematics, since it facilitates definitions of quantitative attributes as such, namely not as being attributes of sensible substances.[13] The possibility of formulating such definitions warrants demonstrative explanations in mathematics since the derivation of mathematical propositions from the definitions of mathematical objects grounds the proved attributes in the essence or being of mathematical objects. Posidonius, by contrast, seems to assume here that for demonstrations to be explanatory their objects have to be real substances.[14]

The different conclusions on the explanatory worth of mathematical demonstrations that Aristotle and Posidonius draw from their common view regarding the substantiality of mathematical objects may be understood in light of their different notions of explanation. In Aristotle's notion, the ontological status of the objects of demonstration has no bearing on the explanatory worth of demonstrations, since by this notion explanation is tantamount to providing grounds or reasons. This notion of explanation does not give rise to the distinction between mathematical and physical demonstrations; by this notion both disciplines can aim at explanation insofar as their definitions can specify the essential feature by virtue of which the conclusion holds. Admittedly, in the *Posterior Analytics* I.13 Aristotle applies the distinction between knowledge of the reason why and knowledge of the fact in differentiating pure mathematics from its subordinate sciences, such as optics, mechanics, and astronomy, claiming that mathematical demonstrations establish the cause or the reason why of the facts that its subordinate sciences establish (78b1–3). Yet, Aristotle's account of the subordinate sciences differs from Posidonius' view in two significant respects: first,

[13] See, *Metaphysics* XIII.3 1077b17–1078a5 and Aristotle's discussion of definitions in *Metaphysics* VII.4 1030a18–27. For the above interpretation of abstraction see Cleary 1985; Lear 1982. For a different view see Mueller 1970.

[14] According to the opening lines of Geminus' epitome, natural philosophy does not study substances alone but also quality and power. Qualities, in the Stoic conception, are corporeal entities that can affect matter causally, Simplicius, *In Cat.* 271.20–2 Kalbfleisch (*SVF* II.383) and 217.32–218.1 Kalbfleisch (*SVF* II.389), and powers are active forces that pervade matter, Sextus Empiricus, *Adv. Math.* IX.75 (*SVF* II.311). Hence, natural philosophy, in Posidonius' view, studies corporeal entities.

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to the extent that mathematics and natural philosophy differ in their explanatory worth it is the former, rather than the latter, that provides explanations. Second, according to the *Posterior Analytics* I.13, physics relies on mathematics for its explanations in cases where the essential feature by virtue of which the conclusion holds is found in the quantitative rather than the physical attributes of sensible objects. Thus, Aristotle's association of explanations with definitions underlies his view regarding the explanatory worth of demonstration in natural philosophy and in mathematics. Further comparison of Posidonius' distinction between natural philosophy and mathematics with Aristotle's distinction between demonstrations of the fact and demonstrations of the reason why suggests that Posidonius' view as to the explanatory worth of mathematical demonstrations is rooted in his dissociation of explanations from definitions.

Aristotle's distinction between demonstration of the fact and demonstration of the reason why does not overlap Posidonius' distinction between demonstrations in the natural sciences and demonstrations in the mathematical sciences. In Aristotle's theory of demonstration, this distinction is tantamount to the distinction between explanatory and non-explanatory demonstrations. Posidonius, however, does not utterly reject the explanatory worth of astronomical demonstrations. According to the above passage, an astronomical demonstration that proves that the earth has a spherical shape *does* provide a certain account (ἀπόδοσις) and the hypothetical method employed in astronomy is said to resemble aitiology. Thus, rather than distinguishing explanatory from non-explanatory demonstrations, Posidonius seems to be drawing a distinction between two types of explanatory accounts. The difference between these accounts may be inferred from the opening sentence of the above passage, where Posidonius states that the natural philosopher searches for the cause, by looking at the productive power (292.9–10 Diels). This statement suggests that Posidonius' distinction between the natural and the mathematical sciences presupposes the Stoic notion of cause, in which causes in the strict sense are factors that bring about their effects by being active.[15] In light of this interpretation, Posidonius' distinction between demonstrations in natural philosophy and in mathematics can be formulated in terms of the distinction between cause and explanation; demonstrations in the natural sciences ground physical phenomena in the productive causes that bring them about,

[15] On this notion of cause see Frede 1987.

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whereas demonstrations in the mathematical sciences establish a variety of other explanatory relations that provide the grounds or reasons of certain phenomena. The distinction between cause and explanation clarifies why, in Posidonius' view, the ontological status of mathematical objects has consequences for the explanatory worth of mathematics. Productive causes that bring about certain effects by their activity can be studied only with regard to real substances identified in Stoicism with bodies that can act or be acted upon.[16] Physical objects are substances of this kind, hence natural philosophy can study causes. Mathematical objects, being attributes, are not real substances, hence mathematics can provide us with mere explanations.[17]

The analysis of the above passage brought to light the assumptions that gave rise to the question whether mathematics, as opposed to other branches of knowledge, is an explanatory science. This question, I argue, stems from the Stoic restriction of the notion of cause to factors that bring about certain effects by being active. This notion of cause underlies the distinction between cause and explanation; that is the distinction between accounts that specify the active factors that bring about certain effects and accounts that specify reasons or grounds. This distinction in turn leads to the restriction of the study of causes to the study of active causes. The second assumption that gave rise to the question of mathematical explanation has its origins in Aristotle's view, in which mathematical objects are not real substances. Taken together with the first assumption, the second assumption implies that mathematics is not an explanatory science as its objects have no causes in the narrow Stoic sense. Further, by bringing to light the dependence of the question of mathematical explanation on the ontological stance regarding the substantiality of mathematical objects, the above analysis provides a plausible answer to the question why Proclus ascribes to Aristotle the view, in which mathematics is not an explanatory science. On

[16] Cf. Diogenes Laertius VII.134 (*SVF* 185).

[17] In the Stoic view, mathematical limits, that is, surfaces, lines and points, are not corporeal but conceptual (see Long/Sedley 1987, 301). Posidonius, however, regards surfaces as existing both in thought and in reality (καὶ κατ' ἐπίνοιαν καὶ καθ' ὑπόστασιν) (Diogenes Laertius VII.135; Fr. 16 Edelstein/Kidd). Yet as I.G. Kidd points out, the textual evidence suggests that Posidonius held this view with regard to surfaces alone; there is no indication that he ascribes real existence to the other mathematical limits (Kidd 1988, II (i), 127). Posidonius' realism with regard to mathematical surfaces has no bearing on the interpretation of the above passage, as the astronomer is not described here as studying surfaces but as studying the attributes of figures and magnitudes.

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the assumption that causes in the strict sense are active factors, Aristotle's ontological stance regarding the substantiality of mathematical objects implies that mathematics is not an explanatory science. In what follows I show that Aristotle's theory of demonstration was indeed interpreted along these lines in late antiquity. Specifically, I show that both the dissociation of causes from reasons or grounds and its implications for the explanatory worth of mathematical demonstrations were integrated into Philoponus' interpretation of the *Posterior Analytics*.[18] I then go on to show that Proclus' account of explanatory demonstrations in mathematics presupposes this interpretation.

## *Cause and Explanation in the Aristotelian Tradition*

The majority of Aristotle's ancient commentators interpreted his contention that mathematical objects are the results of abstraction (τὰ ἐξ ἀφαιρεσέως) as implying that mathematical objects result from the mental act of separating forms from matter and hence exist merely in thought and not in reality.[19] The implication of this interpretation for the question whether mathematics is an explanatory science is explicitly drawn in Philoponus' commentary on Aristotle's *On the Soul*. In distinguishing physics from mathematics, Philoponus claims that physical definitions refer to the matter of physical substances, their form, and the cause by virtue of which their form is realized in matter (55.31–56.2 Hayduck), whereas mathematical definitions refer only to the form:

> The mathematician gives the definitions of abstracted forms in themselves, without taking matter into account, but he gives these [definitions] in themselves. For this reason he does not mention the cause in the definition; for if he defined the cause, clearly he would also have taken the matter into account. Thus, since he

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[18] We have no explicit evidence to the effect that Philoponus' interpretation of Aristotle's theory of demonstration was held also in Proclus' time. However, as its title indicates, Philoponus' commentary on the *Posterior Analytics* is based on Ammonius' – Proclus' disciple – lectures. Thus Proclus' interpretation of the *Posterior Analytics* may rub off on Philoponus' commentary. However it may be, I appeal to Philoponus' commentary here, in showing that the distinction between cause and explanation was read into Aristotle's theory of demonstration in late antiquity. My argument in the last section of this paper, which shows that Proclus shares with Philoponus the presuppositions that underlie the latter's interpretation of the *Posterior Analytics* does not rest on assumption that Philoponus presents Proclus' view in his commentary.

[19] See Mueller 1990.

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> does not discuss the matter he does not mention the cause. For example, what is a triangle? A shape contained by three lines; what is a circle? A shape contained by one line. In these [definitions] the matter is not mentioned and hence neither is the cause through which this form is in this matter. Unless perhaps he gives the cause of those characteristics holding in themselves for shapes: for instance, why does a triangle have its angles equal to two right angles? (57.35–58.6 Hayduck).

This passage indicates that Philoponus, like Posidonius, regards the question of the ontological status of mathematical objects as closely related to the question whether mathematics is an explanatory science. Posidonius' view that a science that deals with immaterial objects cannot study the cause is formulated here in terms of Aristotle's distinction between form and matter. According to Philoponus, physics, which deals with forms embodied in matter, studies the cause, whereas mathematics, which studies forms in separation from matter, does not. The notion of cause that Philoponus uses here is narrow; it refers primarily to the cause of the realization of form in matter. Although there is no indication in this passage that this notion of cause is identical with the Stoic notion of active causes, this passage suggests that Philoponus uses here the term "cause" in two different meanings, corresponding to the distinction between cause and explanation. In the last sentence of the above passage, Philoponus qualifies his contention that mathematicians do not discuss the cause, saying that they perhaps (ἐὰν μὴ ἄρα) do so when proving that the sum of the interior angles of a triangle is equal to two right angles. It is unlikely that this cause is identical with the cause studied in physics. Physics studies the cause of the realization of form in matter, but mathematics, which considers forms separately from matter, cannot possibly study this cause. An examination of Philoponus' commentary on the *Posterior Analytics* supports this interpretation.

Philoponus' analysis of physical definitions into form and matter is reflected in his interpretation of Aristotelian demonstrations. In addressing the question discussed in the *Posterior Analytics* II.2–10, whether definitions can be demonstrated, Philoponus draws a distinction between two types of definitions: formal and material.[20] According to this distinction, formal definitions include the essential attributes of the defined object, whereas material definitions include the attributes

[20] The attribution of the commentary on the second book of the *Posterior Analytics* to Philoponus is doubted. This, however, does not affect my argument, as all the references I make here to the commentary on the second book accord with views expressed in Philoponus' other commentaries.

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that are present in matter (364.16–18 Wallies). He states further that formal definitions are the indemonstrable principles of demonstrations, whereas material definitions are the conclusions of demonstrations (365.25ff. Wallies). For instance, in Philoponus' view the first premise of the following demonstration is a formal definition and the conclusion is a material definition:

(1) Anger is a desire for revenge
(2) A desire for revenge is boiling of the blood around the heart
Therefore (3) Anger is boiling of the blood around the heart.

The three propositions stated in this demonstration are the constituents of Philoponus' example of an adequate physical definition presented in his commentary on Aristotle's *On the Soul*. Here, Philoponus argues that the definition "Anger is a desire for revenge" is empty or dialectical since it does not include all the attributes of the object. The adequate definition of anger, according to Philoponus, is the following definition, which refers to the form, the matter, and the cause: "Anger is boiling of the blood around the heart caused by a desire for revenge" (43.28–44.8 Hayduck). The correspondence between this definition and the above example of demonstration may suggest that in Philoponus' view demonstrative explanation, analogously to physical causes, accounts for the relationship between formal definitions and material definitions. If this interpretation is correct, the first premise and the second premise of the above syllogism do not express the same relation. The first premise, being a formal definition, expresses a predicative relation between "anger" and "desire for revenge". The second premise, specifying the cause of the realization of form in matter, though it has a predicative form, expresses in fact the causal relation between the effect "desire for revenge" and the cause "boiling of the blood around the heart". In this interpretation, then, the conclusion "anger is boiling of the blood around the heart" is derived from two relations: the transitivity of the predicative relation stated in the premises and the causal relation that underlies the predicative formulation of the second premise.

Explicit evidence to this effect is found in Philoponus' comments on the *Posterior Analytics* I.4. In this chapter Aristotle presents four senses in which one thing holds for another "in itself". The first two senses are predicative: in the first sense, a predicate holds for a subject in itself if it is part of the definition of the subject; in the second sense, a predicate holds for a subject in itself if the subject is part of the definition of the predicate. In addition to these senses, Aristotle presents two more senses of "in itself": the third sense distinguishes substances that exist

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in themselves from attributes that depend on substances for their being. The fourth sense distinguishes a causal relation among events from an incidental relation among events. In his comments on this fourfold distinction Philoponus argues that only the first two senses contribute to the demonstrative method (65.10–11 Wallies), yet he also regards the fourth (the causal) as relevant to the theory of demonstration. According to Philoponus, the causal sense of "in itself", though it does not contribute to the formation of the premises of demonstration, contributes to the "production of the whole syllogism" (65.15 Wallies). More precisely, Philoponus argues that the causal sense of "in itself" expresses the relation between the cause, taken as the middle term of demonstration, and the conclusion. In this discussion, then, Philoponus employs two different senses of "in itself": one sense accounts for the predicative relations expressed in the premises of demonstration and the other accounts for the relation between the middle term and the conclusion. According to Philoponus, the premises of demonstration are "in itself" in one of the two predicative senses delineated by Aristotle. Their predicate is either part of the definition of their subject, or their subject is part of the definition of their predicate. By contrast, the middle term and the conclusion of a demonstration are related according to the fourth sense of "in itself"; that is, they are related as cause and effect. Thus in Philoponus' view, the derivation of the demonstrative conclusion is not solely based on the transitivity of the predicative relation stated in the premises. In addition to the transitivity of the predicative relation, the demonstrative derivation is based on causal relations between the middle term and the conclusion. Such a distinction between logical relations and extra-logical or causal relations is explicitly drawn at the beginning of Philoponus' introduction to his commentary on the second book of the *Posterior Analytics*:

> In the first book of the *Posterior Analytics* he showed how there is a demonstration and what is a demonstration and through what premises it has come about, and he showed further how a demonstrative syllogism differs from other syllogisms and that in other syllogisms the middle term is the cause of the conclusion and not of the thing and in a demonstrative syllogism the middle term is the cause both of the conclusion and of the thing (αἴτιος καὶ τοῦ συμπεράσματος καὶ τοῦ πράγματος) (334.1–8 Wallies).

It follows then that Philoponus conceives of demonstrative derivation as resting on two relations: the transitivity of the predicative relation stated in the premises and the causal relation between the middle term and the conclusion. This distinction is tantamount to the distinction

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between explanation (an account that states the reasons or grounds of certain conclusions) and cause (the factor that brings about a certain effect). Philoponus' account of the difference between physical and mathematical objects implies that only physical demonstrations, where formal and material definitions can be given and the cause of the realization of form in matter can be sought, are explanatory in the strict sense; they provide us with both causes and logical grounds. Mathematical demonstrations, by contrast, can provide merely "the cause of the conclusion" and not the "cause of the thing", to use Philoponus' terminology, because the latter cannot be sought with regard to mathematical objects, which lack a material aspect. Indeed, the requirement that the middle term of demonstrations should serve as the cause of the conclusion does not feature in Philoponus' demonstrative reformulation of Euclid's proof of proposition I.32, found in his comments on the *Posterior Analytics* I.9. Here Philoponus focuses only on the following characteristics of Aristotelian demonstrations: (1) the syllogistic form, (2) the derivation of the conclusion from definitions, and (3) the dependence of the proof on principles proper (οἰκεῖος) to the discipline (116.7–22 Wallies). Philoponus' omission of the requirement that demonstrations should specify the cause, rather than being careless, accords with his conception of mathematical objects, his notion of cause, and the distinction between logical and causal relations. More specifically, the assumptions that mathematical objects are abstractions from matter; that causes in the strict sense account for the realization of form in matter; and that explanatory demonstrations are based on both logical and causal relations all imply that mathematical demonstrations can provide the reasons or grounds of the conclusion but not its cause. In this view, then, the question whether the middle term of mathematical demonstration is the cause is simply irrelevant.

The above examination indicates that the two assumptions that underlie Posidonius' distinction between physical and mathematical demonstrations were incorporated into Philoponus' interpretation of Aristotle's theory of demonstration. Like Posidonius, Philoponus excludes explanations from mathematics on the basis of their ontological status, or more precisely their immateriality. Associating the ontological status of mathematical objects with the explanatory worth of mathematical demonstrations presupposes a narrow notion of cause; in Philoponus' view causes in the strict sense refer to factors that account for the realization of form in matter, whereas in Posidonius' view they refer to active factors. Despite this difference, the restriction of the notion of cause leads in both cases to the distinction between causes and

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grounds or reasons and the identification of explanations in the strict sense with the former. Further, it follows from the above analysis that in Philoponus' view it is physics rather than mathematics that fits better with Aristotle's notion of explanatory demonstrations. Philoponus' analysis of Aristotelian demonstrations into formal definitions, which are the indemonstrable premises, material definitions, which are the demonstrative conclusions, and the middle term, which is the cause, corresponds with his analysis of physical objects into form, matter, and cause. Thus, all the three elements that constitute in Philoponus' view an Aristotelian demonstration, and most notably the cause, can be found only in physical demonstrations; mathematical demonstrations, resting on formal definitions alone, do not properly fit into this model. Having shown how the view that mathematics is not an explanatory science came to be associated with Aristotle, I turn to examine Proclus' account of mathematical explanation in light of the assumptions that gave rise to this view.

### *Proclus' Notion of Cause and Demonstrative Explanation*[21]

In chapter VI of the first prologue to his commentary on Euclid's *Elements*, Proclus contrasts his view that mathematical objects exist in the soul prior to sense objects with the view that they are derived from sense objects by abstraction or collection (12.2–9 Friedlein). The latter ontological view, as the above analysis has shown, led Posidonius and Philoponus to exclude explanations from mathematics. The second argument that Proclus presents in support of his ontological stance bears on the question of demonstrative explanation:

> Again, second, if we collect the definitions (λόγοι) of mathematics from below and from sense objects, must we not say that those demonstrations which are composed from sense objects and not those composed from more universal and simpler forms are better? For we say that the causes[22] are absolutely appropriate to the demonstrations that lead to the pursuit of the sought. If, then, the particulars are causes of the universals and sense objects are causes of the objects of discursive reason, by what means can we refer the term of demonstration to the more

[21] I do not venture to offer here an exhaustive account of Proclus' notion of causality. I confine my discussion to the aspects of this notion that feature in the commentary on the *Elements*.

[22] In Glenn Morrow's translation all references to causes (αἴτια) in this passage are translated as "premises". This translation obscures the implications of this argument for explanatory demonstrations.

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> universal rather than to the particulars and render the substance of the objects of discursive reason, which is prior to sense objects, more fitting to demonstrations? (13.26–14.11 Friedlein)

The major claim that Proclus makes here is that those who view mathematical objects as abstractions from sense objects cannot warrant explanatory demonstrations. He alludes here to the *Posterior Analytics* I.24, where Aristotle discusses the question whether universal demonstrations are better than particular demonstrations.[23] In this discussion Aristotle does indeed claim that demonstrations that ground their conclusions in universal terms are more explanatory than demonstrations that refer to particular terms. However, in this discussion he also dissociates the question of the ontological status of universals from the question of the cognitive worth of universal demonstrations (85b15–18). Thus, although Aristotle would not deny that universals are derived from the particulars, he would reject Proclus' contention that these universals lack an explanatory worth. In Aristotle's view the priority of the particulars in the cognitive process that leads to the universal is not an explanatory priority. Proclus, by contrast, assumes here that the priority of the particulars does amount to an explanatory priority. Proclus' argument, then, does not point to a genuine inconsistency in Aristotle's theory of demonstration. Rather, in this argument Proclus reads into Aristotle's theory of demonstration his notion of cause. Whereas in Proclus' argument the term cause refers to the source from which universals are derived, Aristotle uses it to refer to the universal term that is synonymously predicated of the subject of the conclusion and has the proved property by virtue of itself (85b35–86a3). Thus, Proclus' argument resembles Posidonius' and Philoponus' arguments for the exclusion of explanations from mathematics; like these philosophers, Proclus argues on the basis of a modification of Aristotle's notion of demonstrative cause that if the objects of demonstrations are abstractions from the particulars then demonstrations cannot be explanatory. The dependence of this argument on such a notion of cause raises the question whether Proclus regards explanatory demonstrations as based on causal relations between a source and its outcome. Further analysis of Proclus' ontological stance answers this question in the affirmative.

In presenting his ontological account of mathematical objects, Proclus does not merely argue for the priority of mathematical objects over sense objects, he conceives of mathematical objects as products of the

[23] *Posterior Analytics* 85b23–86a3.

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soul's activity, or in his terms as projections (προβολαί) of the forms that pre-exist in the soul (13.7–12 Friedlein).[24] An analysis of Proclus' first argument in support of this view implies that projection is a causal relation that subsists between an efficient cause and its effect. In this argument, Proclus attempts to show that his view, in which mathematical objects are derived from the soul, can account for the exactness and certainty of mathematics (12.9–13.26 Friedlein). He dismisses the alternative view that mathematical objects are derived from sense objects, claiming that this would render sense objects much more exact than they are (12.16–17 Friedlein). In substantiating this conclusion Proclus presents various examples to indicate that sense objects lack mathematical precision; he asks, for instance, where can we find among sense objects equal radii or right angles? Viewed from an Aristotelian standpoint, this argument does not seem to furnish a sound refutation of the abstractionist view. Entities such as equal radii can be derived from sense objects by abstraction that disregards the imprecise attributes of sense objects. The abstractionist view, then, does not imply that the precision of mathematical objects should be present in their source, namely in sense objects. This contention, rather than being a necessary consequence of the abstractionist view, is the assumption on which Proclus' argument is based. In explaining why precise entities cannot be derived from imprecise entities, Proclus says: "for every cause of steadfast knowledge has itself such a characteristic to a greater degree" (13.5–6 Friedlein). This contention is an application of proposition 18 of the *Elements of Theology*: "Everything that furnishes (χορηγοῦν) others in virtue of its being has itself primarily that characteristic, of which it gives a share to the recipients (χορηγουμένοις)". This proposition refers to Proclus' notion of production in virtue of being, which accounts for the necessary relation between cause and effect and their similarity.[25] The notion of production in virtue of being is grounded in the Neoplatonic notion of substance, in which real substances have two activities, an inner activity which is identical with their complete actualization or essence and an external activity leading to the necessary production of lower grades of reality. The similes of fire and the sun, often used by Proclus and other Neoplatonists, illustrate how through these activities a cause produces its effects in virtue of its being. Heat is the complete

[24] Although Proclus uses here the general term "mathematical objects", his argument does not hold for all mathematical objects, but only for those mathematical objects that the soul projects onto the imagination.

[25] On this notion see Trouillard 1958.

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actualization of fire, and light is the complete actualization of the sun. The very actualization of these activities results in external activities: heating other objects in the case of fire and illuminating other objects in the case of the sun. In other words, fire and the sun produce in virtue of their being, as the actualization of their inner activity leads in turn to the external activity that causally affects other entities. This notion of production resembles the Stoic notion of cause, in viewing causes as active factors,[26] but it is modeled, as Anthony Lloyd has shown, on Aristotle's notion of physical causation, found especially in *Physics* III.3.[27] It is thus a physical notion of causality, whose scope of application has been extended to metaphysics. Proclus alludes to this notion of causality in his argument discussed above when he claims that if the essence of mathematical *logoi* does not pre-exist in the soul, their existence would be accidental (ηὐτοματισμένη) (13.18–21 Friedlein). Proclus assumes here that necessary production depends on the presence of the essence of the produced entity in its producer. The notion of production in virtue of being, which guarantees necessary production, explains this dependence. Hence, in viewing mathematical objects as projections of pre-existent forms, Proclus in fact conceives of them as effects of productive causes. In so doing he subjects mathematical objects to the notion of causality that governs, in his philosophy, all realms of being. I now show that this notion of causality underlies Proclus' conception of demonstrative explanation as well.

In concluding the first chapter of the second prologue, where Proclus presents in detail his conception of geometrical objects, in which they are projections of discursive reason-principles (λόγοι) onto the imagination,[28] he defends his view against possible charges:

> Such are things we have to say about geometrical matter. We are not ignorant of what the philosopher Porphyry in his miscellaneous writings and the majority of the Platonists contend, but we believe that our view accords better with both geometrical methods of reasoning (ταῖς γεωμετρικαῖς ἐφόδοις) and Plato who calls the

[26] Unlike the Stoics, Neoplatonists consider causes immaterial. A clear formulation of this view is found in proposition 80 of *The Elements of Theology*.

[27] Lloyd 1990, 99–100.

[28] In the commentary on Euclid's *Elements* Proclus does not draw a sharp distinction between forms (εἴδη) and reason-principles (λόγοι). But according to his commentary on Plato's *Parmenides*, λόγοι are the causes of things that have no paradigmatic causes, such as the parts of animals and accidents. Yet the λόγοι depend on a higher cause, which Proclus identifies with Nature (*In Parm.* 825.36–827.25 Cousin). On the relationship between immanent forms and λόγοι see Helmig 2006, 270 n. 51.

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> subjects of geometry discursive. For these two accord with each other, because on the one hand the causes of geometrical forms, in accordance with which discursive reason projects the demonstrations, pre-exist in discursive reason and on the other hand each of these divided and compounded shapes have been projected onto the imagination. (56.23–57.8 Friedlein)

It is clear from this passage that one of Proclus' objectives in regarding geometrical objects as projections of discursive forms onto the imagination is to account for geometrical reasoning. It is also clear from this passage that Proclus associates demonstrations with the projective activity of discursive reason, and causes with the forms in accordance with which discursive reason projects them. In appealing to these contentions in defense of his ontological stance, Proclus follows his teacher Syrianus, who also regards this ontological account of geometrical objects as guaranteeing explanatory demonstrations.[29] In his comments on *Metaphysics* XIII.2, Syrianus argues that one cannot hold an abstractionist view of mathematical objects and at the same time claim that the premises of explanatory demonstrations are the causes of the things (αἴτια τῶν πραγμάτων) (91.13–17 Kroll). Syrianus' view on this matter, which also appears in Proclus' second prologue,[30] is that geometry deals with the objects of imagination insofar as they co-exist with the reason-principles of discursive reason, and that these reason-principles are the causes of demonstration (91.29–32 Kroll). Consequently, Proclus and Syrianus conceive explanatory demonstrations as based on the notion of causality, to which reality, in their view, is subject. This conception of explanatory demonstrations implies that demonstrative derivation is based on logical necessity, but also on causal necessity, for which the notion of production in virtue of being accounts.[31] This conclusion finds support in book IV of Proclus' commentary on Plato's *Parmenides*:

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[29] This interpretation lends further support to Dominique O'Meara's contention that one of the factors that led Proclus to write a commentary on a geometrical rather than on an arithmetical work is geometry's demonstrative character. O'Meara 1989, 166–169.

[30] *In Eucl.* 54.23–6 Friedlein.

[31] This conclusion accords with Dominic O'Meara's account, according to which Proclus attempted to apply demonstrative methods to metaphysics. (O'Meara 1998) As the object of metaphysics – being as being – is not a genus, demonstrations in metaphysics cannot be based on predicative relations, but on the relation of ἀφ᾽ἑνὸς καὶ πρὸς ἕν, which Proclus identifies with the causal relation of procession and reversion. Thus the application of demonstrative methods to metaphysics requires a causal interpretation of demonstrative derivation.

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> Those things which our discursive reason cannot study at the divine levels, it may observe at these levels as an image […]. If one wondered how causes are present everywhere in the effects, one would look at the perfect image of this [principle] in logical matters. For the genera are absolutely predicated of the things of which the species are predicated, and the species [is present] together with the genus, but the genus [is present] also without the species. (926. 21–39 Cousin)

In this passage Proclus distinguishes logical relations from causal relations and ascribes priority to the latter. Thus Proclus, like Philoponus, regards logical inferences as based on predicative relations between genera and species and on causal relations between causes and effects. But in viewing predication as an image of causality, Proclus does not merely assume that predicative formulations admit causal interpretation, he assumes that causal relations underlie predicative relations. This assumption is central to Proclus' philosophy. In fact, the two sections of the *Elements of Theology*, which E. R. Dodds entitles "Of the Grades of Causality" and "Of Wholes and Parts" (props. 56–74) may be understood as an attempt to substantiate it. For instance, the asymmetry between genera and species, to which Proclus refers in the above passage, is formulated in causal terms in propositions 57 and 60 of the *Elements of Theology*. These propositions imply that productive activity of the more efficacious causes extends lower down in the scale of being than the productive activity of less efficacious causes, so that the more efficacious causes also produce the effects of the less efficacious causes but not vice versa.[32] Thus, just as the genus entails the species but not vice versa, so the more efficacious causes entail the less efficacious causes but not vice versa. The question whether Proclus' theory of causality succeeds in replacing a logical theory deserves a separate study.[33] For our present concern, Proclus' sheer attempt at this replacement suffices to indicate that he conceives of demonstrative explanation as based on causal relations.

The examination of Proclus' conception of demonstrative explanation shows that Proclus shares with Posidonius and Philoponus the assumptions that give rise to the question of mathematical explanation. He (1) assumes that an abstractionist conception of mathematical objects cannot account for explanatory demonstrations in mathematics; (2) conceives of causes as factors that bring about their effects by being active; (3) distinguishes logical derivation from causal derivation; and, finally, (4) regards demonstrative explanation as based on causes rather

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[32] For the rationale behind this conception see Lloyd 1976.
[33] The foundations for such a study were laid down in Lloyd 1955–6.

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than on reasons or grounds. The fundamental assumption that facilitates demonstrative explanation in Proclus' philosophy of mathematics is the rejection of the abstractionist conception of mathematical objects and its replacement by a conception in which geometry studies the reason-principles that pre-exist in discursive reason by means of their projections onto the imagination. In espousing this ontological stance, Proclus, following Syrianus, rejects one of the fundamental assumptions that gave rise to the exclusion of explanation from mathematics: the identification of mathematical objects with abstractions. Yet the notion of explanation which Proclus presupposes is similar to the one adopted by his opponents in being restricted to active factors. Proclus' explanatory reformulations of the Euclidean proofs discussed above reflect this presupposition; by grounding these proofs in kinematic constructions, Proclus accounts for the proved propositions in terms of active causes.[34]

In the first section of this article I argued that Proclus' kinematic construction is aimed at accommodating Euclid's proofs to Aristotle's requirement that demonstrative explanation should establish essential relations between subject and attributes. In concluding this paper I try to show in what sense Proclus' notion of causality accounts for essential predications.

Proclus' conception of geometrical objects implies that geometry should study two objects: the reason-principle that pre-exists in discursive reason, and its immanent manifestation in the imagination. Geometry, according to Proclus, should study both objects as discursive reason cannot apprehend the forms it contains without the aid of the imagination. In the commentary on the *Elements*, Proclus focuses on the role of his conception of geometrical objects in accounting for constructions and visual representations (56.8–15 Friedlein). He does not spell out the consequences of this view for demonstrative explanation, although he claims that it accounts for geometrical methods of reasoning (56.23–57.8 Friedlein). These consequences, I surmise, can be reconstructed from the *Elements of Theology*.

The distinction between the transcendent objects of discursive reason and the immanent objects of the imagination corresponds to the distinction drawn in proposition 65 of the *Elements of Theology* between subsistence as a cause (κατ' αἰτίαν) and the image-like (εἰκονικῶς)

[34] The other metaphysical assumptions that underlie Proclus' kinematic construction are discussed in Harari 2006, 383–8. See also Nikulin 2002, 245–254.

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subsistence by participation (κατὰ μέθεχιν).[35] This amounts to the distinction between self-subsistent characteristics and these characteristics as subsisting in certain substrates, namely as predicates.[36] The logical implications of this distinction can be understood from proposition 71 of the *Elements of Theology*:

> All [characters] having in the principal causes a more universal and higher rank become in the effects, due to the irradiations derived from these [causes], a kind of substrate for the imparting of [the characters] of the more particular [causes]. And while the irradiations from the higher [causes] bear (ὑποδέχονται) the processions from the secondary [causes], the latter are placed upon the former. (66–31–68.4 Dodds)

According to this passage, predicative relations are the reverse of causal relations; that is, higher causes serve as the subjects of lower causes, so that entities resulting from this causal process are viewed as composed of layers of ever-decreasing characteristics. For instance, a living organism may be viewed as composed of prime-matter, which is the effect of the One; body, which is the effect of Being; and soul, which is the effect of the Soul. The relationship between the lower layer, viewed as a substrate, and the upper layer, viewed as a predicate, is analogous to the relationship between a genus and its qualifying differentia. In this account the more universal term serves as a subject and the more particular term serves as a predicate. Such a relation does not seem to be necessary or essential: Matter may be shaped, but it is not necessarily shaped; likewise a body may be animated, but it is not necessarily animated.[37] Nevertheless, in Proclus' view the layers of which entities in the lower realm of reality are composed are not accidentally related. Their relationship is an image of the causal relations subsisting in the

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[35] For this correspondence see Dodds 1963, 237. The logical characteristics of the transcendent and immanent entities are discussed in Harari 2006, 364–371.

[36] For the sake of clarity and convenience I use the term predication loosely, in referring to what Proclus calls "participation". The difference between Aristotelian predication and participation becomes clear later on.

[37] The relationship between the substrate and its qualifying attributes is similar to the second sense of in itself predication, presented in the *Posterior Analytics* I.4. In this sense a predicates holds for a subject in itself if the subject is part of the definition of the predicate. This sense of in itself predication raises the same problems that arose with regard to Proclus' conception of the relationship between the substrate and its qualifying attributes. In the *Posterior Analytics* I.4, a predication in which "line" is the subject and "straight" is the predicate exemplifies this sense of in itself predication. The question in what sense and how this relation is necessary or essential is one of the major problems in contemporary study of Aristotle's *Posterior Analytics*.

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higher realms of reality. When the relationship between the substrate and its qualifying attributes is understood as an image of causal relations, the substrate acquires the role of a cause, and the qualifying attributes the role of its effects. Viewed from this perspective, the relationship between the substrate and its attributes is no longer accidental; it is the necessary relation between a cause that produces in virtue of its being and its effects.[38] This analysis clarifies why Proclus claims that geometry studies both the object of discursive reason and the object of the imagination. In light of the above reconstruction, the projections onto the imagination present a relation between a certain substrate and its qualifying attribute, but they do not represent it as a necessary relation. The study of these projections as images of the objects of discursive reason brings to light the causal relations between the substrate and its qualifying attributes, thereby explaining the content presented in the imagination. Formulated in Aristotelian terms, the projections onto the imagination provide the fact (ὅτι) and their study as images of the forms of discursive reason provides the reason why (διότι). Despite the equivalence with Aristotle's theory of demonstration, the notion of predication on which Proclus' account is based differs from Aristotle's notion of essential predication, inasmuch as Proclus' notion of causality differs from Aristotle's. Both Aristotle and Proclus conceive of explanatory demonstrations as proofs which show that a subject has a certain predicate in virtue of its being. Yet whereas for Aristotle this locution means "in virtue of its essences as expressed in its definition *per genus et differentias*", for Proclus it means "in virtue of its productive cause". Accordingly, in attempting to render explanatory Euclid's proof for the equality of the sum of the interior angles of a triangle to two right angles, and other related proofs, Proclus does not refer to the definition of a triangle but to its mode of production.

The examination of Proclus' account of demonstrative explanation in light of the Stoic restriction of causes to active factors, which became an integral part of the Neoplatonic notion of causation and the ancient interpretation of Aristotle's theory of demonstration sheds fresh light on the history of the question of mathematical explanation[39] and on Pro-

[38] This view has its origins in Plotinus' criticism of the privileged status of the differentia in Aristotle's *Categories*. Plotinus solves this problem by viewing differentiae as activities of substances (e.g., *Enn*. VI.2.3). For a detailed discussion of Plotinus' view see Lloyd 1990, 88–95.

[39] In particular, it refines Paolo Mancosu's historical account of the question of mathematical explanation, which traces the discussions of this question to Aristotle's theory of demonstration (Mancosu 2001, 101; 2000, 108–111). My analy-

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clus' philosophy of mathematics. First, the above analysis shows that the question of mathematical explanation does not simply arise from Aristotle's theory of demonstration but also from two conceptions of Stoic origin: the narrowing down of causes to active factors, and the assumption that abstract entities cannot act or be acted upon. These assumptions cast doubt on the explanatory worth of mathematics, as a science that studies abstract entities. Further, this study brings to light one of Proclus' important motivations for rejecting the abstractionist conception of mathematical objects. Undoubtedly, Proclus' adherence to Plato's ontology played an important part in his preference for a realist account of mathematical objects. Still, the consequences of Proclus' ontological stance for the status of mathematics as an explanatory science are of no lesser importance. Within the context of the ancient interpretation of Aristotle's conception of mathematical objects and his theory of demonstration, the abstractionist conception of mathematical objects entails that mathematics is not an explanatory science. Proclus' alternative ontological account, in which mathematical objects are projections onto the imagination of forms that pre-exist in discursive reason, secures the status of mathematics as an explanatory science, hence as a science in the strict sense. Finally, when the various references to demonstrative explanation scattered throughout Proclus' commentary on the *Elements* are understood in light of the ancient interpretation of Aristotle's *Posterior Analytics* and Proclus' notion of causality, they no longer appear as insignificant and perhaps erroneous allusions to Aristotle's theory of demonstration. Proclus' employment of kinematic constructions in his explanatory reformulations of certain Euclidean proofs, his argument to the effect that the abstractionist conception of mathematical objects is inconsistent with the theory of demonstration, and his contention that the forms in discursive reason are the causes of demonstration, presuppose a consistent alternative to Aristotle's theory of demonstration, which rests on the assumptions that causes necessarily bring about their effects by being active and that logical relations are images of causal relations. Placed in its appropriate context, Proclus'

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sis shows that in late antiquity, this question arises from the incorporation of Stoic notions into Aristotle's theory demonstration. In his analysis of relationship between philosophy of mathematics and mathematical practice in the 17th century Mancosu has shown that concerns over the explanatory worth of mathematical proofs played an instrumental role in the introduction of procedures based on motion and genetic definitions to mathematical proofs (Mancosu 1996, 92–102). Proclus, as my analysis shows, adopts a similar approach in securing the explanatory worth of mathematical proofs.

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commentary on Euclid's *Elements* appears to offer an ontological justification and a consistent account of mathematical explanation.

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