Math. Struct. in Comp. Science (2000), *vol.* 10, *pp.* 205–231. Printed in the United Kingdom
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Unique decomposition categories, Geometry of Interaction and combinatory logic

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Received 5 March 1999; revised 10 September 1999

Dedicated to Jim Lambek on the occasion of his 75th birthday

In another paper (Abramsky *et al.* 1999), we have developed Abramsky's analysis of Girard's Geometry of Interaction programme in detail. In this paper, our goal is to study the dataflow based computational aspects of that analysis. We introduce unique decomposition categories that provide a suitable categorical framework for such computational analysis. The current study also serves to establish connections with the work on proof nets and paths by Girard and Danos and Regnier in this categorical setting. The latter goal is partially achieved here by the presentation of categorical models for dynamic algebras.

1. Introduction

Girard initiated his 'Geometry of Interaction' programme in Girard (1989) and continued his investigation in a series of further papers (Girard 1989a; Girard 1988; Girard 1995). Geometry of Interaction (GoI) provides a mathematical analysis of the cut elimination process. From a computational point of view this is the analysis of β-reduction and has proved quite fruitful in the analysis of optimal reduction strategies for the λ-calculus. In Girard's terms, GoI provides a semantics free from the twin drawbacks of *reductionism* and *subjectivism*. In his approach, Girard uses operator algebras (specifically, a C^*-algebra Λ^*) to interpret multiplicatives, quantifiers and exponentials. The formula $EX(u, \sigma) = (1-\sigma^2)u(1-\sigma u)^{-1}(1-\sigma^2)$ gives the interpretation of the elimination of cuts (represented by σ) in a proof represented by u. Termination of the normalization/cut elimination process is interpreted as the nilpotency of σu, and the part $u(1-\sigma u)^{-1}$ is the candidate for the execution.

In subsequent work, Danos and Regnier (Danos 1990; Danos and Regnier 1993; Danos and Regnier 1994; Regnier 1992) have applied the execution formula to proof nets and λ-terms, and presented the ideas in the GoI programme in terms of properties of paths on proof nets and λ-terms. Danos and Regnier's work has thus helped to illustrate cut elimination as a process of information flow in proof nets.

https://doi.org/10.1017/S0960129599003035 Published online by Cambridge University Press

Abramsky and Jagadeesan gave an analysis of the geometry of interaction programme in terms of Kahn-style dataflow (Abramsky and Jagadeesan 1994). Subsequently, Abramsky provided a categorical notion of the geometry of interaction situation based on the concept of traced monoidal categories with some additional structure, and showed how to define models of combinatory logic from a geometry of interaction situation (Abramsky 1997). In Abramsky *et al.* (1999) we developed this analysis in detail.

We introduce unique decomposition categories that provide the necessary algebraic structure for our purposes. In this paper our goal is twofold. On the one hand, we undertake a computational (dataflow, token pushing) analysis of Abramsky's programme, which is implicit in the general framework. This provides an algebraic setting for an important class of examples and axiomatizes the 'particle-style' GoI in the terminology of Abramsky (1996). On the other hand, we define categorical models of dynamic algebras and interpret the notion of path (in dynamic algebras) as morphisms in a category. This establishes connections with the work of Girard and Danos and Regnier on geometry of interaction, and provides a categorical interpretation for the path-based work of Danos and Regnier on nets. Furthermore, the categorical model captures the original C^*-algebraic model for GoI introduced by Girard in Girard (1989a). This latter point will be further explored in future studies.

The paper is organized as follows. In Section 2 we introduce unique decomposition categories and study some of their properties that are important for our purposes, for example the existence of trace on such categories. We next consider a special class of UDCs, namely, the partially additive categories of Manes and Arbib. Most of the UDC examples we study fall in this class and enjoy interesting properties, for example we show that every partially additive category is traced and the trace is constructed in a canonical form. In Section 3 we review the general GoI construction first introduced by Abramsky (Abramsky 1997) and further elaborated in Abramsky *et al.* (1999). The main contribution of this paper is explained in Section 4. In this section we consider UDCs and construct a linear combinatory algebra from such categories. This is followed by several examples in Section 5. Another contribution of this paper is the introduction of categorical models of dynamic algebras, which we call *dynamic situations*. We construct dynamic situations in Section 6 and show that Girard's original operator theoretic model of a dynamic algebra (Girard 1989a) and Danos' small model (Danos 1990) can be obtained by using the categories $\mathbf{Hilb}_2$ and **PInj**, respectively.

2. Unique Decomposition Categories

Unique Decomposition Categories (UDCs) are categories whose hom-sets are enriched with a partial infinitary sum Σ. These categories are generalizations of the partially additive categories of Manes and Arbib (Manes and Arbib 1986). However, some of the ideas used in the definition of UDCs can be traced back to early works of Kuroš (Kuroš 1963). Unique decomposition categories carry the necessary algebraic structure to capture the main example of GoI, namely the category of sets and partial injective functions, which is not a partially additive category. We first introduce sets with infinite partial sums on them, which will be used in the definition of UDCs.

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Definition 2.1. A Σ-*monoid* consists of a pair (M, Σ) where M is a non-empty set and Σ is a partial operation on the countable families in M (we say that $\{x_i\}_{i\in I}$ is summable if $\Sigma_{i\in I} x_i$ is defined), subject to the following axioms:

1. *Partition-Associativity Axiom.* If $\{x_i\}_{i\in I}$ is a countable family and if $\{I_j\}_{j\in J}$ is a (countable) partition of I, then $\Sigma_{i\in I} x_i = \Sigma_{j\in J}(\Sigma_{i\in I_j} x_i)$. The equality here is a strong equality.
2. *Unary Sum Axiom.* Any family $\{x_i\}_{i\in I}$ in which I is a singleton is summable and $\Sigma_{i\in I} x_i = x_j$ if $I = \{j\}$.

We observe the following facts about Σ-monoids:

(i) Axioms 1 and 2 imply that the empty family is summable. We denote $\Sigma_{i\in\varnothing} x_i$ by 0, which is the additive identity for summation. (In fact, deletion or insertion of countably many zeros leaves a sum unchanged.)
(ii) Axiom 1 implies the obvious equations of commutativity and associativity for the sum (when defined). More generally, $\Sigma_i x_{\varphi(i)}$ is defined for any permutation φ of I whenever $\Sigma_i x_i$ exists.
(iii) When Σ is totally defined, (M, Σ_2) forms an Abelian monoid, where Σ_2 is the restriction of Σ to binary families, that is, $\Sigma_2 : M \times M \longrightarrow M$.
(iv) There are no additive inverses. Indeed, let $x + y = 0$; then $x = x + (y + x) + \cdots = (x + y) + (x + y) + \cdots = 0$.

Similar axiomatizations can be found in Bourbaki (1966, Chapter 3, Section 5), where infinite sums for commutative Hausdorff topological groups are defined. Kuroš (Kuroš 1963) also gives an axiomatization of infinite sums partially defined over sets, specifically hom-sets of categories. Axiom II in Kuroš (1963) is the same as our first axiom. Higgs (Higgs 1980) uses similar axioms, though our first axiom corresponds to his second and third axioms, and is stronger. We first encountered such structures in the work of Manes and Benson (Manes and Benson 1985), where they are called *positive monoids*.

A doubly indexed family f in a Σ-monoid M, $f : I \times J \longrightarrow M$, is denoted by $\{f_{ij}\}_{i\in I, j\in J}$, or simply $\{f_{ij}\}$ if the index sets are clear from the context. Such a family is summable iff $\Sigma_{i\in I}(\Sigma_{j\in J} f_{ij})$ exists, and in that case $\Sigma_{i,j} f_{ij} = \Sigma_{i\in I}(\Sigma_{j\in J} f_{ij})$. It follows, using Axiom 1, that for a summable family $\{f_{ij}\}$, $\Sigma_{i\in I}(\Sigma_{j\in J} f_{ij}) = \Sigma_{j\in J}(\Sigma_{i\in I} f_{ij})$.

Example 2.2.

(i) $M = \mathbf{PInj}(X, Y)$, the set of partial injective functions from X to Y. A family $\{f_i\}_{i\in I} \in \mathbf{PInj}(X, Y)$ is said to be summable iff $Dom(f_i) \cap Dom(f_j) = \varnothing$ and $Codom(f_i) \cap Codom(f_j) = \varnothing$ for all $i \neq j$. In that case, $(\Sigma_i f_i)(x) = f_j(x)$ if $x \in Dom(f_j)$ for some $j \in I$, and undefined otherwise.
(ii) $M = \mathbf{Pfn}(X, Y)$, the set of partial functions from X to Y. A family $\{f_i\}_{i\in I} \in M$ is summable iff $Dom(f_i) \cap Dom(f_j) = \varnothing$ for all $i \neq j$. In that case, $(\Sigma_i f_i)(x) = f_j(x)$ if $x \in Dom(f_j)$ for some $j \in I$, and undefined otherwise.
(iii) Let $M = \mathbf{Pfn}(X, Y)$ and define a family $\{f_i\}_{i\in I}$ to be summable when $f_i(x) = f_j(x)$ for all $x \in Dom(f_i) \cap Dom(f_j)$ and all $i, j \in I$. This additive structure is called *the overlap sum* and will be denoted by Σ^{ov}. For a summable family $\{f_i\}$, $(\Sigma^{ov} f_i)(x) = f_j(x)$ if there exists a $j \in I$ such that $x \in Dom(f_j)$, and undefined otherwise. Notice that in sharp contrast to the previous example the overlap sum is countable idempotent, that

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is, for a family $\{f_i\}_{i\in I}$ with $f_i = f$ for all $i \in I$, $\Sigma_{i\in I} f_i = f$. However, such a family is clearly not even summable in the additive structure given in the previous example.

(iv) $M = \mathbf{Rel}_+(X, Y)$, the set of relations from a set X to a set Y. Any family $\{R_i\}_{i\in I} \in \mathbf{Rel}_+(X, Y)$ is summable with $\Sigma_i R_i = \bigcup_{i\in I} R_i$.

(v) A countably complete poset with Σ as supremum is a Σ-monoid in which all families are summable and the sum is countable idempotent, that is, for $x_i = x$ for all $i \in I$, $\Sigma_{i\in I} x_i = x$. A related *non-example* is ω-complete posets with summable families defined as countable chains and Σ as supremum. For example, let $x, y \leqslant z$ with x, y incomparable, then we have $x + (y + z)$ is defined but $x + y + z$ is not.

(vi) Let M be a poset satisfying the *countable bounded completeness*, that is, given a countable subset D of M, if D has an upper bound, then it has a least upper bound. Let the summable families be the bounded families and Σ be the supremum. Then, (M, Σ) is a Σ-monoid. The additive identity is, of course, $sup(\varnothing)$.

(vii) Let $M = [0, 1]$ and given $\{x_i\}_{i\in I}$, define $\Sigma_i x_i = min(S, 1)$ for I finite, where S is the usual sum in real numbers, and for I infinite let $\Sigma_i x_i$ be the limit of the net of finite partial sums. (M, Σ) forms a Σ-monoid.

Definition 2.3. The category of Σ-monoids, Σ**Mon**, has as objects, Σ-monoids (M, Σ). Its arrows $(M, \Sigma) \xrightarrow{f} (N, \Sigma')$ are maps from M to N that preserve the sum, in the sense that for all summable families $\{x_i\}_{i\in I}$ in M, $\{f(x_i)\}_{i\in I}$ is summable in N and $f(\Sigma_{i\in I} x_i) = \Sigma'_{i\in I} f(x_i)$. Composition and identities are inherited from **Sets**.

Observe that Σ-**Mon** has finite products: given (M, Σ) and (M', Σ'), their product is $(M \times M', \Sigma'')$ where $\Sigma''_{i\in I}(x_i, y_i) = (\Sigma x_i, \Sigma' y_i)$ for all summable families $\{(x_i, y_i)\}_{i\in I}$ in $M \times M'$. The zero object $\mathbf{0}$ is $(\{0\}, \Sigma)$ in which all families are summable, with sum equal to 0. In particular, Σ**Mon** is a symmetric monoidal category with product as the tensor.

A Σ**Mon**-category is a category enriched in Σ**Mon**; that is, the hom-sets are enriched with an additive structure such that composition distributes over addition from left and right. Note that such categories have non-empty hom-sets and automatically have zero morphisms, namely $0_{XY} : X \longrightarrow Y = \Sigma_{i\in\varnothing} f_i$ for $f_i \in \mathscr{C}(X, Y)$. Note that this does not imply the existence of the zero object; however, existence of the initial or terminal object will guarantee the existence of a zero object in the presence of zero morphisms.

Definition 2.4. A *unique decomposition category* (UDC) $\mathscr{C}$ is a symmetric monoidal Σ**Mon**-category that satisfies the following axiom[†]:

(A) For all $j \in I$ there are morphisms *quasi injection*: $\iota_j : X_j \longrightarrow \otimes_I X_i$, and *quasi projection*: $\rho_j : \otimes_I X_i \longrightarrow X_j$, such that

1. $\rho_k \iota_j = 1_{X_j}$ if $j = k$ and $0_{X_j X_k}$ otherwise.
2. $\sum_{i\in I} \iota_i \rho_i = 1_{\otimes_I X_i}$.

[†] We use the following notational conventions: $I.X$ for $X^{\otimes I}$, $\otimes_I X_i$ for $\otimes_{i\in I} X_i$ and $\{X_i\}_I$ for $\{X_i\}_{i\in I}$. All index sets are finite when we discuss UDCs, unless explicitly stated otherwise.

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Proposition 2.5 (Matricial representation). Given $f : \otimes_J X_j \longrightarrow \otimes_I Y_i$ in a UDC with $|I| = m$ and $|J| = n$, there exists a unique family $\{f_{ij}\}_{i\in I, j\in J} : X_j \longrightarrow Y_i$ with $f = \sum_{i\in I, j\in J} \iota_i f_{ij} \rho_j$, namely, $f_{ij} = \rho_i f \iota_j$.

Proof. $f = 1f1 = (\sum_i \iota_i\rho_i)f(\sum_j \iota_j\rho_j) = (\sum_i \iota_i\rho_i f)(\sum_j \iota_j\rho_j) = \sum_{ij} \iota_i\rho_i f \iota_j\rho_j = \sum_{ij} \iota_i f_{ij}\rho_j$.

For uniqueness, suppose there is another family $\{g_{kl}\}$ such that $f = \sum_{kl} \iota_k g_{kl} \rho_l$. Then $f_{ij} = \rho_i f \iota_j = \sum_{kl} \rho_i \iota_k g_{kl} \rho_l \iota_j = g_{ij}$ for all i, j. □

Based on this proposition, every morphism $f : \otimes_J X_j \longrightarrow \otimes_I Y_i$ can be represented by its components. We will use the corresponding matrices to represent morphisms, for example f above (with $|I| = m$ and $|J| = n$) is represented by an $m \times n$ matrix:

$$f = \begin{bmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & \vdots & \vdots \\ f_{m1} & \cdots & f_{mn} \end{bmatrix}$$

Given $f : \otimes_K X_k \longrightarrow \otimes_J Y_j$ and $g : \otimes_J Y_j \longrightarrow \otimes_I Z_i$, let $h = gf$, then $h_{ik} = \rho_i h \iota_k = \rho_i(gf)\iota_k = \rho_i g(\sum_{j\in J} \iota_j\rho_j)f\iota_k = \sum_{j\in J}(\rho_i g \iota_j\rho_j)f\iota_k = \sum_{j\in J}(g_{ij}\rho_j f \iota_k) = \sum_{j\in J} g_{ij} f_{jk}$.

Therefore, the composition of morphisms in a UDC corresponds to matrix multiplication of their matricial representations.

Remark. Note that although any morphism $f : \otimes_J X_j \longrightarrow \otimes_I Y_i$ can be represented by the unique family $\{f_{ij}\}$ of its components, the converse is not necessarily true, that is to say, given a family $\{f_{ij}\}$ with I, J finite there may not be a morphism $f : \otimes_J X_j \longrightarrow \otimes_I Y_i$ satisfying $f = \sum_{ij} \iota_i f_{ij} \rho_j$. However, when such an f exists, it will be unique.

Definition 2.6. A functor $T : \mathscr{C} \longrightarrow \mathscr{D}$ between two UDCs is said to be *additive* if $T_{X,Y} : \mathscr{C}(X, Y) \longrightarrow \mathscr{D}(TX, TY)$ is a **ΣMon** morphism for all X, Y, objects in $\mathscr{C}$. That is, for every summable family $\{f_i\}$ in $\mathscr{C}(X, Y)$, we have $\{T_{X,Y}(f_i)\}$ is summable in $\mathscr{D}(TX, TY)$ and $T_{X,Y}(\Sigma_i f_i) = \Sigma_i T_{X,Y}(f_i)$.

Example 2.7.

(i) Consider the category **PInj** of sets and partial injective functions. Define $X \otimes Y = X \uplus Y$: note that this does not give a coproduct, indeed **PInj** does not have coproducts. The partially additive structure is given in Example 2.2. Define $\rho_j : \otimes_{i\in I} X_i \longrightarrow X_j$ by $\rho_j(x, j) = x$ and $\rho_j(x, i)$ undefined for $i \neq j$. We have $\iota_j : X_j \longrightarrow \otimes_{i\in I} X_i$ by $\iota_j(x) = (x, j)$. It can be easily seen that $\rho_k \iota_j = 1_{X_j}$ if $j = k$ and $0_{X_j X_k}$ otherwise. Note that $Dom(\iota_i\rho_i) \cap Dom(\iota_j\rho_j) = \varnothing$ and $Codom(\iota_i\rho_i) \cap Codom(\iota_j\rho_j) = \varnothing$ for $i \neq j$, and hence the family $\{\iota_i\rho_i\}_{i\in I}$ is summable. Also, $(\sum_i \iota_i\rho_i)((x, j)) = \iota_j(x) = (x, j) = 1_{\otimes_I X_i}((x, j))$.

(ii) Consider the category $\mathbf{Pfn}_{ov}$ with sets as objects and partial functions as morphisms. Define $X \otimes Y = X \uplus Y$, which is in fact the coproduct in this category. The partially additive structure is given in Example 2.2.(iii). Define quasi projections and quasi injections as in the case of **PInj**. It can be easily seen that $\mathbf{Pfn}_{ov}$ is a UDC.

(iii) Consider the category **SRel** (Panangaden 1997; Abramsky 1996) of stochastic relations, with measurable spaces $(X, \mathscr{F}_X)$ as objects and stochastic kernels as arrows. An arrow

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$f : (X, \mathscr{F}_X) \longrightarrow (Y, \mathscr{F}_Y)$ is a map $f : X \times \mathscr{F}_Y \longrightarrow [0,1]$ such that $f(\cdot, B) : X \longrightarrow [0,1]$ is a measurable function for fixed $B \in \mathscr{F}_Y$ and $f(x, \cdot) : \mathscr{F}_Y \longrightarrow [0,1]$ is a subprobability measure (that is, σ-additive, set function, $f(x, \varnothing) = 0$ and $f(x, Y) \leqslant 1$). The identity morphism $id_X : (X, \mathscr{F}_X) \longrightarrow (X, \mathscr{F}_X)$ is $id_X : X \times \mathscr{F}_X \longrightarrow [0,1]$ and $id_X(x, A) = \delta(x, A) = 1$ if $x \in A$ and 0 if $x \notin A$.

For A fixed, $\delta(x, A)$ is the characteristic function of A and for x fixed, $\delta(x, A)$ is the Dirac distribution. Finally, composition is defined as: Given $f : (X, \mathscr{F}_X) \longrightarrow (Y, \mathscr{F}_Y)$ and $g : (Y, \mathscr{F}_Y) \longrightarrow (Z, \mathscr{F}_Z)$, $gf : (X, \mathscr{F}_X) \longrightarrow (Z, \mathscr{F}_Z)$ is $gf(x, C) = \int_Y g(y, C) f(x, dy)$, where we are using $f(x, \cdot)$ as the measure for the integration and the function being integrated is the measurable function $g(y, C)$.

Let tensor be the coproduct (defined as the disjoint union of sets and the σ-field on the union). A family $\{f_i\}_{i \in I} \in \mathbf{SRel}(X, Y)$ is summable if $\Sigma_i f_i(x, Y) \leqslant 1$ for all $x \in X$. In that case, $(\Sigma_i f_i)(x, B) = \Sigma_i f_i(x, B)$ where the latter is the usual sum of real numbers. $\iota_j = in_j : X_j \longrightarrow \coprod_I X_i$ is defined by $in_j(x, \uplus_I A_i) = \delta(x, A_j)$ and $\rho_j : \coprod_I X_i \longrightarrow X_j$ is defined by $\rho((x,k), A_j) = \delta_{kj}\delta(x, A_j)$.

The category **SRel** is a partially additive category and we will show later that any partially additive category is a UDC, hence we have not included the verification of axiom A of Definition 2.4.

Theorem 2.8. Let $\mathscr{C}$ be a category with countable biproducts. Then $\mathscr{C}$ is a UDC (with biproduct as tensor) in which all families of morphisms are summable.

Proof. Let $\mathscr{C}$ be a category with countable biproducts. $\mathscr{C}$ has zero object and hence families of zero morphisms. Given a family $\{f_i\}_I$ of morphisms in $\mathscr{C}(X, Y)$ with countable I, define $\sum_{i \in I} f_i = \nabla_Y (\oplus_{i \in I} f_i) \Delta_X$ where $\Delta_X : X \longrightarrow I.X$ and $\nabla_Y : I.Y \longrightarrow Y$ are defined by: $\pi_i \Delta_X = 1_X$ and $\nabla_Y in_i = 1_Y$ for all $i \in I$ and $\sum_{i \in \varnothing} f_i = 0_{XY}$. This makes every hom-set $\mathscr{C}(X, Y)$ into a commutative monoid with $\sum_{i \in \varnothing} f_i$ as the additive identity. More generally, using the associativity and symmetry isomorphisms of biproduct it can be shown that every hom-set $\mathscr{C}(X, Y)$ is a Σ-monoid. Also, composition distributes over addition. Details appear in Haghverdi (1999). Clearly all families are summable.

We need to verify axiom A of Definition 2.4. Given any family $\{X_i\}_I$ in $\mathscr{C}$, let $\iota_j = in_j$ and $\rho_j = \pi_j$ for all $j \in I$. Here in_j and π_j are the biproduct morphisms of $(\oplus_I X_i, \{\pi_j\}, \{in_j\})$. By the definition of biproduct, we have $\pi_k in_j = 1_{X_j}$ if $j = k$ and $0_{X_j X_k}$ otherwise. We also have $\pi_j(\sum_{i \in I} in_i \pi_i) = \pi_j$ for all $j \in I$, and therefore $\sum_{i \in I} in_i \pi_i = 1_{\oplus_I X_i}$. □

Proposition 2.9. Let $\mathscr{C}$ be a UDC. If all finite families of morphisms in $\mathscr{C}$ are summable, $\mathscr{C}$ has finite biproducts.

Proof. Let $\mathscr{C}$ be a UDC with all finite families summable. We show that given a family $\{X_i\}_I$, we have $(\otimes_I X_i, \{\iota_j\}_I)$ is a coproduct. Given any object Y and a family $\{f_j\}_I$ with $f_j : X_j \longrightarrow Y$, let $f = \sum_j f_j \rho_j$ (f exists since all finite families are summable). $(\sum_j f_j \rho_j)\iota_j = \sum_j (f_j \rho_j \iota_j) = f_j$ for all $j \in I$. Let $h : \otimes_I X_i \longrightarrow Y$ be such that $h\iota_j = f_j$ for all $j \in I$. Then there exists a unique family $h_j : X_j \longrightarrow Y$ such that $h = \sum_j h_j \rho_j$.

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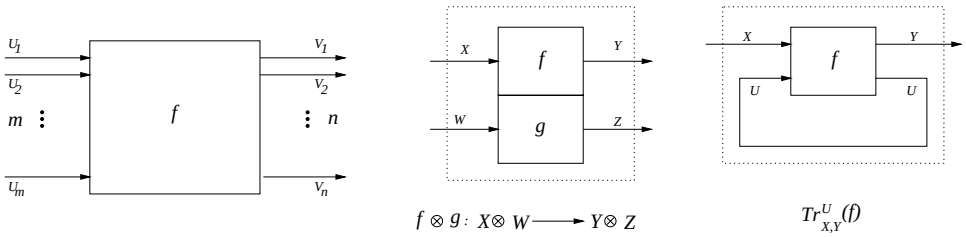


Fig. 1. Graphical representation of f and I/O interfaces

$f_j = h\iota_j = h_j$ for all $j \in I$. Thus $f = h$ and there exists a unique f such that $f\iota_j = f_j$ for all $j \in I$.

Similarly, it can be shown that given a family $\{X_i\}_I$, we have $(\otimes_I X_i, \{\rho_j\}_I)$ is a product and the canonical map from coproduct to product is the identity. Therefore $(\otimes_I X_i, \{\rho_j\}_I, \{\iota_j\}_I)$ is a biproduct. □

Lemma 2.10. Let $\mathscr{C}$ be a UDC and $f : X \otimes U \longrightarrow X \otimes U$ be given by $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$. Then $\sum_{n\in\omega} f^n$, if it exists, is given by $\begin{bmatrix} E & F \\ G & H \end{bmatrix}$, where:

- $E = \sum_{m\in\omega}(A + \sum_{n\in\omega} BD^nC)^m$
- $F = \sum_{l\in\omega} EBD^l$
- $G = \sum_{l\in\omega} D^lCE$
- $H = \sum_{l\in\omega} D^l + \sum_{p,q\in\omega} D^pCEBD^q$

Proof. See the Appendix. □

We recall the definition of traced symmetric monoidal categories of Joyal *et al.* (1996). For readability, and without loss of generality, we consider strict monoidal categories. It is well known that every monoidal category is equivalent to a strict one.

Definition 2.11. A *traced symmetric monoidal category* is a symmetric monoidal category $(\mathscr{C}, \otimes, I, \sigma)$ with a family of functions $Tr^U_{X,Y} : \mathscr{C}(X \otimes U, Y \otimes U) \longrightarrow \mathscr{C}(X, Y)$ pictured in Figure 1, called a *trace*, subject to the following conditions:

(i) **Natural** in X, $Tr^U_{X,Y}(f)g = Tr^U_{X',Y}(f(g\otimes 1_U))$ where $f : X\otimes U \longrightarrow Y\otimes U, g : X' \longrightarrow X$.
(ii) **Natural** in Y, $gTr^U_{X,Y}(f) = Tr^U_{X,Y'}((g\otimes 1_U)f)$ where $f : X\otimes U \longrightarrow Y\otimes U, g : Y \longrightarrow Y'$.
(iii) **Dinatural** in U, $Tr^U_{X,Y}((1_Y \otimes g)f) = Tr^{U'}_{X,Y}(f(1_X \otimes g))$ where $f : X \otimes U \longrightarrow Y \otimes U'$, $g : U' \longrightarrow U$.
(iv) **Vanishing**, $Tr^I_{X,Y}(f) = f$ and $Tr^{U\otimes V}_{X,Y}(g) = Tr^U_{X,Y}(Tr^V_{X\otimes U, Y\otimes U}(g))$ for $f : X\otimes I \longrightarrow Y\otimes I$ and $g : X \otimes U \otimes V \longrightarrow Y \otimes U \otimes V$.
(v) **Superposing**, $g \otimes Tr^U_{X,Y}(f) = Tr^U_{W\otimes X, Z\otimes Y}(g \otimes f)$, for $f : X \otimes U \longrightarrow Y \otimes U$ and $g : W \longrightarrow Z$.
(vi) **Yanking**, $Tr^U_{U,U}(\sigma_{U,U}) = 1_U$.

The axiomatization given here differs from those given in Abramsky (1996) and Joyal *et al.* (1996). It can be shown that the two versions are equivalent, see the Appendix for a proof.

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Notation: We introduce the following graphical notation: an arrow $U_1 \otimes \cdots \otimes U_m \xrightarrow{f} V_1 \otimes \cdots \otimes V_n$ is represented as a box as in Figure 1. We sometimes emphasize the I/O interface of a morphism using a dotted box as in Figure 1. We will omit the labels on the lines when it is clear.

Proposition 2.12 (Generalized Yanking). Let $\mathscr{C}$ be a traced symmetric monoidal category. Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be given. Then

$$gf = Tr^{Y}_{X,Z}(\sigma_{Y,Z}(f \otimes g)).$$

In particular, one can give an equivalent axiomatization of traced symmetric monoidal categories with Yanking replaced by Generalized Yanking (Haghverdi 1999).

Proposition 2.13 (Execution/Trace Formula). Let $\mathscr{C}$ be a unique decomposition category such that for every X, Y, and U and $f : X \otimes U \longrightarrow Y \otimes U$, the sum $f_{11} + \sum_{n=0}^{\infty} f_{12} f_{22}^{n} f_{21}$ exists, where f_{ij} are the components of f. Then $\mathscr{C}$ is traced and

$$Tr^{U}_{X,Y}(f) = f_{11} + \sum_{n=0}^{\infty} f_{12} f_{22}^{n} f_{21}.$$

Before proceeding to the proof of this proposition, let us look at some trace computation examples. Let $\mathscr{C}$ be a traced UDC and hence given any $f : X \otimes U \longrightarrow Y \otimes U$, we have $Tr^{U}_{X,Y}(f)$ exists. Consider the following examples:

(i) Let $f : X \otimes U \longrightarrow Y \otimes U$ be given by $\begin{bmatrix} g & 0 \\ h & 0 \end{bmatrix}$. Then

$$Tr^{U}_{X,Y}(f) = Tr^{U}_{X,Y}\left(\begin{bmatrix} g & 0 \\ h & 0 \end{bmatrix}\right) = g + \textstyle\sum_n 00^n h = g + 0h = g + 0 = g.$$

(ii) Let $f : X \otimes U \longrightarrow Y \otimes U$ be given by $\begin{bmatrix} g & 0 \\ 0 & h \end{bmatrix}$. Then

$$Tr^{U}_{X,Y}(f) = Tr^{U}_{X,Y}\left(\begin{bmatrix} g & 0 \\ 0 & h \end{bmatrix}\right) = g + \textstyle\sum_n 0h^n 0 = g + 0 = g.$$

(iii) Let $f : X \otimes U \longrightarrow Y \otimes U$ be given by $\begin{bmatrix} 0 & g \\ h & 0 \end{bmatrix}$. Then

$$Tr^{U}_{X,Y}(f) = Tr^{U}_{X,Y}\left(\begin{bmatrix} 0 & g \\ h & 0 \end{bmatrix}\right) = 0 + \textstyle\sum_n g0^n h = 0 + g1h = gh.$$

Proof of Proposition 2.13. We verify Vanishing II, Superposing and Yanking. The verification of other axioms is similar and can be found in Haghverdi (1999). All indices range over the set of natural numbers.

Vanishing II: $Tr^{U\otimes V}_{X,Y}(g) = Tr^{U}_{X,Y}(Tr^{V}_{X\otimes U, Y\otimes U}(g))$ where $g : X \otimes U \otimes V \longrightarrow Y \otimes U \otimes V$.

Let g be given by $\begin{bmatrix} L & M & N \\ P & A & B \\ Q & C & D \end{bmatrix}$.

Then $Tr^{U}Tr^{V}\left(\begin{bmatrix} L & M & N \\ P & A & B \\ Q & C & D \end{bmatrix}\right) = Tr^{U}\left(\begin{bmatrix} L & M \\ P & A \end{bmatrix} + \textstyle\sum_n \begin{bmatrix} N \\ B \end{bmatrix} D^n \begin{bmatrix} Q & C \end{bmatrix}\right)$

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$$= Tr^U \begin{bmatrix} L + \sum_n ND^nQ & M + \sum_n ND^nC \\ P + \sum_n BD^nQ & A + \sum_n BD^nC \end{bmatrix}$$

$$= L + \sum_n ND^nQ + (M + \sum_n ND^nC)(\sum_m (A + \sum_n BD^nC)^m)(P + \sum_n BD^nQ)$$

$$= L + \sum_n ND^nQ + (M + \sum_n ND^nC)E(P + \sum_n BD^nQ) \text{ for } E \text{ as in Lemma 2.10}$$

$$= L + \sum_n ND^nQ + MEP + ME\sum_n BD^nQ + \sum_n ND^nCEP + \sum_m ND^mCE\sum_n BD^nQ$$

$$= L + MEP + \sum_n MEBD^nQ + \sum_n ND^nCEP + N(\sum_n D^n + \sum_{m,n} D^mCEBD^n)Q$$

$$= L + MEP + MFQ + NGP + NHQ \text{ for } E, F, G \text{ and } H \text{ as in Lemma 2.10}$$

$$= L + \begin{bmatrix} M & N \end{bmatrix} \begin{bmatrix} E & F \\ G & H \end{bmatrix} \begin{bmatrix} P \\ Q \end{bmatrix}$$

$$= L + \sum_n \begin{bmatrix} M & N \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}^n \begin{bmatrix} P \\ Q \end{bmatrix} = Tr^{U\otimes V} \begin{bmatrix} L & M & N \\ P & A & B \\ Q & C & D \end{bmatrix} = Tr^{U\otimes V}_{X,Y}(g).$$

Superposing: $g \otimes Tr^U_{X,Y}(f) = Tr^U_{W\otimes X, Z\otimes Y}(g \otimes f)$ where $f : X \otimes U \longrightarrow Y \otimes U$ and $g : W \longrightarrow Z$.

$$Tr^U_{W\otimes X,Z\otimes Y}(g \otimes f) = Tr^U_{W\otimes X,Z\otimes Y}\left(\begin{bmatrix} g & 0 & 0 \\ 0 & f_{11} & f_{12} \\ 0 & f_{21} & f_{22} \end{bmatrix}\right) = \begin{bmatrix} g & 0 \\ 0 & f_{11} \end{bmatrix} + \sum_n \begin{bmatrix} 0 \\ f_{12} \end{bmatrix} f^n_{22}$$

$$\begin{bmatrix} 0 & f_{21} \end{bmatrix} = \begin{bmatrix} g & 0 \\ 0 & f_{11} + \sum_n f_{12} f^n_{22} f_{21} \end{bmatrix} = \begin{bmatrix} g & 0 \\ 0 & Tr^U_{X,Y}(f) \end{bmatrix} = g \otimes Tr^U_{X,Y}(f).$$

Yanking: $Tr^U_{U,U}(\sigma_{U,U}) = Tr^U_{U,U}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = 1_U$. □

2.1. *Partially additive categories*

In this section we discuss a special class of UDCs called partially additive categories. Partially additive categories were defined and used by Manes and Arbib to provide an additive semantics for programming languages (Manes and Arbib 1986). Our interest in partially additive categories is primarily due to the fact that they provide a canonical method for fixed-point constructions. These in turn yield canonical constructions for trace and composition in geometry of interaction categories.

Definition 2.14. A *partially additive monoid* is a Σ-monoid (M, Σ), subject to an additional axiom:

— *Limit Axiom.* If $\{x_i\}_{i\in I}$ is a countable family and if $\{x_i\}_{i\in F}$ is summable for every finite subset F of I, then $\{x_i\}_{i\in I}$ is summable.

Clearly, every partially additive monoid is a Σ-monoid. The converse, however, is not true. Consider the set $\mathbb{R}^+$ of all non-negative real numbers and define $\Sigma_{i\in I} x_i$ to be the arithmetic sum for I finite and the limit of the net of finite partial sums for I infinite. The summable families are absolutely convergent series. This is an example of a Σ-monoid

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that is not a partially additive monoid, because the limit axiom fails – just consider the family $\{x_i\}_{i\in I}$ with I countable infinite and $x_i = 1$ for all $i \in I$. All Σ-monoids in Example 2.2. except (vi) are partially additive monoids.

As with Σ-monoids, partially additive monoids and sum preserving maps form a symmetric monoidal category, **PAMon** with product as tensor. Likewise, a **PAMon**-category is a category enriched in **PAMon**.

Definition 2.15. Let $\mathscr{C}$ be a **PAMon**-category with countable coproducts $\coprod_{i\in I} X_i$. We define *quasi projections* as: $\rho_k in_j = 1_{X_j}$ if $k = j$ and $0_{X_j X_k}$ otherwise. Note that ρ_k exists for all $k \in I$ since $\mathscr{C}$ has zero morphisms.

Definition 2.16. A *partially additive category* $\mathscr{C}$ is a **PAMon**-category with countable coproducts that satisfies the following axioms:

1. *Compatible Sum Axiom*: If $\{f_i\}_{i\in I} \in \mathscr{C}(X, Y)$ is a countable family and there exists $f : X \longrightarrow I.Y$ such that $\rho_i f = f_i$ (we say the f_i are *compatible*), then $\sum f_i$ exists.
2. *Untying Axiom*: If $f + g : X \longrightarrow Y$ exists, then so does $in_1 f + in_2 g : X \longrightarrow Y + Y$.

Proposition 2.17. Given $f : X \longrightarrow \coprod_I Y_i$ in a partially additive category, there exists a unique family $f_i : X \longrightarrow Y_i$ with $f = \sum in_i f_i$, namely, $f_i = \rho_i f$.

Proof. See Manes and Arbib (1986, Proposition 15, page 80). □

Theorem 2.18 (Manes and Arbib). Given a map $f : X \longrightarrow Y \coprod X$ in a partially additive category, the sum

$$f^\dagger = \sum_{n=0}^{\infty} f_1 f_2^n : X \longrightarrow Y$$

exists, where $f_1 : X \longrightarrow Y$ and $f_2 : X \longrightarrow X$ are the components of f.

Proof. See Manes and Arbib (1986 Theorem 24, page 83). □

Proposition 2.19. Every PAC is a UDC with coproduct as tensor.

Proof. Let $\mathscr{C}$ be a PAC. Let the tensor product be coproduct and given a family $\{X_i\}_I$ in $\mathscr{C}$ choose $\iota_j = in_j$ and ρ_j as defined in Definition 2.15. Clearly $\mathscr{C}$ is a **ΣMon** enriched category since every partially additive monoid is already a Σ-monoid. To show that $\mathscr{C}$ is a UDC, we need to verify axiom A of Definition 2.4. Observe $\rho_k in_j = 1_{X_j}$ for $j = k$ and 0 otherwise by definition of ρ_k. Consider $1 : \coprod_I X_i \longrightarrow \coprod_I X_i$, by Proposition 2.17, there exists a unique family, namely $\rho_i = \rho_i 1$, such that $1 = \sum_i in_i \rho_i$, and hence axiom A holds. □

Proposition 2.20. Every partially additive category is a traced symmetric monoidal category, where given $f : X \coprod U \longrightarrow Y \coprod U$,

$$Tr^U_{X,Y} = f_{11} + \sum_n f_{12} f_{22}^n f_{21}$$

and f_{ij} are the components of f.

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Proof. Given $f : X \coprod U \longrightarrow Y \coprod U$, let $f_1 \stackrel{\Delta}{=} f in_1 : X \longrightarrow Y \coprod U$ and $f_2 \stackrel{\Delta}{=} f in_2 : U \longrightarrow Y \coprod U$. By Theorem 2.18, $f_2^\dagger = \sum_{n=0}^{\infty} f_{12} f_{22}^n$. Also, by Proposition 2.17, f_1 can be written as $f_1 = in_1 f_{11} + in_2 f_{21}$. Define $Tr_{X,Y}^U(f) = [1_Y, f_2^\dagger] f_1$. Then $Tr_{X,Y}^U(f) = [1_Y, f_2^\dagger](in_1 f_{11} + in_2 f_{21}) = f_{11} + (\sum_{n=0}^{\infty} f_{12} f_{22}^n) f_{21} = f_{11} + \sum_{n=0}^{\infty} f_{12} f_{22}^n f_{21}$. Hence by Proposition 2.13, $\mathscr{C}$ is traced. □

The condition for the existence of a trace in a UDC can be significantly simplified in the presence of the limit axiom.

Proposition 2.21. Let $\mathscr{C}$ be a UDC in which every hom-set is a partially additive monoid. If for every X, Y, U and $f : X \otimes U \longrightarrow Y \otimes U$, the sum $f_{12} + g f_{22}$ exists for every $g : U \longrightarrow Y$ and the sum $f_{11} + h$ exists, where $h = \sum_{n=0}^{\infty} f_{12} f_{22}^n f_{21}$, then $\mathscr{C}$ is traced with $Tr(f) = f_{11} + \sum_{n=0}^{\infty} f_{12} f_{22}^n f_{21}$. Here f_{11}, f_{12}, f_{21} and f_{22} are the components of f.

Proof. Note that f_{12} is summable by the unary sum axiom. Given that $f_{12} + g f_{22}$ exists for all $g : U \longrightarrow Y$, we let $g = f_{12}$, and hence $f_{12} + f_{12} f_{22}$ exists. Next let $g = f_{12} + f_{12} f_{22}$ and hence $f_{12} + f_{12} f_{22} + f_{12} f_{22}^2$ exists. Proceeding in this way, we conclude that the sum $\sum_{k=0}^{n} f_{12} f_{22}^k$ exists for every n, and hence by the limit axiom $\sum_{n=0}^{\infty} f_{12} f_{22}^n$ exists. Note that $h = \sum_n f_{12} f_{22}^n f_{21} = (\sum_n f_{12} f_{22}^n) f_{21}$ exists, since composition distributes over sum.

Finally $Tr_{X,Y}^U(f) = f_{11} + \sum_n f_{12} f_{22}^n f_{21} = f_{11} + h$ exists by hypothesis, and therefore $\mathscr{C}$ is traced by Proposition 2.13. □

Recall that as a result of Theorem 2.8 any category $\mathscr{C}$ with countable biproducts is a UDC. However, since in such a UDC all families are summable, the limit axiom is trivially true and we can state a stronger result.

Proposition 2.22. Let $\mathscr{C}$ be a category with countable biproducts. Then $\mathscr{C}$ is a PAC and all families in $\mathscr{C}$ are summable.

Example 2.23.

(i) Given $f : X \otimes U \longrightarrow Y \otimes U$ in **PInj**, it can be shown that the sum $f_{11} + \sum_n f_{12} f_{22}^n f_{21}$ exists. That is to say, the functions appearing in the sum have pairwise disjoint domains and codomains (Haghverdi 1999).

(ii) **SRel** is a PAC (Panangaden 1997; Haghverdi 1999) with sum as defined in Example 2.7., and hence by Propositions 2.19 and 2.20, is a traced UDC. The trace is given by the same formula as in Proposition 2.20. See Abramsky (1996) for an equivalent formula for trace in **SRel**.

(iii) The category of sets and binary relations $\mathbf{Rel}_+$ is a PAC. The additive structure is as defined in Example 2.2.

(iv) The category of sets and partial functions **Pfn** is a PAC with sum as in Example 2.2.

(v) Consider the category $\mathbf{Pfn}_M$ whose objects are sets, and an arrow from X to Y is a map $f : X \longrightarrow Y \times M$ where M is a monoid. $1_X : X \longrightarrow X$ is $1_X(x) = (x, e)$, and given $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$, we have $gf(x) = (z, ba)$ where $f(x) = (y, a)$ and $g(y) = (z, b)$. $\mathbf{Pfn}_M$ is a PAC (Manes and Arbib 1986) with the additive structure as in **Pfn**.

These examples are studied further in Section 5.

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3. General Geometry of Interaction construction

In this section we review the general GoI construction. For details the reader is referred to Abramsky *et al.* (1999). Linear Combinatory Algebras (LCA) and GoI Situations were first introduced by Abramsky in Abramsky (1997), and further elaborated in Abramsky *et al.* (1999). This section is mainly taken from Abramsky *et al.* (1999).

Definition 3.1. A *Linear Combinatory Algebra* $(\mathscr{A}, \cdot, !)$ consists of the following data:

— An applicative structure $(\mathscr{A}, \cdot)$
— A unary operator $! : \mathscr{A} \to \mathscr{A}$
— Distinguished elements $B, C, I, K, W, D, \delta, F$ of $\mathscr{A}$,

satisfying the following identities (we associate $\cdot$ to the left and write xy for $x \cdot y$, $x \cdot !y = x \cdot (!y)$, and so on) for all variables $x, y, z \in \mathscr{A}$.

1. $Bxyz = x(yz)$ Composition, Cut
2. $Cxyz = (xz)y$ Exchange
3. $Ix = x$ Identity
4. $Kx!y = x$ Weakening
5. $Wx!y = x!y!y$ Contraction
6. $Dx!y = xy$ Dereliction
7. $\delta !x = !!x$ Comultiplication
8. $F!x!y = !(xy)$ Monoidal Functoriality

The LCA corresponds to a Hilbert style axiomatization of the $\{!, \multimap\}$ fragment of linear logic (Abramsky 1997).

Definition 3.2. A *Standard Combinatory Algebra* consists of a pair $(\mathscr{A}, \cdot_s)$ where $\mathscr{A}$ is a non-empty set and $\cdot_s$ is a binary operation on $\mathscr{A}$, and B_s, C_s, I_s, K_s and W_s are distinguished elements of $\mathscr{A}$ satisfying the following identities for all x, y, z variables ranging over $\mathscr{A}$:

1. $B_s \cdot_s x \cdot_s y \cdot_s z = x \cdot_s (y \cdot_s z)$
2. $C_s \cdot_s x \cdot_s y \cdot_s z = (x \cdot_s z) \cdot_s y$
3. $I_s \cdot_s x = x$
4. $K_s \cdot_s x \cdot_s y = x$
5. $W_s \cdot_s x \cdot_s y = x \cdot_s y \cdot_s y$

Let $(\mathscr{A}, \cdot, !)$ be a linear combinatory algebra. We define a binary operation $\cdot_s$ on $\mathscr{A}$ as follows: for $\alpha, \beta \in \mathscr{A}$, $\alpha \cdot_s \beta =_{def} \alpha \cdot !\beta$ [†]. Next consider the following elements of $\mathscr{A}$:

1. $B_s =_{def} C \cdot (B \cdot (B \cdot B \cdot B) \cdot (D \cdot I)) \cdot (C \cdot ((B \cdot B) \cdot F) \cdot \delta)$
2. $C_s =_{def} D \cdot C$
3. $I_s =_{def} D \cdot I$
4. $K_s =_{def} D \cdot K$
5. $W_s =_{def} D \cdot W$

Proposition 3.3. Let $(\mathscr{A}, \cdot, !)$ be a linear combinatory algebra. Then, $(\mathscr{A}, \cdot_s)$, with $\cdot_s$ and the elements B_s, C_s, I_s, K_s, W_s as defined above, is a standard combinatory algebra.

4. Combinators in UDCs: dataflow view

Linear combinatory algebras can be constructed using traced monoidal categories. This relates the notion of trace to computation and yields new models of untyped combinatory logic. In this section we show how to define combinators in a unique decomposition category and construct a linear combinatory algebra from such a category.

[†] The same observation has been made by Girard (see Girard (1987, page 17)).

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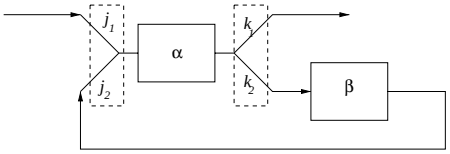


Fig. 2. Binary operation $\alpha.\beta$

Definition 4.1. A *GoI Situation* is a triple $(\mathscr{C}, T, U)$ where:

1. $\mathscr{C}$ is a traced symmetric monoidal category.
2. $T : \mathscr{C} \longrightarrow \mathscr{C}$ is a traced symmetric monoidal functor with the following retractions:
   - $TT \lhd T$ (e, e') (Comultiplication)
   - $Id \lhd T$ (l, l') (Dereliction)
   - $T \otimes T \lhd T$ (c, c') (Contraction)
   - $\mathscr{K}_I \lhd T$ (w, w') (Weakening). Here $\mathscr{K}_I$ is the constant I functor.

   Note that the retraction morphisms above are monoidal natural transformations.
3. U is an object of $\mathscr{C}$, called *a reflexive object*, with the following retractions:
   - $U \otimes U \lhd U$ (j, k)
   - $I \lhd U$
   - $TU \lhd U$ (u, v).

Let $\mathscr{C}$ be a UDC, $T : \mathscr{C} \longrightarrow \mathscr{C}$ an endofunctor and U a reflexive object in $\mathscr{C}$ such that $(\mathscr{C}, T, U)$ forms a GoI Situation. Consider the set $\mathscr{C}(U, U)$. We define the binary operation $\cdot$ and unary operation $!$ on $\mathscr{C}(U, U)$ as follows: given $\alpha, \beta \in \mathscr{C}(U, U)$, we have $\alpha.\beta = Tr^U_{U,U}((1_U \otimes \beta)(k\alpha j))$ and $!\alpha = uT(\alpha)v$. In matricial form we have

$$\alpha.\beta = Tr^U_{U,U}\left(\begin{bmatrix} 1 & 0 \\ 0 & \beta \end{bmatrix}\begin{bmatrix} k_1\alpha j_1 & k_1\alpha j_2 \\ k_2\alpha j_1 & k_2\alpha j_2 \end{bmatrix}\right) = k_1\alpha j_1 + \sum_n k_1\alpha j_2(\beta k_2\alpha j_2)^n(\beta k_2\alpha j_1)$$

where $j = \begin{bmatrix} j_1 & j_2 \end{bmatrix}$ and $k = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$. This operation is represented diagrammatically in Figure 2. Note that we have omitted the boxes for j_1, j_2, k_1 and k_2 in order not to over-complicate the figure. Recall that $kj = 1_{U\otimes U}$, and hence $k_m j_n = 1_U$ if $m = n$ and 0 otherwise. We will next define the combinators.

4.1. *Sum-over-paths: dataflow interpretation of combinators*

We will explain the process of defining combinators I, K and W in detail. We purposely use both diagrammatic order $(f\,;g)$ and algebraic order notation (gf). So (from the above) $j_n;k_m = 1$ if $m = n$, and 0 otherwise.

Combinator I:

I must satisfy the equation $I.x = x$. We view the right-hand side of this equation as the required specification; the path from environment Env to x is labelled as 1_U, and, similarly, the path from x to Env is labelled 1_U (see Figure 3(b)). Note that we think

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of Env as the object U, and the connection from Env to x is thus a morphism from U to U, namely, 1_U in this case. On the other hand, we view the left-hand side of the equation $I \cdot x = x$ as the existing structure (circuitry). In this circuit the box representing the combinator I is initially empty and our goal is to design the internal circuitry of I in such a way that we meet the required specification. We proceed as follows (see Figure 3(a)):

(1) According to the required specification there must be a path from Env to x, hence we put a path with an as yet unknown label **1** through the box. This provides us with the connection from Env to x, namely the path labelled $j_1; \mathbf{1}; k_2$ that will take us from Env to x. Now, the specification also requires that the label of this path be 1_U. Hence, the unknown label, **1**, must be chosen in such a way that $j_1; \mathbf{1}; k_2 = 1_U$. Therefore $\mathbf{1} = k_1; j_2$ (using the equations above).

(2) Similarly, according to the required specification there must be a path from x to Env, and hence we put a path labelled **2** through the box for I. Thus we have the path labelled $j_2; \mathbf{2}; k_1$ from x to Env. Moreover, according to the specification this path must have 1_U as its label; that is, $j_2; \mathbf{2}; k_1 = 1_U$ and hence $\mathbf{2} = k_2; j_1$.

This completes our design for the box I. We have two paths inside the box in parallel with labels $k_1; j_2$ and $k_2; j_1$. We combine this information in a sum, which we call *sum-over-paths* formula. Hence, $I = k_1; j_2 + k_2; j_1$, or with categorical order of composition, $I = j_2 k_1 + j_1 k_2$. The design procedure can be represented compactly in a tabular form: here the first and second columns represent the required specification read from the right-hand side of the equation. The third column represents the design inside the box with unknown labels (**1** and **2**). Finally, the last column provides the last piece of information needed, that is the unknown labels. These can be found by comparing the second and third columns.

Required Specification			
Path	*Label*	*Design*	*Solution*
$Env \longrightarrow x$	1_U	$j_1; \mathbf{1}; k_2$	$\mathbf{1} = k_1; j_2$
$x \longrightarrow Env$	1_U	$j_2; \mathbf{2}; k_1$	$\mathbf{2} = k_2; j_1$

We now verify the correctness of our design for I, namely, we would like to verify $I \cdot x = x$ with $I = j_2 k_1 + j_1 k_2$. We will use a dataflow (particle) argument for this verification as follows. Suppose a particle coming from the environment (see Figure 3(a)) arrives at the left-hand port of the box I. This would be on the path labelled j_1. Now there are two different paths to choose and only one must be chosen. Since the label of the path on which the particle arrives is j_1, and noting that $j_1; k_2 = 0$ (representing a disconnection) and $j_1; k_1 = 1_U$ (representing connectivity with label 1_U), our particle will choose path **1** (inside the box), labelled $k_1; j_2$, and hence will arrive at the right-hand port of the box with a label $j_1; k_1; j_2 = j_2$. At this point the particle is faced with another choice, namely one between paths labelled k_1 and k_2, respectively. However, as the particle has arrived at this point with a label j_2, and since $j_2; k_1 = 0$ represents lack of connectivity, the path labelled k_2 will be chosen, which takes the particle to the entry of box x. The label of the

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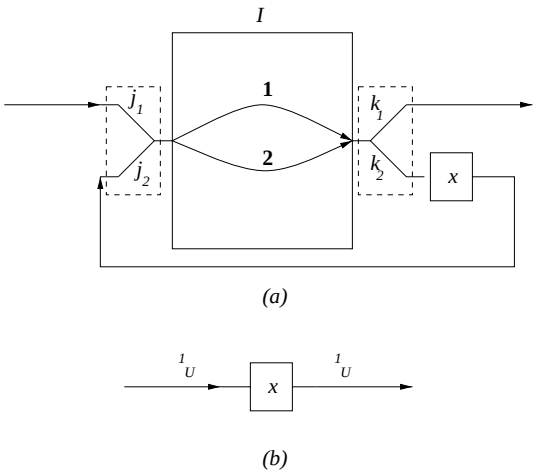


Fig. 3. Identity combinator I

path from *Env* to x that the particle has taken is $j_1;k_1;j_2;k_2 = 1_U$, as we expect from the required specification, (see Figure 3(b)). Our particle now passes through x and finds itself on the exit port of x, and following the path from there leading to the box for I, arrives once again at the left port of the box for I, faced with two choices. This time it chooses path **2** labelled $k_2;j_1$ because $j_2;k_1 = 0$ represents a disconnection and $j_2;k_2 = 1_U$. This way the particle arrives at the right port of the box with a label $j_2;k_2;j_1 = j_1$. This time the way to go will be via the path labelled k_1, which will lead the particle to *Env* with a label $j_1;k_1 = 1_U$, also in agreement with the required specification. This completes our verification process and establishes the correctness of our design.

Combinator K:

K must satisfy the equation $K \cdot x \cdot !y = x$. Viewing the right-hand side of this equation as the required specification for the design of K, and the left-hand side of the equation as the given circuitry with box K to be designed, we obtain the following information:

Required Specification			
Path	*Label*	*Design*	*Solution*
$Env \longrightarrow x$	1_U	$j_1;j_1;\mathbf{1};k_2$	$\mathbf{1} = k_1;k_1;j_2$
$x \longrightarrow Env$	1_U	$j_2;\mathbf{2};k_1;k_1$	$\mathbf{2} = k_2;j_1;j_1$

Using reasoning similar to the case of combinator I, we find that $K = j_2k_1^2 + j_1^2k_2$. We verify the equation of K by following a particle coming from *Env* through the box. Such a particle will necessarily follow the path labelled $j_1;j_1$, and will choose path **1**, labelled $k_1;k_1;j_2$, because $j_1;k_2 = 0$ and $j_1;k_1 = 1_U$. At this point the particle arrives at the junction of k_1 and k_2, and since $j_2;k_2 = 1_U$ and $j_2;k_1 = 0$, it will choose the path labelled by k_2 leading to x. This results in a path with the label $j_1;j_1;k_1;k_1;j_2;k_2 = 1_U$, yielding the desired specification for the path from *Env* to x. Upon exiting the box

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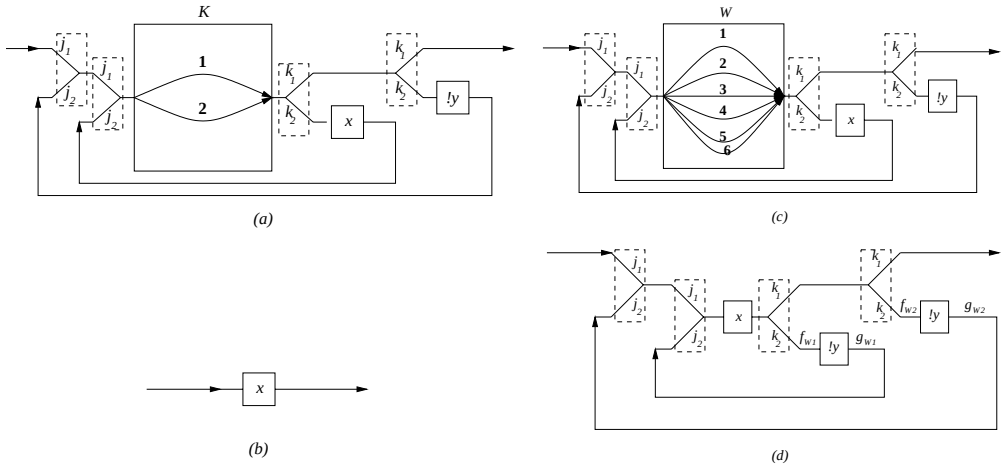


Fig. 4. Weakening and contraction combinators K and W

x, the particle arrives on the path labelled by j_2, and by the same reasoning chooses path **2** and subsequently $k_1;k_1$, exiting to the environment, with the resulting label being $j_2;k_2;j_1;j_1;k_1;k_1 = 1_U$, as required by the specification.

Combinator W:

W must satisfy the equation $W{\cdot}x{\cdot}!y = x{\cdot}!y{\cdot}!y$. Proceeding as in the previous cases:

Required Specification		*Design*	*Solution*
Path	*Label*		
$Env \longrightarrow x$	$j_1;j_1$	$j_1;j_1;\mathbf{1};k_2$	$\mathbf{1} = k_1;k_1;j_1;j_1;j_2$
$x \longrightarrow !y$ (first copy)	$k_2;f_{W1}$	$j_2;\mathbf{2};k_1;k_2$	$\mathbf{2} = k_2;k_2;f_{W1};j_2;j_1$
$x \longrightarrow !y$ (second copy)	$k_1;k_2;f_{W2}$	$j_2;\mathbf{3};k_1;k_2$	$\mathbf{3} = k_2;k_1;k_2;f_{W2};j_2;j_1$
$!y$ (first copy) $\longrightarrow x$	$g_{W1};j_2$	$j_2;j_1;\mathbf{4};k_2$	$\mathbf{4} = k_1;k_2;g_{W1};j_2;j_2$
$!y$ (second copy) $\longrightarrow x$	$g_{W2};j_2;j_1$	$j_2;j_1;\mathbf{5};k_2$	$\mathbf{5} = k_1;k_2;g_{W2};j_2;j_1;j_2$
$x \longrightarrow Env$	$k_1;k_1$	$j_2;\mathbf{6};k_1;k_1$	$\mathbf{6} = k_2;k_1;k_1;j_1;j_1$

The first copy of $!y$ is the one on the left in Figure 4(d). Hence $W = j_2j_1^2k_1^2 + j_1j_2f_{W1}k_2^2 + j_1j_2f_{W2}k_2k_1k_2 + j_2^2g_{W1}k_2k_1 + j_2j_1j_2g_{W2}k_2k_1 + j_1^2k_1^2k_2$. Where $f_{W1} = uc_U(v \otimes v)\imath_1$, $f_{W2} = uc_U(v \otimes v)\imath_2$, $g_{W1} = \rho_1(u \otimes u)c'_Uv$, and $g_{W2} = \rho_2(u \otimes u)c'_Uv$. The verification is done similar to the previous two cases.

Similarly, we define the remaining combinators:

$B = j_2j_1k_1^3 + j_1^3k_1k_2 + j_2^2k_1k_2k_1 + j_1j_2j_1k_2^2 + j_1^2j_2k_2^2k_1 + j_1j_2^2k_2k_1^2$

$C = j_2j_1^2k_1^3 + j_1^3k_1^2k_2 + j_1j_2k_2k_1k_2 + j_2j_1j_2k_2k_1 + j_1^2j_2k_2^2 + j_2^2k_2k_1^2$

$D = j_2j_1k_1^2 + j_1^2k_1k_2 + j_1j_2f_Dk_2^2 + j_2^2g_Dk_2k_1$,

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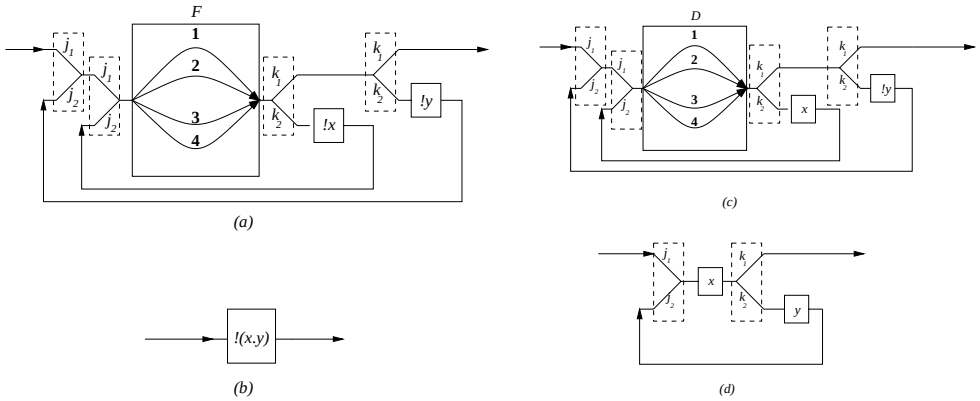


Fig. 5. Functoriality and dereliction combinators F and D

$\delta = j_2 f_\delta k_1 + j_1 g_\delta k_2$, $F = j_2 f_{F1} k_1^2 + j_1^2 g_{F1} k_2 + j_1 j_2 g_{F2} k_2 + j_2 f_{F2} k_2 k_1$ (see Figure 5 for F and D).

Here $f_D = u l_U, g_D = l'_U v, f_\delta = u e_U T(v) v, g_\delta = u T(u) e'_U v$, $f_{F1} = u T(j) \varphi(v \otimes v) \iota_1$, $g_{F1} = \rho_1(u \otimes u) \varphi^{-1} T(k) v$, $f_{F2} = u T(j) \varphi(v \otimes v) \iota_2$, $g_{F2} = \rho_2(u \otimes u) \varphi^{-1} T(k) v$, and φ is the component of the monoidal functor T. Note that the maps f_- and g_- defined above are used in encodings and decodings required in the definition of the combinators.

Theorem 4.2. Let $\mathscr{C}$ be a UDC, T an endofunctor on $\mathscr{C}$, and U an object in $\mathscr{C}$ such that $(\mathscr{C}, T, U)$ is a GoI Situation. Let the distinguished elements $I, B, C, K, W, \delta, D, F$ of $\mathscr{C}(U, U)$ be as defined above. Then

(i) $(\mathscr{C}(U,U), \cdot, !)$ is an LCA.
(ii) Combinators respect the dataflow (particle) semantics above.

Proof.

The summability of terms used in the definition of combinators follows from their correspondence to morphisms in the model category. For example, consider

$$j \sigma k = \begin{bmatrix} j_1 & j_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = j_2 k_1 + j_1 k_2 = I.$$

and similarly for other combinators (Haghverdi 1999).

The verification procedure in this section constitutes a proof. However, a purely algebraic proof can also be given. Here we show it for I, and for details refer the reader to Haghverdi (1999). We would like to prove that $I \cdot x = x$ for all $x \in \mathscr{C}(U, U)$ with I given as $I = j_2 k_1 + j_1 k_2$. Now $I \cdot x = Tr^U_{U,U} \left(\begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) = Tr^U_{U,U} \left(\begin{bmatrix} 0 & 1 \\ x & 0 \end{bmatrix} \right) =$ $0 + x = x$. □

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5. Examples and computations

In this section we consider the UDC **PInj** and give a concrete GoI Situation and investigate the combinators. This is followed by a brief look at the PAC examples **Pfn**, $\mathbf{Rel}_+$, and **SRel**. We will use $\mathbb{N}$ and $\mathbb{C}$ to denote the set of natural and complex numbers, respectively.

5.1. PInj

Consider the category **PInj** of sets and partial injective functions. We have already seen that **PInj** is a UDC. It can be easily checked that $(\mathbf{PInj}, \mathbb{N} \times -, \mathbb{N})$ is a GoI Situation.

— $\mathbb{N} + \mathbb{N} \lhd \mathbb{N}(j,k)$ is given as follows: $j : \mathbb{N} + \mathbb{N} \longrightarrow \mathbb{N}, j(1,n) = 2n, j(2,n) = 2n+1$ and $k : \mathbb{N} \longrightarrow \mathbb{N} + \mathbb{N}, k(n) = (1, n/2)$ for n even, and $(2,(n-1)/2)$ for n odd. Define $j_1 = j\iota_1, j_2 = j\iota_2, k_1 = \rho_1 k, k_2 = \rho_2 k$ where ρ_i and ι_i are as defined in Example 2.7.

— $\mathbb{N} \times \mathbb{N} \lhd \mathbb{N}(u,v)$ is defined as: $u(m,n) =< m,n >= \frac{(m+n+1)(m+n)}{2}$ (Cantor surjective pairing) and v as its inverse, $v(n) = (n_1, n_2)$ with $< n_1, n_2 >= n$.

— The maps f_{Wi}, g_{Wi} for $i = 1,2$ are defined as follows:
Let $n =< n_1, n_2 >$, $f_{W1}(n) =< 2n_1, n_2 >$, $f_{W2}(n) =< 2n_1+1, n_2 >$, $g_{W1}(n) =< n_1/2, n_2 >$ if n_1 is even, and undefined otherwise, and finally $g_{W2}(n) =< (n_1-1)/2, n_2 >$ for n_1 odd and undefined otherwise.

— Recall that $I = j_2k_1 + j_1k_2$. Hence, $I(n) = n+1$ for n even and $n-1$ for n odd.

— $K = j_2k_1^2 + j_1^2k_2$. Hence, $K(n) = n/2 + 1$ if $n \overset{4}{\equiv} 0$, $2n-2$ if n is odd and undefined otherwise.

— $W = j_2j_1^2k_1^2 + j_1j_2f_{W1}k_2^2 + j_1j_2f_{W2}k_2k_1k_2 + j_2^2g_{W1}k_2k_1 + j_2j_1j_2g_{W2}k_2k_1 + j_1^2k_1^2k_2$, with f_{W1}, f_{W2}, g_{W1}, and g_{W2} as above.

$$W(n) = \begin{cases} (2n+1) & \text{if } n \overset{4}{\equiv} 0, \\ 4(< m_1/2, m_2 >) + 3 & \text{if } n \overset{4}{\equiv} 2,\ m_1 \text{ even},\ (n-2)/4 =< m_1, m_2 >, \\ 8(< (m_1-1)/2, m_2 >) + 5 & \text{if } n \overset{4}{\equiv} 2,\ m_1 \text{ odd},\ (n-2)/4 =< m_1, m_2 >, \\ (n-1)/2 & \text{if } n \overset{8}{\equiv} 1, \\ 4(< 2m_1+1, m_2 >) + 2 & \text{if } n \overset{8}{\equiv} 5,\ (n-5)/8 =< m_1, m_2 >, \\ 4(< 2m_1, m_2 >) + 2 & \text{if } n \overset{4}{\equiv} 3,\ (n-3)/4 =< m_1, m_2 >, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

We next consider a computation. Suppose we would like to evaluate $K\cdot x\cdot !y$. We first consider $kKj = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \begin{bmatrix} j_2k_1^2 + j_1^2k_2 \end{bmatrix} \begin{bmatrix} j_1 & j_2 \end{bmatrix} = \begin{bmatrix} 0 & j_1 \\ k_1 & 0 \end{bmatrix}$. Now $K\cdot x = Tr\left(\begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix} \begin{bmatrix} 0 & j_1 \\ k_1 & 0 \end{bmatrix}\right) = Tr\left(\begin{bmatrix} 0 & j_1 \\ xk_1 & 0 \end{bmatrix}\right) = 0 + \sum_{n=0}^{\infty} j_1 0^n x k_1 = j_1 x k_1$.

Next consider $k(K\cdot x)j = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \begin{bmatrix} j_1xk_1 \end{bmatrix} \begin{bmatrix} j_1 & j_2 \end{bmatrix} = \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}$.

Now $K\cdot x\cdot !y = (K\cdot x)\cdot !y = Tr\left(\begin{bmatrix} 1 & 0 \\ 0 & !y \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\right) = Tr\left(\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\right) = x + 0 = x$.

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5.2. **Pfn**

Recall that **Pfn** is a PAC and hence is a traced UDC. The standard trace formula for **Pfn** can be shown to be the same as the trace formula for **Pfn** given in Abramsky (1996). $(\mathbf{Pfn}, \mathbb{N} \times -, \mathbb{N})$ forms a GoI Situation and the choice of morphisms made above for **PInj** works for **Pfn**.

5.3. $\mathbf{Rel}_+$

Recall that the category $\mathbf{Rel}_+$ of sets and binary relations is a PAC (see Example 2.23.). It can be easily shown that $(\mathbf{Rel}_+, \mathbb{N} \times -, \mathbb{N})$ forms a GoI Situation. In particular, the morphisms in the case of **PInj** are relations, and hence they work for this case too.

5.4. **SRel**

The category **SRel** of stochastic relations is another example of a PAC. We define $T : \mathbf{SRel} \longrightarrow \mathbf{SRel}$ as $T(X, \mathscr{F}_X) = (\mathbb{N} \times X, \mathscr{F}_{\mathbb{N} \times X})$ where $\mathscr{F}_{\mathbb{N} \times X}$ is the σ-field on $X \uplus X \uplus X \cdots$ (ω copies). For a given $f : (X, \mathscr{F}_X) \longrightarrow (Y, \mathscr{F}_Y)$, $Tf((n, x), \biguplus B_i) = f(x, B_n)$. Let $U = \mathbb{N}^\infty$ with the Baire metric, which induces a topology on $\mathbb{N}^\infty$ that corresponds to the product topology obtained from $\mathbb{N}$ with discrete topology. It can be shown that $(\mathbf{SRel}, T, \mathbb{N}^\infty)$ forms a GoI situation. Here the morphisms j, k, u and v are defined as follows. $j : \mathbb{N}^\infty + \mathbb{N}^\infty \longrightarrow \mathbb{N}^\infty$ is defined by $j((L, \vec{n}), A) = \delta(0.\vec{n}, A)$, $j((R, \vec{n}), A) = \delta(1.\vec{n}, A)$, and $k : \mathbb{N}^\infty \longrightarrow \mathbb{N}^\infty + \mathbb{N}^\infty$ is defined by $k(0.\vec{n}, A_L \uplus A_R) = \delta(\vec{n}, A_L)$, $k(1.\vec{n}, A_L \uplus A_R) = \delta(\vec{n}, A_R)$, $k(l.\vec{n}, A) = 0$ for all $l \geqslant 2$. Also, $u : \mathbb{N} \times \mathbb{N}^\infty \longrightarrow \mathbb{N}^\infty$ is defined by $u((i, \vec{n}), A) = \delta(i.\vec{n}, A)$, and $v : \mathbb{N}^\infty \longrightarrow \mathbb{N} \times \mathbb{N}^\infty$ by $v((i.\vec{n}), \uplus_i A_i) = \delta(\vec{n}, A_i)$. More details can be found in Haghverdi (1999) and Abramsky *et al.* (1999).

5.5. $\mathbf{Hilb}_2$

This example will provide the connection to operator algebraic models. Given a set X, let $\ell_2(X)$ be the set of all complex valued functions a on X for which the (unordered) sum $\sum_{x \in X} |a(x)|^2$ is finite. $\ell_2(X)$ is a Hilbert space and its norm is given by $||a|| = (\sum_{x \in X} |a(x)|^2)^{1/2}$ and its inner product is given by $< a, b >= \sum_{x \in X} a(x)\overline{b(x)}$ for $a, b \in \ell_2(X)$ (Kadison and Ringrose 1983).

Barr observed that this construction can be made into a functor ℓ_2 (Barr 1992). There is a contravariant faithful functor $\ell_2 : \mathbf{PInj}^{op} \longrightarrow \mathbf{Hilb}$ where **Hilb** is the category that has Hilbert spaces as objects and linear contractions (norm $\leqslant 1$) as morphisms. For a set X, we have $\ell_2(X)$ is defined as above, and given $f : X \longrightarrow Y$ in **PInj**, we have $\ell_2(f) : \ell_2(Y) \longrightarrow \ell_2(X)$ is defined by

$$\ell_2(f)(b)(x) = \begin{cases} b(f(x)) & \text{if } x \in Dom(f), \\ 0 & \text{otherwise.} \end{cases}$$

Hence, this gives a correspondence between partial injective functions and partial isometries on Hilbert spaces (see also Girard (1995) and Abramsky (1996)).

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It can be shown that $\ell_2(X \times Y) \cong \ell_2(X) \otimes \ell_2(Y)$, where $\otimes$ is the tensor product of Hilbert spaces (see Kadison and Ringrose (1983, page 142) for a proof). Also $\ell_2(X \uplus Y) \cong \ell_2(X) \oplus \ell_2(Y)$ where $\oplus$ is the direct sum of Hilbert spaces. To see this, let $a \in \ell_2(X)$ and $b \in \ell_2(Y)$, define $q_{a,b} : X \uplus Y \longrightarrow \mathbb{C}$ by $q_{a,b}(x,1) = a(x)$ and $q_{a,b}(y,2) = b(y)$, clearly, $q_{a,b} \in \ell_2(X \uplus Y)$. The linear transformation $U : \ell_2(X \uplus Y) \longrightarrow \ell_2(X) \oplus \ell_2(Y)$ given by $U(q_{a,b}) = (a,b)$ is an isomorphism.

Consider the category $\mathbf{Hilb}_2$ whose objects are of the form $\ell_2(X)$ for a set X and a morphism $u : \ell_2(X) \longrightarrow \ell_2(Y)$ is of the form $\ell_2(f)$ for some partial injective function $f : Y \longrightarrow X$ (recall that ℓ_2 is contravariant). Hence $\mathbf{Hilb}_2$ is a subcategory of **Hilb**.

For $\ell_2(X)$ and $\ell_2(Y)$ in $\mathbf{Hilb}_2$, the Hilbert space tensor product $\ell_2(X) \otimes \ell_2(Y)$ yields a tensor product in $\mathbf{Hilb}_2$. This follows from the isomorphism $\ell_2(X) \otimes \ell_2(Y) \cong \ell_2(X \times Y)$ and the fact that $X \times Y$ is a tensor product in **PInj**.

Similarly, for $\ell_2(X)$ and $\ell_2(Y)$ in $\mathbf{Hilb}_2$, the direct sum $\ell_2(X) \oplus \ell_2(Y)$ yields a tensor product in $\mathbf{Hilb}_2$. This follows from the isomorphism $\ell_2(X) \oplus \ell_2(Y) \cong \ell_2(X \uplus Y)$ and the fact that $X \uplus Y$ is a tensor product in **PInj**. Notice, however, that although $\ell_2(X) \oplus \ell_2(Y)$ is the direct sum (biproduct) of the Hilbert spaces $\ell_2(X)$ and $\ell_2(Y)$ in **Hilb**, it is only a tensor product in $\mathbf{Hilb}_2$ as otherwise this would imply that $X \uplus Y$ is the coproduct in **PInj**, which is a contradiction.

The structure on **PInj** makes $\mathbf{Hilb}_2$ into a UDC as follows. We define a sum for operators in $\mathbf{Hilb}_2(\ell_2(X), \ell_2(Y))$. Given a family $\{\ell_2(f_i)\}_I \in \mathbf{Hilb}_2(\ell_2(X), \ell_2(Y))$ with $\{f_i\} \in \mathbf{PInj}(Y, X)$, we say that $\{\ell_2(f_i)\}$ is summable if $\{f_i\}$ is summable in **PInj** and in that case $\sum_i \ell_2(f_i) \stackrel{def}{=} \ell_2(\sum_i f_i)$. Clearly this definition makes ℓ_2 an additive functor.

Moreover, $\mathbf{Hilb}_2$ is traced. Given $u : \ell_2(X) \oplus \ell_2(U) \longrightarrow \ell_2(Y) \oplus \ell_2(U)$, $Tr(u) = \ell_2(Tr^U_{Y,X}(f))$ where $u = \ell_2(f)$ with $f : Y \uplus U \longrightarrow X \uplus U$.

We close this section by a discussion of Abramsky and Jagadeesan's construction in Abramsky and Jagadeesan (1994). The construction in Abramsky and Jagadeesan (1994) uses subcategories of the category **Predom** of ω-complete posets and continuous maps, called $\mathcal{GI}$-categories, to construct a model of linear logic that is a *-autonomous category with products. Also one defines a functor T_D :**Dom**$\longrightarrow$ **Dom** by $T_D(E) = (D+1+(E \times E))_\perp$. Here **Dom** is the category of domains and continuous maps. For every D, the functor T_D has an initial algebra (hence a fixed-object), which is denoted by $(\mathcal{T}D, fold_D)$ where $fold_D : (D+1+(\mathcal{T}D \times \mathcal{T}D))_\perp \longrightarrow \mathcal{T}D$ is an isomorphism, whose inverse we call $unfold_D$.

Next, a type-free $\mathcal{GI}$-model is used to give an interpretation of geometry of interaction. A type-free model consists of an object D in a $\mathcal{GI}$-category such that it is a domain and satisfies some retractions analogous to those in Definition 4.1 (3).

This construction does not fit into a traced UDC based model. Notice that although the category **Dom** is a traced symmetric monoidal category, the trace is not of the canonical form used in traced UDCs; rather, it is induced by the fixed-point operator in **Dom**. However, we can show that (**Dom**, T, D) forms a GoI Situation, where D is a type-free $\mathcal{GI}$-model and T :**Dom**$\longrightarrow$**Dom** is defined by $T(A) = \mathcal{T}A$, that is, it sends a domain A to the fixed-object $\mathcal{T}A$ of the functor T_A. Given $f : A \longrightarrow B$, $T(f) = fold_B \circ (f + 1 + (\varphi \times \varphi))_\perp \circ unfold_A$ where φ is the unique map from $(\mathcal{T}A, fold_A)$ to $(\mathcal{T}B, \delta)$ (recall that $(\mathcal{T}A, fold_A)$ is an initial algebra.)

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The necessary retractions are given in Abramsky and Jagadeesan (1994) (see pages 66 and 68). We just need to prove $TT \lhd T\,(e, e')$ (Comultiplication).

Define $e_A : TT(A) \longrightarrow T(A) = \alpha \circ [1, \bot, \bot]_\bot \circ unfold$ and $e'_A : T(A) \longrightarrow TT(A) = fold \circ up \circ in_1$ where $((-)_\bot, up, \mu, t)$ is Moggi's (Moggi 1991) strong monad of lifting and given a domain A, $\alpha_A : A_\bot \longrightarrow A$ is its structure map.

6. Dynamic algebra

We present a categorical model for dynamic algebras and show that the existing *small model* in the literature is a special case of this categorical model. The definition of dynamic algebra given below follows Mackie (1994) and Danos (1990).

Definition 6.1. A *Dynamic Algebra*, denoted by Λ^*, is a single sorted Σ-algebra (L, Σ). We write x and y for the variables, and $0, 1, p, q, r, s, t, d \in \Sigma$ for the constants of the theory. There is an associative multiplication operator $\cdot : L \times L \longrightarrow L$ (which we will omit), which has unit 1 and absorbing element 0. The theory is equipped with an involution $* : L \longrightarrow L$ and an exponential operator $! : L \longrightarrow L$. The following equations define the properties that we require:

$$\begin{array}{cc} 0^* = !(0) = 0 & 1^* = !(1) = 1 \\ 0x = x0 = 0 & 1x = x1 = x \\ !(x)^* = !(x^*) & (xy)^* = y^*x^* \\ (x^*)^* = x & !(x)!(y) = !(xy). \end{array}$$

Annihilation There are six constants to consider: p and q are the multiplicative coefficients, r and s are the contraction coefficients, d is the dereliction coefficient, and t is the side-door coefficient. The following are the annihilation equations:

$$\begin{array}{cc} p^*p = q^*q = 1 & q^*p = p^*q = 0 \\ r^*r = s^*s = 1 & s^*r = r^*s = 0 \\ d^*d = 1 & t^*t = 1. \end{array}$$

Communication The exponential coefficients (r, s, t, d) interact with the exponential operator as follows:

$$\begin{array}{cc} !(x)r = r!(x) & !(x)s = s!(x) \\ !(x)t = t!!(x) & !(x)d = dx. \end{array}$$

The terms built out of this theory are called *monomials*. To write polynomials we need the notion of a sum of monomials. We denote this formal sum by $+$ and give the equations for it:

$$\begin{array}{cc} x + y = y + x & x + 0 = x \\ (x + y)z = (xz + yz) & z(x + y) = (zx + zy) \\ (x + y)^* = x^* + y^* & !(x + y) = !x + !y. \end{array}$$

Definition 6.2 (Dynamic situation). A *dynamic situation* $\mathscr{S}$ is a tuple $(\mathscr{C}, T, U, (-)^*)$ where

1. $\mathscr{C}$ is a traced UDC,
2. $T : \mathscr{C} \longrightarrow \mathscr{C}$ is an additive functor,

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3. $(-)^* : \mathscr{C}^{op} \longrightarrow \mathscr{C}$ is an additive functor that is strictly involutive and the identity on objects[†],
4. U is an object of $\mathscr{C}$,

such that the following conditions hold:

- $(\mathscr{C}, T, U)$ forms a GoI Situation, with all the retractions of the form (f, f^*),
- $TU \cong U$,
- $T(-)^* = (-)^* T$ amd $\iota_i^* = \rho_i$ for all $i \in I$ and finite set I.

Theorem 6.3. A dynamic situation $\mathscr{S}$ is a (sound) model of Λ^*.

Proof. We let the carrier of the model be $\mathscr{C}(U, U)$. The constants of Λ^* are interpreted as follows (see Definition 4.1 for the morphisms used below):
$[\![0]\!] = 0_{UU}$ where 0_{UU} is the zero morphism of the Σ-Monoid $\mathscr{C}(U, U)$, $[\![1]\!] = 1_U$, $[\![p]\!] = j_1$, $[\![q]\!] = j_2$, $[\![r]\!] = uc_U\iota_1 v$, $[\![s]\!] = uc_U\iota_2 v$, $[\![t]\!] = ue_U T(v)v$, $[\![d]\!] = ul_U$. $[\![x.y]\!] = [\![x]\!][\![y]\!]$, $[\![x^*]\!] = [\![x]\!]^*$, $[\![!x]\!] = uT([\![x]\!])v$ and the formal sum is interpreted by the sum on the hom-set $\mathscr{C}(U, U)$.

The first and the last group of equations in Definition 6.1 follow from the additivity and functoriality of T and $(-)^*$ and the fact that they commute, together with the properties of the additive structure on $\mathscr{C}(U, U)$. The annihilation and communication equations follow from the properties of maps and naturality of natural transformations in Definition 4.1. The details can be found in Haghverdi (1999). □

Note that the formal sum in a dynamic algebra gets interpreted by a partial sum. However, this is enough in all the applications as demonstrated by the case of the *small model* in Malacaria and Regnier (1991).

Proposition 6.4. $(\mathbf{PInj}, \mathbb{N} \times -, \mathbb{N}, (-)^{-1})$ is a dynamic situation.

This model is known as the *small model* in Danos (1990) and Regnier (1992).

By using the category $\mathbf{Hilb}_2$ as the underlying category, and the functor ℓ_2, and by taking the $(-)^*$ to be the adjoint operation, we get the following result.

Proposition 6.5. $(\mathbf{Hilb}_2, \ell_2(\mathbb{N}) \otimes -, \ell_2(\mathbb{N}), (-)^*)$ is a dynamic situation.

This model corresponds to Girard's original operator theoretic model of a dynamic algebra presented in Girard (1989a).

Proposition 6.6. $(\mathbf{Rel}_+, \mathbb{N} \times -, \mathbb{N}, (-)^{-1})$ is a dynamic situation.

Note that we can relax the condition on $(-)^*$, since this operation is applied to terms of a dynamic algebra that are built from the constants using the binary and unary operations. Hence it would be sufficient to define $(-)^*$ on the constants of the theory.

Consider the subset $S = \{0, 1, \iota_1, \iota_2, \rho_1, \rho_2, j_1, j_2, k_1, k_2, u, v, e_U, e'_U, l_U, l'_U, c_U, c'_U\}$ of $\mathscr{C}(U, U)$. We define an operation $(-)^*$ on the elements of S as follows: $0^* = 0, 1^* = 1, \iota_1^* = \rho_1, \iota_2^* = \rho_2, \rho_1^* = \iota_1, \rho_2^* = \iota_2, j_1^* = k_1,\ j_2^* = k_2, k_1^* = j_1, k_2^* = j_2, u^* = v, v^* = u, e_U^* =$

[†] A category $\mathscr{C}$ with such a functor is known as a *-category (Abramsky *et al.* 1997).

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$e'_U, (e'_U)^* = e_U$, $l^*_U = l'_U, (l'_U)^* = l_U, c^*_U = c'_U$ and $(c'_U)^* = c_U$. Clearly $(f^*)^* = f$ for every $f \in S$. The operation $(-)^*$ can be extended to terms (finite words) generated on S using the composition operation by an inductive definition: $(\alpha f)^* = f^*\alpha^*$ for any term α and any $f \in S$. We also define $T(f)^* = T(f^*)$ for all $f \in S$ and $(\alpha + \beta)^* = \alpha^* + \beta^*$ for all terms α and β provided all the sums exist.

$[\![x^*]\!]$ is defined inductively as follows: $[\![a^*]\!] = [\![a]\!]^*$ for all $a \in \{0, 1, p, q, r, s, t, d\}$ where the * on the right-hand side is the one defined above. Hence we have, $[\![0^*]\!] = 0_{UU}$ and $[\![1^*]\!] = 1_U$, $[\![p^*]\!] = k_1$, $[\![q^*]\!] = k_2$, $[\![r^*]\!] = u\rho_1 c'_U v$, $[\![s^*]\!] = u\rho_2 c'_U v$, $[\![t^*]\!] = uT(u)e'_U v$, $[\![d^*]\!] = l'_U v$.

With $(-)^*$ defined as given above we have the following propositions.

Proposition 6.7. $(\mathbf{Pfn}, \mathbb{N} \times -, \mathbb{N}, (-)^*)$ is a dynamic situation.

Proposition 6.8. $(\mathbf{SRel}, \mathbb{N} \times -, \mathbb{N}^\infty, (-)^*)$ is a dynamic situation.

7. Conclusion and work in progress

In this paper we have studied the dataflow computational aspects of Abramsky's analysis of Girard's GoI programme. We have shown that UDCs are a suitable categorical framework for such a computational analysis. Furthermore, such categories were used to construct categorical models of dynamic algebras. Currently, we are studying the relationship between different UDC-based models and the use of the sum-over-paths formula in characterising strongly normalising terms. In general we are trying to relate the properties of λ-terms to properties of the sum obtained as a result of the standard translation to combinatory logic. Moreover, we are working on the GoI programme in the categorical models described in this paper. This includes the interpretation of Girard's, and Danos and Regnier's work within categorical models and, specifically, relating the notion of path in a proof net to the paths in our models.

We have presented a categorical model based on the category of Hilbert spaces, namely the image of the ℓ_2 functor. We are currently investigating other possibilities on categories that appear in functional analysis, for example categories of Banach spaces such as $\mathbf{Ban}_1$ of real Banach spaces and linear maps with norm less than or equal to 1, and $\mathbf{Ban}_\infty$ of real Banach spaces and linear bounded maps. This would help to provide a functional analytic analysis of combinators in particular and β-reduction in general, and to deepen our understanding of cut elimination/proof normalization processes by making use of techniques and structures in operator theory.

More interestingly, we encounter a partial trace operator (Abramsky *et al.* 1997) for **Hilb**. The existence conditions for trace when related to the properties of the λ-terms under consideration is expected to shed new light on the mathematical analysis of β-reduction.

The geometry of interaction provides a suitable semantics basis for explaining and improving Lamping's optimal graph-reduction implementation of the λ-calculus. The connections have been explained in Gonthier *et al.* (1992). The investigation of connections and relations between optimal reduction strategies and geometry of interaction offers ideas for efficient and correct implementations. An exciting research direction is the application of the ideas in this paper to the analysis of optimal reduction strategies.

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In particular, recent work of Asperti and Mairson (Asperti and Mairson 1998) offers interesting examples amenable to the token pushing analysis developed in this article. We intend to examine this further in future articles. We thank the anonymous referee for directing our attention to this article.

Appendix A.

Proof of Lemma 2.10. Let f be as in the statement of the lemma. Then we can represent f as a directed graph where the vertices are labelled by X and U, and the edges are labelled by the components of the matricial representation of f, see the figure below. With this representation, f^n corresponds to the matrix that represents the paths of length n between the vertices in the graph, and $\sum_{n\in\omega} f^n$ is the sum over all paths of length n for all $n \in \omega$.

The components of $\sum_{n\in\omega} f^n$ can be obtained by simply following the edges on the graph in the figure below.

- $X \longrightarrow X, \quad E = \sum_{m\in\omega}(A + \sum_{n\in\omega} BD^nC)^m$
- $X \longrightarrow U, \quad G = \sum_{l\in\omega} D^lCE$
- $U \longrightarrow X, \quad F = \sum_{l\in\omega} EBD^l$
- $U \longrightarrow U, \quad H = \sum_{l\in\omega} D^l + \sum_{p,q\in\omega} D^pCEBD^q$

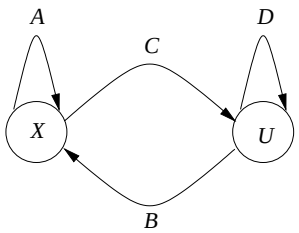


Fig. 6. Graphical representation of f

□

Proposition A.1. The set of axioms for traced symmetric monoidal categories given in Joyal *et al.* (1996) and Abramsky (1996) is equivalent to one where superposing is replaced by the following axiom.

Superposing:

$$g \otimes Tr^U_{X,Y}(f) = Tr^U_{W\otimes X, Z\otimes Y}(g \otimes f)$$

for $f : X \otimes U \longrightarrow Y \otimes U$ and $g : W \longrightarrow Z$.

Proof. Let $\mathscr{C}$ be a symmetric monoidal category with a trace operator subject to naturality, dinaturality, vanishing and yanking axioms. Then the following are equivalent:

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(1) $\quad Tr^U_{X,Y}(f) \otimes g = Tr^U_{X\otimes W, Y\otimes Z}((1_Y \otimes \sigma_{U,Z})(f \otimes g)(1_X \otimes \sigma_{W,U}))$

(2) $\quad g \otimes Tr^U_{X,Y}(f) = Tr^U_{W\otimes X, Z\otimes Y}(g \otimes f)$

for $f : X \otimes U \longrightarrow Y \otimes U$ and $g : W \longrightarrow Z$.

(1) $\Longrightarrow$ (2):

$$\begin{aligned}
g \otimes Tr^U_{X,Y}(f) &= \sigma_{Y,Z}(Tr^U_{X,Y}(f) \otimes g)\sigma_{W,X} \\
&= \sigma_{Y,Z}(Tr^U_{X\otimes W, Y\otimes Z}((1_Y \otimes \sigma_{U,Z})(f \otimes g)(1_X \otimes \sigma_{W,U})))\sigma_{W,X} \\
&= Tr^U_{W\otimes X, Z\otimes Y}((\sigma_{Y,Z} \otimes 1_U)(1_Y \otimes \sigma_{U,Z})(f \otimes g)(1_X \otimes \sigma_{W,U})(\sigma_{W,X} \otimes 1_U)) \\
&= Tr^U_{W\otimes X, Z\otimes Y}((\sigma_{Y,Z} \otimes 1_U)(1_Y \otimes \sigma_{U,Z})(1_Y \otimes \sigma_{Z,U})(\sigma_{Z,Y} \otimes 1_U)(g \otimes f) \\
&\quad (\sigma_{X,W} \otimes 1_U)(1_X \otimes \sigma_{U,W})(1_X \otimes \sigma_{W,U})(\sigma_{W,X} \otimes 1_U)) \\
&= Tr^U_{W\otimes X, Z\otimes Y}(g \otimes f).
\end{aligned}$$

(2) $\Longrightarrow$ (1):

$$\begin{aligned}
Tr^U_{X,Y}(f) \otimes g &= \sigma_{Z,Y}(g \otimes Tr^U_{X,Y}(f))\sigma_{X,W} \\
&= \sigma_{Z,Y}(Tr^U_{W\otimes X, Z\otimes Y}(g \otimes f))\sigma_{X,W} \\
&= Tr^U_{X\otimes W, Y\otimes Z}((\sigma_{Z,Y} \otimes 1_U)(g \otimes f)(\sigma_{X,W} \otimes 1_U)) \\
&= Tr^U_{X\otimes W, Y\otimes Z}((\sigma_{Z,Y} \otimes 1_U)(\sigma_{Y,Z} \otimes 1_U)(1_Y \otimes \sigma_{U,Z})(f \otimes g) \\
&\quad (1_X \otimes \sigma_{W,U})(\sigma_{W,X} \otimes 1_U)(\sigma_{X,W} \otimes 1_U)) \\
&= Tr^U_{X\otimes W, Y\otimes Z}((1_Y \otimes \sigma_{U,Z})(f \otimes g)(1_X \otimes \sigma_{W,U}))
\end{aligned}$$

□

Acknowledgement

I would like to thank P. J. Scott, R. Blute and T. Giordano for useful discussions, S. Abramsky for introducing me to his categorical analysis of Girard's programme and the anonymous referee for helpful suggestions and comments.

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