Synthese (2012) 186:387–409
DOI 10.1007/s11229-011-9986-5

Diagrams as sketches

Brice Halimi

Received: 8 October 2009 / Accepted: 7 August 2010 / Published online: 31 August 2011
© Springer Science+Business Media B.V. 2011

Abstract This article puts forward the notion of "evolving diagram" as an important case of mathematical diagram. An evolving diagram combines, through a dynamic graphical enrichment, the representation of an object and the representation of a piece of reasoning based on the representation of that object. Evolving diagrams can be illustrated in particular with category-theoretic diagrams (hereafter "diagrams*") in the context of "sketch theory," a branch of modern category theory. It is argued that sketch theory provides a diagrammatic* theory of diagrams*, that it helps to overcome the rivalry between set theory and category theory as a general semantical framework, and that it suggests a more flexible understanding of the opposition between formal proofs and diagrammatic reasoning. Thus, the aim of the paper is twofold. First, it claims that diagrams* provide a clear example of evolving diagrams, and shed light on them as a general phenomenon. Second, in return, it uses sketches, understood as evolving diagrams, to show how diagrams* in general should be re-evaluated positively.

Keywords Mathematical diagrams · Pictorialism · Categorical diagrams · Sketch theory · Formal proof · Semantics

A mathematical diagram can be used to represent a mathematical object. This can take on differents aspects, from conveying partial information about an object (as in the case of the diagram showing that an object satisfies a universal property, in algebra

B. Halimi (✉)
Département de Philosophie (IREPH), Université Paris Ouest , Nanterre, France
e-mail: bhalimi@u-paris10.fr

B. Halimi
SPHERE (UMR 7219), Université Paris-Diderot, Paris, France

and category theory), to classifying it as a mathematical structure of such and such kind (as in the case of Hasse diagrams for ordered sets). A mathematical diagram may also be used (as in the case of Venn diagrams) to carry out a mathematical construction and so represent a piece of mathematical reasoning about a given object. Of course, this raises the question of what a mathematical object, information, structure, or reasoning is. What is a mathematical diagram apart from being a diagram pertaining to something mathematical? I am not sure that a fully general characterization of mathematical diagrams as such is possible, but still I think that many examples of mathematical diagrams combine the two features that I have just distinguished, namely the representation of an object and the representation of a piece of reasoning based on the representation of that object.

One of the best examples of the latter feature is the Euclidean figure, which in a proof is often supplemented by the auxiliary lines and circles that the proof calls for. I would like to defend the idea that this is in fact a quite general fact: many mathematical diagrams do not represent mathematical objects once and for all; rather, they can be manipulated and enriched for the purpose of unfolding mathematical properties of the respective objects which they represent.

Here are a few examples. A classical proof that the sum of the angles of a triangle ABC is two right angles, consists in extending one side of the triangle, say AC, into a line, and drawing the parallel (Δ) to BC passing through A. The conclusion then follows from the observation that the angle made by (Δ) and (AB) is equal to $\widehat{ABC}$, and that the angle made by (AC) and (Δ) is equal to $\widehat{ACB}$. The auxiliary features which are added to the original representation of the triangle and lead to the proof are grounded in the original representation of the triangle.

Another example is the proof that $\sum_1^n k = n(n+1)/2$. The proof consists in first laying out n rows of unit dots, the top one with one single dot, the last nth one with n dots. The figure thereby obtained is the representation of the sum to calculate. The sum is completed by going diagonally from the dot on the first row to the last dot on the last row. The diagonal is the hypotenuse of a right triangle whose area is $n^2/2$. The conclusion follows from the observation that the difference between the area and the sum under consideration is composed of n half dots. Once again, the proof is not based only on the original representation of the sum, but needs a further construction, based on the original representation of the sum. The steps of that construction represent the corresponding steps of the proof: first, thinking of the sum as an area, then approximating the sum through a figure whose area is well known, then finally assessing the difference between the sum and its approximation.

The '5-lemma,' in algebra, provides us with yet another case.[1] The proof of that lemma is a typical example of 'diagram chasing,' where exactness conditions or assumptions of injectivity or surjectivity are key. 'It is a long and involved argument, but it is actually almost self-proving. […] At each stage there is really only one thing to do,'[2] which means: only one path to follow at each step. It is thus very natural, in the course of the proof, to gradually adorn the original data—five vertical arrows along

[1] See for example Osborne (2000, p. 23), for a detailed exposition.

[2] Osborne (2000, p. 24).

with horizontal arrows making up a full two-line commutative diagram—with all the stops of an inevitable journey. Each move encapsulates the assumptions that ensure the possibility to carry it out. For example, starting with some element $b \in B$, I know that there is an element $a \in A$ such that $b = f(a)$, since $f : A \to B$ is supposed to be onto, which prompts a move backwards (so to speak), from B to A, and so on. Adding a (relative to the base point b) is not tacking it on to the original diagram as though it were an external item, but contributing to the construction of the overall path that is sought in order to conclude the proof within that diagram. That construction is an intrinsic one, whose possibility is ensured by the very features of the diagram. This is why Osborne can say, as a telling figure of speech, that the lemma is 'almost self-proving.' The resultant diagram is an enriched diagram compared to the initial representation of the configuration, but supported by the intrinsic resources contained in that initial representation—which is why we are dealing with a mathematical proof. The gradually enriched diagram adds to the diagram representing some object (or some configuration of objects) the representation of the steps of a proof based on that original representation. We do not have two diagrams here, but just a single one, which is both a starting point and the embodiment of the successive updates involved in its completion.

Though one cannot claim of course these examples exhibit what is common to mathematical diagrams in general, they do seem illustrate an important kind, whose essential feature is the combination of a visual representation (an original diagram) and of a specific completion of it (corresponding to stages of a proof). Let's refer to diagrams of this kind as *evolving diagrams*. An evolving diagram shows a property of an object by a graphical enrichment, so that the proof of the property lies at the same level as that of the characterization of the object. The representation of a mathematical object, in such a case, is intended to be enriched with respect to a partial aspect of it (as when I extend one of the sides of the triangle into a line), while keeping the overall description of the same object (the triangle, to be specific). Indeed, the final diagram does not generally represent some new object, different from the original one. In the case for example of the proof about the sum of the angles of a triangle, we end up in fact with a growing system of geometrical objects (including additional straight lines on top of the triangle), but this does not preclude the diagram from being about the original triangle all the way throughout the proof: the diagram only evolves, and not the specific object that it represents.

Dealing with something visual is critical here, precisely because it is something that one can consider both all at once and in detail, whereas in the case of a symbolic proof, which is something to be read linearly (it can be reread, but still linearly), the initial description of the object remains distinct from the proof that starts from it. Thus it is essentially because of its visual nature that a diagram makes it possible to represent a proof at the same level as that of the structure the proof relates to. Another way to put it is the following. Usually, the characterization of the mathematical object or situation at stake, with all the assumptions made about it, are the first lines of a proof that consists in a chain of lines, the last line being the proof's conclusion (usually, hypotheses occur at some intermediate lines, and are not called up at the very beginning, but this is not relevant here). A chain of lines is not a line. In the case of a diagram, on the contrary, we do have successive steps, but the starting point (the diagram that represents the

object under study) and the end point (the complete diagram that supports the whole proof) are on a par with each other.

It should be stressed that such a mathematical diagram, in spite of being visual, should not be viewed primarily as a picture, because, most often, it works only on the condition of being completed or transformed, hence somehow modified. It is not a mere picture, but rather a dynamic representation, i.e., a representation which allows us to represent the steps of its own transformation (the transformation of the original representation of an object being the proof that this object has such and such property). Diagrams in the sense that I have singled out are not static finished products. They must be looked at in their making. This is a key feature that the pictorial conception of diagrams (i.e., their conception as pictures) put forward by James R. Brown seems to neglect. Brown, indeed, clearly identifies a mathematical diagram with a picture. Concerning the diagrammatic proof of $1 + 2 + \cdots + n = \frac{n^2}{2} + \frac{n}{2}$, he argues: 'We can in special cases correctly infer theories from pictures, that is, from visualizable situations. An intuition is at work and from this intuition we can grasp the truth of the theorem […] one sees a diagram (sense perception) that induces an intuition (mathematical perception) of something very different. This is what happens when a picture is not merely a heuristic aid, but an actual proof.'[3] The problem of such a conception of diagrams is that no real explanation is given of the shift from a sense perception to a 'mathematical perception.' But Brown's main point is here only to distinguish mathematical intuition from raw sense perception. Furthermore, it could be argued that in some important cases a mathematical diagram does work more as a fixed picture than as anything else.

My aim anyway is only to focus on evolving diagrams, that cover not all but at least significant cases of mathematical diagrams. The feature that those diagrams have in common is the dynamic combination, at the same level, of the representation of an object and of the representation of a proof about that object. That feature, which can be traced back to some proofs in Euclid, is general enough, and, I believe, can be illustrated in the more technical context of modern category theory. The rest of this paper will be devoted to that illustration, which has to do with 'sketch theory.'

Sketches were introduced by Charles Ehresmann[4] in the late sixties, in the field of algebraic and differential topology. Briefly, a sketch is a category-theoretic diagram that represents a given kind of structures, for example monoids, or groups, or fields. The sketch of monoids is a diagram representing the structure common to all monoids, the sketch of groups is a diagram representing the structure common to all groups, and so on. As we will see, this is so in such a way that the proof of a property of monoids (a property following from the definition of monoids) amounts to adding vertices and arrows to the original sketch of monoids. To that extent, sketches are both a nice illustration of the dynamic feature of certain diagrams, and a mathematical reflection upon that feature, a systematic way to turn many structures or theories into a diagrammatic presentation.

[3] Brown (2005, pp. 65–66).

[4] Ehresmann (1968).

1 Sketches

1.1 Categorical diagrams

In sketch theory, diagrams are not just visual representations of mathematical objects, but become mathematical objects in their own right. This is certainly something peculiar to category theory. Generally speaking, it could seem that category-theoretic diagrams are diagrams of their own very special kind and do not exemplify anything general about mathematical diagrams.

On that score, a caveat is in order here. From now on, category-theoretic diagrams will be called diagrams*, in order to distinguish them from diagrams in the general or philosophical sense (even though any ambiguity could be easily cleared up in most cases). This is only for a matter of clarification. This does not detract from their being diagrams in a more general sense, as we will see. Indeed, I will defend the view that diagrams* (and in particular sketch-theoretic diagrams*) are diagrams of the kind that I have singled out, namely evolving diagrams.

Several slogans, in fact, have been associated with diagrams*. Let me first summarize three of them, which are central. Diagrams* are obviously pervasive throughout all of category theory, and for this very reason, category theory could seem doomed to treat diagrams* as a mere tool rather than as an object itself. Diagrams* would be the means to convey mathematical content, but precisely for that reason they would be built-in to the very framework of category theory, rather than something that category theory could speak of. In contrast with that first characterization, I would like to underscore that category theory is not only a diagrammatic* theory, but also provides for a theory of diagrams*. Precisely, as we will see, so called sketch theory can be (partly) described as a diagrammatic* theory of diagrams*.

Along with this feature comes another one. Category theory can be understood to provide a semantics for mathematics alternative to the standard one provided by set theory. From a category-theoretic point of view, as Lawvere has shown it, theories (at least, algebraic theories) can be taken as categories, and their models become functors on those categories. In particular, topoi become the natural generalization of the category of sets, and so the natural means to interpret mathematical theories. Structures of the same type are considered collectively as a category, and models are considered collectively as a category of functors, related to each other through diagrams*, which does not fit the set theoretic way of putting things. I would like to defend the idea that sketch theory helps to overcome in an interesting way the seeming rivalry between set theory and category theory as a general semantical framework.

It remains that one of the main ideas underlying sketch theory is to develop the category-theoretic point of view, by considering a formal theory as being already in itself a genuine mathematical structure, namely a complex diagram* laid out in some base category. Such a twist inevitably induces some redistribution of the usual places of syntax and semantics that has to be made precise. My goal is to show that sketches exhibit diagrams* as being a midpoint (or, as we will see, a flipping point) between syntax (formal theories) and semantics (functorial models). This brings us to a third common idea about diagrams*. It is often claimed that using diagrams*

generally contrasts with more formal conceptions of mathematical reasoning. I hope to show that sketch theory casts doubt on such a claim, to the extent that a sketch can be seen both as a formal proof and as the schematic encapsulation of a concrete reasoning.

My aim in this paper, then, will be twofold. First, I would like to use category theory to illustrate and capture in a systematic way the feature of the kind of mathematical diagrams that I am interested in: sketch-theoretic diagrams* provide a clear example of evolving diagrams, and may shed light on them as a general phenomenon. Second, I would like, in return, to use sketches, understood as evolving diagrams, to dispel the three slogans about diagrams* that I have just summarized.

1.2 Categorical preliminaries

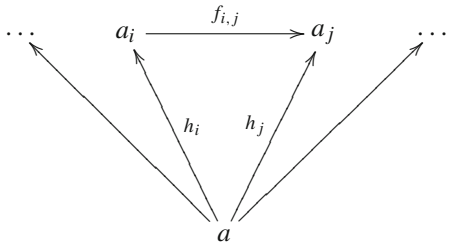
First of all, let me review some basic notions of category theory.

Category This is a collection of objects $a, b, \ldots$ and of arrows $f : a \to b$ between these objects. To any object a corresponds an identity arrow $1_a : a \to a$, and any two arrows $f : a \to b$ and $g : b \to c$ are composable in a natural way, composition being associative. For example, the category *Set* has all sets as objects, and all functions as arrows.

Functor This is a map $F : A \to B$ between two categories such that to any object a of A corresponds an object $F(a)$ of B, and to any arrow $f : a \to b$ in A corresponds an arrow $F(f) : F(a) \to F(b)$ in B, in such a way that $F(1_a) = 1_{F(a)}$ and $F(f \circ g) = F(f) \circ F(g)$.

*Diagram** Formally, a diagram* in a category A is a functor $D : I \to A$ from an indexing graph[5] I called the *scheme* of the diagram* D. Intuitively, a diagram* in a category A is the image of a diagram* in the former sense, that is a graph composed of a subcollection of the objects and arrows of A, along with commutativity conditions on those arrows.

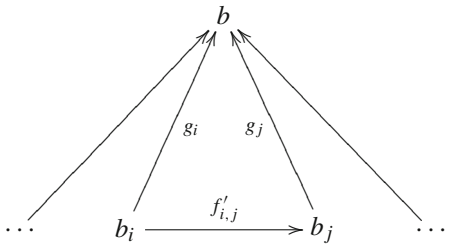
Cone A cone over a diagram* D is roughly a diagram* made up of a bundle of outgoing arrows starting from one object a (called the vertex) and going to all the objects of D and commuting with each arrow in D:



$$f_{i,j} \circ h_i = h_j.$$

[5] A graph is just a collection of objects and arrows between some of those objects. The notion of graph is a little more general than that of category, but the definition of a functor can be generalized accordingly, so that a functor has a graph as its domain or as its codomain.

Cocone A cocone is the same thing, but with ingoing arrows to the vertex instead of outgoing arrows from it:

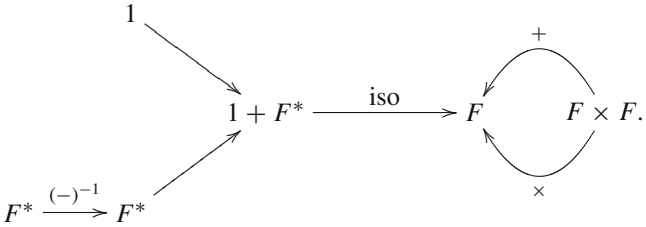


$$g_j \circ f'_{i,j} = g_i.$$

Limit cone A limit cone over a diagram* D is a cone $(f_i : a \to a_i)_{i \in I}$ over D that mediates any other cone $(g_i : b \to a_i)_{i \in I}$ over D in that there is a mediating arrow $h : b \to a$ such that $g_i = f_i \circ h$ for any $i \in I$. An analogous definition defines a *limit cocone*.

1.3 Categories of sketches

As I said, sketches were introduced as a natural way to bring out and to describe certain 'types' of structures. A sketch, broadly speaking, consists in a graph with prescribed diagrams*. An example is more telling than anything else, as for instance the sketch S_f of commutative field:



(this sketch is only an outline, because the arrow $F^* \xrightarrow{-1} F^*$ is underspecified, but I will give the detailed sketch of monoids later). The cocone $1 \longrightarrow 1 + F^* \longleftarrow F^*$ should be explicitly singled out as a distinguished one, to express the fact that F is the disjoint union of 0 and the set of invertibles F^*.

More precisely,[6] a *sketch* is a quadruple $S = \langle |S|, D, C, C' \rangle$, where $|S|$ is a graph, D a collection of diagrams* in $|S|$ (called *distinguished diagrams**), C a collection of cones (called *distinguished cones*) in $|S|$, and C' a collection of cocones (called *distinguished cocones*) in $|S|$.

A *realization* of a given sketch S is a functor from the underlying graph $|S|$ of S into some category A that turns every distinguished diagram* of D into a commuta-

[6] See Makkai and Paré (1989). See also Adámek and Rosický (1994).

tive[7] diagram* in A, every distinguished cone of C into a limit cone in A and every distinguished cocone of C' into a limit cocone in A. In the case where the *realization category* A is Set, a realization of a sketch is called a *model* of this sketch. A field is nothing but a model of the sketch of fields, a group, a model of the sketch of groups, and so forth.

A *morphism of sketches* $X : \langle |S_1|, D_1, C_1, C'_1\rangle \to \langle |S_2|, D_2, C_2, C'_2\rangle$ is simply a functor $X : |S_1| \to |S_2|$ such that $X \circ d_1 \in D_2$ for any $d_1 \in D_1$, $X \circ c_1 \in C_2$ for any $c_1 \in C_1$ and $X \circ c'_1 \in C'_2$ for any $c'_1 \in C'_1$.

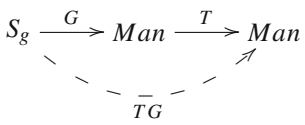
Thus a sketch such as S_f is really to be conceived of as a sketch in the litteral sense of the word, i.e., as a tracing pattern (in an ambient category which doesn't matter very much), that each of its realizations traces in some given category A (for example in Set). All the realizations of S in Set are considered as a whole, and form a full subcategory Set^S of $Set^{|S|}$. For instance, commutative fields are not so much considered as individual models (in the Tarskian sense) of the formal first order theory T_f (which is still the case when one says for example that each model has another one as an elementary extension), as they as introduced as making up collectively a category Set^{S_f} which becomes the proper object under study.

1.4 Sketches

There is a first score on which sketch theory has obvious advantages. Indeed, sketches are an interesting alternative way of presenting a theory. The main advantage of a sketch, as compared with a formal theory, is that it is finite and can be written down explicitly. (From a different point of view, it is less economical: for example, the linguistic presentation of group theory requires one sort of objects and three axioms, whereas the sketch of groups requires 4 objects, 16 morphisms and 19 commutativity conditions.) This 'syntactical' side of sketches has to be elaborated on, in order to understand how not only axioms but formal proofs are transposed into the 'setting' of sketches.

Another convenient aspect of sketches becomes apparent when we consider the sketch of groups. This sketch can be realized (projected, traced) either in the category of sets (and this gives a group in the classical sense of the word), or in the category of topological spaces (and this gives a topological group), or again in the category Man of differentiable manifolds (and this time this gives a Lie group). Prima facie, the gain of such a consideration is to handle a generalized object, which gathers the different cases of group, topological group and Lie group: these different structures correspond in fact to the realizations of the same *type* of structures into different categories. This is but one example speaking to the generality achieved by sketch theory. There is another advantage to underline. Let's consider a Lie group, that is a manifold endowed with an algebraic group structure. This group, say G, gives birth naturally to another Lie group, because the tangent bundle of G, TG, is not only a manifold, but also a group: the group structure is passed on from G to TG. It is a bit tedious to show this with a 'pen to paper' proof. But it becomes quite easy using sketches: just look at

[7] A diagram* is said to be *commutative* when any two paths in that diagram* with same source and target can be identified (as composite arrows).



as soon it has been established once and for all that T is a group structure preserving functor. Indeed, TG becomes $T \circ G$, hence a realization of S_g, and consequently a group. Furthermore, it is a realization of S_g in Man, and so a Lie group.

These beginnings of sketch theory in differential topology suggest a new kind of union of algebra and geometry, quite different from the idea of there being some kind of dictionary between the two. The two are not related by stating some form of translation between algebraic and geometrical concepts, but rather by delimiting different forms of superposition of algebraic and geometrical features. In pursuing this goal, sketches stand out as particularly convenient, insofar as it is very natural to add extra-structure on the basis of a tracing pattern. Such a versatility lies at the level of the realizations of a sketch, and on this score can be related to the 'semantical' side of sketches.

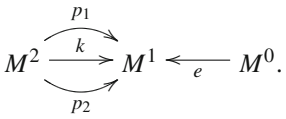
Hence two sides are attached to sketches, a syntactical one and a semantical one. Let's now consider them in turn, so as to understand the way they relate to each other.

2 Sketches, proofs and types

The nice thing about sketches is that a proof becomes itself a sketch that constitutes a specific enrichment of the original sketch of a theory—hence the link with evolving diagrams. Following Coppey (1992), let's consider the theory of monoids, and successive sketches of it.

2.1 A basic sketch, S_0

A starting point to represent the structure of monoids is the following sketch S_0:



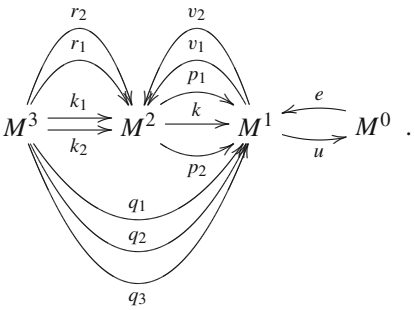
This sketch works as a first order multisorted language, where:

- sorts = objects of S_0;
- function symbols $f : X \to Y$ are arrows of S_0;
- $\forall x : X \quad f'(f(x)) = g(x)$ iff $f'f = g$ in S_0.

In set-theoretic terms, M^1 stands here for the underlying set of a monoid, M^0 for a singleton $\{*\}$, and e for the map such that $e(*)$ is some distinguished element of M^1.

2.2 The sketch of monoids, S_1

A monoid is a set M endowed (a) with an associative binary law $k(x, y)$ and (b) with a distinguished element e such that, for any $x \in M, k(x, e) = k(e, x) = x$. The corresponding sketch S_1 is obtained from S_0 by implementing the conditions (a) and (b):



Here, as above, the p_i's are the natural projections (or have to be realized as such), which is directly expressed by the choice of $M^1 \xleftarrow{p_1} M^2 \xrightarrow{p_2} M^1$ as a distinguished cone. It is the same thing for the q_i's. Now the transition from S_0 to S_1 consists in the addition of arrows (defined through certain equalities) and the addition of axioms (being equalities between the defined arrows).
Added arrows:

(r_1) $p_1r_1 = q_1$, $p_2r_2 = q_2$ (that is, $r_1 : (x, y, z) \mapsto (x, y)$);
(r_2) $p_1r_1 = q_2$, $p_2r_2 = q_3$ (that is, $r_2 : (x, y, z) \mapsto (y, z)$);
(k_1) $p_1k_1 = kr_1$, $p_2k_1 = q_3$ (that is, $k_1 : (x, y, z) \mapsto (xy, z)$);
(k_2) $p_1k_2 = q_1$, $p_2k_2 = kr_2$ (that is, $k_2 : (x, y, z) \mapsto (x, yz)$);
(v_1) $p_1v_1 = 1_{M^1}$, $p_2v_1 = eu$ (that is, $v_1 : x \mapsto (x, e)$);
(v_2) $p_1v_2 = eu$, $p_2v_2 = 1_{M^1}$ (that is, $v_2 : x \mapsto (e, x)$);
(u) u is the only possible map from M^1 to M^0.

Added axioms:
$kv_1 = kv_2 = 1_{M^1}$;
$kk_1 = kk_2$.

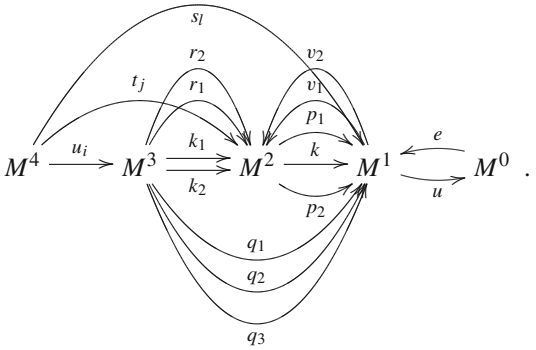
The equalities satisfied by the r_i's and the k_i's, as well as the added axioms, amount to commutativity conditions which are expressed by putting forward some distinguished diagrams*. For example, stating that $kk_1 = kk_2$ amounts to picking $M^3 \underset{k_2}{\overset{k_1}{\rightrightarrows}} M^2 \xrightarrow{k} M^1$ as a distinguished diagram*.

2.3 A richer sketch of monoids, S_1'

Let's pursue Coppey's example. Given a monoid $(M, \cdot)$, the 3-associativity of. entails its 4-associativity, which may be expressed as:

$$(x(yz))t = x(y(zt)) \tag{1}$$

Expressing (1) is not possible using S_1 alone. It requires a new sketch S_1' which supplements S_1 with a new vertex M^4, new arrows $u_i : M^4 \to M^3$, $t_j : M^4 \to M^2$, $s_l : M^4 \to M^1$, and new equalities about these arrows (to the effect that, for example, s_1 is the map $(x, y, z, t) \mapsto x$—the consequences with respect to the other arrows are similar). Once again, the equalities are in fact laid down by mentioning distinguished diagrams*. The underlying graph $|S_1'|$ of the resulting sketch S_1' is:



Now it should be stressed that although both S_1 and S_1' have the same models, neither S_1 nor S_1' allows us to establish (1). Establishing (1) requires the addition of some new arrows $k_i' : M^4 \to M^3 (i = 1, 2, 3)$ defined by equalities in such a way that k_1', for example, represents the map $(x, y, z, t) \mapsto (xy, z, t)$. Then (1) is $k(k_1 k_2') = k(k_2 k_3')$, and may be proved *provided* the ambient category has finite products (see Appendix A). Consequently, 4-associativity results from 3-associativity through a progression starting at S_1, passing through S_1', and arriving at a new sketch S_1'', which is the sketch with finite products generated by S_1 (i.e., the underlying graph $|S_1''|$ of S_1'' is the free category with finite products generated by $|S_1'|$). From all this, it follows that the proof of (1) holding in any monoid amounts to the construction of the sequence $S_1 \longrightarrow S_1' \longrightarrow S_1''$.

Two conclusions: first of all, a proof lies in the completion of a sketch. Secondly, the models (realizations in *Set*) of S_1, S_1' and S_1'' are the same, in spite of the fact that nothing in S_1 or S_1' allows us to deduce (1). This means that the category *Set* automatically adds to any model of S_1 what is needed to get a model of S_1''. This is due to a property of *Set* (having finite products) that the sketch-theoretic presentation helps to make explicit. I will get back to this in a moment. To tackle the issue of the semantical aspect of sketches, I now consider the example of monoids in another way, following another example given by Coppey (1992).

2.4 An alternative sketch of monoids

As already said, a monoid is a set M endowed with an associative binary law $k(x, y)$ and with a distinguished element e such that: $\forall x \in M\ k(x, e) = k(e, x) = x$. The

corresponding sketch is S_1. But a monoid can equivalently been defined as a set M endowed with an associative binary law $k(x, y)$ such that:

$$\exists e \in M\ \forall x \in M\ k(x, e) = k(e, x) = x \tag{2}$$

It turns out that (2) is sketchable, by a sketch S_2 (see Appendix B). The models of S_1 and S_2 are of course the same. But now let's look at their realizations in K-Vect (the category of K-vector spaces). Let F_1 be a realization of S_1 in K-Vect. One has $F_1(S_1):\ M \times M \xrightarrow{k} M \xleftarrow{e} M^0$ with $M^0 = (0)$, so $e = 0$ and k is K-linear. Hence: $k((x, y)) = k((x, 0) + (0, y)) = k(x, e) + k(e, y) = x + y$. This means that M has only one possible monoid structure.

On the other hand, let's consider a model of S_2 in K-Vect. By associativity, one gets: $k(k(x, e), e) = k(x, k(e, e)) = k(x, e)$, so $\lambda : M \to M,\ x \mapsto k(x, e)$ is idempotent and K-linear: it is a projector on M. The same holds of $\mu : M \to M,\ x \mapsto k(e, x)$. Besides: $k(k(x, y), z) = \lambda(\lambda(x) + \mu(y)) + \mu(z) = \lambda(x) + (\lambda\mu)(y) + \mu(z)$ and $k(x, k(y, z)) = \lambda(x) + \mu(\lambda(y) + \mu(z)) = \lambda(x) + (\mu\lambda)(y) + \mu(z)$, therefore $\lambda\mu = \mu\lambda$. Conversely, for any couple (λ, μ) of commuting projectors on M, $k(x, y) := \lambda(x) + \mu(y)$ is a binary law on M turning M into a model of S_2. So there are realizations of S_2 in K-Vect which are not realizations of S_1. The reason why is that there is no canonical choice for the complement of a linear subspace. What is left open when switching from a realization of S_1 to a realization of S_2, is precisely such a choice for the complement of Φ (see Appendix B for details).

From all this, it follows that S_1 and S_2 have the same realizations in *Set*, but not in K-Vect. So there is a difference between *Set* and K-Vect, which needs some explanation.

2.5 Sketch of sketches

As we have just seen, *Set* and K-Vect do not have the same properties. Let's try to express this difference in the framework of sketches. Any category may be seen as a sketch (with no distinguished cones and cocones), so more generally let's try to represent diagrammatically* properties which a sketch might or might not have.

The first thing that needs to be said is that, under some cardinality constraints, sketches may themselves be sketched; let's call σ the sketch of sketches (see Appendix C). Now, any sketch is the same thing as a model of the sketch of sketches, in the same way as any group is a model of the sketch of groups. So, to every sketch S corresponds a unique model $F_S : \sigma \to Set$ of σ. The sketch S and the corresponding model F_S are really the same thing, viewed either as a structure or as a functor.

Now, let's take an example: let's say that a sketch is a *limit sketch* if its distinguished cones and cocones are all limit ones. Along the same lines as for σ, it is possible to draw the sketch of all limit sketches, let's call it λ. Since any limit sketch is a sketch, there is an obvious sketch inclusion $p_l : \sigma \to \lambda$ ('l' as '*l*imit sketch'): the sketch of limit sketches represents indeed an enrichment of the sketch of sketches. Indeed, take a limit sketch S, turned into a model $F_S : \lambda \to Set$ of λ. Then, by composition with

p_l, one gets a functor $F_S \circ p_l : \sigma \to Set$, which is nothing else but a model of σ, that is, a sketch.

For any limit sketch $T : \lambda \to Set$ (we identify here T and F_T), $T \circ p_l$ is the same thing as T, but viewed as an ordinary sketch. And it is clear that any sketch S is a limit sketch if and only if F_S factorizes through p_l. Here *being a limit sketch* is an example of a property that a sketch may or not have, and it is this property that p_l stands for. It can be shown (see Appendix D) that, for any sketch S, there is a smallest limit sketch extending S: it is called 'the *type*[8] of S w.r.t. the property p_l,' or 'the p_l-type of S,' and it has the same models as S.

The construction of the type of a sketch (for a given property of sketches) is very general. Let's go back to the original case of S_2 as another illustration of the idea of type. Indeed, Set has the property of the uniqueness of the complement, whereas K-Vect does not. Such a property may be seen as a property uc of Set as a sketch. If again we further identify that property with a certain sketch morphism $p_{uc} : \sigma \to \sigma_{uc}$, we can get the following more accurate formulation:

$$T_{uc}(K\text{-Vect}) = Set.$$

Thus Set is the completion of K-Vect w.r.t. the property p_{uc}, as $T_l(S)$ was the completion of S w.r.t. the property p_l. This also explains why S_1 and S_2 have the same realizations in Set.

So, to sum up, it might be said that a type arises as the completion of a realization category, viewed itself as a sketch, in order to ensure a certain mathematical fact (such as the uniqueness of the complement).

3 Types of sketches

Let's now pause to take stock of the two dimensions (syntactical and semantical) along which a sketch can be developed.

3.1 The sketch of monoids

To this end, let's return to our example of monoids. We interpret the sketch S_1 of monoids by looking to models of this sketch in Set. The fact that we are dealing with limit cones and working within Set *adds* arrows to the original schema of S_1. As a matter of fact, to any cone in S_1 there will correspond in Set a limit cone c, with all the cones sharing the same basis as c, and all the mediating arrows ϕ induced between such cones and c. This addition of limits and colimits of specified forms forces 4-associativity (1) in any model of S_1 on top of the equalities explicitly true in S_1. The question is then: for which property p are there enough arrows in $T_p(S_1)$ so that (1) is true in any realization of $T_p(S_1)$?

[8] The term 'type' is Ehresmann's. I wish to thank René Guitart for his very helpful explanation concerning this notion of type in the context of sketch theory.

In other words, if a property p of Set is what is required in order to complete the proof of some statement (1) holding in all the models of a given sketch S, then by definition $T_p(S)$ will enjoy that property p and will be minimal for that property. To be specific, the property p corresponding to (1) is the property fp of *having finite products*. The sketch Set has the property fp, and that is why all the models of S_1 are already all the models of S_1''.

But then it is natural to look at Set not as a category where a sketch like S_1 is interpreted, but as a sketch on its own. Such a twist only requires us to consider any model $S \to Set$ of S, not any more as a model, but as a sketch morphism. And then it becomes natural to replace Set with the more accurate fp-type of S_1, to the extent that $T_{fp}(S_1)$ only retains what is strictly necessary to the proof of (1). So $S_1'' = T_{fp}(S_1)$ is the semantical completion of S_1, but at the same time the explicit schema of the proof of (1), the explicit representation of (1) as being deducible from S_1, in the sense that the validity of (1) becomes an obvious part of S_1''. The sketch S_1'' shows exactly what has to be added to S_1 in order to actually derive (1) from the theory presented by S_1. Hence, we can write:

$$S_1'' = T_{fp}(S_1) = T_{(1)}(S_1).$$

Compare this to the notion of formal theory as it is usually understood in model theory: if you add to a formal theory one of its deductive consequences, it does not change anything, because a theory is generally taken to be a deductively closed set of sentences—otherwise it becomes sensitive to the particular axiomatization which is chosen. So there is no way to pinpoint the step at which some theorem is obtained (unless one wants to focus on the sentences of some given complexity that can be deduced from the axioms, but this still is not relative to a single sentence). On the contrary, the transition from S towards $T_\theta(S)$ is each time adjusted to the proof of a certain theorem θ. One stops at $T_\theta(S)$, without being obliged to get to any wider type of S, inasmuch as one is interested only in establishing θ specifically. In this sense, a type constitutes a local feature, because it adds only what is crucial to the purpose of a peculiar proof, that of θ. To construct this proof amounts to drawing a kind of resolution, a finite path $S \to \cdots \to T_\theta(S)$ between S and $T_\theta(S)$.

One must acknowledge that the category-theoretic presentation of theories dates back to Lawvere's dissertation[9] and has also been developed, from a more type-theoretic point of view, by Lambek.[10] Lawvere's seminal idea has opened up a whole

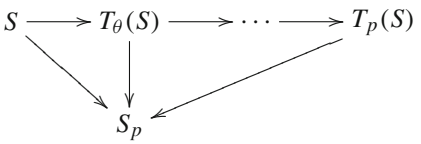
[9] In the second chapter of Lawvere (1963), Lawvere reformulates the notion of algebraic theory through a category $\mathcal{S}_0$ with finite products, whose objects are the natural numbers, and in which the product of n objects is their arithmetical sum. Any algebraic theory in a category $\mathbb{A}$ with finite products becomes a finite-product preserving functor $A : \mathcal{S}_0 \to \mathbb{A}$. The idea is to represent every n-ary operation symbol of an algebraic theory (in the usual sense) as a morphism $n \to 1$ in $\mathcal{S}_0$. Lawvere, in particular, is interested in finite presentations of algebraic theories, and such a finite presentation can be seen as a sketch whose underlying graph is finite (even though it might generate an infinite category). Hence the advantage of dealing with graphs rather than categories as domains of sketches: it enables us, in somes cases, to manipulate diagrams* on a finite graph instead of infinite diagrams* on an infinite category. Many thanks to an anonymous referee for pointing this out to me.

[10] One of the purposes of Lambek and Scott (1999) is to associate an internal language with each cartesian closed category with weak natural numbers object, and to look at this category as a typed λ-calculus. More

avenue for a new conception of syntax and semantics of mathematical theories. Sketch theory adds to it a more structured presentation of theories, using gradually enrichable, adjustable diagrams*, instead of a fixed underlying category.

3.2 The two sides of a type

Going beyond the particular case of monoids, let a *p-model* of a sketch S be the realization of S in any category enjoying the property p and suppose that we want to prove that some theorem θ is true in every p-model of S. What we have to do then is to construct a sketch $T_\theta(S)$ inserted between S and $T_p(S)$, where it is *obviously* true that θ is true in any p-model of $T_\theta(S)$ (think of S_1'' in the case of (1)). Then one gets:



,

where S_p is of type p (that is, has, as a category, the property p). Since there are two morphisms $S \to T_\theta(S)$ and $T_\theta(S) \to T_p(S)$, $S_p^{T_p(S)} \subseteq S_p^{T_\theta(S)} \subseteq S_p^S$. On the other hand, $S_p^S = S_p^{T_p(S)}$ (because S_p is supposed to be of type p). Therefore:

$$S_p^S = S_p^{T_\theta(S)},$$

which means that θ is true in any p-model of S, and that is what was to be proved. As Peter Freyd puts it, the task of the mathematician is to make trivially trivial what is trivial. Here $T_\theta(S)$ is what discharges this task.

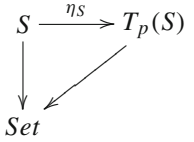
So the demonstration that some theorem θ is valid in all the p-models of S amounts to inserting an arrow $\phi_\theta : S \to T_\theta(S)$ into the canonical arrow $S \xrightarrow{\eta_S} T_p(S)$ and to constructing a finite path

$$S \to \cdots \to T_\theta(S) \to \cdots \to T_p(S)$$

which may be conceived as an analysis of η_S with respect to the particular theorem θ. Each intermediate arrow corresponding to a step in the proof, in such a way that it is obvious in the end that any model of $T_\theta(S)$ satisfies θ. Once again, an example of that configuration is supplied by the sequence $S_1 \to S_1' \to S_1''$ about (1) (over the realization category *Set*).

Footnote 10 continued
precisely, it is to establish a functorial equivalence between the category of cartesian closed categories with weak natural numbers object, and the category of typed λ-calculi.

Hence a type such as $T_p(S)$ summarizes a proof-theoretic construction starting from a given sketch S. The diagram* corresponding to this *syntactical* status of $T_p(S)$ is:



,

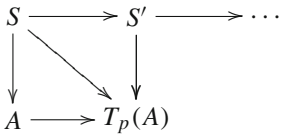
where *Set* is supposed to be of type p (otherwise *Set* has to be replaced by $T_p(Set)$).

The upshot of all this is that a proof may appear as a *specific* extension of a sketch: the addition of the vertices and arrows that make possible the proof of a given theorem. This holds more generally about many mathematical diagrams, the ones that I have singled out as 'evolving diagrams.' As soon as one works with an evolving diagram, one enriches it (think again of the constructions added to the diagram representing a triangle, in a classical geometrical proof), which means that the description that one starts with is gradually changed. In that case, a diagram does not merely suggest a picture of an object, but allows for the construction of a new proof. And for this it has to be open to gradual enrichment, with all the steps of the enrichment recorded within the original diagram: as we said at the beginning, it is a way to represent a proof at the same level as that of the structure the proof relates to.

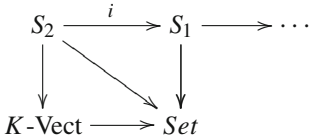
But this is exactly what a diagram* qua sketch is intended for: a sketch is a tracing pattern, upon which you can calk some extra-structure before tracing it, that is, realizing it. Actually, two levels stand out in sketch theory: each mathematical theory is represented through the complex diagram* of its sketch, and then a diagram* can be drawn whose vertices are themselves sketches. In fact, these two levels connect, since a morphism between two sketches means basically that the first one can be transformed or enriched into the second. Thus sketches are a powerful way to turn theories (or the structures that correspond to it) into diagrams*, and then to pinpoint a local proof-theoretic fact through the factorization of an arrow, that is, through a device which is located at the same level as that of the theory (or the structure) that this proof-theoretical fact is about—and, from the latter point of view, sketches are a very nice example of evolving diagrams. So, in spite of its possible abstractness, sketch theory is a general framework to build proofs into diagrams* which, as a result, are also diagrams in the sense of evolving diagrams. Sketch theory shows in particular that mathematical diagrams cannot be confined to a mere heuristic role; they often constitute the very locus of a proof. Furthermore, it helps us to see that making trivially trivial what is trivial is not (at least, not always) the mere presentation of a picture, contrary to Brown's suggestion.

Now, as stressed above, in addition to being a sketch morphism, $\phi_\theta : S \to T_\theta(S)$ may be seen as a realization of S into $T_\theta(S)$. Then $T_\theta(S)$ becomes a category in which S can be adequately realized from the point of view of θ, because $T_\theta(S)$ is complete with respect to the relevant limits. In that respect, $T_\theta(S)$ turns out to be the good semantics as soon as one is interested in the realizations of S in which θ is true. As a matter of fact, $T_\theta(S)$, in a minimal way, gets the proof of θ to work, since it is universal as a solution to the problem of finding a realization category A such that

any realization of S in A satisfies θ.[11] This can be read as implying that Set is not necessarily the best semantics: everything depends on the theorem θ which is as stake. In the cases of some theorems θ, indeed, Set turns out to be richer than $T_\theta(S)$, hence to be richer than necessary: the property that makes θ true is then put together with other properties of Set. Or Set may also not be rich enough. With respect to some other theorems, though, Set will turn out to be the most accurate realization category possible. In this way sketch theory enables us to carry out a rational category-theoretic arbitration between the category of sets and other realization categories, which is more constructive than simply setting up category theory and set theory as two competing frameworks for mathematics. The picture corresponding to that *semantical* function of a type $T_p(S)$ is:



where p is the minimal property confusing S and S', that is, making p-models of S and p-models of S' equal. For example, to get back to (2):

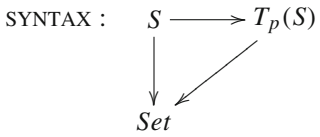


The inclusion map i explains the fact that any realization F of S_1 is a realization $F \circ i$ of S_2, while there are realizations of S_2 in K-Vect that are not realizations of S_1. For that reason, one can write:

$$Set = T_{p_{(2)}}(S_2),$$

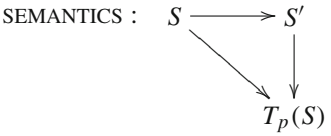
where $p_{(2)}$ is the minimal property confusing S_1 and S_2.

To sum up, the morphism $\eta_S : S \to T_p(S)$ may be drawn vertically (that's the semantical aspect of the type) or horizontally (that's the syntactical aspect of the type). Hence the following flipping diagram*:



[11] This means that, given any such realization category A and any realization $F : S \to A$ of S in A, there is a sketch morphism $i_\theta(F) : T_\theta(S) \to A$ such that $F = i_\theta(F) \circ \phi_\theta$.

(where p is a property needed to prove some theorem θ in any model of S);

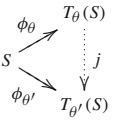
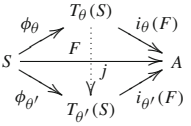


(where S' is an enrichment of S which p confuses with S).

You only have to turn the arrow around to go from one type to the other. This is the sense in which a sketch exhibits the status of diagrams* as being halfway between syntax (formal proofs) and semantics (structures and types of structures). More than that, a sketch-theoretic diagram* (that is, a diagram* whose vertices are themselves sketches) may be conceived of as a diagrammatic representation of the connection between syntax and semantics, to the extent that any sketch morphism can be viewed either as a (semantical) realization, or as a syntactical (proof-theoretic) enrichment.[12]

3.3 Conclusion

To summarize, it appears, first, that sketch theory provides a diagrammatic* theory of diagrams*, since *any* diagram* may be seen as a sketch. Second, sketch theory helps to reconcile the set theoretic and the category-theoretic viewpoints, because after all *Set* remains the intended realization category, and because (which is better) the predominance of *Set* is each time explained in terms of completeness with respect to the appropriate type: the fact that $T_p(Set) = Set$, where p is a relevant property for the proof in question, is what justifies to settle in *Set*. In that respect, sketch theory allows us to understand, depending on the context at stake, why the category of sets is (or is not) called forward as a suitable mathematical universe, and this is something quite different from a mere alternative proposal to replace set theory. Third, sketch theory suggests a more flexible understanding of the prima facie opposition between formal proofs and diagrammatic reasoning: sketches offer a kind of reconciliation between these two ways of expressing mathematics. This is tied up with the fact that sketch theory allows for a kind of flipping between syntax and semantics. So in the end the

[12] Consider from this perspective a sketchable formal theory S (identified with its sketch), and two sentences θ and θ' of the language of S. The fact that $S \vdash \theta' \rightarrow \theta$ is rendered by a sketch morphism $j : T_\theta(S) \rightarrow T_{\theta'}(S)$ such that  commutes. Then, semantically, any realization F of S in a category A satisfying θ will satisfy θ'. Indeed, let's consider:  . If F satisfies θ', i.e., if $F = i_{\theta'}(F) \circ \phi_{\theta'}$, then $F = i_{\theta'}(F) \circ j \circ \phi_\theta$, and so F factors through ϕ_θ, which means that F satisfies θ. As one can see, syntactical and semantical considerations are expressed with the same diagram.

very three slogans that I identified earlier are, if not disposed of, at least properly qualified.

Still, is the sketch of monoids really handy for proving something about monoids? How do we resist the temptation to go back to the first order theory of monoids as the setting where we investigate what is true of monoids? I think that these questions require us to consider the yet more critical status of diagrams. A mathematical diagram supports a proof procedure, but generally is also aimed at giving to the mathematician some kind of reassurance about what he is doing. A diagram helps us to do mathematics and at the same time to represent what we are doing when completing such or such mathematical proof or activity. Of course, diagrams are various. A diagram such as a figure in Euclidean geometry falls mainly within the first aspect: the fulfilment of a mathematical action, the embodiment of a mathematical practice. On the contrary, a diagram such as a sketch (in the technical sense) corresponds mainly to the second aspect: the recording of what makes a proof possible in a category of realizations. But any diagram certainly fulfills some mixture of both contrasting functions. Sketch theory only gives a clearer expression of the second one.

The reassessment of diagrams* (and the strong qualification of the main slogans about them) that I have just put forward, is tightly tied up with the consideration of sketches as evolving diagrams in the general sense brought up at the beginning. The common feature of the mathematical diagrams of the kind I have confined myself to in this paper consists in combining the representation of an object and the representation of a construction based on the representation of that object. Sketches are diagrams in this sense: a theory, represented through its sketch, is in that case the original object, and the p-type of this sketch is the construction based on the representation of that object. Hence all diagrams*, as sketches, are evolving diagrams, even though of course it is not true that any evolving diagram is a diagram*. Sketch theory is just the way to bring that feature to light: it is the theory of diagrams* as evolving diagrams, where enrichment is represented as a sketch morphism. That is about the first slogan. Next, if types of sketches are a way to overcome the sterile confrontation between set theory and category theory as a general framework, this is based on the fact that the type of a sketch is nothing but the completion, each time in a specific perspective, of that sketch viewed as an evolving diagram. Finally, sketches elude the opposition drawn between syntax and semantics, precisely because evolving diagrams are a way to incorporate a formal proof into the semantical structure that it pertains to. So, sketches, as diagrams*, are a nice example that illuminates the characteristic feature of evolving diagrams, and, as evolving diagrams, they show how diagrams* in general should be re-evaluated positively—which concludes the twofold perspective of this paper.

One last point: one might worry that sketch-theoretic diagrams* (diagrams* with sketches as vertices) represent some kind of higher level diagram*, and that this could be the beginning of a regress. But it is nothing of the sort, because, as we have seen, the construction of a diagram* whose vertices are themselves diagrams* is nothing but the representation of the enrichment of an initial diagram*. And since the enrichment of a diagram representing some mathematical object is really part of this diagram as a fuller representation of the same object, one never departs from the single level of all diagrams*.

Appendices

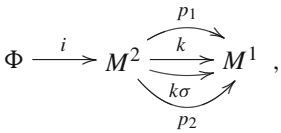
A Proof of (1)

$k(k_1k'_2) = k(k_2k'_3)$ (1) results from the following:

- $k(k_1k'_2) = (kk_1)k'_2$ and $k(k_2k'_2) = (kk_2)k'_2$ hold in any category (because two consecutive arrows are always composable, and because composition is always associative), and $kk_1 = kk_2$ is true in S_1, so $k(k_1k'_2) = k(k_2k'_2)$.
- $k_2k'_2 = k_2k'_3$ (that is $(x, (yz)t) = (x, y(zt))$) holds *provided* the ambient category has finite products[13]:
  - $p_1k_2k'_2 = q_1k'_2 = s_1 = q_1k'_3 = p_1k_2k'_3$;
  - $p_2k_2k'_2 = kr_2k'_2 = kk_1u_1 = kr_2k'_3 = p_2k_2k'_3$.
- Thus $k(k_2k'_2) = k(k_2k'_3)$, and so finally $k(k_1k'_2) = k(k_2k'_2) = k(k_2k'_3)$.

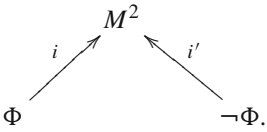
B Sketch of (2)

To sketch the statement (2), one goes back to S_1, one erases M^0 and e, and one introduces a cone with a vertex Φ, namely the subobject of M^2 corresponding to the condition '$k(x, e) = k(e, x) = x$':



where σ is a new arrow defined by $p_1\sigma = p_2$, $p_2\sigma = p_1$ and bound by the new axiom $k\sigma i = ki = p_1i$. This results in a new sketch S_Φ, and any model of S_1 may be extended in a unique way to a model of S_Φ (turning i into an equalizer of the arrows p_1, k and $k\sigma$).

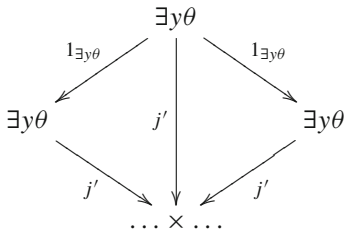
The construction of $\neg\Phi$ from Φ consists in the complement of Φ as subobject of M^2, that is in the following distinguished cocone:



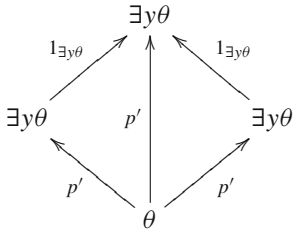
Thus one gets a new sketch S'_Φ, and any model of S_Φ may be extended in a unique way to a model of S'_Φ (turning M^2 into a partition).

The construction of $\exists y\theta$ from an object θ and an arrow $j : \theta \to \cdots \times Y \times \cdots$ consists in representing $\exists y\theta$ as the image of θ by the projection p forgetting y, that is in introducing the following distinguished cone:

[13] See Coppey (1992, p. 17).



and the following cocone:

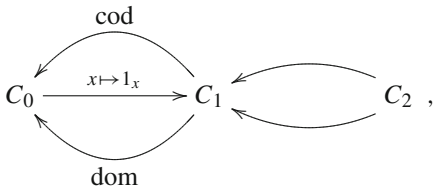


while stating: $j'p' = pj$. Any model of these two cones turns j' into a monomorphism and p' into an epimorphism, and so interprets $\exists y\theta$ as the image of θ by the projection p forgetting y.

The sketching of negation and existential quantification allows us finally to sketch the axiom $\exists e\ \forall x\ k(x,e) = k(e,x) = x$, which is the same as $\exists e \neg \exists x \neg (k(x,e) = k(e,x) = x)$. Thus one gets a sketch $S''_\Phi = S_2$ of (2).

C Sketch of sketches

One may sketch what a category is, through a sketch S_{cat} as follows:



where C_0 represents the collection of objects, C_1 the collection of arrows and $C_2 = \{(f,g) \in C_1 \times C_1 : \mathrm{cod}(f) = \mathrm{dom}(g)\}$. Now for the cardinality constraints: given a cardinal α, a diagram* $X : I \to S$ is said to be *of size* $< \alpha$ if I has less than α arrows. A sketch $S = \langle |S|, (d_i)_{i\in I}, (c_j)_{j\in J}, (c'_k)_{k\in K} \rangle$ is said to be *of size* $< \alpha$ if each d_i, c_j and c'_k is of size $< \alpha$ and if I, J and K are of cardinality $< \alpha$. In this way, α-small categories can be sketched.

In the same way, α-small cones and α-small cocones can be sketched. Finally we arrive at a sketch, say σ_α, which sketches α-small sketches themselves. On pain of falling under a variant of Russell's paradox, σ_α cannot be itself an α-small sketch. In

the paper, for the sake of simplicity, I have dropped the subscript 'α', and spoken of the sketch of sketches and of the sketch of limit sketches while referring in fact, respectively, to the sketch of α-small sketches and to the sketch of α-small limit sketches.

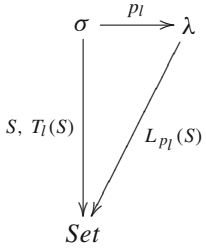
As already said in the paper, since by definition an α-small sketch is the same thing as a model of the sketch of α-small sketches, any α-small sketch S corresponds to a unique model $F_S : \sigma_\alpha \to Set$. The functor F_S itself lives in a convenient big category BIG, and the correspondence $S \mapsto F_S$ is functorial. Indeed, all α-small sketches S and morphisms thereof make up a category Sk_α, and any sketch morphism $X : S \to S'$ between two α-small sketches S and S' in that category gives rise to a corresponding morphism $F_S \to F_{S'}$ in BIG.

D Type of a sketch

Recall that $p_l : \sigma \to \lambda$ is the sketch injection from the sketch of (α-small) sketches to the sketch of (α-small) limit sketches. It is clear that the map $f \mapsto f \circ p_l$ defines a functor Set^{p_l} which is the forgetful functor $Set^\lambda \hookrightarrow Set^\sigma$ downgrading any limit sketch (that is, any model of λ) into an ordinary sketch (that is, a model of σ). The main fact[14] concerning Set^{p_l} is that it has a left adjoint $L_{p_l} : Set^\sigma \to Set^\lambda$.

A functor $F : A \to B$ is said to be *left adjoint* to $G : B \to A$ if, for any objects a in A and b in B, there is a bijection $\eta_{a,b}$ between the arrows $F(a) \to b$ in B and the arrows $a \to G(b)$ in A, so that the family $\langle \eta_{a,b} \rangle$ is natural in a and b. The latter condition means that, for any three arrows $\alpha : a \to a'$ in A, $f : F(a) \to b$ and $f' : F(a') \to b$ in B, $f = f' \circ F(\alpha)$ implies $\eta_{a,b}(f) = \eta_{a',b}(f') \circ \alpha$, and that an analogous condition holds about b. In particular, for $b = F(a)$ and the identity arrow $1_{F(a)}$ in B, $\eta_{a,F(a)}(1_{F(a)})$ is an arrow $\eta_a : a \to (GF)(a)$ in A. The family $\langle \eta_a \rangle$ inherits natural commutativity conditions from the former family, and is called the unit of the adjunction. In the case of Set^{p_l} and L_{p_l}, the family of bijections is $\langle \eta_{S,S'} \rangle$, where each $\eta_{S,S'}$ is a bijection between the collection of all arrows $L_{p_l}(S) \to S'$ in the category of limit sketches, and the collection of all arrows $S \to Set^{p_l}(S')$ in the category of sketches. Accordingly, the unit of the adjunction is the family $\langle \eta_S := \eta_{S,L_{p_l}(S)}(1_{L_{p_l}(S)}) : S \to Set^{p_l}(L_{p_l}(S)) \rangle$ of arrows in Set^σ. (Note that we keep writing 'S' whereas we should write 'F_S' for the model of σ associated to a sketch S.)

Let's write $T_l(S) := Set^{p_l}(L_{p_l}(S)) = L_{p_l}(S) \circ p_l : \sigma \to Set$. This defines a functor $T_l = Set^{p_l} \circ L_{p_l} : Set^\sigma \to Set^\sigma$:



$$L_{p_l} : Set^\sigma \to Set^\lambda.$$

[14] This result is a variant of the adjoint functor theorem. A first version of this result is due to Kennison (1968) and Foltz (1969).

The unit of the adjunction between L_{p_l} and Set^{p_l}, $\langle \eta_S : S \to T_l(S) \rangle$ $(S \in Set^{\sigma})$, is such that, for every (α-small) sketch S and every sketch morphism $f : S \to T$ with T being an (α-small) limit sketch, there is a unique limit sketch morphism g such that $f = Set^{p_l}(g) \circ \eta_S$. (As a matter of fact, the naturality of $\langle \eta_S \rangle$ guarantees that $\eta_T \circ f = T_l(f) \circ \eta_S$; since $T = Set^{p_l}(T)$, $g = L_{p_l}(f)$ will do.) Owing to that universality condition, $T_l(S)$ is the limit sketch generated by S. As mentioned in the paper, it is called 'the *type* of S w.r.t. the property p_l.'

Besides, since Set_α (the category of sets of cardinality $< \alpha$) happens to be an α-small sketch, on can take $T = Set_\alpha = Set^{p_l}(Set_\alpha)$. Then the adjunction between L_{p_l} and Set^{p_l} is a bijection between the maps from S to Set_α and the maps from $T_l(S)$ to Set_α in the category Sk_α, and this holds for any α. Thus S and $T_l(S)$ have the same category of models.

References

Adámek, J., & Rosický, J. (1994). *Locally presentable and accessible categories*, volume 189 of *Lecture notes*. Cambridge: Cambridge University Press.

Brown, J. R. (2005). Naturalism, pictures, and platonic intuitions. In P. Mancosu, K. F. Jørgensen, & S. A. Pedersen (Eds.), *Visualization, explanation and reasoning styles in mathematics* (Chap. 3, pp. 57–74). Berlin: Springer.

Coppey, L. (1992). Esquisses et types. *Diagrammes, 27*, LC1–LC33.

Ehresmann, C. (1968). Esquisses et types de structures algébriques. *Bulletin of the Polytechnic Institute of Iasi, XIV*, 1–14.

Foltz, F. (1969). Sur la catégorie des foncteurs dominés. *Cahiers de topologie et géométrie différentielle catégoriques, XI*(2), 101–130.

Kennison, J. F. (1968). On limit-preserving functors. *Illinois Journal of Mathematics, 12*, 616–619.

Lambek, J., & Scott, P. (1999). *Introduction to higher order categorical logic*, volume 7 of *Cambridge studies in advanced mathematics*. Cambridge: Cambridge University Press.

Lawvere, F. W. (1963). *Functorial semantics of algebraic theories and some algebraic problems in the context of functorial semantics of algebraic theories*. PhD thesis, Columbia University.

Makkai, M., & Paré, R. (1989). *Accessible categories: The foundations of categorical model theory*, volume 104 of *Contemporary mathematics*. Providence: American Mathematical Society.

Osborne, M. S. (2000). *Homological algebra*. New York: Springer.